Strong Asymptotics for Sobolev Orthogonal Polynomials *

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Abstract

In this paper we obtain the strong asymptotics for the sequence of orthogonal polynomials with respect to the inner product

$$\langle f, g \rangle_S = \sum_{k=0}^{m} \int_{\Delta_k} f^{(k)}(x) g^{(k)}(x) d\mu_k(x)$$

where $\{\mu_k\}_{k=0}^m$, with $m \in \mathbb{Z}_+$, are measures supported on $[-1, 1]$ which satisfy Szegő's condition.

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1 Introduction

Let $\{\mu_k\}_{k=0}^m$,with $m \in \mathbb{Z}_+$, be a set of $m+1$ finite positive Borel measures, such that at least one of them has infinitely many points of increase and for each $k = 0, ..., m$ the support Δ_k of μ_k is a compact subset of the real line $\mathbb{R}$. If $f^{(k)}$ denotes the k-th derivative of the function f, then the expression

$$\langle f, g \rangle_S = \sum_{k=0}^{m} \int_{\Delta_k} f^{(k)}(x) g^{(k)}(x) d\mu_k(x) \quad (1)$$

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defines an inner product in the linear space $\mathbf{P}$ of polynomials with real coefficients. A completion of $\mathbf{P}$ with respect to the norm $\|f\|_S = \langle f, f\rangle_S^{1/2}$ leads to the suitable Sobolev space of functions.

The Gram-Schmidt process with respect to (1) applied to the canonical basis of $\mathbf{P}$ generates the orthonormal sequence of polynomials $\{q_n\}$, $n = 0, 1, \ldots$, $\deg q_n = n$; we denote the corresponding monic polynomials by $Q_n(x) = x^n + \ldots$, so that

$$q_n(z) = \frac{Q_n(z)}{\|Q_n\|_S}\ , \ n = 0, 1, \ldots .$$

As usual, we will call these sequences *Sobolev Orthogonal Polynomials*.

During the nineties a very active research on Sobolev orthogonal polynomials has been carried out, although the major advances corresponded initially to specific measures and to the algebraic aspect of the theory. For a historical review of this period the reader is referred to [1] and [8].

The analytic results are rather recent. Unlike the standard orthogonality with respect to a measure, in this case the orthogonal polynomials neither satisfy a three-term recurrence relation nor their zeros are necessarily contained in the convex hull of $\bigcup_{k=0}^{m} \Delta_k$. Thus, we are deprived of two fundamental tools in the study of the asymptotic behavior of the polynomials when the degree tends to infinity, and the traditional analytic methods have no immediate extension to the Sobolev case.

The first important asymptotic results appeared in [6] and [4] and corresponded to the so-called discrete case (that is, when $\{\mu_k\}_{k=1}^{m}$ have at most a finite number of mass points). In the non-discrete or continuous case, the most general results obtained so far deal with $m = 1$. In [3], the potential theoretic approach is used for the study of the asymptotic distribution of zeros and critical points of the Sobolev orthogonal polynomials, under the additional assumption that both μ_0 and μ_1 are supported on $\mathbb{R}$ and belong to the class **Reg** (see the definition in the monograph [11]). In [7], the strong asymptotics for the sequence of Sobolev orthogonal polynomials was obtained, assuming that the measures μ_0 and μ_1 are supported on a smooth Jordan curve or arc in the complex plane and satisfy Szegő's condition.

Possibly, the only known results concerning the non-discrete case and arbitrary $m \in \mathbb{Z}_+$ are contained in [2] and [5]. In [2], the moment problem associated to (1) is investigated and the necessary and sufficient conditions for the existence of a solution of the so-called Sobolev moment problem are established. In [5], for a wide class of Sobolev orthogonal polynomials, it is proved that the zeros are contained in a compact subset of the complex plane; the asymptotic zero distribution is obtained and, with this information, the n-th root asymptotic behavior outside the compact set containing all the zeros is given. A sufficient condition for this asymptotic behavior is what the authors called the *lower sequential domination* of the Sobolev inner product: $\Delta_k \subset \Delta_{k-1}$ and $d\mu_k = f_{k-1} d\mu_{k-1}$, $f_{k-1} \in L_\infty(\mu_{k-1})$, $k = 1, \ldots, m$.

