Middle-East Journal of Scientific Research 13 (Mathematical Applications in Engineering): 70-76, 2013
ISSN 1990-9233

DOI: 10.5829/idosi.mejsr.2013.13.mae.99924

Weight Martingale-Ergodic and Ergodic-Martingale Theorems

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Abstract: In this paper we prove weighted martingale-ergodic and weighted ergodic-martingale theorems. Furthermore, analogous dominant and maximal inequalities for weighted martingale ergodic sequences and weighted ergodic martingale averages are also obtained.

Key words: Ergodic averages . martingale . ergodic-martingale . martingale-ergodic averages

INTRODUCTION

General theories unifying ergodic averages and martingales were reported by Kachurovskii [1-3]. Four different variants for theories unifying ergodic averages and martingales have been reported in [4-7]. Besides, one parameter weighted ergodic theorem and multiparameter weighted ergodic theorems have been investigated by Baxter J.H. Olsen [8] and R.L. Jones, J.H. Olsen [9], respectively. In [10], M. Lin and M. Weber considered weighted ergodic theorems and strong laws of large numbers. General ergodic theory is reported in [11].

In this paper we prove weighted martingale-ergodic and weighted ergodic-martingale theorems. Furthermore, analogous dominant and maximal inequalities for weighted martingale ergodic sequences and weighted ergodic martingale averages are also obtained.

Preliminaries: Let $(\Omega, \Sigma, \lambda)$ be a space with a finite measure, $L_0 = L_0(\Omega)$ be a space of complex measurable functions on Ω,

$$L_p = \{f \in L_0 : \int_\Omega |f|^p d\lambda < \infty\},$$

$p \geq 1$, with the norm

$$\| f \|_p = \left(\int_\Omega |f|^p d\lambda \right)^{\frac{1}{p}}$$

if $1 \leq p < \infty$, $\| f \|_\infty = \sup\{|f(\omega)| : \omega \in \Omega\}$ if $p = \infty$.

Let $\{\mathcal{A}_n\}_{n=1}^\infty$ be a monotone sequence of σ-subalgebras of Σ, $\mathcal{A}_n \uparrow \mathcal{A}_\infty$ (or $\mathcal{A}_n \downarrow \mathcal{A}_\infty$) $E: L_p \to L_p$ be the expectation operator, $T: L_p \to L_p$ be the Dunford-Schwartz operator. Put

$$S_n(f,T) = \frac{1}{n}\sum_{k=1}^{n} T^k f, f^* = \lim_{n\to\infty} S_n(f,T), f_\infty^* = E(f^*|\mathcal{A}_\infty),$$
$$S_n(f,T) = \frac{1}{n}\sum_{k=1}^{n} T^k f, f^* = \lim_{n\to\infty} S_n(f,T), f_\infty^* = E(f^*|\mathcal{A}_\infty)$$

In [1] it is proved the following

Theorem 1.1

1) Let $f \in L_p, p \in [1, \infty)$. Then $E(S_n(f,T)|\mathcal{A}_n) \to f_\infty^*$ in L_p, in this case $\| f_\infty^* \|_p \leq \| f \|_p$.

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2) Let $f \in L_p, p \in (1, \infty)$. Then

$$E(S_n(f,T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^* \text{ in } L_p.$$

3) Let $f \in L_1$, $\sup_{n\geq 1} |S_n(f,T)| \in L_1$. Then $E(S_n(f,T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_0.

Put $f_\infty = E(f|\mathcal{A}_\infty)$, $S_n(f_\infty, T) = \frac{1}{n}\sum_{k=0}^{n-1} T^k f_\infty$, $f_\infty^* = \lim_{n\to\infty} S_n(f_\infty, T)$.

