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A Nonparametric Dimension Test of the Term Structure

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Javier Gil-Bazo and Gonzalo Rubio

Abstract

In an economy with multiple sources of risk, the short-term interest rate does not capture all the information that determines the conditional distribution of bond yields. This is also true for path-dependent term structure models. In either case, the current short rate level is not a sufficient statistic for the conditional density of future short rates. This paper studies the empirical relevance of both issues from a time-series nonparametric perspective. The analysis is formulated as a test for the dependence of the short rate drift and diffusion on variables other than the short rate, and exploits Ait-Sahalia, Bickel, and Stocker (2001) dimension reduction method. The paper explores the finite sample performance of the method and applies the test to US interest rate data. Results reject a single-factor Markovian model, although conclusions are sensitive to the choice of additional conditioning variables.

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1 Introduction

Equilibrium term structure models typically start by characterizing the dynamic behavior of some latent variable or risk factor in a representative agent economy, and then derive the equilibrium dynamics of the short term interest rate. On the other hand, arbitrage-free models start by specifying the short rate process directly. In either case, assuming some functional form for the market price of risk, it is possible to derive closed form or numerical solutions to the prices of bonds of all maturities as a function of the state in the economy. The first term structure models such as Vasicek (1977), Merton (1973) or Cox, Ingersoll and Ross (1985b) (CIR) considered the existence of a single risk factor in the economy which identifies exactly with the short term interest rate. Although tractable and elegant, the performance of these models is poor. For instance, Brown and Dybvig (1986), Gibbons and Ramaswamy (1993), or Pearson and Sun (1994), show that the CIR model does not simultaneously explain the time-series and cross-section behavior of the yield curve. However, the conclusions of these tests apply only to the specific models considered and therefore do not provide further insight on how the short rate process should be modeled.

Another strand of empirical research takes a different approach. Rather than testing for a given term structure model, it studies the adequacy of *different specifications* for the short rate process in a time-series framework. For instance, Chan, Karolyi, Longstaff, and Sanders (1992) find that the data do not support a linear instantaneous short rate variance, which challenges models in which the diffusion coefficient of the short rate continuous-time process equals the square root of a linear function in the state variable vector (in fact they estimate a constant elasticity of variance of 1.5 rather than 0.5).

Similar in spirit, Ait-Sahalia (1996) points out another potential caveat: the linear drift assumption. He develops a test for short rate models which entails comparing a nonparametric estimate of the short rate marginal density to its parametric counterpart. Ait-Sahalia's test is conveniently based on the long-run rather than the conditional density since the latter is generally not known in closed-form for most diffusions. He rejects all linear-drift short rate models[1].

Finally, Stanton (1997) has proposed a nonparametric method for estimating the short rate process based on the short rate conditional distribution. He shows how to obtain discrete-time approximations to the drift and diffusion functions up to any desired level of accuracy as conditional moments of short rate changes. Using kernel smoothing to estimate conditional moments Stanton's paper confirms that the short rate drift is nonlinear[2].

An important feature of all models considered by Chan, Karolyi, Longstaff and Sanders (1992), Ait-Sahalia (1996), and Stanton (1997) is that the current short rate itself contains all relevant information regarding its conditional distribution. These models therefore implicitly assume that either the short rate process is orthogonal to other risk factors or that the state variable vector contains a single variable. Both assumptions are at odds with the evidence that interest rates (including the short rate) are driven by multiple risk factors (see Litterman and Scheinkman (1991)) and with the vast amount of theoretical multifactor models which allow for the presence of multiple sources of risk in the economy, with the short rate being a function of unobservable risk factors. Nevertheless, it is not so clear that single-factor models should be discarded. On one hand, the empirical performance of multifactor models is mixed (see Dai and Singleton (2000) or De Jong (1999)). On the other hand, these

[1] Pritsker (1998) however has shown that Ait-Sahalia's (1996) test presents poor size properties. Overrejection occurs because of the joint interaction of three effects: the short rate is highly autocorrelated, the test is based on the marginal density, and the test is asymptotic and not enough observations are available.

[2] Although Stanton (1997) finds that the short rate drift appears to be nonlinear, Chapman and Pearson (2000) show that this nonlinearity can be attributed to the sample being truncated at its maximum, which causes the short rate drift to appear well below zero for high short rate levels.

tests are typically based on a specific term structure model -usually of the tractable affine class studied by Duffie and Kan (1996)- so the single-factor hypothesis cannot be rejected independently.

In this paper we extend the "unconstrained" specification approach of the above mentioned papers to test for the empirical validity of the one dimensional assumption. From a time-series viewpoint, this assumption implies that the short rate dynamics is not driven by factors affecting the term structure other than the short rate itself. Since the state of the short rate process -i.e., its conditional density at any point in time- is completely characterized by its drift and diffusion coefficients, testing for the dependence of the short rate process on additional latent variables amounts to testing for the dependence of the short rate drift and diffusion on variables other than the short rate itself[3]. In particular, we test for the dependence of the short rate drift and diffusion on the term spread and a proxy for the curvature of the yield curve, as well as on the short rate level. This is motivated by the evidence that at least three factors -which induce changes in its level, slope, and curvature- drive the yield curve[4].

Another situation in which the current short term interest rate is not a sufficient statistic for the state of the system occurs when the yield curve dynamics depend not only on the current yield curve but also on the whole history of past realizations of the term structure. This is the case for the Heath, Jarrow, Morton (1992) (HJM) model. In fact, Cheyette (1992) and Ritchken-Sankarasubramanian (1995) have identified a particular class of single-factor models within the HJM framework which admit a path-independent two-state variable representation, where the second variable captures all path-dependent information. We therefore propose to test directly whether the short rate conditional mean and volatility are non-Markovian, namely whether they depend on past short rate realizations as well as on the current short rate level.

In order to conduct both tests, a dimension reduction method proposed by Ait-Sahalia, Bickel and Stoker (2001) is employed. The method compares nonparametrically estimated conditional expectations, and hence can be used in conjunction with Stanton's (1997) short rate drift and diffusion estimators. In particular we test for expected short rate changes conditional on the current short rate level departing significantly from expected short rate changes conditional on the current short rate level and other variables. If the null is rejected, then we can conclude that the short rate is not a sufficient statistic to characterize its conditional density. The dependence of the diffusion on additional state variables can be tested in an analogous way. This approach however entails two difficulties:

1. If, as suggested by Chapman and Pearson (2000), Stanton's short rate drift estimator displays spurious nonlinearities at high current short rate levels, then the test properties in finite samples could depart from the asymptotic properties. Nevertheless, simulation work presented in Section 2 shows that the bias in Stanton's drift estimator appears to be small when one is estimating close enough to the center of the data and the true drift is close to linear. Our conclusions are therefore constrained to the short rate level being near the center of the empirical distribution.

2. Although the asymptotic distribution of the test statistic is not affected by serial dependence in the data, the strong persistence found in interest rate series suggests

[3]Note that apart from the problem of modeling the short rate process, whether the diffusion coefficient depends on variables other than the short rate has important implications regarding interest rate risk hedging and the pricing of derivative assets on interest rates. As for the short rate drift, testing for the dependence on a second state variable seems like an important previous step before investigating whether the short rate drift is linear.

[4]This intuition, originally derived from principal component analysis, has found theoretical and empirical support within the affine class of term structure models (see for instance De Jong, 1999).

that Monte Carlo experiments should be carried out in order to evaluate the size and power of the test in this setting.

Our paper is close in spirit to Jeffrey, Linton, and Nguyen (2000). In their paper, the authors construct a test for the restrictions imposed through Ito's lemma on the dynamics of the whole yield curve by the assumption of a single risk factor that follows a Markov diffusion. While our paper shows how to test for the dependence of the short rate dynamics on the rest of the term structure, their test exploits information on the nature of that dependence. In this sense, their approach is potentially more efficient than ours. Furthermore, in their test, the alternative hypothesis is not specified. On the other hand, our test can be extended to other null hypotheses. It may be used, for instance, to test the null hypothesis of a two-factor term structure against a three-factor model, while the null hypothesis in their test is always a univariate Markov diffusion for the risk factor. Another advantage is that the test we propose can be performed independently on the drift or on the diffusion. This is desirable given the lower rate of convergence of the nonparametric drift estimator.