By analogy, we say that the Sobolev inner product (1) is *upper sequentially dominated* if $\Delta_k \subset \Delta_m$, $k = 0, \ldots, m-1$, and Δ_m is a compact interval. Clearly, the case when all the measures involved in the inner product are equal satisfies both domination conditions.

The purpose of this paper is to extend the results of [7] to arbitrary $m \in \mathbb{Z}_+$, assuming that the Sobolev inner product (1) is upper sequentially dominated. Without loss of generality, in what follows we suppose that $\Delta_m = [-1, 1]$.

In the next section the main results are stated; the necessary background is gathered in Section 3. Finally, the last two sections are devoted to the proofs.

2 Asymptotics of Sobolev polynomials

Before stating the main results, we introduce some notation and definitions. Denote by $\Delta = [-1,1]$, $\Omega = \overline{\mathbb{C}} \setminus \Delta$ and $\varphi(z)$ the conformal mapping of Ω onto the exterior of the circle $|z| = 1/2$; that is, $\varphi(z) = (z + \sqrt{z^2-1})/2$, where the square root is chosen so that $|z+\sqrt{z^2-1}| > 1$ for $z \in \Omega$.

Let $\mu = \rho(x)\,dx + \mu_s$ be the Lebesgue-Radon-Nicodym decomposition of the measure μ, so that $\rho(x)$ is integrable and non-negative almost everywhere on Δ and μ_s is singular (with respect to the Lebesgue measure) on Δ. The measure μ is said to belong to the Szegő class on Δ (and we denote this fact by $\mu \in S(\Delta)$) if $\operatorname{supp}\mu \subset \Delta$ and

$$\int_\Delta \frac{\log \rho(x)}{\sqrt{1-x^2}}\,dx > -\infty\,. \tag{2}$$

Obviously, this implies that $\rho(x) > 0$ almost everywhere on Δ.

For a finite positive Borel measure μ with infinitely many points of increase, we denote by $P_n(\mu; z)$ the nth monic orthogonal polynomial with respect to μ, and

$$\gamma_n = \gamma_n(\mu) = \int |P_n(\mu;x)|^2\,d\mu(x)\,. \tag{3}$$

Then, $p_n(\mu;z) = \gamma_n(\mu)^{-1/2} P_n(\mu;z)$ stands for the corresponding orthonormal polynomial. When μ is absolutely continuous with respect to the Lebesgue measure and $\mu' = \rho$, we use sometimes ρ instead of μ in the notation of the norms and polynomials.

Each measure μ_k, , $k = 0, 1, \ldots, m$, yields the inner product

$$\langle f, g\rangle_k = \int f\overline{g}\,d\mu_k\,, \quad k = 0, 1, \ldots, m\,,$$

and the norm $\|f\|_k^2 = \langle f, f\rangle_k^{1/2}$. With this notation, the sequence of monic polynomials Q_n is orthogonal with respect to the inner product

$$\langle f, g\rangle_S = \sum_{k=0}^{m} \langle f^{(k)}, g^{(k)}\rangle_k\,.$$

Our first result establishes the asymptotic behavior of the Sobolev norms

$$\kappa_n = \langle Q_n, Q_n\rangle_S = \min\left\{ \langle Q, Q\rangle_S : Q(x) = x^n + \sum_{i=0}^{n-1} c_i x^i \right\}. \tag{4}$$

We have

Theorem 1 *If the inner product (1) is upper sequentially dominated, $d\mu_m(x) = \rho(x)dx$ with $\rho \in S([-1,1])$ and $\{\mu_k\}_{k=0}^{m-1}$ are finite positive Borel measures whose supports are contained in $[-1,1]$, then*

$$\lim_{n\to\infty} \frac{\kappa_n}{n^{2m}\,\gamma_{n-m}(\rho)} = 1\,, \tag{5}$$

and

$$\lim_{n\to\infty} \frac{Q_n^{(m)}(z)}{n^m\, P_{n-m}(\rho;z)} = 1\,, \tag{6}$$

uniformly on compact subsets of Ω.

