Transitory minimal solutions of hypergeometric recursions and pseudoconvergence of associated continued fractions

Alfredo Deaño
Departamento de Matemáticas
Univ. Carlos III de Madrid. 28911-Leganés (Madrid), Spain

Javier Segura
Departamento de Matemáticas, Estadística y Computación,
Univ. Cantabria, 39005-Santander, Spain.
e-mail: adeano@math.uc3m.es, javier.segura@unican.es

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Abstract

Three term recurrence relations $y_{n+1}+b_n y_n+a_n y_{n-1}=0$ can be used for computing recursively a great number of special functions. Depending on the asymptotic nature of the function to be computed, different recursion directions need to be considered: backward for minimal solutions and forward for dominant solutions. However, some solutions interchange their role for finite values of n with respect to their asymptotic behaviour and certain dominant solutions may transitorily behave as minimal. This phenomenon, related to Gautschi's anomalous convergence of the continued fraction for ratios of confluent hypergeometric functions, is shown to be a general situation which takes place for recurrences with a_n negative and b_n changing sign once. We analyze the anomalous convergence of the associated continued fractions for a number of different recurrence relations (modified Bessel functions, confluent and Gauss hypergeometric functions) and discuss the implication of such transitory behaviour on the numerical stability of recursion.

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1 Introduction

Three term recurrence relations (TTRRs)

$$y_{n+1} + b_n y_n + a_n y_{n-1} = 0 \tag{1.1}$$

are satisfied by a large number of special functions, including confluent and Gauss hypergeometric functions.

For computing a solution of a TTRR different strategies should be considered depending on the asymptotic character of the solution. When the recurrence admits a minimal solution, f_n ($\lim_{n\to\infty} f_n/g_n = 0$ for any g_n independent of f_n) forward computation of f_n (increasing n) is a bad conditioned process, at least asymptotically, and only backward recursion should be considered. Contrary, for dominant solutions forward recursion is the right choice. A simple recipe for computing with TTRR would be: if one direction fails, the opposite will usually work well.

This simple recipe is true asymptotically when there exists a minimal solution; however, some results indicate that it may dramatically fail for finite n. Gautschi's anomalous convergence [3] and the instability of certain confluent hypergeometric recurrences found by N.M. Temme [8] point in this direction.

Pincherle's theorem [2] states that a three term recurrence relation (1.1) admits a minimal solution if and only if the associated continued fraction

$$H(k) \equiv \frac{-a_k}{b_k+}\frac{-a_{k+1}}{b_{k+1}+}\frac{-a_{k+2}}{b_{k+2}+}\cdots \tag{1.2}$$

converges; the continued fraction converges to the ratio of minimal solutions f_k/f_{k-1}. In 1977, W. Gautschi found that the continued fraction $H(k)$ associated to the recurrence for $f_n = {}_1\mathrm{F}_1(a+n; c+n; x)$ (which is the minimal solution) initially appears to converge to a value different from f_k/f_{k-1}, particularly for large x.

Gautschi's result and Pincherle's theorem suggest that there may exist dominant solutions of the recurrence that behave as a minimal solution transitorily. As we will see, $g_n = (-1)^n\Gamma(c+n)U(a+n, c+n, x)$ is a transitory minimal solution of the confluent recurrence and, indeed, the approximants of the CF $H(k)$ initially tend to the ratio g_k/g_{k-1} particularly when x is large.

Far from being a special case, we will show that the existence of transitory minimal solutions (for short, pseudominimal solutions) is a quite ubiquitous property. Other examples are provided by the modified Bessel function recurrence, the recurrence satisfied by the confluent family ${}_1\mathrm{F}_1(a+n; c; x)$ (the case described in [8]) as well as some Gauss hypergeometric recursions. We will restrict the analysis to real variables.

The paper is organized as follows. In Section 2, we reinterpret Gautschi's anomalous convergence [3] in terms of the existence of transitory minimal solutions. We identify the minimal solution and a transitory minimal solution which, together with asymptotic expansions (large x) for these solutions allows us to obtain explicit approximations for the accuracy of the continued fraction; we prove that the smallest relative error when the CF approaches the ratio g_k/g_{k-1} decreases exponentially as x increases. In Section 3 we establish that these transitory behaviours are common to a wide family of recurrences, which we call symmetrical; the simplest example is provided by modified Bessel functions. By using the characterization of symmetrical recurrences and, in particular, the modified Bessel function case, we identify additional examples of transitory behaviour. In Section 4 we provide additional examples of recurrences exhibiting transitory behaviour. In 4.1 the instabilities described by N.M. Temme [8] in the recurrent evaluation of the confluent hypergeometric functions $U(a+n, c, x)$ are explained in terms of the existence of pseudominimal solutions; in 4.2 we provide examples of continued fractions associated to Gauss hypergeometric functions and we will obtain Gautschi's phenomenon as the confluent limit of a Gauss hypergeometric case. The last section describes the implications of anomalous transitory behaviours in the numerical computation through three-term recurrence relations when finite precision arithmetic is used.

In the sequel, the recurrence satisfied by a set of hypergeometric functions $y_n = {}_2\mathrm{F}_1(a+\epsilon_1 n, b+\epsilon_2 n; c+\epsilon_3 n; x)$ with ϵ_i integer numbers (not all equal to zero), will be named $(\epsilon_1\,\epsilon_2\,\epsilon_3)$-recurrence, as done in [4]. The same notation is adopted for the recurrences for confluent functions $M(a+\epsilon_1 n, c+\epsilon_3 n, z)$. Because we will restrict to $|\epsilon_i| \leq 1$, we will further simplify the notation by writing only the signs of ϵ_i.

2 Gautschi's anomalous convergence revisited

Let us study in detail the case of the recurrence satisfied by $y_n = {}_1\mathrm{F}_1(a+n; c+n; x) \equiv M(a+n, c+n, x)$ and the associated continued fraction.

The $(+\,+)$ confluent recurrence relation $y_{n+1} + b_n y_n + a_n y_{n-1} = 0$ has coefficients

$$b_n = -\frac{(c+n)(1-c-n+x)}{(a+n)x}\,, a_n = -\frac{(c+n)(c+n-1)}{(a+n)x}, \tag{2.1}$$

and two independent solutions are

$$f_n = M(a+n, c+n, x), \qquad g_n = (-1)^n \Gamma(c+n) U(a+n, c+n, x), \tag{2.2}$$

where $U(a+n, c+n, x)$ is a second solution of the confluent differential equation [1, Eq. 13.1.3]. As it is well known [7], f_n is the minimal solution and therefore the associated continued fraction $H(k)$ converges to the ratio f_k/f_{k-1}. However, as Gautschi described [3], for large x the CF initially appears to converge to a different value. The next result [6, Theorem 7.24] gives an specific meaning to this anomalous behaviour when x is large.

Theorem 1 *Let a, c and k be fixed parameters, $c + k - 1 \neq 0, -1, -2, \ldots$, the continued fraction (1.2) with the coefficients (2.1) corresponds at $x = 0$ to the ratio*

$$\frac{{}_1F_1(a+k; c+k; x)}{{}_1F_1(a+k-1; c+k-1; x)} \tag{2.3}$$

and converges to this function for $x \in \mathbb{R}$.

At $x = \infty$ the continued fraction (1.2) corresponds to the following ratio of formal series

$$-\frac{c+k-1}{x} \frac{{}_2F_0(a+k, a-c+1;; -1/x)}{{}_2F_0(a+k-1, a-c+1;; -1/x)}. \tag{2.4}$$

Therefore, the ratio

$$-(c+k-1)\frac{U(a+k, c+k, x)}{U(a+k-1, c+k-1, x)}. \tag{2.5}$$

corresponds asymptotically (as $x \to \infty$) to the continued fraction (1.2).

Theorem 1 is essentially Theorem 7.24 of [6], but with the inclusion of the U functions.

In the first part, correspondence means that the Taylor series around $x = 0$ of the approximants of the CF coincides with the (convergent) Taylor series of the ratio of ${}_1F_1$ confluent series (to higher order as higher approximants are considered). In the second case, the correspondence for the ratio of ${}_2F_0$ series is in powers of x^{-1} and the formal expansion is divergent. Finally, by asymptotic correspondence we mean that, by considering the asymptotic expansions as $x \to \infty$, there is correspondence in powers of x^{-1}. Indeed, the complete asymptotic expansion of the U function can be expressed in terms of ${}_2F_0$ divergent series because [1, Eq. 13.5.2]:

$$U(\alpha, \gamma, x) = x^{-\alpha}\left(\sum_{j=0}^{J-1} \frac{(\alpha)_j(\alpha-\gamma+1)_j}{j!}(-x)^{-j} + O(x^{-J})\right). \tag{2.6}$$

Theorem 1 explains why for large enough x the continued fraction $H(k)$ gives asymptotic estimations for the ratio of the dominant solutions g_n (2.2); however, because the continued fraction converges to the minimal solution, this can only be true for a finite number of approximants. An initial apparent convergence to g_k/g_{k-1} is possible while for a large enough number of approximants the CF will finally approach f_k/f_{k-1}.

2.1 Error estimation of CFs from solutions of recurrence relations

A more quantitative description can be obtained by analyzing the relation between the continued fraction and backward recursion [2, 5, 6]. This will enable us to estimate the relative accuracy of the CF [5], both for approximating ratios of transitory minimal solutions (pseudoconvergent regime) and ratios of the true minimal solution f_n.

