GROMOV HYPERBOLICITY THROUGH DECOMPOSITION OF METRIC SPACES II

Ana Portilla, José M. Rodríguez[1] **and Eva Tourís**[2]

AMS Subj. Classification: 30F20, 30F45

Key Words: Gromov hyperbolicity, metric space, decomposition.

§0. Abstract

In this paper we study the hyperbolicity in the Gromov sense of metric spaces. We deduce the hyperbolicity of a space from the hyperbolicity of its "building block components", which can be joined following an arbitrary scheme. These results are especially valuable since they simplify notably the topology and allow to obtain global results from local information. Some interesting theorems about the role of punctures and funnels on the hyperbolicity of Riemann surfaces can be deduced from the conclusions of this paper.

§1. Introduction

A good way to understand the important connections between graphs and Potential Theory on Riemannian manifolds (see e.g. [ARY], [CFPR], [FR], [HS], [K1], [K2], [S]) is to study the Gromov hyperbolic spaces. This approach allows to establish a general setting to work simultaneously with graphs and manifolds, in the context of metric spaces. Besides, the idea of Gromov hyperbolicity grasps the essence of negatively curved spaces, and has been successfully used in the theory of groups (see e.g. [GH] and the references therein).

Although there exist some interesting examples of hyperbolic spaces (see the examples after Definition 1), the literature gives no good guide about how to determine whether or not a space is hyperbolic. This limitation can be somehow got round, since the theory allows to obtain powerful results about non-hyperbolic spaces which have hyperbolic universal coverings. As topological "obstacles" may prevent a space from being hyperbolic, the possibility of studying its universal covering instead, which is always free of obstacles, implies a substantial simplification, and sometimes let us extract important information about the space itself (see [P]).

However, as was stated above, the characterization of hyperbolic spaces remains open. Recently, some interesting results about the hyperbolicity of Euclidean bounded domains with their quasihyperbolic metric have made significant progress in this direction (see [BHK] and the references therein).

Originally, we were interested in studying when non-exceptional Riemann surfaces equipped with its Poincaré metric were Gromov hyperbolic. However, we have proved several theorems on hyperbolicity for general metric spaces, which are interesting by themselves and have important consequences for Riemann surfaces (see [PRT]). Although one should expect Gromov hyperbolicity in non-exceptional Riemann surfaces due to its constant curvature -1, this turns out to be untrue in general, since topological obstacles can impede it: for instance, the two-dimensional jungle-gym (a $\mathbf{Z}^2$-covering of a torus with genus two) is not hyperbolic. Let us recall that in the case of modulated plane domains, quasihyperbolic metric and Poincaré metric are equivalent. One can find results on hyperbolicity of Riemann surfaces in [RT] and [PRT].

[1]The research of the second author was partially supported by a grant from DGI (BFM 2000-0022) Spain.
[2]The research of the third author was supported by a grant from DGI (BFM 2000-0022) Spain.

Here we present the outline of the main results. We refer to the next sections for the definitions and the precise statements of the theorems.

The main aim in this paper is obtaining global results on hyperbolicity from local information. That was the idea that lead us to think of a space X as the union of some "pieces" or "building block components" $\{X_n\}_{n\in\Lambda}$.

Our first theorem (see Theorem 1) states that if the above mentioned pieces X_n are joined together following a tree-like design (that is, avoiding the creation of extra topological obstacles), then the uniform hyperbolicity of the pieces guarantees the hyperbolicity of the global space X.

However, if pieces are joined together in a general graph-like style (that is to say, the hypothesis on simple topological connections is removed), the uniform hyperbolicity of pieces is no longer enough to guarantee the hyperbolicity of the global space X. But, surprisingly, if Y is a graph that models appropriately the connections among uniformly hyperbolic pieces X_n's, the hyperbolicity of Y let us assure the hyperbolicity of X. (This fact turns out to be obvious when Y is a tree.)

It is noticeable that the graph Y must comply with some metrical requests in order to be an acceptable model for the connections among the pieces X_n. However, Y is not required at all to model the subspaces themselves (which might be arbitrarily wide far away from the connections). Taking advantage of these facts, Theorem 2 provides a much more general frame, since it does not require that the space Y used as a model to stick the pieces together is a graph.

When applied to Riemann surfaces these theorems let us deduce interesting consequences. In [PRT, Theorems 3.2 and 3.4] we work on the role of punctures and funnels of a Riemann surface in its hyperbolicity. These results allow, in many cases, to forget punctures and funnels in order to analyze the hyperbolicity of a Riemann surface; this fact can be a significant simplification in the topology of the surface, and therefore makes easier the study of its hyperbolicity.

It is a remarkable fact that the constants appearing in the theorems of this paper depend just on a small number of parameters. This is a common place in the theory of hyperbolic spaces (see e.g. theorems A, B and C).

Notations. We denote by X or X_n geodesic metric spaces. By d_X, L_X and B_X we shall denote, respectively, the distance, the length and the balls in the metric of X.

Finally, we denote by k_i positive constants which can assume different values in different theorems.

Acknowledgements. We would like to thank Professors J. L. Fernández and M. V. Melián for some useful discussions.

§2. Results in metric spaces

In our study of hyperbolic Gromov spaces we use the notations of [GH]. We give now the basic facts about these spaces. We refer to [GH] for more background and further results.

Definition 1. Let us fix a point w in a metric space (X,d). We define the *Gromov product* of $x,y\in X$ with respect to the point w as

$$(x|y)_w := \frac{1}{2}\left(d(x,w)+d(y,w)-d(x,y)\right)\geq 0\,.$$

We say that the metric space (X,d) is *δ-hyperbolic* ($\delta \geq 0$) if

$$(x|z)_w \geq \min\big\{(x|y)_w, (y|z)_w\big\} - \delta\,,$$

for every $x,y,z,w \in X$. We say that X is *hyperbolic* (in the Gromov sense) if the value of δ is not important.

It is convenient to remark that this definition of hyperbolicity is not universally accepted, since sometimes the word hyperbolic refers to negative curvature or to the existence of Green's function. However, in this paper we only use the word *hyperbolic* in the sense of Definition 1.

Examples: (1) Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic (see e.g. [GH, p.29]).

(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k$, with $k > 0$, is hyperbolic (see e.g. [GH, p.52]).

(3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [GH, p.29]).

Definition 2. If $\gamma : [a,b] \longrightarrow X$ is a continuous curve in a metric space (X,d), we can define the length of γ as

$$L(\gamma) := \sup\Big\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}),\gamma(t_i)) \,:\, a = t_0 < t_1 < \cdots < t_n = b \Big\}\,.$$

We say that γ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t),\gamma(s)) = |t-s|$ for every $s,t \in [a,b]$. We say that X is a *geodesic metric space* if for every $x,y \in X$ there exists a geodesic joining x and y; we denote by $[x,y]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected.

Definition 3. If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$, with $J_j \subseteq X$, we say that J is *δ-thin* if for every $x \in J_i$ we have that $d(x, \cup_{j\neq i} J_j) \leq \delta$. If $x_1,x_2,x_3 \in X$, a *geodesic triangle* $T = \{x_1,x_2,x_3\}$ is the union of three geodesics $[x_1,x_2]$, $[x_2,x_3]$ and $[x_3,x_1]$. The space X is *δ-thin* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin.

If we have a triangle with two identical vertices, we call it a "bigon". Obviously, every bigon in a δ-thin space is δ-thin.

Definition 4. Given a geodesic triangle $T = \{x,y,z\}$ in a geodesic metric space X, let T_E be a Euclidean triangle with sides of the same length than T. Since there is no possible confusion, we will use the same notation for the corresponding points in T and T_E. The maximum inscribed circle in T_E meets the side $[x,y]$ (respectively $[y,z]$, $[z,x]$) in a point z' (respectively x', y') such that $d(x,z') = d(x,y')$, $d(y,x') = d(y,z')$ and $d(z,x') = d(z,y')$. We call the points x',y',z', the internal points of $\{x,y,z\}$. There is a unique isometry f of the triangle $\{x,y,z\}$ onto a tripod (a tree with one vertex w of degree 3, and three vertices x'',y'',z'' of degree one, such that $d(x'',w) = d(x,z') = d(x,y')$, $d(y'',w) = d(y,x') = d(y,z')$ and $d(z'',w) = d(z,x') = d(z,y')$). The triangle $\{x,y,z\}$ is *δ-fine* if $f(p) = f(q)$ implies that $d(p,q) \leq \delta$. The space X is *δ-fine* if every geodesic triangle in X is δ-fine.

A basic result is that hyperbolicity is equivalent to Rips condition and to be fine:

Theorem A. ([GH, p.41]) *Let us consider a geodesic metric space X.*

(1) *If X is δ-hyperbolic, then it is 4δ-thin and 4δ-fine.*

(2) *If X is δ-thin, then it is 4δ-hyperbolic and 4δ-fine.*

(3) *If X is δ-fine, then it is 2δ-hyperbolic and δ-thin.*

We present now the class of maps which play the main role in the theory.

Definition 5. A function between two metric spaces $f : X \longrightarrow Y$ is a *quasi-isometry* if there are constants $a \geq 1$, $b \geq 0$ with

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a d_X(x_1, x_2) + b\,, \qquad \text{for every } x_1, x_2 \in X.$$

A such function is called an (a, b)-*quasi-isometry.* We say that the image of f is ε-*full* (for some $\varepsilon \geq 0$) if for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq \varepsilon$. We say that X and Y are *quasi-isometrically equivalents* if there exists a quasi-isometry between X and Y, with image ε-full, for some $\varepsilon \geq 0$. An (a, b)-*quasigeodesic* in X is an (a, b)-quasi-isometry between an interval of $\mathbf{R}$ and X. An (a, b)-*quasigeodesic segment* in X is an (a, b)-quasi-isometry between a compact interval of $\mathbf{R}$ and X.

Let us observe that a quasi-isometry can be discontinuous.

Remark. It is well known (see e.g. [K1], [K2]) that quasi-isometrical equivalence is an equivalence relation. In fact, if $f : X \longrightarrow Y$ is an (a, b)-quasi-isometry with image ε-full, then there exists a function $g : Y \longrightarrow X$ which is an $(a, 2a\varepsilon + ab)$-quasi-isometry. In particular, if f is a surjective (a, b)-quasi-isometry, then g is an (a, ab)-quasi-isometry (in this case we can choose as $g(y)$ any point in $f^{-1}(y)$).

Quasi-isometries are important since they are the maps which preserve hyperbolicity:

Theorem B. ([GH, p.88]) *Let us consider an (a, b)-quasi-isometry between two geodesic metric spaces $f : X \longrightarrow Y$. If Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which only depends on δ, a and b. Besides, if the image of f is ε-full for some $\varepsilon \geq 0$, then X is hyperbolic if and only if Y is hyperbolic.*

It is well-known that if f is not ε-full, the hyperbolicity of X does not imply the hyperbolicity of Y: it is enough to consider the inclusion of $\mathbf{R}$ in $\mathbf{R}^2$ (which is indeed an isometry).

Definition 6. Let us consider $H > 0$, a metric space X, and subsets $Y, Z \subseteq X$. The set $V_H(Y) := \{x \in X : d(x, Y) \leq H\}$ is called the *H-neighborhood* of Y in X. The *Hausdorff distance* of Y to Z is defined by $\mathcal{H}(Y, Z) := \inf\{H > 0 : Y \subseteq V_H(Z),\ Z \subseteq V_H(Y)\}$.

The following is a beautiful and useful result:

Theorem C. ([GH, p.87]) *For each $\delta \geq 0$, $a \geq 1$ and $b \geq 0$, there exists a constant $H = H(\delta, a, b)$ with the following property:*

Let us consider a δ-hyperbolic geodesic metric space X and an (a, b)-quasigeodesic g starting in x and finishing in y. If γ is a geodesic joining x and y, then $\mathcal{H}(g, \gamma) \leq H$.

This property is known as geodesic stability. Mario Bonk has proved that, in fact, geodesic stability is equivalent to hyperbolicity [B].

Along this paper we will work with topological subspaces of a geodesic metric space X. There is a natural way to define a distance in these spaces:

Definition 7. If X_0 is a path-connected subset of a geodesic metric space (X,d), then we associate to it the *restricted distance*

$$d_{X_0}(x,y) := d_X|_{X_0}(x,y) := \inf\big\{L(\gamma):\, \gamma\subset X_0 \text{ is a continuous curve joining } x \text{ and } y\big\} \ge d_X(x,y)\,.$$

Theorem 1 below allows to transfer the study of the hyperbolicity of a certain space X to their "building block components" X_n determined by the following definition.