In practice, the most natural situation is when the measures $\{\mu_k\}_{k=0}^m$ are supported on the same interval Δ. In this case we have:

Theorem 2 *If the finite positive and absolutely continuous Borel measures* $\mu_k \in S([-1,1])$, $k=0,1,\ldots,m$, *then*

$$\lim_{n\to\infty} \frac{Q_n^{(k)}(z)}{n^k\, P_{n-k}(\mu_m;z)} = \frac{1}{[\varphi'(z)]^{m-k}}\,, \quad k=0,1,\ldots,m\,, \tag{7}$$

uniformly on compact subsets of Ω.

Observe that the right hand side of (7) is a non-vanishing analytic function in Ω; thus, by Hurwitz's theorem the zeros of Q_n cannot accumulate in Ω. Furthermore, with account of the well-known ratio asymptotics of $P_n(\mu_m;z)$ and (7), the ratio asymptotics of Q_n is straightforward.

Corollary 1 *With the assumptions of Theorem 2:*

1. *The zeros of the Sobolev orthogonal polynomials* Q_n *accumulate on* $\Delta=[-1,1]$.

2. *For* $0\le k_1,k_2\le m$ *and* $d\in \mathbb{Z}_+$,

$$\lim_{n\to\infty} \frac{Q_{n+d}^{(k_1)}(z)}{n^{k_1-k_2}\, Q_n^{(k_2)}(z)} = [\varphi(z)]^{d+k_2-k_1}\ [\varphi'(z)]^{k_1-k_2}\,,$$

uniformly on compact subsets of Ω.

3 Extremal problem for analytic functions

It is well known that the Hardy spaces constitute a natural analytic framework for the study of the asymptotic properties of orthogonal polynomials. A comprehensive account on these topics can be found in the monograph [10].

If a measure μ belongs to the Szegő class on $\Delta=[-1,1]$, then there exists a unique function $\mathcal{R}(z)$ holomorphic in $\Omega=\mathbb{C}\setminus\Delta$ satisfying:

1. $\mathcal{R}(z)\neq 0$ for $z\in\Omega$,

2. $\mathcal{R}(\infty)>0$,

3. for almost every $x\in(-1,\ 1)$

$$\lim_{y\to 0} |\mathcal{R}(x+iy)| = \rho(x)\,.$$

The construction of $\mathcal{R}$ is straightforward if we notice that $\ln|\mathcal{R}|$ is the solution of the Dirichlet problem in Ω with boundary condition $\ln\rho$ on Δ (and its existence is guaranteed by the Szegő's condition). Let $g(z,\infty)$ be the Green function of Ω with pole at infinity, and Γ_r be the level curve $\Gamma_r = \{z \in \Omega : g(z,\infty) = r\}$ with $r > 0$.

An analytic function f in Ω is said to be of class $E^1(\Omega)$ (see e.g. [10], Ch. 10) if

$$\sup_{r>0} \int_{\Gamma_r} |f(z)|^2 |dz| < +\infty \,.$$

By $E^2(\Omega,\rho)$, we denote the space of functions f analytic in Ω such that $|f^2(z)\mathcal{R}(z)| \in E^1(\Omega)$. Any $f \in E^2(\Omega,\rho)$ has non-tangential limits

$$f_+(x) = \lim_{y\to 0^+} f(x+iy)\,, \quad f_-(x) = \lim_{y\to 0^-} f(x+iy)\,,$$

for almost all $x \in (-1,1)$, and can be recovered from its boundary values using Cauchy integrals. Thus, the following Lemma (see [9], Corollary 7.4) is straightforward:

Lemma 1 *Given a weight $\rho \in S(\Delta)$ and a compact subset $\Sigma \subset \Omega$, there exists a constant $M = M(\Sigma)$ such that*

$$\max_{z\in\Sigma} |f(z)|^2 \leq M \int_\Delta \left\{|f_+(x)|^2 + |f_-(x)|^2\right\} \rho(x) dx \quad \text{for all } f \in E^2(\Omega,\rho)\,.$$

