Realization of minimal C^*-dynamical systems in terms of Cuntz-Pimsner algebras

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Dedicated to Klaus Fredenhagen on his 60th birthday

Abstract

In the present article we provide several constructions of C^*-dynamical systems $(\mathcal{F}, \mathcal{G}, \beta)$ with a compact group $\mathcal{G}$ in terms of Cuntz-Pimsner algebras. These systems have a minimal relative commutant of the fixed-point algebra $\mathcal{A} := \mathcal{F}^{\mathcal{G}}$ in $\mathcal{F}$, i.e. $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$, where $\mathcal{Z}$ is the center of $\mathcal{A}$, which is assumed to be nontrivial. In addition, we show in our models that the group action $\beta \colon \mathcal{G} \to \mathrm{Aut}\,\mathcal{F}$ has full spectrum, i.e. any unitary irreducible representation of $\mathcal{G}$ is carried by a $\beta_{\mathcal{G}}$-invariant Hilbert space within $\mathcal{F}$.

First, we give several constructions of minimal C^*-dynamical systems in terms of a single Cuntz-Pimsner algebra $\mathcal{F} = \mathcal{O}_{\mathfrak{H}}$ associated to a suitable $\mathcal{Z}$-bimodule $\mathfrak{H}$. These examples are labeled by the action of a discrete Abelian group $\mathfrak{C}$ (which we call the chain group) on $\mathcal{Z}$ and by the choice of a suitable class of finite dimensional representations of $\mathcal{G}$. Second, we present a more elaborate construction, where now the C^*-algebra $\mathcal{F}$ is generated by a family of Cuntz-Pimsner algebras. Here the product of the elements in different algebras is twisted by the chain group action. We specify the various constructions of C^*-dynamical systems for the group $\mathcal{G} = \mathrm{SU}(N)$, $N \geq 2$.

Keywords: C^*-dynamical systems, minimal relative commutant, Cuntz-Pimsner algebra, Hilbert bimodule, duals of compact groups, tensor categories, non-simple unit
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Contents

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1 Introduction

Duality of groups plays a central role in abstract harmonic analysis. Its aim is to reconstruct a group $\mathcal{G}$ from its dual $\widehat{\mathcal{G}}$, i.e. from the set (of equivalence classes) of continuous, unitary and irreducible representations, endowed with a proper algebraic and topological structure. The most famous duality result is Pontryagin's duality theorem for locally compact *Abelian* groups. For compact, not necessarily Abelian, groups there exist also classical results due to Tannaka and Krein (see [18, 19]). Motivated by a long standing problem in quantum field theory, Doplicher and Roberts came up with a new duality for compact groups (see [13] as well as Müger's appendix in [17]). In the proof of the existence of a compact gauge group of internal gauge symmetries using only a natural set of axioms for the algebra of observables, they placed the duality of compact groups in the framework of C*-algebras. In this situation, the C*-algebra of local observables $\mathcal{A}$ specifies a categorical structure generalizing the representation category of a compact group. The objects of this category are no longer finite-dimensional Hilbert spaces (as in the classical results by Tannaka and Krein), but only a certain semigroup $\mathcal{T}$ of unital endomorphisms of the C*-algebra $\mathcal{A}$. In this setting, $\mathcal{A}$ has a trivial center, i.e. $\mathcal{Z} := \mathcal{Z}(\mathcal{A}) = \mathbb{C}\mathbb{1}$. The arrows of the category are the intertwining operators between these endomorphisms: for any pair of endomorphisms $\sigma, \rho \in \mathcal{T}$ one defines[1]

$$(\rho,\, \sigma) := \{X \in \mathcal{A} \mid X\rho(A) = \sigma(A)X \;,\; A \in \mathcal{A}\}. \tag{1}$$

This category is a natural example of a tensor C*-category, where the norm of the arrows is the C*-norm in $\mathcal{A}$. The tensor product of objects is defined as composition of endomorphisms $\rho, \sigma \mapsto \rho \circ \sigma$ and for arrows $X_i \in (\rho_i, \sigma_i)$, $i = 1, 2$, one defines the tensor product by

$$X_1 \times X_2 := X_1 \rho_1(X_2)\,.$$

The unit object ι is the identity endomorphism, which is simple iff $\mathcal{A}$ has a trivial center (since $(\iota, \iota) = \mathcal{Z}$). If $\mathcal{A}$ has a trivial center, then the representation category of $\mathcal{G}$ embeds as a full subcategory into the tensor C*-category of endomorphisms of $\mathcal{A}$. The concrete group dual can be described in terms of an essentially unique C*-dynamical system $(\mathcal{F}, \mathcal{G}, \beta)$, where $\mathcal{F}$ is a unital C*-algebra containing the original algebra $\mathcal{A}$, and the action of the compact group $\beta \colon \mathcal{G} \to \operatorname{Aut}\mathcal{F}$ has full spectrum. This means that for any element in the dual $D \in \widehat{\mathcal{G}}$ there is a $\beta_{\mathcal{G}}$-invariant Hilbert space $\mathcal{H}_D$ in $\mathcal{F}$ such that $\beta_{\mathcal{G}}\restriction\mathcal{H}_D \in D$. (Recall that the scalar product of any pair of elements $\psi, \psi' \in \mathcal{H}_D$ is defined as $\langle \psi, \psi' \rangle := \psi^*\psi' \in \mathbb{C}\mathbb{1}$ and any orthonormal basis in $\mathcal{H}_D$ is a set of orthogonal isometries $\{\psi_i\}_i$. The support of $\mathcal{H}_D$ the projection given by the sum of the

[1] In this article we will write the set arrows $\mathrm{Hom}(\rho, \sigma)$ simply by (ρ, σ) for each pair ρ, σ of objects.

end projections, i.e. $\operatorname{supp}\mathcal{H}_D = \sum_i \psi_i\psi_i^*$.) Moreover, $\mathcal{A}$ is the fixed point algebra of the C*-dynamical system, i.e. $\mathcal{A} = \mathcal{F}^{\mathcal{G}}$ and one has that the relative commutant of $\mathcal{A}$ in $\mathcal{F}$ is minimal, i.e. $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$. This clearly implies $\mathcal{Z} = \mathbb{C}\mathbb{1}$. The C*-algebra $\mathcal{F}$ can also be seen as a crossed product of $\mathcal{A}$ by the semigroup $\mathcal{T}$ of endomorphisms of $\mathcal{A}$ (cf. [12]): the endomorphisms $\rho \in \mathcal{T}$ (which are inner in $\mathcal{A}$) may be implemented in terms of an orthonormal basis $\{\psi_i\}_i \subset \mathcal{H}$ in $\mathcal{F}$. The endomorphism is unital iff the corresponding implementing Hilbert space in $\mathcal{F}$ has support $\mathbb{1}$.

In a series of articles by Baumgärtel and the first author (cf. [3, 4, 5]) the duality of compact groups has been generalized to the case where $\mathcal{A}$ has a nontrivial center, i.e. $\mathcal{Z} \supsetneq \mathbb{C}\mathbb{1}$, and the relative commutant of $\mathcal{A}$ in $\mathcal{F}$ remains minimal, i.e.

$$\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}\,. \tag{2}$$

(We always have the inclusion $\mathcal{Z} \subseteq \mathcal{A}' \cap \mathcal{F}$.) We define a *Hilbert C*-system* to be a *C**-dynamical system $(\mathcal{F}, \mathcal{G}, \beta)$ with a group action that has full spectrum and for which the Hilbert spaces in $\mathcal{F}$ carrying the irreducible representations of $\mathcal{G}$ have support $\mathbb{1}$ (see Section 2.1 for a precise definition). These particular C*-dynamical systems have a rich structured and many relevant properties hold, for instance, a Parseval like identity (cf. [5, Section 2]). Moreover, there is an abstract characterization by means of a suitable *non full* inclusion of *C**-categories $\mathcal{T}_{\mathbb{C}} \subset \mathcal{T}$, where $\mathcal{T}_{\mathbb{C}}$ is a symmetric tensor category with simple unit, conjugates, subobjects and direct sums (cf. [5]). A similar construction appeared in by Müger in [24], using crossed products of braided tensor *-categories with simple units w.r.t. a full symmetric subcategory.

The C*-dynamical systems $(\mathcal{F}, \mathcal{G}, \beta)$ in this more general context provide natural examples of tensor C*-categories with a nonsimple unit, since $(\iota, \iota) = \mathcal{Z}$. The analysis of these kind of categories demands the extension of basic notions. For example, a new definition of irreducible object is needed (cf. [4, 5]). In this case the intertwiner space $(\iota, \iota) \supsetneq \mathbb{C}\mathbb{1}$ is a unital Abelian *C**-algebra and an object $\rho \in \mathcal{T}$ is said to be irreducible if the following condition holds:

$$(\rho, \rho) = 1_\rho \times (\iota, \iota)\,, \tag{3}$$

where 1_ρ is the unit of the *C**-algebra (ρ, ρ). In other words, (ρ, ρ) is generated by 1_ρ as a (ι, ι)-module. Another new property that appears in the context of non-simple units is the action of a discrete Abelian group on (ι, ι). To any irreducible object ρ one can associate an automorphism $\alpha_\rho \in \operatorname{Aut}\mathcal{Z}$ by means of

$$1_\rho \otimes Z = \alpha_\rho(Z) \otimes 1_\rho \quad , \quad Z \in \mathcal{Z} \quad . \tag{4}$$

Using this family of automorphisms $\{\alpha_\rho\}_\rho$ we define an equivalence relation on $\widehat{\mathcal{G}}$, the dual of the compact group $\mathcal{G}$, and the corresponding equivalence classes become the elements of a discrete Abelian group $\mathfrak{C}(\mathcal{G})$, which we call the chain group of $\mathcal{G}$. The chain group is isomorphic to the character group of the center of $\mathcal{G}$ and the map $\rho \mapsto \alpha_\rho$ induces an action of the chain group on $\mathcal{Z}$,

$$\alpha\colon \mathfrak{C}(\mathcal{G}) \to \operatorname{Aut}\mathcal{Z}\,, \tag{5}$$

(see Section 2.2). The obstruction to have $\mathcal{T}$ symmetric is encoded in the action α: $\mathcal{T}$ is symmetric if and only if α is trivial (cf. [5, Section 7]).

These structures are so involved that it is a difficult task to produce explicit examples of Hilbert *C**-systems with non-simple unit. Indeed, up to now it has been done only for *Abelian* groups and in the setting of the *C**-algebras of the canonical commutation resp. anticommutation relations in [1, 2]. Some *indirect* examples based on the abstract characterization in terms of the inclusion of *C**-categories $\mathcal{T}_{\mathbb{C}} \subset \mathcal{T}$, can be found [5, Section 6].

The aim of the present article is to provide a large class of minimal *C**-dynamical systems and Hilbert *C**-systems for compact *non-Abelian* groups. These examples are labeled by the action of the chain group on the unital Abelian C^*-algebra $\mathcal{Z}$ given in (5). A crucial part of our examples are the Cuntz-Pimsner algebras introduced by Pimsner in his seminal article [27]. This is a new family of C^*-algebras $\mathcal{O}_{\mathcal{M}}$ that are naturally generated by a Hilbert bimodule $\mathcal{M}$ over a C^*-algebra $\mathcal{A}$. These algebras generalize Cuntz-Krieger algebras as well as crossed-products by the group $\mathbb{Z}$. In Pimsner's construction $\mathcal{O}_{\mathcal{M}}$ is given as a quotient of a Toeplitz like algebra acting on a concrete Fock space associated to $\mathcal{M}$. An alternative abstract approach to Cuntz-Pimsner algebras in terms of C^*-categories is given in [10, 20, 28]. In our models we Cuntz-Pimsner algebras $\mathcal{O}_{\mathfrak{H}}$ associated to a certain free $\mathcal{Z}$-bimodules $\mathfrak{H} = \mathcal{H} \otimes \mathcal{Z}$. The factor $\mathcal{H}$ denotes a generating finite dimensional Hilbert space with an orthonormal basis specified by isometries $\{\psi_i\}_i$. The left $\mathcal{Z}$-action of the bimodule is defined in terms of the chain group action (5).

1.1 Main results

To state our first main result we need to introduce the family $\mathfrak{G}_0$ of all finite-dimensional representations V of the compact group $\mathcal{G}$ that satisfy the following two properties: first, V admits an irreducible subrepresentation of dimension or multiplicity ≥ 2 and, second, there is a natural number $n \in \mathbb{N}$ such that $\overset{n}{\otimes} V$ contains the trivial representation ι, i.e. $\iota \prec \overset{n}{\otimes} V$. Then we show:

Main Theorem 1 (Theorem 4.9) *Let $\mathcal{G}$ be a compact group, $\mathcal{Z}$ a unital Abelian C^*-algebra and consider a fixed chain group action $\alpha\colon\ \mathfrak{C}(\mathcal{G}) \to \mathrm{Aut}\,(\mathcal{Z})$. Then for any $V \in \mathfrak{G}_0$ there exists a $\mathcal{Z}$-bimodule $\mathfrak{H}_V = \mathcal{H}_V \otimes \mathcal{Z}$ with left $\mathcal{Z}$-action given in terms of α and a C^*-dynamical system $(\mathcal{O}_{\mathfrak{H}_V}, \mathcal{G}, \beta_V)$, satisfying the following properties:*

- *(i) $(\mathcal{O}_{\mathfrak{H}_V}, \mathcal{G}, \beta_V)$ is minimal, i.e. $\mathcal{A}_V' \cap \mathcal{O}_{\mathfrak{H}_V} = \mathcal{Z}$, where $\mathcal{A}_V := \mathcal{O}_{\mathfrak{H}_V}^{\mathcal{G}}$ is the corresponding fixed-point algebra.*
- *(ii) The Abelian C^*-algebra $\mathcal{Z}$ coincides with the center of the fixed-point algebra $\mathcal{A}_V$, i.e. $\mathcal{A}_V' \cap \mathcal{A}_V = \mathcal{Z}$.*

Moreover, if $\mathcal{G}$ is a compact Lie group, then the Hilbert spectrum of $(\mathcal{O}_{\mathfrak{H}_V}, \mathcal{G}, \beta_V)$ is full, i.e. for each irreducible class $D \in \widehat{\mathcal{G}}$ there is an invariant Hilbert space $\mathcal{H}_D \subset \mathcal{O}_{\mathfrak{H}_V}$ (in this case not necessarily of support $\mathbb{1}$) such that $\beta_V \restriction \mathcal{H}_D$ specifies an irreducible representation of class D.

An important step in the proof is to show that the corresponding bimodules $\mathfrak{H}_V$ are nonsingular. This notion was introduced in [10] and is important for analyzing the relative commutants in the corresponding Cuntz-Pimsner algebras (see Section 3 for further details). We give a characterization of the class of nonsingular bimodules that will appear in this article (cf. Proposition 3.12). The preceding theorem may be applied to the group $\mathrm{SU}(N)$ in order to define a corresponding minimal C^*-dynamical system with full spectrum (cf. Example 4.10).

To present examples of minimal C^*-dynamical systems with full spectrum, where the Hilbert spaces in $\mathcal{F}$ that carry the irreducible representations of the group have support $\mathbb{1}$, we need a more elaborate construction: to begin with, we introduce a C^*-algebra generated by a family of Cuntz-Pimsner algebras that are labeled by *any* family $\mathfrak{G}$ of unitary, finite-dimensional representations of $\mathcal{G}$ (see Subsection 5.1 for precise presentation of this algebra). This construction is interesting in itself and can be performed for coefficient algebras $\mathfrak{R}$ which are not necessarily Abelian. Concretely we show:

Main Theorem 2 (Theorem 5.6) *Let $\mathcal{G}$ be a compact group, $\mathfrak{R}$ a unital C^*-algebra and $\alpha\colon \mathfrak{C}(\mathcal{G}) \to \mathrm{Aut}\,\mathfrak{R}$ a fixed action of the chain group $\mathfrak{C}(\mathcal{G})$. Then, for every set $\mathfrak{G}$ of finite-dimensional representations of $\mathcal{G}$, there exists a universal C^*-algebra $\mathfrak{R} \rtimes^{\alpha} \mathfrak{G}$ generated by $\mathfrak{R}$ and the Cuntz-Pimsner algebras $\{\mathcal{O}_{\mathfrak{H}_V}\}_{V \in \mathfrak{G}}$, where the product of the elements in the different* *algebras is twisted by the chain group action α.*

The C^*-algebra $\mathfrak{R} \rtimes^\alpha \mathfrak{G}$ (which we will also denote simply by $\mathcal{F}$) generalizes some well-known constructions, obtained for particular choices of the family of representations $\mathfrak{G}$, such as Cuntz-Pimsner algebras, crossed products by single endomorphisms (*à la* Stacey) or crossed products by Abelian groups. Hilbert space representations of $\mathfrak{R} \rtimes^\alpha \mathfrak{G}$ are labeled by covariant representations of the C^*-dynamical system $(\mathfrak{R}, \mathfrak{C}(\mathcal{G}), \alpha)$.

Now, we restrict the result of the Main Theorem 2 to the case $\mathfrak{G} = \mathfrak{G}_0$ with Abelian coefficient algebra $\mathfrak{R} = \mathcal{Z}$. The C*-algebra $\mathcal{F} = \mathcal{Z} \rtimes^\alpha \mathfrak{G}_0$ specifies prototypes of Hilbert C^*-systems for non-Abelian groups in the context of non-simple units satisfying all the required properties:

Main Theorem 3 (Theorem 5.14) *Let $\mathcal{G}$ be a compact group, $\mathcal{Z}$ a unital Abelian C^*-algebra and $\alpha\colon \mathfrak{C}(\mathcal{G}) \to \mathrm{Aut}\,\mathcal{Z}$ a fixed chain group action. Given the set of finite-dimensional representations $\mathfrak{G}_0$ introduced above and the C^*-algebra $\mathcal{F} := \mathcal{Z} \rtimes^\alpha \mathfrak{G}_0$ of the preceding theorem, there exists a minimal C^*-dynamical system $(\mathcal{F}, \mathcal{G}, \beta)$, i.e. $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$, where $\mathcal{A}$ is the corresponding fixed point algebra. Moreover, $\mathcal{Z}$ coincides with the center of $\mathcal{A}$, i.e. $\mathcal{Z} = \mathcal{A}' \cap \mathcal{A}$, and for any $V \in \mathfrak{G}_0$ the Hilbert space $\mathcal{H}_V \subset \mathcal{O}_{\mathfrak{H}_V} \subset \mathcal{F}$ has support $\mathbb{1}$.*

We may apply the preceding theorem to the group $\mathcal{G}$:=SU(2). Here we choose as the family of finite-dimensional representations $\mathfrak{G}_0$ all irreducible representations of $\mathcal{G}$ with dimension ≥ 2. This gives an explicit example of a Hilbert C^*-system for SU(2) (cf. Example 5.15).

The structure of the article is as follows: In Section 2 we present the main definitions and results concerning Hilbert C^*-systems and the chain group. In Section 3 we recall the main features of Cuntz-Pimsner algebras that will be needed later. In the following section we present a family of minimal C^*-dynamical systems for a compact group $\mathcal{G}$ and a single Cuntz-Pimsner algebra. This family of examples is labeled by the chain group action (5) and the elements of a suitable class $\mathfrak{G}_0$ of finite-dimensional representations of $\mathcal{G}$. In Section 5 we construct first a C^*-algebra $\mathcal{F}$ generated by the Cuntz-Pimsner algebras $\{\mathcal{O}_{\mathfrak{H}_V}\}_{V \in \mathfrak{G}_0}$ as described above. Then we show that with $\mathcal{F}$ we can construct a Hilbert C^*-system in a natural way. We conclude this article with a short appendix restating some of the previous concrete results in terms of tensor categories of Hilbert bimodules.