Definition 8. We say that a geodesic metric space X has a *decomposition*, if there exists a family of geodesic metric spaces $\{X_n\}_{n\in\Lambda}$ with $X=\cup_{n\in\Lambda}X_n$ and $X_n\cap X_m=\cup_{i\in I_{nm}}\eta_{nm}^i$, where for each $n\in\Lambda$, $\{\eta_{nm}^i\}_{m,i}$ are pairwise disjoint closed subsets of X_n ($\eta_{nm}^i=\varnothing$ is allowed); furthermore any geodesic segment in X meets at most a finite number of η_{nm}^i's.

We say that X_n, with $n\in\Lambda$, is a (k_1,k_2,k_3)-*tree-piece* if it satisfies the following properties:

(a) $\sharp I_{nm}\le 1$ (then we can write $\eta_{nm}^i=\eta_{nm}$); if $\sharp I_{nm}=1$, then $X\setminus\eta_{nm}$ is not connected and a,b are in different connected components of $X\setminus\eta_{nm}$ for any $a\in X_n\setminus\eta_{nm}$, $b\in X_m\setminus\eta_{nm}$.

(b) $\operatorname{diam}_{X_n}(\eta_{nm})\le k_1$ for every $m\ne n$, and there exists $A_n\subseteq\Lambda$, such that $\operatorname{diam}_{X_n}(\eta_{nm})\le k_2\, d_{X_n}(\eta_{nm},\eta_{nk})$ if $m\ne k$ and $m,k\in A_n$, and $\sum_{m\notin A_n}\operatorname{diam}_{X_n}(\eta_{nm})\le k_3$.

We say that a geodesic metric space X has a *tree-decomposition* if it has a decomposition and there exist positive constants k_1, k_2, k_3, such that every X_n, with $n\in\Lambda$, is a (k_1,k_2,k_3)-tree-piece.

We wish to emphasize that condition $\operatorname{diam}_{X_n}(\eta_{nm})\le k_1$ is not very restrictive: if the space is "wide" at every point (in the sense of long injectivity radius, as in the case of simply connected spaces) or "narrow" at every point (as in the case of trees), it is easier to study its hyperbolicity; if we can found narrow parts (as η_{nm}) and wide parts, the problem is more difficult and interesting.

Remarks.

1. Obviously, condition (b) is required only for $\eta_{nm},\eta_{nk}\ne\varnothing$.

2. The sets Λ and A_n do not need to be countable.

3. The hypothesis $\operatorname{diam}_{X_n}(\eta_{nm})\le k_2\, d_{X_n}(\eta_{nm},\eta_{nk})$ holds if we have $d_{X_n}(\eta_{nm},\eta_{nk})\ge k_2'$, since $\operatorname{diam}_{X_n}(\eta_{nm})\le k_1$.

4. Condition (a) for every $n\in\Lambda$ guarantees that the graph $R=(V,E)$ constructed in the following way is a tree: $V=\cup_{n\in\Lambda}\{v_n\}$ and $[v_n,v_m]\in E$ if and only if $\eta_{nm}\ne\varnothing$.

The following result is an improvement of Theorem 2.4 in [RT], using a completely different line of argument; furthermore, this proof provides an explicit expression for the constants involved. It will be used in the proof of Theorem 2 and can be applied to the study of the hyperbolicity of Riemann surfaces (see [PRT, Propositions 3.1 and 3.2]).

Theorem 1. *Let us consider a geodesic metric space X with a tree-decomposition $\{X_n\}_{n\in\Lambda}$. Then X is δ-hyperbolic if and only if there exists a constant k_4 such that X_n is k_4-hyperbolic for every $n\in\Lambda$.*

Furthermore, if X_n is k_4-hyperbolic, we can take $\delta = 4(2k_1+4k_4+2H(k_4, 2\max\{1,k_2\}, 4k_1+2k_3))$, where H is the constant in Theorem C; if X is δ-hyperbolic, we can take $k_4 = 16\max\{1,k_2\}\big(2\delta+k_1+k_3+H(\delta, 2\max\{1,k_2\}, 2k_1+2k_3)\big)$.

Observe that the sets η_{nm} do not need to be connected and therefore we can create a finite number of "handles" each time we paste two pieces.

The conclusion of Theorem 1 is not true without hypothesis (b) in Definition 8, as it is shown in following examples:

The set $Q = \{z = x+iy : x \ge 0, y \ge 0\}$, with its Euclidean distance, is not hyperbolic, but Q is the union of the 1-thin pieces $X_n = \{z \in Q : n-1 \le |z| \le n\}$.

Let us consider any funnel F with boundary γ. The results on [RT] gives that F is hyperbolic. However, it is the union of the non-uniformly hyperbolic pieces $X_n = \{z \in F : n-1 \le d(z,\gamma) \le n\}$ (the hyperbolicity constant of X_n is comparable with $L(\partial X_n)$).

The proof of Theorem 1 gives the following results.

Corollary 1. *Let us consider a geodesic metric space X with a decomposition $\{X_n\}_{n\in\Lambda}$. Let us assume that, for some fixed $n_0 \in \Lambda$, X_{n_0} is a (k_1,k_2,k_3)-tree-piece and it is k_4-hyperbolic. If T is a geodesic triangle in X and X_{n_0} intersects at least two sides of T, then $X_{n_0} \cap T$ is δ^*-thin, with $\delta^* := 3k_1/2 + 4k_4 + 2H(k_4, 2\max\{1,k_2\}, 4k_1+2k_3)$.*

Corollary 2. *Let us consider a δ-hyperbolic geodesic metric space X with a decomposition $\{X_n\}_{n\in\Lambda}$. Let us assume that, for some fixed $n_0 \in \Lambda$, X_{n_0} is a (k_1,k_2,k_3)-tree-piece. Then X_{n_0} is δ_0-thin, with $\delta_0 := 4\max\{1,k_2\}\big(2\delta+k_1+k_3+H(\delta, 2\max\{1,k_2\}, 2k_1+2k_3)\big)$.*

In order to prove Theorem 1 we need some technical results.

Lemma 1. *Let us consider a geodesic metric space X and a geodesic $\eta = [x_0, x_{2n+1}] = \cup_{j=1}^{2n+1}[x_{j-1}, x_j]$. For each $1 \le j \le n$, let us consider a continuous curve η'_j joining x_{2j-1} and x_{2j}, such that $L(\eta'_j) \le a$ for every $1 \le j \le n$ and $L(\eta'_j) \le bL([x_{2j}, x_{2j+1}])$ for every $1 \le j \le n-1$. If η' is the curve obtained from η by replacing $[x_{2j-1}, x_{2j}]$ by η'_j, then η' is a continuous $(2\max\{1,b\}, 2a)$-quasigeodesic with its arc-length parametrization.*

Proof of Lemma 1. Let us consider the arc-length parametrizations $\eta : [0,l] \longrightarrow X$ and $\eta' : [0,l'] \longrightarrow X$. We can write $[0,l] = \cup_{j=1}^{2n+1}[t_{j-1}, t_j]$ and $[0,l'] = \cup_{j=1}^{2n+1}[l_{j-1}, l_j]$, such that $\eta'(l_j) = \eta(t_j) = x_j$ for every $0 \le j \le 2n+1$, $\eta'([l_{2j}, l_{2j+1}]) = [x_{2j}, x_{2j+1}]$ for every $0 \le j \le n$ and $\eta'([l_{2j-1}, l_{2j}]) = \eta'_j$ for every $1 \le j \le n$. The hypothesis give that $l_{2j} - l_{2j-1} \le a$ for every $1 \le j \le n$ and $l_{2j} - l_{2j-1} \le b(l_{2j+1} - l_{2j})$ for every $1 \le j \le n-1$.

Since we consider η' with its arc-length parametrization, then, for every $s,t \in [0,l']$, we have $d(\eta'(t), \eta'(s)) \le L(\eta'([s,t])) = |t-s|$.

If $s,t \in [l_{2j}, l_{2j+1}]$, then $d(\eta'(t), \eta'(s)) = |t-s|$ for every $0 \le j \le n$.

If $s \in [l_{2i}, l_{2i+1}]$ and $t \in [l_{2j}, l_{2j+1}]$, without loss of generality, we can assume that $i < j$; then there exist $s' \in [t_{2i}, t_{2i+1}]$ and $t' \in [t_{2j}, t_{2j+1}]$ such that

$$\begin{aligned} d(\eta'(t),\eta'(s)) &= d(\eta(t'),\eta(s')) \ge t_{2i+1} - s' + \sum_{k=i+1}^{j-1} (t_{2k+1} - t_{2k}) + t' - t_{2j} \\ &= l_{2i+1} - s + \frac{1}{2} \sum_{k=i+1}^{j-1} \big(l_{2k+1} - l_{2k} + l_{2k+1} - l_{2k}\big) + t - l_{2j} \\ &\ge \frac{1}{2}\Big(l_{2i+1} - s + b^{-1} \sum_{k=i+1}^{j-1} (l_{2k} - l_{2k-1}) + \sum_{k=i+1}^{j-1} (l_{2k+1} - l_{2k}) \\ &\quad + l_{2j} - l_{2j-1} - a + t - l_{2j}\Big) \ge \frac{1}{2} \min\{1, b^{-1}\}(t-s) - \frac{a}{2}. \end{aligned}$$

In the general case, if $s, t \in [0, l']$ there exist $s^* \in [l_{2i}, l_{2i+1}]$ and $t^* \in [l_{2j}, l_{2j+1}]$, with $|s - s^*| \le a/2$ and $|t - t^*| \le a/2$. Hence

$$\begin{aligned} d(\eta'(t),\eta'(s)) &\ge d(\eta'(t^*),\eta'(s^*)) - a \ge \frac{1}{2} \min\{1, b^{-1}\}|t^* - s^*| - \frac{3a}{2} \\ &\ge \frac{1}{2} \min\{1, b^{-1}\}|t - s| - 2a. \qquad \square \end{aligned}$$

Lemma 2. *Let us consider a geodesic metric space X and a continuous (a,b)-quasigeodesic with its arc-length parametrization $\eta : [0,l] \longrightarrow X$, such that $[0,l] = \cup_{j=1}^{2n+1} [t_{j-1}, t_j]$. For each $1 \le j \le n$, let us consider a continuous curve η'_j joining $\eta(t_{2j-1})$ and $\eta(t_{2j})$ such that $\sum_{j=1}^{n} L(\eta'_j) \le c$. If η' is the curve obtained from η by substituting $\eta([t_{2j-1}, t_{2j}])$ by η'_j, then η' is a continuous $(a, b+(1+3a^{-1})c/2)$-quasigeodesic with its arc-length parametrization.*

Proof of Lemma 2. Let us consider the arc-length parametrization $\eta' : [0, l'] \longrightarrow X$. We can write $[0, l'] = \cup_{j=1}^{2n+1} [l_{j-1}, l_j]$, such that $\eta'(l_j) = \eta(t_j)$ for every $0 \le j \le 2n+1$, $\eta'([l_{2j}, l_{2j+1}]) = \eta([t_{2j}, t_{2j+1}])$ for every $0 \le j \le n$ and $\eta'([l_{2j-1}, l_{2j}]) = \eta'_j$ for every $1 \le j \le n$. We have that $\sum_{j=1}^{n} (l_{2j} - l_{2j-1}) \le c$.

Since we consider η' with its arc-length parametrization, then, for every $s, t \in [0, l']$, we have that $d(\eta'(t), \eta'(s)) \le L(\eta'([s,t])) = |t - s|$. In order to prove the other inequality, we have:

If $s, t \in [l_{2j}, l_{2j+1}]$, then $d(\eta'(t), \eta'(s)) \ge a^{-1}|t - s| - b$ for every $0 \le j \le n$.

