



Duality of compact groups and Hilbert C*-systems for C*-algebras with a nontrivial center

HELLMUT BAUMGÄRTEL

Mathematical Institute, University of Potsdam,
Am Neuen Palais 10, PF 601 553,
D-14415 Potsdam, Germany.
baumg@rz.uni-potsdam.de

FERNANDO LLEDÓ

Institute for Pure and Applied Mathematics,
RWTH-Aachen, Templergraben 55,
D-52062 Aachen, Germany.
lledo@iram.rwth-aachen.de

February 1, 2008

Dedicated to Detlev Buchholz on his 60th birthday.

Abstract

In this paper we present a duality theory for compact groups in the case when the C*-algebra \mathcal{A} , the fixed point algebra of the corresponding Hilbert C*-system $(\mathcal{F}, \mathcal{G})$, has a nontrivial center $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$ and the relative commutant satisfies the minimality condition

$$\mathcal{A}' \cap \mathcal{F} = \mathcal{Z},$$

as well as a technical condition called regularity. The abstract characterization of the mentioned Hilbert C*-system is expressed by means of an inclusion of C*-categories $\mathcal{T}_{\mathcal{C}} < \mathcal{T}$, where $\mathcal{T}_{\mathcal{C}}$ is a suitable DR-category and \mathcal{T} a full subcategory of the category of endomorphisms of \mathcal{A} . Both categories have the same objects and the arrows of \mathcal{T} can be generated from the arrows of $\mathcal{T}_{\mathcal{C}}$ and the center \mathcal{Z} .

A crucial new element that appears in the present analysis is an abelian group $\mathfrak{C}(\mathcal{G})$, which we call the chain group of \mathcal{G} , and that can be constructed from certain equivalence relation defined on $\widehat{\mathcal{G}}$, the dual object of \mathcal{G} . The chain group, which is isomorphic to the character group of the center of \mathcal{G} , determines the action of irreducible endomorphisms of \mathcal{A} when restricted to \mathcal{Z} . Moreover, $\mathfrak{C}(\mathcal{G})$ encodes the possibility of defining a symmetry ϵ also for the larger category \mathcal{T} of the previous inclusion.

1 Introduction

The superselection theory in algebraic quantum field theory, as stated by the Doplicher-Haag-Roberts (DHR) selection criterion [31, 14, 15], led to a profound body of work, culminating in the general Doplicher-Roberts (DR) duality theory for compact groups [21]. The DHR criterion selects a distinguished class of “admissible” representations of a quasilocal algebra \mathcal{A} of observables, which has trivial center $\mathcal{Z} := \mathcal{Z}(\mathcal{A}) = \mathbb{C}\mathbb{1}$. This corresponds to the selection of a so-called DR-category \mathcal{T} , which is a full subcategory of the category of endomorphisms of the C*-algebra \mathcal{A} (see Definition 3.18 below). Furthermore, from this endomorphism category \mathcal{T} the DR-analysis constructs a C*-algebra $\mathcal{F} \supset \mathcal{A}$ together with a compact group action $\alpha : \mathcal{G} \ni g \rightarrow \alpha_g \in \text{Aut } \mathcal{F}$ such that:

- \mathcal{A} is the fixed point algebra of this action

- \mathcal{T} coincides with the category of all “canonical endomorphisms” of \mathcal{A} , associated with the pair $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ (cf. Subsection 3.2).

\mathcal{F} is called a Hilbert extension of \mathcal{A} in [11]. Physically, \mathcal{F} is identified as a field algebra and \mathcal{G} with a global gauge group of the system. The pair $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$, which we call *Hilbert C*-system* (cf. Definition 2.1; the name *crossed product* is also used), is uniquely determined by \mathcal{T} up to \mathcal{A} -module isomorphisms. Conversely, $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ determines uniquely its category of all canonical endomorphisms. Therefore $\{\mathcal{T}, \mathcal{A}\}$ can be seen as the abstract side of the representation category of a compact group, while $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ corresponds to the concrete side of the representation category of \mathcal{G} , and, roughly, any irreducible representations of \mathcal{G} is explicitly realized within the Hilbert C*-system. One can state the equivalence of the “selection principle”, given by \mathcal{T} and the “symmetry principle”, given by the compact group \mathcal{G} . This is one of the crucial theorems of the Doplicher-Roberts theory.

In the DR-theory the center \mathcal{Z} of the C*-algebra \mathcal{A} plays a peculiar role: as stated above, if \mathcal{A} corresponds to the inductive limit of a net of local C*-algebras indexed by open and bounded regions of Minkowski space, then the triviality of the center of \mathcal{A} is a consequence of standard assumptions on the net of local C*-algebras. But, in general, the C*-algebra appearing in the DR-theorem does not need to be a quasilocal algebra and, in fact, one has to assume explicitly that $\mathcal{Z} = \mathbb{C}\mathbf{1}$ in this context (see [21, Theorem 6.1]). Finally, we quote from the introduction of the article [21]: “There is, however, no known analogue of Theorem 4.1 of [20] for a C*-algebra with a non-trivial center and hence nothing resembling a “duality” in this more general setting.”

The aim of the present paper is to show that *there is* a duality theory for compact groups in the case of a nontrivial center, if the relative commutant of the corresponding Hilbert C*-system satisfies the following minimality condition:

$$\mathcal{A}' \cap \mathcal{F} = \mathcal{Z} \tag{1}$$

(cf. Theorem 4.14). The essence of the previous result is that now the abstract characterization of the mentioned Hilbert C*-system is expressed by means of an inclusion of C*-categories $\mathcal{T}_{\mathcal{C}} < \mathcal{T}$, where $\mathcal{T}_{\mathcal{C}}$ is a suitable DR-category and \mathcal{T} a full subcategory of the category of endomorphisms of \mathcal{A} . Both categories have the same objects and the arrows of \mathcal{T} can be generated from the arrows of $\mathcal{T}_{\mathcal{C}}$ and the center \mathcal{Z} .

Several new elements appear in the generalization of the DR-theory studied here. The crucial one is an abelian group $\mathfrak{C}(\mathcal{G})$, which we call the *chain group* of \mathcal{G} , and that can be constructed from certain equivalence relation defined on $\widehat{\mathcal{G}}$, the dual object of the compact group \mathcal{G} . The chain group, which is interesting in itself and isomorphic to the character group of the center of \mathcal{G} , determines the action of irreducible endomorphisms of \mathcal{A} when restricted to the center $\mathcal{Z}(\mathcal{A})$. Moreover, $\mathfrak{C}(\mathcal{G})$ appears explicitly in the construction of a family of examples realizing the inclusion of categories $\mathcal{T}_{\mathcal{C}} < \mathcal{T}$ mentioned above (cf. Section 6). Finally, the chain group encodes also the possibility of defining a symmetry ϵ also for the larger category \mathcal{T} of the previous inclusion.

There are several reasons that motivate the generalization of the DR-theory for systems satisfying the minimality condition (1) for the relative commutant:

- (i) In this context there is a nice intrinsic characterization of the Hilbert C*-systems satisfying (1) and a further technical condition called regularity (cf. Theorems 4.11 and 4.14). One can also prove several results in the spirit of the DR-theory: for example, the category \mathcal{T} is isomorphic to a subcategory $\mathcal{M}_{\mathcal{G}}$ of the category of free Hilbert \mathcal{Z} -bimodules generated by the algebraic Hilbert spaces in $\mathcal{T}_{\mathcal{G}}$ (cf. Proposition 4.4).
- (ii) In the context of compact groups, the equation (1) is also convenient for technical reasons. The minimality of the relative commutant implies that irreducible endomorphisms are mu-

tually disjoint (cf. Proposition 4.3) and this fact is crucial to have a nice decomposition of objects in terms of irreducible ones (cf. Proposition 4.6).

- (iii) The nontriviality of the center gives also the possibility to a more geometrical interpretation of the DR-theory. Indeed, from Gelfand's theorem we have $\mathcal{Z} \cong C(\Gamma)$, Γ a compact Hausdorff space, and in certain situations the Hilbert C^* -system $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ is a direct integral over Γ , where the Hilbert C^* -system corresponding to a.e. base point $\lambda \in \Gamma$ is of a DR-type with the same group \mathcal{G} . Here the chain group plays again an important role. This more geometrical line of research has led to recent developments in the context of vector bundles (cf. [42, 43, 41]).
- (iv) There are physically relevant examples that satisfy the condition (1). For example, this equation is presented in [36] as a “new principle”. Moreover, the elements of the center \mathcal{Z} of \mathcal{A} may be interpreted as classical observables contained in the quasilocal algebra.
- (v) The present generalization of the DR-theory in the context minimal and regular Hilbert C^* -systems has also found application in the context of superselection theory for systems carrying quantum constraints (see [5] as well as [27, 30] for a C^* -algebraic formulation of the theory of quantum constraints).

The paper is structured in 9 sections: in Section 2 we introduce the notion of a Hilbert C^* -system (cf. Definition 2.1) and give a detailed account of its properties. Hilbert C^* -systems are special types of C^* -dynamical systems $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ that, in addition, contain the information of the representation category of \mathcal{G} . They also satisfy important properties, which are interesting in themselves, as for example: the fixed point algebra \mathcal{A} is simple if \mathcal{F} is simple (cf. Subsection 3.4 for further results on the ideal structure of these algebras); one can naturally introduce spectral subspaces of \mathcal{F} and prove Parseval-type equations for a suitable \mathcal{A} -valued scalar product on \mathcal{F} (cf. Proposition 2.5). Finally, Hilbert C^* -systems provide a natural and concrete frame to describe the DR-theory as well as the generalization to the nontrivial center situation that we study here. In Section 3 we study the important relation between two C^* -categories $\mathcal{T}_{\mathcal{G}}$ and \mathcal{T} that are naturally associated with a Hilbert C^* -system. In general, $\mathcal{T}_{\mathcal{G}}$ is a subcategory of \mathcal{T} and this inclusion turns out to be characteristic for the inverse result stated in Theorem 4.14. In Section 4 the main duality theorems are stated in the context of minimal and regular Hilbert C^* -systems. The next section defines the notion of an irreducible object and introduces the chain group of \mathcal{G} , denoted by $\mathfrak{C}(\mathcal{G})$. We give examples of chain groups for several finite and compact Lie groups and show that the chain group is isomorphic to the character group of the center of \mathcal{G} (see also [37]). There is a close relation between the chain group and the set of irreducible canonical endomorphisms: an irreducible canonical endomorphism of \mathcal{A} restricted to the center \mathcal{Z} turns out to be an automorphism of \mathcal{Z} . We show that there is a group homomorphism between the chain group and the subgroup of $\text{aut } \mathcal{Z}$ generated by irreducible objects (cf. Theorem 5.7). One of the typical difficulties in the context of a nontrivial center is that \mathcal{Z} is not stable under the action of a general canonical endomorphism σ , i.e.

$$\sigma(\mathcal{Z}) \not\subset \mathcal{Z}.$$

In this section we also give an explicit formula in terms of isotypical projections that describes the action of reducible endomorphisms restricted to the center (cf. Theorem 5.9). In Section 6 we construct a family of examples that satisfy the requirements of the pair of categories $\mathcal{T}_{\mathfrak{c}} < \mathcal{T}$ considered in Theorem 4.14. In Section 7 we analyze the situation where the homomorphism between the chain group and the subgroup of $\text{aut } \mathcal{Z}$ generated by irreducible objects is trivial. In this case \mathcal{Z} becomes the common center of \mathcal{A} and \mathcal{F} . We can therefore decompose these algebras, which in this section are assumed to be separable, w.r.t. \mathcal{Z} . Then the Hilbert C^* -system $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$

becomes a direct integral over $\Gamma := \text{spec } \mathcal{Z}$ and the fibre Hilbert C*-system corresponding to the base point $\lambda \in \Gamma$ is of a DR-type with the same group \mathcal{G} . That means, in particular, that the fixed point algebra associated with a.e. λ has a trivial center. Another simplifying condition of the present situation is the fact that any canonical endomorphism acts trivially on the center, i.e. $\rho \upharpoonright \mathcal{Z} = \text{id} \upharpoonright \mathcal{Z}$. Moreover, we show that in this case the minimality condition already implies the regularity of the corresponding Hilbert C*-system (cf. Corollary 7.7). The special situation studied in this section is also related to the notion of extension of C*-categories by abelian C*-algebras (cf. [41]).

Some conclusions connecting the present analysis to related lines of research are stated in Section 8. Finally, the paper contains an appendix recalling the decomposition of a C*-algebra w.r.t. its center.

2 Basic properties of Hilbert C*-systems

In this section we summarize the structures from superselection theory which we need. For proofs, we refer to the literature if possible, otherwise proofs are included in this paper.

Below \mathcal{F} will always denote a unital C*-algebra. A Hilbert space $\mathcal{H} \subset \mathcal{F}$ is called *algebraic* if the scalar product $\langle \cdot, \cdot \rangle$ of \mathcal{H} is given by $\langle A, B \rangle \mathbb{1} := A^*B$ for $A, B \in \mathcal{H}$. Henceforth, we consider only finite-dimensional algebraic Hilbert spaces. The support $\text{supp } \mathcal{H}$ of \mathcal{H} is defined by $\text{supp } \mathcal{H} := \sum_{j=1}^d \Phi_j \Phi_j^*$, where $\{\Phi_j \mid j = 1, \dots, d\}$ is any orthonormal basis of \mathcal{H} . Unless otherwise specified, we assume below that each considered algebraic Hilbert space \mathcal{H} satisfies $\text{supp } \mathcal{H} = \mathbb{1}$.

We also fix a compact C*-dynamical system $\{\mathcal{F}, \mathcal{G}, \alpha\}$, i.e. \mathcal{G} is a compact group and $\alpha : \mathcal{G} \ni g \rightarrow \alpha_g \in \text{Aut } \mathcal{F}$ is a pointwise norm-continuous morphism. For $D \in \widehat{\mathcal{G}}$ (the dual of \mathcal{G}) its *spectral projection* $\Pi_D \in \mathcal{L}(\mathcal{F})$ is defined by

$$\Pi_D(F) := \int_{\mathcal{G}} \overline{\chi_D(g)} \alpha_g(F) dg \quad \text{for all } F \in \mathcal{F}, \quad (2)$$

$$\text{where: } \chi_D(g) := \dim D \cdot \text{Tr } \pi(g), \quad \pi \in D,$$

and dg is the normalized Haar measure of the compact group \mathcal{G} . The spectrum of $\alpha_{\mathcal{G}}$ can then be defined by

$$\text{spec } \alpha_{\mathcal{G}} := \left\{ D \in \widehat{\mathcal{G}} \mid \Pi_D \neq 0 \right\}.$$

Note that $\text{spec } \alpha_{\mathcal{G}}$ coincides with the so-called Arveson spectrum of $\alpha_{\mathcal{G}}$ (see e.g. [1]).

Our central object of study is:

2.1 Definition *The compact C*-dynamical system $\{\mathcal{F}, \mathcal{G}, \alpha\}$ is called a **Hilbert C*-system** if for each $D \in \widehat{\mathcal{G}}$ there is an algebraic Hilbert space $\mathcal{H}_D \subset \Pi_D \mathcal{F}$, such that $\alpha_{\mathcal{G}}$ acts invariantly on \mathcal{H}_D , and the unitary representation $\alpha_{\mathcal{G}} \upharpoonright \mathcal{H}_D$ is in the equivalence class $D \in \widehat{\mathcal{G}}$.*

We are mainly interested in Hilbert C*-systems whose fixed point algebras coincide such that they appear as extensions of it.

2.2 Definition *A Hilbert C*-system $\{\mathcal{F}, \mathcal{G}, \alpha\}$ is called a **Hilbert extension** of a C*-algebra $A \subset \mathcal{F}$ if A is the fixed point algebra of \mathcal{G} . Two Hilbert extensions $\{\mathcal{F}_i, \mathcal{G}, \alpha^i\}$, $i = 1, 2$ of A (w.r.t. the same group \mathcal{G}) are called **A-module isomorphic** if there is an isomorphism $\tau : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ such that $\tau(A) = A$ for $A \in A$, and τ intertwines the group actions, i.e. $\tau \circ \alpha_g^1 = \alpha_g^2 \circ \tau$, $g \in \mathcal{G}$.*

2.3 Remark (i) For a Hilbert C*-system $\{\mathcal{F}, \mathcal{G}, \alpha\}$ one has $\text{spec } \alpha_{\mathcal{G}} = \widehat{\mathcal{G}}$ and the morphism $\alpha : \mathcal{G} \rightarrow \text{Aut } \mathcal{F}$ is necessarily faithful. So, since \mathcal{G} is compact and $\text{Aut } \mathcal{F}$ is Hausdorff w.r.t. the topology of pointwise norm-convergence, α is a homeomorphism of \mathcal{G} onto its image. Thus \mathcal{G} and $\alpha_{\mathcal{G}}$ are isomorphic as topological groups.

- (ii) Group automorphisms of \mathcal{G} lead to \mathcal{A} -module isomorphic Hilbert extensions of \mathcal{A} , i.e. if $\{\mathcal{F}, \mathcal{G}, \alpha\}$ is a Hilbert extension of \mathcal{A} and ξ an automorphism of \mathcal{G} , then the Hilbert extensions $\{\mathcal{F}, \mathcal{G}, \alpha\}$ and $\{\mathcal{F}, \mathcal{G}, \alpha \circ \xi\}$ are \mathcal{A} -module isomorphic.

Therefore, the Hilbert C*-system $\{\mathcal{F}, \mathcal{G}, \alpha\}$ depends, up to \mathcal{A} -module isomorphisms, only on $\alpha_{\mathcal{G}}$, which is isomorphic to \mathcal{G} . In other words, up to \mathcal{A} -module isomorphism we may identify \mathcal{G} and $\alpha_{\mathcal{G}} \subset \text{Aut } \mathcal{F}$ neglecting the action α which has no relevance from this point of view. Therefore in the following, unless it is otherwise specified, we use the notation $\{\mathcal{F}, \mathcal{G}\}$ for a Hilbert extension of \mathcal{A} , where $\mathcal{G} \subset \text{Aut } \mathcal{F}$.

- (iii) As mentioned above, Hilbert C*-systems arise in DHR-superselection theory (cf. [11, 2]). Mathematically, there are constructions by means of tensor products \mathcal{B} of Cuntz algebras $\mathcal{O}_{\mathcal{H}_u}$, $\mathcal{B} = \otimes_{u \in \text{Ob } \mathcal{R}} \mathcal{O}_{\mathcal{H}_u}$, where \mathcal{R} is a category whose objects u are finite-dimensional continuous unitary representations of a compact group \mathcal{G} on Hilbert spaces \mathcal{H}_u with $\dim \mathcal{H}_u > 1$ and whose arrows are the corresponding intertwining operators (cf. [18, Section 7]). In these examples the center \mathcal{Z} of the fixed point algebra \mathcal{A} is trivial.

Further examples in the context of the CAR-algebra with an abelian group $\mathcal{G} = \mathbb{T}$ and nontrivial center \mathcal{Z} are given in [4]. In Section 6 we construct a family of examples of minimal and regular Hilbert C*-systems for nonabelian groups and with nontrivial \mathcal{Z} .

2.4 Remark A Hilbert C*-system is a very highly structured object;- below we list some important properties (for details, consult [2, 11]):

- (i) Given two \mathcal{G} -invariant algebraic Hilbert spaces $\mathcal{H}, \mathcal{K} \subset \mathcal{F}$, then $\text{span}(\mathcal{H} \cdot \mathcal{K})$ is also a \mathcal{G} -invariant algebraic Hilbert space which we will briefly denote by $\mathcal{H} \cdot \mathcal{K}$. It carries the tensor product of the representations of \mathcal{G} carried by \mathcal{H} and \mathcal{K} .
- (ii) Let \mathcal{H}, \mathcal{K} as before but not necessarily of support $\mathbb{1}$: There is a natural isometric embedding of $\mathcal{L}(\mathcal{H}, \mathcal{K})$ into \mathcal{F} given by

$$\mathcal{L}(\mathcal{H}, \mathcal{K}) \ni T \rightarrow \mathcal{J}(T) := \sum_{j,k} t_{jk} \Psi_j \Phi_k^*, \quad t_{jk} \in \mathbb{C},$$

where $\{\Phi_k\}_k$ resp. $\{\Psi_j\}_j$ are orthonormal basis of \mathcal{H} resp. \mathcal{K} and where

$$T(\Phi_k) = \sum_j t_{jk} \Psi_j,$$

i.e. $(t_{j,k})$ is the matrix of T w.r.t. these orthonormal basis. One has

$$T(\Phi) = \mathcal{J}(T) \cdot \Phi, \quad \Phi \in \mathcal{H}.$$

For simplicity of notation we will often put $\widehat{T} := \mathcal{J}(T)$. Moreover, we have $\widehat{T} \in \mathcal{A}$ iff $T \in \mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{K})$, where $\mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{K})$ denotes the linear subspace of $\mathcal{L}(\mathcal{H}, \mathcal{K})$ consisting of all intertwining operators of the representations of \mathcal{G} on \mathcal{H} and \mathcal{K} (cf. [11, p. 222]).

- (iii) Generally for a Hilbert C*-system, the assignment $D \rightarrow \mathcal{H}_D$ is not unique. If $U \in \mathcal{A}$ is unitary then also $U\mathcal{H}_D \subset \Pi_D \mathcal{F}$ is an \mathcal{G} -invariant algebraic Hilbert space carrying a representation in D . Note that each \mathcal{G} -invariant algebraic Hilbert space \mathcal{K} which carries a representation of D is of this form, i.e. there is a unitary $V \in \mathcal{A}$ such that $\mathcal{K} = V\mathcal{H}_D$.
- (iv) There is a useful partial order on the \mathcal{G} -invariant algebraic Hilbert spaces. We define $\mathcal{H} < \mathcal{K}$ to mean that there is an orthoprojection E on \mathcal{K} such that $E\mathcal{K}$ is invariant w.r.t. \mathcal{G} and the representation $\mathcal{G} \upharpoonright \mathcal{H}$ is unitarily equivalent to $\mathcal{G} \upharpoonright E\mathcal{K}$.

Note that $\mathcal{H} < \mathcal{K}$ iff there is an isometry $V \in \mathcal{A}$ such that $VV^* =: E$ is a projection of \mathcal{K} , i.e. $V\mathcal{H} = E\mathcal{K}$ (use (ii)).

(v) Given a Hilbert C*-system $\{\mathcal{F}, \mathcal{G}\}$ a useful *-subalgebra of \mathcal{F} is

$$\mathcal{F}_{\text{fin}} := \left\{ F \in \mathcal{F} \mid \Pi_D F \neq 0 \text{ for only finitely many } D \in \widehat{\mathcal{G}} \right\}$$

which is dense in \mathcal{F} w.r.t. the C*-norm (cf. [39]).

(vi) The spectral projections satisfy:

$$\begin{aligned} \Pi_{D_1} \Pi_{D_2} &= \Pi_{D_2} \Pi_{D_1} = \delta_{D_1 D_2} \Pi_{D_1} \\ \|\Pi_D\| &\leq d(D)^{3/2}, \quad d(D) := \dim(\mathcal{H}_D), \\ \Pi_D \mathcal{F} &= \text{span}(\mathcal{A} \mathcal{H}_D), \\ \Pi_D(AFB) &= A \cdot \Pi_D(F) \cdot B, \quad A, B \in \mathcal{A}, F \in \mathcal{F}, \\ \mathcal{A} &= \Pi_\iota \mathcal{F}, \end{aligned}$$

where $\iota \in \widehat{\mathcal{G}}$ denotes the trivial representation of \mathcal{G} .

(vii) In \mathcal{F} there is an \mathcal{A} -scalar product given by $\langle F, G \rangle_{\mathcal{A}} := \Pi_\iota F G^*$, w.r.t. which the spectral projections are symmetric, i.e. $\langle \Pi_D F, G \rangle_{\mathcal{A}} = \langle F, \Pi_D G \rangle_{\mathcal{A}}$ for all $F, G \in \mathcal{F}, D \in \widehat{\mathcal{G}}$. Using the \mathcal{A} -scalar product one can define a norm on \mathcal{F} , called the \mathcal{A} -norm

$$\|F\|_{\mathcal{A}} := \|\langle F, F \rangle\|^{1/2}, \quad F \in \mathcal{F}.$$

Note that $\|F\|_{\mathcal{A}} \leq \|F\|$ and that \mathcal{F} in general is not closed w.r.t. the \mathcal{A} -norm.

The following result confirms the importance and naturalness of the previously defined norm $\|\cdot\|_{\mathcal{A}}$ in the context of Hilbert C*-systems. This norm plays also a fundamental role in the so-called inverse superselection theory which reconstructs the Hilbert C*-system from the data \mathcal{A} and a suitable family of endomorphisms of \mathcal{A} (cf. [3, 2, 8]).

2.5 Proposition *Let $\{\mathcal{F}, \mathcal{G}\}$ be a Hilbert C*-system, then for each $F \in \mathcal{F}$ we have*

$$F = \sum_{D \in \widehat{\mathcal{G}}} \Pi_D F \tag{3}$$

where the sum on the right hand side is convergent w.r.t. the \mathcal{A} -norm and we have Parseval's equation:

$$\langle F, F \rangle_{\mathcal{A}} = \sum_{D \in \widehat{\mathcal{G}}} \langle \Pi_D F, \Pi_D F \rangle_{\mathcal{A}}. \tag{4}$$

Proof: Let $\Gamma \subset \widehat{\mathcal{G}}, \text{card } \Gamma < \infty$. The set $\{\Gamma\}$ of all such subsets of $\widehat{\mathcal{G}}$ is a directed net. The assertion (3) means

$$\sum_{D \in \widehat{\mathcal{G}}} \Pi_D F := \lim_{\Gamma \rightarrow \widehat{\mathcal{G}}} F_\Gamma,$$

where

$$F_\Gamma := \sum_{D \in \Gamma} \Pi_D F,$$

and “lim” means convergence w.r.t. the \mathcal{A} -norm. On the other hand, if Γ is fixed, we put

$$G_\Gamma = G_\Gamma(C_D, D \in \Gamma) := \sum_{D \in \Gamma} C_D, \quad C_D \in \Pi_D \mathcal{F}.$$

Then $G_\Gamma \in \mathcal{F}_{\text{fin}}$. By a simple calculation one obtains

$$\langle F - G_\Gamma, F - G_\Gamma \rangle_{\mathcal{A}} = \langle F, F \rangle_{\mathcal{A}} - \sum_{D \in \Gamma} \langle \Pi_D F, \Pi_D F \rangle_{\mathcal{A}} + \sum_{D \in \Gamma} \langle \Pi_D F - C_D, \Pi_D F - C_D \rangle_{\mathcal{A}}.$$

Since

$$\sum_{D \in \Gamma} \langle \Pi_D F - C_D, \Pi_D F - C_D \rangle_{\mathcal{A}} \geq 0$$

we obtain

$$\begin{aligned} \langle F - G_\Gamma, F - G_\Gamma \rangle_{\mathcal{A}} &\geq \langle F, F \rangle_{\mathcal{A}} - \sum_{D \in \Gamma} \langle \Pi_D F, \Pi_D F \rangle_{\mathcal{A}} \\ &= \left\langle F - \sum_{D \in \Gamma} \Pi_D F, F - \sum_{D \in \Gamma} \Pi_D F \right\rangle_{\mathcal{A}} \geq 0. \end{aligned} \quad (5)$$

Therefore

$$\langle F, F \rangle_{\mathcal{A}} \geq \sum_{D \in \Gamma} \langle \Pi_D F, \Pi_D F \rangle_{\mathcal{A}}$$

Since $\|X\| \geq |X|_{\mathcal{A}}$ for all $X \in \mathcal{F}$ we have

$$\|F - G_\Gamma\| \geq |F - G_\Gamma|_{\mathcal{A}} \geq |F - F_\Gamma|_{\mathcal{A}}.$$

According to Shiga's theorem (see [39]) the left hand side can be chosen arbitrary small for suitable Γ and suitable coefficients C_D . Hence $|F - F_\Gamma|_{\mathcal{A}} \rightarrow 0$ for $\Gamma \rightarrow \widehat{\mathcal{G}}$ follows. This is (3) and this implies

$$\lim_{\Gamma \rightarrow \widehat{\mathcal{G}}} \left\| \langle F, F \rangle_{\mathcal{A}} - \sum_{D \in \Gamma} \langle \Pi_D F, \Pi_D F \rangle_{\mathcal{A}} \right\| = 0,$$

which proves (4). ■

Note that (3) does not in general converge w.r.t. the C^* -norm $\|\cdot\|$.