1.2 Outlook

Doplicher and Roberts show in the setting of the new duality of compact groups that essentially every concrete dual of a compact group $\mathcal{G}$ may be realized in a natural way within a C*-algebra $\mathcal{F}$, which is the C*-tensor product of Cuntz algebras (cf. [11]). Under additional assumptions it is shown that the corresponding fixed point algebra is simple and therefore must have a trivial center. The results in this paper generalize this situation. In fact, one may also realize concrete group duals within the C*-algebra $\mathcal{F} := \mathcal{Z} \rtimes^\alpha \mathfrak{G}_0$ constructed in the Main Theorem 3, where now the corresponding fixed point algebra has a nontrivial center $\mathcal{Z}$. If $\mathcal{Z} = \mathbb{C}\mathbb{1}$, then $\mathcal{Z} \rtimes^\alpha \mathfrak{G}$ reduces to the tensor product of Cuntz algebras labeled by the finite dimensional representations of the compact group contained in $\mathfrak{G}$.

As mentioned above our models provide natural examples of tensor C*-categories with a non-simple unit. These structures have been studied recently in several problems in mathematics and mathematical physics: in the general context of 2-categories (see [33] an references cited therein), in the study of group duality and vector bundles [30, 31], and in the context of superselection theory in the presence of quantum constraints [2]. Finally, algebras of quantum observables with nontrivial center $\mathcal{Z}$ also appear in lower dimensional quantum field theories with braiding symmetry (see e.g. [15], [23, §2]). In particular, in the latter reference the vacuum representation of the global observable algebra is not faithful and maps central elements to scalars. In the mathematical setting of this article, the analogue of the observable algebra is analyzed without making use of Hilbert space representations that trivialize the center. Moreover, the representation theory of a compact group is described by endomorphisms (i.e. the analogue of superselection sectors) that preserve the center. It is clear that our models do not fit completely in the frame given by lower dimensional quantum field theories, since, for example, we do not use any braiding symmetry. Nevertheless, we hope that some pieces of the analysis considered here can also be applied. E.g. the generalization of the notion of irreducible objects and the analysis of their restriction to the center $\mathcal{Z}$ that in our context led to the definition of the chain group or the importance of Cuntz-Pimsner algebras associated to $\mathcal{Z}$-bimodules.

2 Hilbert C^*-systems and the chain group

For convenience of the reader we recall the main definitions and results concerning Hilbert C^*-systems that will be used later in the construction of the examples. We will also introduce the notion of the chain group associated to a compact group which will be crucial in the specification of the examples. For a more detailed analysis of Hilbert C^*-systems we refer to [5, Sections 2 and 3] and [6, Chapter 10]).

2.1 Hilbert C^*-systems

Roughly speaking, a Hilbert C^*-system is a special type of C^*-dynamical system $\{\mathcal{F},\mathcal{G},\beta\}$ that, in addition, contains the information of the representation category of the compact group $\mathcal{G}$. $\mathcal{F}$ denotes a unital C^*-algebra and $\beta\colon \mathcal{G}\ni g\mapsto \beta_g\in \operatorname{Aut}\mathcal{F}$ is a pointwise norm-continuous morphism. Moreover, the representations of $\mathcal{G}$ are carried by the algebraic Hilbert spaces, i.e. Hilbert spaces $\mathcal{H}\subset\mathcal{F}$, where the scalar product $\langle\cdot,\cdot\rangle$ of $\mathcal{H}$ is given by $\langle A,B\rangle\mathbb{1} := A^*B$ for $A,\ B\in\mathcal{H}$. (Algebraic Hilbert spaces are also called in the literature Hilbert spaces in C*-algebras.) Henceforth, we consider only finite-dimensional algebraic Hilbert spaces. The support $\operatorname{supp}\mathcal{H}$ of $\mathcal{H}$ is defined by $\operatorname{supp}\mathcal{H} := \sum_{j=1}^d \psi_j\psi_j^*$, where $\{\psi_j \mid j=1,\ldots,d\}$ is any orthonormal basis of $\mathcal{H}$.

To give a precise definition of a Hilbert C^*-system we need to introduce the spectral projections: for $D\in\widehat{\mathcal{G}}$ (the dual of $\mathcal{G}$) its *spectral projection* $\Pi_D\in\mathcal{L}(\mathcal{F})$ is defined by

$$\begin{aligned} \Pi_D(F) &:= \int_{\mathcal{G}} \overline{\chi_D(g)}\,\beta_g(F)\,dg \quad \text{for all} \quad F\in\mathcal{F}, \qquad (6)\\ \text{where} \qquad \chi_D(g) &:= \dim D\cdot \operatorname{Tr} U_D(g), \quad U_D\in D\,, \end{aligned}$$

is the so-called modified character of the class D and dg is the normalized Haar measure of the compact group $\mathcal{G}$. For the trivial representation $\iota\in\widehat{\mathcal{G}}$, we put

$$\mathcal{A} := \Pi_\iota\,\mathcal{F} = \Big\{F\in\mathcal{F}\mid g(F)=F,\quad g\in\mathcal{G}\Big\}\,,$$

i.e. $\mathcal{A}=\mathcal{F}^{\mathcal{G}}$ is the fixed-point algebra in $\mathcal{F}$ w.r.t. $\mathcal{G}$. We denote by $\mathcal{Z}=\mathcal{Z}(\mathcal{A})$ the center of $\mathcal{A}$, which we assume to be nontrivial.

Definition 2.1 *The C^*-dynamical system $\{\mathcal{F},\mathcal{G},\beta\}$ with compact group $\mathcal{G}$ is called a* **Hilbert C^*-system** *if it has full Hilbert spectrum, i.e. for each $D\in\widehat{\mathcal{G}}$ there is a β-stable Hilbert space $\mathcal{H}_D\subset\Pi_D\mathcal{F}$, with support $\mathbb{1}$ and the unitary representation $\beta_{\mathcal{G}}\restriction\mathcal{H}_D$ is in the equivalence class $D\in\widehat{\mathcal{G}}$. A Hilbert C^*-system is called* **minimal** *if*

$$\mathcal{A}'\cap\mathcal{F}=\mathcal{Z}\,,$$

where $\mathcal{Z}$ is the center of the fixed-point algebra $\mathcal{A}:=\mathcal{F}^{\mathcal{G}}$.

Since we can identify $\mathcal{G}$ with $\beta_{\mathcal{G}} \subseteq \mathrm{Aut}\,\mathcal{F}$ we will often denote the Hilbert C^*-system simply by $\{\mathcal{F}, \mathcal{G}\}$.

Remark 2.2 Some families of examples of minimal Hilbert C^*-systems with fixed point algebra $\mathcal{A}_{\mathbb{C}} \otimes \mathcal{Z}$, where $\mathcal{A}_{\mathbb{C}}$ has trivial center, were constructed indirectly in [5, Section 6]. Some explicit examples in the context of the CAR/CCR-algebra with an Abelian group are given in [1] and [2, Section V].

To each $\mathcal{G}$-invariant algebraic Hilbert space $\mathcal{H} \subset \mathcal{F}$ there is assigned a corresponding inner endomorphism $\rho_{\mathcal{H}} \in \mathrm{End}\,\mathcal{F}$ given by

$$\rho_{\mathcal{H}}(F) := \sum_{j=1}^{d(\mathcal{H})} \psi_j F \psi_j^* \,,$$

where $\{\psi_j \,\big|\, j = 1, \ldots, d(\mathcal{H})\}$ is any orthonormal basis of $\mathcal{H}$. It is easy to see that $\mathcal{A}$ is stable under the inner endomorphism ρ. We call **canonical endomorphism** the restriction of $\rho_{\mathcal{H}}$ to $\mathcal{A}$, i.e. $\rho_{\mathcal{H}}\restriction\mathcal{A} \in \mathrm{End}\,\mathcal{A}$. By abuse of notation we will also denote it simply by $\rho_{\mathcal{H}}$. Let $\mathcal{Z}$ denote the center of $\mathcal{A}$; we say that an endomorphism ρ is *irreducible* if

$$(\rho, \rho) = \rho(\mathcal{Z})\,.$$

In the nontrivial center situation canonical endomorphisms do not characterize the algebraic Hilbert spaces anymore. In fact, the natural generalization in this context is the following notion of free Hilbert $\mathcal{Z}$-bimodule: let $\mathcal{H}$ be a $\mathcal{G}$-invariant algebraic Hilbert space in $\mathcal{F}$ of finite dimension d. Then we define first the free right $\mathcal{Z}$-module $\mathfrak{H}$ by extension

$$\mathfrak{H} := \mathcal{H}\mathcal{Z} = \left\{ \sum_{i=1}^{d} \psi_i \, Z_i \mid Z_i \in \mathcal{Z} \right\}, \tag{7}$$

where $\Psi := \{\psi_i\}_{i=1}^d$ is an orthonormal basis in $\mathcal{H}$. In other words, the set Ψ becomes a module basis of $\mathfrak{H}$ and $\dim_{\mathcal{Z}}\mathfrak{H} = d$. For $H_1, H_2 \in \mathfrak{H}$ put

$$\langle H_1, H_2 \rangle_{\mathfrak{H}} := H_1^* H_2 \in \mathcal{Z}\,.$$

Then, $\{\mathfrak{H}, \langle \cdot, \cdot \rangle_{\mathfrak{H}}\}$ is a Hilbert right $\mathcal{Z}$-module or a Hilbert $\mathcal{Z}$-module, for short. Now the canonical endomorphism can be also written as

$$\rho_{\mathcal{H}}(A) := \sum_{j=1}^{d} \varphi_j A \varphi_j^* \,, \quad A \in \mathcal{A}\,,$$

where $\{\varphi_i\}_{i=1}^d$ is *any* orthonormal basis of the $\mathcal{Z}$-module $\mathfrak{H}$. Hence we actually have $\rho_{\mathcal{H}} = \rho_{\mathfrak{H}}$ and it is easy to show that

$$H \in \mathfrak{H} \quad \text{iff} \quad H\,A = \rho_{\mathfrak{H}}(A)\,H\,.$$

In other words $\rho_{\mathfrak{H}}$ characterizes uniquely the Hilbert $\mathcal{Z}$-module $\mathfrak{H}$. Moreover, since for any canonical endomorphism $\rho = \rho_{\mathcal{H}}$ we have that $\mathcal{Z} \subset (\rho, \rho)$, it is easy to see that there is a canonical left action of $\mathcal{Z}$ on $\mathfrak{H}$. Concretely, there is a natural *-homomorphism $\mathcal{Z} \to \mathcal{L}(\mathfrak{H})$, where $\mathcal{L}(\mathfrak{H})$ is the set of $\mathcal{Z}$-module morphisms (see [3, Sections 3 and 4] for more details). Hence $\mathfrak{H}$ becomes a $\mathcal{Z}$-bimodule.

We conclude stating the isomorphism between the category of canonical endomorphisms and the corresponding category of free $\mathcal{Z}$-bimodules (cf. [5, Proposition 4.4] and [3, Section 4]).

Proposition 2.3 *Let $\{\mathcal{F}, \mathcal{G}\}$ be a given minimal Hilbert C*-system, where the fixed point algebra $\mathcal{A}$ has center $\mathcal{Z}$. Then the category $\mathcal{T}$ of all canonical endomorphisms of $\{\mathcal{F}, \mathcal{G}\}$ is isomorphic to the subcategory $\mathcal{M}_{\mathcal{G}}$ of the category of free Hilbert $\mathcal{Z}$-bimodules with objects $\mathfrak{H} = \mathcal{H}\mathcal{Z}$, where $\mathcal{H}$ is a $\mathcal{G}$-invariant algebraic Hilbert space with* $\mathrm{supp}\mathcal{H}_\sigma = \mathbb{1}$*, and the arrows given by the corresponding $\mathcal{G}$-invariant module morphisms $\mathcal{L}(\mathfrak{H}_1, \mathfrak{H}_2; \mathcal{G})$.*

The bijection of objects is given by $\rho_{\mathcal{H}} \leftrightarrow \mathfrak{H} = \mathcal{H}\mathcal{Z}$ which satisfies the conditions

$$\begin{aligned} \rho_{\mathcal{H}} = (\mathrm{Ad}\, V) \circ \rho_1 + (\mathrm{Ad}\, W) \circ \rho_2 \quad &\longleftrightarrow \quad \mathfrak{H} = V\mathfrak{H}_1 + W\mathfrak{H}_2 \\ \rho_1 \circ \rho_2 \quad &\longleftrightarrow \quad \mathfrak{H}_1 \cdot \mathfrak{H}_2 \ , \end{aligned}$$

*where $V, W \in \mathcal{A}$ are isometries with $VV^* + WW^* = \mathbb{1}$ and the latter product is the inner tensor product of the Hilbert $\mathcal{Z}$-modules w.r.t. the *-homomorphism $\mathcal{Z} \to \mathcal{L}(\mathfrak{H}_2)$. The bijection on arrows is defined by*

$$\mathcal{J}\colon\ \mathcal{L}(\mathfrak{H}_1, \mathfrak{H}_2; \mathcal{G}) \to (\rho_1, \rho_2) \quad \text{with} \quad \mathcal{J}(T) := \sum_{j,k} \psi_j Z_{j,k} \varphi_k^* \ .$$

Here $\{\psi_j\}_j$, $\{\varphi_k\}_k$ are orthonormal basis of $\mathfrak{H}_2, \mathfrak{H}_1$, respectively, and $(Z_{j,k})_{j,k}$ is the matrix of the right $\mathcal{Z}$-linear operator T from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ which intertwines the $\mathcal{G}$-actions.

The preceding proposition shows that the canonical endomorphisms uniquely determine the corresponding $\mathcal{Z}$-bimodules, but not the choice of the generating algebraic Hilbert spaces. The assumption of the minimality condition in Definition 2.1 is crucial here. From the point of view of the $\mathcal{Z}$-bimodules it is natural to consider next the following property of Hilbert C*-systems: the existence of a special choice of algebraic Hilbert spaces within the modules that define the canonical endomorphisms and which is compatible with products.

Definition 2.4 *A Hilbert C*-system $\{\mathcal{F}, \mathcal{G}\}$ is called* **regular** *if there is an assignment $\mathcal{T} \ni \sigma \to \mathcal{H}_\sigma$, where $\mathcal{H}_\sigma$ is a $\mathcal{G}$-invariant algebraic Hilbert space with* $\mathrm{supp}\mathcal{H}_\sigma = \mathbb{1}$ *and $\sigma = \rho_{\mathcal{H}_\sigma}$ (i.e. σ is the canonical endomorphism of the algebraic Hilbert space $\mathcal{H}_\sigma$), which is compatible with products:*

$$\sigma \circ \tau \mapsto \mathcal{H}_\sigma \cdot \mathcal{H}_\tau \ .$$

Remark 2.5 In a minimal Hilbert C*-system regularity means that there is a “generating” Hilbert space $\mathcal{H}_\tau \subset \mathfrak{H}_\tau$ for each τ (with $\mathfrak{H}_\tau = \mathcal{H}_\tau\mathcal{Z}$) such that the compatibility relation for products stated in Definition 2.4 holds. If a Hilbert C*-system is minimal and $\mathcal{Z} = \mathbb{C}\mathbb{1}$ then it is necessarily regular.

2.2 The chain group

In the present section we recall the main motivations and definitions concerning the chain group associated with a compact group $\mathcal{G}$. For proofs and more details see [5, Section 5] (see also [25]).

One of the fundamental new aspects of superselection theory with a nontrivial center $\mathcal{Z}$ is the fact that irreducible canonical endomorphisms act as (nontrivial) automorphisms on $\mathcal{Z}$. In fact, let $D \in \widehat{\mathcal{G}}$ (the dual of $\mathcal{G}$) and denote by $\rho_D := \rho_{\mathcal{H}_D}$ the corresponding irreducible canonical endomorphism. Then, to any class D we can associate the following automorphism on $\mathcal{Z}$:

$$\widehat{\mathcal{G}} \ni D \mapsto \alpha_D := \rho_D{\restriction}\mathcal{Z} \in \mathrm{Aut}\,\mathcal{Z} \ . \tag{8}$$

This observation allows one to introduce a natural equivalence relation in the dual $\widehat{\mathcal{G}}$ which, roughly speaking, relates elements $D, D' \in \widehat{\mathcal{G}}$ if there is a “chain of tensor products” of elements in $\widehat{\mathcal{G}}$ containing D and D' (see Theorem 2.10 and Remark 2.11 below).

First, we need to recall the algebraic structure of $\widehat{\mathcal{G}}$: denote by $\times$ the natural operation on subsets of $\widehat{\mathcal{G}}$ associated with the decomposition of the tensor products of irreducible representations. For any $D \in \widehat{\mathcal{G}}$ let U_D be an irreducible representation in the class D. Then we define

$$D_1 \times D_2 := \{D \in \widehat{\mathcal{G}} \mid U_D \prec U_{D_1} \otimes U_{D_2}\} .$$

For $\Gamma, \Gamma_1, \Gamma_2 \subset \widehat{\mathcal{G}}$ put

$$\Gamma_1 \times \Gamma_2 = \cup\{D_1 \times D_2 \mid D_i \in \Gamma_i,\, i = 1, 2\} \quad \text{and} \quad D \times \Gamma = \{D\} \times \Gamma .$$

Moreover if $\overline{D} \in \widehat{\mathcal{G}}$ denotes the conjugate class to $D \in \widehat{\mathcal{G}}$ we put $\overline{\Gamma} = \{\overline{D} \mid D \in \Gamma\}$. Recall in particular that if $D \in D_0 \times D_1$, $D' \in D_0' \times D_1'$, then $D \times D' \subset D_0 \times D_1 \times D_0' \times D_1'$ or that the trivial representation ι is contained in $\Gamma \times \overline{\Gamma}$ (cf. [19, Definition 27.35] for further details).

We can now make precise the previous idea:

Definition 2.6 *The elements $D, D' \in \widehat{\mathcal{G}}$ are called equivalent, $D \approx D'$, if there exist $D_1, \ldots, D_n \in \widehat{\mathcal{G}}$ such that*

$$D, D' \in D_1 \times \cdots \times D_n .$$

The preceding definition is an equivalence relation in $\widehat{\mathcal{G}}$ and we denote by square brackets $[\cdot]$ the corresponding chain equivalence classes. We denote the factor space by

$$\mathfrak{C}(\mathcal{G}) := \widehat{\mathcal{G}}/\approx \ .$$

By definition *any* pair $D, D' \in D_0 \times D_1$ satisfies $D \approx D'$. Therefore for $D_0, D_1 \in \widehat{\mathcal{G}}$ we have that $D_0 \times D_1$ also specifies an element of $\mathfrak{C}(\mathcal{G})$ and we can simply put

$$[D_0 \times D_1] := [D] ,$$

where D is any element in $D_0 \times D_1$.

We will define on $\mathfrak{C}(\mathcal{G})$ a product $\boxtimes$ (see Eq. (10) below) so that $(\mathfrak{C}(\mathcal{G}), \boxtimes)$ becomes an Abelian group which for simplicity we call **chain group**. Moreover, the chain group can be related to the character group of the center $\mathcal{C}$ of $\mathcal{G}$. For this recall also the notion of conjugacy class of a representation (cf. [16]): let $D \in \widehat{\mathcal{G}}$ and U_D any representer in D. By Schur's Lemma we have

$$U_D \restriction \mathcal{C} = \Upsilon_D \cdot \mathbb{1} , \tag{9}$$

and it can be easily seen that Υ_D is a character on the center $\mathcal{C}$ of $\mathcal{G}$ which only depends on D, i.e. $\Upsilon_D \in \widehat{\mathcal{C}}$.

Theorem 2.7 *Let $\mathcal{G}$ be a compact non-Abelian group and denote its center by $\mathcal{C}$.*

(i) The set $\mathfrak{C}(\mathcal{G})$ becomes an Abelian group w.r.t. the following multiplication: for $D_0, D_1 \in \widehat{\mathcal{G}}$ put

$$[D_0] \boxtimes [D_1] := [D_0 \times D_1] . \tag{10}$$

(ii) The conjugacy classes Υ_D (cf. Eq. (9)) depend on the chain equivalence class $[D]$. The chain group and the character group of the center of $\mathcal{G}$ are isomorphic. The isomorphism is given by

$$\eta : \mathfrak{C}(\mathcal{G}) \to \widehat{\mathcal{C}} \quad \text{with} \quad \eta([D]) := \Upsilon_{[D]} ,$$

where $\Upsilon_{[D]}$ is the conjugacy class associated with $[D] \in \mathfrak{C}(\mathcal{G})$.