Theorem 1.2: [2]

1) Let $f \in L_p, p \in [1, \infty)$. Then $S_n(E(f|\mathcal{A}_n), T) \to f_\infty^*$ in L_p as $n \to \infty$.
2) Let $f \in L_p, p \in (1, \infty)$. Then

$$S_n(E(f|\mathcal{A}_n), T) \xrightarrow{(o)} f_\infty^* \text{ in } L_p.$$

3) Let $f \in L_1, sup\ |E(f|\mathcal{A}_n)| \in L_1$. Then $S_n(E(f|\mathcal{A}_n), T) \xrightarrow{(o)} f_\infty^*$ in L_0.

Lemma 1: [2, 12] Let $\{f_n\} \in L_p, p \in [1, \infty)$ and $f_n \to f^*$ in L_p, as $n \to \infty$. Then $E(f_n|\mathcal{A}_n) \to E(f^*|\mathcal{A}_\infty)$ in L_p as $n \to \infty$.

Lemma 2: [2, 12] Let $f_n \xrightarrow{(o)} f^*$ as $n \to \infty$ and $\sup_{n\geq 1} |f_n| \in L_1$. Then

$$E(f_n|\mathcal{A}_n) \xrightarrow{(o)} E(f^*|\mathcal{A}_\infty) \text{ in } L_0.$$

Lemma 3: [11] Let $f_n \xrightarrow{(o)} f^*$ as $n \to \infty$ and $\sup_{n\geq 1} |f_n| \in L_1$. Then $S_n(f_n, T) \xrightarrow{(o)} f^*$ in L_0 as $n \to \infty$ where $f^* = \lim_n S_n(f, T)$.

Definition 1.3: The sequence numbers $\alpha(k)$ is called Besicovich sequences such that given $\varepsilon > 0$, there is a trigonometric polynomial $\psi_\varepsilon(k)$ such that $\lim_{n\to\infty} \sup_n \frac{1}{n}\sum_{k=0}^{n-1} |\alpha(k) - \psi_\varepsilon(k)| < \varepsilon$.
A sequence $\alpha(k)$ is called bounded Besicovich if $\{\alpha(k)\} \in l^\infty$. In [9] it was proved the following

Theorem 1.4: Let T be denoted the Dunford-Schwartz operator. Then there exists

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \alpha(k) T^k(f)$$

a.e. for every $f \in L_p, 1 < p < \infty$ and all bounded Besicovich sequences $\alpha(k)$. In this case, if $p \in (1, \infty)$, then $A_n(f,T) = \frac{1}{n}\sum_{k=0}^{n-1} \alpha(k) T^k(f)$ (o) -converges in L_p and if $p = 1$, then $A_n(f,T)$ (o)-converges in L_0.
In this paper we prove theorems analogous to 1.1 and 1.2 in the case of weighted averages.

WEIGHTED MARTINGALE-ERGODIC THEOREMS

Let $\{\mathcal{A}_n\}_{n=1}^\infty$ be a monotone sequence of σ-subalgebras of Σ, $\mathcal{A}_n \uparrow \mathcal{A}_\infty$ (or $\mathcal{A}_n \downarrow \mathcal{A}_\infty$), $\alpha(k)$ be a Besicovich bounded sequence, $T: L_p \to L_p$ be the Dunford-Schwartz operator. We put,

$$A_n(f,T) = \frac{1}{n}\sum_{k=0}^{n-1} \alpha(k) T^k(f), f^* = \lim_{n\to\infty} A_n(f,T), f_\infty^* = E(f^*|\mathcal{A}_\infty).$$

Theorem 2.1

1) Let $f \in L_p, p \in (1, \infty)$. Then $E(A_n(f,T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_p.
2) If $f \in L_1$ and $\sup_{n\geq 1} A_n(f,T) \in L_1$, then $E(A_n(f,T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_0.

