FORMULATION OF HYBRID 3D IMAGE SEGMENTATION ALGORITHM BASED PARTIAL DIFFERENTIAL EQUATION

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Abstract

Segmentation is an important tool for analysis and understanding of most images encountered in science and engineering. One of the best segmentation methods that can perform 3D segmentation is the level-set method which has its mathematical foundation in partial differential equation (PDE). Owing to its complex nature, it exhibits a level of unacceptable sluggishness on implementation hence a need to hasten up the process by hybridizing it with a faster region-based segmentation method which is inherently a logical approach to segmentation pivoted on thresholding but not as good in segmentation as the former. This work presents a mathematical hybrid of the two methods that is hoped to produce a better segmentation result.

***Keywords**: 3D segmentation, level-set, region-growing, partial differential equation (PDE), and hybrid segmentation.*

1 Introduction

Segmentation is a key step in image processing as it aids image analysis and understanding among others. There are lots of methods and techniques in segmentation ranging from simple thresholding to complex contour tracing methods. The result of different methods varies in degree of their accuracy. Literature review shows that no single method is entirely satisfactory in all aspects of image processing requirements, for instance, the level-set method [1],[2],[3] produces good segmentation results but has complex mathematical algorithm there consuming lots of implementation time [4] which made it unsuitable for real-time video image processing. Therefore, we look into hybrid technique for evolving a segmentation method that will satisfy most image processing needs.

Our choice of a hybrid of level-set and region-growing methods was occasioned by the mutual divergent characteristics of the two, that is, while region-growing is fast [5], could be semi-automated, but has poor edge detection capability especially in weak-edged images leading to over segmentation [4]. As for the level-set method, it is slow, largely manual based, but able to segment any kind of image including 3 dimensional (3D) images. More so, the region-growing is mostly based on logic and determination of accurate threshold(s), and on the other hand the level-set method involves many mathematical methods especially partial differential equation (PDE).

2 Level-Set Method

Given $\mathbb{X}(s,t) = [X(s,t), Y(s,t)]$ is a curve moving in time (t) with curve parameterization being (s). If N is the moving curve inward normal, and c is the curvature, let the curve develop along it normal direction according to Partial Differential Equation PDE [6].

$$\frac{\partial \mathbb{X}}{\partial t} = V(c)N \dots\dots\dots (i)$$

in which $V(c)$ is the speed function. Having a level-set function $\emptyset(x,y,t)$ with the contour $\mathbb{X}(s,t)$ as its zero level set, the situation is described by:

$$\emptyset(\mathbb{X}(s,t),t) = 0 \dots\dots\dots (ii)$$

Differentiating this with respect to t and applying the chain rule [7] results in

$$\emptyset_t + \nabla\emptyset(\mathbb{X}(s,t),t)\cdot\mathbb{X}'(s,t) = 0 \dots\dots\dots (iii)$$

$$\frac{\partial \emptyset}{\partial t} + \nabla \emptyset(\mathbb{X}(s,t),t) \cdot \frac{\partial \mathbb{X}}{\partial t} = 0 \dots \dots \dots (iv)$$

$$\frac{\partial \emptyset}{\partial t} + \nabla \emptyset \frac{\partial \mathbb{X}}{\partial t} = 0 \dots \dots \dots (v)$$

where $\nabla \emptyset$ is vector normal and $\frac{\nabla \emptyset}{|\nabla \emptyset|} = N \dots \dots \dots (vi)$