2.6 Corollary (i) Each $F \in \mathcal{F}$ is uniquely determined by its projections $\Pi_D F$, $D \in \widehat{\mathcal{G}}$, i.e. $F = 0$ iff $\Pi_D F = 0$ for all $D \in \widehat{\mathcal{G}}$.

(ii) We have that $|\Pi_D|_{\mathcal{A}} = 1$ for all $D \in \widehat{\mathcal{G}}$, where $|\cdot|_{\mathcal{A}}$ denotes the operator norm of Π_D w.r.t. the norm $|\cdot|_{\mathcal{A}}$ in \mathcal{F} .

3 Two natural examples of C^* -categories associated with a Hilbert C^* -system

In the following we introduce two important examples of C^* -categories that naturally appear in the context of Hilbert C^* -systems. For the general definition and further properties of tensor C^* -categories we refer to [21, 35]. We mention only that the notion of an **irreducible object** introduced in [7, Section 5] (see also [8]) can be defined for arbitrary tensor C^* -categories \mathfrak{T} : $\rho \in \text{Ob } \mathfrak{T}$ is called **irreducible** if

$$(\rho, \rho) = 1_\rho \times (\iota, \iota), \quad (6)$$

where ι denotes the unit for the tensor product of objects, 1_ρ is the unit of the unital C^* -algebra (ρ, ρ) and \times is the tensor product of arrows. We denote the set of all irreducible objects in \mathfrak{T} by **Irr** \mathfrak{T} .

3.1 The category $\mathcal{T}_{\mathcal{G}}$ of all \mathcal{G} -invariant algebraic Hilbert spaces

The \mathcal{G} -invariant algebraic Hilbert spaces \mathcal{H} of $\{\mathcal{F}, \mathcal{G}\}$, satisfying $\text{supp } \mathcal{H} = \mathbb{1}$, form the objects of a C^* -category $\mathcal{T}_{\mathcal{G}}$ whose arrows are given by $(\mathcal{H}, \mathcal{K}) := \mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{K})) \subset \mathcal{A}$. The tensor product of objects is given by the product in \mathcal{F} , the unit object is $\iota := \mathbb{C}\mathbb{1}$ and $(\iota, \iota) = \mathbb{C}\mathbb{1}$. The composition of arrows $\mathcal{J}(T) \in (\mathcal{H}, \mathcal{H}')$, $\mathcal{J}(S) \in (\mathcal{K}, \mathcal{K}')$ is given by $\mathcal{J}(T \otimes S) \in (\mathcal{H}\mathcal{K}, \mathcal{H}'\mathcal{K}')$, where $T \otimes S \in \mathcal{L}_{\mathcal{G}}(\mathcal{H} \otimes \mathcal{K}, \mathcal{H}' \otimes \mathcal{K}')$. \mathcal{H} is irreducible iff $(\mathcal{H}, \mathcal{H}) = \mathbb{C}\mathbb{1}$ (Schur's lemma).

We will focus next on the additional structure of $\mathcal{T}_{\mathcal{G}}$. For this recall the partial order in $\text{Ob } \mathcal{T}_{\mathcal{G}}$ given in Remark 2.4 (iv). If $\mathcal{K} \in \text{Ob } \mathcal{T}_{\mathcal{G}}$ is given, an object $\mathcal{H} < \mathcal{K}$ is called a *subobject* of \mathcal{K} . If $E \in \mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathcal{K}))$ is an orthoprojection $0 < E < \mathbb{1}$, i.e. E is a reducing projection for the representation of \mathcal{G} on \mathcal{K} , then the question arises whether there is an object \mathcal{H} such that the representations on \mathcal{H} and $E\mathcal{K}$ are unitarily equivalent. This suggests the concept of *closedness* of $\mathcal{T}_{\mathcal{G}}$ w.r.t. subobjects.

3.1 Definition *The category $\mathcal{T}_{\mathcal{G}}$ is closed w.r.t. subobjects if to each $\mathcal{K} \in \text{Ob } \mathcal{T}_{\mathcal{G}}$ and to each nontrivial orthoprojection $E \in \mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathcal{K}))$ there is an isometry $V \in \mathcal{A}$ with $VV^* = E$. (In this case $\mathcal{H} := V^*\mathcal{K}$ is a subobject $\mathcal{H} < \mathcal{K}$ assigned to E .)*

Second, if $V, W \in \mathcal{A}$ are isometries with $VV^* + WW^* = \mathbb{1}$ and $\mathcal{H}, \mathcal{K} \in \text{Ob } \mathcal{T}_{\mathcal{G}}$ then we call the algebraic Hilbert space $V\mathcal{H} + W\mathcal{K}$ of support $\mathbb{1}$ a *direct sum* of \mathcal{H} and \mathcal{K} . It is \mathcal{G} -invariant and carries the direct sum of the representations on \mathcal{H} and \mathcal{K} . Therefore we define

3.2 Definition *The category $\mathcal{T}_{\mathcal{G}}$ is closed w.r.t. direct sums if to each $\mathcal{H}_1, \mathcal{H}_2 \in \text{Ob } \mathcal{T}_{\mathcal{G}}$ there is an object $\mathcal{K} \in \text{Ob } \mathcal{T}_{\mathcal{G}}$ and there are isometries $V_1, V_2 \in \mathcal{A}$ with $V_1V_1^* + V_2V_2^* = \mathbb{1}$ such that $\mathcal{K} = V_1\mathcal{H}_1 + V_2\mathcal{H}_2$.*

Since $\text{Ob } \mathcal{T}_{\mathcal{G}}$ contains *all* \mathcal{G} -invariant algebraic Hilbert spaces, $\mathcal{T}_{\mathcal{G}}$ is always closed w.r.t. direct sums provided that \mathcal{A} contains a pair V, W of isometries with $VV^* + WW^* = \mathbb{1}$. This condition for \mathcal{A} we call *Property B*. It will play an important role in the rest of the paper. We have

3.3 Proposition *If \mathcal{A} satisfies Property B then $\mathcal{T}_{\mathcal{G}}$ is closed w.r.t. direct sums.*

3.4 Proposition (i) *If the category $\mathcal{T}_{\mathcal{G}}$ is closed w.r.t. direct sums, then it is closed w.r.t. subobjects.*

(ii) *Let \mathcal{G} be nonabelian. If the category $\mathcal{T}_{\mathcal{G}}$ is closed w.r.t. subobjects, then it is closed w.r.t. direct sums.*

Proof: (i) First we assume that $\mathcal{T}_{\mathcal{G}}$ is closed w.r.t. direct sums. Note that in this case for any $n \in \mathbb{N}$ there are isometries $W_j \in \mathcal{A}, j = 1, 2, \dots, n$, such that $\sum_{j=1}^n W_jW_j^* = \mathbb{1}$. Then to each finite-dimensional representation U there is a \mathcal{G} -invariant algebraic Hilbert space \mathcal{K} such that U is realized on \mathcal{K} , because $U = \bigoplus_{D \in \widehat{\mathcal{G}}} m_D \cdot U_D, U_D \in D$, and in \mathcal{A} there are isometries $W_{D,l}, l = 1, 2, \dots, m_D$ (m_D being the multiplicity of U_D in the decomposition of U) such that $\sum_{D,l} W_{D,l}W_{D,l}^* = \mathbb{1}$. Therefore $\mathcal{K} := \sum_{D,l} W_{D,l}\mathcal{H}_D$ is an object from $\mathcal{T}_{\mathcal{G}}$ and carries exactly the representation U .

Now let $\mathcal{K} \in \text{Ob } \mathcal{T}_{\mathcal{G}}$ and $E \in \mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathcal{K})), 0 < E < \mathbb{1}$, a reducing projection, i.e. $E\mathcal{K} \subset \mathcal{K}$ is a reducing subspace that carries a certain representation of \mathcal{G} . Note that $\text{supp } E\mathcal{K} \neq \mathbb{1}$. Nevertheless there is an object $\mathcal{H} \in \text{Ob } \mathcal{T}_{\mathcal{G}}$ which carries this representation: choose in $E\mathcal{K}$ and \mathcal{H} the (orthonormal) basis $\{\Phi_j\}_j$ of $E\mathcal{K}$, $\{\Psi_j\}_j$ of \mathcal{H} in such a way that the representation matrices coincide. Put $A := \sum_j \Phi_j\Psi_j^*$. Then $A \in \mathcal{A}$, $A^*A = \mathbb{1}$ and $AA^* = E$ follows, i.e. \mathcal{H} is a subobject of \mathcal{K} w.r.t. E .

(ii) Now we assume that $\mathcal{T}_{\mathcal{G}}$ is closed w.r.t. subobjects. Then choose $\mathcal{H}_{D_1}, \mathcal{H}_{D_2} \in \text{Ob } \mathcal{T}_{\mathcal{G}}$, whose dimensions are larger than 1. Then $\mathcal{K} := \mathcal{H}_{D_1} \cdot \mathcal{H}_{D_2} \in \text{Ob } \mathcal{T}_{\mathcal{G}}$ and it carries the reducible

representation $U_{D_1} \otimes U_{D_2}$, i.e. there is a projection E , $0 < E < \mathbb{1}$, $E \in \mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathcal{K}))$. Then to E and $\mathbb{1} - E$ there correspond isometries $V, W \in \mathcal{A}$ with $VV^* + WW^* = \mathbb{1}$, hence $\mathcal{T}_{\mathcal{G}}$ is closed w.r.t. direct sums. \blacksquare

3.5 Remark Note that if the group \mathcal{G} is a compact abelian then $\widehat{\mathcal{G}}$ is a discrete abelian *group*, the character group. Pontryagin's duality theorem shows that in this case the notions of direct sums and subobjects are irrelevant for the duality theory (see also Remark 4.15). If the compact group is non abelian the duality theory changes radically and closure under direct sums and subobjects become essential properties.

3.2 The category \mathcal{T} of all canonical endomorphisms

3.6 Definition To each \mathcal{G} -invariant algebraic Hilbert space $\mathcal{H} \subset \mathcal{F}$ there is assigned a corresponding inner endomorphism $\rho_{\mathcal{H}} \in \text{End } \mathcal{F}$ given by

$$\rho_{\mathcal{H}}(F) := \sum_{j=1}^{d(\mathcal{H})} \Phi_j F \Phi_j^*,$$

where $\{\Phi_j \mid j = 1, \dots, d(\mathcal{H})\}$ is any orthonormal basis of \mathcal{H} . We call **canonical endomorphism** the restriction of $\rho_{\mathcal{H}}$ to \mathcal{A} , i.e. $\rho_{\mathcal{H}} \upharpoonright \mathcal{A} \in \text{end } \mathcal{A}$.

3.7 Remark (i) Note that the definition of the canonical endomorphisms uses terms of \mathcal{F} explicitly. Therefore, the question arises whether the inner endomorphisms $\rho_{\mathcal{H}}$ can be characterized by intrinsic properties of their restriction to \mathcal{A} (see the beginning of Section 4 below). This interplay between the inner and the canonical endomorphisms $\rho_{\mathcal{H}}$ resp. $\rho_{\mathcal{H}} \upharpoonright \mathcal{A}$ plays an essential role in the DR-theory. Below, we omit the restriction symbol and regard the $\rho_{\mathcal{H}}$ also as endomorphisms of \mathcal{A} . We will identify the set of canonical endomorphisms of \mathcal{A} as the objects of a very important category with interesting closure properties.

- (ii) If the emphasis is only on the class $D \in \widehat{\mathcal{G}}$ and not on its corresponding algebraic Hilbert space \mathcal{H}_D , we will write ρ_D instead of $\rho_{\mathcal{H}_D}$.
- (iii) Note that $\Phi A = \rho_{\mathcal{H}}(A)\Phi$ for all $\Phi \in \mathcal{H}$ and $A \in \mathcal{A}$.
- (iv) Note that the identity endomorphism ι is assigned to $\mathcal{H} = \mathbb{C}\mathbb{1}$, i.e. $\rho_{\mathbb{C}\mathbb{1}} := \iota$.
- (v) Let \mathcal{H}, \mathcal{K} be as before, then $\rho_{\mathcal{H}} \circ \rho_{\mathcal{K}} = \rho_{\mathcal{H} \cdot \mathcal{K}}$.
- (vi) Whilst an invariant algebraic Hilbert space uniquely determines its canonical endomorphism, in general the converse does not hold.

3.8 Proposition Let \mathcal{H}, \mathcal{K} be \mathcal{G} -invariant algebraic Hilbert spaces. Then: $\rho_{\mathcal{H}} \upharpoonright \mathcal{A} = \rho_{\mathcal{K}} \upharpoonright \mathcal{A}$ iff $\Psi^* \Phi \in \mathcal{A}' \cap \mathcal{F}$ for all $\Phi \in \mathcal{H}, \Psi \in \mathcal{K}$.

Proof: It is straightforward to check the condition for orthonormal basis of \mathcal{H} and \mathcal{K} . \blacksquare

3.9 Definition Let $\{\mathcal{F}, \mathcal{G}\}$ be a Hilbert C^* -system with fixed point algebra \mathcal{A} . The **intertwiner space** of canonical endomorphisms σ, τ is:

$$(\sigma, \tau) := \{X \in \mathcal{A} \mid X\sigma(A) = \tau(A)X \text{ for all } A \in \mathcal{A}\}$$

and this is a complex Banach space. We will say that $\sigma, \tau \in \text{End } \mathcal{A}$ are **mutually disjoint** if $(\sigma, \tau) = \{0\}$ when $\sigma \neq \tau$.

We denote by \mathcal{T} the category with objects consisting of the canonical endomorphisms $\rho_{\mathcal{H}}$ for \mathcal{G} -invariant algebraic Hilbert spaces $\mathcal{H} \subset \mathcal{F}$ with $\text{supp } \mathcal{H} = \mathbb{1}$ and with arrows given by the intertwiner spaces.

3.10 Remark (i) \mathcal{T} is the second example of a tensor C^* -category. The tensor product of objects is given by composition of endomorphisms (see Remark 3.7(v)) and $\iota = \text{id}$. The composition of arrows is defined as follows: For $A \in (\sigma, \sigma')$, $B \in (\tau, \tau')$, we put $A \times B := A\sigma(B) \in (\sigma\tau, \sigma'\tau')$.

(ii) We have $(\iota, \iota) = \mathcal{Z} := \text{center of } \mathcal{A}$ and from the Definition in (6) we have $\rho_{\mathcal{H}} \in \text{Ob } \mathcal{T}$ is **irreducible** if $(\rho_{\mathcal{H}}, \rho_{\mathcal{H}}) = \rho_{\mathcal{H}}(\mathcal{Z})$. Note that this corresponds precisely to the case where \mathcal{G} acts irreducibly on \mathcal{H} (see [8, Subsection 3.1] and [7, Section 5]). We denote the set of irreducible objects in \mathcal{T} by $\mathbf{Irr } \mathcal{T}$.

(iii) Recall the isometry $\mathcal{J} : \mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{K}) \longrightarrow \mathcal{A}$ encountered in Remark 2.4(ii). We claim that its image is in fact contained in $(\rho_{\mathcal{H}}, \rho_{\mathcal{K}})$. To see this, let $\Phi \in \mathcal{H}$, $A \in \mathcal{A}$ and $T \in \mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{K})$. Then putting $\widehat{T} := \mathcal{J}(T)$ we have

$$\widehat{T}\rho_{\mathcal{H}}(A)\Phi = \widehat{T}\Phi \cdot A = T(\Phi) \cdot A = \rho_{\mathcal{K}}(A)T(\Phi) = \rho_{\mathcal{K}}(A)\widehat{T} \cdot \Phi$$

hence

$$\widehat{T}\rho_{\mathcal{H}}(A) = \rho_{\mathcal{K}}(A)\widehat{T}$$

i.e. $\widehat{T} \in (\rho_{\mathcal{H}}, \rho_{\mathcal{K}})$ or

$$(\mathcal{H}, \mathcal{K}) = \mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{K})) \subseteq (\rho_{\mathcal{H}}, \rho_{\mathcal{K}}).$$

In general, the inclusion is proper. Note finally, that if $A = \mathcal{J}(T)$, $B = \mathcal{J}(S)$ for $T \in \mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{H}')$, $S \in \mathcal{L}_{\mathcal{G}}(\mathcal{K}, \mathcal{K}')$, then $A \times B = \mathcal{J}(T \otimes S)$, i.e. \times restricted to the intertwiner spaces $(\mathcal{H}, \mathcal{K})$ etc., coincides with the composition of arrows of $\mathcal{T}_{\mathcal{G}}$.

(iv) Recall that $\mathcal{H} < \mathcal{K}$ iff there is an isometry $V \in \mathcal{A}$ such that $VV^* =: E$ is a projection of \mathcal{K} i.e. $V\mathcal{H} = EK$. In this case we have $V \in (\rho_{\mathcal{H}}, \rho_{\mathcal{K}})$ and $E \in (\rho_{\mathcal{K}}, \rho_{\mathcal{K}})$. Moreover, E does not belong to the center of \mathcal{A} .

There is an important connection between the categories $\mathcal{T}_{\mathcal{G}}$ and \mathcal{T} in the case of a trivial relative commutant $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$ (see [18, Lemma 2.4]). Note that in this case \mathcal{A} must have a trivial center $\mathcal{Z} = \mathbb{C}\mathbb{1}$.

3.11 Proposition *There is a faithful functor from the categories $\mathcal{T}_{\mathcal{G}}$ to \mathcal{T} which is a bijection of objects. In general the functor is not full, but if the relative commutant satisfies $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$, then $\mathcal{T}_{\mathcal{G}}$ and \mathcal{T} are isomorphic categories.*

Proof: For the objects the functor is specified by $\mathcal{H} \rightarrow \rho_{\mathcal{H}}$ and for the arrows by $(\mathcal{H}, \mathcal{K}) \ni A \rightarrow A \in (\rho_{\mathcal{H}}, \rho_{\mathcal{K}})$. Note that the compatibility of functor w.r.t. the composition of arrows follows from Remark 3.10 (iii). For the second assertion use Proposition 3.8 and that $\Phi^*A\Psi \in \mathbb{C}\mathbb{1}$ for $\Phi \in \mathcal{K}, \Psi \in \mathcal{H}$. ■

We now want to exhibit closure properties of \mathcal{T} similarly as for $\mathcal{T}_{\mathcal{G}}$.

3.12 Definition (i) $\tau \in \text{Ob } \mathcal{T}$ is a **subobject** of $\sigma \in \text{Ob } \mathcal{T}$, denoted $\tau < \sigma$, if there there is an isometry $V \in (\tau, \sigma)$. In this case $\tau(\cdot) = V^*\sigma(\cdot)V$ and $VV^* =: E \in (\sigma, \sigma)$ follow.

(ii) $\rho \in \text{Ob } \mathcal{T}$ is a **direct sum** of $\sigma, \tau \in \text{Ob } \mathcal{T}$, if there are isometries $V \in (\sigma, \rho)$, $W \in (\tau, \rho)$ with $VV^* + WW^* = \mathbb{1}$ such that

$$\rho(\cdot) = V\sigma(\cdot)V^* + W\tau(\cdot)W^* =: \sigma \oplus \tau.$$

3.13 Remark (i) The subobject relation $\tau < \sigma$ is again a partial order: let $\tau < \sigma$ and $\sigma < \mu$, so that there are isometries $V \in (\tau, \sigma)$ and $W \in (\sigma, \mu)$. Then $WV \in (\tau, \mu)$ is also an isometry, i.e. $\tau < \mu$.

(ii) Note that if $\tau = \rho_{\mathcal{H}}$, $\sigma = \rho_{\mathcal{K}}$ for \mathcal{G} -invariant algebraic Hilbert spaces \mathcal{H}, \mathcal{K} satisfying $\mathcal{H} < \mathcal{K}$, then a fortiori $\tau < \sigma$. However if τ, σ are given and one only knows that there are algebraic Hilbert spaces $\mathcal{H} < \mathcal{K}$, then the transitivity property may not hold in general.

(iii) If $\sigma := \rho_{\mathcal{H}}$, $\tau := \rho_{\mathcal{K}}$ then $\rho = \rho_{\mathcal{L}}$ where $\mathcal{L} := V\mathcal{H} + W\mathcal{K}$.

(iv) A direct sum $\sigma \oplus \tau$, defined above (where a priori ρ is not necessarily an object of \mathcal{T}) with isometries $V, W \in \mathcal{A}$, $VV^* + WW^* = \mathbb{1}$ is only unique up to unitary equivalence, i.e. if ρ, ρ' are direct sums of σ and τ , then there is a unitary $U \in (\rho, \rho')$.

The closedness of \mathcal{T} w.r.t. direct sums is defined by the closedness of $\mathcal{T}_{\mathcal{G}}$ w.r.t. direct sums. The closedness of \mathcal{T} w.r.t. subobjects is defined by the closedness of $\mathcal{T}_{\mathcal{G}}$ w.r.t. subobjects in the following sense: if

$$\rho = \rho_{\mathcal{H}} \in \text{Ob } \mathcal{T} \tag{7}$$

is given, then for all \mathcal{H} satisfying (7) and to each nontrivial projection $E \in \mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathcal{H}))$ there is an isometry $V_{\mathcal{H}} \in \mathcal{A}$ with $V_{\mathcal{H}}V_{\mathcal{H}}^* = E$.

This means

3.14 Proposition *If \mathcal{A} satisfies Property B then \mathcal{T} is closed w.r.t. direct sums and subobjects.*

3.3 Permutation and conjugation structures on $\mathcal{T}_{\mathcal{G}}$: DR-categories

To complete the analysis of the categories $\mathcal{T}_{\mathcal{G}}$ and \mathcal{T} we will recall briefly their permutation and conjugation structure. First, we will consider these structures on $\mathcal{T}_{\mathcal{G}}$ (cf. Remark 3.17 (ii) below). We assume in this subsection that the fixed point algebra \mathcal{A} of $\{\mathcal{F}, \mathcal{G}\}$ satisfies Property B.

3.15 Proposition (Permutation structure) *$\mathcal{T}_{\mathcal{G}}$ has a permutation structure, i.e. a map*

$$\text{Ob } \mathcal{T}_{\mathcal{G}} \ni \mathcal{H}, \mathcal{K} \rightarrow \epsilon(\mathcal{H}, \mathcal{K}) \in (\mathcal{H}\mathcal{K}, \mathcal{K}\mathcal{H}),$$

where $\epsilon(\mathcal{H}, \mathcal{K})$ is unitary and satisfies

$$(i) \quad \epsilon(\mathcal{H}, \mathcal{K}) \cdot \epsilon(\mathcal{K}, \mathcal{H}) = \mathbb{1}.$$

$$(ii) \quad \epsilon(\mathbb{C}\mathbb{1}, \mathcal{H}) = \epsilon(\mathcal{H}, \mathbb{C}\mathbb{1}) = \mathbb{1}.$$

$$(iii) \quad \epsilon(\mathcal{H}_1\mathcal{H}_2, \mathcal{H}_3) = \epsilon(\mathcal{H}_1, \mathcal{H}_3) \cdot \rho_{\mathcal{H}_1}(\epsilon(\mathcal{H}_2, \mathcal{H}_3)) \quad , \quad \mathcal{H}_i \in \text{Ob } \mathcal{T}_{\mathcal{G}}, \quad i = 1, 2, 3.$$

$$(iv) \quad \epsilon(\mathcal{H}', \mathcal{K}') A \times B = B \times A \epsilon(\mathcal{H}, \mathcal{K}), \quad A \in (\mathcal{H}, \mathcal{H}'), \quad B \in (\mathcal{K}, \mathcal{K}').$$

Tensor categories that have a map $\epsilon(\cdot, \cdot)$ satisfying the properties (i)-(iv) adapted from above are called *symmetric* (cf. [21, p. 160]). The map $\epsilon(\cdot, \cdot)$ is also called a *permutator* or *symmetry*.

3.16 Proposition (Conjugation structure) *$\mathcal{T}_{\mathcal{G}}$ has a conjugation structure, i.e. to each $\mathcal{H} \in \text{Ob } \mathcal{T}_{\mathcal{G}}$ there is a conjugated algebraic Hilbert space $\overline{\mathcal{H}} \in \text{Ob } \mathcal{T}_{\mathcal{G}}$ carrying the corresponding conjugated representation and there are conjugates $R_{\mathcal{H}} \in (\mathbb{C}\mathbb{1}, \overline{\mathcal{H}}\mathcal{H})$, $S_{\mathcal{H}} = \epsilon(\overline{\mathcal{H}}, \mathcal{H}) R_{\mathcal{H}}$ such that*

$$S_{\mathcal{H}}^* \rho_{\mathcal{H}}(R_{\mathcal{H}}) = \mathbb{1} \quad \text{and} \quad R_{\mathcal{H}}^* \rho_{\overline{\mathcal{H}}}(S_{\mathcal{H}}) = \mathbb{1}.$$

3.17 Remark (i) A permutator (or symmetry) $\epsilon(\cdot, \cdot)$ as in Proposition 3.15 is given by

$$\epsilon(\mathcal{H}, \mathcal{K}) := \mathcal{J}(\theta(\mathcal{H}, \mathcal{K})),$$

where θ denotes the flip operator of the tensor product $\mathcal{H} \otimes \mathcal{K}$. Let $\{\Phi_i\}_i, \{\Psi_k\}_k$ be orthonormal basis of the algebraic Hilbert spaces \mathcal{H} and \mathcal{K} , respectively. Then

$$\epsilon(\mathcal{H}, \mathcal{K}) = \sum_{i,k} \Psi_k \Phi_i \Psi_k^* \Phi_i^*.$$

If \mathcal{H} carries the direct sum of irreducible representations in $\{D_j\}_j \subset \widehat{\mathcal{G}}$, then $\overline{\mathcal{H}}$ carries the corresponding direct sum of the conjugated representations in $\{\overline{D_j}\}_j \subset \widehat{\mathcal{G}}$. Denote by $\{\overline{\Phi}_i\}_i$ the conjugated basis of $\overline{\mathcal{H}}$ w.r.t. $\{\Phi_i\}_i$, then we have the relation

$$R_{\mathcal{H}} = \sum_i \overline{\Phi}_i \Phi_i.$$

- (ii) It is possible to use the functor in Proposition 3.11 to transfer the corresponding permutation and conjugation structure to \mathcal{T} . Note, nevertheless, that if the inclusion $(\mathcal{H}, \mathcal{K}) \subseteq (\rho_{\mathcal{H}}, \rho_{\mathcal{K}})$ is proper (cf. Remark 3.10 (iii)), then the property corresponding to (iv) in Proposition 3.15 is valid only for a smaller set of arrows.

We can now sum up the rich structure of the category $\mathcal{T}_{\mathcal{G}}$ in the notion of a (Doplicher/Roberts) DR-category (cf. [21]).