Remark 2.8 (i) A complete proof of the preceding theorem is given in Theorem 5.5 of [5]. The injectivity of the mapping η was proven for the first time in [25].

(ii) The preceding theorem shows that the equivalence relation defined in $\widehat{\mathcal{G}}$ gives a direct way to reconstruct (via Pontryagin's duality) the center $\mathcal{C}(\mathcal{G})$ of a compact group from the representation ring of $\mathcal{G}$.

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{C}(\mathcal{G}) \\ \widehat{\mathcal{G}} \arrow[r, "\text{factorization by } \approx"'] & \mathfrak{C}(\mathcal{G}) \arrow[u, "\text{Pontryagin}"'] \end{tikzcd}$$

For further results in this direction see [25, 32].

Example 2.9 A simple example that illustrates how to construct a chain group is given by $\mathcal{G} = \mathrm{SU}(2)$. Denote by

$$l \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots\} = \widehat{\mathrm{SU}(2)}$$

the class specified by the usual representation $V^{(l)}$ of SU(2) on the space of complex polynomials of degree $\leq 2l$ which has dimension $2l+1$. Then the decomposition theory for the tensor products $V^{(l)} \otimes V^{(l')}$ (cf. [19, Theorem 29.26]) gives

$$l \times l' = \left\{ |l - l'|, |l - l'| + 1, \dots, l + l' \right\}, \quad l, l' \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots\}.$$

This decomposition structure implies that

$$l \approx l' \quad \text{iff} \quad l, l' \text{ are both integers or both half-integers}.$$

We can finally conclude that

$$\mathfrak{C}(\mathrm{SU}(2)) = \left\{[0], [\tfrac{1}{2}]\right\} \cong \mathbb{Z}_2 \cong \widehat{\mathcal{C}(\mathrm{SU}(2))}.$$

For other examples with finite and compact Lie groups see [5, Subsection 5.1].

We will now concentrate on the relation of the chain group $\mathfrak{C}(\mathcal{G})$, associated with the compact group $\mathcal{G}$ of a Hilbert C*-system $\{\mathcal{F}, \mathcal{G}\}$, with the irreducible canonical endomorphisms restricted to $\mathcal{Z}$. In particular recall the automorphisms on $\mathcal{Z}$ given in Eq. (8) by $\alpha_D := \rho_D \restriction \mathcal{Z} \in \mathrm{Aut}\, \mathcal{Z}$ which are associated with any class $D \in \widehat{\mathcal{G}}$. For a complete proof of the next theorem see Theorem 5.7 of [5].

Theorem 2.10 *(i) Let $D, D' \in \widehat{\mathcal{G}}$ be equivalent, i.e. $D \approx D'$. Then $\alpha_D = \alpha_{D'}$ and we can associate the automorphism $\alpha_{[D]} \in \mathrm{Aut}\, \mathcal{Z}$ with the chain group element $[D] \in \mathfrak{C}(\mathcal{G})$.*

(ii) There is a natural group homomorphism from the chain group to the automorphism group generated by the irreducible endomorphisms restricted to $\mathcal{Z}$:

$$\mathfrak{C}(\mathcal{G}) \ni [D] \mapsto \alpha_{[D]} \in \mathrm{Aut}\, \mathcal{Z}. \tag{11}$$

Remark 2.11 Note that the chain group and in particular Theorem 2.10 (i) completes the picture of the action of the irreducible canonical endomorphisms on the center $\mathcal{Z}$ of the fixed-point algebra $\mathcal{A}$ (recall also Eq. (8)). Indeed, we may summarize this action by means of the following diagram

$$\begin{array}{ccccc} \widehat{\mathcal{G}} & \to & \mathfrak{C}(\mathcal{G}) & \to & \mathrm{Aut}\ \mathcal{Z} \\ D & \mapsto & [D] & \mapsto & \alpha_{[D]} \end{array}$$

Theorem 2.12 *Let ρ be a (reducible) canonical endomorphism. Then its action on $\mathcal{Z}$ can be described by means of the following formula*

$$\rho(Z) = \sum_{[D]\in\mathfrak{C}(\mathcal{G})} \alpha_{[D]}(Z)\cdot E_{[D]} \ , \quad Z\in\mathcal{Z}\,,$$

where $E_{D'}\in\mathcal{A}$ is the isotypical projection w.r.t. $D'\in\widehat{\mathcal{G}}$ and $E_{[D]} := \sum_{D'\,\in\,[D]} E_{D'}$. For $n\geq 2$ we have

$$\rho^n(Z) = \sum_{[D_1],\dots,[D_n]} \alpha_{[D_1\times\cdots\times D_n]}(Z)\cdot E_{[D_1]}\cdot\rho(E_{[D_2]})\cdot\ldots\cdot\rho^{n-1}(E_{[D_n]}) \ , \quad Z\in\mathcal{Z}\,.$$

Proof: The first equation is shown in [5, Theorem 5.9]. From this one can easily check the expression for higher powers of $\rho\restriction\mathcal{Z}$. ■

3 Cuntz-Pimsner algebras

In the present section, we introduce some basic properties of the Cuntz-Pimsner algebra (also called Cuntz-Krieger-Pimsner algebra). The basic reference for this topic is [27]. In the present paper we will also use the alternative categorical approach of [10].

3.1 Basic definitions

Let $\mathfrak{R}$ be a C^*-algebra, $\mathfrak{H}$ a countably generated Hilbert $\mathfrak{R}$-bimodule. We assume that $\mathfrak{H}$ is full as a right Hilbert $\mathfrak{R}$-module. We denote by $\mathcal{L}(\mathfrak{H},\mathfrak{H})$ the C^*-algebra of adjointable, right $\mathfrak{R}$-module operators on $\mathfrak{H}$, and by $\mathcal{K}(\mathfrak{H},\mathfrak{H})\subseteq\mathcal{L}(\mathfrak{H},\mathfrak{H})$ the ideal of compact operators generated by the maps

$$\theta_{\psi,\psi'}\in\mathcal{L}(\mathfrak{H},\mathfrak{H}) \ , \ \ \psi,\psi'\in\mathfrak{H} \ , \quad \text{with} \quad \theta_{\psi,\psi'}(\varphi) := \psi\left\langle\psi',\varphi\right\rangle \ , \ \varphi\in\mathfrak{H} \ , \tag{12}$$

where $\langle\cdot,\cdot\rangle$ is the $\mathfrak{R}$-valued scalar product defined on $\mathfrak{H}$. We also denote by

$$\alpha\colon\mathfrak{R}\to\mathcal{L}(\mathfrak{H},\mathfrak{H})$$

the left $\mathfrak{R}$-action on $\mathfrak{H}$, which we assume to be non-degenerate.

Let now $\mathfrak{R}$ be unital, and $\mathfrak{H}$ finitely generated as a right Hilbert $\mathfrak{R}$-module by a subset $\Psi := \{\psi_l\}_{l=1}^n$. In this case, $\mathcal{L}(\mathfrak{H},\mathfrak{H}) = \mathcal{K}(\mathfrak{H},\mathfrak{H})$. We consider the $*$-algebra ${}^0\mathcal{O}_\mathfrak{H}$ generated by a subalgebra *-isomorphic to $\mathfrak{R}$ and Ψ, subject to the relations[2]

$$\begin{aligned} \psi_l^*\psi_m &= \langle\psi_l,\psi_m\rangle \qquad (13)\\ A\psi_l &= \alpha(A)\psi_l \ , \quad A\in\mathfrak{R} \\ \sum_l \psi_l\psi_l^* &= \mathbb{1}\,. \qquad (14) \end{aligned}$$

Note that (13) implies $\psi^*\psi' = \langle\psi,\psi'\rangle$ as well as $\psi'\psi^*\varphi = \theta_{\psi',\psi}(\varphi)$, $\psi',\psi^*,\varphi\in\mathfrak{H}$. Therefore one has the natural identification

$$\theta_{\psi',\psi} = \psi'\psi^* \quad . \tag{15}$$

The isomorphism class of ${}^0\mathcal{O}_\mathfrak{H}$ does not depend on the choice of the set of generators and it can be proven that there is a unique C^*-norm on ${}^0\mathcal{O}_\mathfrak{H}$ such that the action of the circle $\mathbb{T} := \{z\in\mathbb{C} \mid |z| = 1\}$ given by

$$\delta\colon\mathbb{T}\to\operatorname{Aut}{}^0\mathcal{O}_\mathfrak{H} \quad , \quad \delta_z(\psi) := z\psi \quad , \quad z\in\mathbb{T}, \ \psi\in\mathfrak{H}\,, \tag{16}$$

[2]With an abuse of notation, we identify $\mathfrak{R}$ with its image in ${}^0\mathcal{O}_\mathfrak{H}$.

extends to an (isometric) automorphic action. The Cuntz-Pimsner algebra $\mathcal{O}_{\mathfrak{H}}$ is by definition the completion of ${}^0\mathcal{O}_{\mathfrak{H}}$ w.r.t. this norm. In the case $\mathfrak{R} = \mathbb{C}$, $\mathfrak{H} = \mathbb{C}^d$, $d \in \mathbb{N}$, we obtain the Cuntz algebra $\mathcal{O}_d$. We denote by

$$\mathcal{O}_{\mathfrak{H}}^k := \left\{ T \in \mathcal{O}_{\mathfrak{H}} \mid \delta_z(T) = z^k T \ , \ z \in \mathbb{T} \right\} \ , \ k \in \mathbb{Z} \ , \tag{17}$$

the spectral subspaces w.r.t. the circle action.

Remark 3.1 For the kind of Hilbert bimodules that we consider there is an alternative way to generate the corresponding Cuntz-Pimsner algebras that we will need in the following sections. From [28] we have that the spectral subspaces are given as the following inductive limit, where the natural inclusions are specified by tensoring from the right by the identity on $\mathfrak{H}$:

$${}^0\mathcal{O}_{\mathfrak{H}}^k = \varinjlim \mathcal{L}(\mathfrak{H}^r, \mathfrak{H}^{r+k}) \, .$$

Moreover, we have the following natural $\mathbb{Z}$ grading

$${}^0\mathcal{O}_{\mathfrak{H}} = \oplus_{k \in \mathbb{Z}} \, {}^0\mathcal{O}_{\mathfrak{H}}^k$$

and $\mathcal{O}_{\mathfrak{H}}$ is the closure of ${}^0\mathcal{O}_{\mathfrak{H}}$ in the unique C^*-norm for which δ is isometric.

In order to discuss universality properties of the Cuntz-Pimsner algebra, we give the following definition:

Definition 3.2 *([10, §2]) Let $\mathfrak{R} \subset \mathfrak{B}$ be a C^*-algebra inclusion. A* **Hilbert $\mathfrak{R}$-bimodule in $\mathfrak{B}$** *is a closed vector space $\mathfrak{H} \subset \mathfrak{B}$, such that $A\psi \in \mathfrak{H}$, $\psi A \in \mathfrak{H}$, $\psi^*\psi' \in \mathfrak{R}$ for every $A \in \mathfrak{R}$, $\psi, \psi' \in \mathfrak{H}$.*

We describe in explicit terms some well-known properties for the case of Hilbert bimodule in C^*-algebras. Our Hilbert bimodule $\mathfrak{H}$ is finitely generated if there is a finite subset $\{\psi_l\}_{l=1}^n \subset \mathfrak{H}$ such that $\psi = \sum_l \psi_l(\psi_l^*\psi)$, $\psi \in \mathfrak{H}$. Moreover, $\mathfrak{H}$ is full if for every $A \in \mathfrak{R}$ there are $\psi, \psi' \in \mathfrak{H}$ such that $A = \psi^*\psi'$ and $\mathfrak{H}$ is non-degenerate if for any $A \in \mathfrak{R}$ with $A\psi = 0$ for all $\psi \in \mathfrak{H}$, we have that $A = 0$. If $\mathfrak{H}$ is finitely generated, then it is trivial to verify that

$$P_{\mathfrak{H}} := \sum_l \psi_l \psi_l^* \ \in \ \mathfrak{B}$$

is a projection, and that it does not depend on the choice of the set of generators. We call $P_{\mathfrak{H}}$ the *support* of $\mathfrak{H}$ in $\mathfrak{B}$.

Proposition 3.3 *([27, Theorem 3.12]) Let $\mathfrak{R} \subset \mathfrak{B}$ be an inclusion of unital C^*-algebras, $\mathfrak{H} \subset \mathfrak{B}$ a non-degenerate and full Hilbert $\mathfrak{R}$-bimodule in $\mathfrak{B}$ with support $\mathbb{1}$. Then, there is a canonical monomorphism $\mathcal{O}_{\mathfrak{H}} \hookrightarrow \mathfrak{B}$.*

Corollary 3.4 *Let $(\mathcal{F}, \mathcal{G}, \beta)$ be a Hilbert C^*-system with fixed-point algebra $\mathcal{A} := \mathcal{F}^{\mathcal{G}}$ and denote by $\mathcal{Z} := \mathcal{A}' \cap \mathcal{A}$ its center. Then, for every $D \in \widehat{\mathcal{G}}$ the Hilbert $\mathcal{Z}$-bimodule $\mathfrak{H}_D := \mathcal{Z}\mathcal{H}_D$ induces a monomorphism $\mathcal{O}_{\mathfrak{H}_D} \hookrightarrow \mathcal{F}$.*

Remark 3.5 (i) Examples of the above universality property can be found in the Cuntz-Pimsner algebra itself. Let $r = 1, 2, \ldots$, and $\mathfrak{H}^r := \mathfrak{H} \otimes_{\mathfrak{R}} \cdots \otimes_{\mathfrak{R}} \mathfrak{H}$ denote the r-fold tensor product with coefficients in $\mathfrak{R}$. Then, there is a natural identification

$$\mathfrak{H}^r \simeq \{\psi_1 \cdot \ldots \cdot \psi_r \in \mathcal{O}_{\mathfrak{H}} \mid \psi_k \in \mathfrak{H}, k = 1, \ldots, r\} \ ,$$

so that every $\mathfrak{H}^r$ is a Hilbert $\mathfrak{R}$-bimodule in $\mathcal{O}_{\mathfrak{H}}$, and there are canonical morphisms $\mathcal{O}_{\mathfrak{H}^r} \to \mathcal{O}_{\mathfrak{H}}$. Note that if there is an element $S \in \mathfrak{H}^r$ such that $\langle S, S \rangle = \mathbb{1}$, then S appears in the Cuntz-Pimsner algebra as an isometry, i.e. $S^*S = \mathbb{1}$.

(ii) The bimodule introduced in (7) is easily seen to be finitely generated, full and non-degenerate.

Let now $r, s \in \mathbb{N}$, and $\mathcal{K}(\mathfrak{H}^r, \mathfrak{H}^s)$ denote the set of compact, right $\mathfrak{R}$-module operators from $\mathfrak{H}^r$ into $\mathfrak{H}^s$. Applying Eq. (15), we may identify

$$\mathcal{K}(\mathfrak{H}^r, \mathfrak{H}^s) \simeq \operatorname{span}\{\psi_1 \cdots \psi_s \cdot \varphi_r^* \cdots \varphi_1^* \mid \psi_h, \varphi_k \in \mathfrak{H}\,;\ h = 1, \ldots, s\,;\ k = 1, \ldots, r\} \ .$$

To simplify notation we introduce the following conventions: let $\mathfrak{H}$ be finitely generated by a set $\Psi = \{\psi_l\}_{l=1}^n$, $s \in \mathbb{N}$, and $L := \{l_1, \ldots, l_s\} \in \{1, \ldots, n\}^s$ a multi-index of length s, i.e. $|L| = s$. We denote

$$\psi_L := \psi_{l_1} \cdots \psi_{l_s} \in \mathfrak{H}^s \ , \tag{18}$$

so that by (14) we find

$$\sum_L \psi_L \psi_L^* = \mathbb{1} \ ,$$

and we may also write

$$\mathcal{K}(\mathfrak{H}^r, \mathfrak{H}^s) \simeq \operatorname{span}\{\psi_L\, A\, \psi_M^* \mid\ |L| = s\,;\ |M| = r\,;\ A \in \mathfrak{R}\} \ . \tag{19}$$

Finally, we need to introduce the notion of a nonsingular bimodule:

Definition 3.6 *A Hilbert $\mathfrak{R}$-bimodule $\mathfrak{H}$ with left action $\alpha\colon \mathfrak{R} \to \mathcal{L}(\mathfrak{H}, \mathfrak{H})$ is called* **nonsingular** *if $\theta_{\psi,\psi} \in \alpha(\mathfrak{R})$ for some $\psi \in \mathfrak{H}$ implies $\psi = 0$.*

Proposition 3.7 *([10, Proposition 3.5]) If $\mathfrak{H}$ is nonsingular and there is an isometry $S \in \mathfrak{H}^n \cap \mathfrak{R}'$ for some $n \in \mathbb{N}$, $n > 1$, then $C^*(S)' \cap \mathcal{O}_{\mathfrak{H}} = \mathfrak{R}$ (where $C^*(S)$ denotes the C^*-algebra generated by S in $\mathcal{O}_{\mathfrak{H}}$).*

3.2 Endomorphisms in Cuntz-Pimsner algebras

It is well-known that a natural endomorphism is defined over the Cuntz algebra $\mathcal{O}_n$, in the following way: if $\{\psi_k\}_{k=1}^n$ is a set of orthonormal isometries generating $\mathcal{O}_n$, then we define $\rho \in \operatorname{End}\mathcal{O}_n$ by

$$\rho(T) := \sum_k \psi_k T \psi_k^* \quad , \quad T \in \mathcal{O}_n \quad . \tag{20}$$

Given a generic Hilbert $\mathfrak{R}$-bimodule $\mathfrak{H}$, it is not possible to define in a consistent way the analogue of (20) over $\mathcal{O}_{\mathfrak{H}}$. In fact, the multiplicativity of ρ is ensured by the fact that $\psi_h^* \psi_k \in \mathcal{O}_n \cap \mathcal{O}_n' = \mathbb{C}\mathbb{1}$, $h, k \in \{1, \ldots, n\}$. Given a set $\{\psi_l\}_l$ of generators for $\mathfrak{H}$, it is not true in general that $\psi_l^* \psi_m \in \mathcal{O}_{\mathfrak{H}} \cap \mathcal{O}_{\mathfrak{H}}'$. Nevertheless, the above obstruction for the existence of the endomorphism ρ disappears if we consider a Hilbert bimodule which is free as a right Hilbert $\mathfrak{R}$-module (cf. Subsection 2.1), i.e.

$$\mathfrak{H} \simeq \mathbb{C}^n \otimes \mathfrak{R} \,.$$

In this case $\mathfrak{H}$ admits a (non-unique) set of generators $\Psi := \{\psi_k\}_{k=1}^n \subset \mathfrak{H}$ such that

$$\langle \psi_h, \psi_k \rangle = \delta_{hk} \mathbb{1} \ .$$

We say in this case that Ψ is a set of *orthonormal generators*. Let $\mathcal{H}_\Psi \subset \mathfrak{H} \subset \mathcal{O}_{\mathfrak{H}}$ be the vector space spanned by elements of Ψ (the latter regarded as a subset of $\mathcal{O}_{\mathfrak{H}}$). According to the terminology of Definition 3.2 we obtain by (13) that $\mathcal{H}_\Psi$ is a Hilbert $\mathbb{C}$-bimodule in $\mathcal{O}_{\mathfrak{H}}$. Such particular cases of Hilbert bimodules in C^*-algebras are called *Hilbert spaces in C^*-algebras* in [11]. As mentioned before we call them *algebraic Hilbert spaces* (cf. Section 2.1 or [5]). It is clear that $\mathcal{H}_\Psi$ depends on the choice of Ψ. Different choices of orthonormal generating sets $\Psi \subset \mathfrak{H}$ correspond to different isomorphisms $\mathfrak{H} \simeq \mathcal{H}_\Psi \otimes \mathfrak{R}$ of right Hilbert $\mathfrak{R}$-modules. From the previous considerations, we obtain:

Lemma 3.8 *Let $\mathfrak{R}$ be a unital C^*-algebra, $\mathfrak{H}$ a Hilbert $\mathfrak{R}$-bimodule, $\mathcal{H}$ a rank n algebraic Hilbert space, $n \in \mathbb{N}$. Given a fixed isomorphism $\mathfrak{H} \simeq \mathcal{H} \otimes \mathfrak{R}$ of right Hilbert $\mathfrak{R}$-modules, there exists an endomorphism $\rho_{\mathcal{H}} \in \operatorname{End} \mathcal{O}_{\mathfrak{H}}$, defined in Eq. (20), where $\{\psi_k\}_{k=1}^n$ is a set of orthonormal generators corresponding to a basis of $\mathcal{H}$. The endomorphism $\rho_{\mathcal{H}}$ does not depend on the choice of the basis of $\mathcal{H}$. Moreover, there is a unital monomorphism $\mathcal{O}_n \hookrightarrow \mathcal{O}_{\mathfrak{H}}$.*

Remark 3.9 Note that $\rho_{\mathcal{H}}$ commutes with the circle action, so that the spectral subspaces introduced in (17) are preserved, i.e. $\rho_{\mathcal{H}}(\mathcal{O}_{\mathfrak{H}}^k) \subset \mathcal{O}_{\mathfrak{H}}^k$, $k \in \mathbb{Z}$.