Another closely related paper is Ait-Sahalia (2000). In this paper the author discusses the difficulties of basing a test for the Markov property on:

$$E[(r_{t+\Delta} - r_t)^p \mid r_t, r_{t-\Delta}, \cdots, r_0] = E[(r_{t+\Delta} - r_t)^p \mid r_t]$$

for all $p = 1, 2, ...$ for which these moments exist, where r_s is the interest rate observed at time s. Such a test is what he calls a "simple testing procedure", and is in fact our approach. He argues that the test is likely to fail due to four problems:

i) the null hypothesis may not be rejected, even if false, if relevant lagged values are not included in the unrestricted regression;

ii) we may fail to reject the Markov property hypothesis if the model is not Markov in a multivariate sense;

iii) we may fail to reject the null hypothesis if we do not apply the test to the relevant moments;

iv) we may erroneously reject the null hypothesis if lagged variables proxy for omitted non-linearities in the mean. Moreover, using discretely sampled data to test for non-linearities is not possible if the transition density is not known in closed form.

Ait-Sahalia derives a test based on the property that multi-period Markov transitions can be computed by iterating the one-period transition densities. In practice, the test implies having to integrate numerically a non-parametrically estimated density.

We are fully aware that failure to reject the null hypothesis using our test cannot be taken as conclusive evidence in favor of the single-factor Markov hypothesis. On the other hand, rejection of the null *for at least some additional variable* and *for at least some conditional moment* can be considered as evidence against a single-factor Markov model. Problems i)-iii) can therefore be solved if the relevant variables are chosen in the unrestricted regression. Finally, the fourth problem is solved in our paper since we use fully nonparametric estimators of the conditional moments and therefore we do not need to model the non-linearity explicitly. Hence, our approach is limited by the researcher's ability to choose the appropriate regressors, although it retains the advantage of simplicity.

The rest of the paper is organized as follows: Section 2 introduces the testing method proposed by Ait-Sahalia et al. (2001), and investigates the properties of Stanton's estimator in finite samples as well as the performance of the test; Section 3 presents the empirical results when the test is applied to US interest rate data; And finally Section 4 summarizes and concludes.

2 A dimension reduction test for the term structure

2.1 The method

The term structure being driven by more than one state variable is consistent with the intuition of Litterman and Scheinkman (1991) that at least three factors induce changes in the level, slope, and curvature of the yield curve. If the short rate process is endogenously determined in equilibrium and the state variable vector contains multiple risk factors, then the short rate conditional distribution at any given point in time will also depend on multiple risk factors. Also, because the values of the state variables reflect in the term structure, the term structure will contain information that affects the conditional density of the short rate and is not captured by the short rate level[5]. Consequently, we think it is important to assess the empirical significance of the multi-dimensional nature of the short rate process and we believe this can be accomplished by testing for the dependence of the short rate conditional distribution on information provided by the term structure of interest rates. Since the short rate drift and diffusion functions determine unambiguously its conditional density, we can base the test upon them. In other words, we can test for the null hypotheses that the short rate drift and diffusion functions depend only on the short rate level, against the alternative hypothesis that they depend also on the slope of the term spread or its curvature. We next explain how to combine a nonparametric dimension reduction test with a nonparametric estimator of the short rate drift and diffusion, in order to accomplish this goal.

The test proposed by Ait-Sahalia et al. (2001) uses kernel methods to estimate the regression under the restricted specification and under the unrestricted alternative. The difference between the restricted and the unrestricted kernel regression is then measured via the residual sum of squares.

It should be noted that although the test in principle applies to a data sample of independent and identically distributed observations, the asymptotic distribution of the test statistic is unchanged by serial dependence in the data provided that this is strictly stationary ergodic and the amount of serial dependence in the data decays sufficiently fast[6]. We will assume that this condition holds true for the data set.

If the sample data consists of $Z_i = (Y_i, V_i, W_i)$, $i = 1, ..., N$ the test answers the question of whether the predictor variables V can be omitted from the regression of Y on (W, V). The regression function of Y on (W, V) is defined by

$$m(w, v) \equiv E(Y \mid W = w, V = v) = \frac{\int y f(y, w, v) dy}{f(w, v)} \tag{1}$$

and the regression function of Y on W by

$$M(w) \equiv E(Y \mid W = w) = \frac{\int y f(y, w) dy}{f(w)} \tag{2}$$

These conditional moments may be consistently estimated using the Nadaraya-Watson kernel regression method:

[5]See for instance De Jong (1999) for a theoretical and empirical analysis of the relationship between unobservable risk factors and observable characteristics of the yield curve for affine models.

[6]More technically, if Z_i is the vector of observations at time i, then it must be the case that:

1. The data $\{Z_i; i = 1, ..., N\}$ are strictly stationary and β -mixing with $\beta_N = O(N^{-k}), k > 19/2$.
2. The joint density $f_{1,j}(\cdot, \cdot)$ of (Z_1, Z_{1+j}) exists for all j and is continuous on $(R \times S)^2$.

$$\hat{m}_h(w,v) \equiv \frac{\sum_{i=1}^{N} K_h(w - W_i, v - V_i)Y_i}{\sum_{i=1}^{N} K_h(w - W_i, v - V_i)} \tag{3}$$

$$\hat{M}_H(w) \equiv \frac{\sum_{i=1}^{N} K_H(w - W_i)Y_i}{\sum_{i=1}^{N} K_H(w - W_i)} \tag{4}$$

where $K_h(u) = h^{-d}K(u/h)$ and $K_H(u) = H^{-d}K(u/H)$, d being the dimension of the vector u that measures the distance of the observed regressor data to the design point. The shape of the kernel weights is determined by K, whereas the size of the weights is parameterized by the bandwidth, denoted by h and H[7].

The test statistic is based on the distance, measured in a mean squared error way, between both regression functions or more precisely their estimates. If we define the following statistic:

$$\tilde{\Gamma} \equiv \frac{1}{N} \sum_{i=1}^{N} \left\{ \hat{m}_h(W_i, V_i) - \hat{M}_H(W_i) \right\}^2 A_i \tag{5}$$

where A_i is the value that a weighting function[8] takes for W_i, V_i, then $\tilde{\Gamma}$ is a consistent estimator of the weighted expected squared difference between $m(W,V)$ and $M(W)$. Under the null hypotheses $\tilde{\Gamma}$ is asymptotically zero.

The distribution of the test statistic is derived under standard assumptions about the density functions and the kernel. Specially relevant are those concerning the kernel function and the bandwidth choice:

1. The kernel K is a bounded function on R, symmetric about 0, with $\int \mid K(z) \mid dz < \infty$, $\int K(z)dz = 1$, $\int z^j K(z)dz = 0$ for $1 \le j \le r$. *Further,*

$$r > 3(p+q)/4 \tag{6}$$

where p and q are the dimensions of W and V *respectively.*

2. *As* $N \to \infty$, the unrestricted bandwidth sequence $h = O(N^{-1/\delta})$ *is such that*

$$2(p+q) < \delta < 2r + (p+q)/2 \tag{7}$$

while the restricted bandwidth $H = O(N^{-1/\delta})$ *satisfies*

$$p < \Delta \le 2r + p \tag{8}$$

as well as

$$\delta p/(p+q) \le \Delta < \delta \tag{9}$$

The authors show that under the null hypothesis that V can be omitted from the regression:

[7] More details on kernel smoothing techniques can be found in Härdle (1990).

[8] This weighting function allows us to test goodness-of-fit for particular value ranges and/or avoid technical problems such as the estimation of conditional expectation in areas of low density.

$$\hat{\tau} \equiv \hat{\sigma}_{11}^{-1}(Nh^{(p+q)/2} \cdot \tilde{\Gamma} - h^{-(p+q)/2}\hat{\gamma}_{12} - h^{(q-p)/2}\hat{\gamma}_{22} - h^{(p+q)/2}H^{-p}\hat{\gamma}_{32}) \to N(0,1) \quad (10)$$

where the critical values are calculated in the following way:

$$\hat{\sigma}_{11}^2 = \frac{2C_{11}}{N}\sum\nolimits_{i=1}^{N}\frac{\hat{\sigma}_h^4(W_i,V_i)A_i^2}{\hat{f}_h(W_i,V_i)}, \qquad \hat{\gamma}_{12} = \frac{C_{12}}{N}\sum\nolimits_{i=1}^{N}\frac{\hat{\sigma}_h^2(W_i,V_i)A_i}{\hat{f}_h(W_i,V_i)}$$
$$\hat{\gamma}_{22} = -\frac{2C_{22}}{N}\sum\nolimits_{i=1}^{N}\frac{\hat{\sigma}_h^2(W_i,V_i)A_i}{\hat{f}_H(W_i,V_i)}, \qquad \hat{\gamma}_{32} = \frac{C_{32}}{N}\sum\nolimits_{i=1}^{N}\frac{\hat{\sigma}_H^2(W_i)\tilde{A}_i}{\hat{f}_H(W_i)}$$

with $\hat{\sigma}_h^2(W_i,V_i)$ and $\hat{\sigma}_H^2(W_i)$ being the conditional variances of Y estimated nonparametrically, and,

$$\tilde{A}_i \equiv \frac{\sum_{j=1}^{N} K_H(w-W_j)A_j}{\sum_{j=1}^{N} K_H(w-W_j)}$$