The m-th approximant to the continued fraction $H(k)$ (1.2) is equal to the ratio of solutions y_k/y_{k-1} which are obtained from the backward application of the recurrence relation with starting values

$$y_{k+m-1} = 1\,, y_{k+m} = 0. \tag{2.7}$$

This is shown by iterating m times the following relation (which is equivalent to the application of one backward step of the recurrence):

$$\frac{y_k}{y_{k-1}} = \frac{-a_k}{b_k + \dfrac{y_{k+1}}{y_k}}. \tag{2.8}$$

Let us denote $N = k + m$. Given an independent pair of solutions of the recurrence relation $\{f_n, g_n\}$, we can write $y_N = \alpha f_N + \beta g_N = 0$, $y_{N-1} = \alpha f_{N-1} + \beta g_{N-1} = 1$. Solving for α and β (which is possible because $\{f_n, g_n\}$ is an independent pair) the m-th approximant to $H(k) = f_k/f_{k-1}$ reads:

$$H_m(k) = \frac{y_k}{y_{k-1}} = \frac{\alpha f_k + \beta g_k}{\alpha f_{k-1} + \beta g_{k-1}} = \frac{g_N f_k - f_N g_k}{g_N f_{k-1} - f_N g_{k-1}}. \tag{2.9}$$

Therefore, the continued fraction converges if and only if either f_N/g_N or g_N/f_N have a finite limit as $N \to +\infty$, which means that the recurrence admits a minimal solution (essentially, this proves Pincherle's theorem [5]).

We observe that when $|f_N/g_N|$ becomes small, the m-th approximant to the continued fraction approaches the ratio f_k/f_{k-1}. In particular, when neither f_k nor f_{k-1} vanish we can compute the relative error

$$\epsilon_r^f(k,m) = 1 - \frac{f_{k-1}}{f_k} H_m(k) = \frac{1/r_k - 1/r_{k-1}}{1/r_N - 1/r_{k-1}} \tag{2.10}$$

where

$$r_n \equiv f_n/g_n. \tag{2.11}$$

Notice that $r_k/r_{k-1} \neq 1$ because the solutions f_n and g_n are independent.

As $|r_N/r_{k-1}|$ becomes small the error $|\epsilon_r^f(k,m)|$ tends to decrease. Of course, when $r_N \to 0$ (f_n minimal) we have true convergence to f_k/f_{k-1} as $N \to \infty$.

Even if $r_N \to 0$ as $N \to +\infty$ (which implies that the CF converges to f_k/f_{k-1}), for a finite number of approximants the CF may appear to converge to another ratio of solutions (that is, it pseudoconverges). Indeed, it may happen that g_n is such that initially r_N ($N = k+m$) increases and becomes much larger than r_{k-1}; in this case, the sequence of approximants will initially approach g_k/g_{k-1} and we will say that g_n is transitorily minimal or pseudominimal. The corresponding expression for the relative error is:

$$\epsilon_r^g(k,m) = \frac{r_k}{r_N}\epsilon_r^f(k,m) = \frac{r_k - r_{k-1}}{r_N - r_{k-1}}. \tag{2.12}$$

$|\epsilon_r^g(k,m)|$ will become small when $|r_N/r_{k-1}|$ is large. This expression will be used for estimating the accuracy of the CF when it approximates a transitory minimal solution (pseudoconvergence regime).

Differently to minimal solutions which are unique up to constant factors, transitory minimal solutions are not unique. However, not all dominant solutions are transitorily dominant: if the CF is transitorily converging to the ratio g_k/g_{k-1}, because $g_k/g_{k-1} \neq f_k/f_{k-1}$, it will not approach $(g_k + Cf_k)/(g_{k-1} + Cf_{k-1})$ for C large enough. However, although transitory minimal solutions are not unique, we can expect that there exists a dominant solution (or several) which is (are) optimally pseudominimal in the sense that the CF appears to converge to this ratio of dominant solutions with the best possible accuracy.

2.2 Asymptotic error estimates for Gautschi's pseudoconvergence

Some asymptotic estimates suffice to predict the convergence properties (transitory or not) of the associated continued fraction. In [7], asymptotic expansions for large x are given for the $M(a,c,x)$ and $U(a,c,x)$ functions, which are uniformly valid with respect to $\mu = a/x$ when c and a are comparable in size. The dominant terms provide the following estimation for $r_N = f_N/g_N$

$$r_N \sim (-1)^N K \frac{x^N}{\Gamma(a+N)} \left(1 + \frac{a+N-1}{x}\right)^{a-c} \left(1 + \frac{a+N}{x}\right)^{a-c+1} \tag{2.13}$$

where $K = e^x x^{2a-c}$. Just by considering the factor $x^N/\Gamma(a+N)$ we see that r_N will initially grow rapidly as N increases, particularly for large x, although $r_N \to 0$ as $N \to \infty$. This points toward initial pseudoconvergence to the ratio of U functions and final convergence to the ratio of M functions.

Figure 1 confirms this situation for the evaluation of the continued fraction associated to the ratio

$$\frac{f_1}{f_0} = \frac{M(a+1,c+1,x)}{M(a,c,x)}, \tag{2.14}$$

which initially converges to

$$\frac{g_1}{g_0} = -c\frac{U(a+1,c+1,x)}{U(a,c,x)}, \tag{2.15}$$

and shows the accuracy of the errors estimated from Eqs. (2.10), (2.12) and (2.13).

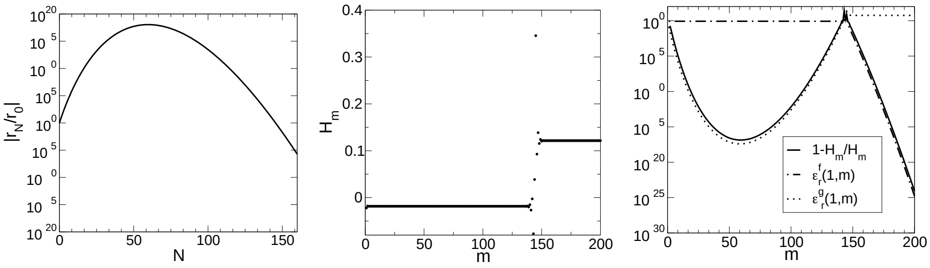

Figure 1 Left: the function $|r_N/r_0|$ is shown for the values $a = 12.4$, $c = 1.3$, $x = 60$. Center: convergence of the successive approximants H_m of the continued fraction for f_1/f_0 (Eq. (2.14)); an abrupt change in the value of the CF is observed when $|r_N/r_0| \simeq 1$. Right, the estimated analytical relative error together with the computed errors obtained from the relative deviation of successive approximants are shown.

The relative error for pseudoconvergence to the wrong limit, $|\epsilon_r^g(1,m)|$ decreases until the maximum r_N $(N = 1+m)$ is reached. For large x, it is easy to obtain the following estimate for the value of N for which $|r_N|$ is maximal:

$$N^* = x - c + \mathcal{O}(1/x). \tag{2.16}$$

The best relative error for pseudoconvergence will then be attained at the $[x-c]$ approximant ($[x-c]$ denoting the integer part of $x-c$). We can estimate the error when x is large and $N = N^* = x - c$ using (2.12); this will estimate the best attainable error in the pseudoconvergent regime. We have

$$|\epsilon_r^g(1,m)| = \left|\frac{r_1 - r_0}{r_{N^*} - r_0}\right| \approx \left|\frac{r_1}{r_{N^*}}\right| \equiv \epsilon_p,$$

where the approximation holds because $|r_{N^*}| >> |r_0|$, $r_1/r_0 \sim -x/a$ and we consider large x. Using (2.13) we have that as $x \to +\infty$

$$\epsilon_p \sim \sqrt{\frac{\pi}{2}} 4^{c-a} \frac{1}{\Gamma(a+1)} x^{a+1/2} e^{-x} (1 + \mathcal{O}(x^{-1})). \tag{2.17}$$

From this error analysis and the comparison with the error estimates, we see that $g_n = (-1)^n \Gamma(c+n) U(a+n, c+n, x)$ is a consistent candidate for being optimally pseudominimal. Indeed, the explicit error estimations agree very well with the relative deviation between successive approximants (Figure 1, right).

After $[x-c]$ iterations, pseudoconvergence worsens. $|r_N/r_0|$ starts to decrease and the successive approximants will eventually start to converge to the ratio of minimal solutions. A rough estimate of the iteration for which this happens can be obtained by considering:

$$|\epsilon_r^f(1,m)| = \left|\frac{1/r_1 - 1/r_0}{1/r_N - 1/r_0}\right| \approx \left|\frac{1}{r_0/r_N - 1}\right| \equiv \epsilon.$$

When $|r_N| < |r_0|$ convergence to f_1/f_0 begins. Taking then $|r_N/r_0| = 1$ in the estimation $r_N/r_0 \approx x^N\Gamma(a)/\Gamma(a+n)$ gives, for large x, $N \sim ex + (a - 1/2)\log(x) + \mathcal{O}(1)$. This is in agreement with the observed numerical behaviour.

A clear and quantitative picture of the "dip-and-peak" effect [3] emerges from the combined used of the error formulas (2.10) and (2.12) and asymptotic approximations for the solutions of the recurrence. The dip (see Figure 1) is reached at $N = [x-c]$, where the best accuracy for pseudoconvergence is reached (2.17) and the peak corresponds to the value of N for which $|r_N| \approx |r_0|$, when the continued fraction starts to converge to the ratio of minimal solutions.