Proof:

1) Since

$$\sup_{n\geq 1}|A_n(f)| = \sup_{n\geq 1}|\frac{1}{n}\sum_{k=0}^{n-1}\alpha(k)T^k(f,T)| \leq \sup_{n\geq 1}\frac{1}{n}\sum_{k=0}^{n-1}|\alpha(k)||T|^k(|f|) \leq b\sup_{n\geq 1}\frac{1}{n}\sum_{k=1}^{n-1}|T|^k(f),$$

where $b = \sup\ |\alpha(k)|$, by the Akcoglu's theorem we have $\sup_{n\geq 1}|\mathcal{A}_n(f,T)| \in L_p$.

According to Theorem 1.2 [9], $A_n(f,T) \xrightarrow{(o)} f^*$. Therefore Lemma 2 implies $E(A_n(f,T)|\mathcal{A}_n) \xrightarrow{(o)} E(f^*|\mathcal{A}_\infty) = f_\infty^*$ in L_0 as n→∞.

Let $\sup_{n\geq 1}|A_n(f,T)| = h$. Then $E(A_n(f,T)|\mathcal{A}_n) \leq E(h|\mathcal{A}_n)$ for all n. By Theorem 2 [13] $\sup_{n\geq 1} E(h|\mathcal{A}_n) \in L_p$. Hence, $\sup_{n\geq 1}|E(A_n(f,T)|\mathcal{A}_n)| \in L_p$. Therefore $E(A_n(f,T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_p.

2) By Theorem 1.4 [9] we have $A_n(f,T) \to f^*$ a.e. As $\sup_{n\geq 1} A_n(f,T) \in L_1$, by Lemma 2 we obtain $E(A_n(f,T)|\mathcal{A}_n) \to f_\infty^*$ a.e. Since the convergence a.e. is the (o)-convergence in L_0, we obtain $E(A_n(f,T)|\mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$ in L_0.

WEIGHTED ERGODIC-MARTINGALE THEOREM

Let, as in Section 2, $\{\mathcal{A}_n\}_{n=1}^{\infty}$ be a monotone sequence of σ-subalgebras of Σ, $\mathcal{A}_n \uparrow \mathcal{A}_\infty$ (or $\mathcal{A}_n \downarrow \mathcal{A}_\infty$), $\alpha(k)$ be a Besicovich bounded sequence, $T: L_p \to L_p$ is the Dunford-Schwartz operator. We put

$$f_\infty = E(f|\mathcal{A}_\infty),$$
$$A_n(f_\infty,T) = \frac{1}{n}\sum_{k=0}^{n-1}\alpha(k)T^k(f_\infty),$$
$$f_\infty^* = \lim_{n\to\infty} A_n(f_\infty,T).$$

Lemma 3.1: Let $f_n \xrightarrow{(o)} f$, $h = \sup_n |f_n| \in L_1$, and $f^* = \lim_{n\to\infty} A_n(f,T)$. Then $A_n(f_n,T) \xrightarrow{(o)} f^*$ in L_0.

Proof: Let $h_n = \sup_{m\geq n}|f_m - f|$.

Obviously,

$$|A_n(f_n,T) - f^*| \leq |A_n(f_n,T) - A_n(f,T)| + |A_n(f,T) - f^*|. |A_n(f_n,T) - f^*| \leq |A_n(f_n,T) - A_n(f,T)| + |A_n(f,T) - f^*|.$$

By theorems 1.2 and 1.4 [9], $A_n(f,T) \xrightarrow{(o)} f^*$ as $n\to\infty$ in L_0. We will prove that

$$A_n(f_n,T) - A_n(f,T) \xrightarrow{(o)} 0 \text{ as } n\to\infty \text{ in } L_0.$$

It is clear,

$$|A_n(f_n,T) - A_n(f,T)| = |A_n(f_n - f,T)| \leq \frac{1}{n}\sum_{k=0}^{n-1}|\alpha(k)||T|^k(|f_n - f|) \leq$$
$$b\cdot\frac{1}{n}\sum_{k=0}^{n-1}|T|^k(|f_n - f|) \leq b\cdot S_n(|f_n - f|,|T|).$$

Since $|f_n - f| \stackrel{(o)}{\to} 0$ and $\sup_n |f_n - f| \le 2h \in L_1$, we obtained from Lemma 3, that $S_n(|f_n - f|, |T|) \stackrel{(o)}{\to} 0$ as n→∞. Therefore $A_n(f_n, T) - A_n(f, T) \stackrel{(o)}{\to} 0$ and $A_n(f_n, T) \stackrel{(o)}{\to} f^*$ in L_0.