Assuming $\emptyset$ is negative inside and positive outside the zero level set, the inward unit normal vector of the level set curve is:

$$N = -\frac{\nabla \emptyset}{|\nabla \emptyset|} \dots \dots \dots (vii)$$

hence $V(c)N$ becomes

$$-\frac{\nabla \emptyset}{|\nabla \emptyset|} V(c) = \frac{\partial \mathbb{X}}{\partial t} \dots \dots \dots (viii)$$

thus

$$\frac{\partial \emptyset}{\partial t} + \nabla \emptyset \frac{\partial \mathbb{X}}{\partial t} = 0 \dots \dots \dots (ix)$$

becomes

$$\frac{\partial \emptyset}{\partial t} = \frac{\nabla \emptyset}{|\nabla \emptyset|} V(c) \nabla \emptyset \dots \dots \dots (x)$$

by substituting (viii) into (ix), therefore

$$\frac{\partial \emptyset}{\partial t} = V(c) |\nabla \emptyset| \dots \dots \dots (xi)$$

where by c is the curvature given by

$$c = \nabla \cdot \frac{\nabla \emptyset}{|\nabla \emptyset|} = \frac{\emptyset_{xx} \emptyset_y^2 - 2\emptyset_x \emptyset_y \emptyset_{xy} + \emptyset_{yy} \emptyset_x^2}{(\emptyset_x^2 + \emptyset_y^2)^{3/2}} \dots \dots \dots (xii)$$

The right hand side of the expression is the approximation of c in 2D [6], and $\emptyset_x, \emptyset_y, \emptyset_{xy}$ are derivatives of $\emptyset$ [8]. Also, ∇ is the curve gradient, and $(\nabla \cdot)$ or (div) is divergence. The objective is to have a mathematical method that will move phi to the boundary of ROI (minimize energy functions), that is C' = 0, which mathematically speaking means $\inf_{a_1, a_2, \emptyset} C'(\emptyset, a_1, a_2)$.

2.1 CHAN-VESE Energy Functional Minimization

The Chan-Vese energy functional is a piecewise constant minimal variance criterion based on the Mumford-Shah functional [9].

$$C(\emptyset, a_1, a_2) = C_1(\emptyset, a_1, a_2) + C_2(\emptyset, a_1, a_2) =$$

$$\int \wedge (I(x,y) - a_1)^2 \, dxdy + \int \vee (I(x,y) - a_2)^2 \, dxdy \dots \dots \dots (xiii)$$

Such that I(x,y) is the image pixel, a_1 & a_2 represents the mean intensities of the interior and exterior of the level-set curve C(Ø) placed around the region of interest. The energy $C(\emptyset, a_1, a_2)$ is minimized when the zero level-set Ø coincides with the region of interest (ROI) boundary thereby separating the whole image into foreground and background with respect to their mean intensities.

Moreover, harder segmentation tasks may be solved by including regularizing terms like length of the level-set curve Ø and area of the region inside the curve Ø. Thus the energy function becomes:

$$C'(\emptyset, a_1, a_2) = \mu(Length\ of\ \emptyset) + v(Area\ of\ \emptyset) + C_1'(\emptyset, a_1, a_2) + C_2'(\emptyset, a_1, a_2) + \lambda_1 \int \wedge (I(x,y) - a_1)^2\, dxdy + \lambda_2 \int \vee (I(x,y) - a_2)^2\, dxdy \ldots\ldots\ldots (xiv)$$

Where μ, v, λ_1, and λ_2 are ≥ 0. The inside of the curve C(Ø) corresponds to $\emptyset(x,y) > 0$ and the outside corresponds to $\emptyset(x,y) < 0$. Also, the Heaviside $H(z)$ and Dirac Delta $\delta(z)$ functions are given by:

$$H(z) = \begin{cases} 1, & z \geq 0 \\ 0, & z < 0 \end{cases} \ldots\ldots\ldots (xv)$$

and

$$\delta_0 = \frac{d}{dx}\left(H(z)\right) \ldots\ldots\ldots (xvi)$$

In order to achieve this, it is necessary to first unify the equations by expressing length of the curve, area of the curve, the expression for inside and outside of the curve, and pixel intensity of inside and outside of the curve in terms of Ø [7],[9],[10],[11]. Hence:

