GROMOV HYPERBOLICITY OF DENJOY DOMAINS WITH HYPERBOLIC AND QUASIHYPERBOLIC METRICS

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Abstract. We derive explicit and simple conditions which in many cases allow one to decide, whether or not a Denjoy domain endowed with the Poincaré or quasihyperbolic metric is Gromov hyperbolic. The criteria are based on the Euclidean size of the complement. As a corollary, the main theorem allows us to deduce the non-hyperbolicity of any periodic Denjoy domain.

1. Introduction

In the 1980s Mikhail Gromov introduced a notion of abstract hyperbolic spaces, which have thereafter been studied and developed by many authors. Initially, the research was mainly centered on hyperbolic group theory, but lately researchers have shown an increasing interest in more direct studies of spaces endowed with metrics used in geometric function theory.

One of the primary questions is naturally whether a metric space (X, d) is hyperbolic in the sense of Gromov or not. The most classical examples, mentioned in every textbook on this topic, are metric trees, the classical Poincaré hyperbolic metric developed in the unit disk and, more generally, simply connected complete Riemannian manifolds with sectional curvature $K \leqslant -k^2 < 0$.

However, it is not easy to determine whether a given space is Gromov hyperbolic or not. In recent years several investigators have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Klein-Hilbert and Kobayashi metrics are Gromov hyperbolic (under particular conditions on the domain of definition, see [8, 14] and [4]); the Gehring-Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [13]). Also, in [15] the hyperbolicity of the conformal modulus metric μ and the related so-called Ferrand metric λ^*, is studied.

Since the Poincaré metric is also the metric giving rise to what is commonly known as the hyperbolic metric when speaking about open domains in the complex plane or in Riemann surfaces, it could be expected that there is a connection between the notions of hyperbolicity. For simply connected subdomains Ω of the complex plane, it follows directly from the Riemann mapping theorem that the metric space (Ω, h_Ω) is in fact

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Gromov hyperbolic. However, as soon as simply connectedness is omitted, there is no immediate answer to whether the space h_Ω is hyperbolic or not. The question has lately been studied in [2] and [19]–[26].

The related quasihyperbolic metric has also recently been a topic of interest regarding the question of Gromov hyperbolicity. In [9], Bonk, Heinonen and Koskela found necessary and sufficient conditions for when a planar domain D endowed with the quasihyperbolic metric is Gromov hyperbolic. This was extended by Balogh and Buckley, [5]: they found two different necessary and sufficient conditions which work in Euclidean spaces of all dimensions and also in metric spaces under some conditions.

In this article we are interested in Denjoy domains. In this case the result of [9] says that the domain is Gromov hyperbolic with respect to the quasihyperbolic metric if and only if the domain is the conformal image of an inner uniform domain (see Section 3). Although this is a very nice characterization, it is somewhat difficult to check that a domain is inner uniform, since we need to construct uniform paths connecting every pair of points.

In this paper we show that it is necessary to look at paths joining only a very small (countable) number of points when we want to determine the Gromov hyperbolicity. This allows us to derive simple and very concrete conditions on when the domain is Gromov hyperbolic. However, the main purpose of the results on the quasihyperbolic metric is that they suggest methods for proving the corresponding results for the hyperbolic metric, which is the main contribution of the paper. To the best of our knowledge, this is the first time that Gromov hyperbolicity of any class of infinitely connected domains has been obtained from conditions on the Euclidean size of the complement of the domain. It means that we are relating Euclidean conditions to properties of non-Euclidean metrics.

The main results in this article are the following:

Theorem 1.1. *Let Ω be a Denjoy domain with $\Omega \cap \mathbb{R} = (-\infty, 0) \cup \bigcup_{n=1}^{\infty}(a_n, b_n)$, $b_n \leqslant a_{n+1}$ for every n, and $\lim_{n\to\infty} a_n = \infty$.*

(1) *The metrics k_Ω and h_Ω are Gromov hyperbolic if*

$$\liminf_{n\to\infty} \frac{b_n - a_n}{a_n} > 0.$$

(2) *The metrics k_Ω and h_Ω are not Gromov hyperbolic if*

$$\lim_{n\to\infty} \frac{b_n - a_n}{a_n} = 0.$$

In the case

$$0 = \liminf_{n\to\infty} \frac{b_n - a_n}{a_n} < \limsup_{n\to\infty} \frac{b_n - a_n}{a_n},$$

which is not covered by the previous theorem, one can construct examples to show that the metrics k_Ω and h_Ω may or may not be Gromov hyperbolic. In this sense our result is optimal.

In this theorem the most relevant and diffcult part is the second one, whereas the first one is a kind of converse. Both of them joined even provide a characterization when the limit exists. Consider the following example: $\Omega := \mathbb{C} \setminus \cup_{n=1}^{\infty}\{(\log n)^\alpha n^\beta A^n\}$ with $\lim_{n\to\infty}(\log n)^\alpha n^\beta A^n = \infty$; Theorem 1.1 gives directly that Ω is hyperbolic if and only if $A > 1$.

The main difficulty in the proof is that it is impossible to determine the precise location of the geodesics with these metrics (we do not even have an explicit expression for the Poincaré density).

It is interesting to note that in the case of Denjoy domains many of the results seem to hold for both the hyperbolic and the quasihyperbolic metrics. In fact, we know of no planar domain which is Gromov hyperbolic with respect to one of these metrics, but not the other.

In the previous theorem, the boundary components have a single accumulation point, at ∞, and the accumulation happens only from one side. It turns out that if this kind of domain is not Gromov hyperbolic, then we cannot mend the situation by adding some boundary to the other side of the accumulation point, as the following theorem shows.

Theorem 1.2. *Let Ω be a Denjoy domain with $(-\infty,0)\subset\Omega$ and let $F\subseteq(-\infty,0]$ be closed. If k_Ω is not Gromov hyperbolic, then neither is $k_{\Omega\setminus F}$; if h_Ω is not Gromov hyperbolic, then neither is $h_{\Omega\setminus F}$.*

One might think that the assumption $F\subseteq(-\infty,0]$ is superfluous; however the following example shows that the conclusion is false in general when we consider a closed set F not contained in the negative half axis: let Ω be as in Theorem 1.1(2) and $F:=[0,\infty)$. Then Ω is not hyperbolic, but $\Omega\setminus F=\mathbb{C}\setminus F$ is hyperbolic, since it is simply connected.

Theorem 1.2, in particular, allows to deduce the same conclusions as Theorem 1.1(2), removing the technical hypothesis $(-\infty,0)\subset\Omega$.

If E_0 is any closed set contained in the open set $\{z=x+iy\in\mathbb{C}:\ x,y\in(0,1)\}$ and $E_{m,n}:=E_0+m+in$, then it is clear that $\mathbb{C}\setminus\cup_{m,n\in\mathbb{Z}}E_{m,n}$ is not hyperbolic, since its isometry group is isomorphic to $\mathbb{Z}^2$ (a non-hyperbolic group).

It might be reasonable to think that any periodic Denjoy domain is hyperbolic, since its isometry group is isomorphic to $\mathbb{Z}$, which is a hyperbolic group. However, in the following example we prove the non-hyperbolicity of any periodic Denjoy domain (as a direct consequence of Theorem 4.6):

Example 1.3. Let $E_0\subset[0,t)$ be closed, $t>0$, set $E_n:=E_0+tn$ for $n\in\mathbb{N}$ or $n\in\mathbb{Z}$, and $\Omega:=\mathbb{C}\setminus\cup_n E_n$. Then h_Ω and k_Ω are not Gromov hyperbolic.