Moreover, $E^2(\Omega,\rho)$ becomes a Hilbert space with the inner product

$$\langle f, g\rangle = \int_\Delta \left\{ f_+(x)\overline{g_+(x)} + f_-(x)\overline{g_-(x)} \right\} \rho(x)\, dx\,.$$

In this space, the following extremal problem can be considered (see [9] and [12]):

$$\nu(\rho) = \inf\left\{\langle F, F\rangle : F \in E^2(\Omega,\rho), F(\infty) = 1\right\}\,. \tag{8}$$

The key fact contained in the Szegő-Kolmogorov-Krein Theorem is that $\mu \in S(\Delta)$ if and only if $\nu(\rho) > 0$, and there exists a unique extremal function $\mathcal{F}(z) = \mathcal{F}(\rho; z)$ solving (8). Furthermore, (see e.g. [9], theorem 6.2),

$$\mathcal{F}^2(z) = \varphi'(z)\frac{\mathcal{R}(\infty)}{\mathcal{R}(z)}\,, \quad z \in \Omega\,, \tag{9}$$

where $\mathcal{R}(z)$ is the holomorphic function introduced above.

Let P_n be the nth monic orthogonal polynomial associated with μ and $\gamma_n(\mu) = \int_\Delta |P_n|^2(x)\rho(x)\,dx$. The next result follows from the classical Bernstein-Szegő's theorem:

Lemma 2 *If $\mu \in S(\Delta)$ and $\rho(x) = \mu'(x)$ then*

$$\lim_{n\to\infty} 4^n \gamma_n(\mu) = \nu(\rho)\,, \tag{10}$$

and

$$\lim_{n\to\infty} \frac{P_n(z)}{\varphi^n(z)} = \mathcal{F}(z), \text{ locally uniformly in } \Omega. \tag{11}$$

4 Proof of Theorem 1

Let $\mathbf{P}^*$ denote the subspace of monic polynomials in $\mathbf{P}$; we introduce the operator $\Pi : \mathbf{P}^* \to \mathbf{P}^*$ which associates to any monic polynomial its monic primitive normalized by 0 at $x = -1$. In other words, if $P(x) = x^n + \ldots$,

$$\Pi(P)(x) \;:=\; (n+1) \int_{-1}^{x} P(t)\, dt\,.$$

For the sake of brevity, we denote

$$\Pi_i = \Pi \circ \cdots \circ \Pi\,,$$

where we have the composition of i operators Π; additionally, Π_0 states for the identity operator in $\mathbf{P}^*$. In consequence, for $P(x) = x^n + \ldots$ the degree of $\Pi_i(P)$ is $n+i$ and

$$\frac{d^k}{dx^k}\Pi_i(P)(x) = \frac{(n+i)!}{(n+i-k)!}\Pi_{i-k}(P)(x)\,, \quad 0 \le k \le n+i\,. \tag{12}$$

Before proving Theorem 1, we establish the following preliminary result:

Lemma 3 *Let $P_n = P_n(\mu;\cdot)$ be the nth monic orthogonal polynomial with respect to a finite positive Borel measures $d\mu(x) = \rho(x)\,dx$, whose support is contained in $[-1,1]$, and $\gamma_n(\mu)$ is as in (3). If $1/\rho \in L^1([-1,1])$, then for any $k \in I\!N$, the sequence of polynomials*

$$\alpha_{n,k}(x) = \frac{n!\,\Pi_k(P_n)(x)}{(n+k)!\,\sqrt{\gamma_n}}\,, \quad n \in I\!N\,,$$

is uniformly bounded on $[-1,1]$ and tends to zero for every $x \in [-1,1]$ as $n \to \infty$.