Length of curve (c) = Length $(\emptyset = 0) = \int |\nabla H(\emptyset(x,y)|\,dxdy =$

$$\int \delta_0\left(\emptyset(x,y)\right)|\nabla\emptyset(x,y)|\,dxdy \ldots\ldots\ldots (xvii)$$

Area inside (c) = Area $(\emptyset \geq 0) = \int H(\emptyset(x,y)dxdy \ldots\ldots\ldots (xviii)$

Inside of the curve = $\int \wedge |I(x,y) - a_1|^2\, dxdy =$

$$\int \wedge |I(x,y) - a_1|^2\, H(\emptyset(x,y))dxdy \ldots\ldots\ldots (xix)$$

Outside of the curve = $\int \vee |I(x,y) - a_2|^2\, dxdy =$

$$\int \wedge |I(x,y) - a_2|^2\, (1 - H\left(\emptyset(x,y)\right))dxdy \ldots\ldots\ldots (xx)$$

From the last expression we have

$$a_1(\emptyset) = \frac{\int I(x,y)H(\emptyset(x,y))dxdy}{\int H(\emptyset(x,y))dxdy} \ldots\ldots\ldots (xxi)$$

$$a_2(\emptyset) = \frac{\int I(x,y)(1 - H\left(\emptyset(x,y)\right))dxdy}{\int(1 - H\left(\emptyset(x,y)\right))dxdy} \ldots\ldots\ldots (xxii)$$

such that

$$\int \wedge |I(x,y) - a_1|^2\, H(\emptyset(x,y))dxdy = 0 \ldots\ldots\ldots (xxiii)$$

Therefore

$$a = \frac{\int I(x,y)\big(H\emptyset(x,y)\big)dxdy)}{\int H\emptyset(x,y)\ dxdy} \ldots\ldots\ldots (xxiv)$$

Thus the energy functional $C'(\emptyset, a_1, a_2)$ can be written as:

$$C'(\emptyset, a_1, a_2) = \mu \int \delta_0\big(\emptyset(x,y)\big)|\nabla\emptyset(x,y)|\,dxdy +$$

$$v\int H\big(\emptyset(x,y)\big)dxdy + \lambda_1 \int \wedge|I(x,y) - a_1|^2\, H\big(\emptyset(x,y)\big)dxdy +$$

$$\lambda_2 \int \wedge|I(x,y) - a_2|^2\,\big(1 - H\big(\emptyset(x,y)\big)\big)dxdy \ldots\ldots\ldots (xxv)$$

Next is to find the minimum, that is, $\underset{a_1,a_2,\emptyset}{inf}\, C'(\emptyset, a_1, a_2)$. This is done by equating the derivatives (in terms of $\emptyset$) of C' to zero while a_1 and a_2 are kept constant. Since our interest is only in derivative of C' with respect to $\emptyset$, it could be said that $C'(\emptyset, a_1, a_2) = C'(\emptyset)$ or simply $\emptyset$ such that $\emptyset$ is equal to $\emptyset_t$ which is itself equal to $\frac{d}{dt}\emptyset$ where t is an artificial time and it is greater than or equal to zero.

Therefore the resulting partial differential equation (PDE) from [9] is:

$$\frac{\partial\emptyset}{\partial t} = \delta_0(\emptyset)\begin{bmatrix} \mu p \left(\int \delta_0(\emptyset)|\nabla\emptyset|\right)^{p-1} div\left(\frac{\nabla\emptyset}{|\nabla\emptyset|}\right) - v - \\ \lambda_1(I(x,y) - a_1)^2 p + \lambda_2(I(x,y) - a_2)^2 \end{bmatrix} = 0 \ldots\ldots\ldots (xxx)$$

Where p = 1, and curvature $k(\emptyset) = div\left(\frac{\nabla\emptyset}{|\nabla\emptyset|}\right) \ldots\ldots\ldots (xxvi)$

then

$$\frac{\partial\emptyset}{\partial t} = \delta_0(\emptyset)\begin{bmatrix} \mu \cdot k(\emptyset) - v - \\ \lambda_1(I(x,y) - a_1)^2 p + \lambda_2(I(x,y) - a_2)^2 \end{bmatrix} = 0 \ldots\ldots\ldots (xxvii)$$