2. Definitions and notation

By $\mathbb{H}^2$ we denote the upper half plane, $\{z\in\mathbb{C}:\ \operatorname{Im}z>0\}$, and by $\mathbb{D}$ the unit disk $\{z\in\mathbb{C}:\ |z|<1\}$. For $D\subset\mathbb{C}$ we denote by ∂D and $\overline{D}$ its boundary and closure, respectively. For $z\in D\subsetneq\mathbb{C}$ we denote by $\delta_D(z)$ the distance to the boundary of D, $\min_{a\in\partial D}|z-a|$. Finally, we denote by c, C, c_j and C_j generic constants which can change their value from line to line and even in the same line.

Recall that a domain $\Omega\subset\mathbb{C}$ is said to be of *hyperbolic type* if it has at least two finite boundary points. The universal cover of such domain is the unit disk $\mathbb{D}$. In Ω we can define the Poincaré metric, i.e. the metric obtained by projecting the metric $ds=2|dz|/(1-|z|^2)$ of the unit disk by any universal covering map $\pi:\mathbb{D}\longrightarrow\Omega$. Equivalently, we can project the metric $ds=|dz|/\operatorname{Im}z$ of the the upper half plane $\mathbb{H}^2$. Therefore, any simply connected subset of Ω is isometric to a subset of $\mathbb{D}$. With this metric, Ω is a geodesically complete Riemannian manifold with constant curvature -1, in particular, Ω is a geodesic metric space. By λ_Ω we denote the density of the Poincaré metric in Ω, i.e. the positive function such that $\lambda_\Omega^2(z)(dx^2+dy^2)$ is the Poincaré metric in Ω. The Poincaré metric is natural and useful in complex analysis; for instance, any holomorphic function between two domains is Lipschitz with constant 1, when we consider the respective Poincaré metrics.

The quasihyperbolic metric is the distance induced by the density $1/\delta_\Omega(z)$. By k_Ω and h_Ω we denote the quasihyperbolic and Poincaré distance in Ω, respectively. Length (of a curve) will be denoted by the symbol $\ell_{d,\Omega}$, where d is the metric with respect to which length is measured. If it is clear which metric or domain is used, either one or both subscripts in $\ell_{d,\Omega}$ might be left out. The subscript Eucl is used to denote the length with respect to the Euclidean metric. Also, as most of the proofs apply to both the quasihyperbolic and the Poincaré metrics, we will use the symbol κ as a "dummy metric" symbol, where it can be replaced by either k or h.

It is well known that for every domain Ω of hyperbolic type

$$\lambda_\Omega(z) \leqslant \frac{2}{\delta_\Omega(z)} \quad \forall\, z \in \Omega, \qquad\qquad \ell_{h,\Omega}(\gamma) \leqslant 2\ell_{k,\Omega}(\gamma) \quad \forall\, \gamma \subset \Omega,$$

and that for all domains $\Omega_1 \subset \Omega_2$ we have $\lambda_{\Omega_1}(z) \geqslant \lambda_{\Omega_2}(z)$ for every $z \in \Omega_1$.

If Ω_0 is an open subset of Ω, we always consider in Ω_0 the usual quasihyperbolic or Poincaré metric, independent of Ω. If E is a relatively closed subset of Ω, we always consider in E the inner metric obtained by the restriction of the quasihyperbolic or Poincaré metric in Ω, that is

$$d_{\Omega|E}(z,w) := \inf \big\{ \ell_{d,\Omega}(\gamma) : \gamma \subset E \text{ is a rectifiable curve joining } z \text{ and } w \big\} \geqslant d_\Omega(z,w)\,.$$

It is clear that $\ell_{\Omega|E}(\gamma) = \ell_\Omega(\gamma)$ for every curve $\gamma \subset E$. We always require that ∂E is a union of pairwise disjoint Lipschitz curves; this fact guarantees that $(E, d_{\Omega|E})$ is a geodesic metric space.

A geodesic metric space (X,d) is said to be *Gromov δ-hyperbolic*, if

$$d(w, [x,z] \cup [z,y]) \leqslant \delta$$

for all $x,y,z \in X$; corresponding geodesic segments $[x,y]$, $[y,z]$ and $[x,z]$; and $w \in [x,y]$. If this inequality holds, we also say that the geodesic triangle is *δ-thin*, so Gromov hyperbolicity can be reformulated by requiring that all geodesic triangles are thin. In order to simplify the notation, we say that d is Gromov-hyperbolic (instead of (X,d) is Gromov-hyperbolic).

A *Denjoy domain* $\Omega \subset \mathbb{C}$ is a domain whose boundary is contained in the real axis. Since $\Omega \cap \mathbb{R}$ is an open set contained in $\mathbb{R}$, it is the union of pairwise disjoint open intervals; as each interval contains a rational number, this union is countable. Hence, we can write $\Omega \cap \mathbb{R} = \cup_{n\in\Lambda}(a_n, b_n)$, where Λ is a countable index set, $\{(a_n,b_n)\}_{n\in\Lambda}$ are pairwise disjoint, and it is possible to have $a_{n_1} = -\infty$ for some $n_1 \in \Lambda$ and/or $b_{n_2} = \infty$ for some $n_2 \in \Lambda$.

In order to study Gromov hyperbolicity, we consider the case where Λ is countably infinite, since if Λ is finite then h_Ω and k_Ω are easily seen to be Gromov hyperbolic by Proposition 3.5, below.

3. Some classes of Denjoy domains which are Gromov hyperbolic

The quasihyperbolic metric is traditionally defined in subdomains of Euclidean n-space $\mathbb{R}^n$, i.e. open and connected subsets $\Omega \subsetneq \mathbb{R}^n$. However, a more abstract setting is also possible, as Bonk, Heinonen and Koskela showed in [9]. They show that if (X, d) is any locally compact, rectifiably connected and noncomplete metric space, then the quasihyperbolic metric k_X can be defined as usual, using the weight $1/\operatorname{dist}(x, \partial X)$.

Given a real number $A \geqslant 1$, a rectifiable curve $\gamma\colon [0,1] \to \Omega$ is called *A-uniform for the metric d* if

$$\begin{aligned}&\ell_d(\gamma) \leqslant A\, d(\gamma(0), \gamma(1)) \quad \text{and}\\ &\min\{\ell_d(\gamma|[0,t]), \ell_d(\gamma|[t,1])\} \leqslant A\ \operatorname{dist}_d(\gamma(t), \partial\Omega), \qquad \text{for all } t \in [0,1].\end{aligned}$$

Moreover, a locally compact, rectifiably connected noncomplete metric space is said to be *A-uniform* if every pair of points can be joined by an A-uniform curve. The abbreviations "A-uniform" and "A-inner uniform" (without mention of the metric) mean A-uniform for the Euclidean metric and Euclidean inner metric, respectively.

Uniform domains are intimately connected to domains which are Gromov hyperbolic with respect to the quasihyperbolic metric (see [9, Theorems 1.12, 11.3]). Specifically, for a Denjoy domain Ω these results imply that k_Ω is Gromov hyperbolic if and only if Ω is the conformal image of an inner uniform.

Here we will use the generalized setting in [9] to show that for Denjoy domains with the quasihyperbolic metric it actually suffices to consider the intersection of the closed upper (or lower) halfplane with the actual domain. The same result holds for the Poincaré metric:

Lemma 3.1. *Let $E \subset \mathbb{R}$ be a closed set with at least two points, and denote by $\Omega = \mathbb{C}\setminus E$ and $\Omega_0 = \Omega \cap \{z \in \mathbb{C} \mid \operatorname{Im} z \geqslant 0\} = \Omega \cap \overline{\mathbb{H}^2}$. Then the metric space Ω_0, with the restriction of the Poincaré or the quasihyperbolic metric in Ω, is δ-Gromov hyperbolic, with some universal constant δ.*

Proof. We deal first with the quasihyperbolic metric. As the upper half-plane is uniform in the classical case, the same curve of uniformity (which is an arc of a circle orthogonal to $\mathbb{R}$) can be shown to be an A-uniform curve in the sense of [9] for the set Ω_0, for some absolute constant A. Hence Ω_0 is A-uniform. By [9, Theorem 3.6] it then follows that the space (Ω_0, k_{Ω_0}) is Gromov hyperbolic. (Note that k_{Ω_0} is the same as k_Ω restricted to Ω_0.)