Theorem 3.2

1) Let $p \in (1, \infty)$. Then $A_n(E(f|\mathcal{A}_n), T) \stackrel{(o)}{\to} f_\infty^*$ in L_p as $n \to \infty$.
2) Let $p = 1$ and $\sup_n |E(f|\mathcal{A}_n)| \le h \in L_1$. Then $A_n(E(f|\mathcal{A}_n), T) \stackrel{(o)}{\to} f_\infty^*$ in L_0.

Proof:

1) According to Theorem 2 [13], $\sup_{n\ge1} E\,(|f||\mathcal{A}_n) \in L_p$. Therefore by Lemma 3.1, we have
$A_n(E(f|\mathcal{A}_n), T) \stackrel{(o)}{\to} f_\infty^*$ in L_0.

Let $\sup_{n\ge1} E\,(|f||\mathcal{A}_n) = h$. Then $E(|f||\mathcal{A}_n) \le h$ and $A_n(E(|f||\mathcal{A}_n), |T|) \le A_n(h, |T|)$. As by the Akcoglu's theorem $\sup_{n\ge1} A_n\,(h, |T|) \in L_p$, we have $\sup_{n\ge1} A_n\,(E(|f||\mathcal{A}_n)|T|) \in L_p$ and $\sup_{n\ge1} |A_n(E(f|\mathcal{A}_n), T)| \in L_p$. Hence, $A_n(E(f|\mathcal{A}_n)T) \stackrel{(o)}{\to} f_\infty^*$.

2) By the Theorem 3 [13], we have

$$E(f|\mathcal{A}_n) \stackrel{(o)}{\to} E(f, \mathcal{A}_\infty)$$

in L_0 As $\sup_{n\ge1} |E(f|\mathcal{A}_n| \in L_1$, by Lemma 3.1 we obtain

$$A_n(E(f|\mathcal{A}_n)T) \stackrel{(o)}{\to} f_\infty^*$$

in L_0.

WEIGHTED DOMINANT AND MAXIMAL INEQUALITIES

Theorem 4.1: Let $p \in (1, \infty)$ and $\mathcal{A}_n \downarrow \mathcal{A}_\infty$. Then for $f \in L_p$ the following inequality holds:

$$\| \sup_{n\ge1} |E(A_n(f, T)|\mathcal{A}_n)| \|_{L_p} \le bq^2 \| f \|_{L_p},$$

where $b = sup_k |\alpha(k)|$.

Proof: Let $g = \sup_{n\ge1} |A_n(f, T)|$. Then

$$|A_n(f, T)| \le g, \quad \sup_{n\ge1} |E(A_n(f, T)|\mathcal{A}_n)| \le \sup_{n\ge1} E\,(g|A_n)\ |A_n(f, T)| \le g, \quad \sup_{n\ge1} |E(A_n(f, T)|\mathcal{A}_n)| \le \sup_{n\ge1} E\,(g|A_n)$$

and therefore

$$\| \sup_{n\ge1} |E(A_n(f, T)|\mathcal{A}_n)| \|_{L_p} \le \| \sup_{n\ge1} E\,(g|\mathcal{A}_n) \|_{L_p}.$$

By the dominant inequality for submartingale [12] we have

$$\| \sup_{n\ge1} E\,(g|\mathcal{A}_n) \|_{L_p} \le q \, \| E(g|\mathcal{A}_1) \|_{L_p}.$$