3.18 Definition An (abstract) tensor C^* -category $\mathcal{T}_{\mathcal{C}}$ with $(\iota, \iota) = \mathbb{C}\mathbb{1}$, closed w.r.t. direct sums and subobjects, equipped with a permutation and a conjugation structure is called an (abstract) **DR-category**.

The category $\mathcal{T}_{\mathcal{G}}$ introduced in Subsection 3.1 is an example of a DR-category. It is a special *Tannaka-Krein* category for the group \mathcal{G} , where the objects and the arrows are embedded in the algebra \mathcal{F} . Moreover, if $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$ (which implies $\mathcal{Z} = \mathbb{C}\mathbb{1}$), the category \mathcal{T} of canonical endomorphisms is another example of a DR-category (cf. Proposition 3.11).

3.19 Remark In the context of the DR-theory we can associate with any $\rho \in \text{Irr } \mathcal{T}_{\mathcal{C}}$ a unique element $D \in \widehat{\mathcal{G}}$, where \mathcal{G} is the group associated with the DR-category $\mathcal{T}_{\mathcal{C}}$. We denote by $\text{Irr}_0 \mathcal{T}_{\mathcal{C}}$ a complete system of irreducible and mutually disjoint objects.

One of the most fundamental results associated with DR-categories is the existence of an integer-valued **dimension function** on the objects of $\mathcal{T}_{\mathcal{C}}$. It is defined as follows: Let $\rho \in \text{Ob } \mathcal{T}_{\mathcal{C}}$ and $R_{\rho} \in (\iota, \overline{\rho})$ a conjugate. Then

$$d(\rho)\mathbb{1} := R_{\rho}^* R_{\rho} \in (\iota, \iota) = \mathbb{C}\mathbb{1}. \quad (8)$$

The dimension function $d(\cdot)$ is independent of the choice of conjugates and gives the same value on unitarily equivalent objects. Moreover, it satisfies the following properties (cf. [21, Sections 2] or [11, Subsection 11.1.6]):

3.20 Proposition Let $\text{Ob } \mathcal{T}_{\mathcal{C}} \ni \rho \rightarrow d(\rho)$ be the dimension function defined above. Then for $\rho, \rho_1, \rho_2 \in \text{Ob } \mathcal{T}_{\mathcal{C}}$ we have

(i) $d(\rho) \in \mathbb{N}$.

(ii) $d(\iota) = 1$ and $d(\overline{\rho}) = d(\rho)$.

$$(iii) \quad d(\rho_1 \circ \rho_2) = d(\rho_1) d(\rho_2) \quad \text{and} \quad d(\rho_1 \oplus \rho_2) = d(\rho_1) + d(\rho_2)$$

$$(iv) \quad \lambda = \bigoplus_{j=1}^r \bigoplus_{l=1}^{m(\rho_j, \lambda)} \rho_{jl}, \text{ with } \rho_{jl} := \rho_j \in \text{Irr}_0 \mathcal{T}_c \text{ (recall Remark 3.19), } l = 1, 2, \dots, m(\rho_j, \lambda) \\ \text{and } d(\lambda) = \sum_{j=1}^r m(\rho_j, \lambda) d(\rho_j), \text{ where } (\rho, \lambda)_c \text{ are algebraic Hilbert spaces and } m(\rho, \lambda) := \\ \dim(\rho, \lambda)_c.$$

3.4 The ideal structure of Hilbert C*-Systems

Given a Hilbert C*-system $\{\mathcal{F}, \mathcal{G}\}$, we will analyze in the present subsection the relation between the ideal structures of \mathcal{F} and of the fixed point algebra \mathcal{A} . It is clear that these must be closely related since \mathcal{F} can be generated from \mathcal{A} and $\{\mathcal{H}_D\}_{D \in \widehat{\mathcal{G}}}$, the latter being a complete system of irreducible algebraic Hilbert spaces with support $\mathbb{1}$ (cf. Definition 2.1 and Remark 2.4 (v),(vi)).

First we introduce the following weaker notions of simplicity which are natural in the context of Hilbert C*-systems.

3.21 Definition $\{\mathcal{F}, \mathcal{G}\}$ denotes a Hilbert C*-system.

- (i) Let $\mathcal{E} \subset \mathcal{F}$ be a closed two-sided ideal in \mathcal{F} , i.e. $\mathcal{E} \triangleleft \mathcal{F}$. We say that \mathcal{E} is **\mathcal{G} -invariant** if $g(\mathcal{E}) \subseteq \mathcal{E}$, $g \in \mathcal{G}$. \mathcal{F} is **\mathcal{G} -simple** if it has no nontrivial \mathcal{G} -invariant closed two-sided ideals.
- (ii) Let $\mathcal{I} \subset \mathcal{A}$ be a closed two-sided ideal in \mathcal{A} , i.e. $\mathcal{I} \triangleleft \mathcal{A}$. We say that \mathcal{I} is **ρ -invariant** if $\rho_D(\mathcal{I}) \subseteq \mathcal{I}$, $D \in \widehat{\mathcal{G}}$, where $\{\rho_D\}_{D \in \widehat{\mathcal{G}}}$ are the canonical endomorphisms associated to a complete system $\{\mathcal{H}_D\}_{D \in \widehat{\mathcal{G}}}$ of irreducible algebraic Hilbert spaces with support $\mathbb{1}$. \mathcal{I} is **ρ -simple** if it has no nontrivial ρ -invariant closed two-sided ideals.

3.22 Remark Note that the notion of ρ -simplicity is independent of the particular choice of the system $\{\mathcal{H}_D\}_{D \in \widehat{\mathcal{G}}}$. In fact, if $\mathcal{I} \triangleleft \mathcal{A}$ is ρ -invariant w.r.t. $\{\rho_D\}_{D \in \widehat{\mathcal{G}}}$, then any other unitary equivalent endomorphism $\rho'_D(\cdot) = V \rho_D(\cdot) V^*$, $D \in \widehat{\mathcal{G}}$ and V a unitary in \mathcal{A} , still satisfies

$$\rho'_D(\mathcal{I}) = V \rho_D(\mathcal{I}) V^* \subseteq V \mathcal{I} V^* \subseteq \mathcal{I}.$$

3.23 Lemma Let $\mathcal{E} \triangleleft \mathcal{F}$ be \mathcal{G} -invariant and Π_D the spectral projections defined in (2). Then \mathcal{E} is Π_D -invariant, i.e. $\Pi_D(\mathcal{E}) \subseteq \mathcal{E}$ for all $D \in \widehat{\mathcal{G}}$.

Proof: Let $E \in \mathcal{E}$. By the definition of spectral projection in (2) we have

$$\Pi_D(E) := \int_{\mathcal{G}} \overline{\chi_D(g)} \alpha_g(E) dg.$$

Since \mathcal{E} is closed, we obtain from the definition of the integral and the pointwise norm continuity of the group action that $\Pi_D(E) \in \mathcal{E}$. ■

3.24 Proposition Let $\{\mathcal{F}, \mathcal{G}\}$ be a Hilbert C*-system with fixed point algebra \mathcal{A} . Then we have the following four implications:

$$\mathcal{F} \text{ is simple} \Rightarrow \mathcal{F} \text{ is } \mathcal{G}\text{-simple} \Rightarrow \mathcal{A} \text{ is simple} \Rightarrow \mathcal{A} \text{ is } \rho\text{-simple} \Rightarrow \mathcal{F} \text{ is } \mathcal{G}\text{-simple}.$$

Proof: The first and third implication above are trivial.

i) To show that the \mathcal{G} -simplicity of \mathcal{F} implies the simplicity of \mathcal{A} assume that \mathcal{A} is not simple: let $\mathcal{I} \triangleleft \mathcal{A}$ be a nontrivial closed 2-sided ideal and consider

$$\mathcal{E}_r := \text{clo}_{\|\cdot\|} \text{span} \{\mathcal{I} \mathcal{H}_D \mid D \in \widehat{\mathcal{G}}\}.$$

\mathcal{E}_r is a closed right ideal in \mathcal{F} : indeed, recall that

$$\mathcal{F} := \text{clo}_{\|\cdot\|} \text{span} \{ \mathcal{A} \mathcal{H}_D \mid D \in \widehat{\mathcal{G}} \}$$

and take $A \in \mathcal{A}$. Then

$$\mathcal{I} \mathcal{H}_D \mathcal{A} \mathcal{H}_{D'} = \underbrace{\mathcal{I} \rho_D(A)}_{\in \mathcal{I}} \mathcal{H}_D \mathcal{H}_{D'} \subset \mathcal{E}_r, \quad D, D' \in \widehat{\mathcal{G}},$$

where the latter inclusion follows from the fact that the tensor product $\mathcal{H}_D \mathcal{H}_{D'}$ can be decomposed in terms of irreducible algebraic Hilbert spaces: indeed, for $H \in \mathcal{H}_D$, $H' \in \mathcal{H}_{D'}$, there are $A_{D,k} \in \mathcal{A}$ such that $H \cdot H' = \sum_{D,k} A_{D,k} \Phi_{D,k}$. Thus we have shown that

$$\text{span} \{ \mathcal{I} \mathcal{H}_D \mid D \in \widehat{\mathcal{G}} \} \cdot \text{span} \{ \mathcal{A} \mathcal{H}_{D'} \mid D' \in \widehat{\mathcal{G}} \} \subset \mathcal{E}_r.$$

Take now $\{F_n\}_n \subset \text{span} \{ \mathcal{A} \mathcal{H}_{D'} \mid D' \in \widehat{\mathcal{G}} \}$ such that $F_n \rightarrow F \in \mathcal{F}$. Then for any $E_0 \in \text{span} \{ \mathcal{I} \mathcal{H}_D \mid D \in \widehat{\mathcal{G}} \}$ we have $E_0 F \in \mathcal{E}_r$, because $\mathcal{E}_r \ni E_0 F_n \rightarrow E_0 F$. Similarly one can show that $EF \in \mathcal{E}_r$ for all $E \in \mathcal{E}_r$, $F \in \mathcal{F}$, hence \mathcal{E}_r is a closed right ideal in \mathcal{F} . This implies that $\mathcal{E} := \mathcal{E}_r \cap \mathcal{E}_r^* \triangleleft \mathcal{F}$ is a nonzero closed 2-sided ideal in \mathcal{F} , which is proper since $\mathbb{1} \notin \mathcal{E}$. Finally, \mathcal{E} is also \mathcal{G} -invariant, because $g(\mathcal{I}) = \mathcal{I} \subseteq \mathcal{A}$ and $g(\mathcal{H}_D) = \mathcal{H}_D$, $g \in \mathcal{G}$, $D \in \widehat{\mathcal{G}}$. Summing up we have shown that if \mathcal{A} is not simple, then \mathcal{F} is not \mathcal{G} -simple.

ii) To show the last implication, assume that \mathcal{F} is not \mathcal{G} -simple: let $\mathcal{E} \triangleleft \mathcal{F}$ be a nontrivial, \mathcal{G} -invariant and closed 2-sided ideal. According to Lemma 3.23 we have that \mathcal{E} is also Π_D -invariant for $D \in \widehat{\mathcal{G}}$ and we define the following closed two-sided ideal in \mathcal{A} :

$$\mathcal{I} := \Pi_\ell(\mathcal{E}) = \mathcal{E} \cap \mathcal{A} \triangleleft \mathcal{A}.$$

We still need to show that \mathcal{I} is ρ -invariant and nontrivial. Since \mathcal{E} is a two-sided ideal in \mathcal{F} we have for any $D \in \widehat{\mathcal{G}}$ and any $X \in \mathcal{I} = \mathcal{E} \cap \mathcal{A}$

$$\rho_D(X) = \sum_k \Phi_{D,k} X \Phi_{D,k}^* \in \mathcal{E} \cap \mathcal{A},$$

where $\{\Phi_{D,k}\}_k$ is an orthonormal basis of \mathcal{H}_D . Thus \mathcal{I} is ρ -invariant. Moreover, \mathcal{I} is proper because \mathcal{E} is proper: $\mathbb{1} \notin \mathcal{I} \subset \mathcal{E}$. To conclude the proof we have to show that $\mathcal{I} \neq \{0\}$. For this choose an element $E' \in \mathcal{E}$ with $E' \neq 0$. Since \mathcal{E} is Π_D -invariant (cf. Lemma 3.23) we have $\Pi_D(E') \in \mathcal{E}$ for all $D \in \widehat{\mathcal{G}}$ and according to Corollary 2.6 (i) there is a $D \in \widehat{\mathcal{G}}$ such that $E := \Pi_D(E') \neq 0$. Then we can write

$$E = \sum_k A_k \Phi_{D,k} \quad \text{for some } A_k \in \mathcal{A}$$

and at least one of the coefficients does not vanish, say $A_{k_0} \neq 0$. Since \mathcal{E} is a two-sided ideal in \mathcal{F} we have $E \Phi_{D,k_0}^* \in \mathcal{E}$. Then we compute

$$\begin{aligned} \Pi_\ell(E \Phi_{D,k_0}^*) &= \int_{\mathcal{G}} g(E \Phi_{D,k_0}^*) dg \\ &= \sum_k A_k \int_{\mathcal{G}} g(\Phi_{D,k}) g(\Phi_{D,k_0}^*) dg \\ &= \sum_k A_k \cdot \sum_{k', i'} \left(\int_{\mathcal{G}} U_{k',k}(g) \overline{U_{i',k_0}(g)} dg \right) \Phi_{D,k'} \Phi_{D,i'}^* \\ &= \sum_k A_k \cdot \sum_{k', i'} \frac{1}{d(D)} \delta_{k', i'} \delta_{k, k_0} \Phi_{D,k'} \Phi_{D,i'}^* \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{d(D)} A_{k_0} \sum_{k'} \Phi_{D,k'} \Phi_{D,k'}^* \\
&= \frac{1}{d(D)} A_{k_0} \neq 0,
\end{aligned}$$

where $d(D)$ is the dimension of the representation $U \in D$. In the previous equations we have used that $\text{supp } \mathcal{H}_D = \mathbb{1}$, the orthogonality of the matrix elements $U_{k',k}(g)$ (recall Peter-Weyl's Theorem [33, Theorem 27.40]) as well as the transformation $g(\Phi_{D,k}) = \sum_{k'} U_{k',k}(g) \Phi_{D,k'}$. We have thus shown that \mathcal{I} is nonzero, since

$$0 \neq \frac{1}{d(D)} A_{k_0} = \Pi_l(E \Phi_{D,k_0}^*) \in \mathcal{I}$$

and the proof is concluded. \blacksquare

3.25 Corollary *If $\mathcal{I} \triangleleft \mathcal{A}$ is ρ -invariant, then the closed right ideal defined by*

$$\mathcal{E} := \text{clo}_{\|\cdot\|} \text{span} \{ \mathcal{I} \mathcal{H}_D \mid D \in \widehat{\mathcal{G}} \}$$

satisfies $\mathcal{E}^ = \mathcal{E}$, hence is a closed two-sided ideal in \mathcal{F} .*

Proof: First we show that $\mathcal{E}^* \subseteq \mathcal{E}$ by using the conjugation structure of $\mathcal{T}_{\mathcal{G}}$: \mathcal{E}^* is generated by $\mathcal{H}_D^* \mathcal{I}$, $D \in \widehat{\mathcal{G}}$. From Remark 3.17 (i) it follows that $\Phi_{D,k}^* = R_D^* \Phi_{\overline{D},k}$ and from this we obtain

$$\mathcal{H}_D^* \mathcal{I} = R_D^* \mathcal{H}_{\overline{D}} \mathcal{I} = R_D^* \rho_{\overline{D}}(\mathcal{I}) \mathcal{H}_{\overline{D}} \subseteq \mathcal{I} \mathcal{H}_{\overline{D}} \subseteq \mathcal{E}.$$

This shows the inclusion $\mathcal{E}^* \subseteq \mathcal{E}$ and from $\mathcal{E}^* \subseteq \mathcal{E} = (\mathcal{E}^*)^* \subseteq \mathcal{E}^*$ we get the equality $\mathcal{E}^* = \mathcal{E}$. \blacksquare

4 Minimal and regular Hilbert C^* -systems

The DR-theorem associates with a DR-category \mathcal{T} an essentially unique compact group \mathcal{G} [21, Theorem 6.1]. In the context of Hilbert C^* -systems we have a bijective correspondence between

$$\{ \mathcal{A}, \mathcal{T} \} \quad \text{and} \quad \{ \mathcal{F}, \mathcal{G} \},$$

where \mathcal{A} is a unital C^* -algebra with trivial center $\mathcal{Z} = \mathcal{A}' \cap \mathcal{A} = \mathbb{C}\mathbb{1}$ (and satisfying Property B) and \mathcal{T} is a DR-category realized as unital endomorphisms of \mathcal{A} . $\{ \mathcal{F}, \mathcal{G} \}$ is a Hilbert extension of \mathcal{A} having trivial relative commutant, i.e. $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$ (see [21, 22, 20, 3]). This correspondence is connected with the second part of Proposition 3.11 which requires a trivial center of \mathcal{A} . The DR-theorem says that in the case of Hilbert extensions of \mathcal{A} with trivial relative commutant, the category \mathcal{T} of all canonical endomorphisms can be indeed characterized intrinsically by their abstract algebraic properties as endomorphisms of \mathcal{A} and a corresponding bijection can be established.

In this section we want to extend such a bijective correspondence to C^* -algebras \mathcal{A} with non-trivial center $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$ and satisfying Property B. A first step in this direction is given in [8]. In this context and due to Proposition 3.11 one has to face the problem that the category $\mathcal{T}_{\mathcal{G}}$ and \mathcal{T} *can not* be isomorphic anymore, since now we have

$$\mathbb{C}\mathbb{1} \subset \mathcal{Z} \subseteq \mathcal{A}' \cap \mathcal{F}.$$

We will investigate in the following the class of Hilbert extensions $\{ \mathcal{F}, \mathcal{G} \}$ with compact group \mathcal{G} and where the relative commutant satisfies the following *minimality* condition

$$\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}. \tag{9}$$

In items (i)-(iv) of the introduction we gave several motivations that justify this choice. Therefore we define

4.1 Definition A Hilbert C^* -system $\{\mathcal{F}, \mathcal{G}\}$ is called **minimal** if the condition

$$\mathcal{A}' \cap \mathcal{F} = \mathcal{Z} \tag{10}$$

is satisfied.

4.2 Remark (i) The adjective minimal comes from the property of the relative commutant. Note that one always has $\mathcal{Z} \subseteq \mathcal{A}' \cap \mathcal{F}$. In the context of the DR-theory one has also minimal Hilbert C^* -systems, because there $\mathcal{Z} = \mathbb{C}\mathbb{1}$ and $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$.

(ii) Let $\{\mathcal{F}, \mathcal{G}\}$ be a C^* -dynamical system with fixed point algebra \mathcal{A} having trivial center $\mathcal{Z}(\mathcal{A}) = \mathbb{C}\mathbb{1}$ and relative commutant satisfying

$$\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}(\mathcal{F}) \quad \text{with} \quad \mathcal{Z}(\mathcal{F}) = \mathcal{F}' \cap \mathcal{F}.$$

Then $\{\mathcal{F}, \mathcal{G}\}$ can be obtained by inducing up from an essentially unique C^* -dynamical system $\{\mathcal{F}_0, \mathcal{G}_0\}$, where \mathcal{G}_0 is a closed subgroup of \mathcal{G} , the fixed point algebra coincides with \mathcal{A} and the relative commutant is trivial, i.e. $\mathcal{A}' \cap \mathcal{F}_0 = \mathbb{C}\mathbb{1}$ (cf. [17, Theorem 1]). For a generalization of this result in the case where $\mathcal{Z}(\mathcal{A})$ is nontrivial and the corresponding relative commutant satisfies

$$\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}(\mathcal{A}) \vee \mathcal{Z}(\mathcal{F})$$

see [42].

4.3 Proposition Let $\{\mathcal{F}, \mathcal{G}\}$ be a given Hilbert C^* -system. Then $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$ iff $(\rho_D, \rho_{D'}) = \{0\}$ for $D \neq D'$, i.e. iff the set $\{\rho_D \mid D \in \widehat{\mathcal{G}}\}$ is mutually disjoint.

Proof: First note that $F \in \mathcal{A}' \cap \mathcal{F}$ iff $\Pi_D F \in (\rho_D, \iota)\mathcal{H}_D$ for all $D \in \widehat{\mathcal{G}}$. Therefore $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$ iff $(\rho_D, \iota) = 0$ for all $D \neq \iota$. But if this is true then also $(\rho_D, \rho_{D'}) = 0$ follows for all $D \neq D'$ (see e.g. [11, p. 193]). ■

Observe that in any Hilbert C^* -system, for each $\tau \in \text{Ob } \mathcal{T}$ the space $\mathfrak{H}_\tau := \mathcal{H}_\tau \mathcal{Z}$, (where \mathcal{H}_τ is a \mathcal{G} -invariant algebraic Hilbert space) is a \mathcal{G} -invariant free right Hilbert \mathcal{Z} -module with inner product given as usual by

$$\langle H_1, H_2 \rangle := H_1^* H_2 \in \mathcal{Z}, \quad H_1, H_2 \in \mathfrak{H}_\tau.$$

Moreover, since for any $\tau \in \text{Ob } \mathcal{T}$, we have that $\mathcal{Z} \subset (\tau, \tau)$, it is easy to see that there is a canonical left action of \mathcal{Z} on \mathfrak{H}_τ . Concretely, there is a natural $*$ -homomorphism $\mathcal{Z} \rightarrow \mathcal{L}_{\mathcal{Z}}(\mathfrak{H})$ (see [7, Sections 3 and 4] for more details). Hence \mathfrak{H}_τ becomes a \mathcal{Z} -bimodule. Next we state the isomorphism between the category of canonical endomorphisms and the corresponding category of \mathcal{Z} -bimodules.

4.4 Proposition Let $\{\mathcal{F}, \mathcal{G}\}$ be a given minimal Hilbert C^* -system, where the fixed point algebra \mathcal{A} has center \mathcal{Z} . Then the category \mathcal{T} of all canonical endomorphisms of $\{\mathcal{F}, \mathcal{G}\}$ is isomorphic to the subcategory $\mathcal{M}_{\mathcal{G}}$ of the category of free Hilbert \mathcal{Z} -bimodules with objects $\mathfrak{H} = \mathcal{H}\mathcal{Z}$, $\mathcal{H} \in \text{Ob } \mathcal{T}_{\mathcal{G}}$, and arrows given by the corresponding \mathcal{G} -invariant module morphisms $\mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1, \mathfrak{H}_2; \mathcal{G})$.

The bijection of objects is given by $\rho_{\mathcal{H}} \leftrightarrow \mathfrak{H} = \mathcal{H}\mathcal{Z}$ which satisfies the conditions

$$\begin{aligned} \rho_{\mathcal{H}} &= (\text{Ad } V) \circ \rho_1 + (\text{Ad } W) \circ \rho_2 \quad \longleftrightarrow \quad \mathfrak{H} = V\mathfrak{H}_1 + W\mathfrak{H}_2 \\ \rho_1 \circ \rho_2 &\quad \longleftrightarrow \quad \mathfrak{H}_1 \cdot \mathfrak{H}_2, \end{aligned}$$

where the latter product is the inner tensor product of the the Hilbert \mathcal{Z} -modules w.r.t. the $*$ -homomorphism $\mathcal{Z} \rightarrow \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_2)$. The bijection on arrows is defined by

$$\mathcal{J}: \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1, \mathfrak{H}_2; \mathcal{G}) \rightarrow (\rho_1, \rho_2) \quad \text{with} \quad \mathcal{J}(T) := \sum_{j,k} \Psi_j Z_{j,k} \Phi_k^* =: \widehat{T}.$$

Here $\{\Psi_j\}_j, \{\Phi_k\}_k$ are orthonormal basis of $\mathfrak{H}_2, \mathfrak{H}_1$, respectively, and $(Z_{j,k})_{j,k}$ is the matrix of the right \mathcal{Z} -linear operator T from \mathfrak{H}_1 to \mathfrak{H}_2 which intertwines the \mathcal{G} -action.

Proof: Note first that the minimality condition and Proposition 3.8 guarantee that the bijection on objects is independent of the choice of the algebraic Hilbert spaces within \mathfrak{H} , provided these define the same canonical endomorphism. The rest of the proof is in Proposition 3.1 and Section 4 of [7]. \blacksquare

4.5 Remark Note that the category $\mathcal{M}_{\mathcal{G}}$ is a tensor C^* -category. This follows from the fact that

$$\widehat{T} Z = Z \widehat{T}, \quad Z \in \mathcal{Z}, \quad T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1, \mathfrak{H}_2; \mathcal{G}),$$

where $\widehat{T} \in \mathcal{A}$ is defined in Remark 2.4 (ii). The previous equation implies

$$T(Z \cdot) = Z T(\cdot), \quad Z \in \mathcal{Z}, \quad T \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1, \mathfrak{H}_2; \mathcal{G}),$$

and by [16, p. 268] this condition guarantees that $\mathcal{M}_{\mathcal{G}}$ is a tensor C^* -category. Note that in general the category of Hilbert \mathcal{Z} -bimodules with the larger arrow sets $\mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1, \mathfrak{H}_2)$ is only a *semitensor C^* -category* (cf. [16, Section 2] for the definition of this notion and further details).¹

The following result recalls the useful decomposition for a general Hilbert \mathcal{Z} -module $\mathfrak{H} \in \text{Ob } \mathcal{M}_{\mathcal{G}}$ in terms of irreducible ones $\mathfrak{H}_D = \mathcal{H}_D \mathcal{Z}$, $D \in \widehat{\mathcal{G}}$. From Proposition 4.4 one has equivalently a decomposition of endomorphism $\rho_{\mathcal{H}} \in \text{Ob } \mathcal{T}$ in terms of the corresponding irreducibles $\rho_D \in \text{Irr } \mathcal{T}$.

4.6 Proposition *Let $\mathfrak{H} \in \text{Ob } \mathcal{M}_{\mathcal{G}}$ be a \mathcal{G} -invariant free Hilbert \mathcal{Z} -module in \mathcal{F} . Then \mathfrak{H} can be decomposed into the following orthogonal direct sum:*

$$\mathfrak{H} = \bigoplus_D (\rho_D, \rho_{\mathcal{H}}) \mathfrak{H}_D.$$

If $\{W_{D,l}\}_{l=1}^{m(D)}$ denotes an orthonormal basis of $(\rho_D, \rho_{\mathcal{H}})$, where $m(D)$ is the multiplicity of $D \in \widehat{\mathcal{G}}$ in the decomposition of $U_{\mathcal{H}}$, then the isotypical projection can be written as

$$P_D := \sum_{l=1}^{m(D)} W_{D,l} W_{D,l}^*.$$

The canonical endomorphism associated with \mathcal{H} is given by

$$\rho_{\mathcal{H}}(A) = \sum_{D,l} W_{D,l} \cdot \rho_D(A) \cdot W_{D,l}^*.$$

Recalling the notion of ρ invariant ideals in \mathcal{A} stated in Definition 3.21 (ii) we have the following immediate consequence of the previous decomposition result for canonical endomorphisms.

4.7 Corollary *Let $\{\mathcal{F}, \mathcal{G}\}$ be a minimal Hilbert C^* -system with fixed point algebra \mathcal{A} and let \mathcal{I} be a closed two-sided ideal in \mathcal{A} . Then \mathcal{I} is ρ -invariant iff $\rho(\mathcal{I}) \subseteq \mathcal{I}$ for all $\rho \in \text{Ob } \mathcal{T}$.*

¹We would like to acknowledge an anonymous referee for mentioning reference [16] to us.