Owing to the fact that:

$$\int \delta_0\big(\emptyset(x,y)\big)|\nabla\emptyset(x,y)|dxdy \ \equiv\ \nabla\cdot\left(\frac{\nabla\emptyset}{|\nabla\emptyset|}\right)$$

$$\equiv div\left(\frac{\nabla\emptyset}{|\nabla\emptyset|}\right) \ldots\ldots\ldots (xxviii)$$

$$H\big(\emptyset(x,y)\big)dxdy:\ \emptyset \geq 0 = 1, H\big(\emptyset(x,y)\big)dxdy:\ \emptyset < 0 = 0 \ldots\ldots\ldots (xxix)$$

Thus

$$v\int H\big(\emptyset(x,y)\big)dxdy = v \ldots\ldots\ldots (xxx)$$

$$\lambda_1 \int \wedge |I(x,y) - a_1|^2 \, H(\emptyset(x,y))dxdy = \lambda_1 (I(x,y) - a_1)^2 \ldots\ldots\ldots (xxxi)$$

$$\lambda_2 \int \wedge |I(x,y) - a_2|^2 \, \left(1 - H(\emptyset(x,y))\right) dxdy =$$

$$\lambda_2 \int \wedge |I(x,y) - a_2|^2 \, (1 - 0) = \lambda_2 (I(x,y) - a_2)^2 \ldots\ldots\ldots (xxxii)$$

Note that; $\emptyset(x,y,0) = \emptyset_0 = \emptyset(x,y)$ *within the image* (Ω) and

$$\frac{\delta(\emptyset)}{|\nabla\emptyset|}\frac{\partial\emptyset}{\partial \vec{n}} = 0 \text{ on the boundary } (\partial\Omega)$$

3 Region Growing

3.1 Homogeneity Condition

$$R = \bigcup_{i=1}^{s} R_i \ldots\ldots\ldots (xxxiii)$$

$$R_i \cap R_j = \emptyset, \ldots\ldots\ldots (xxxiv)$$

where R is universal set representing the whole image, *where* $i \neq j$ and (s) is the total numbers of regions in an image whose character.

H (R_i) = TRUE

H (R_I U R_J) = FALSE owing to the adjacency of R_i and R_j, hence, they are not homogeneous.

3.2 Thresholding

Thresholding is the transformation of an input image (I) to an output (segmented) binary image (B) as follows:

$$B(x,y) = \begin{cases} 1 \text{ for } I(x,y) \geq T \text{ (foreground)} \\ 0 \text{ for } I(x,y) < T \text{ (background)} \end{cases} \ldots\ldots\ldots (xxxv)$$

where T is threshold, B(x, y) is output pixel, and I(x, y) pixel intensity value.

In order to grow the region from a seed point call a pixel, edge significance $E(x,y)_{i,j}$ can be evaluated by first evaluating the crack-edge value $C(x,y)_{i,j}$ using;

$$C(x,y)_{i,j} = |I(x,y)_i - I(x,y)_j| \ldots\ldots\ldots (xxxvi)$$

and then;

$$E(x,y)_{i,j} = \begin{cases} 0 \text{ if } C(x,y)_{i,j} < T \\ \quad 1 \text{ otherwise} \end{cases} \ldots\ldots\ldots (xxxvii)$$

Here, letters (i & j) represent foreground and background respectively, and $I(x,y)_i$is the pixel intensity value of image in the foreground, and $I(x,y)_j$is the intensity value of the neighbouring pixel. $E(x,y)_{i,j}$ = 1 indicates a significant edge, $E(x,y)_{i,j}$ = 0 indicates a weak edge [6].