Since the conditional expectation operator is contracting in L_p we obtain, that

$$\| E(g|\mathcal{A}_1) \|_{L_p} \le \| g \|_{L_p}.$$

Therefore

$$\| \sup_{n\ge1} |E(A_n(f, T)|\mathcal{A}_n)| \|_{L_p} \le q \, \| \sup_{n\ge1} |A_n(f, T)| \|_{L_p}.$$

It follows from Akcoglu's theorem that $\| \sup_{n\geq 1} | A_n(f,T)| \|_{L_p} \leq b \cdot q \| f \|_{L_p}$. Thus, for $f \in L_p$ the inequality

$$\| \sup_{n\geq 1} | E(A_n(f,T)|\mathcal{A}_n)| \|_{L_p} \leq bq^2 \| f \|_{L_p}$$

holds.

Theorem 4.2: Let $f \in L_p, p \in (1,\infty)$ and $\mathcal{A}_n \downarrow \mathcal{A}_\infty$. Then for any $\varepsilon > 0$ the following inequality holds

$$\lambda\{\sup_{n\geq 1} | E(A_n(f,T)|\mathcal{A}_n|)\} \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \cdot b^p \cdot \| f \|_{L_p}^p,$$

where $b = sup_k |\alpha(k)|$..

Proof: Let $g = \sup_{n\geq 1} | A_n(f,T)|$. Then

$\lambda\{\sup\ | E(A_n(f,T)|\mathcal{A}_n)| \geq \varepsilon\} \leq \lambda\{\sup\ E\ (g|\mathcal{A}_n) \geq \varepsilon\}$. $\lambda\{\sup\ | E(A_n(f,T)|\mathcal{A}_n)| \geq \varepsilon\} \leq \lambda\{\sup\ E\ (g|\mathcal{A}_n) \geq \varepsilon\}$.

By the maximal inequality for for submartingale $E(g|\mathcal{A}_n)$ [12] we have, that

$$\lambda\{E(g|A_n) \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \| E(g \| \mathcal{A}_1) \|_{L_p}^p.$$

Applying the contracting property of the conditional expectation operator in L_p, we obtain

$$\| E(g|\mathcal{A}_n) \|_{L_p}^p \leq \| g \|_{L_p}^p = \| \sup_{n\geq 1} | A_n(f,T)| \|_{L_p}^p \leq \| \sup_{n\geq 1} A\,(|f|,|T|) \|_{L_p}^p.$$

Now applying the inequality $\| \sup_{n\geq 1} A_n\,(|f|,|T|) \|_{L_p} \leq b \cdot q \| f \|_{L_p}$, we have

$$\lambda\{\sup_{n\geq 1} | E(A_n(f,T)|\mathcal{A}_n)| \geq \varepsilon\} \leq \tfrac{1}{\varepsilon^p} b^p \cdot q^p \cdot \| f \|_{L_p}^p . b = sup_k |\alpha(k)|.$$

The following theorems can be proved analogously to theorems 4.1 and 4.2.

Theorem 4.3: Let $p \in (1,\infty)$. Then for $f \in L_p$ the following inequality holds

$$\| \sup_{n\geq 1} | A_n(E(f|\mathcal{A}_n)T)| \|_{L_p} \leq b \cdot q^2 \| f \|_{L_p},$$

where $b = sup_k |\alpha(k)|$.

Theorem 4.4: Let $p \in (1,\infty)$. Then for any $\varepsilon > 0$ and $f \in L_p$ the following inequality holds

$$\lambda\{\sup_{n\geq 1} | A_n(E(f|\mathcal{A}_n),T)| \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} b^p \cdot q^p \| f \|_{L_p}^p,$$

where $b = sup_k |\alpha(k)|$.