The Proposition 4.4 above shows that the canonical endomorphisms uniquely fix the corresponding \mathcal{Z} -modules, but not the choice of the generating algebraic Hilbert spaces. The assumption of the minimality condition (10) is crucial here. From the point of view of the \mathcal{Z} -modules it is natural to consider next the following property of Hilbert C^* -systems: the existence of a special choice of algebraic Hilbert spaces within the modules that define the canonical endomorphisms and which is compatible with products.

4.8 Definition *A Hilbert C^* -system $\{\mathcal{F}, \mathcal{G}\}$ is called **regular** if there is an assignment*

$$\text{Ob } \mathcal{T} \ni \sigma \rightarrow \mathcal{H}_\sigma \in \text{Ob } \mathcal{T}_G$$

such that $\sigma = \rho_{\mathcal{H}_\sigma}$, i.e. σ is the canonical endomorphism of the algebraic Hilbert space \mathcal{H}_σ , and which is compatible with products:

$$\sigma \circ \tau \rightarrow \mathcal{H}_\sigma \cdot \mathcal{H}_\tau .$$

4.9 Remark (i) In a minimal Hilbert C^* -system regularity means that there is a “generating” Hilbert space $\mathcal{H}_\tau \subset \mathfrak{H}_\tau$ for each τ (with $\mathfrak{H}_\tau = \mathcal{H}_\tau \mathcal{Z}$) such that the compatibility relation for products stated in Definition 4.8 holds. If a Hilbert C^* -system is minimal and $\mathcal{Z} = \mathbb{C}\mathbb{1}$ then it is necessarily regular.

(ii) Note that the minimality condition and the compactness of the group imply that the Hilbert modules associated with objects in \mathcal{T} are free. From the point of view of crossed products by endomorphisms considered in [43] this corresponds to a special case (cf. [43, Example 4.1]). Nevertheless, even in this particular situation with a compact group, there are cases where still one can not associate a symmetry to the larger category \mathcal{T} . In the context of Hilbert C^* -modules this means that the left action *does not* coincide with the corresponding right action (in contrast with the situation considered in [43, Section 4]). The existence of a symmetry is related to the nontriviality of the chain group homomorphism to be introduced in the next section (see also Proposition 5.10, Remark 7.3 (i) and Section 8).

4.10 Lemma *Let $\{\mathcal{F}, \mathcal{G}\}$ be a minimal and regular Hilbert C^* -system. For $\sigma, \tau \in \text{Ob } \mathcal{T}$ and $\rho, \rho' \in \text{Irr } \mathcal{T}$ put*

$$(\sigma, \tau)_\mathbb{C} := \mathcal{J}(\mathcal{L}_G(\mathcal{H}_\sigma, \mathcal{H}_\tau)) \subseteq (\sigma, \tau) , \quad (\rho, \tau)_\mathbb{C} := \mathcal{J}(\mathcal{L}_G(\mathcal{H}_\rho, \mathcal{H}_\tau)) \subseteq (\rho, \tau) , \text{ etc. ,}$$

(cf. Remark 3.10 (iii)). If $\{W_{\tau, \rho, k}\}_k \subset (\rho, \tau)_\mathbb{C}$, $\{W_{\sigma, \rho', k'}\}_{k'} \subset (\rho', \sigma)_\mathbb{C}$ are orthonormal basis, then

$$\{W_{\tau, \rho, k} \cdot W_{\sigma, \rho', k'}^*\}_{\rho, k, k'}$$

is an orthonormal basis of $(\sigma, \tau)_\mathbb{C}$, where the ρ 's are those irreducibles appearing in the decomposition of σ and τ (cf. Proposition 4.6).

Proof: From the orthonormality relations of $\{W_{\tau, \rho, k}\}_k$ and $\{W_{\sigma, \rho', k'}\}_{k'}$ and from the disjointness relation for irreducible objects in Proposition 4.3, it follows directly that $\{W_{\tau, \rho, k} \cdot W_{\sigma, \rho', k'}^*\}_{\rho, k, k'}$ is an orthonormal system in $(\sigma, \tau)_\mathbb{C}$. For any $A \in (\sigma, \tau)_\mathbb{C}$ put

$$\lambda_{\rho, k, k'} := W_{\tau, \rho, k}^* A W_{\sigma, \rho, k'} \in (\rho, \rho)_\mathbb{C} = \mathbb{C}\mathbb{1} ,$$

where the last equation follows from the fact that the algebraic Hilbert space corresponding to ρ carries an irreducible representation. Then it is immediate to verify that

$$A = \sum_{\rho, k, k'} \lambda_{\rho, k, k'} W_{\tau, \rho, k} W_{\sigma, \rho, k'}^* ,$$

hence $\{W_{\tau, \rho, k} \cdot W_{\sigma, \rho, k'}^*\}_{\rho, k, k'}$ is an orthonormal basis in $(\sigma, \tau)_\mathbb{C}$. ■

4.11 Theorem *Let $\{\mathcal{F}, \mathcal{G}\}$ be a minimal and regular Hilbert C^* -system (where the fixed point algebra \mathcal{A} satisfies Property B). Then \mathcal{T} contains a DR-subcategory $\mathcal{T}_{\mathbb{C}}$ with the same objects, $\text{Ob } \mathcal{T}_{\mathbb{C}} = \text{Ob } \mathcal{T}$, and arrows $(\sigma, \tau)_{\mathbb{C}} \subseteq (\sigma, \tau)$ such that:*

$$(\sigma, \tau) = (\sigma, \tau)_{\mathbb{C}} \sigma(\mathcal{Z}) = \tau(\mathcal{Z}) (\sigma, \tau)_{\mathbb{C}} \sigma(\mathcal{Z}). \quad (11)$$

Proof: Let \mathcal{H}, \mathcal{K} are \mathcal{G} -invariant algebraic Hilbert spaces. Recall that the isometry $\mathcal{J} : \mathcal{L}(\mathcal{H}, \mathcal{K}) \rightarrow \mathcal{F}$ has the property

$$(\mathcal{H}, \mathcal{K}) = \mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{K})) \subseteq (\rho_{\mathcal{H}}, \rho_{\mathcal{K}}),$$

(cf. Remark 3.10 (iii)). Now let $\sigma \rightarrow \mathcal{H}_{\sigma}$ be the assignment given in Definition 4.8 for regular Hilbert C^* -systems, and put

$$(\sigma, \tau)_{\mathbb{C}} := (\mathcal{H}_{\sigma}, \mathcal{H}_{\tau}).$$

Then the definitions of the symmetry $\epsilon(\cdot, \cdot)$, the conjugates $\bar{\sigma}$ and their intertwiners R_{σ}, S_{σ} are as follows (cf. Remark 3.17 (i)):

$$\epsilon(\sigma, \tau) := \sum_{j,k} \Psi_k \Phi_j \Psi_k^* \Phi_j^*,$$

where $\{\Phi_j\}_j$ resp. $\{\Psi_k\}_k$ are orthonormal basis of \mathcal{H}_{σ} resp. \mathcal{H}_{τ} . We also define: $\bar{\sigma} := \rho_{\mathcal{K}}$, where \mathcal{K} carries the representation of \mathcal{G} which is conjugated to the representation on \mathcal{H}_{σ} . Recall that all finite-dimensional representations of \mathcal{G} can be realized by some \mathcal{G} -invariant algebraic Hilbert space. Moreover, \mathcal{K} is chosen as the ‘‘distinguished’’ Hilbert space according to the assumption of regularity. Let $\{\Omega_j\}_j$ be an orthonormal basis of \mathcal{K} and put

$$R_{\sigma} := \sum_j \Omega_j \Phi_j, \quad S_{\sigma} := \epsilon(\bar{\sigma}, \sigma) R_{\sigma}.$$

With these choices it is easy to verify that $\mathcal{T}_{\mathbb{C}}$ is indeed a DR-subcategory of \mathcal{T} (cf. Definition 3.18).

It remains to show Eq. (11). The inclusion $(\sigma, \tau) \supseteq (\sigma, \tau)_{\mathbb{C}} \sigma(\mathcal{Z})$ follows immediately from the fact that (σ, τ) is a right $\sigma(\mathcal{Z})$ -module. To show the reverse inclusion let $\{W_{\tau, \rho, k}\}_k \subset (\rho, \tau)_{\mathbb{C}}$, $\{W_{\sigma, \rho', k'}\}_{k'} \subset (\rho', \sigma)_{\mathbb{C}}$ be orthonormal basis as in Lemma 4.10. Take $A \in (\sigma, \tau)$ and define

$$\rho(Z_{\rho, k, k'}) := W_{\tau, \rho, k}^* A W_{\sigma, \rho, k'} \in (\rho, \rho) = \rho(\mathcal{Z}).$$

Then

$$\begin{aligned} A &= \sum_{\rho, k, k'} W_{\tau, \rho, k} \rho(Z_{\rho, k, k'}) W_{\sigma, \rho, k'}^* \\ &= \sum_{\rho, k, k'} W_{\tau, \rho, k} W_{\sigma, \rho, k'}^* \sigma(Z_{\rho, k, k'}) \in (\sigma, \tau)_{\mathbb{C}} \sigma(\mathcal{Z}), \end{aligned}$$

and the proof of Eq. (11) is completed. ■

4.12 Remark (i) Recall that the category $\mathcal{T}_{\mathcal{G}}$ introduced in Subsection 3.1 is an example of a DR-category (recall Remark 3.10) and in fact it plays the role of the subcategory $\mathcal{T}_{\mathbb{C}}$ as a subcategory of the in general larger category \mathcal{T} of canonical endomorphisms of the minimal and regular Hilbert C^* -system.

(ii) The assumptions of the previous theorem imply that each basis of $(\sigma, \tau)_{\mathbb{C}}$ is simultaneously a module basis of (σ, τ) modulo $\sigma(\mathcal{Z})$ as a right module, i.e. the module (σ, τ) is free.

- (iii) For simplicity we will sometimes call a DR-subcategory $\mathcal{T}_{\mathbb{C}}$ of \mathcal{T} satisfying the properties of Theorem 4.11 **admissible**.
- (iv) Note that the properties (P.2)-(P.4) (with the exception of P.2.6) in [8, Section 2] are contained in the assumption that $\mathcal{T}_{\mathbb{C}}$ is a DR-subcategory of \mathcal{T} (cf. Definition 3.18).

Let $\text{Ob } \mathcal{T} \ni \rho \rightarrow V_{\rho} \in (\rho, \rho)$ be a choice of unitaries such that

$$V_{\rho\circ\sigma} = V_{\rho} \times V_{\sigma}. \quad (12)$$

Note that (12) implies that $V_{\iota} = \mathbb{1}$, because $V_{\sigma} = V_{\iota\sigma} = V_{\iota} \times V_{\sigma} = V_{\iota}V_{\sigma}$. This choice allows one to define from the subcategory $\mathcal{T}_{\mathbb{C}}$ of \mathcal{T} another subcategory $\mathcal{T}'_{\mathbb{C}}$ of \mathcal{T} with the same properties as $\mathcal{T}_{\mathbb{C}}$. To do this, put

$$(\rho, \sigma)'_{\mathbb{C}} := V_{\sigma}(\rho, \sigma)_{\mathbb{C}}V_{\rho}^* \subset (\rho, \sigma) \quad (13)$$

and for the corresponding permutation structure $\epsilon'(\rho, \sigma)$ for $\mathcal{T}'_{\mathbb{C}}$ one takes:

$$\epsilon'(\rho, \sigma) := (V_{\sigma} \times V_{\rho}) \cdot \epsilon(\rho, \sigma) \cdot (V_{\rho} \times V_{\sigma})^*. \quad (14)$$

It is easy to check that ϵ' defines a permutation structure (cf. Proposition 3.15). The corresponding conjugates R'_{ρ} are defined by

$$R'_{\rho} := V_{\bar{\rho}\rho}R_{\rho}, \quad S'_{\rho} := \epsilon'(\rho, \bar{\rho})R'_{\rho} \quad (15)$$

(recall also Proposition 3.16). Then it is straightforward to verify that the new subcategory $\mathcal{T}'_{\mathbb{C}}$ also satisfies the assumptions of Theorem 4.11.

This suggests the following definition of equivalence between subcategories.

4.13 Definition *Two admissible DR-subcategories $\mathcal{T}_{\mathbb{C}}$ and $\mathcal{T}'_{\mathbb{C}}$ of \mathcal{T} are called **equivalent** if there is an assignment*

$$\text{Ob } \mathcal{T} \ni \rho \rightarrow V_{\rho} \in (\rho, \rho), \quad \text{with } V_{\rho} \text{ unitary satisfying } V_{\rho\circ\sigma} = V_{\rho} \times V_{\sigma},$$

and such that the equations (13), (14) and (15) hold.

The converse of Theorem 4.11 gives our main duality theorem. The proof, which is constructive, is divided into several steps (see Subsection 4.1 below).

4.14 Theorem *Let \mathcal{T} be a tensor C^* -category of unital endomorphisms of \mathcal{A} and let $\mathcal{T}_{\mathbb{C}}$ be an admissible DR-subcategory. Then there is a minimal and regular Hilbert extension $\{\mathcal{F}, \mathcal{G}\}$ of \mathcal{A} , where \mathcal{G} is the compact group assigned to the DR-category $\mathcal{T}_{\mathbb{C}}$ and \mathcal{T} is isomorphic to the category of all canonical endomorphisms of $\{\mathcal{F}, \mathcal{G}\}$. Moreover, if $\mathcal{T}_{\mathbb{C}}, \mathcal{T}'_{\mathbb{C}}$ are two admissible DR-subcategories of \mathcal{T} , then the corresponding Hilbert extensions are \mathcal{A} -module isomorphic iff $\mathcal{T}_{\mathbb{C}}$ is equivalent to $\mathcal{T}'_{\mathbb{C}}$.*

The previous result says that for minimal and regular Hilbert C^* -systems there is an intrinsic characterization of the category of all canonical endomorphisms in terms of \mathcal{A} only. Moreover, up to \mathcal{A} -module isomorphisms, there is a bijection between minimal and regular Hilbert extensions and tensor C^* -categories \mathcal{T} of unital endomorphisms of \mathcal{A} with admissible DR-subcategories.

Note that Theorem 4.14 is an immediate generalization of the DR-theorem (mentioned at the beginning of this section) for the case that $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$, i.e. if $\mathcal{Z} = \mathbb{C}\mathbb{1}$ then \mathcal{T} itself is admissible (hence a DR-category) and the corresponding Hilbert extensions have trivial relative commutant. Notice also that from the assumption of the existence of an admissible DR-subcategory it follows that \mathcal{A} satisfies Property B.

4.15 Remark (Abelian groups) In the case that \mathcal{G} is abelian and compact the preceding structure simplifies radically (see Remark 3.5). Specifically, $\widehat{\mathcal{G}}$ is a discrete abelian group (the character group), each \mathcal{H}_D , $D \in \widehat{\mathcal{G}}$, is one-dimensional with a generating unitary U_D , hence the canonical endomorphisms $\rho_{\mathcal{H}_D}$ (denoted briefly by ρ_D , see Remark 3.7(ii)) are in fact automorphisms (necessarily outer on \mathcal{A}). Since $\rho_{D_1} \circ \rho_{D_2} = \rho_{D_1 D_2}$ in this case the set Γ of all canonical endomorphisms $\rho_{\mathcal{H}_D}$ is a group with the property

$$\widehat{\mathcal{G}} \cong \Gamma / \text{int } \mathcal{A}.$$

Therefore, it is not necessary to take into account direct sums or subobjects in this case and one can drop Property B as an assumption on \mathcal{A} .

If in addition $\mathcal{Z} = \mathbb{C}\mathbb{1}$ the permutators ϵ (restricted to $\widehat{\mathcal{G}} \times \widehat{\mathcal{G}}$) are elements of the second cohomology group $H^2(\widehat{\mathcal{G}})$ and

$$U_{D_1} \cdot U_{D_2} = \omega(D_1, D_2) U_{D_1 \circ D_2},$$

where

$$\epsilon(D_1, D_2) = \frac{\omega(D_1, D_2)}{\omega(D_2, D_1)}$$

and ω is a corresponding 2-cocycle. The field algebra \mathcal{F} is just the ω -twisted discrete crossed product of \mathcal{A} with $\widehat{\mathcal{G}}$ (see e.g. [2, p. 86 ff.] for details). For the case $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$ see [9]. (The minimal case is not specially mentioned there.)

4.1 Proof of Theorem 4.14

The construction of the Hilbert extension $\{\mathcal{F}, \mathcal{G}\}$ associated with the pair $\mathcal{T}_c < \mathcal{T}$, where \mathcal{T}_c is an admissible DR-subcategory, is done in several steps which are adapted from [3, Sections 3-6] and [8, 7]. The strategy is to define first a left \mathcal{A} -module \mathcal{F}_0 using the dimension function defined in (8). The structure of \mathcal{F}_0 is then gradually enriched by making use of the properties of \mathcal{T}_c .

1. Step: *The bimodule \mathcal{F}_0 , algebraic Hilbert spaces and free Hilbert \mathcal{Z} -modules.*

To each $\rho \in \text{Irr}_0 \mathcal{T}$ (cf. Remark 3.19) we assign a Hilbert space \mathcal{H}_ρ with $\dim \mathcal{H}_\rho = d(\rho)$ and, using orthonormal bases $\{\Phi_{\rho j}\}_j$ of \mathcal{H}_ρ , we define the left \mathcal{A} -module

$$\mathcal{F}_0 := \left\{ \sum_{\rho, j} A_{\rho j} \Phi_{\rho j} \mid A_{\rho j} \in \mathcal{A}, \text{ finite sum} \right\},$$

where the $\{\Phi_{\rho j}\}_{\rho j}$ form an \mathcal{A} -module basis of \mathcal{F}_0 . \mathcal{F}_0 is independent of the special choice of the bases $\{\Phi_{\rho j}\}_j$ of \mathcal{H}_ρ and putting $\Phi_{\rho j} A := \rho(A) \Phi_{\rho j}$, \mathcal{F}_0 turns out to be a bimodule.

Next we introduce Hilbert spaces for any object in the category. For this purpose recall that $\rho < \alpha$ means that ρ is a subobject of α and that $(\rho, \alpha)_c$ is an algebraic Hilbert space in \mathcal{A} whose dimension coincides with the multiplicity of ρ in the decomposition of α (cf. Proposition 3.20 (iv)). Then we have

$$\mathcal{H}_\alpha := \bigoplus_{\rho < \alpha} (\rho, \alpha)_c \mathcal{H}_\rho \quad \text{and} \quad \mathcal{H}_\alpha \subset \mathcal{F}_0, \alpha \in \text{Ob } \mathcal{T}, \quad (16)$$

as well as the right- \mathcal{Z} -Hilbert modules

$$\mathfrak{H}_\rho := \mathcal{H}_\rho \mathcal{Z} = \mathcal{Z} \mathcal{H}_\rho \quad \text{and} \quad \mathfrak{H}_\alpha := \bigoplus_{\rho < \alpha} (\rho, \alpha) \mathfrak{H}_\rho = \mathcal{H}_\alpha \mathcal{Z},$$

with the corresponding \mathcal{Z} -scalar product

$$\langle X, Y \rangle_\alpha := \sum_{\rho, j} \rho^{-1}(X_{\rho j}^* Y_{\rho j}), \text{ where}$$

$$X = \sum_{\rho,j} X_{\rho j} \Phi_{\rho j}, \quad X_{\rho j} \in (\rho, \alpha), \quad Y = \sum_{\rho,j} Y_{\rho j} \Phi_{\rho j}, \quad Y_{\rho j} \in (\rho, \alpha).$$

The preceding comments show that we have established the following functor \mathfrak{F} between the categories \mathcal{T} (resp. $\mathcal{T}_{\mathbb{C}}$) and the corresponding category of Hilbert \mathcal{Z} -modules (resp. Hilbert spaces); (cf. e.g. [7, Section 4] and [3, Corollary 3.3]).

4.16 Lemma *The functor \mathfrak{F} given by*

$$\text{Ob } \mathcal{T} \ni \alpha \mapsto \mathfrak{H}_{\alpha} \subset \mathcal{F}_0 \quad \text{and} \quad (\alpha, \beta) \ni A \mapsto \mathfrak{F}(A) \in \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_{\alpha} \rightarrow \mathfrak{H}_{\beta}),$$

where $\mathfrak{F}(A)X := AX$, $X \in \mathfrak{H}_{\alpha}$, defines an isomorphism between the corresponding categories and $\mathfrak{F}(A^*)$ is the module adjoint w.r.t. $\langle \cdot, \cdot \rangle_{\alpha}$. Similarly, one can apply \mathfrak{F} to $\mathcal{T}_{\mathbb{C}}$ in order to obtain the associated subcategory of algebraic Hilbert spaces \mathcal{H}_{α} and arrows $\mathfrak{F}((\alpha, \beta)_{\mathbb{C}}) \subset \mathcal{L}(\mathcal{H}_{\alpha} \rightarrow \mathcal{H}_{\beta})$.

Proof: Similar as in [7, p. 791 ff]. ■

2. Step: *Product and *-structure on \mathcal{F}_0 .*

We can now apply results in [3] to the category $\mathfrak{F}(\mathcal{T}_{\mathbb{C}})$

4.17 Lemma *There exists a product structure on \mathcal{F}_0 with the properties*

$$\begin{aligned} \text{span} \{ \Phi \cdot \Psi \mid \Phi \in \mathcal{H}_{\alpha}, \Psi \in \mathcal{H}_{\beta} \} &= \mathcal{H}_{\alpha\beta}, \\ \epsilon(\alpha, \beta) \Phi \Psi &= \Psi \Phi, \quad \Phi \in \mathcal{H}_{\alpha}, \Psi \in \mathcal{H}_{\beta}, \\ \langle XY, X'Y' \rangle_{\alpha\beta} &= \langle X, X' \rangle_{\alpha} \cdot \langle Y, Y' \rangle_{\beta}, \quad X, X' \in \mathcal{H}_{\alpha}, Y, Y' \in \mathcal{H}_{\beta}. \end{aligned}$$

Note that for orthonormal bases $\{\Phi_j\}_j, \{\Psi_k\}_k$ of $\mathcal{H}_{\alpha}, \mathcal{H}_{\beta}$, respectively, we obtain from Lemma 4.17 that

$$\epsilon(\alpha, \beta) = \sum_{j,k} \Psi_k \Phi_j \Psi_k^* \Phi_j^*.$$

As in [3, Section 5] we introduce the notion of a conjugated basis $\Phi_{\bar{\alpha}j}$ of $\mathcal{H}_{\bar{\alpha}}$ w.r.t. an orthonormal basis $\Phi_{\alpha j}$ of \mathcal{H}_{α} such that $R_{\alpha} = \sum_j \Phi_{\bar{\alpha}j} \Phi_{\alpha j}$. This is necessary in order to put a compatible *-structure on \mathcal{F}_0 .

4.18 Lemma *Let $\Phi_{\bar{\rho},j}$ be a conjugated basis corresponding to the basis $\Phi_{\rho,j}$, $\rho \in \text{Irr}_0 \mathcal{T}$, and define $\Phi_{\rho j}^* := R_{\rho}^* \Phi_{\bar{\rho}j}$, $j = 1, 2, \dots, d(\rho)$. Then \mathcal{F}_0 turns into a *-algebra. The Hilbert spaces \mathcal{H}_{α} and the corresponding modules \mathfrak{H}_{α} are algebraic, i.e.*

$$\langle X, Y \rangle_{\alpha} = X^* Y, \quad X, Y \in \mathfrak{H}_{\alpha}.$$

The objects $\alpha \in \text{Ob } \mathcal{T}$ are identified as canonical endomorphisms

$$\alpha(A) = \sum_{j=1}^{d(\alpha)} \Phi_{\alpha j} A \Phi_{\alpha j}^*.$$

3. Step: *C*-norm and completion of \mathcal{F}_0 .*

In \mathcal{F}_0 one has natural projections Π_ρ onto the ρ -component of the decomposition:

$$\Pi_\rho \left(\sum_{\sigma,j} A_{\sigma j} \Phi_{\sigma j} \right) := \sum_{j=1}^{d(\rho)} A_{\rho j} \Phi_{\rho j}, \quad \rho \in \text{Irr}_0 \mathcal{T}.$$

To put a C*-norm $\|\cdot\|_*$ we argue as in [3, Section 6]. Its construction is essentially based on the following \mathcal{A} -scalar product on \mathcal{F}_0

$$\langle F, G \rangle := \sum_{\rho,j} \frac{1}{d(\rho)} A_{\rho j} B_{\rho j}^*, \text{ for } F := \sum_{\rho,j} A_{\rho j} \Phi_{\rho j}, \text{ } G := \sum_{\rho,j} B_{\rho j} \Phi_{\rho j},$$

4.19 Lemma *The scalar product $\langle \cdot, \cdot \rangle$ satisfies $\langle F, G \rangle = \Pi_\nu F G^*$ and Π_ρ is selfadjoint w.r.t. $\langle \cdot, \cdot \rangle$. The projections Π_ρ and the scalar product have continuous extensions to $\mathcal{F} := \text{clo}_{\|\cdot\|_*} \mathcal{F}_0$ and $\Pi_\rho \mathcal{F} = \text{span} \{ \mathcal{A} \mathcal{H}_\rho \}$.*

4. Step: *The compact group \mathcal{G} .*

Finally, the symmetry group w.r.t. $\langle \cdot, \cdot \rangle$ is defined by the subgroup of all automorphisms $g \in \text{aut } \mathcal{F}$ satisfying $\langle gF_1, gF_2 \rangle = \langle F_1, F_2 \rangle$. It leads to

4.20 Lemma *The symmetry group coincides with the stabilizer $\text{stab } \mathcal{A}$ of \mathcal{A} and the modules \mathfrak{H}_α are invariant w.r.t. $\text{stab } \mathcal{A}$.*

Proof: Use [7, Lemma 7.1] (cf. also with [3, Section 6]). ■

This suggests to consider the subgroup $\mathcal{G} \subseteq \text{stab } \mathcal{A}$ consisting of all elements of $\text{stab } \mathcal{A}$ leaving even the Hilbert spaces \mathcal{H}_α invariant. Then it turns out that the pair $\{\mathcal{F}, \mathcal{G}\}$ is a Hilbert extension of \mathcal{A} .

4.21 Lemma *\mathcal{G} is compact and the spectrum $\text{spec } \mathcal{G}$ on \mathcal{F} coincides with the dual $\hat{\mathcal{G}}$. For $\rho \in \text{Irr } \mathcal{T}$ the Hilbert spaces \mathcal{H}_ρ are irreducible w.r.t. \mathcal{G} , i.e. there is a bijection $\text{Irr}_0 \mathcal{T} \ni \rho \leftrightarrow D \in \hat{\mathcal{G}}$. Moreover \mathcal{A} coincides with the fixed point algebra of the action of \mathcal{G} in \mathcal{F} and $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$.*

4.22 Remark The DR-Theorem shows that any DR-category $\mathcal{T}_\mathcal{C}$ has an unique (modulo isomorphisms) compact group \mathcal{G}_{DR} associated with it. We will show here that \mathcal{G}_{DR} coincides with the compact group \mathcal{G} obtained as a subgroup of $\text{stab } \mathcal{A}$ in the previous lemma.