The constants C_{ij} are determined by the choice of kernel. In our application:

$$C_{12} = 1/(2\sqrt{\pi})^2, C_{22} = 1/\sqrt{2\pi}, C_{32} = 1/(2\sqrt{\pi}), C_{11} = 1/(2\sqrt{2\pi})^2$$

With respect to the estimation method for the short rate drift and diffusion, Stanton (1997) considers a diffusion process for the short rate, r_t, which satisfies the stochastic differential equation.

$$dr_t = \mu(r_t)dt + \sigma(r_t)dZ_t$$

where dZ_t is a standard Brownian motion, and shows that first-order approximations to μ and $V \equiv \sigma^2$ can be obtained as the conditional first and second moments of the Euler discretization of the short rate process:

$$\mu = \frac{1}{\Delta}E_t\left[r_{t+\Delta} - r_t\right] + O(\Delta), \quad (11)$$
$$V = \frac{1}{\Delta}E_t\left[(r_{t+\Delta} - r_t)^2\right] + O(\Delta) \quad (12)$$

where Δ is the interval between the times when r_t and $r_{t+\Delta}$ are observed. As the observation frequency increases to infinity, $\Delta \to 0$, the approximations converge to the actual values of the drift and diffusion functions. Finally, the conditional expectations can be estimated using the Nadaraya-Watson method as in (3) or (4).

Assuming that first-order discrete approximations are accurate enough, the hypothesis that the short rate drift/diffusion at any given point in time depends on the short rate, versus the hypothesis that it depends on both the short rate level and a second state variable[9], S_t, can be formulated in the following terms:

[9]Stanton's approach to estimating the drift and diffusion have been previously extended to a bivariate setting by Boudoukh, Richardson, Stanton and Whitelaw (1998), although they do not provide a formal test for the double dependence.

- H1:

$$E\left[r_{t+\Delta} - r_t \mid r_t\right] = E\left[r_{t+\Delta} - r_t \mid r_t, S_t\right], \tag{13}$$

- H2:

$$E\left[(r_{t+\Delta} - r_t)^2 \mid r_t\right] = E\left[(r_{t+\Delta} - r_t)^2 \mid r_t, S_t\right]. \tag{14}$$

Note that these hypotheses can be tested in a nonparametric way using the testing method described above. It should be noted that, as Bandi and Phillips (2003) point out, Stanton's (1997) estimator assumes a time-invariant density for the short rate process, and hence is not robust to non-stationarity. In their paper they propose an alternative nonparametric estimation method that only requires recurrence, i.e., the process visits any level in its range an infinite number of times over time. Since we are using Ait-Sahalia et al. (2001) test we employ kernel drift and diffusion estimators as in Stanton (1997). This means that we could fail to reject the null if the true process is not stationary, since under the alternative hypothesis we only assume dependence on r_t and S_t.

In our application, the following bandwidth functions are used for the unrestricted and restricted regressions:

$$h = h_0 N^{-1/\delta} \quad \text{with} \quad \delta = 4.75 \tag{15}$$

$$H = H_0 N^{-1/\Delta} \quad \text{with} \quad \Delta = 4.25 \tag{16}$$

Also, the test is performed using an independent Gaussian kernel:

$$K_H(w - W_i) = H^{-1} \tfrac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}(\tfrac{w-W_i}{H})^2)$$

$$K_h(w - W_i, v - V_i) = h^{-2} \tfrac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}(\tfrac{w-W_i}{h})^2) \times \tfrac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}(\tfrac{v-V_i}{h})^2)$$

Finally, we choose to standardize regressors before applying the test. This has two potential advantages. First, it allows us to use more similar bandwidths for different regressors, which simplifies our search for the bandwidths with the most desirable properties. Second, it is crucial to standardize both regressors in the bidimensional regression since we use the same bandwidth (h) for smoothing both samples. If we do not standardize both regressors and one series has much higher variance, using the same bandwidth will most likely result in oversmoothing one series and undersmoothing the other. Clearly, this can potentially bias inferences based on the test.

Throughout the paper, we focus our attention on the explanatory power of three additional regressors.

First, the term spread as the difference between the ten-year yield and the three-month rate, which we consider a proxy for the slope of the yield curve.

Second, a proxy for the curvature of the yield curve, which we define as the difference between the ten-year yield and twice the one-year yield plus three-month rate.

Third, as pointed out in the introduction, deviations from the Markovian structure (as is the case with HJM) may also result in the short rate level not being a sufficient statistic for the state of the system. In fact, the time series literature has provided evidence that the short rate process may indeed be non-Markovian[10]. We therefore explore whether the state of the short rate process depends on lagged values many periods apart. In order to conduct the test, we build on the fractional integration literature. Fractional integration enables a parsimonious modelling of long memory processes (for a review of fractional integration and long memory processes, see Baillie (1996)). Take for instance a fractional white noise process, defined as

$$(1-L)^d (y_t - \mu) = \varepsilon_t, \tag{17}$$

where L is the usual lag operator, $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma^2$, and $E(\varepsilon_t, \varepsilon_s) = 0$ for $s \neq t$, and where the fractional parameter is possibly noninteger. The process admits an infinite autorregresive representation given by

$$y_t = \sum_{k=1}^{\infty} \pi_k y_{t-k} + \varepsilon_t, \tag{18}$$

where the weights, π_k, are obtained from the expansion of $(1-L)^d$

$$(1-L)^d = 1 - dL + \frac{1}{2!} d\,(d-1)\,L^2 + -\frac{1}{3!} d\,(d-1)\,(d-2)\,L^3 + \cdots \tag{19}$$

Motivated by this literature, we construct a variable which is a weighted average of lagged short rate values, where the weights correspond to the autorregresive representation of a fractionally integrated process (18) truncated at $k = 150$ (three years). We then test whether the resulting variable affects the state of the short rate process significantly. In order to perform the test we apply the dimension reduction test to the new variable. Our approach can hence be regarded as a semiparametric test for long memory. The test is conducted for $d = 0.01$ following Duan and Jacobs (2001), who estimated a value of d close to 0.01 in a fractionally integrated GARCH model.

In any case, the validity of our proposal depends both on the finite sample properties of Stanton's (1997) estimator as well as on the performance of Ait-Sahalia et al. (2001) asymptotic test in finite samples. In the following sections we study both issues.

2.2 Finite-sample properties of Stanton's (1997) estimator

Let us start by applying Stanton's (1997) estimation method to our data set. For the application, a series of daily observations of annualized discount rates on US Treasury Bills with three months to maturity (secondary market closing bid rates), as well as on one-year and 10-year US Government Bonds (Treasury constant maturity rates) was obtained from the Internet Site of the Federal Reserve. The rates correspond to secondary market closing bid rates. From the daily series, we constructed a weekly series in which each observation corresponds to the rate quoted on Wednesday (the day of the week with the least number of missing observations). Finally, discount rates were converted to annualized continuously compounded yields. The resulting series covers the period from February 1962 to December 2002 – a total of 2,101 observations– and is displayed on the top panel in Figure 1. The middle panel displays the slope of the yield curve as proxied by the difference between the

[10] Duan and Jacobs (2001) study the presence of short and long memory in the daily short rate series in a parametric framework. Using a GARCH model, they find that lags beyond a week do not improve the empirical fit of the model. They also extend Backus and Zin (1993) fractionally integrated approach to conditional heteroscedasticity and report a significant long memory component.

10-year yield and the three-month yield. Finally, the bottom panel shows the series of the curvature of the yield curve as proxied by the difference between the 10-year to one-year term spread and the one-year to three-month term spread. Table 1 shows descriptive statistics of the data set and Figure 2 shows the slow decay in the autocorrelation function for the three-month yield series, which is a characteristic of long memory processes.