3.3 Amplimorphisms and their associated Cuntz-Pimsner algebras

Let $\mathfrak{R}$ be a unital C^*-algebra with identity $\mathbb{1}$. The C^*-category **bimod**$(\mathfrak{R})$ having as objects the Hilbert $\mathfrak{R}$-bimodules which are finitely generated and projective as right Hilbert $\mathfrak{R}$-modules can be described by means of *amplimorphisms*, i.e. C^*-algebra morphisms of the form

$$\alpha \colon \mathfrak{R} \to \mathbb{M}_n \otimes \mathfrak{R} \quad ,$$

where $\mathbb{M}_n$ denotes the C^*-algebra of $n \times n$-matrices. The Hilbert $\mathfrak{R}$-bimodule associated with α is defined by

$$\mathfrak{H}_\alpha := \{\psi \in \mathbb{C}^n \otimes \mathfrak{R} \mid \alpha(\mathbb{1})\psi = \psi\}$$

with left and right $\mathfrak{R}$-actions defined as follows: if $\psi := \{A_l\}_{l=1}^n$, $\psi' := \{A'_m\}_{m=1}^n \in \mathfrak{H}_\alpha$, we have

$$\psi A := \{A_l A\}_l \in \mathfrak{H} \quad , \quad A\psi := \alpha(A)\psi \in \mathfrak{H} \quad , \quad \langle \psi, \psi' \rangle := \sum_l A_l^* A'_l \in \mathfrak{R} \quad .$$

The following notion of diagonal bimodule will play an important role in the construction of examples presented in the following two sections.

Definition 3.10 *A Hilbert $\mathfrak{R}$-bimodule $\mathfrak{H} = \mathfrak{H}_\alpha$, $\alpha \in$ **ampl**$(\mathfrak{R})$, is called* **diagonal of rank** n *if $\alpha(A) := \operatorname{diag}(\alpha_1(A), \ldots, \alpha_n(A))$, $A \in \mathfrak{R}$, where $\alpha_1, \ldots, \alpha_n \in \operatorname{Aut} \mathfrak{R}$, and $\operatorname{diag}(\cdot)$ denotes the diagonal matrix in $\mathbb{M}_n \otimes \mathfrak{R}$.*

It is clear that a diagonal bimodule $\mathfrak{H}$ is free as a right Hilbert $\mathcal{Z}$-module. In this case the Cuntz-Pimsner algebra $\mathcal{O}_{\mathfrak{H}}$ is specified by the orthonormal basis $\{\psi_k\}_{k=1}^n \subset \mathfrak{H}$ that satisfies the following simple relations:

$$\psi_h^* \psi_k = \delta_{hk} \mathbb{1} \quad , \quad Z\psi_k = \psi_k \alpha_k(Z) \quad , \quad \sum_k \psi_k \psi_k^* = \mathbb{1} \quad , \tag{21}$$

$h, k = 1, \ldots, n$. Note that $\rho_{\mathcal{H}}(Z)\psi_k = \psi_k Z = \alpha_k^{-1}(Z)\psi_k$, $k = 1, \ldots, n$. Let us consider the projections $E_k := \psi_k \psi_k^* \in \mathcal{K}(\mathfrak{H})$. Then

$$\rho_{\mathcal{H}}(Z) = \sum_k \alpha_k^{-1}(Z) E_k \tag{22}$$

It will be useful for the construction of the class of examples to give a characterization of diagonal nonsingular bimodules in the sense of Definition 3.6 in order to apply Proposition 3.7. For this purpose we will apply Gelfand's theorem and identify the unital abelian C*-algebra $\mathcal{Z}$ with continuous functions on the compact Hausdorff space $\Omega := \operatorname{spec} \mathcal{Z}$. Moreover, to any automorphism $\alpha_k \in \operatorname{Aut} \mathcal{Z}$ there corresponds a homeomorphism f_k of Ω such that for any $Z \in \mathcal{Z}$ we have the relation

$$\left(\alpha_k^{-1} Z\right)(\omega) = Z(f_k(\omega)) \, , \quad \omega \in \Omega \, , k = 1, \ldots, n \, .$$

Lemma 3.11 *Let $\mathcal{Z}$ be an Abelian, unital C*-algebra and $\mathfrak{H} = \mathfrak{H}_\alpha$ a diagonal free $\mathcal{Z}$-bimodule of rank n. If $\{\psi_k\}_{k=1}^n$ is an orthonormal set of generators for $\mathfrak{H}$ and $\varphi := \sum_h \psi_h Z_h \in \mathfrak{H}$, then the equation $\theta_{\varphi,\varphi} = \alpha(Z)$ for some $Z \in \mathcal{Z}$ is equivalent to the equations*

$$\begin{aligned} Z_h Z_k^* &= 0 \ , \quad h \neq k \ , \quad h,k = 1,\ldots,n \ , && (23) \\ Z_k Z_k^* &= \alpha_k(Z) \ , \quad k = 1,\ldots,n \, . && (24) \end{aligned}$$

Proof: Evaluating the equation $\theta_{\varphi,\varphi} = \alpha(Z)$ on the basis elements we obtain

$$\sum_h \psi_h Z_h Z_k^* = \psi_k \alpha_k(Z) \ , \quad k = 1,\ldots,n \ ,$$

which implies the statement. ■

Proposition 3.12 *Let $\mathcal{Z}$ be an Abelian, unital C*-algebra and $\mathfrak{H} = \mathfrak{H}_\alpha$ a diagonal free $\mathcal{Z}$-bimodule of rank n. Then, $\mathfrak{H}$ is nonsingular iff there is a pair of indices $h,k \in \{1,\ldots,n\}$, $h \neq k$, such that $\alpha_h = \alpha_k \in \mathrm{Aut}\,\mathcal{Z}$.*

Proof: 1. Using Gelfand's theorem recall that to any automorphism α_i there corresponds a homeomorphism f_i on $\Omega := \mathrm{spec}\,\mathcal{Z}$. Assume that there exist a pair of indices $h,k \in \{1,\ldots,n\}$, $h \neq k$, such that $f_h(\omega) = f_k(\omega)$, $\omega \in \Omega$. Suppose that there are $Z, Z_1,\ldots,Z_n \in \mathcal{Z}$ as above with $Z \neq 0$ and such that the equation $\theta_{\varphi,\varphi} = \alpha(Z)$ holds, i.e.

$$\begin{aligned} Z_h(\omega)\overline{Z_k}(\omega) &= 0 \ , \quad \omega \in \Omega \ , \ h \neq k && (25) \\ |Z_k|^2 (f_k(\omega)) &= Z(\omega) \ , \quad h,k = 1,\ldots,n \, . && (26) \end{aligned}$$

Then for $\omega \in \Omega$ with $Z(\omega) \neq 0$ we get from Eq. (26) that

$$Z_k(f_k(\omega)) \neq 0 \neq Z_h(f_h(\omega))$$

which contradicts Eq. (25). Thus $Z = 0$, hence $Z_k = 0$, $k = 1,\ldots n$, which implies that $\varphi = 0$. This shows that the bimodule is nonsingular.

2. For the reverse implication we show that if there is a point $\omega \in \Omega$ such that for all pairs of indices (h,k), $h \neq k$, we have $f_h(\omega) \neq f_k(\omega)$, then $\mathfrak{H}$ is singular (i.e. there exist $Z_1,\ldots,Z_n \in \mathcal{Z}$ (not all equal to 0) and a $Z \in \mathcal{Z}$ such that Eqs. (25) and (26) hold; note that in this case it follows that $Z \neq 0$).

We will show next that since Ω is a compact Hausdorff space and the rank of the bimodule is finite there is a neighborhood W of ω such that $f_k(W) \cap f_h(W) = \emptyset$ for all (h,k), $h \neq k$. Indeed, for any pair (h,k), $h \neq k$ since $f_h(\omega) \neq f_k(\omega)$ there exist open neighborhoods W_{hk} resp. W_{kh} of $f_h(\omega)$ resp. $f_k(\omega)$ with $W_{hk} \cap W_{kh} = \emptyset$. Therefore we can define the open neighborhood of ω by

$$W := \bigcap_{h \neq k} \left(f_h^{-1}(W_{hk}) \cap f_k^{-1}(W_{kh}) \right) \ ,$$

which satisfies the required properties.

Let ω and W be as in the preceding paragraph and define $Z(\cdot)$ as a positive continuous function with support contained in W. Putting

$$Z_k(\omega) := \sqrt{Z\left(f_k^{-1}(\omega)\right)}$$

we obtain continuous functions $Z, Z_1,\ldots,Z_n \in \mathcal{Z}$ satisfying Eqs. (25) and (26). Hence $\mathfrak{H}$ is singular and the proof is concluded. ■

Remark 3.13 (i) We note that if $\mathfrak{H}$ is a diagonal bimodule, then the left $\mathfrak{R}$-module action on $\mathfrak{H}$ is injective (i.e. $\alpha(A) = 0 \Rightarrow A = 0$, $A \in \mathfrak{R}$) and non-degenerate ($\alpha(\mathbb{1}) = \mathbb{1}_n$, where $\mathbb{1}_n$ is the identity of $\mathbb{M}_n \otimes \mathfrak{R}$).

(ii) The Cuntz-Pimsner algebras $\mathcal{O}_{\mathfrak{H}}$ considered in this paper are nuclear. For a proof of this fact in the context of a more general class of Cuntz-Pimsner algebras see [22].

4 Examples of minimal C^*-dynamical systems

Let $\mathcal{G}$ be a compact group. We denote by $\widehat{\mathcal{G}}$ its dual object and by $\mathfrak{C}(\mathcal{G})$ its chain group (cf. Subsection 2.2). Given a unital (not necessarily Abelian) C^*-algebra $\mathfrak{R}$, we consider a fixed chain group action

$$\mathfrak{C}(\mathcal{G}) \ni [D] \mapsto \alpha_{[D]} \in \operatorname{Aut}\mathfrak{R}\,.$$

Later on we will restrict to the case where $\mathfrak{R} = \mathcal{Z}$ is Abelian.

Let V be a unitary representation of $\mathcal{G}$ on a finite-dimensional Hilbert space $\mathcal{H}$. In general, V may be reducible, hence we can decompose it as a direct sum

$$V = \sum_{D_i} E_{D_i} V \cong \oplus_i m_i V_{D_i} \quad \text{on} \quad \mathcal{H} = \sum_{D_i} E_{D_i} \mathcal{H} \cong \oplus_i m_i \mathcal{H}_{D_i}\,,$$

where m_i is the multiplicity of the irreducible representation V_{D_i} in V and $E_{D_i} \in (V,V)$ is the isotypical projection corresponding to $D_i \in \widehat{\mathcal{G}}$. A useful orthonormal basis adapted to the previous decomposition is given by

$$\Psi := \left\{\psi_{D,l,i} \mid D \in \widehat{\mathcal{G}}\,;\ i = 1,\ldots,d = \dim D\,;\ l = 1,\ldots,m_D\right\}\,. \tag{27}$$

Next we consider the free right $\mathfrak{R}$-module $\mathfrak{H}$ generated $\mathfrak{R}$ and Ψ as in Eq. (7). Moreover, $\mathfrak{H}$ becomes a bimodule if we define the left action of $\mathfrak{R}$ as

$$B\,\psi_{D,l,i} := \psi_{D,l,i}\,\alpha^{-1}_{[D]}(B)\,, \tag{28}$$

where $[D] \in \mathfrak{C}(\mathcal{G})$ is the chain equivalence class corresponding to $D \in \widehat{\mathcal{G}}$ (cf. Subsection 2.2). By construction $\mathfrak{H}$ is a diagonal bimodule in the sense of Definition 3.10.

Let $\mathcal{O}_{\mathfrak{H}}$ be the Cuntz-Pimsner algebra generated by Ψ and $\mathfrak{R}$ (cf. Section 3). Then, $\mathcal{H}$ is an algebraic Hilbert space in $\mathcal{O}_{\mathfrak{H}}$ with support $\mathbb{1}$. With respect to the basis (27), the isotypical projections may be written as

$$E_D = \sum_{l,i} \psi_{D,l,i}\psi^*_{D,l,i} \tag{29}$$

and since the support of $\mathcal{H}$ is $\mathbb{1}$ we also have the relation

$$\sum_{D,l,i} \psi_{D,l,i}\psi^*_{D,l,i} = \mathbb{1}\,.$$

From Eq. (29), we have that $E_D B = B E_D$ as well as $E_{[D]}B = BE_{[D]}$ (recall from Theorem 2.12 that $E_{[D]} := \sum\limits_{D' \in [D]} E_{D'}$).

Remark 4.1 If the representation V is irreducible (e.g., in the case of the defining representation of $\mathrm{SU}(N)$), then for any $T \in \mathcal{K}(\mathfrak{H})$ we have simply

$$T(B\psi) = B\,T(\psi)\,, \quad B \in \mathfrak{R}\,, \psi \in \mathfrak{H}\,. \tag{30}$$

In fact, if $V \in D \in \widehat{\mathcal{G}}$, then we have for the generators

$$T(B\psi_i) = T(\psi_i\alpha^{-1}_{[D]}(B)) = T(\psi_i)\alpha^{-1}_{[D]}(B) = BT(\psi_i)\,.$$

Note that Eq. (30) is no longer true in the reducible case.

To define an action of $\mathcal{G}$ on $\mathcal{O}_{\mathfrak{H}}$ it is enough to specify it by means of the representation V on the generating module $\mathfrak{H}$:

$$g(\psi B) := (V(g)\psi)B \;, \quad \psi \in \mathcal{H}\,,\; B \in \mathfrak{R}\,,\; g \in \mathcal{G}\,. \tag{31}$$

Note that this immediately implies that $g(B\psi) = Bg(\psi)$, $\psi \in \mathcal{H}$, $B \in \mathcal{B}$, in fact

$$g(B\psi_{D,\,l,\,i}) = g(\psi_{D,\,l,\,i}\alpha_{[D]}^{-1}(B)) = (V(g)\psi_{D,\,l,\,i})\alpha_{[D]}^{-1}(B)) = Bg(\psi_{D,\,l,\,i}) \;. \tag{32}$$

Then g extends to an automorphism $g \in \mathrm{Aut}\,\mathcal{O}_{\mathfrak{H}}$ and $(\mathcal{O}_{\mathfrak{H}}, \mathcal{G})$ becomes a C^*-dynamical system. We denote its fixed-point algebra by

$$\mathcal{A} := (\mathcal{O}_{\mathfrak{H}})^{\mathcal{G}} = \{A \in \mathcal{O}_{\mathfrak{H}} \mid g(A) = A\,,\; g \in \mathcal{G}\}\,.$$

Let now $\rho_{\mathcal{H}} \in \mathrm{End}\,\mathcal{O}_{\mathfrak{H}}$ be the endomorphism induced by an orthonormal basis $\{\psi_i\}_i^d \subset \mathcal{H}$, according to Lemma 3.8. As we mentioned in Subsection 2.1 we call canonical endomorphism the restriction of $\rho_{\mathcal{H}}$ to $\mathcal{A}$, i.e.

$$\rho := \rho_{\mathcal{H}}\restriction\mathcal{A} \in \mathrm{End}\,\mathcal{A}\;.$$

Lemma 4.2 *Let $\mathcal{A}$ be the fixed-point algebra of the C^*-dynamical system $(\mathcal{O}_{\mathfrak{H}}, \mathcal{G})$ and ρ be the endomorphism introduced before. Then*

(i) $\rho^n(\mathfrak{R}) := \{\rho^n(B) \mid B \in \mathfrak{R}\} \subseteq \mathcal{A}$, $n \in \mathbb{N}$.

(ii) $\rho(B)B' = B'\rho(B)$, $B, B' \in \mathfrak{R}$.