For any $\alpha \in \text{Ob } \mathcal{T}_\mathcal{C} = \text{Ob } \mathcal{T}$ we can assign a Hilbert space as in the first step above:

$$\alpha \rightarrow \mathcal{H}_\alpha := \bigoplus_{\rho < \alpha} (\rho, \alpha)_\mathcal{C} \mathcal{H}_\rho.$$

These Hilbert spaces together with the corresponding arrows (cf. Lemma 4.16) defines a TK-category \mathcal{T}_{TK} for \mathcal{G}_{DR} and by construction it is clear that we have the isomorphism $\mathcal{T}_{\text{TK}} \cong \mathcal{T}_\mathcal{C}$. But at the same time \mathcal{T}_{TK} is a Tannaka-Krein category for \mathcal{G} , since by Lemma 4.21, \mathcal{G} acts on \mathcal{H}_ρ irreducibly, $\rho \in \text{Irr } \mathcal{T}$, and invariantly on \mathcal{H}_α , $\alpha \in \text{Ob } \mathcal{T}$ (recall that $(\rho, \alpha)_\mathcal{C} \subset \mathcal{A}$ and $\mathcal{G} \subset \text{stab } \mathcal{A}$). Therefore \mathcal{G}_{DR} and \mathcal{G} are isomorphic, because they have the same TK-category.

5. Step: *Uniqueness of the Hilbert extension.*

First assume that the subcategories $\mathcal{T}_\mathcal{C}$ and $\mathcal{T}'_\mathcal{C}$ are equivalent. We consider the Hilbert extension \mathcal{F} assigned to $\mathcal{T}_\mathcal{C}$. The corresponding invariant Hilbert spaces are given by (16). Now we change these Hilbert spaces by

$$\mathcal{H}_\alpha \rightarrow V_\alpha \mathcal{H}_\alpha =: \mathcal{H}'_\alpha.$$

Using the function \mathfrak{F} of Lemma 4.16 so that $\mathcal{L}_{\mathcal{G}}(\mathcal{H}_{\alpha} \rightarrow \mathcal{H}_{\beta}) := \mathfrak{F}((\alpha, \beta)_{\mathbb{C}}) \cong (\alpha, \beta)_{\mathbb{C}}$ we obtain

$$\mathcal{L}_{\mathcal{G}}(V_{\alpha}\mathcal{H}_{\alpha} \rightarrow V_{\beta}\mathcal{H}_{\beta}) = V_{\beta}\mathcal{L}_{\mathcal{G}}(\mathcal{H}_{\alpha} \rightarrow \mathcal{H}_{\beta})V_{\alpha}^{*} \cong (\alpha, \beta)_{\mathbb{C}}'. \quad (17)$$

Further, w.r.t. the ‘‘new Hilbert spaces’’ we obtain the ‘primed’ permutators and conjugates of the second subcategory. This means, it is sufficient to prove that if the subcategory $\mathcal{T}_{\mathbb{C}}$ is given, then two Hilbert extensions, assigned to $(\mathcal{T}, \mathcal{T}_{\mathbb{C}})$ according to the first part of the theorem, are always \mathcal{A} -module isomorphic. Now let $\mathcal{F}_1, \mathcal{F}_2$ be two Hilbert extensions assigned to $\mathcal{T}_{\mathbb{C}}$. For $\rho \in \text{Irr}_0 \mathcal{T}$ let $\{\Phi_{\rho j}^1\}_j, \{\Phi_{\rho j}^2\}_k$ be orthonormal bases of the Hilbert spaces $\mathcal{H}_{\rho}^1, \mathcal{H}_{\rho}^2$, respectively. Then

$$\Phi_{\rho j}^r \cdot \Phi_{\sigma k}^r = \sum_{\tau, l} K_{\rho j \sigma k}^{\tau l} \Phi_{\tau l}^r, \quad K_{\rho j \sigma k}^{\tau l} \in (\tau, \rho \sigma)_{\mathbb{C}}, \quad r = 1, 2.$$

Therefore the definition

$$\mathcal{J}\left(\sum_{\rho, j} A_{\rho j} \Phi_{\rho j}^1\right) := \sum_{\rho, j} A_{\rho j} \Phi_{\rho j}^2$$

is easily seen to extend to an \mathcal{A} -module isomorphism from \mathcal{F}_1 onto \mathcal{F}_2 (see [11, p. 203 ff.]).

Second, we assume that the Hilbert extensions $\mathcal{F}_1, \mathcal{F}_2$ assigned to $\mathcal{T}_{\mathbb{C}}^1, \mathcal{T}_{\mathbb{C}}^2$, respectively, are \mathcal{A} -module isomorphic. The \mathcal{G} -invariant Hilbert spaces are given by (16). Now let \mathcal{J} be an \mathcal{A} -module isomorphism $\mathcal{J}: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ so that

$$\mathcal{J}(\mathcal{H}_{\alpha}^1) = \bigoplus_{\rho < \alpha} (\rho, \alpha)_{\mathbb{C}}^1 \mathcal{J}(\mathcal{H}_{\rho}^1)$$

and again the $\mathcal{J}(\mathcal{H}_{\alpha}^1)$ form a system of \mathcal{G} -invariant Hilbert spaces in \mathcal{F}_2 . Further we have the system \mathcal{H}_{α}^2 in \mathcal{F}_2 . That is, to each α we obtain two \mathcal{G} -invariant Hilbert spaces \mathcal{H}_{α}^2 and $\mathcal{J}(\mathcal{H}_{\alpha}^1)$ that are contained in the Hilbert module \mathfrak{H}_{α}^2 . Let $\{\Phi_{\alpha, j}\}_j, \{\Psi_{\alpha, j}\}_j$ be orthonormal bases of $\mathcal{J}(\mathcal{H}_{\alpha}^1), \mathcal{H}_{\alpha}^2$, respectively. Then obviously $V_{\alpha} := \sum_j \Psi_{\alpha, j} \Phi_{\alpha, j}^{*}$ is a unitary with $V_{\alpha} \in (\alpha, \alpha)$ and $\mathcal{H}_{\alpha}^2 = V_{\alpha} \mathcal{J}(\mathcal{H}_{\alpha}^1)$. Further, for $X \in \mathcal{H}_{\alpha}^1, Y \in \mathcal{H}_{\beta}^1$ (hence $XY \in \mathcal{H}_{\alpha\beta}^1$) we have

$$V_{\alpha} \mathcal{J}(X) V_{\beta} \mathcal{J}(Y) = V_{\alpha} \alpha(V_{\beta}) \mathcal{J}(XY) = V_{\alpha\beta} \mathcal{J}(XY),$$

and this implies $V_{\alpha\beta} = V_{\alpha} \times V_{\beta}$. Finally, we argue as in (17) to obtain

$$V_{\beta} (\alpha, \beta)_{\mathbb{C}}^1 V_{\alpha} = (\alpha, \beta)_{\mathbb{C}}^2.$$

and the latter equation implies Eqs. (13)-(15).

5 Minimal Hilbert C*-systems for nonabelian groups

Let $\{\mathcal{F}, \mathcal{G}\}$ be a minimal Hilbert C*-system with \mathcal{G} nonabelian and such that the fixed point algebra \mathcal{A} satisfies Property B. Recall from Remark 3.10 (i) that $\rho \in \text{Irr } \mathcal{T}$ if $(\rho, \rho) = \rho(\mathcal{Z})$. In this case one has trivially that $\mathcal{Z} \subseteq \rho(\mathcal{Z})$. We will next show that for irreducible endomorphisms the previous inclusion is actually an equality.

5.1 Proposition *For any $\rho \in \text{Irr } \mathcal{T}$ we have $\rho(\mathcal{Z}) = \mathcal{Z}$ and $\rho \upharpoonright \mathcal{Z}$ is an automorphism of \mathcal{Z} , i.e. $\rho \in \text{aut } \mathcal{Z}$.*

Proof: Choose an irreducible endomorphisms $\rho \in \text{Irr } \mathcal{T}$ and recall that in this case the conjugated endomorphism is also irreducible, i.e. $(\bar{\rho}, \bar{\rho}) = \bar{\rho}(\mathcal{Z})$. From Proposition 3.16 we have that the conjugates R_{ρ} and $S_{\rho} := \epsilon(\bar{\rho}, \rho) R_{\rho}$ satisfy the relations

$$R_{\rho}^{*} \bar{\rho}(S_{\rho}) = \mathbb{1} \quad \text{and} \quad S_{\rho}^{*} \rho(R_{\rho}) = \mathbb{1},$$

and we can define in terms of these the following two vector space isomorphisms (see e.g. [34]):

$$(\bar{\rho}, \bar{\rho}) \ni Y \rightarrow Y^* R_\rho \in (\iota, \bar{\rho}\rho) \quad \text{and} \quad (\iota, \bar{\rho}\rho) \ni X \rightarrow S_\rho^* \rho(X) \in (\rho, \rho).$$

Composing these isomorphisms in the case of an irreducible pair $\rho, \bar{\rho}$ we obtain

$$(\bar{\rho}, \bar{\rho}) \ni \bar{\rho}(Z) \rightarrow S_\rho^* \rho(\bar{\rho}(Z)^* R_\rho) = Z^* \in (\rho, \rho)$$

i.e.

$$(\bar{\rho}, \bar{\rho}) \ni \bar{\rho}(Z) \rightarrow Z^* \in (\rho, \rho)$$

is a vector space isomorphism from $(\bar{\rho}, \bar{\rho})$ onto (ρ, ρ) . Therefore $\rho(\mathcal{Z}) = \mathcal{Z}$ and $\rho \upharpoonright \mathcal{Z}$ is an automorphism of \mathcal{Z} . \blacksquare

From the previous proposition it follows immediately:

5.2 Corollary (i) *Let $\lambda = \rho_1 \circ \rho_2 \circ \dots \circ \rho_n$, where $\rho_i \in \text{Irr } \mathcal{T}$, $i = 1, \dots, n$. Then $\lambda(\mathcal{Z}) = \mathcal{Z}$ and $\lambda \upharpoonright \mathcal{Z} \in \text{aut } \mathcal{Z}$.*

(ii) *For any unitary $U \in \mathcal{A}$ we have $(\text{ad } U \circ \rho) \upharpoonright \mathcal{Z} = \rho \upharpoonright \mathcal{Z}$, $\rho \in \text{Irr } \mathcal{T}$.*

From the previous result we can now introduce the following automorphism on \mathcal{Z} which only depends on the class $D \in \widehat{\mathcal{G}}$:

$$\widehat{\mathcal{G}} \ni D \rightarrow \alpha_D := \rho_D \upharpoonright \mathcal{Z} \in \text{aut } \mathcal{Z}. \quad (18)$$

We will introduce next an equivalence relation² in $\widehat{\mathcal{G}}$ which, roughly speaking, relates elements $D, D' \in \widehat{\mathcal{G}}$ if there is a ‘‘chain of tensor products’’ of elements in $\widehat{\mathcal{G}}$ containing D and D' . This equivalence relation appears naturally when considering the action of irreducible canonical endomorphisms on \mathcal{Z} (see Theorem 5.7 and Remark 5.8 below).

To make the preceding notion precise recall the algebraic structure of $\widehat{\mathcal{G}}$: denote by \times the natural operation on subsets of $\widehat{\mathcal{G}}$ associated with the decomposition of the tensor products of irreducible representations: for $D_1, D_2 \in \widehat{\mathcal{G}}$ the set $D_1 \times D_2$ contains the corresponding classes that appear in the decomposition of $U_{D_1} \otimes U_{D_2}$, where $U_{D_i} \in D_i$, $i = 1, 2$, is any representant of the corresponding class. That means that if

$$U_{D_1} \otimes U_{D_2} = \bigoplus_{k=1}^s m_k U_{D'_k} \quad (m_k \equiv \text{multiplicities}),$$

then $D_1 \times D_2 = \{D'_1, \dots, D'_s\}$. For $\Gamma, \Gamma_1, \Gamma_2 \subset \widehat{\mathcal{G}}$ put

$$\Gamma_1 \times \Gamma_2 = \cup \{D_1 \times D_2 \mid D_i \in \Gamma_i, i = 1, 2\} \quad \text{and} \quad D \times \Gamma = \{D\} \times \Gamma.$$

Further if $\bar{D} \in \widehat{\mathcal{G}}$ denotes the conjugate class to $D \in \widehat{\mathcal{G}}$ denote by $\bar{\Gamma} = \{\bar{D} \mid D \in \Gamma\}$. Recall in particular that if $D \in D_0 \times D_1$, $D' \in D'_0 \times D'_1$, then $D \times D' \subset D_0 \times D_1 \times D'_0 \times D'_1$ or that $1 \in \Gamma \times \bar{\Gamma}$ (cf. [33, Definition 27.35] for further details).

We can now make precise the previous idea:

5.3 Definition *$D, D' \in \widehat{\mathcal{G}}$ are called equivalent, $D \approx D'$, if there exist $D_1, \dots, D_n \in \widehat{\mathcal{G}}$ such that*

$$D, D' \in D_1 \times \dots \times D_n.$$

²We would like to acknowledge an observation of P.A. Zito concerning the equivalence relation that served to simplify the presentation below.

5.4 Remark (i) It is easy to check that the relation \approx is an equivalence relation in $\widehat{\mathcal{G}}$. Indeed, reflexivity and symmetry are trivial. To show transitivity let $D \approx D'$ as above and $D' \approx D''$, i.e. $D', D'' \in D'_1 \times \dots \times D'_k$ for some $D'_1, \dots, D'_k \in \widehat{\mathcal{G}}$. We will show that D, D'' is contained in the larger chain

$$\Gamma := D_1 \times \dots \times D_n \times \overline{D_1 \times \dots \times D_n} \times D'_1 \times \dots \times D'_k.$$

Indeed, $D'' \in \Gamma$, because $1 \in D_1 \times \dots \times D_n \times \overline{D_1 \times \dots \times D_n}$ and $D'' \in D'_1 \times \dots \times D'_k$. Further, since $\overline{D'} \in \overline{D_1 \times \dots \times D_n}$ and $D' \in D'_1 \times \dots \times D'_k$ we also have that $1 \in \overline{D_1 \times \dots \times D_n} \times D'_1 \times \dots \times D'_k$, and therefore $D \in \Gamma$ (cf. [33, Theorem 27.38]).

Denote by square brackets $[\cdot]$ the corresponding chain equivalence classes and by $\mathfrak{C}(\mathcal{G})$ the set of these equivalence classes, i.e.

$$\mathfrak{C}(\mathcal{G}) := \{[D] \mid D \in \widehat{\mathcal{G}}\}.$$

(ii) Note that *any* pair $D, D' \in D_0 \times D_1$ satisfies by definition $D \approx D'$. Therefore for $D_0, D_1 \in \widehat{\mathcal{G}}$ we have that $D_0 \times D_1$ also specifies an element of $\mathfrak{C}(\mathcal{G})$ and we can simply put

$$[D_0 \times D_1] := [D],$$

where D is any element in $D_0 \times D_1$.

(iii) It is also possible to formulate the previous equivalence relation entirely in terms of characters.

We will define on $\mathfrak{C}(\mathcal{G})$ a product \boxtimes (see Eq. (20) below) so that $(\mathfrak{C}(\mathcal{G}), \boxtimes)$ becomes an abelian group which for simplicity we call **chain group**. Moreover, the chain group can be related to the character group of the center \mathcal{C} of \mathcal{G} . For this recall also the notion of conjugacy class of a representation (cf. [26]): let $D \in \widehat{\mathcal{G}}$ and U_D any representant in D . By Schur's Lemma we have

$$U_D \upharpoonright \mathcal{C} = \Upsilon_D \cdot \mathbb{1}, \quad (19)$$

and it can be easily seen that Υ_D is a character on the center \mathcal{C} of \mathcal{G} which only depends on D , i.e. $\Upsilon_D \in \widehat{\mathcal{C}}$.

5.5 Theorem *Let \mathcal{G} be a compact nonabelian group and denote its center by \mathcal{C} .*

(i) *The set $\mathfrak{C}(\mathcal{G})$ becomes an abelian group w.r.t. the following multiplication: for $D_0, D_1 \in \widehat{\mathcal{G}}$ put*

$$[D_0] \boxtimes [D_1] := [D_0 \times D_1] \quad (20)$$

(recall Remark 5.4 (ii)).

(ii) *The conjugacy classes Υ_D (cf. Eq. (19)) depend on the chain equivalence class $[D]$. The chain group and the character group of the center of \mathcal{G} are isomorphic. The isomorphism is given by*

$$\eta : \mathfrak{C}(\mathcal{G}) \rightarrow \widehat{\mathcal{C}} \quad \text{with} \quad \eta([D]) := \Upsilon_{[D]},$$

where $\Upsilon_{[D]}$ is the conjugacy class associated with $[D] \in \mathfrak{C}(\mathcal{G})$.

Proof: (i) Recall first that by Remark 5.4 (ii) the r.h.s. of (20) is well defined. We need to verify next that the l.h.s. of (20) is independent of the representants $D_0, D_1 \in \widehat{\mathcal{G}}$: suppose that $D_0 \approx D'_0$

as well as $D_1 \approx D'_1$ and we will show that any $D \in D_0 \times D_1$ is related to any $D' \in D'_0 \times D'_1$, i.e. $D \approx D'$, so that they specify the same equivalence class. Let

$$D_i, D'_i \in D_{i_1} \times \dots \times D_{i_{k_i}}, \quad i = 0, 1,$$

for some $D_{i_1}, \dots, D_{i_{k_i}} \in \widehat{\mathcal{G}}$. Then

$$D \in D_0 \times D_1 \subset D_{0_1} \times \dots \times D_{0_{k_0}} \times D_{1_1} \times \dots \times D_{1_{k_1}} \supset D'_0 \times D'_1 \ni D'.$$

This shows that $D \approx D'$ and the product \boxtimes is well defined.

The neutral element of $\mathfrak{C}(\mathcal{G})$ is given by the class generated by the trivial representation 1. The inverse element to $[D]$ is given by $[\overline{D}]$, because $1 \in D \times \overline{D}$. Finally the commutativity of the product is guaranteed by the equation $D_0 \times D_1 = D_1 \times D_0$ (see [33, Theorem 27.38] for details).

(ii) To show that the mapping η is well defined note that for the tensor product of *irreducible* representations U_{D_i} , $i = 1, \dots, n$, we have

$$U_{D_1} \otimes \dots \otimes U_{D_n} \upharpoonright \mathcal{C} = \Upsilon_{D_1} \cdot \dots \cdot \Upsilon_{D_n} \mathbb{1}. \quad (21)$$

Even more, any irreducible representation appearing in the decomposition of $U_{D_1} \otimes \dots \otimes U_{D_n}$ defines the same conjugacy class $\Upsilon_{D_1} \cdot \dots \cdot \Upsilon_{D_n}$. This shows that if $D \approx D'$, then $\Upsilon_D = \Upsilon_{D'}$ and that the η is a group homomorphism from $\mathfrak{C}(\mathcal{G})$ to $\widehat{\mathcal{C}}$:

$$\Upsilon_{[D_1] \boxtimes [D_2]} = \Upsilon_{[D_1 \boxtimes D_2]} = \Upsilon_{[D_1]} \cdot \Upsilon_{[D_2]}.$$

Similarly we can consider the group homomorphism $\widehat{\eta}: \mathcal{C} \rightarrow \widehat{\mathfrak{C}(\mathcal{G})}$ given by

$$\mathcal{C} \ni c \mapsto \widehat{\eta}_c, \quad \text{where} \quad \widehat{\eta}_c([D]) := \Upsilon_{[D]}(c), \quad [D] \in \mathfrak{C}(\mathcal{G}).$$

The homomorphism $\widehat{\eta}$ is injective because if $\widehat{\eta}_{c_0}([D]) = 1$ for all $[D] \in \mathfrak{C}(\mathcal{G})$, then $U_D(c_0) = \mathbb{1}$ for all $D \in \widehat{\mathcal{G}}$, hence $c_0 = e$ (recall from Gelfand-Raikov's theorem [32, Theorem 22.12] that the continuous irreducible unitary representations of \mathcal{G} separate the points of the group).

Finally, the surjectivity of $\widehat{\eta}$ is an application of Tannaka's duality theorem (cf. Theorem 30.40 in [33]). To sketch the argument we need to recall the following facts. Let \mathcal{T}_{TK} be a Tannaka-Krein category associated with the compact group \mathcal{G} . We denote by $\{\mathcal{H}_D\}_{D \in \widehat{\mathcal{G}}}$ a complete set of irreducible objects. A representation \mathfrak{r} of \mathcal{T}_{TK} is an assignment

$$\text{Ob } \mathcal{T}_{\text{TK}} \ni \mathcal{H} \rightarrow \mathfrak{r}(\mathcal{H}) \in \mathcal{U}(\mathcal{H}) \quad (\text{Unitary operators on } \mathcal{H}),$$

which is compatible with the direct sums, the tensor products, the conjugation structure and the arrows of \mathcal{T}_{TK} (see properties $T_1 - T_6$ in 30.34 of [33] for further details). Now any character $\chi \in \widehat{\mathfrak{C}(\mathcal{G})}$ specifies the following assignment

$$\mathcal{H}_D \rightarrow c_\chi(\mathcal{H}_D) := \chi([D]) \cdot \mathbb{1}_{\mathcal{H}_D} \in \mathcal{U}(\mathcal{H}_D), \quad D \in \widehat{\mathcal{G}}.$$

Taking into account direct sums, tensor products, the conjugation structure and the arrows of \mathcal{T}_{TK} we may extend $c_\chi(\cdot)$ to a representation of \mathcal{T}_{TK} . By Tannaka's duality theorem $c_\chi(\cdot)$ specifies an element of \mathcal{G} . Even more, $c_\chi(\cdot) \in \mathcal{C}$, because

$$c_\chi(\mathcal{H}_D) \cdot g(\mathcal{H}_D) = g(\mathcal{H}_D) \cdot c_\chi(\mathcal{H}_D), \quad D \in \widehat{\mathcal{G}}$$

implies

$$c_\chi(\mathcal{H}) \cdot g(\mathcal{H}) = g(\mathcal{H}) \cdot c_\chi(\mathcal{H}), \quad \mathcal{H} \in \text{Ob } \mathcal{T}_{\text{TK}}.$$

Therefore $\widehat{\eta}_{c_\chi} = \chi$ and we have shown that \mathcal{C} and $\widehat{\mathfrak{C}(\mathcal{G})}$ are isomorphic. Pontryagin's duality theorem concludes the proof. \blacksquare

5.6 Remark (i) The injectivity of the mapping η in the previous theorem was stated as a conjecture in the first version of this paper (see Conjecture 5.10 in [6]). This conjecture was then proved by M. Müger in [37]. For the sake of completeness we have included a simple proof of this property. We also refer to [37] for further nice consequences of the isomorphism $\mathfrak{C}(\mathcal{G}) \cong \widehat{\mathcal{C}}$ in the context of fusion categories.

(ii) We will leave for the end of this section the computation of chain groups associated with several finite and compact Lie groups. In all the examples we will show explicitly that the chain group $\mathfrak{C}(\mathcal{G})$ is isomorphic to the character group $\widehat{\mathcal{C}(\mathcal{G})}$ of the center of \mathcal{G} .

We will now concentrate on the relation of the chain group $\mathfrak{C}(\mathcal{G})$, associated with the group \mathcal{G} of a Hilbert C^* -system $\{\mathcal{F}, \mathcal{G}\}$, with the irreducible canonical endomorphisms restricted to \mathcal{Z} . In particular recall the automorphisms on \mathcal{Z} given in Eq. (18) by $\alpha_D := \rho_D \upharpoonright \mathcal{Z} \in \text{aut } \mathcal{Z}$ which are associated with any class $D \in \widehat{\mathcal{G}}$.

5.7 Theorem (i) *Let $D, D' \in \widehat{\mathcal{G}}$ be equivalent, i.e. $D \approx D'$. Then $\alpha_D = \alpha_{D'}$ and we can associate the automorphism $\alpha_{[D]} \in \text{aut } \mathcal{Z}$ with the chain group element $[D] \in \mathfrak{C}(\mathcal{G})$.*

(ii) *There is a natural group homomorphism between the chain group and the automorphism group generated by the irreducible endomorphisms restricted to \mathcal{Z} (cf. Proposition 5.1):*

$$\mathfrak{C}(\mathcal{G}) \ni [D] \mapsto \alpha_{[D]} \in \text{aut } \mathcal{Z}. \quad (22)$$

Proof: (i) First we show that if $\lambda = \rho_1 \circ \dots \circ \rho_n$ is a finite product of irreducibles whose dimensions are larger than one, then $\lambda \upharpoonright \mathcal{Z} = \alpha_D$ for all $D \in \widehat{\mathcal{G}}$ appearing in the decomposition of λ : for this recall from Proposition 4.6 that λ can be decomposed as

$$\lambda(\cdot) = \sum_{D,j} W_{D,j} \rho_D(\cdot) W_{D,j}^*, \quad (23)$$

where the ρ_D 's, $D \in \widehat{\mathcal{G}}$, are irreducible and $\{W_{D,j}\}_j$ is an orthonormal basis of (ρ_D, λ) . Then by Corollary 5.2 (i) and Proposition 5.1 we have $\lambda(Z) \in \mathcal{Z}$ for all $Z \in \mathcal{Z}$ as well as

$$\lambda(Z) = \sum_{D,j} W_{D,j} \alpha_D(Z) W_{D,j}^* = \sum_D \alpha_D(Z) \sum_j W_{D,j} W_{D,j}^* = \sum_D \alpha_D(Z) E_D,$$

where

$$E_D := \sum_j W_{D,j} W_{D,j}^* \quad (24)$$

is the so-called isotypical projection w.r.t. $D \in \widehat{\mathcal{G}}$. Therefore $\sum_D \alpha_D(Z) E_D = \lambda(Z)$ or $\sum_D (\alpha_D(Z) - \lambda(Z)) E_D = 0$ and this implies $(\alpha_D(Z) - \lambda(Z)) E_D = 0$ for all D appearing in the decomposition of λ . Using (24) and the orthogonality relations of the $W_{D,j}$'s we obtain finally $\alpha_D = \lambda \upharpoonright \mathcal{Z}$, hence all irreducibles ρ_D occurring in the decomposition of λ coincide on \mathcal{Z} .

Second, let $D \approx D'$, i.e. $D, D' \in D_1 \times \dots \times D_n$ for some $D_1, \dots, D_n \in \widehat{\mathcal{G}}$ (cf. Definition 5.3). For the endomorphisms this implies that ρ_D and $\rho_{D'}$ appear in the decomposition of $\rho_{D_1} \circ \dots \circ \rho_{D_n}$. The arguments of the first part of the proof show that

$$\alpha_D = \alpha_{D'}.$$

(ii) The homomorphism property follows immediately from the arguments of the proof of part (i) since for any $D_1, D_2 \in \widehat{\mathcal{G}}$ we have

$$\alpha_{[D_1]} \circ \alpha_{[D_2]} = \alpha_{[D_1 \times D_2]} = \alpha_{[D_1] \boxtimes [D_2]}$$

(recall Theorem 5.5 (i)). ■

5.8 Remark Note that the chain group and in particular Theorem 5.7 (i) completes the picture of the action of the irreducible canonical endomorphisms on the center \mathcal{Z} of the fixed point algebra \mathcal{A} (recall also Eq. (18)). Indeed, we may summarize this action by means of the following diagram

$$\begin{array}{ccccc} \widehat{\mathcal{G}} & \rightarrow & \mathfrak{C}(\mathcal{G}) & \rightarrow & \text{aut } \mathcal{Z} \\ D & \mapsto & [D] & \mapsto & \alpha_{[D]} \end{array}$$

5.9 Theorem *Let $\lambda \in \text{Ob } \mathcal{T}$. Then its action on \mathcal{Z} can be described by means of the following formula*

$$\lambda(Z) = \sum_{[D]} \alpha_{[D]}(Z) \cdot \left(\sum_{D' \in [D]} E_{D'} \right), \quad Z \in \mathcal{Z},$$

where $E_{D'}$ is the isotypical projection w.r.t. $D' \in \widehat{\mathcal{G}}$.