If $s \in [l_{2i}, l_{2i+1}]$ and $t \in [l_{2j}, l_{2j+1}]$, without loss of generality we can assume that $i < j$; then there exist $s' \in [t_{2i}, t_{2i+1}]$ and $t' \in [t_{2j}, t_{2j+1}]$ such that

$$\begin{aligned} d(\eta'(t),\eta'(s)) &= d(\eta(t'),\eta(s')) \ge a^{-1}|t' - s'| - b \\ &= a^{-1}\Big(t' - t_{2j} + \sum_{k=2i+1}^{2j-1} (t_{k+1} - t_k) + t_{2i+1} - s'\Big) - b \\ &\ge a^{-1}\Big(t - l_{2j} + \sum_{k=2i+1}^{2j-1} (l_{k+1} - l_k) + l_{2i+1} - s\Big) - (b + a^{-1}c) \\ &= a^{-1}(t - s) - (b + a^{-1}c). \end{aligned}$$

In the general case, if $s, t \in [0, l']$, there exist $s^* \in [l_{2i}, l_{2i+1}]$ and $t^* \in [l_{2j}, l_{2j+1}]$, with $|s - s^*| + |t - t^*| \le c/2$. Hence

$$\begin{aligned} d(\eta'(t),\eta'(s)) &\ge d(\eta'(t^*),\eta'(s^*)) - c/2 \ge a^{-1}|t^* - s^*| - (b + a^{-1}c + c/2) \\ &\ge a^{-1}|t - s| - (b + 3a^{-1}c/2 + c/2). \qquad \square \end{aligned}$$

Lemma 3. *Let us consider an (a, b)-quasigeodesic $q_1 : [\alpha, \beta] \longrightarrow X$ and two continuous curves with arc-length parametrization $q_0 : [\alpha - d_1, \alpha] \longrightarrow X$, $q_2 : [\beta, \beta + d_2] \longrightarrow X$, verifying $q_0(\alpha) = q_1(\alpha)$ and $q_2(\beta) = q_1(\beta)$. Then the curve $q := q_0 \cup q_1 \cup q_2$ is an $(a, b + (1 + a^{-1})(d_1 + d_2))$-quasigeodesic.*

Proof of Lemma 3. We consider the case $s \in [\alpha - d_1, \alpha]$ and $t \in [\beta, \beta + d_2]$, since the other cases are easier.

$$\begin{aligned} d(q(t), q(s)) &\le d(q(t), q_1(\beta)) + d(q_1(\beta), q_1(\alpha)) + d(q_1(\alpha), q(s)) \\ &\le d_2 + a(\beta - \alpha) + b + d_1 \le a(t - s) + b + d_1 + d_2 \,, \\ d(q(t), q(s)) &\ge d(q_1(\beta), q_1(\alpha)) - d(q(t), q_1(\beta)) - d(q_1(\alpha), q(s)) \\ &\ge a^{-1}(\beta - \alpha) - b - d_1 - d_2 \ge a^{-1}(t - s) - a^{-1}(d_1 + d_2) - b - d_1 - d_2 \,. \qquad \square \end{aligned}$$

Definition 9. Let us consider three quasigeodesic segments J_1 joining x_1 and x_2', J_2 joining x_2 and x_3', J_3 joining x_3 and x_1', in a metric space. We say that $T = \{J_1, J_2, J_3\}$ is an (a, b, c)-*quasigeodesic triangle* if J_1, J_2, J_3 are (a, b)-quasigeodesics and $d(x_i, x_i') \le c$ for $1 \le i \le 3$.

Lemma 4. *For each $\delta, b, c \ge 0$ and $a \ge 1$, there exists a constant $K = K(\delta, a, b, c)$ with the following property:*

If X is a δ-hyperbolic geodesic metric space and $T \subseteq X$ is an (a, b, c)-quasigeodesic triangle, then T is K-thin. Furthermore, $K = 4\delta + c + 2H(\delta, a, b + 2c)$, where H is the constant in Theorem C.

Proof of Lemma 4. We consider three geodesic segments $[x_2', x_2]$, $[x_3', x_3]$ and $[x_1', x_1]$. By Lemma 3 (with $d_1 = 0$ and $d_2 \le c$), the curves $s(x_1, x_2) := J_1 \cup [x_2', x_2]$, $s(x_2, x_3) := J_2 \cup [x_3', x_3]$ and $s(x_3, x_1) := J_3 \cup [x_1', x_1]$ are $(a, b + 2c)$-quasigeodesics. By Theorem C, there exist geodesics $\{[x_1, x_2], [x_2, x_3], [x_3, x_1]\}$ with $\mathcal{H}(s(x_i, x_j), [x_i, x_j]) \le H$, for some constant $H = H(\delta, a, b + 2c)$.

We prove now that the $(a, b + 2c, 0)$-quasigeodesic triangle $T' = \{J_1, J_2, J_3\}$ is $(4\delta + 2H)$-thin. Let us consider any permutation $\{x_i, x_j, x_k\}$ of $\{x_1, x_2, x_3\}$ and $x \in s(x_i, x_j)$; then there exists $x' \in [x_i, x_j]$ with $d(x, x') \le H$.

Since the geodesics $\{[x_1, x_2], [x_2, x_3], [x_3, x_1]\}$ are a geodesic triangle 4δ-thin, there exists $y' \in [x_j, x_k] \cup [x_k, x_i]$ with $d(x', y') \le 4\delta$. Now we can choose $y \in s(x_j, x_k) \cup s(x_k, x_i)$ with $d(y, y') \le H$. Hence, T' is $(4\delta + 2H)$-thin.

Consequently, T is K-thin, with $K := 4\delta + c + 2H$, since $[x_2', x_2]$, $[x_3', x_3]$ and $[x_1', x_1]$ have length less or equal than c. $\square$

Definition 10. Let us assume that we have a triangle T (not necessarily geodesic) with vertices $\{x_1, x_2, x_3\}$; we denote by $x_i x_j$ the side of T joining x_i with x_j. We consider now another triangle T' with vertices $\{x_1', x_2', x_3'\}$ such that $x_i' x_j'$ is obtained by a certain kind of modification of $x_i x_j$. We say that $z \in T$ and $z' \in T'$ are in *corresponding sides* if $z \in x_i x_j$ and $z' \in x_i' x_j'$ for some i, j.

Proof of Theorem 1. Let us assume that X_n is k_4-hyperbolic for every $n \in \Lambda$.

We consider a geodesic triangle $T = \{a, b, c\}$ in X. We fix $z \in T$; if z belongs to two sides of T, there is nothing to prove; if z only belongs to one side of T, we denote by A the union of the sides of T which does not intersect z. Without loss of generality we can assume that $z \in [a, b]$.

If $T \subseteq X_n$ for some n, then T is $4k_4$-thin, by Theorem A.

We assume that T intersects several X_n's. We intend to study T in each of those X_n's separately.

Let us fix $n \in \Lambda$. We consider first the case in which every side of T intersect X_n. We construct a quasigeodesic triangle $T_n \subseteq X_n$ modifying T in the following way: If $[a,b] \subseteq X_n$, we consider $[a_n, b_n] = [a,b]$. If $[a,b]$ is not contained in X_n, then we consider $g : [0,l] \longrightarrow X$ as an oriented geodesic joining a and b. By hypothesis, the geodesic segment g meets at most a finite number of η_{nm}'s. Let us define

$$t_0 := \min\{0 \le t \le l : g(t) \in X_n\}, \qquad t_l := \max\{0 \le t \le l : g(t) \in X_n\}.$$

First of all, let us assume that g meets $\cup_{m\in A_n}\eta_{nm}$. We define

$$t_1^1 := \min\{t_0 \le t \le t_l : g(t) \in \cup_{m\in A_n}\eta_{nm}\}.$$

There exists this minimum since g is a continuous function in a compact interval and $g \cap (\cup_{m\in A_n}\eta_{nm})$ is a compact set: each η_{nm} is a closed set and g meets at most a finite number of η_{nm}'s.

Then there is $m_1 \in A_n$ such that $g(t_1^1) \in \eta_{nm_1}$, and we define

$$t_1^2 := \max\{t_0 \le t \le t_l : g(t) \in \eta_{nm_1}\}.$$

In a similar way, we define recursively

$$t_i^1 := \min\{t_{i-1}^2 < t \le t_l : g(t) \in \cup_{m\in A_n}\eta_{nm}\};$$

if $g(t_i^1) \in \eta_{nm_i}$, for some $m_i \in A_n$, we take

$$t_i^2 := \max\{t_{i-1}^2 < t \le t_l : g(t) \in \eta_{nm_i}\}.$$

We can continue this choice for $1 \le i \le r$. We define g' as the restriction of g to the closed set $[t_0, t_1^1] \cup [t_1^2, t_2^1] \cup \cdots \cup [t_{r-1}^2, t_r^1] \cup [t_r^2, t_l]$. Observe that $g' \subseteq X_n$. Now, let us choose geodesics g_i in X_n connecting $g(t_i^1)$ and $g(t_i^2)$. We define $\gamma := g' \cup g_1 \cup g_2 \cup \cdots \cup g_r$. By Lemma 1, we have that $\gamma : [0,L] \longrightarrow X_n$ is a continuous $(2\max\{1,k_2\}, 2k_1)$-quasigeodesic with its arc-length parametrization (observe that $\gamma(0) = g(t_0)$ and $\gamma(L) = g(t_l)$).

If g does not meet $\cup_{m\in A_n}\eta_{nm}$ (or if $t_i^1 = t_i^2$ for $1 \le i \le r$), we take $\gamma = g$.

We assume now that γ meets $\cup_{m\notin A_n}\eta_{nm}$. If we repeat the previous argument, then we can find a $m^1 \notin A_n$ for which we have

$$s_1^1 := \min\{0 \le s \le L : \gamma(s) \in \eta_{nm^1}\}, \qquad s_1^2 := \max\{0 \le s \le L : \gamma(s) \in \eta_{nm^1}\}.$$

In a similar way, there exist $m^2, \dots, m^j$ for which we define recursively for $i = 2, \dots, j$,

$$s_i^1 := \min\{s_{i-1}^2 < s \le L : \eta(s) \in \eta_{nm^i}\}, \qquad s_i^2 := \max\{s_{i-1}^2 < s \le L : \eta(s) \in \eta_{nm^i}\}.$$

We define γ' as a restriction of γ to the closed set $[0, s_1^1] \cup [s_1^2, s_2^1] \cup \cdots \cup [s_j^2, L]$; we also have $\gamma' \subseteq X_n$. Now, let us choose geodesics h_i in X_n connecting $\gamma(s_i^1)$ and $\gamma(s_i^2)$. We define $\alpha_1 := \gamma' \cup h_1 \cup h_2 \cup \cdots \cup h_j$.

If $\alpha_1 : [0,l_1] \longrightarrow X_n$ is its arc-length parametrization, Lemma 2 gives that α_1 is a $(2\max\{1,k_2\}, 2k_1 + 2k_3)$-quasigeodesic.

If γ does not meet $\cup_{m\notin A_n}\eta_{nm}$ (or if $s_i^1 = s_i^2$ for $1 \le i \le j$), we take $\alpha_1 = \gamma$.

In a similar way, we construct the quasigeodesics $\alpha_2 : [0,l_2] \longrightarrow X_n$ and $\alpha_3 : [0,l_3] \longrightarrow X_n$ corresponding to the sides $[b,c]$ and $[c,a]$ respectively.

Observe that if $\alpha_1(l_1) \neq \alpha_2(0)$, then both points belong to some η_{nm}, since we have a tree-decomposition; condition (b) gives that $d_{X_n}(\alpha_1(l_1), \alpha_2(0)) \le k_1$. The same is true if $\alpha_2(l_2) \neq \alpha_3(0)$, and if $\alpha_3(l_3) \neq \alpha_1(0)$. Hence $T_n := \alpha_1 \cup \alpha_2 \cup \alpha_3$ is a $(2\max\{1,k_2\}, 2k_1+2k_3, k_1)$-quasigeodesic triangle. Lemma 4 gives that T_n is δ_1-thin, with $\delta_1 = k_1 + 4k_4 + 2H(k_4, 2\max\{1,k_2\}, 4k_1 + 2k_3)$, where H is the constant in Theorem C.

If $z \in X_n$, without loss of generality we can assume that $z \in \alpha_1$; if $A' := \alpha_2 \cup \alpha_3$, then there exists $z' \in A'$ with $d_{X_n}(z,A') = d_{X_n}(z,z') \le \delta_1$. If $z' \in A$, then $d_X(z,A) \le \delta_1$. If $z' \notin A$, then, there exists $z_0 \in A$ such that $d_{X_n}(z_0,z') \le k_1/2$; then, $d_X(z,A) \le d_{X_n}(z,z') + d_{X_n}(z',z_0) \le \delta_1 + k_1/2$.