We also have that Ω_0 is hyperbolic with the restriction of the Poincaré metric h_Ω, since it is isometric to a geodesically convex subset of the unit disk (in fact, for every pair of points in Ω_0, there is just one geodesic contained in Ω_0 joining them). Therefore, Ω_0 has $\log\big(1+\sqrt{2}\big)$-thin triangles, as the unit disk does (see, e.g. [3, p. 130]). ☐

Definition 3.2. Let Ω be a Denjoy domain of hyperbolic type. Then we have $\Omega \cap \mathbb{R} = \cup_{n\geqslant 0}(a_n, b_n)$ for some pairwise disjoint intervals. We say that a curve in Ω is a *fundamental geodesic* if it is a geodesic (with respect to the metric considered in Ω) joining (a_0, b_0) and (a_n, b_n), $n > 0$, which is contained in the closed halfplane $\overline{\mathbb{H}^2} = \{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0\}$. We denote by γ_n a fundamental geodesic corresponding to n. Some examples are shown in Figure 1.

The next result was proven for the hyperbolic metric in [2, Theorem 5.1]. In view of Lemma 3.1 one can check that the same proof carries over to the quasihyperbolic metric.

By a *bigon* we mean a polygon with two edges.

We say that an inequality holds *quantitatively*, if it holds with a constant depending only on the constants in the assumptions.

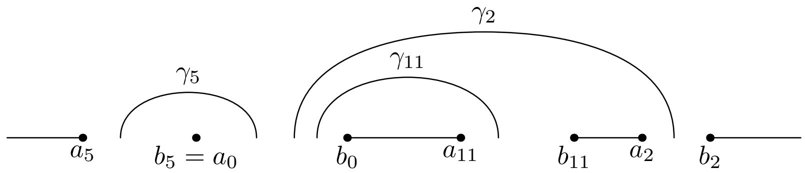

FIGURE 1. Fundamental geodesics

Theorem 3.3. *Let Ω be a Denjoy domain of hyperbolic type and denote by κ_Ω the Poincaré or the quasihyperbolic metric. Then the following conditions are quantitatively equivalent:*

(1) *κ_Ω is δ-hyperbolic.*
(2) *There exists a constant c_1 such that for every choice of fundamental geodesics $\{\gamma_n\}_{n=1}^\infty$ we have $\kappa_\Omega(z, \mathbb{R}) \leqslant c_1$ for every $z \in \cup_{n\geqslant 1}\gamma_n$.*
(3) *There exists a constant c_2 such that for a fixed choice of fundamental geodesics $\{\gamma_n\}_{n=1}^\infty$ we have $\kappa_\Omega(z, \mathbb{R}) \leqslant c_2$ for every $z \in \cup_{n\geqslant 1}\gamma_n$.*
(4) *There exists a constant c_3 such that every geodesic bigon in Ω with vertices in $\mathbb{R}$ is c_3-thin.*

Note that the case $\Omega \cap \mathbb{R} = \cup_{n=0}^N (a_n, b_n)$ is also covered by the theorem.

Corollary 3.4. *Let Ω be a Denjoy domain of hyperbolic type and denote by κ_Ω either the Poincaré or the quasihyperbolic metric. If there exist a constant C and a sequence of fundamental geodesics $\{\gamma_n\}_{n\geqslant 1}$ with $\ell_{\kappa,\Omega}(\gamma_n) \leqslant C$, then κ_Ω is δ-Gromov hyperbolic, and δ depends only on C.*

If Ω has only finitely many boundary components, then it is always Gromov hyperbolic, in a quantitative way:

Proposition 3.5. *Let Ω be a Denjoy domain of hyperbolic type with $\Omega \cap \mathbb{R} = \cup_{n=1}^N (a_n, b_n)$, and denote by κ_Ω either the Poincaré or the quasihyperbolic metric. Then κ_Ω is δ-Gromov hyperbolic, where δ is a constant which only depends on N and*

$$c_0 = \sup_n \kappa_\Omega\big((a_n, b_n), (a_{n+1}, b_{n+1})\big).$$

Note that we do not require $b_n \leqslant a_{n+1}$ (although the intervals $\{(a_n, b_n)\}_n$ are, as always, pairwise disjoint).

Proof. Let us consider the shortest geodesics g_n^* with respect to κ_Ω joining (a_n, b_n) and (a_{n+1}, b_{n+1}) in $\Omega^+ := \Omega \cap \overline{\mathbb{H}^2}$. Then $\ell_\Omega(g_n^*) \leqslant \ell_\Omega(g_n) \leqslant c_0$ for $0 \leqslant n \leqslant N-1$.

By Theorem 3.3, we only need to prove that there exists a constant c, which only depends on c_0 and N, such that $\kappa_\Omega(z, \mathbb{R}) \leqslant c$ for every $z \in \cup_{n=1}^N \gamma_n$.

For each $0 \leqslant n \leqslant N-1$, let us consider the geodesic polygon P in Ω^+, with the following sides: $\gamma_n, g_0^*, \dots, g_{n-1}^*$, and the geodesics joining their endpoints which are contained in $(a_0, b_0), \dots, (a_n, b_n)$. Since $(\Omega^+, \kappa_\Omega)$ is δ_0-Gromov hyperbolic, where δ_0 is a constant which only depends on c_0, by Lemma 3.1, and P is a geodesic polygon in Ω^+ with at most $2N+2$ sides, P is $2N\delta_0$-thin. Therefore, given any $z \in \gamma_n$, there exists a point $w \in \cup_{k=0}^{N-1} g_k^* \cup \mathbb{R}$ with $\kappa_\Omega(z, w) \leqslant 2N\delta_0$. Since $\ell_\Omega(g_k^*) \leqslant c_0$ for $0 \leqslant k \leqslant N-1$, there exists $x \in \mathbb{R}$ with $\kappa_\Omega(x, w) \leqslant c_0/2$. Hence, $\kappa_\Omega(z, \mathbb{R}) \leqslant \kappa_\Omega(z, x) \leqslant 2N\delta_0 + c_0/2$, and we conclude that κ_Ω is δ-Gromov hyperbolic. □

Theorem 3.6. *Let Ω be a Denjoy domain with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$, $(a_0, b_0) = (-\infty, 0)$ and $b_n \leqslant a_{n+1}$ for every n. Suppose that $b_n \geqslant K a_n$ for a fixed $K > 1$ and every n. Then h_Ω and k_Ω are δ-Gromov hyperbolic, with δ depending only on K.*

Proof. Fix n and consider the domain

$$\Omega_n = \frac{1}{a_n}\,\Omega = \left\{ \frac{x}{a_n} \mid x \in \Omega \right\}.$$

If we define $D := \mathbb{C} \setminus ([0,1] \cup [K, \infty))$, then $D \subset \Omega_n$, and $\ell_{k,\Omega_n}(\gamma) \leqslant \ell_{k,D}(\gamma)$ for every curve $\gamma \subset D$. The circle $\sigma := S^1(0, (1+K)/2)$ goes around the boundary component $[0,1]$ in D and has finite quasihyperbolic length:

$$\ell_{k,D}(\sigma) \leqslant \int_\sigma \frac{|dz|}{(K-1)/2} = 2\pi\,\frac{K+1}{K-1}\,.$$