Proof: First note that for a general $\lambda \in \text{Ob } \mathcal{T}$ the equation (23) is still valid (cf. Proposition 4.6). From this we have for $Z \in \mathcal{Z}$

$$\lambda(Z) = \sum_D \alpha_D(Z) \sum_j W_{D,j} W_{D,j}^* = \sum_D \alpha_D(Z) E_D = \sum_{[D]} \alpha_{[D]}(Z) \cdot \left(\sum_{D' \in [D]} E_{D'} \right),$$

where for the last equation we have used Theorem 5.7 (i). ■

Next we show how a nontrivial chain group homomorphism (22) acts as an obstruction to the existence of a symmetry associated with the larger category \mathcal{T} .

Let $\mathcal{H}, \widetilde{\mathcal{H}} \in \text{Irr } \mathcal{T}_{\mathcal{G}}$ be irreducible algebraic Hilbert spaces and $\mathfrak{H} = \mathcal{H}\mathcal{Z}, \widetilde{\mathfrak{H}} = \widetilde{\mathcal{H}}\mathcal{Z} \in \text{Irr } \mathcal{M}_{\mathcal{G}}$ the corresponding free \mathcal{Z} -modules. By Proposition 4.4 we associate with them the irreducible endomorphisms $\rho, \widetilde{\rho} \in \text{Irr } \mathcal{T}$ and denote the automorphisms of their restriction to \mathcal{Z} by

$$\alpha := \rho \upharpoonright \mathcal{Z} \quad \text{and} \quad \widetilde{\alpha} := \widetilde{\rho} \upharpoonright \mathcal{Z}.$$

If $\{\Phi_i\}_i$ and $\{\widetilde{\Phi}_j\}_j$ are orthonormal basis of \mathcal{H} resp. $\widetilde{\mathcal{H}}$, then

$$\Psi_i := \sum_{i'} \Phi_{i'} Z_{i'i} \quad \text{and} \quad \widetilde{\Psi}_j := \sum_{j'} \widetilde{\Phi}_{j'} \widetilde{Z}_{j'j} \quad Z_{i'i}, \widetilde{Z}_{j'j} \in \mathcal{Z}, \quad (25)$$

are arbitrary orthonormal basis of the corresponding modules \mathfrak{H} resp. $\widetilde{\mathfrak{H}}$, where

$$\mathfrak{Z} := (Z_{i'i})_{i',i} \quad \text{and} \quad \widetilde{\mathfrak{Z}} := (\widetilde{Z}_{j'j})_{j',j} \in \text{Mat}(\mathcal{Z}) \quad \text{satisfy} \quad \mathfrak{Z}^* \mathfrak{Z} = \mathfrak{Z} \mathfrak{Z}^* = \mathbb{1} = \widetilde{\mathfrak{Z}}^* \widetilde{\mathfrak{Z}} = \widetilde{\mathfrak{Z}} \widetilde{\mathfrak{Z}}^*.$$

5.10 Proposition *With the previous notation define*

$$\epsilon(\mathcal{H}, \widetilde{\mathcal{H}}) := \sum_{i,j} \widetilde{\Phi}_j \Phi_i \widetilde{\Phi}_j^* \Phi_i^* \quad \text{and} \quad \epsilon(\mathfrak{H}, \widetilde{\mathfrak{H}}) := \sum_{i,j} \widetilde{\Psi}_j \Psi_i \widetilde{\Psi}_j^* \Psi_i^*.$$

Then we have

$$\epsilon(\mathcal{H}, \widetilde{\mathcal{H}}) = \epsilon(\mathfrak{H}, \widetilde{\mathfrak{H}}) \quad \text{iff} \quad \widetilde{\alpha}(\mathfrak{Z}) \mathfrak{Z}^* = \mathbb{1} \quad \text{and} \quad \widetilde{\mathfrak{Z}} \alpha(\widetilde{\mathfrak{Z}}^*) = \mathbb{1},$$

where $\widetilde{\alpha}(\mathfrak{Z}) := (\widetilde{\alpha}(Z_{i'i}))_{i',i} \in \text{Mat}(\mathcal{Z})$ (similarly for $\alpha(\widetilde{\mathfrak{Z}}^*)$).

Proof: Using Eq. (25) we have

$$\begin{aligned} \epsilon(\mathfrak{H}, \widetilde{\mathfrak{H}}) &= \sum_{i,j} \widetilde{\Psi}_j \Psi_i \widetilde{\Psi}_j^* \Psi_i^* = \sum_{i,j,i',j',i'',j''} \widetilde{\Phi}_{j'} \widetilde{Z}_{j'j} \Phi_{i'} Z_{i'i} \widetilde{Z}_{j''j}^* \widetilde{\Phi}_{j''}^* Z_{i''i}^* \Phi_{i''}^* \\ &= \sum_{i,j,i',j',i'',j''} \widetilde{\Phi}_{j'} \Phi_{i'} \alpha^{-1}(\widetilde{Z}_{j'j}) Z_{i'i} \widetilde{Z}_{j''j}^* \widetilde{\alpha}^{-1}(Z_{i''i}^*) \widetilde{\Phi}_{j''}^* \Phi_{i''}^* \\ &\stackrel{!}{=} \sum_{i,j} \widetilde{\Phi}_j \Phi_i \widetilde{\Phi}_j^* \Phi_i^* = \epsilon(\mathcal{H}, \widetilde{\mathcal{H}}) \end{aligned}$$

Multiplying the previous equation from the left with $\Phi_{i_0}^* \tilde{\Phi}_{j_0}^*$ and from the right with $\Phi_{i_0} \tilde{\Phi}_{j_0}$ we obtain

$$\sum_{i,j} \alpha^{-1}(\tilde{Z}_{j_0 j}) \tilde{Z}_{j_0 j}^* Z_{i_0 i} \tilde{\alpha}^{-1}(Z_{i_0 i}^*) = \delta_{j_0 j_0} \delta_{i_0 i_0} .$$

That means

$$\sum_j \tilde{Z}_{j_0 j} \alpha(\tilde{Z}_{j_0 j}^*) = \delta_{j_0 j_0} \quad \text{and} \quad \sum_i \tilde{\alpha}(Z_{i_0 i}) Z_{i_0 i}^* = \delta_{i_0 i_0} .$$

and the proof is concluded. ■

5.1 Examples of chain groups for some finite and compact Lie groups

We will give next several examples of chain groups associated with nonabelian finite and compact Lie groups. We will also show that in all the examples considered the chain group is isomorphic to the character group of the center. Note also that if the group is abelian, then one can identify the chain group with the corresponding character group.

If \mathcal{G} is the group we will denote its center by $\mathcal{C}(\mathcal{G})$ and the corresponding chain group by $\mathfrak{C}(\mathcal{G})$.

Compact Lie groups: We begin with the case $\mathcal{G} = \text{SU}(2)$. Denote by

$$l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\} = \widehat{\text{SU}(2)}$$

the class specified by the usual representation $T^{(l)}$ of $\text{SU}(2)$ on the space of complex polynomials of degree $\leq 2l$ which has dimension $2l+1$. Then the decomposition theory for the tensor products $T^{(l)} \otimes T^{(l')}$ (cf. [33, Theorem 29.26]) gives

$$l \times l' = \{ |l-l'|, |l-l'|+1, \dots, l+l' \} , \quad l, l' \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\} .$$

This decomposition structure implies that

$$l \approx l' \quad \text{iff} \quad l, l' \text{ are both integers or both half-integers .}$$

We can finally conclude that

$$\mathfrak{C}(\text{SU}(2)) = \{ [0], [\frac{1}{2}] \} \cong \mathbb{Z}_2 \cong \widehat{\mathcal{C}(\text{SU}(2))} .$$

Using Brauer-Weyl theory one can directly establish for $\mathcal{G} = \text{SU}(N)$ the isomorphism between the corresponding chain group and the character group of the center.³

Similarly one can proceed in other examples. Using well-known results on the decomposition of the tensor product of irreducible representations (see e.g. [33, Section 29]) we list the following further examples of chain groups.

(i) If $\mathcal{G} = \text{U}(2)$ we have that its dual is given by the following labels:

$$\widehat{\text{U}(2)} = \{ (m, l) \mid l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}, m \in \mathbb{Z} \text{ and } m+2l \text{ even} \} .$$

Then we compute

$$\mathfrak{C}(\text{U}(2)) = \{ [(m_+, 0)], [(m_-, \frac{1}{2})] \mid m_+/m_- \text{ is even/odd} \} \cong \mathbb{Z} \cong \widehat{\mathcal{C}(\text{U}(2))} .$$

³Christoph Schweigert, private communication.

(ii) If $\mathcal{G} = \mathrm{O}(3)$ we have

$$\widehat{\mathrm{O}(3)} = \{ (0, l), (1, l) \mid l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\} \}$$

and

$$\mathfrak{C}(\mathrm{O}(3)) = \{ [(0, 0)], [(1, 0)] \} \cong \mathbb{Z}_2 \cong \widehat{\mathfrak{C}(\mathrm{O}(3))}.$$

(iii) For $\mathcal{G} = \mathrm{SO}(3)$ recall that

$$\widehat{\mathrm{SO}(3)} = \{0, 1, 2, \dots\}.$$

In this case the corresponding center as well as the chain group are trivial:

$$\mathfrak{C}(\mathrm{SO}(3)) = \{[0]\} \cong \widehat{\mathfrak{C}(\mathrm{SO}(3))}.$$

Finite groups: We consider first the family of dihedral groups $\mathcal{G} = \mathbb{D}_{2m}$, $m \geq 2$. The group \mathbb{D}_{2m} has order $2m$ and is generated by two elements a, b that satisfy the relations

$$a^m = b^2 = e \quad \text{and} \quad bab = a^{m-1}.$$

We consider first the case where $m = 2l$, $l \in \mathbb{N}$, is even. Then the center of \mathbb{D}_{2m} is $\mathfrak{C}(\mathbb{D}_{2m}) = \{e, a^l\} \cong \mathbb{Z}_2$ and its dual is given by

$$\widehat{\mathbb{D}_{2m}} = \{1, \chi_1, \chi_2, \chi_3\} \cup \{D_k \mid k = 1, \dots, \frac{m-2}{2}\},$$

where $\{1, \chi_1, \chi_2, \chi_3\}$ are 1-dimensional representations and $\{D_k\}_{k=1}^{\frac{m-2}{2}}$ are 2-dimensional representations. From the results concerning the decomposition of tensor products of irreducible representations stated in [33, §27.62 (d)] we conclude:

$$\mathfrak{C}(\mathbb{D}_{2m}) = \{[1], [D_1]\} \cong \mathbb{Z}_2 \cong \widehat{\mathfrak{C}(\mathbb{D}_{2m})}. \quad (26)$$

To check the details of the previous example \mathbb{D}_{2m} , $m = 2l$, $l \in \mathbb{N}$, it is useful to distinguish further between the cases l even or odd. Indeed, if l is even, then

$$1 \approx \chi_1 \approx \chi_2 \approx \chi_3 \approx D_k, \quad k \text{ even} \quad \text{and} \quad D_1 \approx D_{k'}, \quad k' \text{ odd}.$$

If l is odd, then the corresponding chain classes have slightly different representants

$$1 \approx \chi_1 \approx D_k, \quad k \text{ even} \quad \text{and} \quad \chi_2 \approx \chi_3 \approx D_1 \approx D_{k'}, \quad k' \text{ odd}.$$

Similarly we can use the results in [33, Section 27] to compute the following family of examples:

(iv) $\mathcal{G} = \mathbb{D}_{2m}$ with m odd. Then the center is trivial, $\mathfrak{C}(\mathbb{D}_{2m}) = \{e\}$, and

$$\widehat{\mathbb{D}_{2m}} = \{1, \chi_1\} \cup \{D_k \mid k = 1, \dots, \frac{m-1}{2}\},$$

where $\{1, \chi_1, \chi_2, \chi_3\}$ are 1-dimensional representations and $\{D_k\}_{k=1}^{\frac{m-1}{2}}$ are 2-dimensional representations. As before we compute

$$\mathfrak{C}(\mathbb{D}_{2m}) = \{[1]\} \cong \widehat{\mathfrak{C}(\mathbb{D}_{2m})}.$$

- (v) Let $\mathcal{G} = \mathbb{Q}_{4m}$ be the generalized quaternion groups which is a group of order $4m$ generated by two elements a, b that satisfy the relations $a^{2m} = b^4 = e$, $b^2 = a^m$ and $bab^{-1} = a^{2m-1}$. Its center is given by $\mathcal{C}(\mathbb{Q}_{4m}) = \{e, a^m\} \cong \mathbb{Z}_2$ and

$$\widehat{\mathbb{Q}_{4m}} = \{1, \chi_1, \chi_2, \chi_3\} \cup \{D_k \mid k = 1, \dots, m-1\},$$

where $\{1, \chi_1, \chi_2, \chi_3\}$ are 1-dimensional representations and $\{D_k\}_{k=1}^{m-1}$ are 2-dimensional representations. Distinguishing again between the cases m even or odd we obtain using [33, §27.62 (e)]

$$\mathfrak{C}(\mathbb{Q}_{4m}) = \{[1], [D_1]\} \cong \mathbb{Z}_2 \cong \widehat{\mathcal{C}(\mathbb{Q}_{4m})}. \quad (27)$$

- (vi) We conclude this list of examples mentioning the cases of the permutation groups $\mathbb{P}_3, \mathbb{P}_4$ and the the alternating group \mathbb{A}_4 which have trivial center. It is straightforward to verify that the corresponding chain groups are also trivial.

5.11 Remark As stated in Corollary 3.2 of [37] the isomorphism between the chain group of \mathcal{G} and the character group of its center shows that the center of a compact group depends only on the representation ring of \mathcal{G} . This is in fact explicitly verified for the groups \mathbb{D}_{8l} and \mathbb{Q}_{8l} , $l \in \mathbb{N}$, which are particularly interesting for this question. Recall that these groups are non-isomorphic but have isomorphic duals (cf. [33, §27.62 (f)]) and therefore isomorphic chain groups. Therefore the centers of \mathbb{D}_{8l} and \mathbb{Q}_{8l} must also be isomorphic (compare with the Eqs. (26) and (27) above).

6 A family of examples

In this section we will give a family of examples of pairs of categories $\mathcal{T}_c < \mathcal{T}$, where \mathcal{T}_c is admissible (recall Remark 4.12 (iii)).

Let \mathcal{A}_c be a unital C^* -algebra with trivial center, $\mathcal{Z}(\mathcal{A}_c) = \mathbb{C}\mathbb{1}$, and satisfying Property B. Denote by \mathfrak{Z} a unital abelian C^* -algebra and define

$$\mathcal{A} := \mathcal{A}_c \otimes \mathfrak{Z},$$

which is again a C^* -algebra with unit $\mathbb{1} \otimes \mathbb{1}$ and center $\mathcal{Z} = \mathcal{Z}(\mathcal{A}) = \mathbb{1} \otimes \mathfrak{Z}$. Let \mathcal{T}_{DR} be a DR-category (recall Definition 3.18) realized as endomorphisms of \mathcal{A}_c . The objects of \mathcal{T}_{DR} are denoted by ρ, σ etc. and the corresponding arrows by (ρ, σ) . Let \mathcal{G} be the compact group associated with \mathcal{T}_{DR} and denote by \mathfrak{C} its corresponding chain group. We consider also a fixed group homomorphism (recall Theorem 5.7)

$$\mathfrak{C} \ni [D] \mapsto \alpha_{[D]} \in \text{aut } \mathcal{Z}. \quad (28)$$

We can now start defining the C^* -category \mathcal{T} realized as endomorphism of the larger algebra \mathcal{A} with nontrivial center \mathcal{Z} . To identify the new objects we proceed in two steps: first we extend irreducible endomorphisms in \mathcal{T}_{DR} to endomorphisms of \mathcal{A} . Second, we use the decomposition result in Proposition 4.6 to extend general objects in \mathcal{T}_{DR} to endomorphisms of \mathcal{A} . The extended endomorphisms of the larger algebra \mathcal{A} are interpreted as new objects of the category \mathcal{T} .

- (a) If ρ is irreducible, $\rho \in \text{Irr } \mathcal{T}_{\text{DR}}$, we define

$$\tilde{\rho} := \rho \otimes \alpha_{[D]} \in \text{end } \mathcal{A},$$

where $D \in \widehat{\mathcal{G}}$ is the corresponding class associated with $\rho \in \text{Irr } \mathcal{T}_{\text{DR}}$ (cf. Remark 3.19).

(b) Let $\tau \in \text{Ob } \mathcal{T}_{\text{DR}}$. According to Proposition 4.6, the endomorphism τ can be decomposed in terms of irreducible objects as

$$\tau(\cdot) = \sum_{\rho, l} W_{\rho, l} \rho(\cdot) W_{\rho, l}^*,$$

where $\{W_{\rho, l}\}_l$ denotes an orthonormal basis of (ρ, τ) and $\rho \in \text{Irr } \mathcal{T}_{\text{DR}}$. We assign to τ the following endomorphism of \mathcal{A}

$$\begin{aligned} \tilde{\tau}(\cdot) &:= \sum_{\rho, l} (W_{\rho, l} \otimes \mathbb{1}) \tilde{\rho}(\cdot) (W_{\rho, l} \otimes \mathbb{1})^* \\ &= \sum_{\rho, l} (W_{\rho, l} \rho(\cdot) W_{\rho, l}^*) \otimes \alpha_{[D]}(\cdot), \end{aligned} \quad (29)$$

where for the second equation we have used the previous item (a).

(c) The arrows in \mathcal{T} are defined as usual

$$(\tilde{\rho}, \tilde{\tau}) := \{A \in \mathcal{A} \mid A \tilde{\rho}(X) = \tilde{\tau}(X) A, \quad X \in \mathcal{A}\}.$$

6.1 Proposition *Let \mathcal{T} be the C^* -category defined by means of (a), (b) and (c) above. Then the objects $\tilde{\rho}, \tilde{\rho}_1, \tilde{\rho}_2$ defined in part (a) satisfy:*

(i) *Irreducibility:* $\tilde{\rho} \in \text{Irr } \mathcal{T}$, i.e. $(\tilde{\rho}, \tilde{\rho}) = \tilde{\rho}(\mathcal{Z}) = \mathcal{Z}$.

(ii) *Pairwise disjointness:* $(\tilde{\rho}_1, \tilde{\rho}_2) = \{0\}$ if $\tilde{\rho}_1 \neq \tilde{\rho}_2$.

Proof: For the proof it is convenient to apply Gelfand's theorem: any $Z \in \mathfrak{Z}$ can be identified with a continuous function over the compact space $\text{spec } \mathfrak{Z}$, $Z(\cdot) \in C(\text{spec } \mathfrak{Z})$, and

$$\mathcal{A} = \mathcal{A}_{\mathbb{C}} \otimes \mathfrak{Z} \cong C(\text{spec } \mathfrak{Z} \rightarrow \mathcal{A}_{\mathbb{C}}).$$

In particular, we will need below that any elementary tensor $A = A_0 \otimes Z \in \mathcal{A}_{\mathbb{C}} \otimes \mathfrak{Z}$ can be expressed as the function $\text{spec } \mathfrak{Z} \ni \mu \rightarrow A(\mu) = Z(\mu)A_0$.

(i) Let $\tilde{\rho} := \rho \otimes \alpha_{[D]} \in \text{Ob } \mathcal{T}$ with $D \in \hat{\mathcal{G}}$ associated with $\rho \in \text{Irr } \mathcal{T}_{\text{DR}}$ by means of the DR-Theorem (cf. Remark 3.19). It is clear that $\tilde{\rho}(\mathcal{Z}) \subseteq (\tilde{\rho}, \tilde{\rho})$, since

$$\tilde{\rho}(\mathcal{Z}) = \mathbb{1} \otimes \alpha_{[D]}(\mathfrak{Z}) = \mathbb{1} \otimes \mathfrak{Z} = \mathcal{Z} \subseteq (\tilde{\rho}, \tilde{\rho}).$$

For the converse inclusion let $A \in (\tilde{\rho}, \tilde{\rho})$. In particular, this implies

$$A \tilde{\rho}(X \otimes \mathbb{1}) = \tilde{\rho}(X \otimes \mathbb{1}) A, \quad X \in \mathcal{A}_{\mathbb{C}}.$$

The previous equation can be rewritten using Gelfand's theorem in terms of functions over $\text{spec } \mathfrak{Z}$ as

$$A(\mu) \rho(X) = \rho(X) A(\mu), \quad X \in \mathcal{A}_{\mathbb{C}}, \quad \mu \in \text{spec } \mathfrak{Z}.$$

Since $\rho \in \text{Irr } \mathcal{T}_{\text{DR}}$, i.e. $(\rho, \rho) = \mathbb{C}\mathbb{1}$, we conclude that $A(\mu) = \lambda(\mu)\mathbb{1}$, where λ is a continuous scalar function on $\text{spec } \mathfrak{Z}$. Applying once more Gelfand's theorem we have

$$A \in \mathbb{1} \otimes \mathfrak{Z} = \mathbb{1} \otimes \alpha_{[D]}(\mathfrak{Z}) = \tilde{\rho}(\mathcal{Z})$$

and we have shown that $\tilde{\rho} \in \text{Irr } \mathcal{T}$.

(ii) Let $\tilde{\rho}_i = \rho_i \otimes \alpha_{[D_i]}$, $\rho_i \in \text{Irr } \mathcal{T}_{\text{DR}}$, $i = 1, 2$, and $\tilde{\rho}_1 \neq \tilde{\rho}_2$. To show the disjointness relation $(\tilde{\rho}_1, \tilde{\rho}_2) = \{0\}$ choose $A \in (\tilde{\rho}_1, \tilde{\rho}_2) \subset \mathcal{A} = \mathcal{A}_{\mathbb{C}} \otimes \mathfrak{Z}$. The intertwiner element A must satisfy in particular

$$A (\rho_1 \otimes \alpha_{[D_1]}) (X \otimes \mathbb{1}) = (\rho_2 \otimes \alpha_{[D_2]}) (X \otimes \mathbb{1}) A, \quad X \in \mathcal{A}_{\mathbb{C}},$$

which in terms of functions over $\text{spec } \mathfrak{Z}$ means

$$A(\mu) \rho_1(X) = \rho_2(X) A(\mu), \quad X \in \mathcal{A}_{\mathbb{C}}, \mu \in \text{spec } \mathfrak{Z}.$$

Since $\rho_1 \neq \rho_2$ are irreducible in \mathcal{T}_{DR} we have $(\rho_1, \rho_2) = \{0\}$, hence $A(\mu) = 0$, $\mu \in \text{spec } \mathfrak{Z}$, and we conclude that $A = 0$. \blacksquare

We can now state the main result of this section, namely the specification of a family of examples satisfying the hypothesis of Theorem 4.14. Note that from the construction prescription in (a) and (b) above, there is a bijective correspondence between $\text{Ob } \mathcal{T}_{\text{DR}}$ (which are realized as endomorphisms of $\mathcal{A}_{\mathbb{C}}$) and $\text{Ob } \mathcal{T}$ (which are realized as endomorphisms of the larger algebra $\mathcal{A} = \mathcal{A}_{\mathbb{C}} \otimes \mathfrak{Z}$).

6.2 Theorem *Let $\mathcal{A}_{\mathbb{C}}$ be a unital C^* -algebra with trivial center (and satisfying Property B). Denote by \mathfrak{Z} a unital abelian C^* -algebra and define*

$$\mathcal{A} := \mathcal{A}_{\mathbb{C}} \otimes \mathfrak{Z} \quad \text{so that} \quad \mathcal{Z} := \mathcal{Z}(\mathcal{A}) = \mathbb{1} \otimes \mathfrak{Z}.$$

If \mathcal{T}_{DR} is a DR-category (cf. Definition 3.18) realized as endomorphisms of $\mathcal{A}_{\mathbb{C}}$, let \mathcal{T} be the C^* -category specified by (a), (b), (c) above. Define the C^* -category $\mathcal{T}_{\mathbb{C}}$ as follows: $\text{Ob } \mathcal{T}_{\mathbb{C}} := \text{Ob } \mathcal{T}$ and

$$(\tilde{\sigma}, \tilde{\tau})_{\mathbb{C}} := (\sigma, \tau) \otimes \mathbb{1} \subset \mathcal{A},$$

where σ, τ are the objects in \mathcal{T}_{DR} corresponding to $\tilde{\sigma}, \tilde{\tau} \in \text{Ob } \mathcal{T}$. Then $\mathcal{T}_{\mathbb{C}}$ is an admissible subcategory of \mathcal{T} , i.e. the arrows satisfy the equation

$$(\tilde{\sigma}, \tilde{\tau}) = (\tilde{\sigma}, \tilde{\tau})_{\mathbb{C}} \tilde{\sigma}(\mathcal{Z}), \quad (30)$$

(cf. Theorem 4.11).

Proof: We need to show Eq. (30), that means that we can generate the “larger” arrows set of \mathcal{T} with the “smaller” arrow set of $\mathcal{T}_{\mathbb{C}}$ and the center \mathcal{Z} . From the decomposition result in Proposition 4.6 it is sufficient to prove the special case

$$(\tilde{\rho}, \tilde{\tau}) = (\tilde{\rho}, \tilde{\tau})_{\mathbb{C}} \tilde{\rho}(\mathcal{Z}), \quad \tilde{\rho} \in \text{Irr } \mathcal{T}, \tilde{\tau} \in \text{Ob } \mathcal{T}.$$

First we show $(\tilde{\rho}, \tilde{\tau}) \supseteq (\tilde{\rho}, \tilde{\tau})_{\mathbb{C}} \tilde{\rho}(\mathcal{Z})$. For this, take an orthonormal basis $\{W_{\rho, l}\}_{l=1}^n \subset (\rho, \tau)$, where $\rho \in \text{Irr } \mathcal{T}_{\text{DR}}$, $\tau \in \text{Ob } \mathcal{T}_{\text{DR}}$. It is enough to show that for any $l = 1, \dots, n$ and any $Z_0 \in \mathfrak{Z}$ the following equation holds:

$$\left((W_{\rho, l} \otimes \mathbb{1}) \tilde{\rho}(\mathbb{1} \otimes Z_0) \right) \tilde{\rho}(X \otimes Z) = \tilde{\tau}(X \otimes Z) \left((W_{\rho, l} \otimes \mathbb{1}) \tilde{\rho}(\mathbb{1} \otimes Z_0) \right), \quad X \in \mathcal{A}_{\mathbb{C}}, Z \in \mathfrak{Z}.$$

Using the definition of irreducible $\tilde{\rho}$ in part (a) above we can rewrite the last equation as

$$W_{\rho, l} \rho(X) \otimes \alpha_{[D]}(Z_0 Z) = \tilde{\tau}(X \otimes Z) (W_{\rho, l} \otimes \alpha_{[D]}(Z_0)), \quad X \in \mathcal{A}_{\mathbb{C}}, Z \in \mathfrak{Z}, \quad (31)$$

where $D \in \hat{\mathcal{G}}$ corresponds to $\rho \in \text{Irr } \mathcal{T}_{\text{DR}}$ according to DR-Theorem. We consider now the expression $\tilde{\tau}(X \otimes Z)$ separately and use Eq. (29) to obtain

$$\tilde{\tau}(X \otimes Z) = \sum_{\rho', l'} (W_{\rho', l'} \rho'(X) W_{\rho', l'}^*) \otimes \alpha_{[D']}(Z),$$

where $\{W_{\rho',l}\}_{l'} \subset (\rho', \tau)$ is an orthonormal basis and $D' \in \widehat{\mathcal{G}}$ corresponds to $\rho' \in \text{Irr } \mathcal{T}_{\text{DR}}$. Inserting this in the r.h.s. of Eq. (31) and using the orthogonality relations $W_{\rho',l'}^* W_{\rho,l} = \delta_{\rho\rho'} \delta_{ll'}$ we obtain

$$\begin{aligned} \widetilde{\tau}(X \otimes Z) (W_{\rho,l} \otimes \alpha_{[D]}(Z_0)) &= \sum_{\rho',l'} (W_{\rho',l'} \rho'(X) W_{\rho',l'}^* W_{\rho,l}) \otimes (\alpha_{[D']}(Z) \alpha_{[D]}(Z_0)) \\ &= W_{\rho,l} \rho(X) \otimes \alpha_{[D]}(Z_0 Z) \end{aligned}$$

which coincides with the l.h.s. of Eq. (31). This concludes the proof of the inclusion $(\widetilde{\rho}, \widetilde{\tau}) \supseteq (\widetilde{\rho}, \widetilde{\tau})_{\mathbb{C}} \widetilde{\rho}(\mathcal{Z})$.