Proof: By the definition of the action of $\mathcal{G}$ on the Cuntz-Pimsner algebra specified in Eq. (31), it follows that $\mathfrak{R} \subseteq \mathcal{A}$. Since ρ is an endomorphism of $\mathcal{A}$ it follows that $\rho^n(\mathfrak{R}) \subseteq \mathcal{A}$. The proof of part (ii) follows immediately from the definition of ρ and Eq. (28). ∎

Lemma 4.3 *Let V be a representation of the compact group $\mathcal{G}$ acting on the algebraic Hilbert space $\mathcal{H}$. Then, the trivial representation ι is contained in the decomposition of $\overset{n}{\otimes} V$, i.e. $\iota \prec \overset{n}{\otimes} V$, iff there exists an isometry $S \in \mathcal{H}^n$ such that $g(S) = S$, $g \in \mathcal{G}$, (i.e. $S \in \mathcal{H}^n \cap \mathcal{A}$).*

Proof: The unitary representation $\overset{n}{\otimes} V$ contains the trivial representation if and only if there exists a (normalized) vector $S \in \mathcal{H}^n$, invariant under $\overset{n}{\otimes} V$. Now, by Remark 3.5 (i), S appears as an isometry in $\mathcal{O}_{\mathfrak{H}}$. Moreover, by the definition of the $\mathcal{G}$-action we have $g(S) = S$, thus $S \in \mathcal{A} \cap \mathcal{H}^n$. Conversely, if $S \in \mathcal{H}^n$ is a $\mathcal{G}$-invariant isometry, then $\mathcal{H}_\iota := \mathbb{C}S \subset \mathcal{H}^n$ carries the trivial representation, hence $\iota \prec \overset{n}{\otimes} V$. ∎

Lemma 4.4 *Let $X \in \mathcal{H}^n$. Then, the equation $XB = \rho^n(B)X$ holds for all $B \in \mathfrak{R}$.*

Proof: Consider $X = \varphi_1 \cdot \ldots \cdot \varphi_n \in \mathcal{H}^n$, $\varphi_k \in \mathcal{H}$, $k = 1, \ldots, n$. Each φ_k can be decomposed in terms of the chain group components, i.e. $\varphi_k = \sum_{[D_{i_k}]} \varphi_{[D_{i_k}]}$, where $\varphi_{[D_{i_k}]} := E_{[D_{i_k}]}\varphi_k$ and $E_{[D]} := \sum\limits_{D' \in [D]} E_{D'}$ (cf. Theorem 2.12). Therefore we have

$$X = \sum_{[D_{i_1}], \ldots, [D_{i_n}]} \varphi_{[D_{i_1}]} \cdot \ldots \cdot \varphi_{[D_{i_n}]}\,.$$

From the definition of the left action in Eq. (28) we have for any $B \in \mathfrak{R}$

$$
\begin{aligned}
XB &= \sum_{[D_{i_1}],\ldots,[D_{i_n}]} \varphi_{[D_{i_1}]} \cdot \ldots \cdot \varphi_{[D_{i_n}]} \cdot B \\
&= \sum_{[D_{i_1}],\ldots,[D_{i_n}]} \alpha_{[D_{i_1} \times \cdots \times D_{i_n}]}(B) \varphi_{[D_{i_1}]} \cdot \ldots \cdot \varphi_{[D_{i_n}]} \\
&= \Big(\alpha_{[D_{i_1} \times \cdots \times D_{i_n}]}(B) \cdot E_{[D_1]} \cdot \rho(E_{[D_2]}) \cdot \ldots \cdot \rho^{n-1}(E_{[D_n]}) \Big) \varphi_1 \cdot \ldots \cdot \varphi_n \\
&= \rho^n(B)\, X \,,
\end{aligned}
$$

where for the last equation we have used Theorem 2.12. ■

Proposition 4.5 *Let $X \in \mathcal{H}^n$. If X is $\mathcal{G}$-invariant, i.e. $X \in \mathcal{A} \cap \mathcal{H}^n$, then $XB = BX$, $B \in \mathfrak{R}$.*

Proof: Let $X = \varphi_1 \cdot \ldots \cdot \varphi_n \in \mathcal{H}^n$, $\varphi_i \in \mathcal{H}$, $i = 1, \ldots, n$. First note that, as in the proof of Lemma 4.4, we have

$$
XB = \sum_{[D_{i_1}],\ldots,[D_{i_n}]} \alpha_{[D_{i_1} \times \cdots \times D_{i_n}]}(B) \varphi_{[D_{i_1}]} \cdot \ldots \cdot \varphi_{[D_{i_n}]} \,, \quad B \in \mathfrak{R}. \tag{33}
$$

Moreover, from the definition of the isotypical projections in Eq. (29) it is clear that $g(E_{D_k}) = E_{D_k}$, $k = 1, \ldots, n$. Therefore, denoting $\varphi_{D_{i_k}} := E_{D_{i_k}} \varphi_k$ we have that

$$
g(X) = X \ , \ g \in \mathcal{G}, \quad \text{iff} \quad g\left(\varphi_{D_{i_1}} \cdot \ldots \cdot \varphi_{D_{i_n}} \right) = \varphi_{D_{i_1}} \cdot \ldots \cdot \varphi_{D_{i_n}} \ , \ g \in \mathcal{G} \,, \tag{34}
$$

for all D_{i_k} appearing in the decomposition of V. (Note that the preceding equation implies in particular the invariance of the corresponding chain group components, i.e. $g(\varphi_{[D_{i_1}]} \cdot \ldots \cdot \varphi_{[D_{i_n}]}) = \varphi_{[D_{i_1}]} \cdot \ldots \cdot \varphi_{[D_{i_n}]}$.) Now, decomposing the tensor product $\mathcal{H}_{D_{i_1}} \cdot \ldots \cdot \mathcal{H}_{D_{i_n}}$ in terms of irreducible components it is clear that Eq. (34) implies that there is a $\mathcal{G}$-invariant isometry carrying the trivial representation, i.e. $\iota \in D_{i_1} \times \cdots \times D_{i_n}$ (cf. Lemma 4.3). By the definition of the chain group (cf. Definition 2.6) this shows that

$$
\alpha_{[D_{i_1} \times \cdots \times D_{i_n}]}(B) = \alpha_{[\iota]}(B) = B.
$$

Therefore Eq. (33) reads $XB = BX$, $B \in \mathfrak{R}$. ■

For the rest of the present section, we assume that the coefficient algebra $\mathfrak{R}$ is Abelian and will write $\mathcal{Z}$ instead of $\mathfrak{R}$.

Proposition 4.6 *Let $T \in \mathcal{L}(\mathfrak{H}^s, \mathfrak{H}^n)$, $s, n \in \mathbb{N}$. If T is $\mathcal{G}$-invariant, i.e. $T \in \mathcal{A} \cap \mathcal{L}(\mathfrak{H}^s, \mathfrak{H}^n)$, then $TZ = ZT$, $Z \in \mathcal{Z}$. Moreover, $\mathcal{Z} \subseteq \mathcal{A}' \cap \mathcal{A}$.*

Proof: We divide the proof in two steps. First, we consider an elementary tensor of the form

$$
X = \varphi_1 \cdot \ldots \cdot \varphi_n \cdot \tilde{\varphi}_1^* \cdot \ldots \cdot \tilde{\varphi}_s^* \ \in \ \mathcal{H}^n (\mathcal{H}^*)^s \ \subset \ \mathcal{L}(\mathfrak{H}^s, \mathfrak{H}^n)
$$

where $\varphi_i, \tilde{\varphi}_k \in \mathcal{H}$, $i, = 1 \ldots n$, $k = 1, \ldots, s$. Recall that $\mathcal{H}^n (\mathcal{H}^*)^s$ carries a unitary representation of $\mathcal{G}$ equivalent to $V^{\otimes^n} \otimes \overline{V}^{\otimes^s}$. Decomposing again each φ_i in terms of the chain group components as in the proof of Lemma 4.4 we get

$$
X = \sum_{\substack{[D_{i_1}],\ldots,[D_{i_n}] \\ [\overline{D}_{k_1}],\ldots,[\overline{D}_{k_s}]}} \varphi_{[D_{i_1}]} \cdot \ldots \cdot \varphi_{[D_{i_n}]} \cdot \varphi_{[\overline{D}_{k_1}]} \cdot \ldots \cdot \varphi_{[\overline{D}_{k_s}]} \,,
$$

where $\varphi_{[D_{i_k}]} := E_{[D_{i_k}]}\varphi_k$. Therefore

$$XZ = \sum_{\substack{[D_{i_1}],\ldots,[D_{i_n}] \\ [\overline{D}_{k_1}],\ldots,[\overline{D}_{k_s}]}} \alpha_{[D_{i_1}\times\cdots\times D_{i_n}\times\overline{D}_{k_1}\times\cdots\times\overline{D}_{k_s}]}(Z)\varphi_{[D_{i_1}]}\cdot\ldots\cdot\varphi_{[\overline{D}_{k_s}]}\,.$$

Suppose now that $g(X) = X$, $g \in \mathcal{G}$. As in the proof of Proposition 4.5 this implies that the product of chain group components are also $\mathcal{G}$-invariant (cf. Eq. (34)). Therefore, Lemma 4.3 implies that $\iota \in D_{i_1}\times\cdots\times D_{i_n}\times\overline{D}_{k_1}\times\cdots\times\overline{D}_{k_s}$, hence

$$\alpha_{[D_{i_1}\times\cdots\times D_{i_n}\times\overline{D}_{k_1}\times\cdots\times\overline{D}_{k_s}]}(Z) = Z\,.$$

This shows that $XZ = ZX$, $Z \in \mathcal{Z}$.

Second, we consider an element of the full intertwiner space $T \in \mathcal{L}(\mathfrak{H}^s, \mathfrak{H}^n)$. Using the identification in (19), we can write T in terms of an orthonormal basis $\{\psi_i\} \subset \mathcal{H}$ as follows

$$T = \sum_{I,J} \psi_I\, T_{IJ}\, \psi_J^*\ ,$$

where, to simplify notation, we put $\psi_I := \psi_{i_1}\cdot\ldots\cdot\psi_{i_n} \in \mathcal{H}^n$, $\psi_J := \psi_{j_1}\cdot\ldots\cdot\psi_{j_s} \in \mathcal{H}^s$, and

$$T_{IJ} = \psi_I^*\, T\, \psi_J \in \mathcal{Z}\quad .$$

Thus, by Lemmas 4.4 and 4.2 (i) we have

$$T = \sum_{I,J} \psi_I\, \psi_J^*\, \rho^s(T_{IJ}) \quad \text{with} \quad \rho^s(T_{IJ}) \in \mathcal{Z}\,.$$

Suppose now that $g(T) = T$, $g \in \mathcal{G}$. Using the group mean given by

$$\mathfrak{m}_{\mathcal{G}}(H) := \int_{\mathcal{G}} g(H)\ dg\ , \quad H \in \mathcal{O}_{\mathfrak{H}}\,, \tag{35}$$

where dg is the normalized Haar measure, we obtain

$$T = \mathfrak{m}_{\mathcal{G}}(T) = \sum_{I,J} X_{IJ}\rho^s(T_{IJ})\,,$$

where $X_{IJ} := \int_{\mathcal{G}} g(\psi_I\psi_J^*)\,dg\ \in \mathcal{H}^n(\mathcal{H}^*)^s$. By definition the X_{IJ} are $\mathcal{G}$-invariant, hence the first part of the proof gives $X_{IJ}Z = ZX_{IJ}$, $Z \in \mathcal{Z}$. Applying Lemma 4.2 (ii) we obtain finally

$$ZT = \sum_{I,J} ZX_{IJ}\rho^s(T_{IJ}) = \sum_{I,J} X_{IJ}\rho^s(T_{IJ})Z = TZ\ , \quad Z \in \mathcal{Z}\,.$$

To show the inclusion $\mathcal{Z} \subseteq \mathcal{A}' \cap \mathcal{A}$ recall that from the definition of the group action in Eq. (31) we already have $\mathcal{Z} \subseteq \mathcal{A}$. Next we show that $\{\mathcal{L}(\mathfrak{H}^s, \mathfrak{H}^n) \cap \mathcal{A}\}_{s,n\in\mathbb{N}}$ is dense in $\mathcal{A}$: first note that since the circle action given in Eq. (16) commutes with the group action, we may decompose $\mathcal{A}$ in the corresponding spectral subspaces (cf. (17)),

$$\mathcal{A} = \oplus_{k\in\mathbb{Z}}\mathcal{A}_k\,.$$

The density follows from the existence of a sequence of norm one projections E_r from $\mathcal{A}_k$ onto $\mathcal{L}(\mathfrak{H}^r, \mathfrak{H}^{r+k}) \cap \mathcal{A}$ pointwise convergent to the identity (cf. the proof of Proposition 3.4 (b) in [10]). Therefore, by the first part of this theorem any $Z \in \mathcal{Z}$ will commute with any $T \in \mathcal{L}(\mathfrak{H}^s, \mathfrak{H}^n) \cap \mathcal{A}$, $s, n \in \mathbb{N}$, hence with any $A \in \mathcal{A}$. This shows that $\mathcal{Z} \subseteq \mathcal{A}' \cap \mathcal{A}$. ■

In this section we have shown that given a chain group action $\alpha\colon\ \mathfrak{C}(\mathcal{G}) \to \mathrm{Aut}\,(\mathcal{Z})$ we can construct for any finite-dimensional representation V of the compact group $\mathcal{G}$ a diagonal Hilbert bimodule $\mathfrak{H}$ and a C^*-dynamical system $(\mathcal{O}_{\mathfrak{H}}, \mathcal{G})$ with the corresponding Cuntz-Pimsner algebra. To show the minimality of $(\mathcal{O}_{\mathfrak{H}}, \mathcal{G})$, i.e. $\mathcal{A}' \cap \mathcal{O}_{\mathfrak{H}} = \mathcal{Z}$, where $\mathcal{A} = \mathcal{O}_{\mathfrak{H}}^{\mathcal{G}}$ is the corresponding fixed point algebra, we need to make some further assumptions on the representation V. Essential facts in the proof of the minimality property are that $\mathfrak{H}$ is nonsingular and that there exists a $\mathcal{G}$-invariant isometry (recall Definition 3.6 and Proposition 3.7). These facts will be guaranteed if we consider finite-dimensional representations in the following class (see also Proposition 3.12):

Definition 4.7 *Let $\mathcal{G}$ be a compact group. We denote by $\mathfrak{G}_0$ the set of all finite-dimensional representations V of $\mathcal{G}$ satisfying the following properties:*

(i) V has an irreducible subrepresentation of dimension or multiplicity ≥ 2.

(ii) There exists an $n \in \mathbb{N}$ such that $\iota \prec \overset{n}{\otimes} V$.

Remark 4.8 If $\mathcal{G}$ is non-Abelian we can use as representation V any irreducible representation of dimension ≥ 2. If $\mathcal{G}$ is Abelian we consider a representation containing some character with multiplicity ≥ 2.

Part (ii) in the preceding definition is satisfied if the representation V on the Hilbert space $\mathcal{H}$ satisfies $\det V = 1$. In this case there exists an isometry $S \in \mathcal{H}^d$, $d = \dim \mathcal{H}$, with $g(S) = S$, $g \in \mathcal{G}$. We may pick S as a normalized vector generating the totally antisymmetric tensor power $\wedge^d \mathcal{H} \subseteq \mathcal{H}^d$. If V does not have determinant 1 we can always consider $V \oplus \overline{\det V}$.

Theorem 4.9 *Let $\mathcal{G}$ be a compact group, $\mathcal{Z}$ a unital Abelian C^*-algebra and consider a fixed chain group action $\alpha\colon\ \mathfrak{C}(\mathcal{G}) \to \mathrm{Aut}\,(\mathcal{Z})$. For any $V \in \mathfrak{G}_0$ there exists a nonsingular diagonal bimodule $\mathfrak{H}_V$ with left $\mathcal{Z}$-action given in terms of α as in Eq. (28) and a C^*-dynamical system $(\mathcal{O}_{\mathfrak{H}_V}, \mathcal{G})$ satisfying the following properties:*

(i) $(\mathcal{O}_{\mathfrak{H}_V}, \mathcal{G})$ is minimal, i.e. $\mathcal{A}_V' \cap \mathcal{O}_{\mathfrak{H}_V} = \mathcal{Z}$, where $\mathcal{A}_V := \mathcal{O}_{\mathfrak{H}_V}^{\mathcal{G}}$ is the corresponding fixed-point algebra.

(ii) The Abelian C^*-algebra $\mathcal{Z}$ coincides with the center of the fixed-point algebra $\mathcal{A}_V$, i.e. $\mathcal{A}_V' \cap \mathcal{A}_V = \mathcal{Z}$.

Moreover, if $\mathcal{G}$ is a compact Lie group, then the Hilbert spectrum of $(\mathcal{O}_{\mathfrak{H}_V}, \mathcal{G})$ is full, i.e. for each $D \in \widehat{\mathcal{G}}$ there is an invariant algebraic Hilbert space $\mathcal{H}_D \subset \mathcal{O}_{\mathfrak{H}_V}$ (in this case not necessarily of support $\mathbb{1}$) such that $\mathcal{G} \restriction \mathcal{H}_D \in D$.

Proof: Consider a representation $V \in \mathfrak{G}_0$ and let $\alpha\colon\ \mathfrak{C}(\mathcal{G}) \to \mathrm{Aut}\,(\mathcal{Z})$ be a fixed chain group action. We construct the diagonal Hilbert $\mathcal{Z}$-bimodule $\mathfrak{H}_V$ with a left action as specified in Eq. (28).

First we show that $\mathfrak{H}_V$ is nonsingular. By part (i) in Definition 4.7 there are indices $i \neq j$ such that for the corresponding orthonormal basis elements adapted to the decomposition of V (cf. Eq. (27)) satisfy $\psi_{D,l,i} \neq \psi_{D,l,j}$ as well as

$$Z\psi_{D,l,i} = \psi_{D,l,i}\alpha_{[D]}^{-1}(Z) \quad \text{and} \quad Z\psi_{D,l,j} = \psi_{D,l,j}\alpha_{[D]}^{-1}(Z)\,, \quad Z \in \mathcal{Z}\,.$$

From Proposition 3.12 we conclude that $\mathfrak{H}_V$ is nonsingular.

We define the C^*-dynamical system $(\mathcal{O}_{\mathfrak{H}_V}, \mathcal{G})$ as in (31). Next we show that $(\mathcal{O}_{\mathfrak{H}_V}, \mathcal{G})$ is minimal (cf. Definition 2.1). By part (ii) in Definition 4.7 and Proposition 3.7 we conclude that $C^*(S)' \cap \mathcal{O}_{\mathfrak{H}_V} = \mathcal{Z}$. But $C^*(S) \subset \mathcal{A}_V$, thus $\mathcal{A}_V' \cap \mathcal{O}_{\mathfrak{H}_V} \subseteq \mathcal{Z}$. The reverse inclusion follows from $\mathcal{Z} \subseteq \mathcal{A}_V' \cap \mathcal{A}_V$ (cf. Proposition 4.6). This proves that $\mathcal{A}_V' \cap \mathcal{O}_{\mathfrak{H}_V} = \mathcal{Z}$, i.e. $(\mathcal{O}_{\mathfrak{H}_V}, \mathcal{G})$ is minimal. Note that the preceding equation also shows that $\mathcal{A}'_V \cap \mathcal{A}_V = \mathcal{Z}$, since $\mathcal{A}'_V \cap \mathcal{A}_V \subseteq \mathcal{A}'_V \cap \mathcal{O}_{\mathfrak{H}_V} = \mathcal{Z}$ (the reverse inclusion $\mathcal{Z} \subseteq \mathcal{A}'_V \cap \mathcal{A}_V$ is proved in Proposition 4.6).

If $\mathcal{G}$ is a compact Lie group, then there is a faithful representation V such that every irreducible is contained in some tensor power $\otimes^n V$, $n \in \mathbb{N}$ (see [8, Theorem III.4.4]). Therefore $(\mathcal{O}_{\mathfrak{H}_V}, \mathcal{G})$ has full Hilbert spectrum. ■

Example 4.10 (Minimal C^*-dynamical systems for SU(N))
Theorem 4.9 can be applied to the group $\mathcal{G} = \mathrm{SU}(N)$. One can use the defining representation V of dimension $N \geq 2$, since it is easy to see that $V \in \mathfrak{G}_0$. In fact, in this case we have

$$\iota \prec \overset{N}{\otimes} V$$

(see also Example 2.9). Therefore the preceding theorem gives a minimal C^*-dynamical system $(\mathcal{O}_{\mathfrak{H}_V}, \mathrm{SU}(N))$ with full spectrum.

5 Construction of Hilbert C^*-systems

In the present section we will construct Hilbert C^*-systems $(\mathcal{F}, \mathcal{G})$ for compact (non-Abelian) groups $\mathcal{G}$. This means that the C^*-dynamical system must encode the categorical structure of the dual of $\mathcal{G}$ (cf. Definition 2.1). The important new feature now is that in $\mathcal{F}$ all irreducible representations must be realized on algebraic Hilbert spaces with support $\mathbb{1}$. As a first step towards this goal we will construct a C^*-algebra $\mathcal{F}$ suitably generated by the Cuntz-Pimsner algebras $\mathcal{O}_{\mathfrak{H}_V}$, where $V \in \mathfrak{G}$ and $\mathfrak{G}$ is a family of finite-dimensional unitary representations of $\mathcal{G}$.

5.1 The C^*-algebra of a chain group action

Let $\mathcal{G}$ be a compact group with chain group $\mathfrak{C}(\mathcal{G})$. We consider a (non necessarily Abelian) C^*-algebra $\mathfrak{R}$ with identity $\mathbb{1}$ and carrying an action of the chain group

$$\alpha\colon \mathfrak{C}(\mathcal{G}) \to \mathrm{Aut}\,\mathfrak{R}\ .$$

With these ingredients, and a family $\mathfrak{G}$ of finite-dimensional unitary representations of $\mathcal{G}$, we will construct a C^*-dynamical system generalizing both Cuntz-Pimsner algebras of the type given in Section 3.3 and crossed-products by Abelian groups (cf. [27]).

The following variation in the notation of the basis elements in this section is convenient for the presentation of the algebra $\mathcal{F}$ given below. Let $V \in \mathfrak{G}$ and consider the decomposition of V into irreducible components. Then, we construct an orthonormal basis for $\mathcal{H}_V$, denoted by

$$\Psi_V := \{\psi_{V,i}\ ,\ i = 1, \ldots, \dim\mathcal{H}_V\}\ ,$$

and such that each $\psi_{V,i}$ transforms according an irreducible subrepresentation of V with class D (see also Section 4):

$$V(g)\psi_{V,i} = V_D(g)\psi_{V,i}\quad ,\quad g \in \mathcal{G}\ .$$

When $\psi_{V,i}$ satisfies the preceding condition, we write

$$\psi_{V,i;(D)} := \psi_{V,i}\ .$$

We emphasize that in this case we are counting the basis elements in a different way as in Eq. (27). In fact, now the element D of the dual, which is written in brackets, does not play the role of an index. This label is used simply to recall the transformation character of the basis element under the action of $\mathcal{G}$.