If only two sides of T intersect X_n, we have the same result since we can see a bigon as a triangle with two equal vertices. These facts prove Corollary 1. We finish now the proof of Theorem 1.

If $A \cap X_n = \varnothing$, then z belongs to some geodesic $g_0 \subseteq g$ joining some η_{mk} with itself such that $A \cap X_m \neq \varnothing$, since we have a tree-decomposition. By (b), there exists $z' \in g_0 \cap \eta_{mk}$ with $d_X(z,z') \le k_1/2$, and then, there exists $z_0 \in A \cap X_m$ such that $d_{X_m}(z_0,z') \le \delta_1 + k_1/2$. Consequently, $d_X(z,A) \le \delta_1 + k_1$, and X is δ-thin with $\delta := 2k_1 + 4k_4 + 2H(k_4, 2\max\{1,k_2\}, 4k_1 + 2k_3)$.

Let us assume that X is δ-hyperbolic.

We prove now that the inclusion $i : X_n \longrightarrow X$ is a $(2\max\{1,k_2\}, 2k_1 + 2k_3)$-quasi-isometry.

Given $x,y \in X_n$, we have that $d_X(x,y) \le d_{X_n}(x,y)$, since there are more curves joining x and y in X than in X_n. In order to prove the other inequality, let us consider a geodesic g in X joining x and y. If $g \subseteq X_n$, then $d_X(x,y) = d_{X_n}(x,y)$. In other case, we can define $t_1^1, t_1^2, \dots, t_r^1, t_r^2, s_1^1, s_1^2, \dots, s_j^1, s_j^2$, and the $(2\max\{1,k_2\}, 2k_1+2k_3)$-quasigeodesic $\alpha_1 : [0,l_1] \longrightarrow X_n$ joining x and y as in the proof of the first part of the theorem. Since α_1 has its arc-length parametrization, $\frac{1}{2}\min\{1,k_2^{-1}\}L(\alpha_1) - 2(k_1 + k_3) \le d_X(\alpha_1(0), \alpha_1(l_1)) = d_X(x,y)$.

Since α_1 is a continuous curve in X_n joining x and y, $d_{X_n}(x,y) \le L(\alpha_1)$, and then

$$\frac{1}{2}\min\{1,k_2^{-1}\} d_{X_n}(x,y) - 2(k_1 + k_3) \le d_X(x,y) \le d_{X_n}(x,y).$$

Hence, if X is δ-hyperbolic, then X_n is $4\max\{1,k_2\}\big(2\delta + k_1 + k_3 + H(\delta, 2\max\{1,k_2\}, 2k_1 + 2k_3)\big)$-thin (see [GH, p.88]).

Let us observe that in this proof of the hyperbolicity of X_n we do not use that the other pieces are tree-pieces; this gives Corollary 2. □

Theorem 2 below let us move the study of the hyperbolicity of a certain space X to another space Y with simpler structure, so long as between them there exists the type of relationship described by the following definition.

Definition 11. We say that two geodesic metric spaces X and Y (in this order) have *comparable decompositions*, if there exist decompositions $\{X_n\}_{n\in\Lambda}$ of X and $\{Y_n\}_{n\in\Lambda}$ of Y, and constants k_i, with the following properties:

(a) If $X_n \cap X_m = \cup_{i\in I_{nm}} \eta^i_{nm}$, then $Y_n \cap Y_m = \cup_{i\in I_{nm}} \sigma^i_{nm}$, and $\sigma^i_{nm} = \varnothing$ if and only if $\eta^i_{nm} = \varnothing$.

(b) For any n, m, i, $\operatorname{diam}_{X_n}(\eta^i_{nm}) \le k_1$ and $\operatorname{diam}_{Y_n}(\sigma^i_{nm}) \le k_1$.

(c) We can split Λ into $F \cup G$ and F into $F_1 \cup F_2$ with:

($c1$) If $n \in G$, X_n is a (k_1, k_2, k_3)-tree-piece.

($c2$) If $n \in F$, $\operatorname{diam}_{X_n}(\eta^i_{nm}) \le k_2\, d_{X_n}(\eta^i_{nm}, \eta^j_{nk})$ and $\operatorname{diam}_{Y_n}(\sigma^i_{nm}) \le k_2\, d_{Y_n}(\sigma^i_{nm}, \sigma^j_{nk})$ if $(m,i) \ne (k,j)$.

($c3$) If $n \in F_1$, for each $\eta^i_{nm} \ne \eta^j_{nk}$, there exists a geodesic γ^{ij}_{mnk} in X_n, joining η^i_{nm} with η^j_{nk}, and a (k_4, b^{ij}_{mnk})-quasi-isometry $f^{ij}_{mnk} : \gamma^{ij}_{mnk} \longrightarrow h^{ij}_{mnk} \subseteq Y_n$, with h^{ij}_{mnk} starting in σ^i_{nm} and finishing in σ^j_{nk}, and $\sum_{n\in F_1} \sum_{m,k,i,j} b^{ij}_{mnk} \le k_5$, such that for any $x, y \in \cup_{m,k,i,j} \gamma^{ij}_{mnk}$, with corresponding points $x', y' \in \cup_{m,k,i,j} h^{ij}_{mnk}$, we have $k_4^{-1} d_{X_n}(x,y) - k_5 \le d_{Y_n}(x', y')$.

($c4$) If $n \in F_2$, there exists a $(k_4, 0)$-quasi-isometry $f_n : X_n \longrightarrow Y_n$, with $f_n(\eta^i_{nm}) \subseteq \sigma^i_{nm}$.

Remarks.

1. Obviously, these conditions are required only for $\eta^i_{nm}, \sigma^i_{nm} \ne \varnothing$.

2. The sets Λ, F, G and I_{nm} do not need to be countable.

3. We obviously have $\eta^i_{nm} = \eta^i_{mn}$ and $I_{nm} = I_{mn}$.

4. The hypothesis ($c2$) trivially holds if for $n \in F$, $d_{X_n}(\eta^i_{nm}, \eta^j_{nk}) \ge k'_2$ and $d_{Y_n}(\sigma^i_{nm}, \sigma^j_{nk}) \ge k'_2$, by ($b$).

5. The hypothesis ($c3$) can be relaxed: let us consider any connected component B_s of $\cup_{n\in F} X_n$; the proof of Theorem 2 gives that it is enough to have $\sum_{n\in F_1^s, m,k,i,j} b^{ij}_{mnk} \le k_5$, for any s, where $F_1^s := \{n \in F_1 : X_n \subseteq B_s\}$ (see the construction of T_2 in the proof of Theorem 2).

6. As a consequence of ($c3$), we have that $k_4^{-1} d_{X_n}(x, \eta^t_{nr}) - k_1 - k_5 \le d_{Y_n}(f^{ij}_{mnk}(x), \sigma^t_{nr})$, for every $x \in \gamma^{ij}_{mnk}$ and r, t.

7. Since condition ($c3$) can be tedious to check, it could be interesting to check instead the following statement which implies ($c3$):

($c3'$) If $n \in F_1$, we have that $k_7^{-1} \le d_{X_n}(\eta^i_{nm}, \eta^j_{nk}) / d_{Y_n}(\sigma^i_{nm}, \sigma^j_{nk}) \le k_7$, $\operatorname{diam}_{X_n}(\cup_{mi} \eta^i_{nm}) \le k_8$ and $\operatorname{diam}_{Y_n}(\cup_{mi} \sigma^i_{nm}) \le k_8$.

In the decomposition of X one can find pieces of two different types: $\{X_n\}_{n\in F}$ and $\{X_n\}_{n\in G}$. The connections among a piece X_n, with $n \in G$, and the rest of the pieces are simple enough for being X_n a tree-piece. The connections of the pieces X_n, with $n \in F$, do not have topological restrictions; therefore, besides (b) and ($c2$) (as in the case $n \in G$), they must be controlled somehow: the conditions ($c3$) and ($c4$) let us assure that the connections between X_n and the rest of the pieces must be alike to the ones in Y_n. Observe that condition ($c3$) involves just a small subset of points of each X_n, with $n \in F_1$.

In spite of lengthening Definition 11, splitting Λ into the union of the three types of sets F_1, F_2 and G is an extremely convenient course of action: on the one hand, the wider the range of possibilities, the easier it will be to fit a certain piece into one of them. On the other hand, the determination of the conditions that X_n must verify when n belongs to F_1, F_2 or G, is not arbitrary at all. In fact, what lies behind is an appropriate modelization for the study of the following problem in Riemann surfaces (see [PRT]): Given a Riemann surface S, another one S^* can be obtained from S by removing a union of simply connected closed sets $\{E_m\}_{m\in M}$. In [PRT] it is proved that S is hyperbolic if and only if S^* is hyperbolic, when $\{E_m\}$ are sufficiently separated. Theorem 2 is used in the proof of the latest statement: The idea is to consider some neighborhoods of $\{E_m\}$ as pieces $\{S_m\}$ (in S^* we take $S_m^* := S_m \setminus E_m$). G is defined as the set of m's belonging to M such that S_m is a tree-piece; F_1 is the rest of indices of M, and F_2 is the set of indices which parametrizes the connected components of $S \setminus \cup_{m\in M} S_m$ (in S^* we take the same connected components). Finally, Definition 11 has been formulated by abstracting the essential properties of pieces in each of the three sets.

Theorem 2. *Let us assume that two geodesic metric spaces X and Y have comparable decompositions. If Y is δ'-hyperbolic and there exists a constant k_6 such that X_n is k_6-hyperbolic for every $n \in \Lambda \setminus F_2$, then X is δ-hyperbolic, with δ a constant which only depends on δ' and k_i.*

There is an explicit expression of δ at the end of the proof of Theorem 2.

It is obvious that $(c4)$ is much more restrictive than $(c3)$; however, it is a small price to pay in return for not having to check the hyperbolicity of pieces in F_2.

We can see this theorem as a version of Theorem B: if $\Lambda = F_1$, condition $(c3)$ says somehow that there is a quasi-isometry of a small subset of X on a subset of Y. From a dual point of view, we can consider that there is a quasi-isometry of a subset of Y on a subset of X; in this case we have the surprising result that the hyperbolicity of the original space implies the hyperbolicity of the final space.

The hyperbolicity of X does not imply the uniform hyperbolicity of X_n in general (this is another difference with Theorem 1). In fact, the hyperbolicity of X does not guarantee the hyperbolicity of each X_n, as it is shown in the following example: let us consider X_1 as the Cayley graph of the group $\mathbf{Z}^2$, and X_2 the tree with a countable number of edges of length 1 with a common vertex v_0; we construct X by gluing in a bijective way each vertex of X_2 (except for v_0) with a vertex of X_1; it is clear that X is hyperbolic since it is bounded, and that X_1 is not hyperbolic. In the same line, it is easy to construct a locally finite graph $X = \cup_n X_n$ with $\lim_{n\to\infty} \delta(X_n) = \infty$.

Next, we provide some conditions which guarantee the hyperbolicity of X_n.

Proposition 1. *Let X be a δ-hyperbolic geodesic metric space with a decomposition as in Definition 11. If for some $n \in \Lambda$ there exist constants $k_7 \geq 1, k_8 \geq 0$, with $d_{X_n}(\eta_{nm}^i, \eta_{nk}^j) \leq k_7\, d_X(\eta_{nm}^i, \eta_{nk}^j) + k_8$, for any m, k, i, j, then X_n is k_6-hyperbolic, with $k_6 := 4k_7\big(4\delta + k_7^{-1}(2k_1+k_8) + 2H(\delta, k_7, k_7^{-1}(2k_1+k_8))\big)$, where H is the constant in Theorem C.*

The following result is weaker than the one in Proposition 1, but it has the advantage that it only involves distances in X_n. In fact, this is the best possible general result involving just distances in X_n; besides it allows to get sharper constants.