To show the reverse inclusion choose $A \in (\widetilde{\rho}, \widetilde{\tau})$ so that

$$\begin{aligned} A \widetilde{\rho}(X \otimes Z) &= \widetilde{\tau}(X \otimes Z) A, \quad X \in \mathcal{A}_{\mathbb{C}}, Z \in \mathfrak{Z} \\ A (\rho(X) \otimes \alpha_{[D]}(Z)) &= \left(\sum_{\rho',l'} (W_{\rho',l'} \rho'(X) W_{\rho',l'}^* \otimes \alpha_{[D']}(Z)) \right) A, \end{aligned}$$

where $\{W_{\rho',l'}\}_{l'} \subset (\rho', \tau)$ is an orthonormal basis as before. Multiplying the previous equation with $W_{\rho,l}^* \otimes \mathbb{1}$ from the left we get

$$((W_{\rho,l}^* \otimes \mathbb{1})A) (\rho(X) \otimes \alpha_{[D]}(Z)) = (\rho(X) \otimes \alpha_{[D]}(Z)) ((W_{\rho,l}^* \otimes \mathbb{1})A), \quad X \in \mathcal{A}_{\mathbb{C}}, Z \in \mathfrak{Z},$$

and this shows that

$$(W_{\rho,l}^* \otimes \mathbb{1})A \in (\widetilde{\rho}, \widetilde{\rho}) = \widetilde{\rho}(\mathbb{1} \otimes \mathfrak{Z}),$$

where for the last equation we have used Proposition 6.1 (i). Therefore, for any $W_{\rho,l}$ there is a $Z_{\rho,l} \in \mathfrak{Z}$ such that $(W_{\rho,l}^* \otimes \mathbb{1})A = \mathbb{1} \otimes \alpha_{[D]}(Z_{\rho,l})$. Multiplying this relation from the left by $W_{\rho,l} \otimes \mathbb{1}$ and summing up w.r.t. l we obtain

$$\sum_l (E_{\rho} \otimes \mathbb{1}) A = \sum_l W_{\rho,l} \otimes \alpha_{[D]}(Z_{\rho,l}),$$

where $E_{\rho} := \sum_l W_{\rho,l} W_{\rho,l}^* \in (\tau, \tau)$ is the isotypical projection w.r.t. $\rho \in \text{Irr } \mathcal{T}_{\text{DR}}$. To conclude the proof recall the disjointness relation in Proposition 6.1 (ii) which implies

$$(W_{\rho',l'}^* \otimes \mathbb{1})A \in (\widetilde{\rho}, \widetilde{\rho}') = \{0\} \quad \text{for all } \widetilde{\rho} \neq \widetilde{\rho}',$$

hence $(E_{\rho'} \otimes \mathbb{1})A = 0$ for all $\widetilde{\rho} \neq \widetilde{\rho}'$. Now from the property $\sum_{\rho} E_{\rho} = \mathbb{1}$ of the isotypical projections we obtain

$$A = \sum_l W_{\rho,l} \otimes \alpha_{[D]}(Z_{\rho,l}) = \sum_l (W_{\rho,l} \otimes \mathbb{1}) (\mathbb{1} \otimes \alpha_{[D]}(Z_{\rho,l})) \in (\widetilde{\rho}, \widetilde{\tau})_{\mathbb{C}} \widetilde{\rho}(\mathcal{Z})$$

and the proof is concluded. ■

We can now apply Theorem 4.14 to the pair of categories $\mathcal{T}_{\mathbb{C}} < \mathcal{T}$ constructed in this section to obtain the following result:

6.3 Proposition *Let $\mathcal{T}_{\mathbb{C}} < \mathcal{T}$ be the pair of C^* -categories constructed in Theorem 6.2, where $\mathcal{T}_{\mathbb{C}}$ is an admissible subcategory of \mathcal{T} . Then there exists an essentially unique minimal and regular Hilbert extension $\{\mathcal{F}, \mathcal{G}\}$ of \mathcal{A} .*

6.4 Remark Note that construction of the inclusion of C^* -categories $\mathcal{T}_{\mathbb{C}} < \mathcal{T}$ in Theorem 6.2 depends crucially on the choice of the chain group homomorphism in Eq. (28). Therefore different choices of this homomorphism will produce different minimal and regular Hilbert extensions.

7 The case of a trivial chain group homomorphism

We will assume in this section that the chain group homomorphism given in Eq. (22) is trivial. We will see that in this case the analysis of minimal Hilbert C^* -systems $\{\mathcal{F}, \mathcal{G}\}$ simplifies considerably. In fact, this assumption implies that any irreducible endomorphism acts trivially on the center \mathcal{Z} and by Proposition 4.6 we finally obtain

$$\rho \upharpoonright \mathcal{Z} = \text{id} \upharpoonright \mathcal{Z} \quad \text{for any } \rho \in \text{Ob } \mathcal{T}. \quad (32)$$

For example, the chain group homomorphism is trivial if the chain group $\mathfrak{C}(\mathcal{G})$ itself is trivial (see the examples in (iii),(iv) and (vi) of Subsection 5.1). This means that any $D \in \widehat{\mathcal{G}}$ lies in the chain equivalence class of the trivial representation.

7.1 Proposition *Let $\{\mathcal{F}, \mathcal{G}\}$ be minimal Hilbert C^* -system with fixed point algebra \mathcal{A} satisfying Property B and center $\mathcal{Z} = \mathcal{A} \cap \mathcal{A}'$. Then the center of \mathcal{F} coincides with \mathcal{Z} , i.e. $\mathcal{F} \cap \mathcal{F}' = \mathcal{Z}$.*

Proof: Let $\rho = \rho_{\mathcal{H}}$ be irreducible, $\rho(A) = \sum_j \Phi_j A \Phi_j^*$ with an orthonormal basis $\{\Phi_j\}_j$ of \mathcal{H} . Since $\rho(\mathcal{Z}) = \mathcal{Z}$ we get $\Phi_j \mathcal{Z} = \mathcal{Z} \Phi_j$ for all j . Further $\mathcal{F} = C^*(\mathcal{A}, \{\mathcal{H}\})$, where \mathcal{H} runs through all irreducible Hilbert spaces. This implies $ZF = FZ$ for all $F \in \mathcal{F}$. Therefore $\mathcal{Z} \subseteq \mathcal{F}' \cap \mathcal{F} \subseteq \mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$, hence $\mathcal{F}' \cap \mathcal{F} = \mathcal{Z}$ follows. \blacksquare

Next we show that in the case of a trivial chain group homomorphism one can still associate a symmetry with the larger category \mathcal{T} . For this purpose, recall from Proposition 4.4 that to any $\rho \in \text{Ob } \mathcal{T}$ there exists a unique free \mathcal{Z} -bimodule $\mathfrak{H}_{\rho} \in \text{Ob } \mathcal{M}_{\mathcal{G}}$.

7.2 Corollary *Each canonical endomorphism of a minimal Hilbert C^* -system $\{\mathcal{F}, \mathcal{G}\}$ is a \mathcal{Z} -module endomorphism. The symmetry $\epsilon(\rho, \sigma)$, $\rho, \sigma \in \text{Ob } \mathcal{T}$, defined for $\{\Phi_i\}_i, \{\Psi_j\}_j$ orthonormal basis in $\mathfrak{H}_{\rho}, \mathfrak{H}_{\sigma}$ by*

$$\epsilon(\rho, \sigma) := \sum_{i,j} \Psi_j \Phi_i \Psi_j^* \Phi_i^* \in (\rho\sigma, \sigma\rho),$$

satisfy the corresponding properties of Proposition 3.15.

Proof: From the result $\mathcal{F}' \cap \mathcal{F} = \mathcal{Z}$ and the definition of a canonical endomorphisms we get immediately that $\rho(AZ) = \rho(A)Z = Z\rho(A)$ for any $\rho \in \text{Ob } \mathcal{T}$ and any $Z \in \mathcal{Z}$, hence the objects in \mathcal{T} are \mathcal{Z} -module endomorphism. In particular, this also implies $\alpha := \rho \upharpoonright \mathcal{Z} = \text{id} \upharpoonright \mathcal{Z}$ for all $\rho \in \text{Ob } \mathcal{T}$. Therefore by Proposition 5.10 we get that the definition of $\epsilon(\cdot, \cdot)$ is independent of the module basis chosen in the Hilbert \mathcal{Z} -module assigned to the objects of \mathcal{T} . The additional properties of $\epsilon(\cdot, \cdot)$ are then verified easily (cf. Proposition 3.15). \blacksquare

7.3 Remark (i) As already mentioned in [7, Remark 6.4] it is not possible in general to associate a symmetry $\epsilon(\cdot, \cdot)$ with the larger category of canonical endomorphisms \mathcal{T} . The reason is that, in general, the formula in the previous corollary is not independent of the module basis chosen (recall Proposition 5.10). Therefore the existence of a symmetry in the present context suggests that the nontriviality of the chain group homomorphism given in Eq. (22) is an obstruction to the existence of a well defined ϵ within the category \mathcal{T} .

(ii) The present section is also related to the notion of extension of C^* -categories \mathcal{C} by abelian C^* -algebras $C(\Gamma)$ studied in [41]. In this reference it is shown that the DR-algebra associated with an object of the extension category \mathcal{C}^{Γ} is a continuous field of DR-algebras corresponding to the initial category \mathcal{C} . (For the construction of DR-algebras associated with suitable C^* -categories see [19].)

The previous corollary means that, in the present situation, the category \mathcal{T} of all canonical endomorphisms is “almost” a DR-category (cf. Definition 3.18): in fact, there is a permutation and a conjugation structure. The only difference is that we have $(\iota, \iota) = \mathcal{Z} \supset \mathbb{C}\mathbb{1}$.

The next theorem shows that, using the central decomposition w.r.t. the common center \mathcal{Z} (cf. Proposition 7.1), a minimal Hilbert C^* -system $\{\mathcal{F}, \mathcal{G}\}$ satisfying $\rho \upharpoonright \mathcal{Z} = \text{id} \upharpoonright \mathcal{Z}$, $\rho \in \text{Ob } \mathcal{T}$, can be considered as a direct integral of Hilbert C^* -systems $\{\mathcal{F}(\lambda), \mathcal{G}\}_\lambda$, $\lambda \in \text{spec } \mathcal{Z}$, with trivial relative commutant and a fiber-independent compact group. Moreover, the Hilbert C^* -systems $\{\mathcal{F}(\lambda), \mathcal{G}\}$ associated with the base points λ are Hilbert extensions of their fixed point algebras $\mathcal{A}(\lambda)$ which carry DR-categories given by the fibre decomposition of the category \mathcal{T} of $\{\mathcal{F}, \mathcal{G}\}$.

We assume for the rest of this section that the C^* -algebras \mathcal{A} and \mathcal{F} are *separable* and *faithfully represented* in some Hilbert space \mathfrak{h} . We put $\mathcal{Z} \cong C(\Gamma)$, $\Gamma := \text{spec } \mathcal{Z}$ (cf. Section 9).

7.4 Theorem *Let $\{\mathcal{F}, \mathcal{G}\}$ be a minimal Hilbert C^* -system with fixed point algebra \mathcal{A} satisfying Property B. \mathcal{Z} is the common center of \mathcal{A} and \mathcal{F} (cf. Proposition 7.1). The fiber C^* -algebras corresponding to the central decomposition w.r.t. \mathcal{Z} (cf. Section 9) are denoted by $\mathcal{A}(\lambda)$, $\mathcal{F}(\lambda)$, $\lambda \in \Gamma$. Then, there is an exceptional Borel set $\Gamma_0 \subset \Gamma$, with $\mu(\Gamma_0) = 0$, such that for all $\lambda \in \Gamma \setminus \Gamma_0$,*

(i) $\mathcal{A}(\lambda) \subset \mathcal{F}(\lambda)$ and $\mathcal{A}(\lambda)$ satisfies Property B (cf. Subsection 3.1).

(ii) Let $\mathcal{H} \subset \mathcal{F}$ be a \mathcal{G} -invariant algebraic Hilbert space with support $\mathbb{1}$. Then the fiber spaces $\mathcal{H}(\lambda) \subset \mathcal{F}(\lambda)$ are again \mathcal{G} -invariant algebraic Hilbert spaces satisfying $\text{supp } \mathcal{H}(\lambda) = \mathbb{1}$. If $\mathfrak{H} = \mathcal{H}\mathcal{Z}$ is the free \mathcal{Z} -bimodule generated by \mathcal{H} , then $\mathfrak{H}(\lambda) = \mathcal{H}(\lambda)$.

(iii) $\{\mathcal{F}(\lambda), \mathcal{G}\}$ is a Hilbert C^* -system. Let $\rho, \sigma \in \text{Ob } \mathcal{T}$ be canonical endomorphisms and denote by $\rho_\lambda, \sigma_\lambda$ their fibre decomposition (cf. Proposition 9.3). Then $\rho_\lambda, \sigma_\lambda$ are canonical endomorphisms associated with $\{\mathcal{F}(\lambda), \mathcal{G}\}$ and the intertwiner space $(\rho_\lambda, \sigma_\lambda)$ is given by $(\rho_\lambda, \sigma_\lambda) = (\rho, \sigma)_\lambda$.

(iv) The category \mathcal{T}_λ of canonical endomorphisms associated with $\{\mathcal{F}(\lambda), \mathcal{G}\}$ is a DR-category, that means in particular $(\iota_\lambda, \iota_\lambda) = \mathbb{C}\mathbb{1}_\lambda$ and the Hilbert C^* -system $\{\mathcal{F}(\lambda), \mathcal{G}\}$ has a trivial relative commutant.

Proof: (i) The inclusion $\mathcal{A}(\lambda) \subset \mathcal{F}(\lambda)$ is obvious from Theorem 9.1 in the appendix. The C^* -algebra \mathcal{A} satisfies Property B, if there exist isometries $V, W \in \mathcal{A}$ satisfying $VV^* + WW^* = \mathbb{1}$. By Theorem 9.1 it follows that the representation of V, W on the fibre spaces satisfy analogous properties.

(ii) Let $\{\Phi_i\}_i$ be an orthonormal basis of the \mathcal{G} -invariant algebraic Hilbert $\mathcal{H} \in \text{Ob } \mathcal{T}_\mathcal{G}$. It transforms according to a unitary representation U of \mathcal{G} . By Theorem 9.1 we have that, for $\lambda \in \Gamma \setminus \Gamma_0$, $\{\Phi_i(\lambda)\}_i$ is an orthonormal basis of $\mathcal{H}(\lambda)$ transforming according to the *same* representation U (hence $\mathcal{H}(\lambda)$ is \mathcal{G} -invariant) and $\text{supp } \mathcal{H}(\lambda) = \mathbb{1}$. Finally, let $\mathfrak{H} = \mathcal{H}\mathcal{Z} \subset \mathcal{F}$ be the free \mathcal{Z} -bimodule generated by \mathcal{H} . Any $H \in \mathfrak{H}$ can be written as $\sum_i \Phi_i Z_i$ for some $Z_i \in \mathcal{Z}$, hence its fibre component becomes

$$H(\lambda) = \sum_i \Phi_i(\lambda) Z_i(\lambda) \in \mathcal{H}(\lambda).$$

This shows $\mathfrak{H}(\lambda) = \mathcal{H}(\lambda)$ for all $\lambda \in \Gamma \setminus \Gamma_0$.

(iii) The first part follows already from (ii). Eq. (32) implies that all canonical endomorphisms of $\{\mathcal{F}, \mathcal{G}\}$ are \mathcal{Z} -module endomorphisms, hence from Proposition 9.3 we have

$$\rho_\lambda(A(\lambda)) = \sum_i \Phi_i(\lambda) A(\lambda) \Phi_i(\lambda)^*,$$

i.e. ρ_λ is canonical w.r.t. $\{\mathcal{F}(\lambda), \mathcal{G}\}$. Finally, it is straightforward to show that $A \in (\rho, \sigma)$ iff $A(\lambda) \in (\rho_\lambda, \sigma_\lambda)$ for all $\lambda \in \Gamma \setminus \Gamma_0$.

(iv) The permutation and conjugation structures of $\mathcal{T}_{\mathcal{G}}$ (recall Subsection 3.3) can be “disintegrated” and the fibre components define the permutation and conjugation structures of \mathcal{T}_{λ} . Finally, $(\iota, \iota) = \mathcal{Z}$ implies $(\iota_{\lambda}, \iota_{\lambda}) = \mathbb{C}\mathbb{1}_{\lambda}$ and $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$ gives $\mathcal{A}(\lambda)' \cap \mathcal{F}(\lambda) = \mathbb{C}\mathbb{1}_{\lambda}$ for all $\lambda \in \Gamma \setminus \Gamma_0$. ■

7.5 Remark Note the item (iii) in the previous theorem implies that the compact group \mathcal{G} is unique for all $\mathcal{F}(\lambda)$, $\lambda \in \Gamma \setminus \Gamma_0$, even if $\text{spec } \mathcal{Z}$ is disconnected.

Furthermore we have the following inverse theorem.

7.6 Theorem *Let \mathcal{A} be a unital C^* -algebra with center \mathcal{Z} and satisfying Property B. Suppose that $\Gamma := \text{spec } \mathcal{Z}$ is connected and let \mathcal{T} be a tensor C^* -category realized as unital endomorphisms of \mathcal{A} and equipped with the following properties:*

- (i) *All $\rho \in \text{Ob } \mathcal{T}$ are \mathcal{Z} -module endomorphisms.*
- (ii) *\mathcal{T} is closed w.r.t. direct sums and subobjects.*
- (iii) *\mathcal{T} is equipped with a permutation and a conjugation structure (cf. Propositions 3.15 and 3.16).*

Then there is a minimal Hilbert extension $\{\mathcal{F}, \mathcal{G}\}$ of \mathcal{A} with $\mathcal{F}' \cap \mathcal{F} = \mathcal{Z}$ and such that \mathcal{T} is isomorphic to the category of all canonical endomorphisms of $\{\mathcal{F}, \mathcal{G}\}$.

Proof: Let $\mathcal{A} = \int_{\Gamma} \mathcal{A}(\lambda) \mu(d\lambda)$ be the central decomposition of \mathcal{A} over the direct integral $L^2(\Gamma, \mu, \mathfrak{f}_{\lambda})$ (cf. Theorem 9.1). Denote by $\mathcal{T}(\lambda)$, $\lambda \in \Gamma \setminus \Gamma_0$, the C^* -category associated with $\mathcal{A}(\lambda)$ given by the fibre decomposition of \mathcal{T} and whose objects ρ_{λ} are realized as endomorphisms of $\mathcal{A}(\lambda)$ (cf. Proposition 9.3 and recall that Γ_0 is the corresponding exceptional set). First we show that $\mathcal{T}(\lambda)$ is a DR-category (recall Definition 3.18): indeed, $(\iota_{\lambda}, \iota_{\lambda}) = \mathbb{C}\mathbb{1}_{\lambda}$ follows from the fact that the C^* -algebras $\mathcal{A}(\lambda)$ have trivial center (cf. Theorem 9.1). Note also that $\rho \in \text{Irr } \mathcal{T}$ iff $(\rho_{\lambda}, \rho_{\lambda}) = \mathbb{C}\mathbb{1}_{\lambda}$, $\lambda \in \Gamma \setminus \Gamma_0$, i.e. $\rho_{\lambda} \in \text{Irr } \mathcal{T}(\lambda)$. Closure under direct sums and subobjects follows from (ii). Recall that from (i), (iii) and Corollary 7.2 we have a well defined permutation structure on \mathcal{T} which can be carried over to $\mathcal{T}(\lambda)$. Similarly we can “disintegrate” the conjugation structure, e.g. we have for the conjugate

$$(\iota, \bar{\rho}) \ni R_{\rho} = \int_{\Gamma} R_{\rho_{\lambda}} \mu(d\lambda)$$

and $R_{\rho_{\lambda}}$ satisfies the corresponding properties on the fibre. This shows that $\mathcal{T}(\lambda)$ is a DR-category for all $\lambda \in \Gamma \setminus \Gamma_0$.

Now, by the DR-theory we have on the one hand that

$$R_{\rho_{\lambda}}^* R_{\rho_{\lambda}} = d_{\rho_{\lambda}} \mathbb{1}_{\lambda},$$

where $d_{\rho_{\lambda}} \in \mathbb{N}$. On the other hand $R_{\rho}^* R_{\rho} \in \mathcal{Z} \cong C(\Gamma)$ and therefore $\lambda \rightarrow d_{\rho_{\lambda}}$ is continuous on Γ . That means that $d_{\rho_{\lambda}} = d_{\rho} \in \mathbb{N}$ is constant over Γ . We use this result to analyze the Hilbert extension $\{\mathcal{F}(\lambda), \mathcal{G}(\lambda)\}$ of $\mathcal{A}(\lambda)$ which satisfies $\mathcal{A}(\lambda)' \cap \mathcal{F}(\lambda) = \mathbb{C}\mathbb{1}_{\lambda}$ (cf. [3]). The existence of this fibre Hilbert C^* -system for all $\lambda \in \Gamma \setminus \Gamma_0$ is guaranteed by the DR-Theorem. For any $\rho_{\lambda} \in \text{Irr } \mathcal{T}(\lambda)$ we consider a d_{ρ} -dimensional algebraic Hilbert space $\mathcal{H}_{\rho_{\lambda}} \equiv \mathcal{H}_{\rho}$, which is constant over Γ and generated by the an orthonormal basis $\{\Phi_{\rho, i}\}_{i=1}^{d_{\rho}}$, i.e.

$$\mathcal{H}_{\rho} = \text{span} \{\Phi_{\rho, i} \mid i = 1, \dots, d_{\rho}\}. \quad (33)$$

We may assume that the isometries $\Phi_{\rho,i}$ are represented on a fixed Hilbert space \mathfrak{h}_0 . For any arbitrary $\tau_\lambda \in \text{Ob } \mathcal{T}(\lambda)$ we associate the algebraic Hilbert space

$$\mathcal{H}_\tau(\lambda) := \bigoplus_{\rho_\lambda \in \text{Irr } \mathcal{T}(\lambda)} (\rho_\lambda, \tau_\lambda) \mathcal{H}_\rho.$$

We have

$$\mathcal{H}_\tau(\lambda) \subset \mathcal{F}_{\text{fin}}(\lambda) := \left\{ \sum_{\rho_\lambda \in \text{Irr } \mathcal{T}(\lambda)} A_{\rho_\lambda, j}(\lambda) \Phi_{\rho, j} \right\} \subset \mathcal{F}(\lambda) := \text{clo}_{\|\cdot\|_\lambda} \mathcal{F}_{\text{fin}}(\lambda)$$

and the elements of the algebraic Hilbert space are bounded operators on $\mathfrak{f}_\lambda \otimes \mathfrak{h}_0$. Put

$$\mathcal{F} := \int_{\Gamma} \mathcal{F}(\lambda) \mu(d\lambda) \subset \mathcal{L}(L^2(\Gamma, \mu, \mathfrak{f}_\lambda \otimes \mathfrak{h}_0))$$

with C^* -norm given by $\|\cdot\| := \text{ess sup}_\lambda \|\cdot\|_\lambda$.

Finally we have to define a compact group action on \mathcal{F} . For this, recall that on each fibre over $\lambda \in \Gamma \setminus \Gamma_0$, the compact group $\mathcal{G}(\lambda)$ acts as follows: each $A(\lambda) \in \mathcal{A}(\lambda)$ is left pointwise invariant under the group action. Moreover, any \mathcal{H}_ρ , $\rho \in \text{Irr } \mathcal{T}$, carries an irreducible representation of $\mathcal{G}(\lambda)$ which does not depend on λ (cf. Eq. (33)). Therefore the action $\mathcal{G}(\lambda)$ is independent of λ , hence we put $\mathcal{G}(\lambda) \equiv \mathcal{G}$, where \mathcal{G} is compact. Since $\{\mathcal{F}(\lambda), \mathcal{G}\}$ is a Hilbert C^* -system it follows immediately that $\{\mathcal{F}, \mathcal{G}\}$ is also a Hilbert C^* -system. We still need to show that it is also minimal. For this take for any $D \in \mathcal{G}$ a canonical endomorphism $\rho_{D, \lambda} \in \text{Irr } \mathcal{T}(\lambda)$, $\lambda \in \Gamma \setminus \Gamma_0$. Since $\rho_{D, \lambda}$ is disjoint from ι_λ for any $D \neq \iota$, i.e. $(\rho_{D, \lambda}, \iota_\lambda) = 0$, we have that the corresponding integrated endomorphism satisfies the same property $(\rho_D, \iota) = 0$ for any $D \neq \iota$. From [11, Lemma 10.1.8] we have that $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$ and the proof is completed. \blacksquare

7.7 Corollary *Each minimal Hilbert C^* -system satisfying $\rho \upharpoonright \mathcal{Z} = \text{id} \upharpoonright \mathcal{Z}$, $\rho \in \text{Ob } \mathcal{T}$, with $\Gamma := \text{spec } \mathcal{Z}$ connected, is regular.*

Proof: First recall from the proof of the previous theorem that the fiber Hilbert C^* -systems have trivial relative commutant. This means that for all $\lambda \in \Gamma \setminus \Gamma_0$ there is a one-to-one correspondence between the fiber endomorphisms and the generating Hilbert spaces,

$$\rho_\lambda \leftrightarrow \mathcal{H}_{\rho_\lambda} \subset \mathcal{F}(\lambda)$$

such that

$$\rho_\lambda \circ \sigma_\lambda \leftrightarrow \mathcal{H}_{\rho_\lambda} \cdot \mathcal{H}_{\sigma_\lambda}.$$

If one chooses for all $\lambda \in \Gamma \setminus \Gamma_0$ a fixed Hilbert space $\mathcal{H}_{\rho_\lambda}$, then these fiber spaces define a Hilbert space $\mathcal{H}_\rho \subset \mathcal{F}$ such that

$$\rho \circ \sigma \leftrightarrow \mathcal{H}_\rho \cdot \mathcal{H}_\sigma.$$

Therefore $\{\mathcal{F}, \mathcal{G}\}$ is regular. \blacksquare

7.8 Remark We will extend in this remark the inverse result stated in Theorem 7.6 to the following situation: let $\{\mathcal{A}, \mathcal{T}\}$ satisfy the assumptions of Theorem 7.6 except that now Γ is a disjoint union of (in general infinite) connected components Γ_a , $a \in \mathbb{A}$, i.e.

$$\Gamma = \dot{\bigcup} \Gamma_a.$$

With each Γ_a , $a \in \mathbb{A}$, we can associate a central orthoprojection P_a , which is defined by means of the following continuous function over Γ :

$$P_a(\lambda) := \begin{cases} 1 & \text{if } \lambda \in \Gamma_a \\ 0 & \text{otherwise.} \end{cases}$$

The projections in $\{P_a\}_{a \in \mathbb{A}}$ are mutually disjoint and satisfy $\sum_a P_a = \mathbb{1}$ (strong operator convergence; to define the previous sum in the infinite case consider a net of projections indexed by the class of all finite subsets of \mathbb{A} partially ordered by inclusion \subseteq , cf. [28, Sections 2.5 and 2.6]). The Hilbert space \mathfrak{h} , on which the algebra \mathcal{A} is represented, decomposes as $\mathfrak{h} = \bigoplus_a \mathfrak{h}_a$ and $(x_a)_{a \in \mathbb{A}} \in \bigoplus_a \mathfrak{h}_a$ if $\sum_a \|x_a\|^2 < \infty$. Therefore we can decompose \mathcal{A} as a direct sum

$$\mathcal{A} = \bigoplus_{a \in \mathbb{A}} \mathcal{A}_a \quad \text{and in particular} \quad \mathcal{Z} = \bigoplus_{a \in \mathbb{A}} \mathcal{Z}_a$$

where $\mathcal{A}_a := \mathcal{A}P_a$ has center $\mathcal{Z}_a := \mathcal{Z}P_a$. Recall that $(A_a)_{a \in \mathbb{A}} \in \bigoplus_a \mathcal{A}_a$ if $\sup\{\|A_a\| \mid a \in \mathbb{A}\} < \infty$.