Consider the shortest fundamental geodesics joining (a_0, b_0) with (a_n, b_n), with the Poincaré and the quasihyperbolic metrics, γ_n^h and γ_n^k, respectively. Then,

$$\ell_{k,\Omega}(\gamma_n^k) = \ell_{k,\Omega_n}\Big(\frac{1}{a_n}\gamma_n^k\Big) \leqslant \ell_{k,\Omega_n}(\sigma) \leqslant \ell_{k,D}(\sigma) \leqslant 2\pi\,\frac{K+1}{K-1}\,,$$

$$\ell_{h,\Omega}(\gamma_n^h) \leqslant \ell_{h,\Omega}(\gamma_n^k) \leqslant 2\,\ell_{k,\Omega}(\gamma_n^k) \leqslant 4\pi\,\frac{K+1}{K-1}\,.$$

Therefore, by Corollary 3.4, h_Ω and k_Ω are δ-Gromov hyperbolic (and δ depends only on K). □

Proof of Theorems 1.1(1). If $\liminf_{n\to\infty}(b_n - a_n)/a_n > 0$, then we can choose $K > 1$ so that $(b_n - a_n)/a_n > K - 1$ for every n, whence $b_n > K a_n$. Thus the previous theorem implies the claims. □

4. Some classes of Denjoy domains which are not Gromov hyperbolic

The following function was introduced by Beardon and Pommerenke [7].

Definition 4.1. For $\Omega \subset \mathbb{C}$ a domain of hyperbolic type, define $\beta_\Omega(z)$ as the function

$$\beta_\Omega(z) := \inf\left\{ \left|\log\left|\frac{z-a}{b-a}\right|\right| :\ a, b \in \partial\Omega,\ |z-a| = \delta_\Omega(z) \right\}.$$

The function β_Ω has a geometric interpretation. We say that a closed set E is *uniformly perfect* if there exists a constant c such that $E \cap \{z \in \mathbb{C} :\ cr \leqslant |z-a| \leqslant r\} \neq \emptyset$ for every $a \in E$ and $0 < r < \infty$ (see [7, 17, 18]). Now we see that β_Ω is bounded precisely when $\mathbb{C} \setminus \Omega$ is uniformly perfect.

Thus it follows from the next theorem, that λ_Ω and $1/\delta_\Omega$ are comparable if and only if $\mathbb{C} \setminus \Omega$ is uniformly perfect.

Theorem 4.2 (Theorem 1, [7])**.** *For every domain $\Omega \subset \mathbb{C}$ of hyperbolic type and for every $z \in \Omega$, we have that*

$$2^{-3/2} \leqslant \lambda_\Omega(z)\,\delta_\Omega(z)\,(k_0 + \beta_\Omega(z)) \leqslant \pi/4\,,$$

where $k_0 = 4 + \log(3 + 2\sqrt{2})$.

Lemma 4.3. *Let γ be a rectifiable curve in a domain $D \subset \mathbb{R}^n$ from $a \in D$ with Euclidean length s. Then:*

(1) $\ell_{k,D}(\gamma) \geqslant \log\left(1 + \frac{s}{\delta_D(a)}\right)$.

(2) *If D is a Denjoy domain of hyperbolic type with $D \cap \mathbb{R} = \cup_{n\geqslant 0}(a_n, b_n)$ and $a \in (a_n, b_n)$, with $b_n - a_n \leqslant r$, then $\ell_{h,D}(\gamma) \geqslant 2^{-3/2} \log\big(1 + k_0^{-1} \log\big(1 + \frac{s}{r}\big)\big)$, where k_0 as in Theorem 4.2.*

Proof. For completeness we give the proof of statment (1) which is well-known. Let $z \in \partial D$ be a point with $\delta_D(a) = |a - z|$. Without loss of generality we assume that $z = 0$. By monotonicity $\ell_{k,D}(\gamma) \geqslant \ell_{k,\mathbb{R}^n\setminus\{0\}}(\gamma)$. Further, it is clear that $\ell_{k,\mathbb{R}^n\setminus\{0\}}(\gamma) \geqslant \ell_{k,\mathbb{R}^n\setminus\{0\}}([|a|, |a| + s])$, whence the first estimate by integrating the density $1/|x|$.

We then prove the second estimate. Without loss of generality we assume that $b_n = 0$. By monotonicity $\ell_{h,D}(\gamma) \geqslant \ell_{h,\mathbb{C}\setminus\{a_n,0\}}(\gamma)$. By [16, Theorem 4.1(ii)] we have that $\lambda_{\mathbb{C}\setminus\{a_n,0\}}(z) \geqslant \lambda_{\mathbb{C}\setminus\{a_n,0\}}(|z|)$ and by [16, Theorem 4.1(i)] that $\lambda_{\mathbb{C}\setminus\{a_n,0\}}(r)$ is a decreasing function in $r \in (0,\infty)$; hence, $\ell_{h,\mathbb{C}\setminus\{a_n,0\}}(\gamma) \geqslant \ell_{h,\mathbb{C}\setminus\{a_n,0\}}([|a_n|, |a_n| + s]) = \ell_{h,\mathbb{C}\setminus\{-1,0\}}([1, 1 + s/|a_n|])$. By Theorem 4.2

$$\begin{aligned}\ell_{h,D}(\gamma) \geqslant \ell_{h,\mathbb{C}\setminus\{-1,0\}}([1, 1 + s/|a_n|]) &\geqslant \int_1^{1+s/|a_n|} \frac{2^{-3/2}\,dx}{x\,(k_0 + \log x)} \\ &= 2^{-3/2} \log\Big(1 + k_0^{-1} \log\Big(1 + \frac{s}{|a_n|}\Big)\Big) \geqslant 2^{-3/2} \log\Big(1 + k_0^{-1} \log\Big(1 + \frac{s}{r}\Big)\Big). \quad \square\end{aligned}$$

Heuristic proof of Theorem 1.1(2), for the quasihyperbolic metric. We use the characterization of Bonk, Heinonen and Koskela [9]. Hence it suffices to show that the domain in not the conformal image of an inner uniform domain. However, we do not know how to deal with the conformal degree of freedom. Let us prove that the domain is not A-inner uniform. We stress that this is not a complete proof of what we want. However, this simple proof provides the outline for how to approach the hyperbolic metric. The proof of that case also works for the quasihyperbolic metric, in fact the latter case is easier. Therefore we will not provide the details for this and instead provide here just the heuristic idea of the proof of non-Gromov hyperbolicity.

So, suppose for a contradiction that the domain is A-inner uniform for some fixed $A > 0$. We define $s_n := \max_{1\leqslant m\leqslant n}(b_m - a_m)$. It is clear that s_n is an increasing sequence and the assumption of the theorem implies that $\lim_{n\to\infty} s_n/a_n = 0$. If we define $g_n := \sqrt{s_n/a_n}$, then $b_m - a_m \leqslant a_n g_n^2$ for every $1 \leqslant m \leqslant n$ and $\lim_{n\to\infty} g_n = 0$.

Since $g_n > 0$, we can choose a subsequence $\{g_{n_k}\}$ with $g_{n_k} \geqslant g_m$ for every $m \geqslant n_k$; consider a fixed n from the sequence $\{n_k\}$. Set $c_n = \frac{b_n+a_n}{2}$, the mid-point of (a_n, b_n). We define $x_n = c_n + ic_n g_n$ and $y_n = c_n - ic_n g_n$. Since $[x_n, y_n] \subset \Omega$, we have $\ell_{\mathrm{Eucl},\Omega}([x_n, y_n]) = 2c_n g_n$. Let γ be an A-inner uniform curve joining x_n and y_n, and let $z \in \gamma \cap \mathbb{R}$. Since $|x_n - z|, |y_n - z| \geqslant c_n g_n$, we conclude by the uniformity of the curve that $\delta_\Omega(z) \geqslant \frac{c_n g_n}{A}$. On the other hand, the uniformity of γ also implies that $|z - c_n| \leqslant 2Ac_n g_n$.