We define a *-algebra ${}^0\mathcal{F} = {}^0\mathcal{F}(\alpha,\mathfrak{G})$, generated by $\mathfrak{R}$ and $\{\Psi_V\}_{V\,\in\,\mathfrak{G}}$, and satisfying the relations

$$\begin{aligned}
\sum_i \psi_{V,i;(D)}\ \psi^*_{V,i;(D)} &= \mathbb{1} &(36)\\
\psi^*_{V,i;(D)}\ \psi_{V,j;(D')} &= \delta_{ij}\mathbb{1} &(37)\\
\psi_{V,i;(D)}B &= \alpha_{[D]}(B)\ \psi_{V,i;(D)}\ ,\ [D]\in\mathfrak{C}(\mathcal{G})\ ,\ B\in\mathfrak{R} &(38)\\
\left[\psi_{V,i;(D)}\ ,\ \psi_{W,j;(D')}\right] = \left[\psi^*_{V,i;(D)}\ ,\ \psi_{W,j;(D')}\right] &= 0\ , \qquad V\neq W\ ,\ V,W\in\mathfrak{G}\ . &(39)
\end{aligned}$$

Remark 5.1 For every $V\in\mathfrak{G}$ put $d(V):=\dim\mathcal{H}_V\in\mathbb{N}$ and denote by $\mathcal{O}_{d(V)}$ the Cuntz algebra generated by $\mathcal{H}_V$, and by ${}^0\mathcal{O}_{d(V)}$ the dense *-subalgebra of $\mathcal{O}_{d(V)}$ algebraically generated by $\mathcal{H}_V$ (cf. Section 3). By universality of the Cuntz relations, we find that for every $V\in\mathfrak{G}$ there is a unital *-monomorphism ${}^0\mathcal{O}_{d(V)}\hookrightarrow{}^0\mathcal{F}$. Moreover, (39) implies that the algebraic tensor product ${}^0\mathcal{O}_{\mathfrak{G}}:=\odot_V\,{}^0\mathcal{O}_{d(V)}$ is embedded in ${}^0\mathcal{F}$. We denote by

$$\mathcal{O}_{\mathfrak{G}}:=\bigotimes_{V\,\in\,\mathfrak{G}}\mathcal{O}_{d(V)} \tag{40}$$

the C^*-tensor product of the corresponding Cuntz algebras (which are nuclear). From Eq. (38) we have that every Hilbert $\mathfrak{R}$-bimodule $\mathfrak{H}_V$ defined in Section 4 is also embedded in ${}^0\mathcal{F}$. Recall from Remark 3.1 that ${}^0\mathcal{O}_{\mathfrak{H}_V}$ is the dense *-subalgebra of $\mathcal{O}_{\mathfrak{H}_V}$ algebraically generated by $\mathfrak{H}_V$. Then, again by universality, we have that ${}^0\mathcal{O}_{\mathfrak{H}_V}$ is embedded in ${}^0\mathcal{F}$. Note, nevertheless, that in general the *-algebras ${}^0\mathcal{O}_{\mathfrak{H}_{V_1}}$, ${}^0\mathcal{O}_{\mathfrak{H}_{V_2}}$, $V_1,V_2\in\mathfrak{G}$, do *not* appear in ${}^0\mathcal{F}$ as a tensor product due to the twist introduced by the chain group action in Eq. (38). In fact, it is easy to check using the previous relations, that if the chain group action described by α is nontrivial, then

$$\psi_2(B\psi_1)=\alpha(B)\psi_1\psi_2\neq(B\psi_1)\psi_2\,,\quad \psi_i\in\mathcal{O}_{\mathfrak{H}_{V_i}},\ i=1,2;\ B\in\mathfrak{R}\,.$$

In the following we will show that ${}^0\mathcal{F}$ admits a nontrivial C^*-norm. Let

$$\mathbb{T}^\infty:=\times_{V\,\in\,\mathfrak{G}}\mathbb{T}\quad,\quad \mathbb{Z}^\infty:=\times_{V\,\in\,\mathfrak{G}}\mathbb{Z}$$

be the product of circles $\mathbb{T}$ (resp. $\mathbb{Z}$) and denote its elements as maps $z\colon\mathfrak{G}\to\mathbb{T}$ (resp. $k\colon\mathfrak{G}\to\mathbb{Z}$). On ${}^0\mathcal{F}$ we have the following natural actions by *-automorphisms

$$\beta\colon\mathcal{G}\to\mathrm{Aut}\,{}^0\mathcal{F}\quad,\quad\beta_g(\psi_{V,i;(D)}):=V(g)\psi_{V,i;(D)}=V_D(g)\psi_{V,i;(D)}\quad\text{and}\quad\beta_g(B):=B \tag{41}$$

$$\delta\colon\mathbb{T}^\infty\to\mathrm{Aut}\,{}^0\mathcal{F}\quad,\quad\delta_z(\psi_{V,i;(D)}):=z(V)\cdot\psi_{V,i;(D)}\quad\text{and}\quad\delta_z(B):=B\,,\ B\in\mathfrak{R}\,. \tag{42}$$

Moreover, we consider the sets

$$\begin{aligned}
\mathbb{Z}^\infty_0 &:= \{k\in\mathbb{Z}^\infty\mid\mathrm{supp}\,(k)<\infty\}\\
\mathbb{N}^\infty_0 &:= \{r\in\mathbb{Z}^\infty_0\mid r(V)\geq 0\,,V\in\mathfrak{G}\}\,.
\end{aligned}$$

For every $k\in\mathbb{Z}^\infty_0$, we define the spectral subspaces w.r.t. the action δ as follows:

$${}^0\mathcal{F}^k:=\left\{T\in{}^0\mathcal{F}\mid\delta_z(T)=\prod_{V\in\ supp(k)}z(V)^{k(V)}\ T\ ,\ \ z\in\mathbb{T}^\infty\right\}\ .$$

Note that these subspaces are a natural generalization of the standard spectral subspace considered for a single Cuntz-Pimsner algebra in (17). If $k_1\neq k_2$, then the corresponding spectral subspaces have a trivial intersection. Moreover, if $0\in\mathbb{Z}^\infty_0$ denotes the zero section, then ${}^0\mathcal{F}^0$ is the fixed-point algebra w.r.t. the $\mathbb{T}^\infty$-action. It is natural to introduce the notation ${}^0\mathcal{O}^k := {}^0\mathcal{F}^k \cap {}^0\mathcal{O}$, $k \in \mathbb{Z}_0^\infty$, where ${}^0\mathcal{O}$ is the algebraic tensor product of the algebraic part of the Cuntz algebras defined in Remark 5.1. We have ${}^0\mathcal{F} = \mathrm{span}\,\{{}^0\mathcal{F}^k \mid k \in \mathbb{Z}_0^\infty\}$ as well as ${}^0\mathcal{O}_{\mathfrak{G}} = \mathrm{span}\,\{{}^0\mathcal{O}^k \mid k \in \mathbb{Z}_0^\infty\}$. Note that for every $V \in \mathfrak{G}$, it turns out that ${}^0\mathcal{O}_{d(V)} \cap {}^0\mathcal{F}^k$ coincides with the spectral subspace ${}^0\mathcal{O}_{d(V)}^{k(V)}$ w.r.t. the standard canonical circle action on ${}^0\mathcal{O}_{d(V)}$. We introduce the projection

$$m_0 : {}^0\mathcal{F} \to {}^0\mathcal{F}^0 \quad , \quad m_0\left(\sum_{k \,\in\, \mathbb{Z}_0^\infty}^{\mathrm{fin}} T_k\right) := T_0\,, \tag{43}$$

where T_k is in the corresponding spectral subspace ${}^0\mathcal{F}^k$, and the symbol $\sum^{\mathrm{fin}}$ indicates a sum over a finite set of indexes. If $\mathcal{V}$ is a finite subset of $\mathfrak{G}$ and $T \in {}^0\mathcal{F}^k$ with $k(V) \neq 0$ for some $V \in \mathcal{V}$, then $m_0(T) = 0$.

For any $r \in \mathbb{N}_0^\infty$ put $\mathcal{V} := \mathit{supp}(r)$ which is a finite subset of $\mathfrak{G}$. Then $\mathcal{H}_\mathcal{V}^r$ denotes the linear span of elements of the form $\prod_{V \,\in\, \mathcal{V}} \varphi_V$, with $\varphi_V \in \mathcal{H}_V^{r(V)}$. It is clear that we may identify $\mathcal{H}_\mathcal{V}^r$ with the tensor product $\otimes_{V \,\in\, \mathcal{V}} \mathcal{H}_V^{r(V)}$ (recall that $\mathcal{H}_V^{r(V)}$ is identified with the tensor product $\otimes^{r(V)}\mathcal{H}_V$). We denote by $\mathcal{L}(\mathcal{H}_\mathcal{V}^r, \mathcal{H}_\mathcal{V}^s)$ the space of linear maps from $\mathcal{H}_\mathcal{V}^r$ into $\mathcal{H}_\mathcal{V}^s$: it coincides with the linear span of elements of the form $\varphi'\varphi^*$, where $\varphi' \in \mathcal{H}_\mathcal{V}^s$, $\varphi \in \mathcal{H}_\mathcal{V}^r$, where $r, s \in \mathbb{N}_0^\infty$ have support $\mathcal{V}$.

Remark 5.2 For every finite $\mathcal{V} \subset \mathfrak{G}$, $k \in \mathbb{Z}_0^\infty$, we consider the vector space

$$\mathfrak{H}_\mathcal{V}^k := \mathrm{span} \prod_{V \in \mathcal{V}} \mathfrak{H}_V^{k(V)}$$

(for $k(V) < 0$, $V \in \mathcal{V}$, we define as usual $\mathfrak{H}_V^{k(V)} := \left(\mathfrak{H}_V^{-k(V)}\right)^*$ which can be interpreted as the dual bimodule of bounded, right $\mathfrak{R}$-module maps from $\mathfrak{H}_V^{-k(V)}$ into $\mathfrak{R}$; for $k(V) = 0$, recall that by definition $\mathfrak{H}_V^0 = \mathfrak{R}$). Now, we may identify $\mathfrak{H}_\mathcal{V}^k$ with the $\mathfrak{R}$-bimodule inner tensor product

$$\bigotimes_{V \,\in\, \mathcal{V}} \mathfrak{H}_V^{k(V)}$$

In the following lemma we give several useful characterizations of the spectral subspaces ${}^0\mathcal{F}^k$. These will play a fundamental role in the rest of this section.

Lemma 5.3 *For every $k \in \mathbb{Z}_0^\infty$, we have*

$$\begin{aligned} {}^0\mathcal{F}^k &= \left\{ \sum_{\mathcal{V}}^{\mathit{fin}} B_\mathcal{V} T_\mathcal{V} \mid B_\mathcal{V} \in \mathfrak{R}\;,\; T_\mathcal{V} \in \mathcal{L}(\mathcal{H}_\mathcal{V}^r, \mathcal{H}_\mathcal{V}^{r+k})\;,\;\; r \in \mathbb{N}_0^\infty\;,\; \mathcal{V} \subseteq \mathfrak{G}_0 \text{ finite} \right\} &(44)\\ &= \left\{ \sum_{\mathcal{V}}^{\mathit{fin}} R_\mathcal{V} \mid R_\mathcal{V} \in \mathcal{L}(\mathfrak{H}_\mathcal{V}^r, \mathfrak{H}_\mathcal{V}^{r+k})\;,\;\; r \in \mathbb{N}_0^\infty\;,\; \mathcal{V} \subseteq \mathfrak{G}_0 \text{ finite} \right\}\,. &(45)\end{aligned}$$

In particular, with the conventions of Remark 5.2 we have $\mathfrak{H}_\mathcal{V}^k \subset {}^0\mathcal{F}^k$. Moreover recalling Remark 3.1 we can also write

$${}^0\mathcal{F}^k = \mathrm{span}\left(\prod_{V \in\; \mathit{supp}(k)} {}^0\mathcal{O}_{\mathfrak{H}_V}^{k(V)}\right)\,.$$

Proof: From the structure of the spectral subspaces of the single Cuntz-Pimsner algebras (cf. Remark 3.1) it is clear that every element of ${}^0\mathcal{F}^k$ is the sum of terms of the form

$$T = B_0 T_{V_1} B_1 \cdots B_{n-1} T_{V_n} B_n\,,$$

where $V_1, \ldots, V_n \in \mathfrak{G}$, $T_{V_i} \in \mathcal{L}(\mathcal{H}_{V_i}^{r(V_i)}, \mathcal{H}_{V_i}^{s(V_i)})$, $s(V_i) = r(V_i) + k(V_i)$, and $B_i \in \mathfrak{R}$. By using Eq. (39), we may change the order of terms T_{V_i}, T_{V_j}, if $i \neq j$, and put together the terms arising from the same representation. Thus, without loss of generality, we may assume that $V_i \neq V_j$ if $i \neq j$. Moreover, every T_{V_i} can be written as a sum of terms of the form $\psi_{V_i}\varphi_{V_i}^*$, where $\psi_{V_i} \in \mathcal{H}_{V_i}^{s(V_i)}$, $\varphi_{V_i} \in \mathcal{H}_{V_i}^{r(V_i)}$. Therefore applying Eq. (38) successively we can write all elements from $\mathfrak{R}$ to the left, i.e. for some $B \in \mathfrak{R}$ we have

$$T = BT' \quad \text{with} \quad T' := T_{V_1} \cdots T_{V_n} \ .$$

Finally, (39) implies that T' can be rewritten as

$$\psi_{V_1}\psi_{V_2} \cdots \psi_{V_n} \cdot \varphi_{V_1}^* \varphi_{V_2}^* \cdots \varphi_{V_n}^* \ .$$

Hence $T' \in \mathcal{L}(\mathcal{H}_{\mathcal{V}}^r, \mathcal{H}_{\mathcal{V}}^{r+k})$, where $\mathcal{V} := \{V_1, \ldots, V_n\}$.

The second characterization of the spectral subspace ${}^0\mathcal{F}^k$ follows from the fact that

$$BT_{\mathcal{V}} \in \mathcal{L}(\mathfrak{H}_{\mathcal{V}}^r, \mathfrak{H}_{\mathcal{V}}^{r+k})$$

for every $B \in \mathfrak{R}$, $T \in \mathcal{L}(\mathcal{H}_{\mathcal{V}}^r, \mathcal{H}_{\mathcal{V}}^{r+k})$.

The last statements in the lemma follow immediately from the structure of the spectral subspaces of the single Cuntz-Pimsner algebras (cf. Remark 3.1). ∎

Definition 5.4 *For every $T \in {}^0\mathcal{F}^k$, $k \in \mathbb{Z}_0^\infty$, we denote by supp$(T)$ the finite subset in $\mathfrak{G}$ given by the union of all $\mathcal{V} \subset \mathfrak{G}$ that appear in the decomposition (44) or (45).*

5.1.1 Crossed products

In the following we will study Hilbert space representations of the *-algebra ${}^0\mathcal{F}$. We begin introducing the notion of covariant representation associated to a family of unitary finite-dimensional representations of the compact group $\mathcal{G}$ and a fixed chain group action on a C^*-algebra.

Definition 5.5 *Let $\mathfrak{G} = \mathfrak{G}(\mathcal{G})$ be a family of finite-dimensional unitary representations of $\mathcal{G}$ and denote by $\mathcal{O}_{\mathfrak{G}}$ the tensor product of the corresponding Cuntz algebras (cf. (40)). Consider a unital C^*-algebra $\mathfrak{R}$ and a fixed chain group action $\alpha\colon \mathfrak{C}(\mathcal{G}) \to \mathrm{Aut}\,\mathfrak{R}$. A* **covariant representation** *of $(\mathfrak{R}, \mathfrak{G}, \mathfrak{C}(\mathcal{G}), \alpha)$ is a triple $(\mathfrak{h}, \pi, \eta)$ where $\pi\colon \mathfrak{R} \to \mathcal{L}(\mathfrak{h})$ and $\eta\colon \mathcal{O}_{\mathfrak{G}} \to \mathcal{L}(\mathfrak{h})$ are non-degenerate Hilbert space representations satisfying*

$$\eta(\psi_{V,i;(D)}) \cdot \pi(B) = \pi \circ \alpha_{[D]}(B) \cdot \eta(\psi_{V,i;(D)}) \ , \quad B \in \mathfrak{R} \ , [D] \in \mathfrak{C}(\mathcal{G}) \ , \psi_{V,i;(D)} \in \mathcal{H}_V \subset \mathcal{O}_{\mathfrak{G}} \ .$$

Here we identify $\psi_{V,i;(D)}$ with the corresponding image in $\mathcal{O}_{\mathfrak{G}}$ according to the natural inclusions $\mathcal{H}_V \hookrightarrow \mathcal{O}_{d(V)} \hookrightarrow \mathcal{O}_{\mathfrak{G}}$. Now, since $\mathcal{O}_{\mathfrak{G}}$ is a tensor product labeled by the different elements in $\mathfrak{G}$ and since η is non-degenerate (i.e. unital), we find that the operators $\eta(\psi_{V,i;(D)})$ satisfy the relations (36), (37) and (39). Moreover from Definition 5.5 it is also clear that the operators $\pi(B)$, $\eta(\psi_{V,i;(D)})$ satisfy the relation (38). Therefore, defining

$$\pi \rtimes \eta\colon \ {}^0\mathcal{F} \to \mathcal{L}(\mathfrak{h}) \ \ , \ \ \pi \rtimes \eta(B\psi_{V,i;(D)}) := \pi(B) \cdot \eta(\psi_{V,i;(D)}) \ , \tag{46}$$

we obtain a representation of ${}^0\mathcal{F}$ on the Hilbert space $\mathfrak{h}$. Let us define the C^*-algebra $\mathcal{F}$ as the closure of ${}^0\mathcal{F}$ w.r.t. the C^*-norm

$$\|F\| := \sup_{(\pi,\eta)} \|(\pi \rtimes \eta)(F)\|_{\mathrm{op}} \ , F \in \ {}^0\mathcal{F} \ . \tag{47}$$

Thus by construction we have that for every covariant representation (π, η), there exists a representation Π of $\mathcal{F}$ extending the former pair, i.e.

$$\Pi\colon \mathcal{F} \to \mathcal{L}(\mathfrak{h}) \quad \text{with} \quad \Pi(T) = \eta(T)\ ,\ \Pi(B) = \pi(B)\ , \quad T \in {}^0\mathcal{O}_{\mathfrak{G}} \subset {}^0\mathcal{F}\ ,\ B \in \mathfrak{R}\,. \tag{48}$$

The proof of the following theorem uses the notion of twisted tensor product introduced by Cuntz in [9, §1].

Theorem 5.6 *Let $\mathcal{G}$ be a compact group, $\mathfrak{R}$ a unital C^*-algebra and $\alpha\colon \mathfrak{C}(\mathcal{G}) \to \mathrm{Aut}\,\mathfrak{R}$ a fixed chain group action. Then, for every set $\mathfrak{G}$ of finite-dimensional representations of $\mathcal{G}$, there exists a universal C^*-algebra*

$$\mathfrak{R} \rtimes^{\alpha} \mathfrak{G}$$

generated by $\mathfrak{R}$ and $\{\psi_{V,i;(D)}\}_{V\,\in\,\mathfrak{G}}$ and satisfying the relations (36)-(39) and such that (41),(42) extend to automorphic actions on $\mathcal{F}$.