Proof. For $k = 1$, fix $x \in [-1,1]$ and define $a(x,t) = \chi_{[-1,x]}(t)/\rho(t)$, where χ_A represents the characteristic function of the set A. Then $\alpha_{n,1}(x)$ is the nth Fourier coefficient of $a(x,\cdot)$ with respect to the orthonormal system $p_n(\rho;\cdot)$:

$$\int_\Delta a(x,t)p_n(\rho;t)\,\rho(t)\,dt = \int_{-1}^{x} p_n(\rho;t)\,dt = \frac{\Pi_1(P_n)(x)}{(n+1)\,\sqrt{\gamma_n}} = \alpha_{n,1}(x)\,.$$

The condition $1/\rho \in L^1([-1,1])$ guarantees that $a(x,\cdot) \in L^2(\rho)$; hence, from Bessel's inequality it follows that

$$\lim_{n\to\infty} \alpha_{n,1}(x) = 0\,, \quad \text{and} \quad |\alpha_{n,1}(x)|^2 \le \int_\Delta \frac{1}{\rho(t)}\,dt \;<\infty\,, \tag{13}$$

which establishes the assertion for $k = 1$.

On the other hand, it is easy to check that

$$\alpha_{n,k+1}(x) = \frac{n!\,\Pi\circ\Pi_k(P_n)(x)}{(n+k+1)!\,\sqrt{\gamma_n}} = \int_\Delta \chi_{[-1,x]}(t)\alpha_{n,k}(t)\,dt\,.$$

Thus, (13), the Lebesgue dominated convergence theorem and simple induction arguments allow to conclude the proof for any $k \in I\!N$. ■

Proof of (5) in Theorem 1. Let $P_n(\mu;\cdot)$ and $\gamma_n(\mu)$ be as in (3); in what follows for the sake of brevity we omit the explicit reference to the measure when $\mu=\mu_m$.

We have that

$$\kappa_n = \|Q_n\|_S^2 = \sum_{k=0}^{m} \|Q_n^{(k)}\|_k^2 = \sum_{k=0}^{m-1} \|Q_n^{(k)}\|_k^2 + \|Q_n^{(m)}\|_m^2 \,.$$

By the extremal property of the orthogonal polynomials,

$$\gamma_{n-m} = \|P_{n-m}\|_m^2 \le \left[\frac{(n-m)!}{n!}\right]^2 \|Q_n^{(m)}\|_m^2 \,,$$

thus,

$$\left[\frac{(n-m)!}{n!}\right]^2 \frac{\kappa_n}{\gamma_{n-m}} \ge 1 + \sum_{k=0}^{m-1} \left[\frac{(n-m)!}{n!}\right]^2 \frac{\|Q_n^{(k)}\|_k^2}{\gamma_{n-m}} \,,$$

so that

$$\liminf_n \left[\frac{(n-m)!}{n!}\right]^2 \frac{\kappa_n}{\gamma_{n-m}} \ge 1 \,. \tag{14}$$

It remains to prove that

$$\limsup_n \left[\frac{(n-m)!}{n!}\right]^2 \frac{\kappa_n}{\gamma_{n-m}} \le 1 \,. \tag{15}$$

Assume initially that $1/\rho \in L^1([-1,1])$. From the extremal property for κ_n and (12) it follows that

$$\begin{aligned}
\kappa_n &= \|Q_n\|_S^2 \le \|\Pi_m(P_{n-m})\|_S^2 = \|\Pi_m^{(m)}(P_{n-m})\|_m^2 + \sum_{k=0}^{m-1} \|\Pi_m^{(k)}(P_{n-m})\|_k^2 \\
&\le \left[\frac{n!}{(n-m)!}\right]^2 \|P_{n-m}\|_m^2 + \sum_{k=0}^{m-1} \left[\frac{n!}{(n-k)!}\right]^2 \|\Pi_{m-k}(P_{n-m})\|_k^2 \,.
\end{aligned}$$

Hence, using the notation introduced in Lemma 3,

$$\left[\frac{(n-m)!}{n!}\right]^2 \frac{\kappa_n}{\gamma_{n-m}} \le 1 + \sum_{k=1}^{m} \|\alpha_{n-m,m-k}\|_k^2 \,, \tag{16}$$

From Lemma 3 via the Lebesgue dominated convergence theorem it follows that

$$\lim_{n\to\infty} \|\alpha_{n-m,m-k}\|_k = 0 \,, \quad k = 0,\ldots,m-1 \,. \tag{17}$$

Consequently, (15) holds when $1/\rho \in L^1([-1,1])$.