From property (i) we can consistently define a family of \mathcal{Z}_a -module endomorphisms $\mathcal{T}_a := \{\rho_a\} \subset \text{end } \mathcal{A}_a$ by means of

$$\rho_a(AP_a) := \rho(AP_a) = \rho(A)P_a \in \mathcal{A}_a, \quad A \in \mathcal{A}.$$

Moreover, since \mathcal{A} satisfies Property B, then \mathcal{A}_a also satisfies this property on \mathfrak{h}_a , i.e. \mathcal{A}_a contains are isometries V_a, W_a satisfying $V_a V_a^* + W_a W_a^* = P_a$. Similarly we can adapt the assumptions (i)-(iii) to the pair $\{\mathcal{A}_a, \mathcal{T}_a\}$, $a \in \mathbb{A}$. By the proof of Theorem 7.6 we can construct Hilbert C*-systems $\{\mathcal{F}_a, \mathcal{G}_a\}$, with \mathcal{G}_a compact and satisfying

$$\mathcal{A}'_a \cap \mathcal{F}_a = \mathcal{Z}_a \quad \text{as well as} \quad \mathcal{F}'_a \cap \mathcal{F}_a = \mathcal{Z}_a, \quad a \in \mathbb{A}.$$

Now, in order to be able to built up from these systems a minimal Hilbert C*-system with a compact group we need to make the following additional assumption (recall Remark 7.5):

Assumption: The compact groups \mathcal{G}_a are mutually isomorphic, i.e.

$$\mathcal{G}_a \cong \mathcal{G}, \quad \text{for some compact group } \mathcal{G}. \quad (34)$$

Under this assumption put finally $\mathcal{F} := \bigoplus_a \mathcal{F}_a$, where \mathcal{G} acts on each component. The Hilbert C*-system $\{\mathcal{F}, \mathcal{G}\}$ satisfies

$$\mathcal{A}' \cap \mathcal{F} = \bigoplus_a \mathcal{Z}_a = \mathcal{Z},$$

hence it is minimal. Finally, note that Corollary 7.7 can be also adapted to the present more general situation satisfying the assumption (34).

7.9 Remark The reason why we need to make the assumption (34) is that we want to reconstruct Hilbert C*-systems $\{\mathcal{F}, \mathcal{G}\}$ of the type studied in Theorem 7.4. A similar situation that considers more general groups, where e.g. the mapping $\Gamma \ni \lambda \rightarrow \mathcal{G}_\lambda$ is not constant, is studied in [43, Section 3, Example 3.1].

8 Conclusions

In the present paper we have described a generalization of the DR-duality theory of compact groups, to the case where the underlying C*-algebra \mathcal{A} has a nontrivial center. The abstract characterization of minimal and regular Hilbert C*-systems with a compact group \mathcal{G} is now given by the inclusion of C*-categories $\mathcal{T}_c < \mathcal{T}$, where \mathcal{T}_c is an admissible DR-subcategory of \mathcal{T} , the latter category being realized as endomorphisms of \mathcal{A} . A crucial new entity that appears when the center \mathcal{Z} of \mathcal{A} is nontrivial is the chain group $\mathcal{C}(\mathcal{G})$, which is an abelian group constructed from a suitable equivalence relation in $\widehat{\mathcal{G}}$ (the dual object of \mathcal{G}) and which is isomorphic to the character group of the center of \mathcal{G} . Our results suggest the following considerations:

- As far as the symmetry ϵ is concerned, the special case studied in Section 7 was also addressed in the context of vector bundles and crossed products by endomorphisms (see e.g. [43, Eq. (3.7) and Section 4]). In the mentioned reference, the existence of a symmetry is guaranteed by the fact that the left and right \mathcal{Z} -actions on (ι, \mathcal{E}) coincide, where \mathcal{E} is a vector bundle over the compact Hausdorff space $\text{spec } \mathcal{Z}$ and (ι, \mathcal{E}) denotes the \mathcal{Z} -bimodule vector bundle morphism from $\iota := \text{spec } \mathcal{Z} \times \mathbb{C}$ into \mathcal{E} . (Vasselli studies also bundles in [43], where left and right actions do not coincide.)

However, the situation analyzed in the present paper can not be fully compared with the case studied in [43]. In the latter paper much more general groups are considered and, in fact, many of them are not even locally compact. For this reason no decomposition theory in terms of irreducible objects is mentioned in that context. It is therefore not clear how the notion of a nontrivial chain group should be extended to the general framework of vector bundles. Recall that the notion of chain group was suggested by the decomposition theory of canonical endomorphisms and their restriction to \mathcal{Z} (cf. Theorem 5.7 (ii)). The nontriviality of the chain group homomorphism Eq. (22) gives an obstruction to the existence of a symmetry associated with the larger category \mathcal{T} (see Proposition 5.10 and Remark 7.3 (i)).

- In lower dimensional quantum field theory models (see e.g. [12, 36, 25] or [23, Chapter 8]), a nontrivial center appears when one constructs the so-called universal algebra. In the case of nets of C^* -algebras indexed by open intervals of S^1 , the universal algebra replaces the notion of quasi-local algebra (inductive limit). (Recall that in this case the index set is not directed. See [24] or [11, Chapter 5].) Although these models do not fit completely within the frame studied in this paper (there is no DR-Theorem and a nontrivial monodromy in two dimensions) we still hope that some pieces of the analysis considered here can be also applied in that situation. E.g. the generalization of the notion of irreducible objects and the analysis of their restriction to the center \mathcal{Z} that in our context led to the definition of the chain group.

9 Appendix: Decomposition of a C^* -algebra w.r.t. its center

For convenience of the reader we recall the following facts: let \mathcal{A} be a unital and separable C^* -algebra, \mathcal{Z} its center and π a faithful representation of \mathcal{A} on a separable Hilbert space \mathfrak{h} , $\pi(\mathcal{A}) \subset \mathcal{L}(\mathfrak{h})$. According to Gelfand's theorem we have $\mathcal{Z} \cong C(\Gamma)$, where $\Gamma := \text{spec } \mathcal{Z}$ is a compact second countable Hausdorff space. Then $\pi \upharpoonright \mathcal{Z}$ defines a distinguished spectral measure $E_\pi(\cdot)$ on the Borel sets $\{\Delta\} \subset \Gamma$ such that

$$\pi(Z) = \int_{\Gamma} Z(\lambda) E_\pi(d\lambda), \quad (35)$$

where $Z(\cdot) \in C(\Gamma)$ is the continuous function corresponding to $Z \in \mathcal{Z}$ (see e.g. M.A. Neumark [38, p. 278]). Since \mathcal{Z} is the center of \mathcal{A} we obtain from (35)

$$E_\pi(\Delta)\pi(A) = \pi(A)E_\pi(\Delta), \quad A \in \mathcal{A}, \Delta \subset \Gamma. \quad (36)$$

Let $\Phi : \mathfrak{h} \rightarrow \widehat{\mathfrak{h}} := L^2(\Gamma, \mu, \mathfrak{f}_\lambda)$ be a unitary spectral transformation assigned to E_π , where μ is a corresponding regular Borel measure on Γ and \mathfrak{f}_λ are the fibre Hilbert spaces (cf. [44, Chapter 14]). (The spectral representation space $\widehat{\mathfrak{h}}$ (direct integral) is also denoted in the literature as $\int_{\Gamma} \mathfrak{f}_\lambda \mu(d\lambda)$.) The transformed projections $E(\Delta)$ on $\widehat{\mathfrak{h}}$ act as multiplication by the corresponding characteristic function $\chi_\Delta(\cdot)$.

Applying the spectral transformation we obtain from the equations (35) and (36) the following inclusions

$$C(\Gamma) \subset \text{ad } \Phi \circ \pi(\mathcal{A}) \subset L^\infty(\Gamma, \mu, \mathcal{L}(f_\lambda)), \quad (37)$$

where $L^\infty(\Gamma, \mu, \mathcal{L}(f_\lambda))$ denotes the von Neumann algebra on $\widehat{\mathfrak{h}}$ of all decomposable operators $B: \widehat{\mathfrak{h}} \rightarrow \widehat{\mathfrak{h}}$ given for almost all $\lambda \in \Gamma$ by

$$(Bg)(\lambda) := B(\lambda)g(\lambda), \quad g \in \widehat{\mathfrak{h}}, B(\lambda) \in \mathcal{L}(f_\lambda).$$

These operators $\lambda \mapsto B(\lambda)$ are called “admissible” (see e.g. [10, Chapter 4]) and the measurable Borel function $\lambda \rightarrow \|B(\lambda)\|_\lambda$ satisfies $\text{ess sup } \|B(\lambda)\|_\lambda < \infty$, where $\|\cdot\|_\lambda$ denotes the operator norm in $\mathcal{L}(f_\lambda)$. Then $\|B\| = \text{ess sup } \|B(\lambda)\|_\lambda$ follows.

The equation (37) means that we have an isomorphism

$$\mathcal{A} \ni A \mapsto A(\cdot), \quad A(\lambda) \in \mathcal{L}(f_\lambda),$$

and that

$$\left((\text{ad } \Phi \circ \pi(A)) \widehat{x} \right) (\lambda) = \left(A(\lambda) \right) \widehat{x}(\lambda), \quad \text{for } \widehat{\mathfrak{h}} \ni \widehat{x} := \Phi(x), x \in \mathfrak{h}.$$

Moreover $C(\Gamma)$ is the center of the C*-algebra $\text{ad } \Phi \circ \pi(\mathcal{A})$ and if $A(\lambda) = 0$ for almost all $\lambda \in \Gamma$, then $A = 0$.

9.1 Theorem *Let (Γ, μ) be the measure space mentioned above. Then, there is an exceptional Borel set $\Gamma_0 \subset \Gamma$, with $\mu(\Gamma_0) = 0$, such that for all $\lambda \in \Gamma \setminus \Gamma_0$,*

(i) *the set*

$$\mathcal{A}_\lambda := \{A(\lambda) \mid A \in \mathcal{A}\}$$

is a well defined C-subalgebra of $\mathcal{L}(f_\lambda)$ and π_λ , given by*

$$\mathcal{A} \ni A \mapsto \pi_\lambda(A) := A(\lambda) \in \mathcal{A}_\lambda,$$

is a representation of \mathcal{A} on f_λ ,

(ii) *\mathcal{A}_λ has trivial center, i.e. $\mathcal{Z}(\mathcal{A}_\lambda) = \mathcal{A}'_\lambda \cap \mathcal{A}_\lambda = \mathbb{C}\mathbb{1}_\lambda$.*

Proof: Part (i) follows from Eq. (37). For simplicity we omit in following the explicit use of the representation π . To show (ii) let $\widetilde{\mathcal{Z}}$ be a separable abelian C*-algebra containing $\mathcal{Z} \cong C(\Gamma)$ and strongly closed in $\mathcal{A}' \cap \mathcal{A}''$, i.e.

$$\mathcal{Z} \subset \widetilde{\mathcal{Z}} \subset \widetilde{\mathcal{Z}}'' = \mathcal{A}' \cap \mathcal{A}''.$$

Then by Gelfand’s theorem we have $\widetilde{\mathcal{Z}} \cong C(\widetilde{\Gamma})$, where $\widetilde{\Gamma}$ is a compact second countable Hausdorff space. Moreover we have $\Gamma \cong \widetilde{\Gamma} / \sim$, where \sim denotes the following equivalence relation: $\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2$ if $Z(\widetilde{\lambda}_1) = Z(\widetilde{\lambda}_2)$ for all $Z \in \widetilde{\mathcal{Z}}$. In other words the elements of $\widetilde{\mathcal{Z}}$ can be identified with functions in $C(\widetilde{\Gamma})$ that are constant on the corresponding equivalence classes (i.e. let $\lambda \in \Gamma$ and denote by $[\lambda]$ the corresponding equivalence class in $\widetilde{\Gamma}$, so that for any $\widetilde{\lambda} \in [\lambda]$ we have $Z(\widetilde{\lambda}) = Z(\lambda)$).

According to M.A. Neumark [38, p. 278] we have also for $\widetilde{\mathcal{Z}}$ a distinguished spectral measure $\widetilde{E}(\cdot)$ on the Borel sets $\widetilde{\Delta}$ in $\widetilde{\Gamma}$ such that

$$\widetilde{\mathcal{Z}} = \int_{\widetilde{\Gamma}} \widetilde{Z}(\widetilde{\lambda}) \widetilde{E}(d\widetilde{\lambda}).$$

The relation of the previous decomposition with the one given in (35) is specified by the following equation: for Δ be a Borel set in Γ we have

$$E(\Delta) = \int_{\widetilde{\Delta}} \widetilde{E}(d\widetilde{\lambda}), \quad \text{where } \widetilde{\Delta} = \bigcup_{\lambda \in \Delta} [\lambda] \text{ is a Borel set in } \widetilde{\Gamma}.$$

Now the central decomposition of the von Neumann algebra \mathcal{A}'' is done over the space $\tilde{\Gamma}$ with regular Borel measure $\tilde{\mu}$ and the fibre von Neumann algebras $\mathcal{A}''(\tilde{\lambda})$ are factors for all $\tilde{\lambda} \in \tilde{\Gamma} \setminus \tilde{\Gamma}_0$, where $\tilde{\Gamma}_0 = \cup_{\lambda \in \Gamma_0} [\lambda]$ and $\tilde{\mu}(\tilde{\Gamma}_0) = \mu(\Gamma_0) = 0$. In the decomposition of $\mathcal{A} \subset \mathcal{A}''$ over $\tilde{\Gamma}$ we have that the algebras $\mathcal{A}_{\tilde{\lambda}}$ coincide on the representatives of the equivalence class $[\lambda]$ with the algebra \mathcal{A}_{λ} , $\lambda \in (\Gamma \setminus \Gamma_0)$. Moreover the functions $\tilde{\lambda} \rightarrow A(\tilde{\lambda}) \in \mathcal{A}_{\tilde{\lambda}}$ have constant values, $A(\tilde{\lambda}) = A(\lambda)$, for all $\tilde{\lambda} \in [\lambda]$.

Finally, for $\tilde{Z} \in \tilde{\mathcal{Z}}$ we have that $\tilde{Z}(\tilde{\lambda})$, $\tilde{\lambda} \in \tilde{\Gamma} \setminus \tilde{\Gamma}_0$, are scalar functions. If, in particular, $Z \in \mathcal{Z}$, then $Z(\tilde{\lambda})$ has a constant value ζ_Z for all $\tilde{\lambda} \in [\lambda]$. Therefore we have

$$Z(\lambda) = \zeta_Z \mathbb{1}_\lambda$$

and the proof is concluded. ■

9.2 Remark We mention here the special case where the spectral measure E_π has homogeneous multiplicity. Then there is a unique fiber Hilbert space \mathfrak{f} and $\hat{\mathfrak{h}} = L^2(\Gamma, \mu, \mathfrak{f})$. Moreover, \mathcal{A}_λ is a C*-algebra on \mathfrak{f} for μ -almost all $\lambda \in \Gamma$ and for $A(\lambda) \in \mathcal{A}_\lambda$ we have

$$\text{ess sup}_{\lambda \in \Gamma} \|A(\lambda)\|_{\mathcal{L}(\mathfrak{f})} < \infty.$$

If we assume that all operator functions $\Gamma \ni \lambda \rightarrow A(\lambda) \in \mathcal{L}(\mathfrak{f})$ are continuous w.r.t. the operator norm $\|\cdot\|_{\mathcal{L}(\mathfrak{f})}$, then also $\lambda \rightarrow \|A(\lambda)\|_{\mathcal{L}(\mathfrak{f})}$ is continuous and

$$\hat{\mathcal{A}} := \text{ad } \Phi \circ \pi(\mathcal{A}) \subset C(\Gamma, \mathcal{L}(\mathfrak{f})) \subset L^\infty(\Gamma, \mu, \mathcal{L}(\mathfrak{f})).$$

No exceptional set is needed and \mathcal{A}_λ is a unital C*-subalgebra of $\mathcal{L}(\mathfrak{f})$, $\{\mathcal{A}_\lambda, \hat{\mathcal{A}}\}$ is a *continuous field* of C*-algebras over Γ and $\hat{\mathcal{A}}$ is simultaneously the C*-algebra defined by this field (see Dixmier [13, p. 218 ff.]). If \mathcal{A}_λ is independent of λ , for example $\mathcal{A}_\lambda = \mathcal{L}(\mathfrak{f})$ (this is true if $\hat{\mathcal{A}} = C(\Gamma, \mathcal{L}(\mathfrak{f})) = C(\Gamma) \otimes \mathcal{L}(\mathfrak{f})$ consists of *all* continuous operator functions on Γ), then the field is trivial (in the sense of Dixmier).

9.1 \mathcal{Z} -module endomorphisms

To keep notation simple we omit the explicit use of the representation π . Recall that a unital endomorphism ρ of \mathcal{A} is called a \mathcal{Z} -module endomorphism if

$$\rho(AZ) = \rho(A)Z, \quad A \in \mathcal{A}, Z \in \mathcal{Z}. \quad (38)$$

The following proposition can be easily verified using the results in this section.

9.3 Proposition *Let ρ be a unital \mathcal{Z} -module endomorphism of \mathcal{A} and let Γ_0 be the exceptional set of Theorem 9.1. Then the family of mappings $\{\rho_\lambda: \mathcal{A}_\lambda \rightarrow \mathcal{A}_\lambda\}_{\lambda \in (\Gamma \setminus \Gamma_0)}$ defined by*

$$\mathcal{A}_\lambda \ni A(\lambda) \rightarrow \rho_\lambda(A(\lambda)) := (\rho(A))(\lambda) \in \mathcal{A}_\lambda$$

is a family of unital endomorphism of \mathcal{A}_λ .

9.4 Remark Note that the family of endomorphisms $\{\rho_\lambda\}_{\lambda \in (\Gamma \setminus \Gamma_0)}$ introduced in the previous proposition satisfies

$$\|\rho_\lambda(A(\lambda))\|_\lambda \leq \|A(\lambda)\|_\lambda \quad \text{and} \quad \text{ess sup}_{\lambda \in (\Gamma \setminus \Gamma_0)} \|\rho_\lambda(A(\lambda))\|_\lambda < \infty, \quad A(\lambda) \in \mathcal{A}(\lambda).$$

If $\{\sigma_\lambda\}_{\lambda \in (\Gamma \setminus \Gamma_0)}$ is *any* family of unital endomorphism of $\{\mathcal{A}_\lambda\}_{\lambda \in (\Gamma \setminus \Gamma_0)}$, then it also satisfies the following boundedness condition:

$$\text{ess sup}_{\lambda \in (\Gamma \setminus \Gamma_0)} \|\sigma_\lambda(A(\lambda))\|_\lambda \leq \text{ess sup}_{\lambda \in (\Gamma \setminus \Gamma_0)} \|A(\lambda)\|_\lambda < \infty, \quad A \in \mathcal{A}.$$

However, the family $\{\sigma_\lambda\}_{\lambda \in (\Gamma \setminus \Gamma_0)}$ *does not* necessarily define a ‘‘global’’ endomorphism σ of \mathcal{A} . But if this is the case, then σ is also a \mathcal{Z} -module endomorphism, because

$$(\sigma(AZ))(\lambda) = \sigma_\lambda((AZ)(\lambda)) = \sigma_\lambda(A(\lambda)Z(\lambda)) = Z(\lambda)\sigma_\lambda(A(\lambda)) = (\sigma(A)Z)(\lambda).$$

Acknowledgments First of all we want to acknowledge Hendrik Grundling for a thorough reading of a first version of the manuscript and for many proposals that improved the presentation of Hilbert C^* -systems. We would also like to thank Michael Müger and Christoph Schweigert for some useful remarks concerning the chain group. Finally, we are grateful to the DFG-Graduiertenkolleg “Hierarchie und Symmetrie in mathematischen Modellen” for supporting two visits of H.B. to the RWTH-Aachen.

References

- [1] H. Baumgärtel, *Fourier transformation of Hilbert C^* -systems with compact groups, by their regular representation*, Math. Nachr. **172** (1995), 15-24.
- [2] H. Baumgärtel, *Operatoralgebraic Methods in Quantum Field Theory. A Series of Lectures*, Akademie Verlag, Berlin, 1995.
- [3] H. Baumgärtel, *A modified approach to the Doplicher/Roberts theorem on the construction of the field algebra and the symmetry group in superselection theory*, Rev. Math. Phys. **9** (1997), 279-313.
- [4] H. Baumgärtel and A.L. Carey, *Hilbert C^* -systems for actions of the circle group*, Rep. Math. Phys. **47** (2001), 349-361.
- [5] H. Baumgärtel and H. Grundling, *Superselection in the presence of constraints*, preprint, arXiv: math-ph/0405038v1, 2004.
- [6] H. Baumgärtel and F. Lledó, *Duality of compact groups and Hilbert C^* -systems*, preprint, arXiv: math.OA/0311170v1, 2003.
- [7] H. Baumgärtel and F. Lledó, *Superselection structures for C^* -algebras with nontrivial center*, Rev. Math. Phys. **9** (1997), 785-819.
- [8] H. Baumgärtel and F. Lledó, *An application of DR-duality theory for compact groups to endomorphism categories of C^* -algebras with nontrivial center*, Fields Inst. Commun. **30** (2001), 1-10.
- [9] H. Baumgärtel and F. Lledó, *Dual group actions on C^* -algebras and their description by Hilbert extensions*, Math. Nachr. **239/240** (2002), 11-27.
- [10] H. Baumgärtel and M. Wollenberg, *Mathematical Scattering Theory*, Birkhäuser Verlag, Basel, 1983.
- [11] H. Baumgärtel and M. Wollenberg, *Causal Nets of Operator Algebras. Mathematical Aspects of Algebraic Quantum Field Theory*, Akademie Verlag, Berlin, 1992.
- [12] D. Buchholz, G. Mack, and I. Todorov, *The current algebra on the circle as a germ of local field theories*, Nucl. Phys. Proc. Suppl. **B5** (1988), 20-56.
- [13] J. Dixmier, *C^* -algebras*, North Holland Publishing co., Amsterdam, 1977.
- [14] S. Doplicher, R. Haag, and J.E. Roberts, *Fields, observables and gauge transformations I*, Commun. Math. Phys. **13** (1969), 1-23.
- [15] S. Doplicher, R. Haag, and J.E. Roberts, *Fields, observables and gauge transformations II*, Commun. Math. Phys. **15** (1969), 173-200.

- [16] S. Doplicher, C. Pinzari, and R. Zuccante, *The C^* -algebra of a Hilbert bimodule*, Bollettino U.M.I. Serie VIII **1-B** (1998), 263-282.
- [17] S. Doplicher and J.E. Roberts, *A remark on compact automorphism groups of C^* -algebras*, J. Funct. Anal. **66** (1986), 67-72.
- [18] S. Doplicher and J.E. Roberts, *Compact group actions on C^* -algebras*, J. Operator Theory **19** (1988), 283-305.
- [19] S. Doplicher and J.E. Roberts, *Duals of compact Lie groups realized in the Cuntz algebras and their actions on C^* -algebras*, J. Funct. Anal. **74** (1987), 96-120.
- [20] S. Doplicher and J.E. Roberts, *Endomorphisms of C^* -algebras, cross products and duality for compact groups*, Ann. Math. **130** (1989), 75-119.
- [21] S. Doplicher and J.E. Roberts, *A new duality for compact groups*, Invent. Math. **98** (1989), 157-218.
- [22] S. Doplicher and J.E. Roberts, *Why there is a field algebra with compact gauge group describing the superselection structure in particle physics*, Commun. Math. Phys. **131** (1990), 51-107.
- [23] D.E. Evans and Y. Kawahigashi, *Quantum Symmetries and Operator Algebras*, Oxford Science Publications, Clarendon Press, Oxford, 1998.
- [24] K. Fredenhagen, *Generalization of the theory of superselection sectors*, In *The Algebraic Theory of Superselection Sectors. Introduction and Recent Results*, (Proceedings, Palermo, 1990), D. Kastler (ed.), World Scientific, Singapore, 1990.
- [25] K. Fredenhagen, K.-H. Rehren, and B. Schroer, *Superselection sectors with braid group statistics and exchange algebras II, Geometric aspects and conformal covariance*, Rev. Math. Phys. **Special Issue** (1992), 113-157.
- [26] J. Fuchs and C. Schweigert, *Symmetries, Lie algebras and representations*, Cambridge University Press, Cambridge, 1997.
- [27] H. Grundling and C.A. Hurst, *Algebraic quantization of systems with a gauge degeneracy*, Commun. Math. Phys. **98** (1985), 369-390.
- [28] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras I*, American Mathematical Society, Rhode Island, 1997.
- [29] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras II*, Academic Press, Orlando, 1986.
- [30] H. Grundling and F. Lledó, *Local quantum constraints*, Rev. Math. Phys. **12** (2000), 1159-1218.
- [31] R. Haag, *Local Quantum Physics*, Springer Verlag, Berlin, 1992.
- [32] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis, Vol. I*, Springer Verlag, Berlin, 1979.
- [33] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis, Vol. II*, Springer Verlag, Berlin, 1994.
- [34] R. Longo and J.E. Roberts, *A theory of dimension, K -Theory* **11** (1997), 103-159.

- [35] S. Mac Lane, *Categories for the Working Mathematician*, Springer, Berlin, 1998.
- [36] G. Mack and V. Schomerus, *Conformal field algebras with quantum symmetry from the theory of superselection sectors*, Commun. Math. Phys. **134** (1990), 139-196.
- [37] M. Müger, *On the center of a compact group*, preprint, arXiv: math.GR/0312257, 2003 (to appear in International Mathematics Research Notices).
- [38] M.A. Neumark, *Normierte Algebren*, Verlag Harry Deutsch, Thun, 1990.
- [39] K. Shiga, *Representation of a compact group on a Banach space*, J. Math. Soc. Japan **7** (1955), 224-248.
- [40] M. Takesaki, *Theory of Operator Algebras I*, Springer Verlag, Berlin, 2002.
- [41] E. Vasselli, *Continuous fields of C^* -algebras arising from extensions of tensor C^* -categories*, J. Funct. Anal. **199** (2003), 122-152.
- [42] E. Vasselli, *Some remarks on group bundles and $C(X)$ -dynamical systems*, preprint, arXiv: math.OA/0301124.
- [43] E. Vasselli, *Crossed products by endomorphisms, vector bundles and group duality*, preprint, arXiv: math.OA/0301214.
- [44] N.R. Wallach, *Real reductive groups II*, Academic Press, Boston, 1992.