After the initial seed-pixel whose intensity is used to compare with neighbouring pixels, once it has merged with other pixels to form a small region, the region intensity mean is then used as the basis for comparison with neighbouring pixels. Curiously, it is the use of region mean that leads to over segmentation in this method owing to the fact that the mean gradually moves away from the threshold as the region continuously absorb neighbouring pixels of quantum different intensity. That is one of the problem our hybrid technique intends to address. Region mean (g_m) is determined by:

$$g_m = \sum_{i=0}^{m} f(x,y)_i \dots\dots\dots (xxxviii)$$

4 Hybrid Segmentation

Assuming a particular image intensity ranges from low intensity (I_L) to high intensity (I_H) in which the image can be divided into binary group of foreground and background with the foreground having high intensity (I_H) and background having low intensity (I_L). If the boundary between (I_L) and (I_H) is marked by a threshold value (T) such that:

$I_H \geq T$ and $I_L < T$

Furthermore, if the two extremes of (I_H) is marked by T_{min} & T_{max}, in order to segment the foreground from the background in this image using a combination of Region Growing and Level-Set in such a way that the process starts with region growing and ends with level-set, then the region is grown from a seed pixel (P_S) in the foreground. Suppose that P_S is T_{max} and it is taken to be the center of a L-by-L matrix, and the search for pixel of similar intensity based on T follows the pattern below:

$$P_{S\,i+n,j+m}\ for -1 \leq n, m \leq 1$$

This is the space search step condition, and;

$$P_{S\,i+n,j+m} \begin{cases} P_S, & for\ \ n = m = 0 \\ P_N, & Otherwise \end{cases} \dots\dots\dots (xxxix)$$

where $i\ and\ j$ are equal being center row and column of the L-by-L matrix, and P_N stands for neighbouring pixels. If L is 3, the space search step according to the condition above is:

(a) Perpendicular Neighbour
$P_{S\,i-1,j}, P_{S\,i+1,j}, P_{S\,i,j-1}, and\ P_{S\,i,j+1}$

(b) Diagonal Neighbour
$P_{S\,i-1,j-1}, P_{S\,i+1,j-1}, P_{S\,i-1,j+1}, and\ P_{S\,i+1,j+1}$

This is similar to front advection in traditional interface tracing with a square velocity $Vs = (1, -1; 1, -1)$. At each step, a test for inclusion into region is conducted by;

$$T_{min} \leq P_{S\,i+n,j+m} \leq T_{max}$$

This is the region growing condition, and all pixel of the initial matrix L-by-L satisfied the condition, n & m are incremented and equally decreased by 1, such that the next space search step condition becomes:

$$P_{S\,i+n,j+m}\ for -2 \leq n, m \leq 2$$

and the space search steps are:

$P_{S_{i-2,j-2}}, P_{S_{i-2,j-2}}, P_{S_{i-2,j}}, P_{S_{i-2,j+1}}, P_{S_{i-2,j+2}}, P_{S_{i-1,j+2}}, P_{S_{i,j+2}}, P_{S_{i+1,j+2}},$

$P_{S_{i+2,j+2}}, P_{S_{i+2,j+1}}, P_{S_{i+2,j}}, P_{S_{i+2,j-1}}, P_{S_{i+2,j-2}}, P_{S_{i+1,j-2}}, P_{S_{i,j-2}}, P_{S_{i-1,j-2}}$

$$T_{min} \le P_{S_{i+n,j+m}} \le T_{max} \begin{cases} TRUE, & H(R_i) \\ FALSE, & H(R_i \cup R_j) \end{cases} \dots\dots\dots (xL)$$

Again, test for inclusion into region will be conducted at each step, and if all get included, an increase in range is initiated until at least a pixel fails the test. When at least one of the pixels fail the test, a one step backward is made and all the pixels at that step are linked together to form the initial foreground boundary or the zero level-set $\emptyset$. However, increase in range is still initiated until at least all pixels in two consecutive steps failed the inclusion test. Again the pixels of the second step that failed the inclusion test are linked together to form an artificial boundary of the image designated as $\partial\Omega'$.