We may assume that n is so large that $c_n > 2Ac_n g_n$. Then z lies in the positive real axis, which means that $z \in (a_m, b_m)$ for some $m \geqslant 1$. If $m \leqslant n$, then we have $b_m - a_m \leqslant s_n = a_n g_n^2 < c_n g_n^2$. For $m > n$ we have $b_m - a_m \leqslant g_m^2 a_m \leqslant g_n^2 a_m$. However, since $a_m < z \leqslant c_n + 2Ac_n g_n < 2c_n$, we obtain $b_m - a_m < 2c_n g_n^2$ also in this case.

Since $\delta_\Omega(z) < \frac{b_m - a_m}{2}$, we conclude that $\frac{c_n g_n}{A} < c_n g_n^2$. Since $g_n \to 0$ and A is a constant, this is a contradiction. Hence the assumption that an A-inner uniform curve exists was false, and we can conclude that the domain is not Gromov hyperbolic. $\square$

For the proof of Theorem 1.1(2) in the hyperbolic case we need the following concepts. A function between two metric spaces $f : X \longrightarrow Y$ is an (α, β)-*quasi-isometry*, $\alpha \geqslant 1$,

$\beta \geqslant 0$, if

$$\frac{1}{\alpha}\, d_X(x_1,x_2) - \beta \leqslant d_Y(f(x_1), f(x_2)) \leqslant \alpha d_X(x_1,x_2) + \beta\,, \qquad \text{for every } x_1,x_2 \in X.$$

An (α,β)-*quasigeodesic* in X is an (α,β)-quasi-isometry between an interval of $\mathbb{R}$ and X.

For future reference we record the following lemma:

Lemma 4.4. *Let us consider a geodesic metric space X and a geodesic $\gamma : I \longrightarrow X$, with I any interval, and $g : I \longrightarrow X$, with $d(g(t),\gamma(t)) \leqslant \varepsilon$ for every $t \in I$. Then g is a $(1,2\varepsilon)$-quasigeodesic.*

Proof. We have for every $s,t \in I$

$$d(g(s),g(t)) \geqslant d(\gamma(s),\gamma(t)) - d(\gamma(s),g(s)) - d(\gamma(t),g(t)) \geqslant |t-s| - 2\varepsilon.$$

The upper bound is similar. □

Proof of Theorem 1.1(2), for the hyperbolic metric. We consider two cases: either $\{b_m - a_m\}_m$ is bounded or unbounded. We start with the latter case.

As in the previous proof, we define $s_n := \max_{1\leqslant m\leqslant n}(b_m - a_m)$ and $g_n := \sqrt{s_n/a_n}$. Then $b_m - a_m \leqslant a_n g_n^2$ for every $1 \leqslant m \leqslant n$ and $\lim_{n\to\infty} g_n = 0$. Since $g_n > 0$, we can choose a subsequence $\{g_{n_k}\}$ with $g_{n_k} \geqslant g_m$ for every $m \geqslant n_k$. Since $\{b_m - a_m\}_m$ is not bounded we may, moreover, choose the sequence so that $g_n^2 = (b_n - a_n)/a_n$ for every $n \in \{n_k\}$. Fix now n from the sequence $\{n_k\}$. As before, we conclude that $b_m - a_m \leqslant a_n g_n^2$ for $m \leqslant n$ and $b_m - a_m \leqslant a_m g_m^2 \leqslant a_m g_n^2$ for $m > n$.

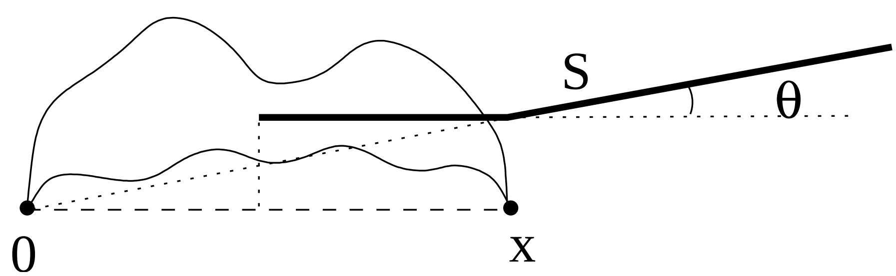

FIGURE 2. The set S

Consider $x \in (a_n,b_n)$ which lies on the shortest fundamental geodesic γ_n joining $(-\infty,0)$ with (a_n,b_n). Define an angle $\theta = \arctan g_n \in (0,\pi/2)$ and a set

$$S = [\tfrac{1}{2}x + ixg_n, x + ixg_n] \cup \{x + ixg_n + te^{\pi i\theta} \mid t \geqslant 0\}.$$

The set S is shown in Figure 2. Notice that any point $\zeta \in S$ satisfies $g_n \operatorname{Re}\zeta \leqslant \operatorname{Im}\zeta \leqslant 2g_n \operatorname{Re}\zeta$. It is clear that γ_n hits the set $S \cup [\frac{1}{2}x + ixg_n, \frac{1}{2}x]$. We claim that it in fact hits S. Assume to the contrary that this is not the case. Then it hits $[\frac{1}{2}x + ixg_n, \frac{1}{2}x]$. Let γ' denote a part of γ_n connecting x and this segment which does not intersect S. Since Ω is a Denjoy domain, we conclude that $b \mapsto \lambda_\Omega(a+ib)$ is decreasing for $b > 0$ (see [16, Theorem 4.1(i)]). Hence $\ell_{h,\Omega}(\gamma') \geqslant \ell_{h,\Omega}([\frac{1}{2}x + ixg_n, x + ixg_n])$. Since the gap size in $[\frac{1}{2}x, x]$ is at most $a_n g_n^2$, we have $\delta_\Omega(w) \leqslant \sqrt{x^2 {g_n}^2 + a_n^2 g_n^4} \leqslant \sqrt{2}\, xg_n$. Since the gap size is smaller than the distance to the boundary, it follows from Theorem 4.2 that

$$\lambda_\Omega(w) \geqslant \frac{C}{\delta_\Omega(w)} \geqslant \frac{C}{xg_n}$$

for $w \in [\frac{1}{2}x + ixg_n, x + ixg_n]$. Multiplying this with the Euclidean length $\frac{1}{2}x$ of the segment gives

$$\ell_{h,\Omega}(\gamma_n) \geqslant \ell_{h,\Omega}([\tfrac{1}{2}x + ixg_n, x + ixg_n]) \geqslant \frac{C}{g_n}.$$

We next construct another path σ and show that it is in the same homotopy class as the supposed geodesic, only shorter. Let z be the midpoint of gap n and let σ be the curve $[z, z + iz] \cup [z + iz, -z + iz] \cup [-z + iz, -z]$. Using $b_n - a_n = a_n g_n^2$ we easily calculate

$$\ell_{h,\Omega}(\sigma) \leqslant 2\ell_{k,\Omega}(\sigma) \leqslant 2 \log\Big(\frac{2z}{a_n g_n^2}\Big) + C \leqslant 4 \log\Big(\frac{1}{g_n}\Big) + C$$

with an absolute constant C. The curve σ joins $(-\infty, 0)$ and (a_n, b_n); therefore $\ell_{h,\Omega}(\gamma_n) \leqslant \ell_{h,\Omega}(\sigma)$. But this contradicts the previously derived bounds for the lengths as $g_n \to 0$.