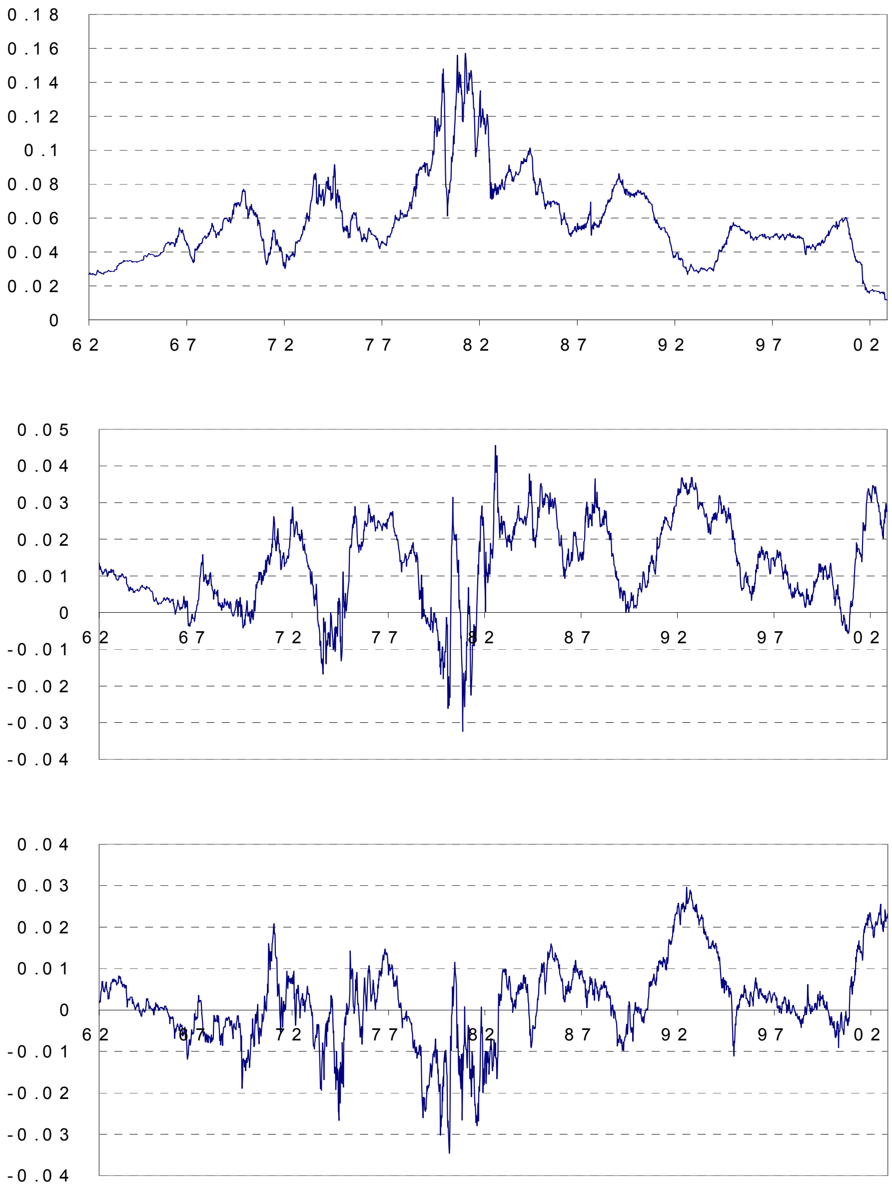
Figures 3 and 4 display kernel estimates of first-order approximations to the short rate drift and diffusion coefficients respectively as a function of the short rate level. The most striking feature of these Figures is that when the conditioning short rate value is beyond 12 percent, the short rate drift decreases dramatically displaying a higher speed of mean reversion. These results are very similar to those obtained by Stanton (1997) using a daily series of the three month Treasury yield over a similar period.

Using simulations, Chapman and Pearson (2000) have shown, however, that non-linearities displayed by Stanton's kernel drift estimator may be spurious. In particular, they simulate a large number of short rate samples following a CIR process (i.e., with linear drift) and apply Stanton's method to each of them. Their results suggest that Stanton's drift estimator is biased for low and high levels of the short rate. They distinguish between two sources of bias near the upper and lower edges of the short rate data:

a) the boundary bias, which skews moment estimates at low density areas towards estimates at the center of the distribution. Because the short rate drift is downward sloping, this effect results in estimates of the drift at high short rate levels appearing higher than true moments. Similarly, for low short rates, the drift tends to be underestimated;

b) the truncation bias: because in finite samples short rate levels above the sample maximum are not observed, changes in the short rate level conditional on the short rate being close to the upper boundary are *necessarily* negative. This pushes the estimated drift below the true drift for high values of the short rate. Analogously, this effect increases the estimated drift conditional on low short rate values.

Figure 1. Time series plot of the short-term interest rate (top panel), the spread between the three-month yield and the 10-year yield (middle panel), and the curvature of the yield curve (bottom panel) for the February 1962-December 2002 period.



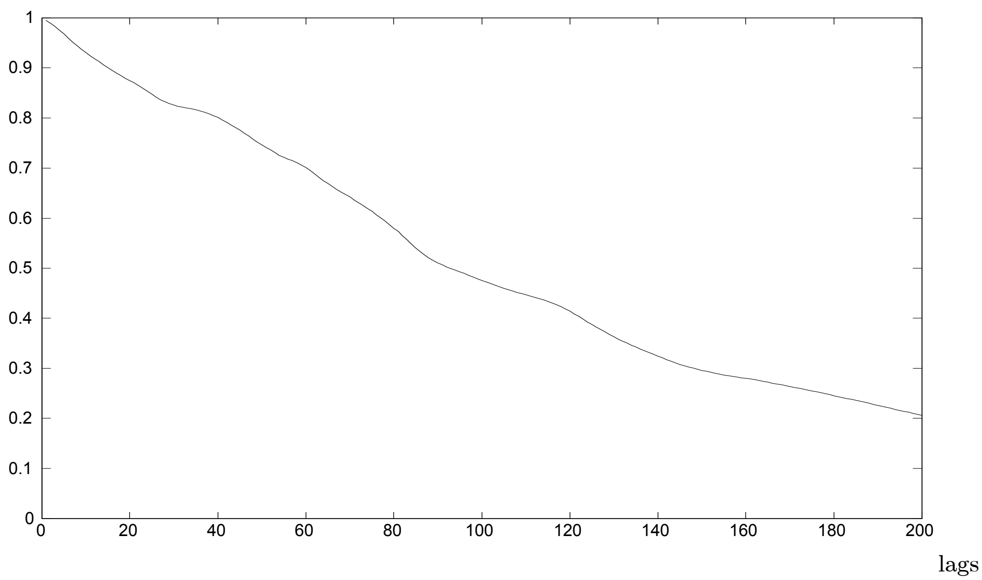
Year

Table 1. Summary statistics of the 3-month T-Bill yield, the yield on the 1-year US Bond, the yield on the 10-year US Bond, the term spread, the curvature of the yield curvature and weekly changes in the 3-month T-Bill rate. The series covers the period from February 1962 to Dec. 2002.

Variable	Mean	Std. Dev.	Autocorr.
3-m. yield	0.0580	0.0246	0.9954
1-yr. yield	0.0636	0.0260	0.9966
10-yr. yield	0.0710	0.0229	0.9979
Spread	0.0130	0.0119	0.9860
Curvature	0.0017	0.0100	0.9814
3-m. weekly changes	-7.01×10^{-6}	0.0024	0.0898

Correlation Matrix	3-m. yield	1-yr. yield	10-yr. yield	Spread	Curvature
3-m. yield	1				
1-yr. yield	0.9875	1			
10-yr. yield	0.8776	0.9213	1		
Spread	-0.3781	-0.2677	0.1121	1	
Curvature	-0.6551	-0.6508	-0.3335	0.7138	1
3-m. weekly changes	-0.0433	-0.0241	-0.0248	0.0418	0.0418

Figure 2. Autocorrelation function for the weekly series of the three-month T-Bill yield.



We simulated 5,000 sample paths of weekly short rate observations following a CIR process with parameters chosen to match true stationary moments with real data, and sample size equal to $2,000$. The combined effect of both biases *averaged across all simulated paths* can be seen on Figures 5 and 6 for bandwidths $H = \hat{\sigma}N^{-1/5}$ (Figure 5) and $H = 4 \cdot \hat{\sigma}N^{-1/5}$ (Figure 6), where $\hat{\sigma}$ and N are the short rate sample standard deviation and size respectively. Although the true drift is linear, the first graph shows the short rate drift as being highly nonlinear, showing fast mean reversion for high short rate levels. This nonlinearity is the consequence of the truncation bias. However, as the bandwidth increases (Figure 6) the boundary bias drives drift estimates towards its sample average, hence pulling the estimated drift upwards for high short rate levels. The boundary bias therefore partially offsets the truncation bias. These graphs closely resemble the results obtained by Chapman and Pearson (2000).