Depending on how deep is the dip, accuracy in the evaluation of the ratio of minimal solutions will suffer loss of accuracy when finite precision arithmetic is used. Typically, the loss of accuracy in the computation of ratios of minimal solutions will be reciprocal to the attainable accuracy in the pseudoconvergent region. We postpone this analysis until section 5.5. By the moment, we do not consider restrictions on the available number of significant digits (we use Maple with a high enough number of digits).

3 Symmetrical recurrences and pseudoconvergence

In this section we will show that the sign properties which take place for the coefficients of the $(++)$ confluent recurrence (negative a_n coefficient and coefficient b_n which changes sign) are clear signatures of transitory anomalous convergence. Also, the fact that the pattern of signs of the minimal solution does not change, as happens for the $(++)$ case, is characteristic of transitory behaviour. When both conditions are satisfied the minimal solution changes its role, causing anomalous behaviour of the CF.

Perron's theorem [2] can be used for determining the pattern of signs of minimal solutions. We write Perron's theorem [2] in a form suitable for analyzing the cases for which the theorem gives positive results regarding the existence of a minimal solution [5].

Theorem 2 ("Intuitive" Perron's theorem) *Let $y_{n+1}+b_ny_n+a_ny_{n-1} = 0$, a_n and b_n being rational functions of n such that $b_n^2 - 4a_n > 0$ for large n; let $\lambda_1(n)$ and $\lambda_2(n)$ be the solutions of $\lambda^2+b_n\lambda+a_n = 0$. If $\lim_{n\to+\infty} |\lambda_1(n)/\lambda_2(n)| \neq 1$ then there exists a pair of independent solutions $\{f_n, g_n\}$ such that*

$$\lim_{n\to+\infty} \frac{1}{\lambda_1(n)}\frac{f_n}{f_{n-1}} = 1\,, \quad \lim_{n\to+\infty} \frac{1}{\lambda_2(n)}\frac{g_n}{g_{n-1}} = 1$$

and the minimal solution is the one corresponding to the smallest $|\lambda(n)|$, namely:

$$\lambda_1(n) = -\frac{2a_n}{b_n + sign(b_n)\sqrt{b_n^2 - 4a_n}}.$$

As a consequence of this result, the characteristic roots $\lambda_1(n)$ and $\lambda_2(n)$ will have opposite patterns of signs because we will assume that $a_n < 0$. This implies that the minimal solution will be asymptotically alternating when $b_n < 0$ for large enough n or it will have constant sign if $b_n > 0$. Any dominant solution will have an asymptotic pattern of signs opposite to that of the minimal solution.

3.1 A canonical example: modified Bessel functions

The simplest examples of recurrences displaying transitory behaviour are those of the form

$$\begin{gathered} y_{n+1} - y_{n-1} = b_n y_n \,, n \in \mathbb{N}, \\ b_n = -b_{-n}, \end{gathered} \tag{3.1}$$

or the slightly more general case $w_{n+1} - a w_{n-1} = B_n w_n \,, n \in \mathbb{N}$, $a > 0$, $B_{n_0+p} = -B_{n_0-p}$, $p, n_0 \in \mathbb{N}$, which can be transformed to the previous case with the changes $w_n = a^{n/2} y_{n-n_0}$, $b_n = a^{-1/2} B_{n-n_0}$. We will say that this type of recurrences are symmetrical around n_0. The value n_0 will play a central role. For simplicity let us consider (3.1), that is, $n_0 = 0$, $a = 1$.

All the solutions of the recurrence (3.1) are symmetrical, that is, they verify $y_n = y_{-n}$ for all $n \in \mathbb{N}$.

Without loss of generality, we can consider that $b_n > 0$ when $n > 0$ (if the sign is opposite, we can consider the recurrence satisfied by $(-1)^n y_n$). When Theorem 2 applies, the minimal solution is alternating for large enough n while the dominant solutions have constant sign. Furthermore, if a solution f_n is alternating as $n \to +\infty$ ($(-1)^n f_n$ with constant sign) then it is alternating for all n and $|f_n|$ decreases as $|n|$ increases; this is seen by applying backward recursion for positive n. In addition, using forward recursion we see that all solutions g_n with $g_0 > 0$, $g_1 > 0$ are positive for all n and therefore dominant; furthermore, $|g_n|$ increases as $|n|$ increases.

Therefore, if g_n is a positive (or negative) dominant solution, $|r_n| = |f_n/g_n|$ reaches its maximum at $n = n_0 = 0$ and it is strictly monotonic for $n > 0$ and $n < 0$. This is the type of situation leading to pseudoconvergence of the associated continued fraction.

For non-integer $n - n_0$ this complete symmetry of the solutions is lost but the change of behaviour around n_0 may remain. The simplest case is that of modified Bessel functions. The recurrence relation

$$y_{\nu+1}(x) - \frac{2\nu}{x} y_\nu(x) - y_{\nu-1}(x) = 0 \tag{3.2}$$

has as a pair of independent solutions $K_\nu(x)$ and $(-1)^{[\nu]} I_\nu(x)$ the first being dominant and the second minimal as $\nu \to +\infty$ (a more standard notation for the minimal solution is $e^{i\pi\nu} I_\nu(x)$, but we prefer to use real notation only). For integer orders, as is true for the general case above described, we have that $K_n = K_{-n}$ and $I_n = I_{-n}$ and the transitory behaviour certainly takes place.

Pseudoconvergence takes place for real orders ν as well. The associated continued fraction, F, reads

$$H(\nu) = \lim_{m\to+\infty} H_m(\nu),\ H_m(\nu) = \frac{1}{b_\nu +}\,\frac{1}{b_{\nu+1} +}\dots\frac{1}{b_{\nu+m-1}}, \tag{3.3}$$

and $b_r = -2r/x$. Although converging to $-I_\nu(x)/I_{\nu-1}(x)$, this CF initially approaches the ratio $K_\nu(x)/K_{\nu-1}(x)$ when ν is negative. After the N-th approximant, $N = [\nu]$, the continued fraction does no longer pseudoconverge to this ratio and, for a large enough number of approximants m, the sequence $\{F_m(\nu)\}$ starts to converge to $-I_\nu(x)/I_{\nu-1}(x)$.

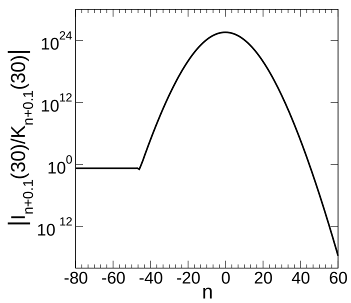

Figure 2 Plot of the ratio $I_{n+0.1}(30)/K_{n+0.1}(30)$ as a function of $n \in \mathbb{Z}$. Several regions can be distinguished

Similarly as happened for the confluent hypergeometric case, the accuracy reached in the pseudoconvergent regime can be very high (see Figure 3, center). Three features take place for the modified Bessel function case which explain the transitory behaviour of the CF (3.3): the central coefficient of the recurrence b_n changes sign at some n_0 ($n_0 = 0$ in this case), the coefficient a_n is negative and the minimal solution keeps the same pattern of signs around n_0 (for the particular case of Fig.2, the minimal solution is alternating when $n > -46$). This features are also shared by the confluent case previously described. In the next section we prove that these three conditions are enough to guarantee the appearance of transitory behaviour.

The transitory behaviour is usually limited to a finite range of parameters. We will not consider the detailed analysis of the parameter regions for which anomalous behaviour takes place but we give some indications for the case of modified Bessel functions. For modified Bessel functions, pseudoconvergence of the CF (3.3) occurs when $\nu < 0$ but the effect tends to disappear for very negative ν because $-I_\nu(x)/I_{\nu-1}(x)$ and $K_\nu(x)/K_{\nu-1}(x)$ become very similar; then, the change from apparent convergence to $K_\nu(x)/K_{\nu-1}(x)$ to true convergence to the ratio of minimal solutions as $\nu \to +\infty$, $-I_\nu(x)/I_{\nu-1}(x)$, becomes less noticeable. This can be observed in Figure 3 (center, right), where the peak preceding convergence to the minimal solution tends to disappear as smaller ν ($\nu < -50$) is considered.

The fact that the ratios $-I_\nu(x)/I_{\nu-1}(x)$, $K_\nu(x)/K_{\nu-1}(x)$ approach each other for very negative $\nu \notin \mathbb{Z}$ can be understood by noticing that

$$I_\nu(x) = I_{-\nu}(x) - \frac{2}{\pi}\sin\nu\pi K_\nu(x) \tag{3.4}$$

together with the fact that $K_\nu(x) = K_{-\nu}(x)$ and $I_\nu(x)/K_\nu(x) \to 0$, $\nu \to \infty$. Additionally, this also shows that $(-1)^{[\nu]}I_\nu(x)$ is dominant as $\nu \to -\infty$ for ν non-integer (the minimal solution as $\nu \to -\infty$ is $(-1)^{[\nu]}I_{-\nu}(x)$).

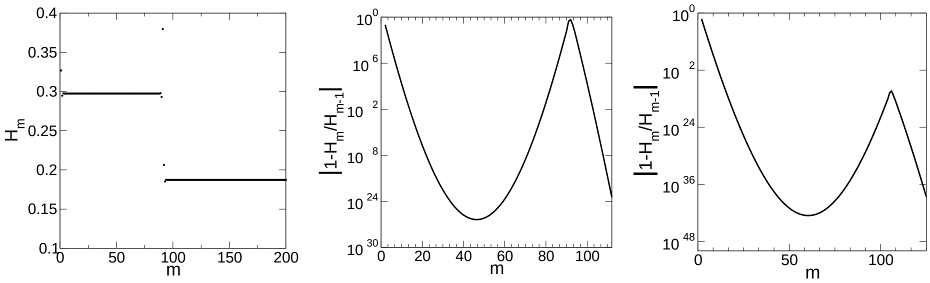

Figure 3 Left: The successive approximants to the CF of Eq. (3.3) for $\nu = -45.9$, $x = 30$. Center: relative deviation $|1 - H_m/H_{m-1}|$ between successive approximants for $\nu = -45.9$, $x = 30$. Right: same but for $\nu = -59.9$, $x = 30$.

