Strong asymptotics for Gegenbauer-Sobolev orthogonal polynomials *

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Abstract

We study the asymptotic behaviour of the monic orthogonal polynomials with respect to the Gegenbauer-Sobolev inner product $(f,g)_S = \langle f,g\rangle + \lambda\langle f',g'\rangle$ where $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)(1-x^2)^{\alpha-1/2}dx$ with $\alpha > -1/2$ and $\lambda > 0$. The asymptotics of the zeros and norms of these polynomials is also established.

The study of the orthogonal polynomials with respect to the inner products that involve derivatives (the so-called Sobolev orthogonal polynomials) has been very active for the last ten years (see [5] for a wide bibliography on this subject). Al though algebraic properties of such polynomials (existence, recurrence relations, etc.) have been widely studied, non-trivial asymptotic results were known only for the case when the measure associated with the derivatives is discrete (see [**?**], [6] and, recently, [1]). The outer as well as inner asymptotics for the orthogonal polynomials with both measures absolutely continuous (excluding the trivial cases) is in general an open question. We will produce here the outer asymptotics of this sequence when both measures correspond to the same Gegenbauer weight, that can be considered as the first non-trivial example.

Let us consider the Gegenbauer-Sobolev inner product

$$(f,g)_S = \int_{-1}^{1} f(x)g(x)(1-x^2)^{\alpha-1/2}dx + \lambda\int_{-1}^{1} f'(x)g'(x)(1-x^2)^{\alpha-1/2}dx$$

with $\alpha > -1/2$ and $\lambda \geq 0$. Clearly, if $\lambda = 0$ we have the classical Gegenbauer inner product, thus in the sequel we suppose $\lambda > 0$. The inner product $(\cdot,\cdot)_S$ is positive definite

* A. Martínez, J. J. Moreno, and H. Pijeira, *"Strong asymptotics for Gegenbauer-Sobolev orthogonal polynomials," J. Comp. Appl. Math.*, **81** (1997), 211–216.

† Partially supported by Junta de Andalucía, Grupo de Investigación FQM 0229.

and thus the corresponding sequence o f orthogonal polynomials exists. Furthermore, if we denote by $Q_n^{(\alpha)}(x)$ the nth monic orthogonal polynomial with respect to $(.,.)_S$ then $\deg Q_n^{(\alpha)}(x) = n$ and $Q_n^{(\alpha)}(-x) = (-1)^n Q_n^{(\alpha)}(x)$. The Gegenbauer-Sobo lev orthogonal polynomials and their algebraic properties have been studied by T.E. Pérez in [8]. These polynomials constitute a particular case of the so-called symmetrically coherent pairs of measures, first studied in [2].

Denote by $C_n^{(\alpha)}(x)$ the monic classical Gegenbauer polynomial of degree n, orthogonal with respect to

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)(1-x^2)^{\alpha-1/2}dx\,, \quad \alpha > -\frac{1}{2}\,.$$

The asymptotic behaviour of $C_n^{(\alpha)}$ is well known, and is easily obtained from [9, (8.21.9)]: uniformly on compact subsets of $\mathbf{C}\setminus[-1,1]$,

$$C_n^{(\alpha)}(x) = 2^{1-2\alpha}(x^2-1)^{-\frac{\alpha}{2}}(\sqrt{x+1}+\sqrt{x-1})^{2\alpha-1}\Phi(x)^{n+1/2}(1+o(1))\,, \tag{1}$$

when $n \to \infty$, where $\Phi(x) = \frac{x+\sqrt{x^2-1}}{2}$, with $\sqrt{x^2-1} > 0$ when $x > 1$.

Then the strong asymptotics of the Gegenbauer-Sobolev polynomials can be derived from the asymptotic behaviour of the ratio $\frac{Q_n^{(\alpha)}(x)}{C_n^{(\alpha)}(x)}$, along with the properties of their zeros. The zeros have been studied by H.G . Meijer in [7] and T.E. Pérez [8], but the asymptotic behaviour of the zeros with the largest absolute value was unknown. From our result, it follows that all the zeros accumulate in $[-1,1]$ when $n \to \infty$.

The following theorem establishes the desired relative asymptotics.

Theorem 1 *With the notation introduced above,*

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha)}(x)}{C_n^{(\alpha)}(x)} = \frac{1}{\Phi'(x)} \tag{2}$$

uniformly on compact subsets of $\Omega = \overline{\mathbf{C}}\setminus[-1,1]$.