Proof: We will divide the proof into several steps:

1. Consider first a covariant representation $(\mathfrak{h}, \pi_0, U)$ of the C^*-dynamical system $(\mathfrak{R}, \mathfrak{C}(\mathcal{G}), \alpha)$ in the usual sense of crossed products by group actions, i.e. $\pi_0\colon \mathfrak{R} \to \mathcal{L}(\mathfrak{h})$ is a non-degenerate representation and $U\colon \mathfrak{C}(\mathcal{G}) \to \mathcal{U}(\mathfrak{h})$ is a unitary representation of the chain group satisfying

$$U_{[D]} \cdot \pi_0(B) = \pi_0 \circ \alpha_{[D]}(B) \cdot U_{[D]}\ , \quad B \in \mathfrak{R}\ ,\ [D] \in \mathfrak{C}(\mathcal{G})\ .$$

It is well-known that $\|B\| = \sup_{\pi_0} \|\pi_0(B)\|$ (see [26]). Moreover, consider a Hilbert space representation $\eta_0\colon \mathcal{O}_{\mathfrak{G}} \to \mathcal{L}(\mathfrak{h}_0)$. By simplicity of the Cuntz algebras, we have that $\mathcal{O}_{\mathfrak{G}}$ is also simple, hence η_0 is faithful.

2. In terms of η_0 and the previous covariant representation π_0 we will introduce next a covariant representation of $(\mathfrak{R}, \mathfrak{G}, \mathfrak{C}(\mathcal{G}), \alpha)$ in the sense of Definition 5.5. We define

$$\pi\colon \mathfrak{R} \to \mathcal{L}(\mathfrak{h} \otimes \mathfrak{h}_0)\quad ,\quad \pi(B) := \pi_0(B) \otimes \mathbb{1}\ ,$$

$$\eta\colon \mathcal{O}_{\mathfrak{G}} \to \mathcal{L}(\mathfrak{h} \otimes \mathfrak{h}_0)\quad ,\quad \eta(\psi_{V,i;(D)}) := U_{[D]} \otimes \eta_0(\psi_{V,i;(D)})\ .$$

Note that with this definition we have

$$\begin{aligned}
\eta(\psi_{V,i;(D)}) \cdot \pi(B) &= \Big(U_{[D]} \otimes \eta_0(\psi_{V,i;(D)})\Big) \cdot \Big(\pi_0(B) \otimes \mathbb{1}\Big) \\
&= \Big(U_{[D]} \cdot \pi_0(B)\Big) \otimes \eta_0(\psi_{V,i;(D)}) \\
&= \Big(\pi_0 \circ \alpha_{[D]}(B) \cdot U_{[D]}\Big) \otimes \eta_0(\psi_{V,i;(D)}) \\
&= \Big(\pi_0 \circ \alpha_{[D]}(B) \otimes \mathbb{1}\Big) \cdot \Big(U_{[D]} \otimes \eta_0(\psi_{V,i;(D)})\Big) \\
&= \Big(\pi \circ \alpha_{[D]}\Big)(B) \cdot \eta(\psi_{V,i;(D)})\,,
\end{aligned}$$

This shows that (π, η) specifies a representation of the relation (38). Furthermore we verify that η is also a representation of the remaining relations:

$$\sum_i \eta(\psi_{V,i;(D)})\eta(\psi_{V,i;(D)})^* = \sum_i \Big(U_{[D]}U_{[D]}^*\Big) \otimes \eta_0(\psi_{V,i;(D)}\psi_{V,i;(D)}^*) = \mathbb{1} \otimes \mathbb{1}\ ,$$

$$\eta(\psi_{V,i;(D)})^*\eta(\psi_{V,j;(D')}) = \Big(U_{[D]}^*U_{[D']}\Big) \otimes \eta_0(\psi_{V,i;(D)}^*\psi_{V,j;(D')}) = \delta_{ij}\ (\mathbb{1} \otimes \mathbb{1})\ ,$$

and

$$\begin{aligned}
\eta(\psi_{V,i;(D)})\eta(\psi_{W,j;(D')}) &= \Big(U_{[D]}\otimes\eta_0(\psi_{V,i;(D)})\Big)\cdot\Big(U_{[D']}\otimes\eta_0(\psi_{W,j;(D')})\Big) \\
&= \Big(U_{[D]}U_{[D']}\Big)\otimes\eta_0(\psi_{V,i;(D)}\psi_{W,j;(D')}) \\
&= \Big(U_{[D']}U_{[D]}\Big)\otimes\eta_0(\psi_{W,j;(D')}\psi_{V,i;(D)}) \\
&= \eta(\psi_{W,j;(D')})\eta(\psi_{V,i;(D)}) \ ,
\end{aligned}$$

where we used (39) and the fact that $\mathfrak{C}(\mathcal{G})$ is Abelian. In the same way, one can verify the relations

$$\eta(\psi_{W,j;(D')})^*\eta(\psi_{V,i;(D)}) = \eta(\psi_{V,i;(D)})\eta(\psi_{W,j;(D')})^* .$$

3. In the preceding step we have shown that for any covariant representation $(\mathfrak{h},\pi_0,U)$ and any representation $\eta_0\colon \mathcal{O}_{\mathfrak{G}}\to\mathcal{L}(\mathfrak{h}_0)$ as in step 1 we have constructed a covariant representation (π,η) of $(\mathfrak{R},\mathfrak{G},\mathfrak{C}(\mathcal{G}),\alpha)$ over the Hilbert space $\mathfrak{h}\otimes\mathfrak{h}_0$. In particular, (π,η) defines also a representation $\pi\rtimes\eta$ of ${}^0\mathcal{F}$ on $\mathfrak{h}\otimes\mathfrak{h}_0$. Therefore, we can introduce the C^*-norm as in Eq. (47)

$$\|F\| := \sup_{(\pi,\eta)} \|(\pi\rtimes\eta)(F)\|_{\mathrm{op}} \ , F\in\ {}^0\mathcal{F} \ ,$$

and complete ${}^0\mathcal{F}$ w.r.t. this norm:

$$\mathfrak{R}\rtimes^{\alpha}\mathfrak{G} := \mathrm{clo}_{\|\cdot\|}\left(\ {}^0\mathcal{F}\right) .$$

(We will sometimes denote $\mathfrak{R}\rtimes^{\alpha}\mathfrak{G}$ simply by $\mathcal{F}$.)

Note that the preceding C^*-norm extends the C^*-norm on $\mathfrak{R}$, since recall that we have

$$\|B\| = \sup_{\pi_0}\|\pi_0(B)\|_{\mathrm{op}} = \sup_{\pi}\|\pi(B)\|_{\mathrm{op}} = \sup_{(\pi,\eta)}\|(\pi\rtimes\eta)(B)\|_{\mathrm{op}} .$$

To show the universal property, let $\mathcal{F}'$ be a C^*-algebra with generators satisfying the relations (36)-(39), and $\Pi'\colon\mathcal{F}'\to\mathcal{L}(\mathfrak{h})$ a faithful, non-degenerate representation. Then, $\mathfrak{R}$ and $\mathcal{O}_{\mathfrak{G}}$ are C^*-subalgebras of $\mathcal{F}'$ and $(\pi' := \Pi'\restriction_{\mathfrak{R}}\ ,\ \eta' := \Pi'\restriction_{\mathcal{O}_{\mathfrak{G}}})$ specifies a covariant representation. Since

$$\|(\pi'\rtimes\eta')(F)\|_{\mathrm{op}} \leq \|F\| \ , \quad F\in\ {}^0\mathcal{F}$$

it is clear that there is a monomorphism $\mathcal{F}\hookrightarrow\mathcal{F}'$.

4. Finally we address the question of the automorphic extensions. Let $g\in\mathcal{G}$ and β_g defined as in (41) in terms of unitary representations. Note that if (π,η) is a covariant representation, then $(\pi,\eta\circ\beta_g)$ is again a covariant representation. From this and the universality of the Cuntz algebra it is clear that β extends to an automorphic action of the compact group $\mathcal{G}$. A similar argument shows that (42) extends to an automorphic action of $\mathbb{T}^\infty$. ■

Remark 5.7 (i) The C^*-algebra $\mathfrak{R}\rtimes^{\alpha}\mathfrak{G}$ generalizes several well-known constructions. If $\mathfrak{G}$ has a unique element V, then $\mathfrak{R}\rtimes^{\alpha}\mathfrak{G}$ coincides with the Cuntz-Pimsner algebra $\mathcal{O}_{\mathfrak{H}_V}$ studied in Section 4. In particular, if $V=D$ is an irreducible representation, then $\mathfrak{R}\rtimes^{\alpha}\mathfrak{G}$ is isomorphic to the Stacey crossed product $\mathfrak{R}\rtimes^{d(D)}_{\alpha}\mathbb{N}$ (see [29]). If $\mathcal{G}$ is Abelian and $\mathfrak{G}=\mathfrak{C}(\mathcal{G})$, then $\mathfrak{R}\rtimes^{\alpha}\mathfrak{G}$ is isomorphic to the usual crossed-product $\mathfrak{R}\rtimes_{\alpha}\mathfrak{C}(\mathcal{G})$ by the $\mathfrak{C}(\mathcal{G})$-action on $\mathfrak{R}$. Finally, if the action of the chain group on $\mathfrak{R}$ is trivial, then $\mathfrak{R}\rtimes^{\alpha}\mathfrak{G}$ reduces to the tensor product $\mathfrak{R}\otimes\mathcal{O}_{\mathfrak{G}}$ (recall that $\mathcal{O}_{\mathfrak{G}}$ is nuclear, thus we do not have to specify the norm of the tensor product).

(ii) Note also that the C^*-algebra $\mathfrak{R} \rtimes^\alpha \mathfrak{G}$ is generated as a C^*-algebra by the family of Cuntz-Pimsner algebras

$$\{\mathcal{O}_{\mathfrak{H}_V}\}_{V \in \mathfrak{G}} ,$$

where the product of the elements in different algebras is twisted by the chain group action. The present construction has also some elements of Cuntz notion of twisted tensor product given in [9]. Finally, note also that the C^*-norm restricted to a single Cuntz-Pimsner algebra is unique.

To simplify notation we write $\mathcal{F} := \mathfrak{R} \rtimes^\alpha \mathfrak{G}$. We denote by $\mathcal{F}^k$, $k \in \mathbb{Z}_0^\infty$, the spectral subspaces of $\mathcal{F}$ w.r.t. the δ-action. It is clear that each ${}^0\mathcal{F}^k$ is a dense subspace of $\mathcal{F}^k$. Moreover, the projection onto the zero spectral subspace in (43) extends in a natural way to $m_0 \colon \mathcal{F} \to \mathcal{F}^0$.

By construction, for every $V \in \mathfrak{G}$ there corresponds an algebraic Hilbert space $\mathcal{H}_V$ with support $\mathbb{1}$ contained in $\mathcal{F}$ (see Eq. (36)). Moreover, from (41) we have that $\beta_g(\psi) = V(g)\psi$, $g \in \mathcal{G}$, $\psi \in \mathcal{H}_V$, and for every $V \in \mathfrak{G}$, there is an endomorphism $\sigma_V \in \operatorname{End}\mathcal{F}$ induced as usual by

$$\sigma_V(T) := \sum_i \psi_{V,i;(D)} T \psi^*_{V,i;(D)} \quad , \quad T \in \mathcal{F} ,$$

where $\left\{\psi_{V,i;(D)}\right\}_i$ is an adapted orthonormal basis of the algebraic Hilbert space $\mathcal{H}_V$. Each σ_V coincides with the canonical endomorphism on the Cuntz algebra $\mathcal{O}_{d(V)} \subset \mathcal{F}$. Let $\mathcal{A} \subset \mathcal{F}$ denote the fixed-point algebra w.r.t. the action β. Since the construction of σ_V does not depend on the choice of the basis ψ_V, we obtain $\sigma_V \circ \beta_g = \beta_g \circ \sigma_V$, $g \in \mathcal{G}$, as in Section 4. This implies that every σ_V restricts to a canonical endomorphism $\rho_V \in \operatorname{End}\mathcal{A}$.

5.2 Minimal and regular C^*-dynamical systems

We now assume that $\mathfrak{R}$ is Abelian and denote it by $\mathcal{Z}$ as in the first part of the paper. Let us consider the class $\mathfrak{G}_0$ of finite-dimensional representations introduced in Definition 4.7. For the choice $\mathfrak{G} := \mathfrak{G}_0$ we obtain from the construction in the preceding subsection a C^*-dynamical system $(\mathcal{F}, \mathcal{G}, \beta)$, $\mathcal{F} := \mathcal{Z} \rtimes^\alpha \mathfrak{G}_0$, with $\mathbb{T}^\infty$-action given by δ (cf. Eq. (42)) and fixed-point algebra $\mathcal{A}$. Recall also that $\mathcal{F}$ contains all Cuntz-Pimsner algebras $\mathcal{O}_{\mathfrak{H}_V}$, $V \in \mathfrak{G}_0$. Moreover, by definition of the group action given in Eq. (41) it is clear that β restricts to each single Cuntz-Pimsner algebra, so that the large C^*-dynamical system $(\mathcal{F}, \mathcal{G}, \beta)$ contains all single C^*-dynamical systems $(\mathcal{O}_{\mathfrak{H}_V}, \mathcal{G})$ with corresponding fixed-point algebra $\mathcal{A}_V$ considered in Theorem 4.9. Note also that we have the following relation between the fixed-point algebras of the single systems and the fixed-point algebra $\mathcal{A}$ of the large C^*-dynamical system:

$$\mathcal{A}_V \subset \mathcal{A} \, , \quad V \in \mathfrak{G}_0 \, .$$

In the present subsection the relative commutant $\mathcal{A}' \cap \mathcal{F}$ will analyzed in detail. We begin stating the following stability result:

Lemma 5.8 *The relative commutant $\mathcal{C} := \mathcal{A}' \cap \mathcal{F}$ of the C^*-dynamical system $(\mathcal{F}, \mathcal{G}, \beta)$ with fixed-point algebra $\mathcal{A}$ is stable under the $\mathbb{T}^\infty$-action given by δ. Therefore $\mathcal{C}$ is generated as a C^*-algebra by the spectral subspaces*

$$\mathcal{C}^k := \mathcal{F}^k \cap \mathcal{C} \, , \quad k \in \mathbb{Z}_0^\infty .$$

Proof: Let $z \in \mathbb{T}^\infty$ and $C \in \mathcal{C}$. Since $\mathcal{A}$ is generated by the corresponding spectral subspaces, it is enough to show that

$$\delta_z(C) A = A \delta_z(C) \quad \text{for any} \quad A \in \mathcal{A}^k \, , \ \ k \in \mathbb{Z}_0^\infty \, .$$

But this follows from

$$\delta_z(C) \prod_V z(V)^{n(V)} A = \delta_z(CA) = \delta_z(AC) = \prod_V z(V)^{n(V)} A \delta_z(C) \,.$$

This shows the stability of $\mathcal{C}$ and the fact that it is generated by the spectral subspaces $\mathcal{C}^k$. ■

In the next proposition, we show that *any* finite-dimensional representation of $\mathcal{G}$ is realized on an algebraic Hilbert space in C^*-algebra $\mathcal{F}$.

Proposition 5.9 *Let V be a finite-dimensional representation of $\mathcal{G}$ (not necessarily belonging to $\mathfrak{G}_0$) on $\mathcal{H}_V$. Then, $\mathcal{H}_V$ can be identified with an algebraic Hilbert space in $\mathcal{F}$ with support E_V. Moreover, $\beta_g(\psi) = V(g)\psi$, $g \in \mathcal{G}$, $\psi \in \mathcal{H}_V$.*

Proof: Let V_D be an irreducible subrepresentation of V of class D. Define $L := V \oplus V_D$ and consider the representation $W := L \oplus \overline{\det L}$. By construction $W \in \mathfrak{G}_0$ and the algebraic Hilbert space $\mathcal{H}_W$ is contained in $\mathcal{F}$ with support $\mathbb{1}$. (Note that $W \in \mathfrak{G}_0$ also in the case where V is an irreducible representation of an Abelian group.) There is a reducing projection E_V associated with the decomposition $\mathcal{H}_W = \mathcal{H}_V \oplus \mathcal{H}_{V_D \oplus \overline{\det L}}$. Thus, $\mathcal{H}_V$ has support E_V in $\mathcal{F}$. Moreover, $\beta_g(\psi) = (V(g) \oplus V_D(g) \oplus \overline{\det L(g)})(E_V\psi) = V(g)\psi$, $g \in \mathcal{G}$, $\psi \in \mathcal{H}_V$. ■

From Definition 4.7 for every $V \in \mathfrak{G}_0$ there exists $n(V) \in \mathbb{N}$ and a $\mathcal{G}$-invariant isometry S_V of $\mathcal{H}_V^{n(V)}$. In the following lemma we will summarize some useful properties of these invariant vectors (see also Lemma 4.3).

Lemma 5.10 *For every $V \in \mathfrak{G}_0$, there exists $S_V \in \mathcal{H}_V^{n(V)} \subset \mathcal{F}$ such that*

- *(i) $S_V^* S_V = \mathbb{1}$, $\beta_g(S_V) = S_V$ and $S_V Z = Z S_V$, $Z \in \mathcal{Z}$, $g \in \mathcal{G}$.*

Moreover,

- *(ii) if $V, W \in \mathfrak{G}_0$, $V \neq W$, then $S_V T = T S_V$, $T \in \mathcal{O}_{\mathfrak{H}_W} \subset \mathcal{F}$.*
- *(iii) if δ is the $\mathbb{T}^\infty$-action on $\mathcal{F}$ (cf. Eq.(42)) and $C \in \mathcal{C} := \mathcal{A}' \cap \mathcal{F}$, then $\delta_z(C) S_V = S_V \delta_z(C)$, $z \in \mathbb{T}^\infty$.*

Proof: The existence of the elements S_V and first two equations in (i) follow from Definition 4.7 (ii) and Lemma 4.3. Therefore $S_V \in \mathcal{A}_V \subset \mathcal{A}$, where $\mathcal{A}_V$ is the fixed-point algebra of the C^*-dynamical system $(\mathcal{O}_{\mathfrak{H}_V}, \mathcal{G})$. The last equation in (i) follows from the equation $\mathcal{Z} = \mathcal{A}_V' \cap \mathcal{A}_V$ (cf. Theorem 4.9 (ii)). Recall that the Cuntz-Pimsner algebra $\mathcal{O}_{\mathfrak{H}_W}$ is generated by $\mathcal{Z}$ and $\mathcal{H}_W$. Therefore the equation in (ii) follows from $S_V \in \mathcal{Z}' \cap \mathcal{A}$ (cf. part (i)) and relation (39). Part (iii) is an immediate consequence of Lemma 5.8 and the fact that $S_V \in \mathcal{A}$. ■

For every finite set $\mathcal{W} \subseteq \mathfrak{G}_0$, we define the isometry

$$S_{\mathcal{W}} := \prod_{W \,\in\, \mathcal{W}} S_W \tag{49}$$

and for every $T \in \mathcal{F}$, $p \in \mathbb{N}$, we put

$$\widetilde{T}_{\mathcal{W},p} := (S_{\mathcal{W}}^p)^* \; T \; S_{\mathcal{W}}^p \;\; \in \mathcal{F} \;. \tag{50}$$

Lemma 5.11 *Let $R \in {}^0\mathcal{F}^k$, $k \in \mathbb{Z}_0^\infty$, and set $\mathcal{V} := supp(R)$ (see Definition 5.4). Then, for every finite set $\mathcal{W} \subseteq \mathfrak{G}_0$ such that $\mathcal{V} \subseteq \mathcal{W}$, and for every $p \in \mathbb{N}$, we have*

$$(S_{\mathcal{W}}^p)^* \; R \; S_{\mathcal{W}}^p = (S_{\mathcal{V}}^p)^* \; R \; S_{\mathcal{V}}^p \;.$$

Proof: Using Lemma 5.3 and linearity it is enough to consider expressions of the form

$$R = ZR_1 \cdots R_n \ , \ Z \in \mathcal{Z} \ , \ R_i \in \mathcal{L}(\mathcal{H}_{V_i}^{r(V_i)}, \mathcal{H}_{V_i}^{r(V_i)+k(V_i)}) \, ,$$

with $\{V_1, \ldots, V_n\} \subseteq supp(R)$. Now, by Lemma 5.10 (ii) it follows that $S_W^* R_i S_W = R_i$ for every $W \neq V_i$. ■

Next we give an approximation result for elements of the form as given in Eq. (50) with $T \in \mathcal{F}^k$ in terms of series of elements in the tensor product of suitable Hilbert bimodules (cf. Remark 5.2).