It is however not possible to assess the true magnitude of the truncation bias by simulating a large number of sample paths, estimating the short rate drift conditional on *the same design points for every sample*, and finally averaging across all estimates associated to each sample path. The reason is that each path will have a different support. For instance, one series will have a sample maximum at 20% while another will always be below 12%. Although it is technically possible to estimate the short rate drift conditional on 15% for both series, this value is in the center of the distribution for the first series and outside the support of the distribution for the second one. Therefore, we are averaging an almost unbiased drift estimate with a biased estimate. In order to overcome this problem we simply propose to simulate series until we have enough of them within particular boundaries. In practice, we simulate sample paths from a CIR process and keep 5,000 whose sample minimum is between 2.60% and 2.80%, and whose sample maximum is between 11.90% and 12.30%. We then estimate the short rate drift for each sample for design points in the interval [2.70%, 12.10%]. We also compute percentiles 2.5%, 5%, 95%, and 97.5% for each sample and the average of those values across all samples. Similar experiments have been conducted by Li, Pearson, and Potesham (2001), who study and propose a solution for the truncation bias in a parametric setting.

Figures 7 and 8 show the results of the experiment for both bandwidths considered. Vertical lines represent the mean percentiles across all simulated paths. For the smaller bandwidth, Stanton's estimator is remarkably unbiased inside the support of the distribution. For the larger bandwidth however, estimates are slightly biased although there is a significant gain in accuracy. In either case, there are almost no signs of the truncation bias which causes the drift to appear nonlinear.

We can therefore consider Stanton's drift estimator to be moderately unbiased as long we are conditioning on short rate level not in the tails of the sample distribution. For this reason, we shall condition all conclusions on the short rate level being in the center of the data. This is accomplished by defining A_i in (5) as an indicator function that equals 1 if the short rate is between its 5th and 95th percentiles of its sample, and zero otherwise. In any case, our simulations show that unbiasedness only comes at the cost of low precision in drift estimates. This can potentially affect the ability of tests designed to distinguish between alternative models based on the short rate drift.

Figure 3. The short rate drift as a function of the short rate level, estimated using Stanton's (1997) first-order approximation and kernel smoothing. A Gaussian kernel has been employed with bandwidth $H = 4 \cdot \hat{\sigma} \cdot N^{-1/5}$.

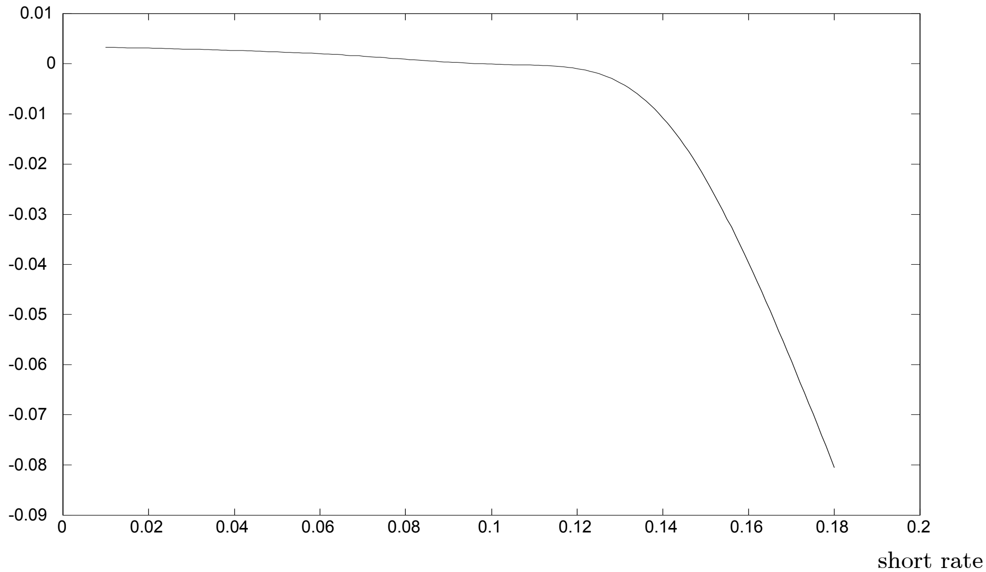


Figure 4. The short rate diffusion as a function of the short rate level, estimated using Stanton's (1997) first-order approximation and kernel smoothing. A Gaussian kernel has been employed with bandwidth $H = 3 \cdot \hat{\sigma} \cdot N^{-1/5}$.

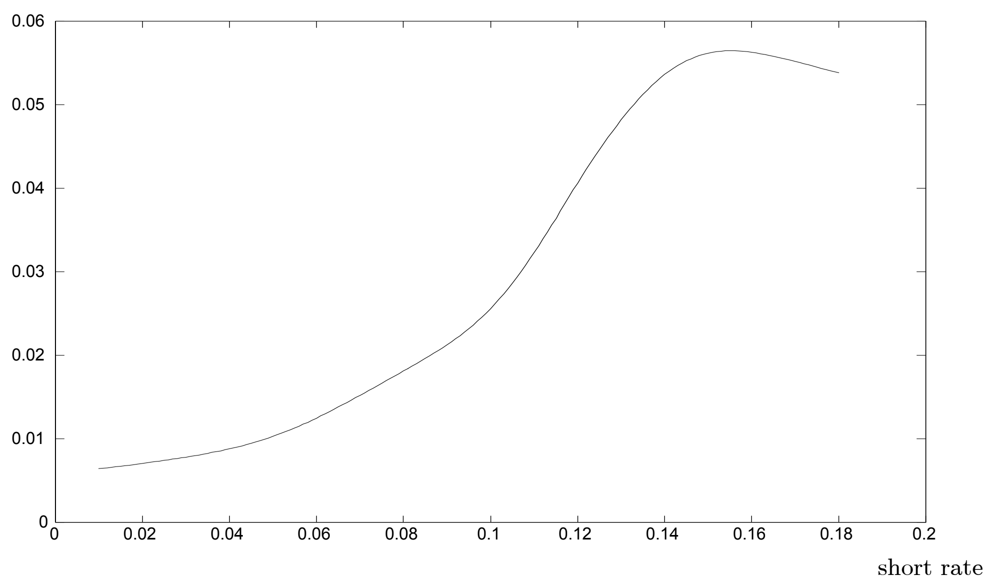


Figure 5. The average estimated drift from 5,000 simulated paths of size 2,000 from a CIR process, versus the true drift of the process, with bandwidth $H = \hat{\sigma} \cdot N^{-1/5}$ where $\hat{\sigma}$ is the sample standard deviation and N is the sample size.

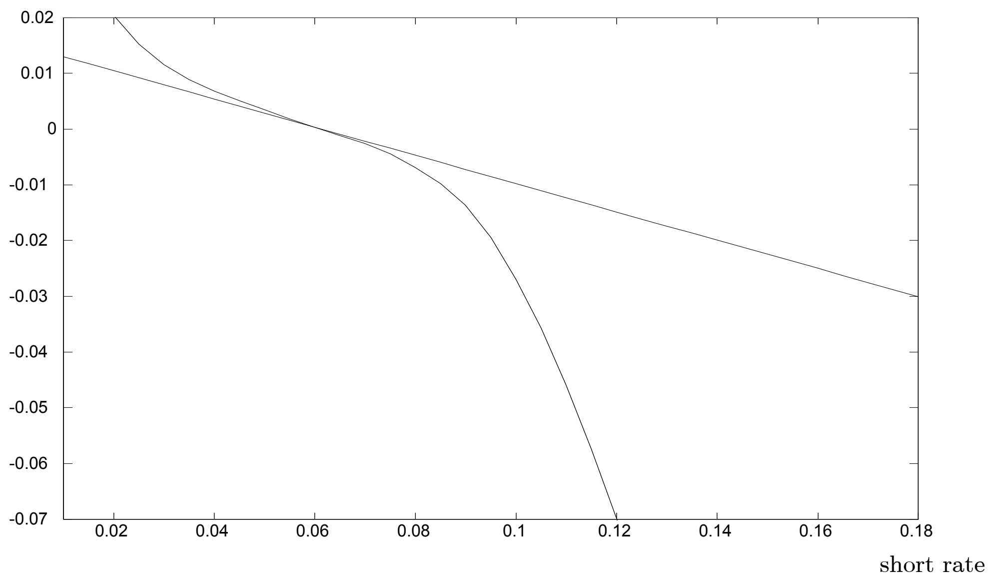


Figure 6. The average estimated drift from 5,000 simulated paths of size 2,000 from a CIR process, versus the true drift of the process, with bandwidth $H = 4 \cdot \hat{\sigma} \cdot N^{-1/5}$ where $\hat{\sigma}$ is the sample standard deviation and N is the sample size.