Proof: Formula (2) is a direct consequence of the two-term relation between the Gegenbauer-Sobolev and the Gegenbauer monic orthogonal polynomials obtained by Iserles et al. in [2] ,

$$Q_n^{(\alpha)}(x) - d_{n-2}(\lambda)Q_{n-2}^{(\alpha)}(x) = C_n^{(\alpha)}(x) - \xi_{n-2}^{(\alpha)}C_{n-2}^{(\alpha)}(x)\,, \tag{3}$$

where $\{d_{n-2}(\lambda)\}$ is a sequence of real positive numbers and

$$\xi_n^{(\alpha)} = \frac{(n+2)(n+1)}{4(n+\alpha+1)(n+\alpha)}\,. \tag{4}$$

With the notation

$$Y_n(x) := \frac{Q_n^{(\alpha)}(x)}{C_n^{(\alpha)}(x)}\,, \qquad \delta_n(x) := d_{n-2}(\lambda)\frac{C_{n-2}^{(\alpha)}(x)}{C_n^{(\alpha)}(x)}\,,$$

$$\beta_n(x) := 1 - \xi_{n-2}^{(\alpha)}\frac{C_{n-2}^{(\alpha)}(x)}{C_n^{(\alpha)}(x)}$$

the equation (3) can be rewritten as

$$Y_n(x) - \delta_n(x) Y_{n-2}(x) = \beta_n(x)\,, \tag{5}$$

which uniquely defines the sequence $\{Y_n\}$ of analytic functions in Ω, with the initial values $Y_0 = Y_1 = 1$.
It is clear that

$$|Y_n(x)| \le |\delta_n(x)||Y_{n-2}(x)| + |\beta_n(x)|\,. \tag{6}$$

In [8] (see also [2]) it was proved that

$$d_n(\lambda) = O(\frac{1}{n^2}), \tag{7}$$

and by (4)

$$\xi_{n-2}^{(\alpha)} \to 1/4. \tag{8}$$

Furthermore, from (1),

$$\frac{C_{n-2}^{(\alpha)}(x)}{C_n^{(\alpha)}(x)} \to \frac{1}{\Phi^2(x)}\,, \tag{9}$$

uniformly on compact subsets of Ω.
Combining (7) and (9) we obtain that there exists $n_0 \in \mathbf{N}$ such that

$$|\delta_n(x)| < \frac{1}{2}\,, \quad n \ge n_0. \tag{10}$$

On the other hand,

$$|\beta_n(x)| \le 1 + \xi_{n-2}^{(\alpha)} \left| \frac{C_{n-2}^{(\alpha)}(x)}{C_n^{(\alpha)}(x)} \right| .$$

Using (8), (9) as well as the inequality $|\Phi(x)|^2 > 1/4$ for $x \notin [-1, 1]$ we deduce the existence of $B > 0$ and $n_1 \in \mathbf{N}$ such that

$$|\beta_n(x)| < B\,, \quad n \ge n_1\,. \tag{11}$$

From (6), (10) and (11) we have for $n \ge n_2 = \max\{n_0, n_1\}$,

$$|Y_n(x)| < \frac{1}{2}|Y_{n-2}(x)| + B\,. \tag{12}$$

Consider the sequence

$$Z_n(x) = \begin{cases} |Y_n(x)|\,, & n \le n_2\,, \\ \frac{1}{2} Z_{n-2}(x) + B\,, & n > n_2\,. \end{cases}$$

For $n > n_2$,

$$Z_{n+2r} = (\frac{1}{2})^r Z_n + 2B(1 - \frac{1}{2^r})\,. \tag{13}$$

Taking limits when $r \to \infty$ in (13) we obtain that $Z_n(x)$ is uniformly bounded for all n sufficiently large. Moreover, $0 < |Y_n(x)| \le Z_n(x)$, for all $n \in \mathbf{N}$. Hence, $Y_n(x)$ is uniformly bounded, and taking limits in (5) we have tha t

$$Y_n(x) = \frac{Q_n^{(\alpha)}(x)}{C_n^{(\alpha)}(x)} \longrightarrow 1 - \frac{1}{4\Phi^2(x)}\,, \tag{14}$$

uniformly on compact subsets of Ω . Notice that for $x \in \Omega$

$$1 - \frac{1}{4\Phi^2(x)} = \frac{\sqrt{x^2-1}}{\Phi(x)} = \frac{1}{\Phi'(x)} \,. \tag{15}$$

From (14) and (15) the result (2) immediately follows. □

Now we study the (Sobolev) norm behaviour of $\{Q_n^{(\alpha)}(x)\}$. With the notation

$$k_n^{(\alpha)} = \langle C_n^{(\alpha)}(x), C_n^{(\alpha)}(x)\rangle = \pi 2^{1-2\alpha-2n} \frac{n!\Gamma(n+2\alpha)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha)} \,,$$

(see [8], [9, (4.7.15)] and

$$\tilde{k}_n^{(\alpha)} = (Q_n^{(\alpha)}(x), Q_n^{(\alpha)}(x))_S$$

the following theorem holds:

Theorem 2

$$k_n^{(\alpha)} + \lambda n^2 k_{n-1}^{(\alpha)} \leq \tilde{k}_n^{(\alpha)} \leq k_n^{(\alpha)} + (\xi_{n-2}^{(\alpha)})^2 k_{n-2}^{(\alpha)} + \lambda n^2 k_{n-1}^{(\alpha)} \,, \quad n \geq 3 \,. \tag{16}$$

where $\xi_{n-2}^{(\alpha)}$ *have been defined in (4).*
In particular,

$$\lim_{n\to\infty} \frac{4^{n-1}\tilde{k}_n^{(\alpha)}}{n^2} = \pi 2^{1-2\alpha}\lambda \,.$$