$$\begin{cases} \begin{array}{c} for\ all\ P_{S_{i+n,j+m}} \ge T, \\ Continue\ Region\ Growing \\ \\ for\ any\ one\ of\ P_{S_{i+n,j+m}} < T, \\ Link\ Previous\ Pixels\ to\ form\ \emptyset, and\ Increase\ range \\ \\ for\ all\ P_{S_{i+n,j+m}} < T,\ \ Stop\ after\ 2\ steps, \\ and\ Link\ Pixels\ of\ the\ second\ step\ to\ form\ \partial\Omega' \end{array} \end{cases} \dots\dots\dots (xLi)$$

Hence, the partial differential equation (PDE) from [9] is applied on $\emptyset$ thus:

$$\frac{\partial\emptyset}{\partial t} = \delta_0(\emptyset) \begin{bmatrix} \mu p \left(\int \delta_0(\emptyset) |\nabla\emptyset| \right)^{p-1} div\left(\frac{\nabla\emptyset}{|\nabla\emptyset|} \right) - v - \\ \lambda_1 (I(x,y) - a_1)^2 p + \lambda_2 (I(x,y) - a_2)^2 \end{bmatrix} = 0 \dots\dots\dots (xLii)$$

where p = 1, and curvature $\kappa(\emptyset) = div\left(\frac{\nabla\emptyset}{|\nabla\emptyset|}\right)$ then

$$\frac{\partial\emptyset}{\partial t} = \delta_0(\emptyset) \begin{bmatrix} \mu \cdot \kappa(\emptyset) - v - \\ \lambda_1 (I(x,y) - a_1)^2 p + \lambda_2 (I(x,y) - a_2)^2 \end{bmatrix} = 0 \dots\dots\dots (xLiii)$$

Thus, a_1 is the average intensity of pixels inside $\emptyset$, and a_2 is the average pixel intensity for pixel between $\emptyset$ and $\partial\Omega'$. What is left is to move $\emptyset$ to the foreground boundary $\partial\Omega$ which lies somewhere between $\emptyset$ and $\partial\Omega'$ to $\partial\Omega'$. The implementation of the level-set then becomes similar to the narrow-band technique, with 2D and 3D approximation of $\kappa(\emptyset)$ given by:

$$\kappa(\emptyset) = c = \frac{\emptyset_{xx}\emptyset_y^2 - 2\emptyset_x\emptyset_y\emptyset_{xy} + \emptyset_{yy}\emptyset_x^2}{(\emptyset_x^2 + \emptyset_y^2)^{3/2}} \dots\dots\dots (xLiv))$$

and

$$\kappa(\emptyset) = \begin{bmatrix} (\emptyset_{yy} + \emptyset_{zz})\emptyset_x^2 + (\emptyset_{xx} + \emptyset_{zz})\emptyset_y^2 \\ +(\emptyset_{xx} + \emptyset_{yy})\emptyset_z^2 - 2\emptyset_x\emptyset_y\emptyset_{xy} - 2\emptyset_x\emptyset_z\emptyset_{xz} - 2\emptyset_y\emptyset_z\emptyset_{yz} \end{bmatrix}$$

$$\times [(\emptyset_x^2 + \emptyset_y^2 + \emptyset_z^2)]^{-3/2} \dots\dots\dots (xLv)$$

according to [6], [7] the different forms of $\emptyset$ in equations (xLiv) and (xL), plus their second order forms are all derivatives of $\emptyset$.

5 Conclusions

A mathematical hybrid of two segmentation methods are here presented. The technique made use of many mathematical and logical approaches but mainly pivoted on partial differential equation (PDE) for evolution of the front advection normal to the curve's curvature and its numerical solution. This technique maintains both the attributes of region-growing, and level-set methods but in a version that extract their strengths and looks more like a narrow-band implementation of level-set [7] with its high dependent on PDE.

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