Proposition 2. *Let X be a δ-hyperbolic geodesic metric space with a decomposition as in Definition 11. If for some $n \in \Lambda$ there exists a positive constant k_7 with $\operatorname{diam}_{X_n}(\cup_{mi} \eta_{nm}^i) \leq k_7$, then X_n is*

$(\delta + 3k_7/2)$-*hyperbolic.*

Corollary 3. *Let us assume that two geodesic metric spaces X and Y have comparable decompositions, that Y is hyperbolic, and that there exists a positive constant k_7 with* $\operatorname{diam}_{X_n}(\cup_{mi}\eta^i_{nm}) \le k_7$ *for every $n \in \Lambda$. Then X is hyperbolic if and only if there exists a constant k_6 such that X_n is k_6-hyperbolic for every $n \in \Lambda$.*

Proof of Theorem 2. Let us consider a geodesic triangle $T = \{a, b, c\}$ in X. It is obvious that if $T \subseteq X_n$ for some $n \in \Lambda \setminus F_2$, then T is $4k_6$-thin by hypothesis. In other case (i.e., whether $T \subseteq X_n$ with $n \in F_2$ or T intersects several X_n's), the main idea of the proof is to choose successively quasigeodesic triangles T_1, T_2, T_3, T_4 in X (closely related to T), which will allow to construct a quasigeodesic triangle T_5 in Y (related to T_4). Since Y is hyperbolic, then T_5 is thin by Lemma 4, and we will use this information in order to obtain that T is also thin. One of the main obstacles is that although X and Y have similar connections among their components, each pair of spaces X_n and Y_n can be very different (in fact, a quasi-isometry might not exist between X_n and Y_n).

Even though the main idea is simple, the proof is long and technical; in order to make the arguments more transparent, we collect some results we need along the proof in technical lemmas. Most of them will be proved in the last section of the paper.

A partial goal is to obtain a triangle T_4 in X easily transformable into another triangle T_5 in Y (in fact, $T \cap X_n$ is contained in $\cup_{m,k,i,j}\gamma^{ij}_{mnk}$ if $n \in \Lambda \setminus F_2$). In order to do this, the first step is to obtain a triangle T_1 in X such that for any $n \in \Lambda \setminus F_2$, each connected component of $T_1 \cap X_n$ is a geodesic in X_n. Recall that although each connected component of $S \cap X_n$ of any side S of T is a geodesic in X_n, there can exist connected components of $T \cap X_n$ (containing a vertex of T) which are not geodesics in X_n.

We start with the construction of T_1.

Let us assume that in the piece X_n there is at least one vertex a of T. If $n \in F_2$, we do not change $T \cap X_n$. (In particular, if $T \subseteq X_n$, with $n \in F_2$, then $T_1 = T$.) Let us consider now $n \in \Lambda \setminus F_2$, and let us call η_a to the connected component of $T \cap X_n$ which contains a.

Case 1. Assume first that η_a only contains a vertex of T. We denote by x_1, x_2 the end points of η_a. Then, we consider a geodesic triangle $T_a = \{a, x_1, x_2\}$ in X_n with $[a, x_1], [a, x_2] \subset T$. Let us denote by a_1 the internal point of T_a in the geodesic $[x_1, x_2]$ in X_n. We define $\eta_{a_1} := [x_1, x_2] = [x_1, a_1] \cup [a_1, x_2] \subset T_a$. If $b \in X_m$ (where m can be either n or not) and the connected component η_b of $T \cap X_m$ which contains b does not contain c, then we can proceed with the vertices b, c in a similar way that with a. In this case, T_1 is defined as the (not necessarily geodesic) triangle connecting the vertices a_1, b_1, c_1, obtained from T by replacing η_a, η_b, η_c by $\eta_{a_1}, \eta_{b_1}, \eta_{c_1}$ respectively.

Case 2. Let us assume now that $b \in \eta_a$ and $c \notin \eta_a$.

Without loss of generality, we can assume that η_a starts in x_1, ends in x_2, and meets a before than b. We consider the quadrilateral $\eta_a \cup [x_1, x_2] \subseteq X_n$ and we draw its diagonal $[a, x_2]$ (we can get a similar result by drawing $[b, x_1]$), obtaining two geodesic triangles in X_n: $T_a = \{a, x_1, x_2\}$ (with the internal points $u_1 \in [a, x_2], u_2 \in [x_1, x_2]$ and $u_3 \in [a, x_1]$), $T_b = \{a, b, x_2\}$ (with the internal points $v_1 \in [a, x_2], v_2 \in [b, x_2]$ and $v_3 \in [a, b]$).

Case 2.1. We consider first the situation $d_{X_n}(x_2, v_1) < d_{X_n}(x_2, u_1)$. We denote by b_1 the point in $[x_1, x_2]$ with $d_{X_n}(x_2, b_1) = d_{X_n}(x_2, v_1)$. If we denote $a_1 := u_2$, we can define $\eta_{a_1} := [x_1, x_2] = [x_1, a_1] \cup [a_1, b_1] \cup [b_1, x_2] \subset T_a$. We define η_{c_1} as in Case 1. Then we construct the triangle T_1 connecting the vertices a_1, b_1, c_1, obtained from T by replacing η_a, η_c by η_{a_1}, η_{c_1} respectively.

Case 2.2. We consider the situation $d_{X_n}(x_2, v_1) \geq d_{X_n}(x_2, u_1)$. If we denote $a_1 := b_1 := u_2$, we can define $\eta_{a_1} := [x_1, x_2] = [x_1, a_1] \cup [a_1, x_2] \subset T_a$. We define η_{c_1} as in Case 1. Then we construct the bigon T_1 connecting the vertices a_1, c_1, obtained from T by replacing η_a, η_c by η_{a_1}, η_{c_1} respectively.

Case 3. Finally, let us assume that $b, c \in \eta_a$. Without loss of generality, we can assume that η_a starts in x_1, ends in x_2, and meets a before than b and meets b before than c.

We consider the pentagon $\eta_a \cup [x_1, x_2] \subseteq X_n$ and we draw its diagonals $[x_1, b]$, $[b, x_2]$, obtaining three geodesic triangles in X_n: $T_a = \{a, b, x_1\}$ (with the internal points $s_1 \in [a, b], s_2 \in [a, x_1]$, $s_3 \in [b, x_1]$), $T_b = \{b, x_1, x_2\}$ (with the internal points $u_1 \in [b, x_2], u_2 \in [b, x_1]$ and $u_3 \in [x_1, x_2]$), $T_c = \{b, c, x_2\}$ (with the internal points $v_1 \in [c, x_2], v_2 \in [b, c]$ and $v_3 \in [b, x_2]$).

Case 3.1. We consider first the situation $d_{X_n}(x_1, s_3) < d_{X_n}(x_1, u_2)$ and $d_{X_n}(x_2, v_3) < d_{X_n}(x_2, u_1)$. We denote by a_1 the point in $[x_1, x_2]$ with $d_{X_n}(x_1, a_1) = d_{X_n}(x_1, s_3)$, and by c_1 the point in $[x_1, x_2]$ with $d_{X_n}(x_2, c_1) = d_{X_n}(x_2, v_3)$. If we denote $b_1 := u_3$, we can define $\eta_{a_1} := [x_1, x_2] = [x_1, a_1] \cup [a_1, b_1] \cup [b_1, c_1] \cup [c_1, x_2] \subset T_b$. Then we construct the triangle T_1 connecting the vertices a_1, b_1, c_1, obtained from T by replacing η_a by η_{a_1}.

Case 3.2. We consider now the situation $d_{X_n}(x_1, s_3) \geq d_{X_n}(x_1, u_2)$ and $d_{X_n}(x_2, v_3) < d_{X_n}(x_2, u_1)$. We define $a_1 := b_1 := u_3$, and we denote by c_1 the point in $[x_1, x_2]$ with $d_{X_n}(x_2, c_1) = d_{X_n}(x_2, v_3)$. We can define $\eta_{a_1} := [x_1, x_2] = [x_1, a_1] \cup [a_1, c_1] \cup [c_1, x_2] \subset T_b$. Then we construct the bigon T_1 connecting the vertices a_1, c_1, obtained from T by replacing η_a by η_{a_1}.

Case 3.3. The situation $d_{X_n}(x_1, s_3) < d_{X_n}(x_1, u_2)$ and $d_{X_n}(x_2, v_3) \geq d_{X_n}(x_2, u_1)$ is symmetric to Case 3.2, changing the roles of a and c.

Case 3.4. Finally, we consider the situation $d_{X_n}(x_1, s_3) \geq d_{X_n}(x_1, u_2)$ and $d_{X_n}(x_2, v_3) \geq d_{X_n}(x_2, u_1)$. In this case, we do not construct the triangle T_1.

Lemma 5. *If T_1 is δ_1-thin, then T is δ_0-thin, with $\delta_0 := \max\{\delta_1 + 16k_6, 18k_6\}$.*

See the proof of Lemma 5 in Section 3.

We have the following elementary fact.

Lemma 6. *Let us consider a metric space X, an interval I, an (a, b)-quasigeodesic $g : I \longrightarrow X$ and a curve $g_1 : I \longrightarrow X$ such that $d(g(t), g_1(t)) \leq c$ for every $t \in I$. Then g_1 is a $(a, b+2c)$-quasigeodesic.*

Proof. For any $s, t \in I$, we have that

$$d(g_1(t), g_1(s)) \leq d(g_1(t), g(t)) + d(g(t), g(s)) + d(g(s), g_1(s)) \leq a|t - s| + b + 2c\,,$$
$$d(g_1(t), g_1(s)) \geq d(g(t), g(s)) - d(g_1(t), g(t)) - d(g(s), g_1(s)) \geq a^{-1}|t - s| - b - 2c\,. \qquad \square$$

Lemma 7. *Each side of T_1 is a $(1, 16k_6)$-quasigeodesic with its arc-length parametrization. Furthermore, each connected component of $T_1 \cap X_n$ is a geodesic in X_n, if $n \in \Lambda \setminus F_2$.*

See the proof of Lemma 7 in Section 3.

As a second step, we split the triangle T_1 in several parts; Corollary 1 will allow to forget the part of T_1 which intersects the pieces X_n with $n \in G$ (see Lemma 8).

We consider the connected components $\{B_s\}_{s\in S}$ of the set $\cup_{n\in F} X_n$. We can study the triangle T_1 in each piece of $\{B_s\}_{s\in S}$ and of $\{X_n\}_{n\in G}$. We denote by T_2 the quasigeodesic triangle $T_1 \cap B_s$, for some fixed $s \in S$; in fact, we should write T_2^s instead of T_2, but our notation is simpler and there will be no place to confusion. Let us observe that T_2 is the union of three sides (possibly not continuous) joining a_2 with b'_2, b_2 with c'_2 and c_2 with a'_2.

Recall that we want to obtain a triangle T_4 in X contained in $\cup_{n,m,k,i,j}\gamma^{ij}_{mnk}$. As a third step, we construct the triangle T_3 in order to remove from T_2 the connected components of $T_2 \cap X_n$ which join some η^i_{nm} with itself.

We define the triangle T_3 in the following way:

Without loss of generality we can consider a side g_1 of T_1 as the oriented curve from a_1 to b_1. We have that $a_2 = g_1(\alpha)$ and $b'_2 = g_1(\beta)$, for some real numbers $\alpha < \beta$. By hypothesis, g_1 meets at most a finite number of η^i_{nm}'s. Let us assume that g_1 meets $\cup_{n,m,i}\eta^i_{nm}$. As we consider $g_1 : [\alpha,\beta] \longrightarrow X$, let us define

$$t^1_1 := \min\{\alpha \le t \le \beta \,:\, g_1(t) \in \cup_{n,m,i}\eta^i_{nm}\}\,.$$

There exists this minimum since g_1 is a continuous function in a compact interval and $g_1 \cap (\cup_{n,m,i}\eta^i_{nm})$ is a compact set: each η^i_{nm} is a closed set and g_1 meets at most a finite number of η^i_{nm}'s.

Then $g_1(t^1_1) \in \eta^{i_1}_{n_1 m_1}$, for some n_1, m_1, i_1, and we define

$$t^2_1 := \max\{\alpha \le t \le \beta \,:\, g_1(t) \in \eta^{i_1}_{n_1 m_1}\}\,.$$

In a similar way, we define recursively

$$t^1_j := \min\{t^2_{j-1} < t \le \beta \,:\, g_1(t) \in \cup_{n,m,i}\eta^i_{nm}\}\,;$$

if $g_1(t^1_j) \in \eta^{i_j}_{n_j m_j}$, for some n_j, m_j, i_j, we take

$$t^2_j := \max\{t^2_{j-1} < t \le \beta \,:\, g_1(t) \in \eta^{i_j}_{n_j m_j}\}\,.$$

We can continue this choice for $1 \le j \le r$. We define $t^2_0 := \alpha$ if $\alpha \ne t^1_1$, and $t^1_{r+1} := \beta$ if $\beta \ne t^2_r$.