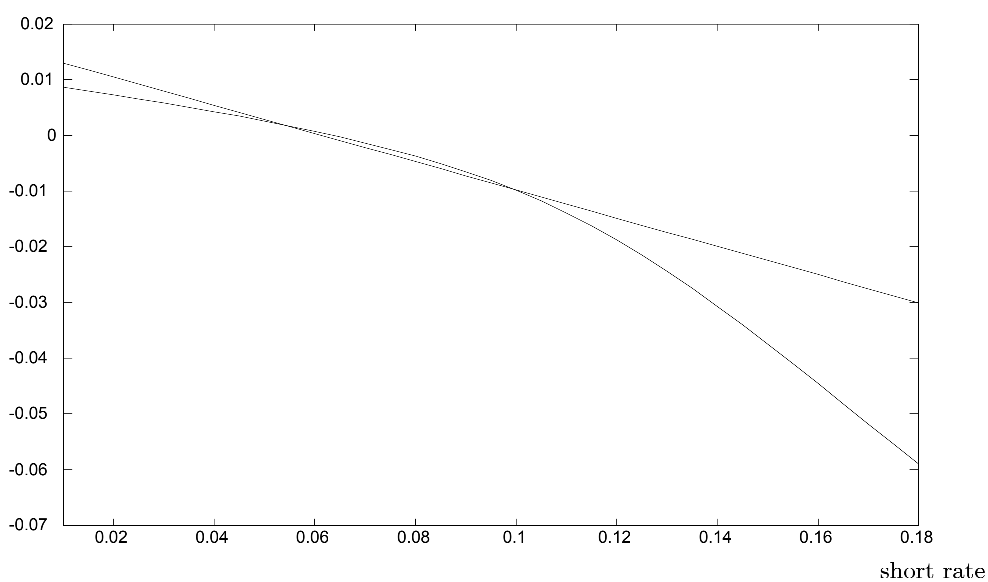


Figure 7. The average estimated drift for 5,000 simulated paths of size 2,000 from a CIR process, and Monte-Carlo 5th and 95th percentiles versus the true drift of the process, with bandwidth $H = \hat{\sigma} \cdot N^{-1/5}$ where $\hat{\sigma}$ is the sample standard deviation and N is the sample size. All samples have a sample minimum within 10 basis points of 2.70% and 20 basis points of 12.10%. Vertical lines (from left to right) mark the average 2.5, 5, 95, and 97.5 percentiles across all paths.

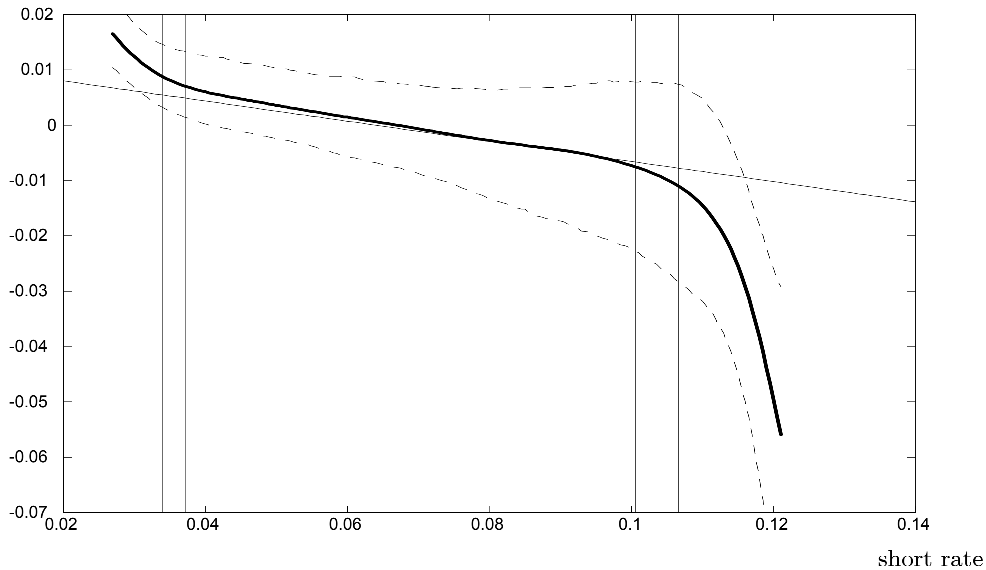
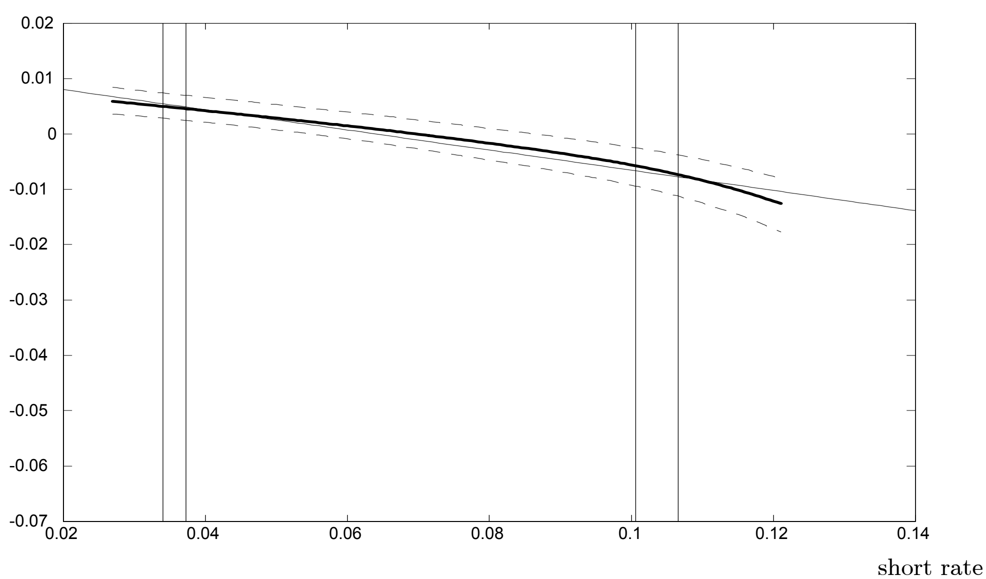


Figure 8. The average estimated drift for 5,000 simulated paths of size 2,000 from a CIR process, and Monte-Carlo 5th and 95th percentiles versus the true drift of the process, with bandwidth $H = 4 \cdot \hat{\sigma} \cdot N^{-1/5}$ where $\hat{\sigma}$ is the sample standard deviation and N is the sample size. All samples have a sample minimum within 10 basis points of 2.70% and 20 basis points of 12.10%. Vertical lines (from left to right) mark the average 2.5, 5, 95, and 97.5 percentiles across all paths.



2.3 Finite sample properties of Ait-Sahalia et al. (2001) test

The second source of concern is the finite sample performance of Ait-Sahalia et al. (2001) asymptotic test when applied to interest rate series. On one hand, interest rate data are highly persistent as the autocorrelation coefficients of Table 1 show. Although serial dependence has no impact on moment estimates of the test statistic asymptotic distribution, it potentially biases inferences from finite samples. Another problem is related to the independence of the regressors. In the course of our experiments, we have found that the size of the test in finite samples quickly departs from the nominal size as the correlation between the regressors increases. In particular, we have considered the same model as in Ait-Sahalia et al. (section 5, 2001): $Y = W\theta + 0.75e^{-0.5W^2}\varepsilon$, with $\theta = 1$, and W, ε distributed as two independent $N(0,1)$. We have then simulated 500 samples of size 1,000 each and we have tested the null hypothesis that $E[Y \mid W = w] = E[Y \mid W = w, V = v]$, where V is also $N(0,1)$. Using the same kernels, the same bandwidths and the same weighting function as in their simulations, but setting the correlation coefficient between W and V at 0.75 instead of assuming independence, we have found that the test rejects the null hypothesis 24% and 35% of times for nominal sizes of 5% and 10%, respectively.

In order to study the adequacy of the test as applied to the interest rate data, we propose to conduct Monte Carlo experiments.

We begin with the size of the test and proceed as follows:

1. We generate a path of 2,000 weekly 3-month interest rates -which we identify with the short rate- following the one-factor CIR process:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dZ_t$$

   with parameters $\kappa = 0.1822$, $\theta = 0.0640$, and $\sigma = 0.0609$.

2. For our simulated path, we compute the corresponding one-year and ten-year yields according to the model (see CIR (1985b)) given the short rate realizations. We also compute the term spread, the curvature of the yield curve, and the moving average in (19).

3. Because in the CIR model (just as in any other one-factor affine term structure model), yields are all perfectly correlated, we add a measurement (independent zero-mean normal) error to the spread and curvature, with standard deviations chosen to match the observed correlations in the data[11]. No measurement error is added to the simulated moving average since its correlation with the short rate is similar to that found in the data (0.9320). We then standardize the regressors and apply the test.