Therefore the supposition that γ_n does not intersect S was wrong, so we conclude that $\gamma_n \cap S \neq \emptyset$. Let now $\zeta \in S \cap \gamma_n$. We claim that $h_\Omega(\zeta, \mathbb{R}) \to \infty$, which means the domain is not Gromov hyperbolic, by Theorem 3.3. Let $\xi \in \Omega \cap \mathbb{R}$; chose m so that $\xi \in (a_m, b_m)$. Let α be a curve joining ξ and ζ.

If $0 < m \leqslant n$, then the size of (a_m, b_m) is at most $a_n g_n^2$, so $\delta_\Omega(\xi) \leqslant a_n g_n^2$. Then α has Euclidean length at least $\operatorname{Im} \zeta \geqslant xg_n$, so by Lemma 4.3, $\ell_{h,\Omega}(\alpha) \geqslant c \log\log(C/g_n)$. As $g_n \to 0$, this bound tends to ∞. If, on the other hand, $m > n$, then the Euclidean length of α is at least

$$d(\xi, \zeta) \geqslant d(\xi, S) \geqslant \xi \sin\theta \geqslant \tfrac{1}{2}\, \xi \tan\theta = \tfrac{1}{2}\, \xi g_n,$$

and the size of the gap is at most $a_m g_n^2$. By Lemma 4.3 this implies that $\ell_{h,\Omega}(\alpha) \geqslant c \log\log(C/g_n)$. As $g_n \to 0$, this bound again tends to ∞.

It remains to consider $m = 0$, i.e., $\xi < 0$. We consider only the case $\zeta \in [\frac{1}{2}x + ixg_n, x + ixg_n]$, since the other case is similar. Now the Euclidean length of α is at least $\frac{1}{2}x$. Since the gap size in $[0, \frac{1}{2}x]$ is at most $a_n g_n^2$, we see that the boundary satisfies the separation condition when $|\operatorname{Im} z| \geqslant a_n g_n^2$ in which case also $\delta_\Omega(z) \geqslant |\operatorname{Im} z| \geqslant a_n g_n^2$. Since $\lambda_\Omega(z)$ is decreasing in $|\operatorname{Im} z|$ (see [16, Theorem 4.1(i)]), we conclude that

$$\lambda_\Omega(z) \geqslant \frac{C}{\max\{|\operatorname{Im} z|, a_n g_n^2\}} \geqslant \frac{C}{\max\{\delta_\Omega(z), a_n g_n^2\}} \tag{4.5}$$

for the points on the curve with $\operatorname{Re} z \in (0, x/2)$. Let α^- be the part of α on which $\delta_\Omega(z) < a_n g_n^2$. If $\ell_{\mathrm{Eucl}}(\alpha^-) > xg_n^{3/2}$, then

$$\ell_{h,\Omega}(\alpha) \geqslant \ell_{h,\Omega}(\alpha^-) \geqslant \frac{xg_n^{3/2}}{a_n g_n^2} > g_n^{-1/2}.$$

If $\ell_{\mathrm{Eucl}}(\alpha^-) \leqslant xg_n^{3/2}$, then $\ell_{\mathrm{Eucl}}(\alpha \setminus \alpha^-) > \frac{1}{2}x - xg_n^{3/2}$. Hence we conclude (as in the proof of part (1) in Lemma 4.3) that

$$\int_\alpha \lambda_\Omega(z)\, |dz| \geqslant C \int_{\delta_\Omega(\zeta) + xg_n^{3/2}}^{x/2} \frac{dt}{t} \geqslant C \log\Big(\frac{x/2}{\sqrt{2}\, a_n g_n + xg_n^{3/2}}\Big) \geqslant C \log\Big(\frac{1}{g_n}\Big) - C.$$

Hence in either case we get a lower bound which tends to infinity as $g_n \to 0$.

This takes care of the case when $\{b_m - a_m\}_m$ is unbounded. Assume next that $\sup_m(b_m - a_m) = M < \infty$. In this case it is difficult to work with bigons, since we do not get a good control on what the geodesics look like; the problem with the previous argument is that we cannot choose $g_{n_k}^2 = (b_{n_k} - a_{n_k})/a_{n_k}$ in our sequence, and consequently we do not get a good bound on the length of the curve σ, as defined above.

To get around this we consider a geodesic triangle. Assume for a contradiction that h_Ω is δ-Gromov hyperbolic. By geodesic stability [10], there exists a number δ' so that every $(\sqrt{2}, 0)$-quasigeodesic triangle is δ'-thin.

Fix $R \gg M^2$ and set $w_\pm = \pm iR$. Let γ_0 be the geodesic segment joining w_+ and w_-. Choose $t > 0$ so large that $h_\Omega(\gamma_0, H_t) > \delta'$, where $H_t = \{z \in \mathbb{C} \,|\, \operatorname{Re} z > t\}$. Let $w \in \Omega \cap \mathbb{R}$ be a point in $H_{2\max\{t,R\}}$, and let $\gamma_+ \subset \overline{\mathbb{H}^2}$ be a geodesic joining w and w_+.

If γ_+ dips below the ray from w through w_+, then we replace the part below the ray by a part of the ray. The resulting curve is denoted by $\tilde{\gamma}_+$. Let us show that $\tilde{\gamma}_+$ is a quasigeodesic. We define a mapping $f\colon \gamma_+ \to \tilde{\gamma}_+$ as follows. If $x \in \gamma_+ \cap \tilde{\gamma}_+$, then $f(x) = x$. If $x \in \gamma_+ \setminus \tilde{\gamma}_+$ then we set $f(x)$ to equal the point on $\tilde{\gamma}_+$ with real part equal to $\operatorname{Re} x$.

Since Ω is a Denjoy domain, the function $b \mapsto \lambda_\Omega(a + ib)$ is decreasing for $b > 0$ (see [16, Theorem 4.1(i)]). Hence $\lambda_\Omega(f(x)) \leqslant \lambda_\Omega(x)$. The arc-length distance element is the vertical projection of the distance element at x to the line through w and w_+: specifically, the distance element (dx, dy) becomes $(dx, \theta dx)$, where θ is the slope of the line. Thus the maximal increase in the distance element is $\sqrt{1 + \theta^2}$. Since the slope of the line lies in the range $[-1, 0)$, we conclude from these facts that $\tilde{\gamma}_+$ is a $(\sqrt{2}, 0)$-quasigeodesic.

Similarly, we construct $\tilde{\gamma}_-$ and conclude that it is a $(\sqrt{2}, 0)$-quasigeodesic. Choose now $\zeta \in \tilde{\gamma}_+ \cap H_{\max\{t,R\}}$ with $\operatorname{Im} \zeta = \sqrt{R}$. Since $\gamma_0 \cup \tilde{\gamma}_+ \cup \tilde{\gamma}_-$ is a $(\sqrt{2}, 0)$-quasigeodesic triangle, it should be possible to to connect ζ with some point in $\gamma_0 \cup \tilde{\gamma}_-$ using a path of length δ'. By the definition of t, $h_\Omega(\zeta, \gamma_0) > \delta'$. If α is a path connecting ζ and γ_-, then it crosses the real axis at some point ξ. If ξ lies in (a_m, b_m), $m > 0$, then $\ell_{h,\Omega}(\alpha) \geqslant C \log\log \frac{\sqrt{R}}{M}$, by Lemma 4.3. Otherwise, $\xi \in (-\infty, 0)$. This case is handled as in the first case of the proof, see the paragraph around (4.5). In each case we see that $h_\Omega(\zeta, \gamma_-) > \delta'$ provided R is large enough. But this means that Ω is not Gromov hyperbolic, which finishes the proof. ☐

In Theorem 1.1(2) the gaps (a_n, b_n) and (a_{n+1}, b_{n+1}) are separated by a boundary component $[b_n, a_{n+1}]$. We easily see from the proofs that it would have made no difference if this boundary component had some gaps, as long as they at most comparable to the lengths of the adjacent gaps, (a_n, b_n) and (a_{n+1}, b_{n+1}). Thus we get the following stronger theorem by the same proofs. (In the proofs we can assume that $(-\infty, 0) \subset \Omega$, by using Theorem 1.2).