We define g_3 (in this case) as the restriction of g_1 to the set $[\alpha, t^1_1] \cup (t^2_1, t^1_2] \cup \cdots \cup (t^2_{r-1}, t^1_r] \cup (t^2_r, \beta]$. If g_1 does not intersects $\cup_{n,m,i}\eta^i_{nm}$, we take $g_3 = g_1$. We define $a_3 := a_2$ if $\alpha < t^1_1$ and $a_3 := g_1(t^2_1)$ if $\alpha = t^1_1$; we define $b'_3 := b'_2$ if $t^2_r < \beta$ and $b'_3 := g_1(t^1_r)$ if $t^2_r = \beta$. g_3 is a left continuous curve between a_3 and b'_3. We consider a similar construction with the other sides of T_2. The triangle T_3 is the union of these three curves.

Lemma 8. *If T_3 is δ_3-thin, then T_1 is $\max\{\delta_3 + k_1, \delta^*\}$-thin, with $\delta^* := 3k_1/2 + 4k_6 + 2H(k_6, 2\max\{1, k_2\}, 4k_1 + 2k_3)$, where H is the constant in Theorem C.*

Proof. We study the triangle T_1 in each piece of $\{B_s\}_{s\in S}$ and of $\{X_n\}_{n\in G}$.

Recall that $(c1)$ gives that for any $n \in G$, X_n is a (k_1, k_2, k_3)-tree-piece. Corollary 1 gives that $T_3 \cap X_n$ is δ^*-thin for every $n \in G$(we can assume that X_n intersects at least two sides of T_3; if X_n had intersected only one side of T_1, this part of T_1 would have been removed during the construction of T_3, since X_n is a tree-piece). We consider now $T_1 \cap B_s$ for each s.

By (b) and the construction of T_3, given any $z \in T_1 \cap B_s$, there exists $z_2 \in T_3$ in the corresponding side of z, with $d_X(z, z_2) \le d_{X_n}(z, z_2) \le k_1$. Then there exists w in the union of the two other sides of T_3 with $d_X(w, z_2) \le \delta_3$. Since $T_3 \subseteq T_1 \cap B_s$, we have the result. □

Lemma 9. *Each side of T_3 is a $(1+k_2, k_1+16k_6)$-quasigeodesic with its arc-length parametrization. Furthermore, each connected component of $T_3 \cap X_n$ is a geodesic in X_n, if $n \in F_1$.*

See the proof of Lemma 9 in Section 3.

Remark. After the construction of T_3 and lemmas 8 and 9, without loss of generality we can assume that there is a unique component B_s, i.e. that T_3 is a $(1 + k_2, k_1 + 16k_6, k_1)$-quasigeodesic triangle in X, with $\Lambda = F$ and $G = \varnothing$.

We construct the triangle T_4 by changing each geodesic segment in T_3 joining η^i_{nm} with η^j_{nk} by a new geodesic γ^{ij}_{mnk}. This triangle and conditions $(c3)$ and $(c4)$ will allow to obtain a triangle T_5 in Y in an obvious way.

These are the details in the construction of T_4:

Each connected component of T_3 is a geodesic segment g^{ij}_{mnk} in some X_n, joining η^i_{nm} with η^j_{nk}. If $n \in F_1$, $(c3)$ gives that for each g^{ij}_{mnk} there exists a geodesic γ^{ij}_{mnk} in X_n, joining η^i_{nm} with η^j_{nk}, and a (k_4, b^{ij}_{mnk})-quasi-isometry $f^{ij}_{mnk} : \gamma^{ij}_{mnk} \longrightarrow h^{ij}_{mnk} \subseteq Y_n$. If $n \in F_2$, we define f^{ij}_{mnk} as the restriction of f_n to g^{ij}_{mnk}, $\gamma^{ij}_{mnk} := g^{ij}_{mnk}$, and $h^{ij}_{mnk} := f^{ij}_{mnk}(\gamma^{ij}_{mnk})$. (Then, f^{ij}_{mnk} is a (k_4, b^{ij}_{mnk})-quasi-isometry, with $b^{ij}_{mnk} := 0$.)

We obtain T_4 in X by replacing each g^{ij}_{mnk} by γ^{ij}_{mnk}. We only need to choose the vertices of T_4, if some vertex of T_3 is in $\cup_{n\in F_1} X_n$:

Let us consider $n \in F_1$ and the arc-length parametrizations $g^{ij}_{mnk} : [0, l] \longrightarrow X$ and $\gamma^{ij}_{mnk} : [0, l'] \longrightarrow X$. We observe first that $(c2)$ gives $l' - l = L_X(\gamma^{ij}_{mnk}) - L_X(g^{ij}_{mnk}) \le \operatorname{diam}_{X_n}(\eta^i_{nm}) + \operatorname{diam}_{X_n}(\eta^j_{nk}) \le 2k_2 L_X(g^{ij}_{mnk}) = 2k_2 l$. Therefore we conclude $l'/l \le 1 + 2k_2$, and symmetrically $l/l' \le 1 + 2k_2$.

Lemma 10. *Let us consider an absolute continuous and bijective function between two intervals $u : I \longrightarrow J$ with $c^{-1} \le |u'| \le c$, and an (a, b)-quasigeodesic $g : J \longrightarrow X$. Then $g \circ u : I \longrightarrow X$ is an (ac, b)-quasigeodesic.*

Proof. We have that $c^{-1}|t - s| \le |u(t) - u(s)| \le c|t - s|$, and hence

$$a^{-1}c^{-1}|t-s| - b \le a^{-1}|u(t)-u(s)| - b \le d(g(u(t)), g(u(s))) \le a|u(t)-u(s)| + b \le ac|t-s| + b\,. \quad \square$$

Lemma 11. *Let us consider two geodesics $\gamma_1 : [0, l_1] \longrightarrow X$ and $\gamma_2 : [0, l_2] \longrightarrow X$ in a δ-fine space X, with $d(\gamma_1(0), \gamma_2(0)) \le c$ and $d(\gamma_1(l_1), \gamma_2(l_2)) \le c$. Then $d(\gamma_1(t), \gamma_2(l_2 t/l_1)) \le 2\delta + 7c$, for $t \in [0, l_1]$.*

See the proof of Lemma 11 in Section 3.

We consider the reparametrization $g_{mnk}^{ij}(lt/l') : [0,l'] \longrightarrow X$ of g_{mnk}^{ij}; recall that $l'/l, l/l' \le 1 + 2k_2$. Using these local reparametrizations, if $G_3 : J_0 \longrightarrow X$ and $G_4 : I_0 \longrightarrow X$ are arc-length parametrizations of T_3 and T_4 (respectively), we can construct a global bijection $u : I_0 \longrightarrow J_0$ (in fact, a continuous juxtaposition of straight lines) with $(1+2k_2)^{-1} \le |u'| \le 1+2k_2$. Since $G_3 \circ u$ and G_4 are defined over the same interval I_0, if $(G_3 \circ u)(t_0)$ is a vertex in T_3, for some $t_0 \in I_0$, we can define $G_4(t_0)$ as its corresponding vertex in T_4. Lemmas 9 and 10 give that if $g_3 = (G_3 \circ u)|_I$, is a side of T_3, for some interval I, then g_3 is a $((1+k_2)(1+2k_2), k_1 + 16k_6)$-quasigeodesic. Observe that $g_4 := G_4|_I$ is an arc-length parametrization of the side of T_4 corresponding to g_3. Since we have (b), Lemma 11 gives that $d_X(g_3(t), g_4(t)) \le 8k_6 + 7k_1$, for every $t \in I$. Then, Lemma 6 implies that g_4 is a $((1+k_2)(1+2k_2), 15k_1 + 32k_6)$-quasigeodesic. Consequently we obtain the following result.

Lemma 12. *Each side of T_4 is a $((1+k_2)(1+2k_2), 15k_1+32k_6)$-quasigeodesic with its arc-length parametrization. Furthermore, each connected component of $T_4 \cap X_n$ is a geodesic in X_n, if $n \in F_1$. If T_4 is δ_4-thin, then T_3 is $(\delta_4 + 14k_1 + 16k_6)$-thin.*

Proof. We have proved the first two statements. In order to prove the last one we only need to remark that for every point in any side of T_3 there is another one in the corresponding side of T_4 which is at distance $7k_1 + 8k_6$ at most; the same result is true if we change the roles of T_3 and T_4. □

Let us observe that if $T \subseteq X_n$, with $n \in F_2$, then $T_4 = T$.

So far, we have modified the original triangle in X to obtain a new one T_4 which can now be easily transformed into a triangle T_5 in Y by replacing $\gamma_{mnk}^{ij} \subseteq X_n$ by $h_{mnk}^{ij} \subseteq Y_n$. We take the *canonical* parametrization $f_{mnk}^{ij}(\gamma_{mnk}^{ij}(t))$ in h_{mnk}^{ij}, where t is the arc-length parameter for γ_{mnk}^{ij}.

Lemma 13. *Each side of T_5 is a (d_1, d_2)-quasigeodesic with its canonical parametrization, where $d_0 := (1+k_2)(1+2k_2)k_4$, $d_1 := d_0(1+k_2)(1+2k_2)$ and*

$$d_2 := \max\left\{k_1 + (1+k_2)k_5,\, k_4(15k_1+32k_6) + k_5,\, d_0^{-1}(17k_1+32k_6) + 2(k_1+k_5) + (1+2k_2)^{-1}k_5\right\}.$$

In fact, the proof of Lemma 13 (see Section 3) gives the following result.

Corollary 4. *For any $x, y \in T_4$ with corresponding points $x', y' \in T_5$, we have that $d_X(x,y) \le d_0\, d_Y(x',y') + 2k_1 + d_0(2(k_1+k_5) + (1+2k_2)^{-1}k_5)$.*

By Lemma 13, the sides of T_5 are (d_1,d_2)-quasigeodesics. By (b) and the construction of T_5, we have that an end point of any side of T_5 has an end point of another side at distance less or equal than k_1. Since Y is δ'-hyperbolic, Lemma 4 gives that T_5 is δ_5-thin with $\delta_5 := 4\delta' + k_1 + 2H(\delta', d_1, d_2 + 2k_1)$. Now Corollary 4 gives that T_4 is δ_4-thin, with $\delta_4 := d_0\delta_5 + 2k_1 + d_0(2(k_1+k_5) + (1+2k_2)^{-1}k_5)$.

Lemma 12 gives that T_3 is δ_3-thin with $\delta_3 := \delta_4 + 14k_1 + 16k_6$. By Lemma 8, we have that T_1 is δ_1-thin with $\delta_1 := \max\{\delta_3 + k_1, \delta^*\}$, where $\delta^* = 3k_1/2 + 4k_6 + 2H(k_6, 2\max\{1,k_2\}, 4k_1 + 2k_3)$. Theorem 2 is now a consequence of Lemma 5, and we have $\delta := 4(\delta_1 + 16k_6)$, since $\delta_1 \ge 2k_6$ (in fact, $\delta_1 \ge \delta_3 \ge 16k_6$). □

Proof of Proposition 1. Firstly we prove that the inclusion $i : X_n \longrightarrow X$ is a $(k_7, k_7^{-1}(2k_1+k_8))$-quasi-isometry.