MULTIPARAMETER WEIGHTED MARTINGALE ERGODIC THEOREMS

In this section we define the d-dimensional case of martingale ergodic averages and consider convergence theorems for such averages.
Let $\{a(k): k \in Z_d^+\}$ be the class of weights and

$$k = (k_1,k_2,\dots,k_d), \boldsymbol{N} = (N_1,N_2,\dots,N_d), |\boldsymbol{N}| = N_1 \cdot N_2 \cdot\ldots\cdot N_d, \boldsymbol{0} = (0,0,\dots,0).$$ $$k = (k_1,k_2,\dots,k_d), \boldsymbol{N} = (N_1,N_2,\dots,N_d), |\boldsymbol{N}| = N_1 \cdot N_2 \cdot\ldots\cdot N_d, \boldsymbol{0} = (0,0,\dots,0).$$

For $N = (N_1, N_2, \dots, N_d)$ $N \to \infty$ means that $Ni \to \infty$ for any $i = 1,2, \dots, d$.

Definition 5.1: [9] The sequence $\{a(k): k \in Z_d^+\}$ is called r-Besicovich if for every $\varepsilon > 0$ there is a sequence of trigonometric polynomials in d variables, $\psi_\varepsilon(k)$, such that $\lim_{|\boldsymbol{N}| \to \infty} \sup \frac{1}{|\boldsymbol{N}|} \sum_{k=0}^{\boldsymbol{N}-1} | a(k) - \psi_\varepsilon(k)| < \varepsilon$.

This class is denoted by $B(r)$. The sequence $\alpha(k)$ is called r-bounded Besicovitch if $\{a(k)\} \in B(r) \cap l^\infty$. If $\{a(k)\} \in B(1) \cap l^\infty$ then $\{\alpha(k)\}$ is called a bounded Besicovich sequence.
Let $T_1, T_1, \dots, T_1$ denote a family of d linear operators in L_p. We consider averages

$$A_{\boldsymbol{N}}(\boldsymbol{T})f = \frac{1}{|\boldsymbol{N}|} \sum_{k=0}^{\boldsymbol{N}} a\,(k) \boldsymbol{T}^k f,$$

where $f \in L_p, \boldsymbol{T}^k = T_1^{k_1} \cdot T_2^{k_2} \cdot \ldots \cdot T_d^{k_d}$
In [9] it is proved the following

Theorem 5.2: Let $T = (T_1, T_1, \dots, T_1)$ denoted d Dunford-Schwartz operators. Then $A_N(T)f$ converges a.e. for every $f \in L_p$, $1 < p \leq \infty$ and all bounded Besicovich sequences $\alpha(k)$.

Let, as in the Section 2 $\{\mathcal{A}_n\}_{n=1}^\infty$ be a monotone sequence of σ-subalgebras of Σ and $\mathcal{A}_n \downarrow \mathcal{A}_\infty$. We put $f^* = \lim_{\boldsymbol{N} \to \infty} A_{\boldsymbol{N}}(\boldsymbol{T})f$, $f_\infty^* = E(f^* | \mathcal{A}_\infty)$.

Theorem 5.3: Let $f \in L_p$, $1 < p \leq \infty$, $\alpha(k)$ be a bounded Besicovich sequence, $\boldsymbol{N}_n = (N_1^n, N_2^n, \dots, N_d^n) \to \infty$ as $n \to \infty$. Then

$$E(A_{\boldsymbol{N}_n}(\boldsymbol{T})f | \mathcal{A}_n) \xrightarrow{(o)} f_\infty^*$$

in L_p.
The proof of Theorem 5.3 is analogous to the proof of Theorem 2.1 1).

Theorem 5.4: Let $\mathcal{A}_l \downarrow \mathcal{A}_\infty$, $f \in L_p$, $1 < p \leq \infty$, $\alpha(k)$ bounded Besicovich sequence, $b = sup_k |\alpha(k)|$, $\boldsymbol{N}_l = (N_1^l, N_2^l, \dots, N_d^l) \to \infty$ as $l \to \infty$. Then

1) $\| \sup_l | E(A_{\boldsymbol{N}_l}(\boldsymbol{T})f | \mathcal{A}_l)| \|_p \leq b \cdot q^{d+1} \| f \|_p$;
2) $\lambda\{\sup_l | E(A_{\boldsymbol{N}_l}(\boldsymbol{T})f | \mathcal{A}_l)| \geq \varepsilon\} \leq b \cdot q^{pd} \frac{\|f\|_p^p}{\varepsilon^p}$.