If ρ is a general weight satisfying Szegő's condition on $[-1,1]$, we take an arbitrary constant $\delta > 0$ (to be fixed later) and define $\tilde\rho(x) := \rho(x) + \delta$. Let $\tilde P_n = P_n(\tilde\mu;\cdot)$ be the nth monic orthogonal polynomial with respect to $d\tilde\mu(x) := \tilde\rho(x)\,dx$, $\tilde\gamma_n := \gamma(\tilde\mu)$, and $\tilde\alpha_{n,k}$ is the sequence defined in Lemma 3 corresponding to the weight $\tilde\rho$. Since $\tilde\rho \ge \rho$, by (16),

$$\left[\frac{(n-m)!}{n!}\right]^2 \frac{\kappa_n}{\gamma_{n-m}} \le \frac{\tilde\gamma_{n-m}}{\gamma_{n-m}} \left(1 + \sum_{k=1}^{m} \|\tilde\alpha_{n-m,m-k}\|_k^2\right) \,.$$

But $1/\tilde{\rho} \in L^1([-1,1])$, and from (17),

$$\lim_{n\to\infty} \|\tilde{\alpha}_{n-m,m-k}\|_k = 0\,, \quad k = 0,\ldots,m-1\,. \tag{18}$$

Consequently,

$$\limsup_{n\to\infty} \left[\frac{(n-m)!}{n!}\right]^2 \frac{\kappa_n}{\gamma_{n-m}} \le \lim_{n\to\infty} \frac{\tilde{\gamma}_{n-m}}{\gamma_{n-m}}\,.$$

Since $\rho,\ \tilde{\rho} \in S(\Delta)$, then (see e.g. [12, theorem 12.7.1])

$$\tilde{\gamma}_{n-m} = \sqrt{\pi}\, 2^{n-m} \exp\left\{-\frac{1}{2\pi}\int_{-1}^{1} \frac{\ln \tilde{\rho}(x)}{\sqrt{1-x^2}}\, dx\right\} (1+o(1))\,,$$

and

$$\gamma_{n-m} = \sqrt{\pi}\, 2^{n-m} \exp\left\{-\frac{1}{2\pi}\int_{-1}^{1} \frac{\ln \rho(x)}{\sqrt{1-x^2}}\, dx\right\} (1+o(1))\,.$$

Hence,

$$\lim_{n\to\infty} \frac{\tilde{\gamma}_{n-m}}{\gamma_{n-m}} = \exp\left\{-\frac{1}{2\pi}\int_{-1}^{1} \frac{\ln \tilde{\rho}(x) - \ln \rho(x)}{\sqrt{1-x^2}}\, dx\right\}\,.$$

It remains to use continuity arguments in the metric given by

$$\operatorname{dist}(\vartheta,\sigma) = \frac{1}{\pi}\int_{-1}^{1} \frac{\ln \vartheta(x) - \ln \sigma(x)}{\sqrt{1-x^2}}\, dx\,,$$

for $\vartheta,\ \sigma \in S(\Delta)$. In fact, by Lebesgue monotone convergence theorem,

$$\operatorname{dist}(\tilde{\rho},\rho) \to 0\,, \quad \text{when} \quad \delta \downarrow 0\,.$$

Thus, for an arbitrary $\varepsilon > 0$, we can choose $\delta > 0$ such that

$$\limsup_{n\to\infty} \left[\frac{(n-m)!}{(n-k)!}\right]^2 \frac{\kappa_n}{\gamma_{n-m}} \le 1+\varepsilon\,.$$

Now, (15) follows from the arbitrariness of $\varepsilon > 0$, and (5) is proved. ∎