Given $x, y \in X_n$, we have that $d_X(x,y) \le d_{X_n}(x,y)$, since there are more curves joining x and y in X than in X_n. In order to prove the other inequality, let us consider a geodesic g in X joining x and y. If $g \subseteq X_n$, then $d_X(x,y) = d_{X_n}(x,y)$. In other case, we have for some m, k, i, j,

$$\begin{aligned} d_X(x,y) &\ge d_{X_n}(x,\eta_{nm}^i) + d_X(\eta_{nm}^i,\eta_{nk}^j) + d_{X_n}(y,\eta_{nk}^j) \\ &\ge d_{X_n}(x,\eta_{nm}^i) + k_7^{-1} d_{X_n}(\eta_{nm}^i,\eta_{nk}^j) + d_{X_n}(y,\eta_{nk}^j) - k_8 k_7^{-1} \\ &\ge k_7^{-1}\big(d_{X_n}(x,\eta_{nm}^i) + \operatorname{diam}_{X_n}(\eta_{nm}^i) + d_{X_n}(\eta_{nm}^i,\eta_{nk}^j) + \operatorname{diam}_{X_n}(\eta_{nk}^j) + d_{X_n}(y,\eta_{nk}^j) - 2k_1 - k_8\big) \\ &\ge k_7^{-1} d_{X_n}(x,y) - k_7^{-1}(2k_1+k_8)\,. \end{aligned}$$

Hence, since X is δ-hyperbolic, then X_n is $k_7\big(4\delta + k_7^{-1}(2k_1+k_8) + 2H(\delta, k_7, k_7^{-1}(2k_1+k_8))\big)$-thin (see [GH, p.88]). □

Proof of Proposition 2. Given $x, y \in X_n$, we have that

$$d_X(x,y) \le d_{X_n}(x,y) \le d_X(x,y) + \operatorname{diam}_{X_n}(\cup_{mi}\eta_{nm}^i) \le d_X(x,y) + k_7\,.$$

If we denote by $(x,y)_w$ and $(x,y)_{w,n}$ the Gromov products in X and X_n respectively, the last inequalities give for any $x, y, w \in X_n$

$$(x,y)_{w,n} - k_7 \le (x,y)_w \le (x,y)_{w,n} + k_7/2\,.$$

Then, we deduce for any $x, y, z, w \in X_n$, that

$$\begin{aligned} (x,z)_{w,n} &\ge (x,z)_w - k_7/2 \ge \min\{(x,y)_w, (y,z)_w\} - \delta - k_7/2 \\ &\ge \min\{(x,y)_{w,n} - k_7, (y,z)_{w,n} - k_7\} - \delta - k_7/2 \ge \min\{(x,y)_{w,n}, (y,z)_{w,n}\} - \delta - 3k_7/2\,. \end{aligned}$$

Hence, X_n is $(\delta + 3k_7/2)$-hyperbolic. □

§3. Proof of technical lemmas

Lemma 5. *For each point z in one side of T, we denote by $A = A(z)$ the union of the two other sides of T. If we are in case 3.4 we have $d_X(z,A) \le 18k_6$. In other case, we have either:*

(1) *$d_X(z,A) \le 12k_6$, or*

(2) *there exists a point $z_1 \in T_1$ with $d_X(z,z_1) \le 8k_6$, and besides z and z_1 are in corresponding sides.*

Moreover, for each point z_1 in one side of T_1 there exists a point $z \in T$ with $d_X(z,z_1) \le 8k_6$, and furthermore z and z_1 are in corresponding sides.

Consequently, if T_1 is δ_1-thin, then T is δ_0-thin, with $\delta_0 := \max\{\delta_1 + 16k_6, 18k_6\}$.

Proof. Recall that if $n \in F_2$, then $T_1 \cap X_n = T \cap X_n$. Consequently, we can assume that the vertices of T belong to $\cup_{n\in\Lambda\setminus F_2} X_n$, since in other case the argument is easier.

If $z \notin \eta_a \cup \eta_b \cup \eta_c$, then $z \in T_1$ and we have (2) with $z_1 = z$. In other case we can assume that $z \in \eta_a$. We consider now the same cases in the construction of T_1 in the proof of Theorem 2.

Case 1. We have that $\eta_a \subseteq X_n$ only contains a vertex of T. Let us denote by a_1, x_1', x_2' the internal points of the geodesics $[x_1,x_2],[a,x_2],[a,x_1]$ in X_n respectively. We have $\eta_{a_1} := [x_1,x_2] = [x_1,a_1] \cup [a_1,x_2] \subset T_a$. Since T_a is $4k_6$-fine in X_n by the hypothesis and Theorem A, if $z \in [x_1,x_2']$ then there exists $z_1 \in [x_1,a_1]$ with $d_X(z_1,z) \le d_{X_n}(z_1,z) \le 4k_6$, and if $z \in [x_1',x_2]$ then there exists $z_1 \in [a_1,x_2]$ with $d_X(z_1,z) \le 4k_6$; then, we have (2). If $z \in [a,x_1']$, we can take $w \in [a,x_2']$ with $d_X(z,w) \le 4k_6$; $z \in [a,x_2']$, we can take $w \in [a,x_1']$ with $d_X(z,w) \le 4k_6$; then, we have (1).

Case 2. We have now that $b \in \eta_a$ and $c \notin \eta_a$.

Case 2.1. We consider the situation $d_{X_n}(x_2,v_1) < d_{X_n}(x_2,u_1)$. We denote by u_1' the point in $[a,v_3] \subset [a,b]$ with $d_{X_n}(a,u_1) = d_{X_n}(a,u_1')$.

(i) If $z \in [x_1,u_3] \subseteq [x_1,a]$, then there exists $z_1 \in [x_1,a_1]$ such that $d_X(z,z_1) \le 4k_6$.

(ii) If $z \in [u_1',v_3]$, then there exists $z_1 \in [a_1,b_1]$ such that $d_X(z,z_1) \le 8k_6$, since the triangles T_a and T_b are $4k_6$-fine.

(iii) If $z \in [x_2,v_2] \subseteq [x_2,b]$, then there exists $z_1 \in [b_1,x_2]$ with $d_X(z,z_1) \le 8k_6$.

In these three cases we have (2).

(iv) If $z \in [a,u_3]$, then there exists $w \in [a,u_1']$ such that $d_X(z,w) \le 8k_6$.

(v) If $z \in [a,u_1']$, then there exists $w \in [a,u_3]$ such that $d_X(z,w) \le 8k_6$.

(vi) If $z \in [b,v_3]$, then there exists $w \in [b,v_2]$ such that $d_X(z,w) \le 4k_6$.

(vii) If $z \in [b,v_2]$, then there exists $w \in [b,v_3]$ such that $d_X(z,w) \le 4k_6$.

In these four cases we have (1).

Case 2.2. We consider the situation $d_{X_n}(x_2,v_1) \ge d_{X_n}(x_2,u_1)$. Let us recall that $a_1 = b_1$. We denote by v_1' the point in $[a,u_3] \subset [a,x_1]$ with $d_{X_n}(a,v_1) = d_{X_n}(a,v_1')$ and by u_1' the point in $[v_2,x_2] \subset [b,x_2]$ with $d_{X_n}(x_2,u_1) = d_{X_n}(x_2,u_1')$.

(i) If $z \in [x_1,u_3] \subseteq [x_1,a]$, then there exists $z_1 \in [x_1,a_1]$ such that $d_X(z,z_1) \le 4k_6$.

(ii) If $z \in [x_2,u_1'] \subseteq [x_2,b]$, then there exists $z_1 \in [b_1,x_2]$ with $d_X(z,z_1) \le 8k_6$.

In these two cases we have (2).

(iii) If $z \in [a,v_1']$, then there exists $w \in [a,v_3]$ such that $d_X(z,w) \le 8k_6$.

(iv) If $z \in [a,v_3]$, then there exists $w \in [a,v_1']$ such that $d_X(z,w) \le 8k_6$.

(v) If $z \in [b,v_3]$, then there exists $w \in [b,v_2]$ such that $d_X(z,w) \le 4k_6$.

(vi) If $z \in [b,v_2]$, then there exists $w \in [b,v_3]$ such that $d_X(z,w) \le 4k_6$.

(vii) If $z \in [u_3,v_1']$, then there exists $w \in [v_2,u_1'] \subset [b,u_1']$ such that $d_X(z,w) \le 8k_6$.

(viii) If $z \in [v_2,u_1']$, then there exists $w \in [v_1',u_3] \subset [a,u_3]$ such that $d_X(z,w) \le 8k_6$.

In these five cases we have (1).

Case 3. We have now that $b, c \in \eta_a$.

Case 3.1. We consider the situation $d_{X_n}(x_1,s_3) < d_{X_n}(x_1,u_2)$ and $d_{X_n}(x_2,v_3) < d_{X_n}(x_2,u_1)$. We denote by u_2' the point in $[b,s_1] \subset [a,b]$ with $d_{X_n}(b,u_2) = d_{X_n}(b,u_2')$, and by u_1' the point in $[b,v_2] \subset [b,c]$ with $d_{X_n}(b,u_1) = d_{X_n}(b,u_1')$.

(i) If $z \in [x_1,s_2] \subseteq [x_1,a]$, then there exists $z_1 \in [x_1,a_1]$ such that $d_X(z,z_1) \le 8k_6$, since the triangles T_a and T_b are $4k_6$-fine.

(ii) If $z \in [s_1,u_2']$, then there exists $z_1 \in [a_1,b_1]$ such that $d_X(z,z_1) \le 8k_6$.

(iii) If $z \in [u_1', v_2]$, then there exists $z_1 \in [b_1, c_1]$ such that $d_X(z, z_1) \leq 8k_6$.

(iv) If $z \in [x_2, v_1] \subseteq [x_2, c]$, then there exists $z_1 \in [c_1, x_2]$ with $d_X(z, z_1) \leq 8k_6$.

In these four cases we have (2).

(v) If $z \in [a, s_2]$, then there exists $w \in [a, s_1]$ such that $d_X(z, w) \leq 4k_6$. We have a similar result if $z \in [a, s_1]$.

(vi) If $z \in [b, u_2']$, then there exists $w \in [b, u_1']$ such that $d_X(z, w) \leq 12k_6$. We have a similar result if $z \in [b, u_1']$.

(vii) If $z \in [c, v_2]$, then there exists $w \in [c, v_1]$ such that $d_X(z, w) \leq 4k_6$. We have a similar result if $z \in [c, v_1]$.

In these three cases we have (1).

Case 3.2. We have the situation $d_{X_n}(x_1, s_3) \geq d_{X_n}(x_1, u_2)$ and $d_{X_n}(x_2, v_3) < d_{X_n}(x_2, u_1)$. We denote by u_2' the point in $[x_1, s_2] \subset [x_1, a]$ with $d_{X_n}(x_1, u_2) = d_{X_n}(x_1, u_2')$, by u_1' the point in $[b, v_2] \subset [b, c]$ with $d_{X_n}(b, u_1) = d_{X_n}(b, u_1')$, and by s_3' the point in $[b, v_2] \subset [b, c]$ with $d_{X_n}(b, s_3) = d_{X_n}(b, s_3')$.

(i) If $z \in [x_1, u_2'] \subseteq [a, c]$, then there exists $z_1 \in [x_1, a_1]$ such that $d_X(z, z_1) \leq 8k_6$.

(ii) If $z \in [u_1', v_2] \subseteq [b, c]$, then there exists $z_1 \in [b_1, c_1]$ such that $d_X(z, z_1) \leq 8k_6$.

(iii) If $z \in [x_2, v_1] \subseteq [a, c]$, then there exists $z_1 \in [c_1, x_2]$ with $d_X(z, z_1) \leq 8k_6$.

In these three cases we have (2).

(iv) If $z \in [a, s_2]$, then there exists $w \in [a, s_1]$ such that $d_X(z, w) \leq 4k_6$. We have a similar result if $z \in [a, s_1]$.

(v) If $z \in [b, s_1]$, then there exists $w \in [b, s_3']$ such that $d_X(z, w) \leq 12k_6$. We have a similar result if $z \in [b, s_3']$.

(vi) If $z \in [c, v_2]$, then there exists $w \in [c, v_1]$ such that $d_X(z, w) \leq 4k_6$. We have a similar result if $z \in [c, v_1]$.

(vii) If $z \in [u_2', s_2]$, then there exists $w \in [u_1', s_3']$ such that $d_X(z, w) \leq 12k_6$. We have a similar result if $z \in [u_1', s_3']$.

In these four cases we have (1).

Case 3.3 is similar to 3.2.

Case 3.4. We have the situation $d_{X_n}(x_1, s_3) \geq d_{X_n}(x_1, u_2)$ and $d_{X_n}(x_2, v_3) \geq d_{X_n}(x_2, u_1)$. Without loss of generality we can assume that $d_{X_n}(b, v_3) \geq d_{X_n}(b, s_3)$, since the other case is similar. We denote by v_3' the point in $[b, u_2] \subset [b, x_1]$ with $d_{X_n}(b, v_3) = d_{X_n}(b, v_3')$, by v_3'' the point in $[x_1, s_2] \subset [x_1, a]$ with $d_{X_n}(x_1, v_3') = d_{X_n}(x_1, v_3'')$, and by s_1' the point in $[b, v_2] \subseteq [b, c]$ with $d_{X_n}(b, s_1) = d_{X_n}(b, s_1')$.