This gradual disappearance of transitory effects due to a change in the behaviour of one of the solutions is also observed in the $(+\,+)$ confluent recurrence and the $(+\,+\,+)$ Gauss recurrence to be studied later. The anomalous convergence properties are then limited to finite range of parameters. Also, anomalous convergence will disappear if the solutions of the recurrence enter an oscillatory region (as happens for the $(+\,0)$ confluent recurrence). As commented before, we will not study in detail which are the ranges of parameters for which pseudoconvergence takes place.

In section 4, we will use the modified Bessel function case as a reference for determining the existence of transitory behaviour.

In the next section, we analyze general conditions (shared by the Bessel case) under which anomalous convergence can be expected.

3.2 General symmetrical recurrences

Let us now consider the more general case of recurrences $y_{n+1}+b_ny_n+a_ny_{n-1} = 0$ with $a_n < 0$ and such that b_n changes sign at n_0 (we call this a general symmetrical recurrence). Furthermore, we will assume that the recurrence has a minimal solution f_n as $n \to +\infty$

When Theorem 2 applies, because $a_n < 0$, the minimal and dominant solutions have opposite patterns of signs for large n (one alternating and the other one with constant sign). When, for large enough n, $b_n > 0$ the minimal solution is alternating and when $b_n < 0$ it has constant sign. Furthermore, it is easy to check that the condition $a_n < 0$ necessary holds when the solutions have this pattern of signs (one with constant signs, the other one alternating).

We will analyze the appearance of the following type of transitory behaviour for the quantity:

$$R_n \equiv |r_n| = |f_n/g_n|, \tag{3.5}$$

f_n being minimal.

Definition 1 *Given a TTRR with f_n the minimal solution as $n \to +\infty$, a dominant solution g_n is transitorily minimal for $n \leq n_t$ if $\{R_n\}$ is an increasing sequence for $n \leq n_t$ ($R_{n-1} < R_n$) and decreasing for $n \geq n_t + 1$ ($R_n > R_{n+1}$)*

We will consider that b_n may change its sign only once. Provided that the pattern of signs of the minimal solution does not change, we will check that transitory behaviour takes place if and only if b_n changes of sign and that, furthermore, the change of sign of b_n coincides with the change of tendency of $R_n = |f_n/g_n|$, g_n being a dominant solution with a pattern of signs contrary to the minimal solution. Therefore, the behaviour observed for the modified Bessel function and the confluent $(++)$ recurrences, are general for symmetrical recurrences.

First, we prove that when transitory minimal solutions exist the change of sign of b_n coincides with the change of behaviour of the functions.

Theorem 3 *Let $y_{n+1} + b_n y_n + a_n y_{n-1} = 0$ with $a_n < 0$. Let f_n and g_n be solutions with fixed and opposite pattern of signs (one of them with constant sign and the other one alternating). If $R_n = |f_n/g_n|$ reaches an absolute extremum at $n = n_0$ and it is strictly monotonic when $n \geq n_0$ and $n \leq n_0$ then the sign of b_n for $n \leq n_0 - 1$ is opposite to the sign for $n \geq n_0 + 1$.*

Proof. Let us assume, for instance, that f_n is alternating and that R_n reaches a maximum at n_0. We define

$$\Delta_n = \left|\frac{g_{n+1}}{g_n}\right| - \left|\frac{f_{n+1}}{f_n}\right| = \frac{g_{n+1}}{g_n} + \frac{f_{n+1}}{f_n}. \tag{3.6}$$

Using the recurrence relation we have

$$\Delta_n + \lambda_n \Delta_{n-1} = -2b_n, \tag{3.7}$$

where

$$\lambda_n \equiv a_n \frac{f_{n-1}}{f_n}\frac{g_{n-1}}{g_n} > 0.$$

Because $R_{n-1} < R_n$ for $n \leq n_0$ and $R_n > R_{n+1}$ for $n \geq n_0$ then $\Delta_n < 0$ if $n \leq n_0 - 1$ and $\Delta_n > 0$ if $n \geq n_0$. From Eq. (3.7) $b_n > 0$ if $n \leq n_0 - 1$ and $b_n < 0$ in $n \geq n_0 + 1$. $\square$

Now, we prove that the existence of transitorily minimal solutions is guaranteed when the minimal solution of a symmetrical recurrence maintains its pattern of signs. We also prove that the change of behaviour of the solutions takes place at the point where b_n changes sign.

In the sequel, we will say that b_n (or any other function depending on n) changes sign at $n = n_0$ if its sign when $n \leq n_0$ (excluding $n = n_0$ when $b_{n_0} = 0$) is opposite to the sign when $n \geq n_0 + 1$.

Theorem 4 *Let $y_{n+1} + b_n y_n + a_n y_{n-1} = 0$ be a recurrence such that, for $n \geq n^-$, $a_n < 0$ and b_n changes sign at $n_0 > n^- + 1$. Suppose that there exists a solution f_n with fixed pattern of signs for all $n \geq n^-$, the pattern being alternating if $b_n < 0$ for large n or with constant sign if $b_n > 0$ for large n (f_n may be minimal). Let g_n be any solution (not minimal) such that*

$$\frac{g_{n_0+1}}{g_{n_0}} = -\gamma \frac{f_{n_0+1}}{f_{n_0}},\ \gamma > 0,$$

and let $R_n = |f_n / g_n|$, then for $n \geq n^-$ the following holds depending on the value γ:

1. *If $\gamma > 1$ then $R_n < R_{n_0}$ if $n \neq n_0$; $R_{n-1} < R_n$ if $n \leq n_0$.*
2. *If $\gamma < 1$ then $R_n < R_{n_0+1}$ if $n \neq n_0 + 1$; $R_n > R_{n+1}$ if $n \geq n_0 + 1$.*
3. *If $\gamma = 1$ then $R_n < R_{n_0} = R_{n_0+1}$ if $n \neq n_0, n_0 + 1$; $R_{n-1} < R_n$ if $n \leq n_0$ and $R_n > R_{n+1}$ if $n \geq n_0 + 1$.*

Proof. Let us consider the case for which $b_n < 0$ for large n (and therefore f_n has alternating sign). Clearly, it is enough to consider the cases with starting values $G_0 \equiv (g_{n_0}, g_{n_0+1}) = (|f_{n_0}|, \lambda |f_{n_0+1}|)$ for the first case and $G_0 = (\lambda |f_{n_0}|, |f_{n_0+1}|)$ for the second, $\lambda > 1$.

We analyze the first possibility; for the second case, the proof is similar and the third case is also obtained in a very similar way. Let us consider the generation of the solution $y_n = g_n - f_n$, which, given G_0, has starting values $y_{n_0} \geq 0$, $y_{n_0+1} \geq 0$ (not both equal to zero). Forward recursion

$$y_{n+1} = -b_n y_n - a_n y_{n-1}$$

for $n \geq n_0 + 1$ provides positive values for y_n, $n \geq n_0 + 2$ because $a_n < 0$, $b_n < 0$ if $n \geq n_0 + 1$. Similarly, the solution $y_n = g_n + f_n$ is also positive. Therefore $g_n - f_n > 0$ and $g_n + f_n > 0$ when $n \geq n_0 + 2$, that is $|f_n| < g_n = |g_n|$ for $n > n_0$ (also for $n = n_0 + 1$ because of the definition of g_{n_0}).

Considering now backward recursion:

$$y_{n-1} = -\frac{1}{a_n}(b_n y_n + y_{n+1}),$$

it is also clear that $g_n - f_n > 0$ and $g_n + f_n > 0$ when $n \leq n_0 - 1$. Therefore, $|f_n| < |g_n|$ if $n < n_0$.

In summary, $R_n = |f_n/g_n| < 1 = R_{n_0}$ for $n \neq n_0$, which completes the proof of the first case.

In addition, because $R_n < R_{n_0}$ when $n < n_0$, then in particular

$$\left| \frac{g_{n_0}}{g_{n_0-1}} \right| > \left| \frac{f_{n_0}}{f_{n_0-1}} \right| .$$

Therefore, we can repeat the same argument as before and we have that $R_n < R_{n_0-1}$, $n < n_0 - 1$, and by induction $R_{n-1} < R_n$, $n \leq n_0$.

The rest of cases can be proved in a similar way. $\square$

When Perron's theorem is positive with respect to the existence of a minimal solution, the f_n in the previous theorem is necessarily minimal because the rest of solutions (all of them dominant) have asymptotically the opposite pattern of signs. On the other hand, it is easy to see that the minimal solutions do not change their pattern of signs (at least when Perron's theorem provides information) for $n \geq n_0$.