Proof
1) Let $g_l = \sup_{m \geq l} | A_{\boldsymbol{N}_m}(\boldsymbol{T})f|$. Then

$$|E(A_{\boldsymbol{N}_l}(\boldsymbol{T})f | \mathcal{A}_l)| \leq E(\sup_{m \geq l} | A_{\boldsymbol{N}_m}(\boldsymbol{T})f | \mathcal{A}_l) = E(g_l | \mathcal{A}_l)$$

and

$$\sup_l | E(A_{\boldsymbol{N}_l}(\boldsymbol{T})f | \mathcal{A}_l)| \leq \sup_l E\,(g_l | \mathcal{A}_l). \quad (1)$$

Therefore

$$\| \sup_l | E(A_{\boldsymbol{N}_l}(\boldsymbol{T})f | \mathcal{A}_l)| \|_p \leq \| \sup_l E\,(g_l | \mathcal{A}_l) \|_p.$$

By the dominant inequality,

$$\| \sup_l E\,(g_l | \mathcal{A}_l) \|_p \leq q \| E(g_1 | \mathcal{A}_1) \|_p.$$

Since the conditional expatiation operators is contracting in L_p, we have

$$\| E(g_1 | \mathcal{A}_1) \|_p \leq \| g_1 \|_p. \quad (2)$$

Thus

$$\| \sup_l | E(A_{\boldsymbol{N}_l}(\boldsymbol{T})f|\mathcal{A}_l)| \|_p \leq q \| g_1 \|_p =$$
$$= q \| \sup_m A_{\boldsymbol{N}_m}(\boldsymbol{T})|f| \|_p \leq q \cdot q^d \cdot b \| f \|_p =$$
$$= b \cdot q^{d+1} \| f \|_p.$$

2) By the inequality (1) we have

$$\lambda\{\sup_l | E(A_{\boldsymbol{N}_l}(\boldsymbol{T})f|\mathcal{A}_l)| \geq \varepsilon\} \leq \lambda\{\sup_l E(g_l|\mathcal{A}_l) \geq \varepsilon\}. \lambda\{\sup_l | E(A_{\boldsymbol{N}_l}(\boldsymbol{T})f|\mathcal{A}_l)| \geq \varepsilon\} \leq \lambda\{\sup_l E(g_l|\mathcal{A}_l) \geq \varepsilon\}.$$

According to the maximal inequality

$$\lambda\{\sup_l E(g_l|\mathcal{A}_l) \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \| E(g_l|\mathcal{A}_l) \|_p^p.$$

Applying inequality (2), we obtain

$$\frac{1}{\varepsilon^p} \| E(g_l|\mathcal{A}_l) \|_p^p \leq \frac{1}{\varepsilon^p} \| g_1 \|_p^p \leq \frac{1}{\varepsilon^p} \| \sup_m A_{\boldsymbol{N}_m}(\boldsymbol{T})|f| \|_p^p.$$

Since $\| \sup_m A_{\boldsymbol{N}_m}(\boldsymbol{T})|f| \|_p^p \leq b^p q^{dp} \| f \|_p$, we obtain

$$\lambda\{\sup_l | E(A_{\boldsymbol{N}_l}(\boldsymbol{T})f|\mathcal{A}_l)| \geq \varepsilon\} \leq b^p \cdot q^{pd} \frac{\| f \|_p^p}{\varepsilon^p}.$$

ACKNOWLEDGEMENT

This work has been supported by IIUM-RMC through project EDW B 11-185-0663.

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