4. We repeat steps 1-3 for 500 sample paths.

Monte Carlo results are reported in Tables 2, 3, and 4. Top panels correspond to results for the drift while bottom panels display results for the diffusion. Each panel reports rejection rates for the one-tail test and 1%, 5% and 10% critical values ($z_{0.01} = 2.32$, $z_{0.05} = 1.64$, and $z_{0.10} = 1.28$), as well as the standard deviation of the computed test statistic, which is 1 asymptotically under the null.

[11] More specifically, the standard deviation of the measurement errors added to the spread and curvature equal 3% and 1% respectively.

Table 2. Size of the test: spread. We simulate 500 short rate paths followinga one-factor CIR model and derive the term spread as the difference between the ten-year yield and the short rate. We then add a measurement error and apply the dimension test to the standardized regressors. Rejection rates are in percent points.

Panel A: Drift	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.50$ $h_0 = 2.45$	$H_0 = 4.00$ $h_0 = 3.95$
1%	31.60	1.40	0.40
5%	54.60	6.40	1.40
10%	68.40	13.20	3.80
Std. Dev.($\hat{\tau}$)	1.11	0.67	0.40

Panel B: Diffusion	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.25$ $h_0 = 2.20$	$H_0 = 4.00$ $h_0 = 3.95$
1%	17.80	1.80	6.20
5%	38.60	5.00	15.20
10%	54.20	8.60	25.40
Std. Dev.($\hat{\tau}$)	1.30	0.67	0.73

Table 3. Size of the test: curvature. We simulate 500 short rate paths followinga one-factor CIR model and derive the curvature of the yield curve as described in the text. We then add a measurement error and apply the dimension test to the standardized regressors. Rejection rates are in percent points.

Panel A: Drift	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.50$ $h_0 = 2.45$	$H_0 = 4.00$ $h_0 = 3.95$
1%	27.40	0.80	0.20
5%	50.00	3.60	1.20
10%	65.20	8.00	3.40
Std. Dev. ($\hat{\tau}$)	1.06	0.58	0.37

Panel B: Diffusion	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.00$ $h_0 = 1.95$	$H_0 = 4.00$ $h_0 = 3.95$
1%	15.20	2.00	15.40
5%	30.60	8.00	34.20
10%	43..40	13.60	49.40
Std. Dev. ($\hat{\tau}$)	1.1535	0.67	0.90

Table 4. Size of the test: long memory. We simulate 500 short rate paths following a one-factor CIR model and derive a weighted average of past short rate realizations as described in the text. Rejection rates are in percent points.

Panel A: Drift	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 1.50$ $h_0 = 1.45$	$H_0 = 2.00$ $h_0 = 1.95$
1%	7.00	1.00	0.60
5%	18.80	7.60	2.40
10%	33.00	16.60	7.60
Std. Dev. ($\hat{\tau}$)	0.72	0.48	0.34

Panel B: Diffusion	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.00$ $h_0 = 1.95$	$H_0 = 4.00$ $h_0 = 3.95$
1%	2.20	1.80	5.80
5%	12.60	8.40	31.00
10%	25.20	24.20	63.40
Std. Dev. ($\hat{\tau}$)	0.59	0.46	0.48

In each experiment, we fix h_0 and H_0 in (15) and (16), which is equivalent to using a plug-in rule to choose the bandwidths. Bandwidths for the bivariate and univariate regression are therefore proportional to $N^{-1/4.25}$ and $N^{-1/4.75}$, respectively. We report simulation results for three pairs of bandwidth parameters (h_0, H_0) : (i) the pair that gives the empirical size closest to the nominal size (the column in the center); (ii) a larger pair of bandwidths [12]; and (iii) tighter bands for comparison purposes. From Tables 2-4, it is clear that values of h_0 and H_0 close to 2 result in the empirical distribution of the test statistic being closest to the asymptotic distribution, while larger and smaller bandwidths produce large distortions. With respect to the test statistic applied to the short rate drift, and focusing exclusively on the bandwidth parameters with the nominal size closest to the empirical size, the largest distortion corresponds to the case when we are testing for the null hypothesis that the short rate drift does not depend on past short rate values and the 10% critical value. In this case, the empirical size is 6% larger than the nominal size.

On the other hand, rejection of the null based on the diffusion cannot be attributable to the test displaying poor size, as long as appropriate bandwidths are employed. There is only one exception: the test rejects the null that the diffusion does not depend on lagged short rate values 24.20% of times at the 10% critical value for the best bandwidth pair.

It is also possible to explore the power of the test to reject the null when the true data generating process is a multi-factor model. We take the following steps:

1. We generate a path of 2,000 weekly realizations of the unobservable factors in a two-factor version of the CIR model as described in Geyer and Pichler (1999) using their parameter estimates.

2. We compute the corresponding three-month, one-year and ten-year yields according to the model (see Geyer and Pichler (1999)) given the factor realizations. We also compute the term spread, the curvature of the yield curve, and the moving average in (19).

[12] Optimal bandwidths -in the sense that the mean squared error is minimized- estimated through cross-validation for our original dataset, are in the neighbourhood of the largest bandwidths reported in the table.

3. Although in the simulated two-factor CIR samples, the short rate is not perfectly correlated with the term spread or the curvature of the yield curve, in order to generate correlations between the regressors closer to those observed in the data, we add a measurement error to the spread and curvature[13]. Again, we standardize the regressors and apply the test.

4. We repeat steps 1-3 for 500 paths.

Results are reported in Tables 5, 6, and 7. Again, top panels correspond to results for the drift, while bottom panels display results for the diffusion. Each panel reports rejection rates for the one-tail test and 1%, 5% and 10% critical values, as well as the standard deviation of the empirical distribution of the test statistic. Results are reported for the same bandwidth parameters (h_0, H_0) as in the previous experiments.

Monte Carlo results for the drift suggest that for bandwidth parameters that give reasonable size, the test lacks all power to reject the single factor hypothesis. Therefore, failure to reject the null may well be attributable to poor power and not necessarily to the one-factor hypothesis being true. This is probably due to lack of precision in the drift estimator, which results in noisy estimates and is penalized by the test. In contrast, it is easier to reject the null when the test is applied to the short rate diffusion. This is especially true for the curvature.

Table 5. Power of the test: spread. We simulate 500 interest rate paths following a two-factor CIR model and compute the term spread as described in the text. We then add a measurement error and apply the dimension test to the standardized regressors. Rejection rates are in percent points.

Panel A: Drift	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.50$ $h_0 = 2.45$	$H_0 = 4.00$ $h_0 = 3.95$
1%	29.80	1.40	0.40
5%	55.20	3.80	2.80
10%	70.20	8.20	4.60
Std. Dev.$(\hat{\tau})$	1.14	0.62	0.45

Panel B: Diffusion	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.25$ $h_0 = 2.20$	$H_0 = 4.00$ $h_0 = 3.95$
1%	26.60	23.00	63.60
5%	48.00	44.80	81.60
10%	64.20	56.80	90.40
Std. Dev. $(\hat{\tau})$	1.95	1.43	1.68

[13]In this case, the standard deviation of the measurement errors added to the spread and curvature equal 0.02 and 0.008 respectively.

Table 6. Power of the test: curvature. We simulate 500 interest rate paths following a two-factor CIR model and compute the term spread as described in the text. We then add a measurement error and apply the dimension test to the standardized regressors. Rejection rates are in percent points.

Panel A: Drift	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.50$ $h_0 = 2.45$	$H_0 = 4.00$ $h_0 = 3.95$
1%	27.20	2.20	1.20
5%	46.00	6.80	3.20
10%	60.00	11.00	6.20
Std. Dev. ($\hat{\tau}$)	1.05	0.62	0.49

Panel B: Diffusion	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.00$ $h_0 = 1.95$	$H_0 = 4.00$ $h_0 = 3.95$
1%	30.40	44.00	92.20
5%	50.80	62.60	98.20
10%	63.40	72.40	99.00
Std. Dev. ($\hat{\tau}$)	1.49	1.71	1.91

Table 7. Power of the test: long memory. We simulate 500 interest rate paths following a two-factor CIR model and compute a weighted average of past short rate realizations as described in the text. We then apply the dimension test to the standardized regressors.Rejection rates are in percent points.