Theorem 5 *let $y_n + b_n y_{n-1} + a_n y_{n-1} = 0$ with negative a_n and such that b_n only changes sign at $n = n_0$. Then, if $b_n < 0$ ($b_n > 0$) for $n > n_0$ and the minimal solution is alternating (of constant sign) for large n, then it is alternating for all $n \geq n_0$ (of constant sign).*

Proof. Let us, for instance, consider the case $b_n > 0$ for large n, then $y_{n-1} = -(y_{n+1} + b_n y_n)/a_n$, which, for $n > n_0$, is positive if y_n and y_{n+1} are positive. And because, if Perron's theorem applies, the solution is positive (or negative) for large n, the theorem is proved. $\square$

The situation in which the pattern for the minimal solution is kept for $n < n_0$ depends on the specific coefficients a_n, b_n, but it is a common situation (as the confluent and the modified Bessel functions case show and as further examples will illustrate). When this takes place, the minimal solution as $n \to +\infty$ does no longer behave like a minimal solution when $n < n_0$ and transitory minimal solutions exist (Theorem 4). From the error analysis of section 2.1, we conclude that when there exist solutions which are transitorily minimal in $n^- \leq n < n_0$ a CF for y_n/y_{n-1} will approximate these solutions instead of the minimal solution for the first $[n_0 - n]$ approximants. The next theorem summarizes part of the results of this section combined with the error analysis of section 2.1:

Theorem 6 *Let $y_{n+1} + b_n y_n + a_n y_{n-1} = 0$, $a_n < 0$ and b_n changing sign at n_0. Let f_n be minimal and with a fixed pattern of signs for $n \geq n^-$ ($n^- < n_0 - 1$) and g_n a second independent solution such that $g_{n_0} f_{n_0+1} + g_{n_0+1} f_{n_0} = 0$. Let k be such that $n^- < k < n_0$. Let $|\epsilon_r^f(k,m)|$ be the relative deviation from f_k/f_{k-1} of the m-th approximant of the CF and $|\epsilon_r^g(k,m)|$ the analogous deviation from g_k/g_{k-1}.*

Let $n_k \in \mathbb{N}$ be the only value greater that n_0 such that $\dfrac{R_n}{R_k} - 1$ changes sign at $n = n_k$. Then:

1. If $m < n_k - k$, $|\epsilon_r^g(k,m)| < |\epsilon_r^f(k,m)|$

2. If $m > n_k - k$, $|\epsilon_r^f(k,m)| < |\epsilon_r^g(k,m)|$

The smallest error for pseudoconvergence can be bounded in the following way:

$$\min_m |\epsilon_r^g(k,m)| \leq \frac{R_k + R_{k-1}}{R_{n_0} - R_{k-1}}. \tag{3.8}$$

We don't prove this theorem, which is a direct consequence of Theorem 4 and the error formulas of Section 2.1.

Part of the previous result concerning the behaviour of the approximants of the associated continued fraction can also be understood from an elementary analysis of the CF. It is easy to see that when the pattern of signs of the minimal solution does not change for $n < n_0$ and b_n changes sign the first approximants can not initially approach the ratios of minimal solutions. As before, let us suppose that b_n is positive for $n \leq n_0$ (and negative for $n > n_0$) and therefore the minimal solution is asymptotically alternating (and we suppose that the pattern is kept for $n < n_0$). Taking into account that in the successive approximants of the continued fraction the coefficients a_n are always negative while the coefficients b_n are positive when $n \leq n_0$ the approximants $H_m \equiv H_m(k)$ of the associated continued fraction are

$$H_m = \frac{\alpha_k}{\beta_k +} \frac{\alpha_{k+1}}{\beta_{k+1} +} \cdots \frac{\alpha_{k+m-1}}{\beta_{k+m-1}},$$

with $\alpha_n = -a_n > 0$, $\beta_n = b_n > 0$ if $n < n_0$. Then, the approximants H_m are interlaced following the scheme

$$0 < H_2 < H_4 < H_6 < \ldots < H_5 < H_3 < H_1$$

when $m \leq n_0 - k + 1$. Therefore, they are all inside a positive interval and they do not approach the ratios for the minimal solution f_n, which is alternating and then $f_k/f_{k-1} < 0$.

Similarly as the existence of minimal solution is an intrinsic property of the recurrence relation, we see that the existence of transitory minimal solutions can be deduced from sign properties of the coefficients of the recurrence relation and of the minimal solution.

4 Further examples

Pseudoconvergence is a quite ubiquitous property of hypergeometric recursions and it is not restricted to Gaustchi's phenomenon [3]. In this section we identify additional examples of pseudoconvergent transitory behaviour by using the results of the previous section. Both the confluent and the Gauss hypergeometric recursion present this type of behaviour. As we will see, Gautschi's case can be also interpreted as the confluent limit of a the Gauss hypergeometric $(+\,+\,+)$ recurrence.

4.1 The confluent recurrences and Temme's numerical instability revisited

As we studied earlier, the recurrence relation satisfied by $f_n = M(a+n, c+n, x)$, which is minimal when $n \to +\infty$, shows pseudoconvergence to a ratio for transitory minimal solutions, particularly when x is large. This can be understood by considering the signs of the coefficients of the recurrence satisfied by f_n (Eq. (2.1)).

For a and c positive, the b_n coefficient changes sign at $n_0 = [x - c]$, and the coefficient a_n is negative. This, together with the fact that the solution f_n maintains its sign pattern, shows that transitory behaviour will take place.

The recurrence for large x resembles that of modified Bessel functions. Let us denote $\lambda = x + 1 - c$ and let us shift n by considering the replacement $\hat{y}_n = y_{n+\lambda}$, then $\hat{y}_n$ satisfies the recurrence

$$\hat{y}_{n+1} + \hat{b}_n \hat{y}_n + \hat{a}_n y_{n-1} = 0 \tag{4.1}$$

with

$$\hat{b}_n = \frac{n}{x}\phi(n,x)\,,\, \hat{a}_n = -\frac{n+x}{x}\phi(n,x)\,,\, \phi(n,x) = \left(1 + \frac{a-c}{n+x+1}\right)^{-1}.$$

Therefore $\hat{b}_n(x) = \frac{n}{x}(1 + \mathcal{O}(x^{-1}))$ and $\hat{a}_n(x) = -1 + \mathcal{O}(x^{-1})$ which, in first order, is essentially the recurrence for modified Bessel functions. Thus, for large enough x, we can expect noticeable transitory behaviour for $\hat{y}_n$ around $n = 0$; equivalently, we can expect transitory behaviour around $n = n_0$ for f_n and g_n, with a reversion in their roles. Thus, the continued fraction for the ratio of minimal solutions f_n/f_{n-1} will display anomalous convergence when $n < n_0$ as we have already shown in section 2 (and similarly as happens for the continued fraction for modified Bessel functions, $I_\nu(x)/I_{\nu-1}(x)$, when $\nu < 0$).

The $(++)$ recurrence is just one of the various hypergeometric recursions where this transitory behaviour takes place. Also the recurrence for the set of functions $y_n = M(a+n, c, x)$ presents this type of behaviour, indeed, the recurrence has coefficients

$$b_n = -\frac{2n + 2a + x - c}{a+n}\,,\, a_n = \frac{a+n-c}{a+n}. \tag{4.2}$$

The coefficient a_n is negative when $c > a + n$; keeping this behaviour for a large range in n requires c large. The b_n coefficient changes sign at $n_0 = [(c - x - 2a)/2]$. A second solution is $f_n = \Gamma(1+a+n-c)\, U(a+n, c, x)$. Perron's theorem is inconclusive with respect to the existence of minimal solutions. From asymptotic information, it is easy to see that f_n is minimal and g_n is dominant [8]. We observe that, around n_0, g_n is positive and f_n alternating. Therefore, transitory behaviour around n_0 will take place.

As before, we can relate this case to modified Bessel functions. The shifted solutions $\hat{y}_n = y_{n+\lambda}$, $\lambda = (c - x - 2a)/2$, satisfy a new recurrence (4.1) with coefficients

$$\hat{b}_n = -\frac{4n}{c-x+2n} = -\frac{4n}{c}(1+\mathcal{O}(c^{-1})),$$
$$\hat{a}_n = -\frac{c+x-2n}{c-x+2n} = -1+\mathcal{O}(c^{-1}), \tag{4.3}$$

and for large c the recurrence is similar to the modified Bessel functions case. Transitory behaviours will occur around n_0 and the associated continued fraction will display pseudoconvergence to the wrong limit, particularly for large c.

This also means, as happens with the Bessel functions and with the $(++)$ recurrence, that, for values smaller that n_0 the minimal solution will cease to behave as minimal and that for such values of n backward recurrence for the minimal solution will be bad conditioned, at least transitorily. Similarly, forward recursion for certain dominant solutions will be bad conditioned. In Section 5.5 we will consider the problem of the condition of the recurrences in more detail. These problems were already noticed by N. M. Temme [8] in connection with the computation of the confluent function $U(a,c,x)$.

Differently from the $(++)$ case, we will not analyze in detail the degree of pseudoconvergence. We postpone this type of analysis, also for the Gauss hypergeometric recursions, for a later paper. Numerical experiments show that the set of functions $g_n = M(a+n,c,x)$ is a transitorily minimal solution companion of the minimal solution $f_n = \Gamma(1+a+n-c)U(a+n,c,x)$, as Figure 4 illustrates. Notice that, indeed, the accuracy of the continued fraction is governed by the value of $|r_N| = |f_N/g_N|$.