Theorem 4.6. *Let Ω be a Denjoy domain with $\Omega \cap \mathbb{R} = \cup_{n=0}^\infty (a_n, b_n)$ and $\limsup_{n\to\infty} a_n = \infty$. Suppose $G\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a function with $\lim_{x\to\infty} G(x) = 0$. If $b_n - a_n \leqslant a_n G(a_n)$ for every $a_n > 0$, then κ_Ω, the hyperbolic or quasihyperbolic metric, is not Gromov hyperbolic.*

The function G plays the role of g_n^2 in the proof of Theorem 1.1(2).

Remark 4.7. The condition $\Omega \cap \mathbb{R} = \cup_{n=0}^\infty (a_n, b_n)$ (without the hypothesis $b_n \leqslant a_{n+1}$ for every n) allows any topological behavior; for instance, $\partial\Omega$ can contain a countable sequence of Cantor sets.

Let $E_0 \subset [0, t)$ be closed, $t > 0$, set $E_n := E_0 + tn$ for $n \in \mathbb{N}$, and $\Omega := \mathbb{C} \setminus \cup_n E_n$. Then Ω satisfies the hypotheses of Theorem 4.6 for $G(x) = t/x$. From this we deduce

Example 1.3, the non-hyperbolicity of periodic Denjoy domain, in the case the index set is $\mathbb{N}$. The case with index set $\mathbb{Z}$ follows from this and Theorem 1.2.

5. On the far side of the accumulation point

Lemma 5.1. *Let Ω be a Denjoy domain with $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$ and $a_0 = -\infty$. If h_Ω is not Gromov hyperbolic, then for every $N > 0$ there exist fundamental geodesics γ_{n_k}, $n_k > N$, such that the hyperbolic distance of the endpoints of γ_{n_k} to $(-\infty, b_0)$ is greater than N, and points $z_k \in \gamma_{n_k}$ with $\lim_{k\to\infty} h_\Omega(z_k, \mathbb{R}) = \infty$.*

Proof. Let us choose fundamental geodesics $\{\gamma_n^0\}$. Since h_Ω is not Gromov hyperbolic, by Theorem 3.3 there exists points $w_k \in \gamma_{n_k}^0$ with $n_k > N$ and $\lim_{k\to\infty} h_\Omega(w_k, \mathbb{R}) = \infty$. Since $\lim_{x\to b_n} h_\Omega(x, (-\infty, b_0)) = \infty$ for every n, there exist $x_0 \in (a_0, b_0)$ and $x_{n_k} \in (a_{n_k}, b_{n_k})$, with $h_\Omega(x_0, (-\infty, b_0)), h_\Omega(x_{n_k}, (-\infty, b_0)) > N$.

Let us consider the fundamental geodesics γ_{n_k} joining x_0 and x_{n_k}, as well as the bordered Riemann surface $X := \Omega \cap \overline{\mathbb{H}^2}$, which as in the proof of Theorem 3.1 can be shown to have $\log\big(1+\sqrt{2}\,\big)$-thin triangles.

Let Q_k be the geodesic quadrilateral given by $\gamma_{n_k}^0$, γ_{n_k} and the two geodesics (contained in (a_0, b_0) and (a_{n_k}, b_{n_k})) joining their endpoints. Since $Q_k \subset X$, it is $2\log\big(1+\sqrt{2}\,\big)$-thin, and there exists $z_k \in \gamma_{n_k} \cup \mathbb{R}$ with $h_\Omega(z_k, w_k) \leqslant 2\log\big(1+\sqrt{2}\,\big)$.

Since $\lim_{k\to\infty} h_\Omega(w_k, \mathbb{R}) = \infty$, we deduce that $z_k \in \gamma_{n_k}$ for every $k \geqslant k_0$ and $\lim_{k\to\infty} h_\Omega(z_k, \mathbb{R}) = \infty$. □

Lemma 5.2 (Lemma 3.1, [1])**.** *Consider an open Riemann surface S of hyperbolic type, a closed non-empty subset C of S, and set $S^* := S \setminus C$. For $\epsilon > 0$ we have $1 < \ell_{S^*}(\gamma)/\ell_S(\gamma) < \coth(\varepsilon/2)$, for every curve $\gamma \subset S$ with finite length in S such that $h_S(\gamma, C) \geqslant \varepsilon$.*

Given a Riemann surface S, a geodesic γ in S, and a continuous unit vector field ξ along γ orthogonal to γ, we define *Fermi coordinates* based on γ as the map $Y(r,t) := \exp_{\gamma(r)} t\xi(r)$.

It is well known that if the curvature is $K \equiv -1$, then the Riemannian metric can be expressed in Fermi coordinates as $ds^2 = dt^2 + \cosh^2 t\, dr^2$ (see e.g. [11, p. 247–248]).

Corollary 5.3. *Consider an open Riemann surface of hyperbolic type S, a closed non-empty subset C of S, and set $S^* := S \setminus C$. For $\epsilon > 0$ and $C_\varepsilon := \{z \in S : h_S(z, C) \geqslant \varepsilon\}$ we have*

$$h_S(z,w) \leqslant h_{S^*}(z,w), \qquad \text{for every } z, w \in S^*,$$

$$h_{S^*}(z,w) \leqslant \coth(\varepsilon/2)\, h_{S|C_\varepsilon}(z,w), \qquad \text{for every } z, w \in C_\varepsilon\,.$$

Furthermore, if S is a Denjoy domain and C is a component of $S \cap \mathbb{R}$ then

$$h_{S^*}(z,w) \leqslant \cosh\varepsilon\, \coth(\varepsilon/2)\, h_S(z,w),$$

for every z, w in the same component of C_ε with $\operatorname{Im} z, \operatorname{Im} w \geqslant 0$.

Proof. The first and second inequalities are direct consequences of Lemma 5.2. In order to prove the third one, it is sufficient to prove that

$$h_{S|C_\varepsilon}(z,w) \leqslant (\cosh\varepsilon)\, h_S(z,w), \tag{5.4}$$

for every z, w in the same component of C_ε with $\operatorname{Im} z, \operatorname{Im} w \geqslant 0$.

Fix z, w in the same component Γ of C_ε. Since $\operatorname{Im} z, \operatorname{Im} w \geqslant 0$ there exists a unique geodesic $\gamma \subset S \cap \overline{\mathbb{H}^2}$ joining z with w.

If $\gamma \subset \Gamma$, then $h_{S|C_\varepsilon}(z,w) = h_S(z,w)$. If γ is not contained in Γ, then it is sufficient to show that there exists a curve η joining z and w in Γ, with $\ell_{h,S}(\eta) \leqslant (\cosh\varepsilon)\,\ell_{h,S}(\gamma)$. In order to prove this, consider the geodesics $\gamma_z, \gamma_w \subset S \cap \overline{\mathbb{H}^2}$ joining z and w with C, and the geodesic $\gamma_0 \subset C$ joining the endpoints of γ_z, γ_w (which are in C).

We denote by P the simply connected closed region with boundary $\gamma \cap \gamma_z \cap \gamma_w \cap \gamma_0$. Since P is simply connected, we can identify it with a domain $P_0 \subset \overline{\mathbb{H}^2}$ using Fermi coordinates based on C.