Proof: We use the extremal property $k_n^{(\alpha)} = \inf\{\langle P, P\rangle : deg\, P = n,$ $P\,\text{monic}\}$:

$$\begin{aligned} \tilde{k}_n^{(\alpha)} &= (Q_n^{(\alpha)}(x), Q_n^{(\alpha)}(x))_S = \\ &= \langle Q_n^{(\alpha)}(x), Q_n^{(\alpha)}(x)\rangle + \lambda \langle (Q_n^{(\alpha)}(x))', (Q_n^{(\alpha)}(x))'\rangle \\ &\geq \langle C_n^{(\alpha)}(x), C_n^{(\alpha)}(x)\rangle + \lambda n^2 \langle C_{n-1}^{(\alpha)}(x), C_{n-1}^{(\alpha)}(x)\rangle = \\ &= k_n^{(\alpha)} + \lambda n^2 k_{n-1}^{(\alpha)} \,. \end{aligned} \tag{17}$$

On the other hand, the polynomials $R_n^{(\alpha)}(x) = C_n^{(\alpha)} - \xi_{n-2}^{(\alpha)} C_{n-2}^{(\alpha)}$ that appear in the right-hand side of (3) satisfy (see (3.3.4) in [8])

$$(R_n^{(\alpha)}(x))' = n C_{n-1}^{(\alpha)}(x) \,, \quad n \geq 2 \,. \tag{18}$$

Hence, the corresponding extremal property of $\tilde{k}_n^{(\alpha)}$ means that

$$\tilde{k}_n^{(\alpha)} \leq (R_n^{(\alpha)}(x), R_n^{(\alpha)}(x))_S = k_n^{(\alpha)} + (\xi_{n-2}^{(\alpha)})^2 k_{n-2}^{(\alpha)} + \lambda n^2 k_{n-1}^{(\alpha)} \,. \tag{19}$$

Using the inequalities (17) and (19) the result (16) follows.
In particular, taking limits in (16) when $n \to \infty$,

$$\lim_{n\to\infty} \frac{4^{n-1}\tilde{k}_n^{(\alpha)}}{n^2} = \lambda \lim_{n\to\infty} 4^{n-1} k_{n-1}^{(\alpha)} \,.$$

It remains to use the explicit formula for $k_n^{(\alpha)}$ in order to obtain the desired asymptotics. $\square$

Finally we make some remarks on the behaviour of the zeros of $Q_n^{(\alpha)}(x)$.

First, strong asymptotics of $Q_n^{(\alpha)}$ implies weak asymptotics. That is, if we associate with each $Q_n^{(\alpha)}(x)$ its enumeration measure (i.e. the discrete unit measure with equal positive masses at their zeros, according to their multipl icity)

$$\mu_n = \frac{1}{n} \sum_{Q_n^{(\alpha)}(\xi)=0} \delta_\xi \,,$$

then

$$d\mu_n(x) \longrightarrow \frac{1}{\pi} \frac{dx}{\sqrt{1-x^2}} \tag{20}$$

(the equilibrium distribution on $[-1,1]$) in the weak-* topology.

Moreover, it is known (see T.E. Pérez [8]) that the zeros of Gegenbauer-Sobolev orthogonal polynomials $Q_n^{(\alpha)}(x)$ are real and simple, and that they interlace with the roots of the Gegenbauer orthogonal polynomials $C_n^{(\alpha)}(x)$. Furthermore, for $\alpha \geq 1/2$ they are all contained in the interval $[-1,1]$ and for $-1/2 < \alpha < 1/2$ there is at most a pair of roots symmetric with respect to the origin outside the interval $[-1,1]$. Theorem 1 implies a stronger assertion:

Corollary 1 *All the roots of* $Q_n^{(\alpha)}(x)$ *accumulate at* $[-1,1]$*, that is,*

$$\bigcap_{n\geq 1} \bigcup_{k=n}^{\infty} \{z : Q_k^{(\alpha)}(z) = 0\} = [-1,1]\,. \tag{21}$$

Proof: It is sufficient to observe that $\frac{Q_n^{(\alpha)}(x)}{C_n^{(\alpha)}(x)}$ is a sequence of analytic functions in $\overline{\mathbf{C}}\backslash[-1,1]$ and $\Phi(x)$ is analytic and has no zeros in $\overline{\mathbf{C}}\backslash[-1,1]$. Hence, the zeros of $Q_n^{(\alpha)}(x)$ cannot accumulate outside $[-1,1]$. On the other hand, (20) shows that they must be dense in [-1,1]. $\square$

Acknowledgments: The authors wish to thank Prof. Francisco Marcellán for introducing us in the topic of the Sobolev orthogonality and for having attracted our attention on the asymptotics problem.

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