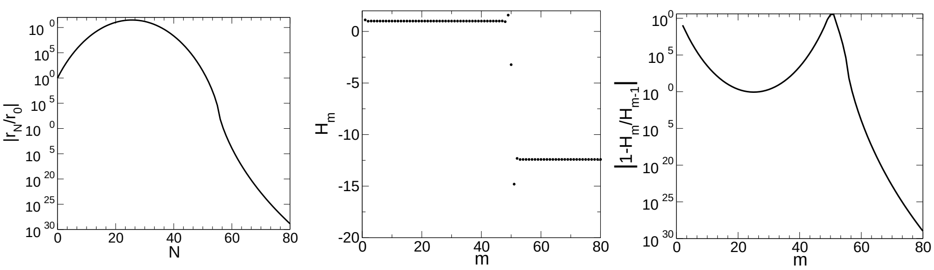

Figure 4 Left: the function r_N is shown for the values $a = 4.4$, $c = 60.3$, $x = 0.3$. Center: successive approximants of the continued fraction. Right, the relative deviation between consecutive approximants.

Let us notice that $\lim_{n\to+\infty} a_n = 1 > 0$, which means that the negativity condition for a_n coefficient is violated for large enough n. However, what is important is that the condition $a_n < 0$ is met in a wide range of n-values around n_0, which is the case when c is large. This, example, however, is indicating that the condition $a_n < 0$ can be relaxed.

4.2 The Gauss hypergeometric case

Transitory behaviour also takes place for recurrences satisfied by families of Gauss hypergeometric functions $y_n = {}_2\mathrm{F}_1(a+\epsilon_1 n, b+\epsilon_2 n; c+\epsilon_3 n; x)$ with ϵ_i integer numbers (not all equal to zero).

In [4], the existence of minimal solutions in the complex plane was investigated when $|\epsilon_i| \leq 1$. It was shown that all the possible cases are reducible to four cases; the cases selected as representative were $(+\,+\,0)$, $(0\,0\,+)$, $(+\,+\,-)$ and $(+\,0\,-)$. As we will next see, the recurrence $(0\,0\,+)$ (or related cases like $(+\,+\,+)$), $(+\,+\,-)$ and $(+\,0\,-)$ are candidates for the appearance of transitory behaviour and they indeed show pseudoconvergence of the associated CF. We will mainly concentrate on the recurrence $(+\,+\,+)$ which has as a limiting case the $(+\,+)$ confluent hypergeometric recursion (that is, Gautschi's pseudoconvergence).

The Gauss hypergeometric functions satisfy recurrence relations with coefficients a_n and b_n with finite limits as $n \to \pm\infty$. Let us denote $\alpha = \lim_{n\to+\infty} a_n$ and $\beta = \lim_{n\to+\infty} b_n$. From Perron's theorem, the recurrence admits a minimal solution when the roots λ_1, λ_2 of the characteristic equation $\lambda^2 + \beta\lambda + \alpha = 0$ have different modulus.

Of the four basic cases, only one has a value α which is positive for all real x, namely, the recurrence $(+\,+\,0)$. Because we are interested in the cases for which $a_n < 0$, we will not consider this recurrence; however, as we already noticed for the confluent $(+\,0)$ recurrence, it is not necessary that $a_n < 0$ for all n but only that this holds around the change of sign for b_n. The other three recurrences verify $\alpha < 0$ in $(0,1)$. Furthermore, in the three cases, the curves in the complex plane $|\lambda_1| = |\lambda_2|$ (which divide the complex plane in disjoint regions with different minimal solutions) intersect the interval $(0,1)$. These intersection points are $x_0 = 1/2$ for the $(0\,0\,+)$ (or the related $(+\,+\,+)$ recurrence), $x_0 = (-5+3\sqrt{3})/4$ for $(+\,+\,-)$ and $x_0 = 3-2\sqrt{2}$ for $(+\,0\,-)$. Therefore these recurrences have the following form:

$$y_{n+1} + (f(x) - g/n + \mathcal{O}(n^{-2}))y_n + (a + \mathcal{O}(n^{-1}))y_{n-1} = 0, \tag{4.4}$$

with $f(x_0) = 0$, $a < 0$; g is a function of all the parameters (except n). The coefficient b_n changes sign at $n_0 \simeq g/f(x)$ which will be large if x is close to x_0. Therefore, for x close enough to x_0 the shifted solutions $\hat{y}_n = y_{n+n_0}$ verify:

$$\hat{y}_{n+1} + n\frac{f(x)}{n_0}(1 + O(n_0^{-1}))y_n + a(1 + \mathcal{O}(n_0^{-1}))y_{n-1} = 0 \tag{4.5}$$

for $|n| << |n_0|$. This is, again, similar to the modified Bessel function case and transitory behaviour leading to pseudoconvergence can be expected.

In this paper we will not describe in detail the $(+\,+\,-)$ and $(+\,0\,-)$ cases, but later we provide numerical evidence for transitory behaviour. We only describe in detail the more simple case of the $(+\,+\,+)$ recurrence. This case is easy to analyze and recovers Gautschi's anomalous convergence by means of a confluent limit.

4.2.1 The Gauss $(+ + +)$ recurrence

For simplicity, we consider a recurrence equivalent to $(+ + +)$, namely, the recurrence relation satisfied by the set

$$y_1(x) = \frac{1}{\Gamma(c+n)}\ {}_2\mathrm{F}_1\left(\begin{array}{c} a+n, b+n \\ c+n \end{array};x\right).$$

The recurrence relation has coefficients

$$a_n = \frac{1}{x(x-1)(a+n)(b+n)}\,,\ b_n = \frac{((a+b+2n-1)x-c-n+1)}{x(x-1)(a+n)(b+n)}. \tag{4.6}$$

We choose the following pair of solutions for $x \in (0,1)$:

$$y_{1,n}(x) = \frac{1}{\Gamma(c+n)}\ {}_2\mathrm{F}_1\left(\begin{array}{c} a+n, b+n \\ c+n \end{array};x\right) \tag{4.7}$$

and

$$y_{4,n}(x) = \frac{(-1)^n}{\Gamma(a+b+1-c+n)}\ {}_2\mathrm{F}_1\left(\begin{array}{c} a+n, b+n \\ a+b+n+1-c \end{array};1-x\right). \tag{4.8}$$

The numbering of solutions is the same as in [4] (but with the extra $1/\Gamma(c+n)$). It was shown [4] that the function $y_{1,n}(x)$ is minimal if $x < 1/2$, while the function $y_{4,n}(x)$ is minimal if $x > 1/2$. At $x = 1/2$, where Perron's theorem is inconclusive, it is easy to check explicitly the character of the solutions. Given:

$$y_n(d,x) \equiv \frac{1}{\Gamma(d+n)}\ {}_2\mathrm{F}_1\left(\begin{array}{c} a+n, b+n \\ d+n \end{array};x\right) = \sum_{k=0}^{\infty} \frac{(a+n)_k(b+n)_k}{k!\Gamma(d+n+k)}x^k \tag{4.9}$$

and taking into account that

$$\lim_{p\to+\infty} \frac{\Gamma(p)}{\Gamma(p+\epsilon)} = 0 \text{ if } \epsilon > 0, \tag{4.10}$$

we have that, if $d_1 > d_2$

$$\lim_{n\to+\infty} \frac{y_n(d_1,x)}{y_n(d_2,x)} = 0 \tag{4.11}$$

for $x \in (0,1)$. Now, because $|y_1(1/2)| = y_n(c, 1/2)$ and $|y_4(1/2)| = y_n(a+b+1-c, 1/2)$, then for $x = 1/2$ and defining

$$\lambda = a+b+1-2c \tag{4.12}$$

we have:

1. If $\lambda > 0$ $y_{1,n}(1/2)$ is dominant and $y_{4,n}(1/2)$ minimal.

2. If $\lambda < 0$ $y_{4,n}(1/2)$ is dominant and $y_{1,n}(1/2)$ minimal.

3. If $\lambda = 0$ there is no minimal solution because $y_{4,n}(1/2) = (-1)^n y_{1,n}(1/2)$.

In any case, $\{y_{1,n}(x), y_{4,n}(x)\}$ is a numerically satisfactory pair in $(0, 1)$.

We observe that $a_n < 0$ if $x \in (0, 1)$, when $a + n > 0$, $b + n > 0$. In addition, the coefficient b_n changes sign at

$$n_0 = \frac{c - 1 - (a + b - 1)x}{2x - 1} = -\frac{1}{2}\left((a + b - 1) + \frac{\lambda}{2x - 1}\right). \tag{4.13}$$

When λ is positive n_0 becomes larger as x approaches $1/2$ with values smaller that $1/2$; contrary, when $\lambda < 0$ and $x \to 1/2^+$ then $n_0 \to +\infty$.

Considering in particular the case $\lambda > 0$, this is consistent with the fact that $y_{1,n}(1/2)$ is dominant and $y_{4,n}(1/2)$ minimal while for $x < 1/2$ their asymptotic behaviour is the opposite: $y_{1,n}(x)$ is minimal and $y_{4,n}(x)$ dominant.

This clarifies the appearance of a transitorily minimal solution for $\lambda > 0$ and $x < 1/2$ but close to $1/2$: $y_{4,n}(x)$ is transitory minimal because it is recessive for forward recursion with respect to $y_{1,n}(x)$ as long as $n < n_0$. The closer x is to $1/2$ the larger n_0 is and when $x = 1/2$ then $n_0 = \infty$ and $y_{4,n}(1/2)$ becomes minimal for all positive n.

Therefore the transitory behaviour is a remnant of the behaviour at $x = 1/2$ (also in the case $\lambda < 0$). An example exhibiting pseudoconvergence is given in Figure 5.