If g is the lift of γ, then $g_1 := g \cap \{(r,t) :\, 0 \leqslant t \leqslant \varepsilon\}$ is the lift of $\gamma \setminus C_\varepsilon$. If $g \cap \{(r,t) : t = \varepsilon\} = \{(r_1,\varepsilon),(r_2,\varepsilon)\}$ (with $r_1 < r_2$), then we define $g_2 := \{(r,\varepsilon) :\, r_1 \leqslant r \leqslant r_2\}$ and $g_0 := \{(r,0) :\, r_1 \leqslant r \leqslant r_2\}$. Notice that in order to prove (5.4) it is sufficient to show that $\ell(g_2) \leqslant (\cosh\varepsilon)\,\ell(g_1)$. But this is a direct consequence of the facts $\ell(g_0) \leqslant \ell(g_1)$ and $\ell(g_2) = (\cosh\varepsilon)\,\ell(g_0)$. □

Proof of Theorem 1.2. Since κ_Ω is not Gromov hyperbolic, by Proposition 3.5, we conclude that Ω has countably infinitely many boundary components: $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$. Without loss of generality we can assume that $(-\infty, 0) \subseteq (a_1, b_1)$.

We first prove that $(\Omega\setminus F, k_{\Omega\setminus F})$ is not Gromov hyperbolic. Let us consider fundamental geodesics γ_n of k_Ω joining the midpoint c_0 of (a_0,b_0) with the midpoint c_n of (a_n,b_n) for $n \geqslant 2$ which are shortest possible. Since γ_n is contained in $\{z \in \mathbb{C} :\, c_0 \leqslant \operatorname{Re} z \leqslant c_n\}$, and $k_{\Omega\setminus F} = k_\Omega$ in $\{z \in \mathbb{C} :\, \operatorname{Re} z \geqslant \inf_{n\geqslant 2} a_n\}$, we deduce that γ_n is also a fundamental geodesic with the metric $k_{\Omega\setminus F}$.

Since k_Ω is not Gromov hyperbolic, there exist points $z_k \in \gamma_{n_k}$ with $\lim_{k\to\infty} k_\Omega(z_k, \mathbb{R}) = \infty$ by Theorem 3.3. Since γ_{n_k} are also fundamental geodesics with the metric $k_{\Omega\setminus F}$, we deduce that $\lim_{k\to\infty} k_{\Omega\setminus F}(z_k,\mathbb{R}) \geqslant \lim_{k\to\infty} k_\Omega(z_k,\mathbb{R}) = \infty$. Consequently, $(\Omega \setminus F, k_{\Omega\setminus F})$ is not Gromov hyperbolic.

We now prove that $(\Omega \setminus F, h_{\Omega\setminus F})$ is not Gromov hyperbolic. Choose $\varepsilon_0 > 0$. Since h_Ω is not Gromov hyperbolic, by Lemma 5.1 there exist fundamental geodesics γ_{n_k} of h_Ω, such that the hyperbolic distance of the endpoints of γ_{n_k} to $(-\infty, b_1)$ is greater than ε_0, and points $z_k \in \gamma_{n_k}$ with $\lim_{k\to\infty} h_\Omega(z_k,\mathbb{R}) = \infty$.

Fix $\varepsilon \in \big(0, \min\{\varepsilon_0, \min_k h_\Omega(z_k,\mathbb{R})\}\big)$. If we define

$$U_\varepsilon := \{z \in \Omega :\, h_\Omega(z, (-\infty, b_1)) \geqslant \varepsilon\},$$

we see that $z_k \in \gamma_{n_k} \cap U_\varepsilon$ for every k. (Notice that $\gamma_{n_k} \cap \partial U_\varepsilon$ has at most two points.) If $\gamma_{n_k} \cap \partial U_\varepsilon$ is empty or a one-point set, we define $g_{n_k} := \gamma_{n_k}$. Since the endpoints of γ_{n_k} are in U_ε, we conclude that $g_{n_k} \subset U_\varepsilon$.

Then we consider the remaining case, $\gamma_{n_k} \cap \partial U_\varepsilon = \{w^1, w^2\}$. If there is an arc α in ∂U_ε joining w^1 and w^2, we define a curve g_{n_k} joining (a_0,b_0) with (a_{n_k}, b_{n_k}) in U_ε, by $g_{n_k} := (\gamma_{n_k} \cap U_\varepsilon) \cup \alpha$. Then γ_{n_k} and g_{n_k} have the same endpoints and are homotopic. If there is not an arc in ∂U_ε joining w^1 and w^2, there are still maximal arcs α, β in ∂U_ε joining w^1 and $\omega^1 \in (a_{m^1}, b_{m^1})$, and w^2 and $\omega^2 \in (a_{m^2}, b_{m^2})$, respectively, and a geodesic η (with respect to h_Ω) in $\Omega\setminus U_\varepsilon$ joining ω^1 and ω^2, such that if $\gamma_{n_k} \cap U_\varepsilon = [z^1, w^1] \cup [z^2, w^2]$, then $[z^1,w^1] \cup \alpha \cup \eta \cup \beta \cup [z^2, w^2]$ has the same endpoints as γ_{n_k}, and they are homotopic.

Since $\varepsilon < h_\Omega(z_k,\mathbb{R})$, we have either $z_k \in [z^1,w^1]$ or $z_k \in [z^2,w^2]$. Without loss of generality we can assume that $z_k \in [z^2,w^2]$. Then we define $g_{n_k} := \beta \cup [z^2,w^2] \subset U_\varepsilon$, which is a curve joining (a_{m^2}, b_{m^2}) with (a_{n_k}, b_{n_k}).

In any case, Lemma 4.4 gives that g_{n_k} is a $(1, 2\varepsilon)$-quasigeodesic with respect to h_Ω. Hence, for every t, s, we have

$$|t-s| - 2\varepsilon \leqslant h_\Omega\big(g_{n_k}(t), g_{n_k}(s)\big) \leqslant |t-s| + 2\varepsilon.$$

Since g_{n_k} is contained in U_ε, Corollary 5.3 implies that

$$\begin{aligned} |t-s| - 2\varepsilon &\leqslant h_\Omega\big(g_{n_k}(t), g_{n_k}(s)\big) < h_{\Omega\setminus F}\big(g_{n_k}(t), g_{n_k}(s)\big) \\ &\leqslant h_{\Omega\setminus(-\infty,0]}\big(g_{n_k}(t), g_{n_k}(s)\big) \\ &\leqslant \cosh\varepsilon\, \coth(\varepsilon/2)\, h_\Omega\big(g_{n_k}(t), g_{n_k}(s)\big) \\ &\leqslant \cosh\varepsilon\, \coth(\varepsilon/2)\, \big(|t-s| + 2\varepsilon\big), \end{aligned}$$

and hence g_{n_k} is a $\big(\cosh\varepsilon \coth(\varepsilon/2), 2\varepsilon\cosh\varepsilon\coth(\varepsilon/2)\big)$-quasigeodesic with respect to $h_{\Omega\setminus F}$.

To get a contradiction, assume that $(\Omega \setminus F, h_{\Omega\setminus F})$ is Gromov hyperbolic. Consider the fundamental geodesic η_{n_k} of $h_{\Omega\setminus F}$ with the same endpoints as g_{n_k}. Then there is a constant C such that the Hausdorff distance of g_{n_k} and η_{n_k} is less than C. Hence, there exist points $w_k \in \eta_{n_k}$ with $h_{\Omega\setminus F}(z_k, w_k) \leqslant C$, and thus

$$\lim_{k\to\infty} h_{\Omega\setminus F}(w_k, \mathbb{R}) \geqslant \lim_{k\to\infty} h_{\Omega\setminus F}(z_k, \mathbb{R}) - C \geqslant \lim_{k\to\infty} h_\Omega(z_k, \mathbb{R}) - C = \infty,$$

which contradicts $h_{\Omega\setminus F}$ being Gromov hyperbolic. □

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