Lemma 5.12 *Let $T \in \mathcal{F}^k$, $k \in \mathbb{Z}_0^\infty$. Then, for every $\varepsilon > 0$ there exist $n(\varepsilon)$, $p(\varepsilon) \in \mathbb{N}$, finite sets $\mathcal{V}_1, \ldots, \mathcal{V}_{n(\varepsilon)} \subseteq \mathfrak{G}_0$, and $\Phi_l \in \mathfrak{H}_{\mathcal{V}_l}^k$ satisfying the following property: defining the finite subset $\mathcal{W} := \bigcup_{l=1}^{n(\varepsilon)} \mathcal{V}_l \subset \mathfrak{G}_0$ we have*

$$\left\| \widetilde{T}_{\mathcal{W},p} - \sum_{l=1}^{n(\varepsilon)} \Phi_l \right\| < \varepsilon \, , \quad \text{for all } \ p > p(\varepsilon) \quad .$$

Proof: Recall that by Lemma 5.3 every element $T \in \mathcal{F}^k$, $k \in \mathbb{Z}_0^\infty$, can be written as

$$T = \sum_{l=1}^{\infty} R_l \quad ,$$

where $R_l \in \mathcal{L}(\mathfrak{H}_{\mathcal{V}_l}^r, \mathfrak{H}_{\mathcal{V}_l}^{r+k})$, $r \in \mathbb{N}_0^\infty$, and $\mathcal{V}_l \subseteq \mathfrak{G}_0$ is a finite set. Take $n(\varepsilon)$ such that

$$\left\| T - \sum_{l=1}^{n(\varepsilon)} R_l \right\| < \varepsilon \ .$$

For $\mathcal{W} := \bigcup_{l}^{n(\varepsilon)} \mathcal{V}_l \subset \mathfrak{G}_0$, $p \in \mathbb{N}$, we have

$$\widetilde{T}_{\mathcal{W},p} = \sum_{l=1}^{\infty} (S_{\mathcal{W}}^p)^* \ R_l \ S_{\mathcal{W}}^p = \sum_{l=1}^{\infty} (S_{\mathcal{V}_l}^p)^* \ R_l \ S_{\mathcal{V}_l}^p \ ,$$

where for the last equation we have used Lemma 5.11. Since

$$R_l = \sum_j^{\text{fin}} \prod_{V \in \mathcal{V}_l} R_{V,j} \quad ,$$

with $R_{V,j} \in \mathcal{L}(\mathfrak{H}_V^{r(V)}, \mathfrak{H}_V^{r(V)\,+\,k(V)})$ we can concentrate our analysis on the single factors appearing in the preceding product. From the proof of Proposition 3.5 in [10] we have that for every $V \in \mathcal{V}_l$, $l = 1, \ldots, n(\varepsilon)$, there exist $p(l) \in \mathbb{N}$ such that, for all $p > p(l)$,

$$(S_V^p)^* \ R_{V,j} \ S_V^p = \varphi_{V,j} \ , \quad \text{for some} \quad \varphi_{V,j} \in \mathfrak{H}_V^{k(V)} \, .$$

This implies that for every $p > p(\varepsilon) := \max \{p(l)\}_{l=1}^{n(\varepsilon)}$,

$$(S_{\mathcal{W}}^p)^* \ R_l \ S_{\mathcal{W}}^p = (S_{\mathcal{V}_l}^p)^* \ R_l \ S_{\mathcal{V}_l}^p = \Phi_l := \sum_j^{\text{fin}} \prod_{V \in \mathcal{V}_l} \varphi_{V,j} \ ,$$

where the r.h.s. of the preceding equality belongs to $\mathfrak{H}^k_{\mathcal{V}_l}$. Finally,

$$\left\| \widetilde{T}_{\mathcal{W},p} - \sum_{l=1}^{n(\varepsilon)} \Phi_l \right\| = \left\| (S^p_{\mathcal{W}})^* \left(T - \sum_{l=1}^{n(\varepsilon)} R_l \right) S^p_{\mathcal{W}} \right\| \leq \left\| T - \sum_{l=1}^{n(\varepsilon)} R_l \right\| < \varepsilon$$

and the proof is concluded. ■

Proposition 5.13 *The C*-dynamical system $(\mathcal{F}, \mathcal{G}, \beta)$ with fixed-point algebra $\mathcal{A}$ constructed in this section is minimal, i.e.*

$$\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}\,.$$

Moreover, $\mathcal{Z}$ coincides with the center of $\mathcal{A}$, i.e. $\mathcal{Z} = \mathcal{A}' \cap \mathcal{A}$.

Proof: Our proof will be divided in two steps. We will apply techniques from [10, Propositions 3.4 and 3.5].

i) The inclusions $\mathcal{Z} \subseteq \mathcal{A}' \cap \mathcal{A} \subseteq \mathcal{A}' \cap \mathcal{F}$: Since $\mathcal{A}$ is stable w.r.t. the $\mathbb{T}^\infty$-action given by δ, it suffices to verify that $Z \in \mathcal{Z}$ commutes with the elements of the spectral subspaces $\mathcal{A}^k := \mathcal{F}^k \cap \mathcal{A}$. If $T \in \mathcal{A}^k$, then there is a sequence $\{T_l\}_l$ of elements of ${}^0\mathcal{F}^k$ approximating T. By Lemma 5.3 we can write $T_l = \sum_{\mathcal{V}}^{\mathrm{fin}} Z^l_{\mathcal{V}} T^l_{\mathcal{V}}$, with $T^l_{\mathcal{V}} \in \mathcal{L}(\mathcal{H}^l_{\mathcal{V}}, \mathcal{H}^{l+k}_{\mathcal{V}})$. Applying the mean $\mathfrak{m}_{\mathcal{G}}$ over the group action (cf. Eq. (35)), we conclude that T is approximated by the sequence

$$\mathfrak{m}_{\mathcal{G}}(T_l) = \sum_{\mathcal{V}}^{\mathrm{fin}} Z^l_{\mathcal{V}}\ \mathfrak{m}_{\mathcal{G}}(T^l_{\mathcal{V}})\ .$$

Note that each $\mathfrak{m}_{\mathcal{G}}(T^l_{\mathcal{V}})$ is a $\mathcal{G}$-invariant element of $\mathcal{L}(\mathcal{H}^l_{\mathcal{V}}, \mathcal{H}^{l+k}_{\mathcal{V}})$. This implies that the set ${}^0\mathcal{A}^k := {}^0\mathcal{F}^k \cap \mathcal{A}$ is dense in $\mathcal{A}^k$. Therefore it is enough to show commutativity w.r.t. ${}^0\mathcal{A}^k$. Let $T = \sum_{\mathcal{V}}^{\mathrm{fin}} Z_{\mathcal{V}} T_{\mathcal{V}}$ be a generic element of ${}^0\mathcal{A}^k$. If $Z \in \mathcal{Z}$, then the same argument as in Proposition 4.6 implies $T_{\mathcal{V}} Z = Z T_{\mathcal{V}}$, so that

$$TZ = \sum_{\mathcal{V}}^{\mathrm{fin}} Z_{\mathcal{V}} T_{\mathcal{V}} Z = \sum_{\mathcal{V}}^{\mathrm{fin}} Z_{\mathcal{V}} Z T_{\mathcal{V}} = ZT\ .$$

The preceding equation and the definition of the group action in Eq. (41) proves the inclusion $\mathcal{Z} \subseteq \mathcal{A}' \cap \mathcal{A} \subseteq \mathcal{A}' \cap \mathcal{F}$.

ii) The inclusion $\mathcal{A}' \cap \mathcal{F} \subseteq \mathcal{Z}$: From Lemma 5.8 we can reduce our analysis to the spectral subspaces of the relative commutant $\mathcal{C} := \mathcal{A}' \cap \mathcal{F}$. Let $C \in \mathcal{C}^k := \mathcal{F}^k \cap \mathcal{C}$, $k \in \mathbb{Z}^\infty_0$. For each finite $\mathcal{W} \subseteq \mathfrak{G}_0$, we consider the isometry $S_{\mathcal{W}}$ defined as in Eq. (49). By Lemma 5.10 (i), $S_{\mathcal{W}}$ belongs to $\mathcal{A}$, thus for every $p \in \mathbb{N}$ we have

$$\widetilde{C}_{\mathcal{W},p} := (S^p_{\mathcal{W}})^*\ C\ S^p_{\mathcal{W}} = C\,.$$

From the approximation result in Lemma 5.12 we conclude that

$$C = \sum_{l=1}^{\infty} \Phi_l\ , \quad \text{where} \quad \Phi_l \in \mathfrak{H}^k_l := \mathfrak{H}^k_{\mathcal{V}_l}\,.$$

If $k(V) = 0$ for every $V \in \mathfrak{G}_0$, then every Φ_l belongs to $\mathcal{Z}$, and we conclude that $\mathcal{C}^0 \subseteq \mathcal{Z}$. Finally we show that $\mathcal{C}^k = 0$ for all $k \neq 0$. For this purpose, take $C \in \mathcal{C}^k$, with $k(V) \neq 0$ for some $V \in \mathfrak{G}_0$ and consider the following equivalence relation for the indices:

$$l \sim m\ :\Leftrightarrow\ k|_{\mathcal{V}_l} = k|_{\mathcal{V}_m}\ .$$

Note that $l \sim m$ if and only if $\Phi_l \Phi_m^* \in \mathfrak{H}_l^k (\mathfrak{H}_m^k)^* \subset \mathcal{F}^0$. Let us consider

$$CC^* = \sum_{l,m=1}^{\infty} \Phi_l \Phi_m^* \quad ;$$

since $CC^* \in \mathcal{C}^0$, by applying the projection onto the zero spectral component (cf. Eq. (43)) we obtain

$$CC^* = m_0(CC^*) = \sum_{l,m} m_0(\Phi_l \Phi_m^*) = \sum_{l \sim m} \Phi_l \Phi_m^* \,. \tag{51}$$

We now apply to the sum (51) the argument used for generic elements of $\mathcal{C}^0$, and conclude that $\Phi_l \Phi_m^* \in \mathcal{Z}$ for every $l \sim m$. In particular, we obtain $\Phi_l \Phi_l^* \in \mathcal{Z}$ for every $l \in \mathbb{N}$. Now, since every $\mathfrak{H}_V$ is a nonsingular bimodule (see Definition 3.6 and Theorem 4.9), the same is true for $\mathfrak{H}_l^k$ (see [10, p. 273]). This implies that $\Phi_l = 0$ for every $l \in \mathbb{N}$. We conclude that if $C \in \mathcal{C}^k$, $k \neq 0$, then $C = 0$ and therefore $\mathcal{C} = \mathcal{C}^0 = \mathcal{Z}$. ■

Theorem 5.14 *Let $\mathcal{G}$ be a compact group, $\mathcal{Z}$ a unital Abelian C^*-algebra and $\alpha\colon \mathfrak{C}(\mathcal{G}) \to \mathrm{Aut}\,\mathcal{Z}$ a fixed chain group action. Given the set of finite-dimensional representations $\mathfrak{G}_0$ as in Definition 4.7, the C^*-dynamical system $(\mathcal{F}, \mathcal{G}, \beta)$ with fixed-point algebra $\mathcal{A}$ constructed in this section is minimal (i.e. $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$) and regular. Moreover, $\mathcal{Z}$ coincides with the center of $\mathcal{A}$, i.e. $\mathcal{Z} = \mathcal{A}' \cap \mathcal{A}$, and every algebraic Hilbert space $\mathcal{H}_V$, $V \in \mathfrak{G}_0$ (cf. Definition 4.7), is contained in $\mathcal{F}$ with support $\mathbb{1}$.*

Proof: In Subsection 5.1 and 5.2 we have specified the construction of a C^*-dynamical system $(\mathcal{F}, \mathcal{G}, \beta)$ where every algebraic Hilbert space $\mathcal{H}_V$, $V \in \mathfrak{G}_0$, is contained in $\mathcal{F}$ with support $\mathbb{1}$ (recall the relations (36)-(39)). The minimality property has been shown in Proposition 5.13.

To show that $(\mathcal{F}, \mathcal{G}, \beta)$ is regular recall Proposition 2.3 and Definition 2.4. By construction, the free $\mathcal{G}$-invariant bimodules $\mathfrak{H}_V$ are generated by the corresponding $\mathcal{G}$-invariant algebraic Hilbert spaces $\mathcal{H}_V$. From relation (38) it is clear that the assignment $\mathfrak{H}_V \mapsto \mathcal{H}_V$ is compatible with products, hence $(\mathcal{F}, \mathcal{G}, \beta)$ is regular. ■

Example 5.15 (Hilbert C*-systems for SU(2))
We will apply here the preceding theorem to the group $\mathcal{G} = \mathrm{SU}\,(2)$ and a given Abelian C^*-algebra $\mathcal{Z}$. Note that the all nontrivial irreducible representations of $\mathcal{G}$ satisfy the properties (i) and (ii) in Definition 4.7. Indeed, any irreducible representation $V^{(l)}$, $l \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$, has dimension $2l + 1 \geq 2$ and the decomposition of the tensor product $V^{(l)} \otimes V^{(l)}$ always contains the trivial representation ι as a subrepresentation (cf. Example 2.9). Therefore we can use $\widehat{\mathcal{G}}$ to construct the C^*-algebra $\mathcal{F}$ as in Subsection 5.1. Note that since the chain group of $\mathrm{SU}\,(2)$ is isomorphic to $\mathbb{Z}_2$, there are only two choices of chain group homomorphism $\alpha\colon \mathbb{Z}_2 \to \mathcal{Z}$.

Applying Theorem 5.14 we obtain a minimal Hilbert C^*-system $(\mathcal{F}, \mathrm{SU}\,(2))$ (i.e. a minimal and regular C^*-dynamical system where all irreducible representations of the group are realized on algebraic Hilbert spaces with support $\mathbb{1}$). Note finally that the trivial representation is realized on the space $\mathbb{C}\mathbb{1} \subset \mathcal{Z}$.

6 Appendix: Tensor categories of Hilbert bimodules

Let $\mathfrak{R}$ be a C^*-algebra, $\mathfrak{H}$ a Hilbert $\mathfrak{R}$-bimodule. We consider the category with objects the internal tensor powers $\mathfrak{H}^r$, $r \in \mathbb{N}_0$, and arrows the spaces $\mathcal{L}(\mathfrak{H}^r, \mathfrak{H}^s)$. It is well-known that such a category is not a tensor category: in fact, in general it is not possible to define in a consistent way the tensor product of right $\mathfrak{R}$-module operators $T \in \mathcal{L}(\mathfrak{H}^r, \mathfrak{H}^s)$, $T' \in \mathcal{L}(\mathfrak{H}^{r'}, \mathfrak{H}^{s'})$ (see [7, VI.13.5]).

Nevertheless, the tensor product $T \otimes T' \in \mathcal{L}(\mathfrak{H}^{r+r'}, \mathfrak{H}^{s+s'})$ makes sense if T, T' are $\mathfrak{R}$-*bimodule operators*, i.e. if they commute with the left action: $TA\psi = AT\psi$, $A \in \mathfrak{R}$, $\psi \in \mathfrak{H}^r$. Thus, the above-mentioned problem of tensoring right $\mathfrak{R}$-module operators vanishes, if we consider an $\mathfrak{R}$-bimodule $\mathfrak{H}$ such that $A\psi = \psi A$, $\psi \in \mathfrak{H}$, $A \in \mathfrak{R}$. Such a bimodule is called *symmetric*. In the context of diagonal bimodules considered in this paper symmetry is guaranteed if we have a trivial chain group action (cf. Eq. (38)).

Let $\mathcal{G}$ be a compact group such that $\mathfrak{H}$ is a $\mathcal{G}$-Hilbert $\mathfrak{R}$-bimodule (in the sense of [7, VIII.20.1], [21, §2]). Suppose that every $g \in \mathcal{G}$ acts on $\mathfrak{H}$ as an $\mathfrak{R}$-bimodule operator. Then, for every $r \in \mathbb{N}_0$ it makes sense to consider the unitary $g_r := g \otimes \ldots \otimes g \in \mathcal{L}(\mathfrak{H}^r, \mathfrak{H}^r)$, so that there is an action $\mathcal{G} \ni g \to g_r \in \mathcal{L}(\mathfrak{H}^r, \mathfrak{H}^r)$, and $\mathfrak{H}^r$ is a $\mathcal{G}$-Hilbert $\mathfrak{R}$-bimodule. We denote by $\mathcal{L}(\mathfrak{H}^r, \mathfrak{H}^s; \mathcal{G})$ the spaces of $\mathcal{G}$-equivariant operators $T \in \mathcal{L}(\mathfrak{H}^r, \mathfrak{H}^s)$ such that $Tg_r\psi = g_sT\psi$, $g \in \mathcal{G}$, $\psi \in \mathfrak{H}^r$. We denote by $\mathbf{tens}(\mathcal{G}, \mathfrak{H})$ the C^*-category with objects $\mathfrak{H}^r$, $r \in \mathbb{N}_0$, and arrows $\mathcal{L}(\mathfrak{H}^r, \mathfrak{H}^s; \mathcal{G})$. For $r = 0$, we define $\mathfrak{H}^0 := \mathfrak{R}$, so that

$$\mathcal{L}(\mathfrak{R}, \mathfrak{R}) = \mathcal{L}(\mathfrak{R}, \mathfrak{R}; \mathcal{G}) = \mathfrak{R} \ . \tag{52}$$

Note that $Z \in \mathcal{L}(\mathfrak{R}, \mathfrak{R})$ is an $\mathfrak{R}$-bimodule operator if and only if Z belongs to the center of $\mathfrak{R}$. It is an interesting question to ask whether there exist non-symmetric, $\mathcal{G}$-Hilbert $\mathfrak{R}$-bimodules $\mathfrak{H}$ such that $\mathbf{tens}(\mathcal{G}, \mathfrak{H})$ is a *tensor* C^*-category (if $\mathfrak{H}$ is symmetric, it is clear that the above question is trivial). From the above considerations, it follows that in order to get a tensor structure on $\mathbf{tens}(\mathcal{G}, \mathfrak{H})$, the following two conditions are needed:

1. if $T \in \mathcal{L}(\mathfrak{H}^r, \mathfrak{H}^s; \mathcal{G})$, then T must be an $\mathfrak{R}$-bimodule operator. In fact, in such a case it makes sense to consider the tensor product $T \otimes T'$, for every $T' \in \mathcal{L}(\mathfrak{H}^{r'}, \mathfrak{H}^{s'}; \mathcal{G})$;

2. as a consequence of the preceding point, from (52) and subsequent remarks we conclude that $\mathfrak{R}$ has to be Abelian.

A $\mathcal{G}$-action satisfying the above properties is called *tensor action*, in the terminology of [30, Definition 5.3]. The next result is just a reformulation of Theorem 4.9, and shows that the class of Hilbert bimodules admitting a tensor action is quite rich.

Proposition 6.1 *Let $\mathcal{G}$ be a compact group, $\mathcal{Z} \neq \mathbb{C}\mathbb{1}$ a nontrivial unital Abelian C^*-algebra with a fixed (nontrivial) action $\alpha\colon\ \mathfrak{C}(\mathcal{G}) \to \mathrm{Aut}\,(\mathcal{Z})$. Then, there exists at least a (nonsymmetric) Hilbert $\mathcal{Z}$-bimodule $\mathfrak{H}$ carrying a tensor $\mathcal{G}$-action by unitary $\mathcal{Z}$-bimodule operators.*

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