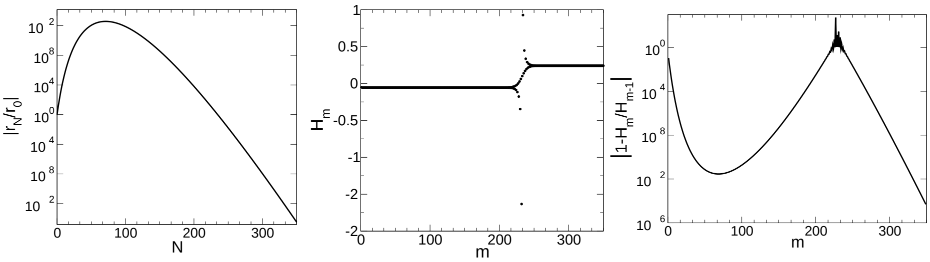

Figure 5 Left: Plot of $|r_N/r_0|$ ($r_N = f_N/g_N$, $f_N = y_{1,N}$, $g_n = y_{4,N}$) for $a = 20$, $b = 15.5$, $c = 0.6$, $x = 0.4$. Center: successive approximants of the associated CF. Right: relative deviation between successive approximants.

In Figure 5, we observe that the reversion of tendency seems to be kept for x not so close to $x = 1/2$, of course depending on the values of the parameters. But, of course, as x approaches $1/2$ pseudoconvergence is more noticeable. This explains the transitory behaviours observed for the $(++)$ confluent recursions, as we next show.

4.2.2 The $(++)$ confluent limit

The pseudoconvergence for the Gauss hypergeometric case can be understood as the consequence of the change of behaviour of the solutions $y_{1,n}$ and $y_{4,n}$ as the line $x = 1/2$ is crossed.

It is tempting to consider the confluent limit

$$M(a,c,x) \equiv \ {}_1\mathrm{F}_1(a;c;x) = \lim_{b\to\infty} \ {}_2\mathrm{F}_1\left(\begin{array}{c} a,b \\ c \end{array} ; \frac{x}{b}\right) \tag{4.14}$$

relating the minimal solutions of the $(++)$ hypergeometric recurrence f_n (Eq. (2.1)) with the minimal solution of the Gauss recurrence $(+\ +\ +)$ for $0 < x < 1/2$, $y_{1,n}(x)$ (Eq. (4.7), but without the factor $1/\Gamma(c+n)$)

$$f_n(x) = \lim_{b\to+\infty} y_{1,n}(x/b). \tag{4.15}$$

We follow the notation $y_{i,n}(x)$ of [4].

Using well known identities satisfied by Gauss hypergeometric functions a second solution of the recurrence satisfied by $y_{1,n}$, $y_{4,n}$ can be written:

$$y_{4,n} = \frac{(-1)^n x^{-(a+n)}\Gamma(1+b)}{\Gamma(a+b+1-c+n)} \ {}_2\mathrm{F}_1\left(\begin{array}{c} a+n, a+1-c \\ a+b+n+1-c \end{array} ; 1-\frac{1}{x}\right), \tag{4.16}$$

where the factors not depending on n can be dropped or they can be included if convenient. The factor $\Gamma(1+b)$ has been included for convenience.

Now, considering the formal identity

$$\lim_{c\to\infty} \ {}_2\mathrm{F}_1\left(\begin{array}{c} a,b \\ c \end{array} ; cz\right) = \ {}_2\mathrm{F}_0(a,b;;z), \tag{4.17}$$

which is true term by term but leads to a divergent series, we have, formally,

$$\begin{aligned} \lim_{b\to\infty} y_{4,n}(x/b) &= (-1)^n x^{-a-n} \ {}_2\mathrm{F}_0\left(a+n, a+1-c;;-\frac{1}{x}\right) \\ &\sim (-1)^n U(a+n, c+n, x) = g_n(x) \text{ as } x \to +\infty, \end{aligned} \tag{4.18}$$

where g_n is a second solution (dominant) of the $(++)$ confluent recurrence.

This explains the behaviour for the $(++)$ confluent recurrence in terms of the Gauss $(+++)$ recurrence. When considering the confluent limit $b \to +\infty$ in (4.15) and (4.18), the transitory behaviour for the Gauss functions close to $x = 1/2$ transforms to transitory behaviour for x large. The minimal solution is the confluent limit of the minimal solution for the Gauss case, while the transitorily minimal solution is the confluent limit of the corresponding transitory solution for the Gauss case. From (4.13), the change of behaviour takes place at

$$\lim_{b\to+\infty} n_0(x/b) = \lim_{b\to+\infty} \frac{c-1-(a+b-1)x/b}{2x/b-1} = x+1-c.$$

For the Gauss case, the identification of a pseudominimal solution for $n < n_0$ was quite obvious: the dominant solution which becomes minimal at the transition point $x = 1/2$ is pseudominimal. A corresponding selection (of course, not unique) for the confluent case becomes now also obvious by taking the confluent limit. Not all cases discussed in this paper are equally clear, particularly the confluent $(+\,0)$ recurrence. For the other two Gauss cases, it appears that similar arguments as for $(+\,+\,+)$ holds while for the Bessel case the identification of a transitory minimal solution was immediate.

4.2.3 Other Gauss recurrences

In this paper, we don't consider a detailed analysis of the recurrences $(+\,0\,-)$ and $(+\,+\,-)$. Numerical experiments show that, indeed, transitory behaviour is present for both recurrences. For example, for $(+\,0\,-)$ we have considered the minimal solution on the left of $x_0 = 3 - 2\sqrt{2}$ (denoted by $y_{2,n}$ in [4]), and the minimal solution on the right (denoted by $y_{3,n}$); convergence to the ratio of dominant solutions $y_{3,n}$ with more than double precision accuracy for the values $x = x_0 - 0.01$, $a = 2.5$, $b = 6.8$, $c = 6.5$ is reached with less than 50 iterations; only after 600 iterations does the CF start to converge to the ratio of minimal solutions $y_{2,n}/y_{2,n-1}$. The $y_{3,n}$ solution appears to be pseudominimal for $x < x_0$ and close to $x_0 = (-5 + 3\sqrt{3})/4$. Experiments also show that for the $(+\,+\,-)$ similar results (for the corresponding solutions $y_{2,n}$ and $y_{3,n}$ in [4]) can be obtained for example for the set $a = 23.5$, $b = 22.8$, $c = 10$, $x = x_0 - 0.01$: better than double precision for the pseudoconvergence regime with less than 40 iterations, convergence to the ratio of minimal solutions after 200 iterations. The detailed analysis of these cases will be considered in later papers.

5 Error analysis for finite precision arithmetic

A problem related to those of convergence and pseudoconvergence of the associated continued fraction are the problems of condition of the recurrence relation and numerical stability for finite precision arithmetic.

The convergence of the continued fraction associated with a TTRR means that the recurrence admits a minimal solution. As discussed before, the convergence of the continued fraction is a consequence of the fact that the ratios of the numerical solutions for the TTRR, y_k, are such that the ratios y_k/y_{k-1} approach the ratios f_k/f_{k-1} when applying backward recurrence starting for any values of y_N and y_{N-1} (not both equal to 0) for high enough $N >> k$. This implies that backward recursion for a minimal solution is a well conditioned process.

Similarly, forward recursion is well conditioned for dominant solutions and, for any *numerical* starting values y_n, y_{n-1} (with $y_n y_{n-1} \neq 0$) we have that

$$\lim_{k\to+\infty} \frac{y_k/y_{k-1}}{g_k/g_{k-1}} = 1, \tag{5.1}$$

where g_k is a dominant solution. For forward recursion, all numerical solutions become dominant for large enough orders k. Forward recursion for minimal solutions should never be used.

This, of course, does not mean that only this asymptotic information is enough for a stable application of a TTRR because, as we have analyzed, the behaviour for finite orders may be opposite to the asymptotic behaviour when the central coefficient in the TTRR changes sign. This may result, for instance, in a loss of precision for the forward (backward) evaluation of dominant solutions (minimal), at least transitorily.

We will first study the possible loss of precision both in the ratios of solutions of consecutive orders y_k/y_{k-1}, focusing on the backward computation of the minimal solution (equivalent to the evaluation of the associated the CF). Finally, we study the errors for the computation of numerical values of the solution y_n, particularly for pseudominimal solutions.

As in the rest of the article, we will consider recurrence relations

$$y_{n+1} + b_n y_n + a_n y_{n-1} = 0$$

with b_n changing sign at n_0 and such that all conditions are met which guarantee the appearance of anomalous behaviour.

5.1 Errors in the numerical evaluation of the continued fraction

Let us first consider the backward evaluation of f_k/f_{k-1} being f_n the minimal solution and starting from f_n, $n = N, N+1$, $N > n_0$; this is equivalent to the computation of the m-the approximant of the associated CF ($m = N - k$). Backward recursion is well conditioned for $n > n_0$ and then we can assume that the numerical solution y_n, $n = n_0, n_0 + 1$ is accurately computed. When fixed precision arithmetic is used, we have

$$y_n = \epsilon_1 r_{n_0} g_n + (1 + \epsilon_2) f_n, n = n_0, n_0 + 1 \tag{5.2}$$

where ϵ_1 and ϵ_2 are of order machine-ϵ and g_n is dominant but transitorily minimal (for $n \leq n_0$). Then, because $R_n = |r_n| = |f_n/g_n|$ decreases as $n < n_0$ decreases, backward recurrence for $n < n_0$ increases the unwanted component g_n. The relative deviation of y_k/y_{k-1} from the ratio of minimal solutions can then be bounded as follows:

$$|\epsilon_k| \equiv \left| 1 - \frac{y_k}{y_{k-1}} \frac{f_{k-1}}{f_k} \right| > F_k \frac{|1 - r_{k-1}/r_k|}{|1 + \epsilon_2| + F_k} \approx F_k \frac{C_k}{1 + F_k}, \tag{5.3}$$

where

$$F_k = |\epsilon_1| \frac{R_{n_0}}{R_{k-1}} \tag{5.4}$$

and $C_k = |1 - r_{k-1}/r_k| = 1 + R_{k-1}/R_k \approx 1$ when f_n and g_n have opposite patterns of signs; in any case, $C_k \neq 0$ because f_n and g_n are independent solutions.