Proof of (6) in Theorem 1. Set

$$\Psi_n(z) = 2^{n-m}\frac{(n-m)!}{n!}\, Q_n^{(m)}(z)\,, \quad \text{and} \quad \Phi_n(z) = 2^{n-m}\,\varphi^{n-m}(z)\,\mathcal{F}(z)\,,$$

where $\mathcal{F}(z)$ is the solution of the extremal problem (8). Using the parallelogram law in $E^2(\Omega,\rho)$ one has

$$\|\Psi_n - \Phi_n\|_m^2 = 2\,\|\Psi_n\|_m^2 + 2\,\|\Phi_n\|_m^2 - \|\Psi_n + \Phi_n\|_m^2\,. \tag{19}$$

Taking into account (8) and that $|\varphi(x)| = 1/2$ for $x \in [-1,1]$, we have

$$\|\Phi_n\|_m^2 = \|\mathcal{F}(z)\|_m^2 = \nu(\rho)\,.$$

On the other hand, by (5),

$$\begin{aligned} \|\Psi_n\|_m^2 &= 4^{n-m}\left[\frac{(n-m)!}{n!}\right]^2 \|Q_n^{(m)}(z)\|_m^2 \\ &\le 4^{n-m}\left[\frac{(n-m)!}{n!}\right]^2 \kappa_n = 4^{n-m}\,\gamma_{n-m}\,[1+o(1)]\,. \end{aligned}$$

With account of (10),

$$\limsup_{n\to\infty} \|\Psi_n\|_m^2 \le \limsup_{n\to\infty} 4^{n-m}\, \gamma_{n-m} = \nu(\rho)\,.$$

Notice that

$$\frac{1}{2}\left(\frac{(n-m)!}{n!}\,\frac{Q_n^{(m)}}{\varphi^{n-m}} + \mathcal{F}\right) \in E^2\left(\Omega, \rho\right)$$

and it is equal to 1 at ∞, so by extremality of $\nu(\rho)$,

$$\|\Psi_n + \Phi_n\|_m^2 = 4\left\|\frac{1}{2}\left(\frac{(n-m)!}{n!}\,\frac{Q_n^{(m)}}{\varphi^{n-m}} + \mathcal{F}\right)\right\|_m^2 \ge 4\nu(\rho)\,.$$

Gathering the inequalities obtained so far, we see that necessarily

$$\lim_{n\to\infty} \|\Psi_n - \Phi_n\|_m^2 = \lim_{n\to\infty} \left\|\frac{(n-m)!}{n!}\,\frac{Q_n^{(m)}}{\varphi^{n-m}} - \mathcal{F}\right\|_m^2 = 0\,.$$

Since

$$\frac{(n-m)!\,Q_n^{(m)}(z)}{n!\,\varphi^{n-m}(z)} - \mathcal{F}(z) \in E^2(\Omega, \rho)\,,$$

by Lemma 1 one has that

$$\lim_{n\to\infty} \frac{(n-m)!}{n!}\,\frac{Q_n^{(m)}(z)}{\varphi^{n-m}(z)} = \mathcal{F}(z)\,, \tag{20}$$

uniformly on compact subsets of Ω. By (11), this is equivalent to (6). ∎

5 Proof of Theorem 2

First, we prove an auxiliary Lemma.

Lemma 4 *With the assumptions of Theorem 2, for any $0 \le k \le n$ and $j \ge k+1$,*

$$\lim_{n\to\infty} \frac{Q_n^{(k)}(z)}{n^j\,\varphi^{n-k}(z)} = 0\,, \tag{21}$$

uniformly on compact subsets of Ω.

Proof. Recall that

$$\frac{4^{n-m}\,\kappa_n}{n^{2m}} = \sum_{k=0}^{m-1}\left\|\frac{2^{n-m}\,Q_n^{(k)}}{n^m}\right\|_k^2 + \left\|\frac{Q_n^{(m)}}{n^m\,\varphi^{n-m}}\right\|_m^2\,.$$

By Theorem 1 and (10),

$$\lim_{n\to\infty} \frac{4^{n-m}\,\kappa_n}{n^{2m}} = \nu(\rho)\,,$$

and by (20),

$$\lim_{n\to\infty}\left\|\frac{Q_n^{(m)}}{n^m\,\varphi^{n-m}}\right\|_m^2=\|\mathcal{F}\|_m^2=\nu(\rho)\,.$$