Panel A: Drift	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 1.50$ $h_0 = 1.45$	$H_0 = 2.00$ $h_0 = 1.95$
1%	9.00	1.40	1.80
5%	21.60	6.00	5.40
10%	34.20	14.20	10.00
Std. Dev. ($\hat{\tau}$)	0.89	0.61	0.53

Panel B: Diffusion	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.00$ $h_0 = 1.95$	$H_0 = 4.00$ $h_0 = 3.95$
1%	35.20	20.20	38.00
5%	54.80	36.20	52.80
10%	67.20	50.00	65.20
Std. Dev. ($\hat{\tau}$)	1.71	1.30	1.12

3 Empirical results

Let us first consider the effect of the term spread on the short rate drift and diffusion. The test statistic $\hat{\tau}$ associated with the short rate drift and its corresponding p-value are shown in Table 8 (panel A) for the same bandwidth parameters as those employed in the Monte Carlo experiments of Subsection 2.3. We are therefore also using a plug-in method to choose the bandwidths for testing purposes. The test for the bandwidth parameters that give the best empirical size indicates that we cannot reject the null hypothesis that expected conditional changes in the three-month rate are fully explained by the current short rate at the 1%, 5% or 10% significance levels. The asymptotic test therefore suggests that the slope of the yield curve contains no predictive power about future weekly changes in interest rates. However, this result is not reliable given the low power of the test to reject the null as discussed above.

As for the test for the short rate diffusion, the associated test statistic $\hat{\tau}$ and its corresponding p-value are shown in Table 8 (panel B) also for the same bandwidths as in the Monte Carlo experiments of Subsection 2.3. The value of the test statistic is significant at the 5% and 10% levels only for the smallest bandwidth considered. Table 2, however, shows that for the smallest bandwidth, the test severely overrejects the null even if the null is true. Consequently, we may not reject the single-factor hypothesis on the basis of this test when the information content of the term spread is considered.

Table 9 presents the test results for the curvature. Although the null that the drift depends only on the short rate is rejected for the smallest bandwidth, the result is most likely driven by poor size of the test in finite samples as seen on Table 3 (Panel A). Also, failure to reject the null for the best bandwidth pair can be explained by extreme low power of the test. Different conclusions can be reached when examining Panel B. The test rejects the null hypothesis of a one-dimensional diffusion function for all critical values and according to Table 3 (Panel B), this result is not explained by poor size of the test in finite samples, but rather may be considered as evidence against a single-factor Markov short rate diffusion.

Finally, Table 10 shows test results for lagged values of the short term interest rate. Similarly to the previous case, Panel A shows that the null corresponding to the drift is only rejected when the bandwidth with worst size (strongest tendency to overrejection) is employed. For more adequate bandwidth parameters, however, we cannot discard the single factor model on the basis of this test. Again, low power is a possible explanation for this result. However, when the dependence of the short rate diffusion on past short rate realizations is considered, the null is rejected at the 1%, 5%, and 10% significance levels. As Table 4 shows, when the single factor model is the true model, the null is rejected only for 1.80% of our simulations at the 1% level. This result can be taken as further evidence against the short rate being driven by a single factor Markovian process, and is consistent with Ait-Sahalia (2000).

Table 8. Test Results: Spread. This Table shows the test statistic values and the corresponding p-values for the null hypothesis that the term spread does not contribute to explaining changes in the drift and diffusion of the short rate process. Regressors have been standardized.

Panel A: Drift	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.50$ $h_0 = 2.45$	$H_0 = 4.00$ $h_0 = 3.95$
$\hat{\tau}$	1.1687	-0.2490	-0.5220
(p-value)	(0.1213)	(0.5983)	(0.6992)

Panel B: Diffusion	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.25$ $h_0 = 2.20$	$H_0 = 4.00$ $h_0 = 3.95$
$\hat{\tau}$	1.8332	1.1128	0.7722
(p-value)	(0.0334)	(0.1329)	(0.2200)

Table 9. Test Results: Curvature. This Table shows the test statistic values and the corresponding p-values for the null hypothesis that the curvature of the yield curve does not contribute to explaining changes in the drift and diffusion of the short rate process. Regressors have been standardized.

Panel A: Drift	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.50$ $h_0 = 2.45$	$H_0 = 4.00$ $h_0 = 3.95$
$\hat{\tau}$	2.3203	1.2610	0.6725
(p-value)	(0.0102)	(0.1037)	(0.2506)

Panel B: Diffusion	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.00$ $h_0 = 1.95$	$H_0 = 4.00$ $h_0 = 3.95$
$\hat{\tau}$	2.8834	2.7931	3.3839
(p-value)	(0.0020)	(0.0026)	(0.0004)

Table 10. Test Results: Long memory. This Table shows the test statistic values and the corresponding p-values for the null hypothesis that past short rate realizations do not contribute to explaining changesin the drift and diffusion of the short rate process. Regressors have been standardized.

Panel A: Drift	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 1.50$ $h_0 = 1.45$	$H_0 = 2.00$ $h_0 = 1.95$
$\hat{\tau}$	2.5678	1.1239	0.8547
(p-value)	(0.0051)	(0.1305)	(0.1964)

Panel B: Diffusion	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.00$ $h_0 = 1.95$	$H_0 = 4.00$ $h_0 = 3.95$
$\hat{\tau}$	3.0221	2.4732	2.5243
(p-value)	(0.0020)	(0.0067)	(0.0058)

To summarize, considering only the tests and bandwidth parameters with lowest size distortions, we may conclude that the *asymptotic* test rejects the null hypothesis that the short rate diffusion does not depend on the curvature of the yield curve or on lagged short rate values. However, we can exploit the *empirical* distribution obtained in Monte Carlo experiments in order to test the null hypothesis that our observations have been generated by a one-factor CIR model. Table 11 displays the critical values for the test when the alternative hypothesis is that the short rate diffusion depends on the curvature of the yield curve (Panel A) and the moving average (Panel B). Test statistics for the real data (from Tables 6 and 7) are also displayed. The null hypothesis is rejected for all bandwidths considered at 5% significance level or at the 5% and 1% significance levels with a single exception, where the null is rejected at the 10% significance level. Empirical test results therefore do not contradict those of the asymptotic test.

Table 11. Empirical Test. This Table shows the critical values from the empirical distribution of the test statistic under the null hypothesis that the curvature of the yield curve and past short rate realizations do not contribute to explaining the short rate diffusion. In each experiment, 1,000 sample paths have been sampled from a one-factor CIR term structure model. Test statistics for the real data are displayed in italics in the bottom line of each panel.

Panel A: Curvature	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.00$ $h_0 = 1.95$	$H_0 = 4.00$ $h_0 = 3.95$
90th percentile	2.3827	1.4836	2.6460
95th percentile	3.0933	1.9541	2.9890
99th percentile	5.3574	3.1673	4.0398
$\hat{\tau}$	*2.8834*	*2.7931*	*3.3839*

Panel B. Long memory	$H_0 = 1.00$ $h_0 = 0.95$	$H_0 = 2.00$ $h_0 = 1.95$	$H_0 = 4.00$ $h_0 = 3.95$
90th percentile	1.7612	1.7156	2.0940
95th percentile	2.1179	1.9413	2.2856
99th percentile	2.9287	2.4033	2.6264
$\hat{\tau}$	*3.0221*	*2.4732*	*2.5243*

4 Summary and conclusions

This paper assesses the ability of general single-factor Markovian processes to fit observed short rate data. Despite evidence that the conditional distribution of bond yields depends on multiple common risk factors, and the large body of theoretical research along this line, the single-factor Markov diffusion process for the short rate has not been directly tested in a non-parametric setting with two exceptions: Ait-Sahalia (2000) and Jeffrey et al. (2000). Compared to those papers, our approach has the virtue of simplicity and potential generality of the null hypothesis, but suffers from the caveat of the alternative hypothesis being specific with respect to the regressors.

The paper first shows how to conduct the analysis in a nonparametric fashion. In particular, Stanton's (1997) estimation method of the short rate conditional distribution is combined with a dimension reduction test developed by Ait-Sahalia, Bickel, and Stoker (2001). Simulation work shows that using kernel smoothing to estimate conditional moments leads to relatively unbiased (albeit perhaps inefficient) drift estimates as long as the conditioning variables are not in the tails of the sample distribution. Additionally, the paper examines, via Monte Carlo experiments, the finite sample performance of Ait-Sahalia et al. (2001) test as applied to this specific setting. We conclude from the analysis that although it is possible to find bandwidth values that lead to appropriate empirical sizes, when applied to the short rate drift, the test lacks all power to reject the null. On the other hand, tests based on the diffusion can more easily reject the null hypothesis of a single factor process.

Applying the test to our data set, we find that the short rate drift does not appear to be determined by any of the variables considered. Unfortunately, given the low power of the test finite samples, this result should not be taken as evidence in favor of a one-dimensional drift function for the short rate. Different conclusions are obtained when examining the short rate diffusion. In particular, the null is rejected when considering the dependence of the short rate diffusion on the curvature of the yield curve and on the past short rate path.

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