The accuracy is essentially determined by the factor F_k and the smaller k is the larger the loss of accuracy. Given the equivalence between backward computation of ratios and approximants of the continues fractions we deduce that the accuracy reachable for the evaluation of f_k/f_{k-1} is approximately machine-ϵ multiplied by R_{n_0}/R_{k-1}; because of the monotonicity, the loss of accuracy increases when smaller values of k, $k < n_0$, are considered. Notice also that the best accuracy reachable in the pseudoconvergent region is well approximated by R_k/R_{n_0}. Therefore, the loss of accuracy in the computation of ratios of minimal solutions is approximately the inverse of the attainable accuracy in the pseudoconvergent region. When the CF pseudoconverges with the full precision available in the computer we can expect total loss of accuracy for the computation of ratios of minimal solutions from the CF, no matter how many approximants are considered.

5.2 Errors in the numerical computation of y_n

Let us consider the forward numerical evaluation of the pseudominimal solution starting with $n < n_0$. We take the starting numerical values $y_n \simeq g_n$, $n = m, m-1$, $m < n_0$, which we write

$$y_n = \epsilon_1 f_n + (1+\epsilon_2) r_m g_n, \tag{5.5}$$

ϵ_1, ϵ_2 being small numbers (of order machine-ϵ). Then, if $R_n = |f_n/g_n|$ initially increases loss of precision in the forward evaluation of the dominant solution will take place. Furthermore, for finite precision arithmetic all the accuracy will be lost when

$$R_m/R_n < |\epsilon_1| \simeq \epsilon, \tag{5.6}$$

ϵ being machine-ϵ, because the second term in (5.5), that is, the term corresponding to the solution we intend to compute, becomes smaller than the first one. This is shown in Figure 6, where loss of precision in the forward evaluation of a dominant (transitorily minimal) solution is observed for $n < n_0$. We compute the recurrence using Fortran in double precision and compare with the results produced by direct computation using Maple.

As illustrated in Figure 6, when not all the accuracy is lost at $n = n_0$, the accuracy in the results is regained for sufficiently large n (left figure, solid line). Even more, also when the results for the dominant solution are totally inaccurate around $n = n_0$, accuracy can also be regained, at least partly (dotted line).

Notice that the ratios y_n/y_{n-1} in fact converge to g_n/g_{n-1} as $n \to +\infty$; therefore, the loss of precision, if any, is due to loss of significant digits in a global factor. This can be understood by writing the numerical values y_{n_0}, y_{n_0+1} as a combination of the independent pair $\{f_n, g_n\}$:

$$\begin{pmatrix} y_{n_0} \\ y_{n_0+1} \end{pmatrix} = \frac{1}{C_{n_0}[f,g]} \left[C_{n_0}[y,g] \begin{pmatrix} f_{n_0} \\ f_{n_0+1} \end{pmatrix} + C_{n_0}[f,y] \begin{pmatrix} g_{n_0} \\ g_{n_0+1} \end{pmatrix} \right], \tag{5.7}$$

where the Casorati determinants are given by

$$C_{n_0}[v,w] \equiv \begin{vmatrix} v_{n_0} & w_{n_0} \\ v_{n_0+1} & w_{n_0+1} \end{vmatrix}. \tag{5.8}$$

Then, if $|r_n| = |f_n/g_n|$ initially increases because f_n is transitorily dominant for $n < n_0$ we will have that the numerical solution (y_{n_0}, y_{n_0+1}) will be nearly collinear to (f_{n_0}, f_{n_0+1}) when $|r_m/r_{n_0}| < \epsilon$ and, for finite precision, loss of significant digits take place in $C_{n_0}[f,y]$ which are kept for all $n > n_0$. Contrary, no digits are lost in $C_{n_0}[y,g]$, particularly because, as before discussed, f_n and g_n typically have opposite sign patterns (and therefore y and g too for $n = n_0, n_0+1$). The loss of precision can be estimated from the ratio of first components in (5.7). The relative accuracy in the second term (which will dominate for $n >> n_0$) at $n = n_0$ is therefore reduced by a factor

$$L = \left| \frac{\epsilon_1 f_{n_0}}{(1+\epsilon_2) r_m g_{n_0}} \right| \simeq |\epsilon_1| \frac{R_{n_0}}{R_m} \tag{5.9}$$

and then the best attainable accuracy for $n >> n_0$ can be estimated by

$$\max\{\epsilon, \epsilon L\} \approx \epsilon \max\{1, \epsilon R_{n_0}/R_m\} \tag{5.10}$$

assuming that the errors at n_0 are the only source of rounding error propagation.

Therefore, as long as

$$R_m/R_{n_0} > \epsilon^2 \tag{5.11}$$

some accuracy will be recovered for $n > n_0$.

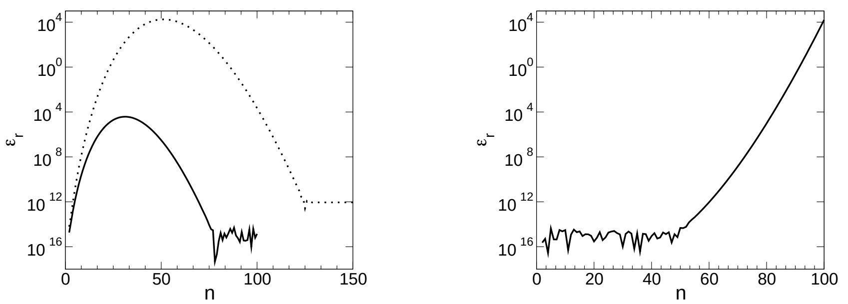

Figure 6. Left: relative errors for the forward computation of the transitorily minimal solution $g_n = (-1)^n \Gamma(c+n) U(a+n, c+n, x)$ for $a = 0.3$, $b = 0.8$. $x = 31$ for the solid line and $x = 51$ for the dotted line. $\epsilon_r = |1 - g_n^F/g_n^M|$ where g_n^F are values computed with Fortran in double precision ($\epsilon \approx 2.2\,10^{-16}$) and g_n^M are values computed with 30 digits in Maple. Right: same for the minimal solution $f_n = M(a+n, c+n, x)$. Parameters: $a = 0.3$, $b = 0.8$, $x = 31$.

This phenomenon of partial recovery of accuracy for finite precision arithmetic, even when the accuracy is completely lost for intermediate values, takes place also when a recurrence with no transitory behaviours and having minimal solutions is first applied in the forward direction for the minimal solution and then in the more natural backward direction. Depending on the number of forward steps all or part of the accuracy will be lost, no matter how many backward steps are applied.

With respect to the minimal (transitorily dominant) solution, as Figure 6 shows, the behaviour is the corresponding to a dominant solution when $n < n_0$, where no loss of precision takes place for forward recursion. For large values of n, accuracy is rapidly lost as is normal for a minimal solution.

For backward recursion, similar situations take place when starting the recurrence from $k > n_0$. The minimal solution will initially be accurately computed until n_0 is reached; then, error will increase for $n < n_0$ (this is the situation described in [8] for the computation of the U function). For the transitorily minimal solution, the backward recurrence is initially bad conditioned until n_0 is reached; then, accuracy for $n < n_0$ can be recovered or not depending on how large is the loss of accuracy at n_0.

It becomes clear that for a safe numerical use of recurrence relations, it is not enough to know the asymptotic behaviour of the solutions but it is also important to study the behaviour for finite orders, particularly when the recurrence $y_{n+1} + b_n y_n + a_n y_{n-1} = 0$ is such that $a_n < 0$ and b_n changes sign. In this cases, it is also important to know if there are transitory minimal solutions.

6 Conclusions

We have analyzed the appearance of transitory minimal solutions for recurrences of the type $y_{n+1} + b_n y_n + a_n y_{n-1} = 0$ with $a_n < 0$ and b_n changing sign once. The related phenomenon of pseudoconvergence of the associated continued fraction has been discussed too. This type of behaviour is present in a considerable number of hypergeometric recurrences. Six different recurrences have been considered, with special emphasis on the Gautschi's anomalous convergence case, for which an accurate description of the accuracy of anomalous convergence is provided, and the related Gauss hypergeometric recurrence.

Extreme care has to be taken when using recurrences and associated continued fractions where b_n changes sign at a given $n = n_0$. The continued fraction may appear to converge with high precision but to a ratio of solutions different from the minimal solution. If finite precision is used, the CF may completely fail to compute ratios of minimal solutions, no matter how many approximants are considered.

Regarding the computation of solutions from these type of recurrence relations, it is concluded that asymptotic information is not sufficient for deciding the condition and stability of the recurrent process and that the usual recipes based on asymptotics fail past the value n_0. The minimal solution, specially when it has a fixed pattern of signs, should not be computed by backward recursion past $n < n_0$ while forward recursion is bad conditioned for some dominant solutions (the transitorily minimal solutions) when $n < n_0$ (although the accuracy may be recovered for large enough n).

For a numerically stable use of these recurrences, it is not only important to identify the minimal solution of the recurrence; also transitorily minimal solutions should be identified and their forward computation avoided for $n < n_0$.

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