Thus,

$$\lim_{n\to\infty}\sum_{k=0}^{m-1}\left\|\frac{2^{m-k}\,Q_n^{(k)}}{n^m}\right\|_k^2=\lim_{n\to\infty}\sum_{k=0}^{m-1}\left\|\frac{Q_n^{(k)}}{n^m\,\varphi^{n-k}}\right\|_k^2=0\,,$$

which is equivalent to

$$\lim_{n\to\infty}\left\|\frac{Q_n^{(k)}}{n^m\,\varphi^{n-k}}\right\|_k^2=0\,,\quad k=0,\dots,m-1\,.$$

Since by assumption, $\mu_k\in S[-1,1]$ for $0\le k\le m$, using Lemma 1, we obtain

$$\lim_{n\to\infty}\frac{Q_n^{(k)}(z)}{n^m\,\varphi^{n-k}(z)}\;=\;0\,,\quad 0\le k\le m\,,\tag{22}$$

uniformly on compact subsets of Ω. Then, by Weierstrass' Theorem we have that

$$\lim_{n\to\infty}\left[\frac{Q_n^{(k)}(z)}{n^m\,\varphi^{n-k}(z)}\right]'\;=\;0\,,\quad 0\le k\le m\,,\tag{23}$$

also uniformly on compact subsets of Ω.

It is easy to check that for $0\le k\le m-1$,

$$\frac{(n-k)\,\varphi'(z)}{n\,\varphi(z)}\left[\frac{Q_n^{(k)}(z)}{n^{m-1}\,\varphi^{n-k}(z)}\right]=\frac{1}{\varphi(z)}\frac{Q_n^{(k+1)}(z)}{n^m\,\varphi^{n-(k+1)}(z)}-\left[\frac{Q_n^{(k)}(z)}{n^m\,\varphi^{n-k}(z)}\right]'\,.$$

Then, using (22),

$$\lim_{n\to\infty}\frac{Q_n^{(k)}(z)}{n^{m-1}\,\varphi^{n-k}(z)}\;=\;0\,,\quad 0\le k\le m-1\,.$$

Repeating this reasoning we conclude the proof. ■

Proof of Theorem 2. For $k=m$, the assertion has been established above (see (20)):

$$\lim_{n\to\infty}\frac{Q_n^{(m)}(z)}{n^m\,\varphi^{n-m}(z)}=\mathcal{F}(z)\,,\quad \text{uniformly on compact subsets of }\Omega.$$

For $1\le k\le m$, the following identity holds:

$$\frac{(n+1-k)\,\varphi'(z)}{n\,\varphi(z)}\frac{Q_n^{(k-1)}(z)}{n^{k-1}\,\varphi^{n+1-k}(z)}=\frac{Q_n^{(k)}(z)}{n^k\,\varphi(z)\,\varphi^{n-k}(z)}-\left[\frac{Q_n^{(k-1)}(z)}{n^k\,\varphi^{n+1-k}(z)}\right]'\,.$$

Furthermore, by Lemma 4 and Weierstrass' Theorem we know that

$$\lim_{n\to\infty}\left[\frac{Q_n^{(k-1)}(z)}{n^k\,\varphi^{n+1-k}(z)}\right]'=0\,,$$

uniformly on compact subsets of Ω.

Assuming that the assertion of Theorem 2 has been proved for a k, $1 \leq k \leq m$, that is,

$$\lim_{n\to\infty} \frac{Q_n^{(k)}(z)}{n^k\, \varphi^{n-k}(z)} = \frac{\mathcal{F}(z)}{[\varphi'(z)]^{n-k}}\,, \quad \text{uniformly on compact subsets of } \Omega\,, \tag{24}$$

this leads us to the result

$$\lim_{n\to\infty} \frac{Q_n^{(k-1)}(z)}{n^{k-1}\, \varphi^{n-k+1}(z)} = \frac{\mathcal{F}(z)}{[\varphi'(z)]^{m-k+1}}\,,$$

uniformly on compact subsets of Ω, which establishes the same assertion for $k-1$. Thus, Theorem 2 is proved completely. ■

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