(i) If $z \in [a, s_2]$, then there exists $w \in [a, s_1]$ such that $d_X(z, w) \leq 4k_6$. We have a similar result if $z \in [a, s_1]$.

(ii) If $z \in [b, s_1]$, then there exists $w \in [b, s_1']$ such that $d_X(z, w) \leq 12k_6$. We have a similar result if $z \in [b, s_1']$.

(iii) If $z \in [c, v_2]$, then there exists $w \in [c, v_1]$ such that $d_X(z, w) \leq 4k_6$. We have a similar result if $z \in [c, v_1]$.

(iv) If $z \in [v_3'', s_2]$, then there exists $w \in [v_2, s_1']$ such that $d_X(z, w) \leq 12k_6$. We have a similar result if $z \in [v_2, s_1']$.

(v) In other case, $z \in [v_3'', v_1] \subseteq [a, c]$. We have that $L_X([v_3'', v_1]) = d_X(v_3'', v_1) \le 12k_6$; consequently $d_X(z, \{v_3'', v_1\}) \le 6k_6$ and $d_X(z, A) \le 6k_6 + 12k_6 = 18k_6$.

This finishes the proof of the first part of the lemma. The proof of the second one follows a similar argument and is easier, since there is no dichotomy.

Finally, let us see that T_1 is δ_1-thin in X implies that T is δ_0-thin in X. We consider $z \in T$; if z satisfies (1), there is nothing to prove. In other case, there exists $z_1 \in T_1$ such that $d_X(z, z_1) \le 8k_6$ and z and z_1 are in corresponding sides. Since T_1 is δ_1-thin in X, there exists $w_1 \in T_1$ with $d_X(z_1, w_1) \le \delta_1$ and w_1 in the union of the two other sides. The second part of the lemma gives that there exists $w \in A$ with $d_X(w_1, w) \le 8k_6$. Therefore $d_X(z, A) \le d_X(z, w) \le \delta_1 + 16k_6$. □

Lemma 7. *Each side of T_1 is a $(1, 16k_6)$-quasigeodesic with its arc-length parametrization. Furthermore, each connected component of $T_1 \cap X_n$ is a geodesic in X_n, if $n \in \Lambda \setminus F_2$.*

Proof. We can assume that the vertices of T belong to $\cup_{n \in \Lambda \setminus F_2} X_n$, since in other case the argument is easier.

The second statement is a direct consequence of the construction of T_1. This first one is a consequence of Lemma 6 and the construction of T_1:

If $g : J \longrightarrow X$ is a geodesic side of T, Lemma 6 gives that it is enough to check that there exists a subinterval $I \subseteq J$ such that $g_1 : I \longrightarrow X$ is the arc-length parametrization for the corresponding side in T_1 of g, and that $d_X(g(t), g_1(t)) \le 8k_6$ for every $t \in I$.

We consider now the same cases in the construction of T_1 in the proof of Theorem 2.

Case 1. If $[x_1, x_2'] \subset g$, then we substitute this interval for $[x_1, a_1]$ in order to obtain g_1, and then we have $d_X(g(t), g_1(t)) \le 4k_6$ in these arcs, since T_a is $4k_6$-fine. The case $[x_2, x_1'] \subset g$ is similar.

Case 2. If $[x_1, u_3] \subset g$, then we substitute this interval for $[x_1, a_1]$ in order to obtain g_1, and then we have $d_X(g(t), g_1(t)) \le 4k_6$ in these arcs. The case $[x_2, v_2] \subset g$ is similar, with constant $8k_6$, since T_a and T_b are $4k_6$-fine.

Case 2.1. If $g = [a, b]$, then $[u_1', v_3] \subset g$ and $g_1 = [a_1, b_1]$. We have $d_X(g(t), g_1(t)) \le 8k_6$ in g_1.

Case 2.2. If $g = [a, b]$, then $a_1 = b_1$ and g_1 is this unique point.

Case 3.1. If $[x_1, s_2] \subset g$, then we substitute this interval for $[x_1, a_1]$ in order to obtain g_1, and then we have $d_X(g(t), g_1(t)) \le 8k_6$ in these arcs. The case $[x_2, v_1] \subset g$ is similar.

If $g = [a, b]$, then $[s_1, u_2'] \subset g$ and $g_1 = [a_1, b_1]$. We have $d_X(g(t), g_1(t)) \le 8k_6$ in g_1. If $g = [b, c]$, then $[u_1', v_2] \subset g$ and $g_1 = [b_1, c_1]$. We have $d_X(g(t), g_1(t)) \le 8k_6$ in g_1.

Case 3.2. If $g = [a, c]$, we have $[x_1, u_2'] \cup [x_2, v_1] \subset g$, and then we substitute these intervals for $[x_1, a_1] \cup [x_2, c_1]$ (respectively) in order to obtain g_1; then we have $d_X(g(t), g_1(t)) \le 8k_6$ in these arcs.

If $g = [b, c]$, then $[u_1', v_2] \subset g$ and $g_1 = [b_1, c_1]$. We have $d_X(g(t), g_1(t)) \le 8k_6$ in g_1.

If $g = [a, b]$, then $a_1 = b_1$ and g_1 is this unique point.

Case 3.3 is similar to 3.2; we do not consider 3.4 since in this case we do not have T_1. □

Lemma 9. *Each side of T_3 is a $(1+k_2, k_1+16k_6)$-quasigeodesic with its arc-length parametrization. Furthermore, each connected component of $T_3 \cap X_n$ is a geodesic in X_n, if $n \in F_1$.*

Proof. We can assume that the vertices of T belong to $\cup_{n\in F_1} X_n$, since in other case the argument is easier.

The second statement is a direct consequence of the construction of T_3 and Lemma 7. In order to see the first one, let us consider an arc-length parametrization $g_1 : [0,l] \longrightarrow X$ of one side of T_1. Without loss of generality we can assume that $g_1(0) = a_2$ and $g_1(l) = b_2'$. g_1 is a $(1, 16k_6)$-quasigeodesic by Lemma 7. We consider now an arc-length parametrization $g_3 : [0,l'] \longrightarrow X$ of the side of T_3 corresponding to g_1. If $g_3 = g_1$, there is nothing to prove.

In other case, if $s,t \in [0,l']$ there exist $s^* \in (t_{i-1}^2, t_i^1]$ and $t^* \in (t_{j-1}^2, t_j^1]$ such that $s = s^* - \sum_{k=1}^{i-1}(t_k^2 - t_k^1)$, $t = t^* - \sum_{k=1}^{j-1}(t_k^2 - t_k^1)$, $g_3(s) = g_1(s^*)$ and $g_3(t) = g_1(t^*)$. Provided that $i = j$, we have that

$$d_X(g_3(t), g_3(s)) = d_X(g_1(t^*), g_1(s^*)) \le |t^* - s^*| = |t-s|\,,$$
$$d_X(g_3(t), g_3(s)) = d_X(g_1(t^*), g_1(s^*)) \ge |t^* - s^*| - 16k_6 = |t-s| - 16k_6\,.$$

Otherwise, we can assume that $i < j$. Then we have that

$$d_X(g_3(t), g_3(s)) = d_X(g_1(t^*), g_1(s^*)) \le t^* - s^* = t - s + \sum_{k=i}^{j-1}(t_k^2 - t_k^1)\,,$$
$$d_X(g_3(t), g_3(s)) = d_X(g_1(t^*), g_1(s^*)) \ge t^* - s^* - 16k_6 \ge t - s - 16k_6\,.$$

Observe that $(c2)$ gives $t_{k+1}^1 - t_k^2 \ge k_2^{-1}(t_k^2 - t_k^1)$. This fact implies that

$$t - s \ge \sum_{k=i}^{j-2}(t_{k+1}^1 - t_k^2) \ge k_2^{-1}\sum_{k=i}^{j-2}(t_k^2 - t_k^1)\,.$$

This inequality and (b) give

$$d_X(g_3(t), g_3(s)) \le t - s + \sum_{k=i}^{j-2}(t_k^2 - t_k^1) + t_{j-1}^2 - t_{j-1}^1 \le (1+k_2)(t-s) + k_1\,. \qquad \square$$

Lemma 11. *Let us consider two geodesics $\gamma_1 : [0,l_1] \longrightarrow X$ and $\gamma_2 : [0,l_2] \longrightarrow X$ in a δ-fine space X, with $d(\gamma_1(0), \gamma_2(0)) \le c$ and $d(\gamma_1(l_1), \gamma_2(l_2)) \le c$. Then $d(\gamma_1(t), \gamma_2(l_2 t/l_1)) \le 2\delta + 7c$, for $t \in [0,l_1]$.*

Proof. Without loss of generality we can assume that $l_1 \le l_2$. We consider the geodesic quadrilateral $Q = \{\gamma_1(0), \gamma_1(l_1), \gamma_2(l_2), \gamma_2(0)\}$ and the geodesic triangles $T_1 = \{\gamma_1(0), \gamma_1(l_1), \gamma_2(0)\}$ (with internal points $p_1 \in \gamma_1$, $p_2 \in [\gamma_1(l_1), \gamma_2(0)]$, $p_3 \in [\gamma_1(0), \gamma_2(0)]$) and $T_3 = \{\gamma_1(l_1), \gamma_2(l_2), \gamma_2(0)\}$ (with internal points $q_1 \in [\gamma_1(l_1), \gamma_2(0)]$, $q_2 \in \gamma_2$, $q_3 \in [\gamma_1(l_1), \gamma_2(l_2)]$).

Let us call q_1' the point in γ_1 with $d(\gamma_1(l_1), q_1') = d(\gamma_1(l_1), q_1) = d(\gamma_1(l_1), q_3) =: v_1$, and p_2' the point in γ_2 with $d(\gamma_2(0), p_2') = d(\gamma_2(0), p_2) = d(\gamma_2(0), p_3) =: u_2$. We define $u_1 := d(\gamma_1(0), p_1) = d(\gamma_1(0), p_3)$, and $v_2 := d(\gamma_2(l_2), q_2) = d(\gamma_2(l_2), q_3)$. Observe that $d(\gamma_1(0), \gamma_2(0)) = u_1 + u_2 \le c$ and $d(\gamma_1(l_1), \gamma_2(l_2)) = v_1 + v_2 \le c$.

We can assume that $u_1 + v_1 \le l_1 = L(\gamma_1)$, since the another case is simpler; this fact implies $u_2 + v_2 \le l_2 = L(\gamma_2)$. Since T_1 and T_3 are δ-fine, we have that $d(\gamma_1(t+u_1), \gamma_2(t+u_2)) \le 2\delta$, for every $t \in [0, l_1 - u_1 - v_1]$.

Observe that $d(\gamma_1(t),\gamma_2(t)) \le 2\delta + c$, for every $t \in [0, l_1 - u_1 - v_1]$:

$$d(\gamma_1(t),\gamma_2(t)) \le d(\gamma_1(t),\gamma_1(t+u_1)) + d(\gamma_1(t+u_1),\gamma_2(t+u_2)) + d(\gamma_2(t+u_2),\gamma_2(t))$$
$$\le u_1 + 2\delta + u_2 \le 2\delta + c\,.$$

If $t \in [l_1 - u_1 - v_1, l_1]$, we have that

$$d(\gamma_1(t),\gamma_2(t)) \le d(\gamma_1(t),\gamma_1(l_1-u_1-v_1)) + d(\gamma_1(l_1-u_1-v_1),\gamma_2(l_1-u_1-v_1))$$
$$+ d(\gamma_2(l_1-u_1-v_1),\gamma_2(t))$$
$$\le u_1 + v_1 + 2\delta + c + u_1 + v_1 \le 2\delta + 5c\,.$$

Then we have $d(\gamma_1(t),\gamma_2(t)) \le 2\delta + 5c$, for every $t \in [0, l_1]$.

The same argument with parametrizations which reverse the orientation, gives $d(\gamma_1(t),\gamma_2(t + l_2 - l_1)) \le 2\delta + 5c$, for every $t \in [0, l_1]$.

Observe now that $t \le l_2 t/l_1 \le t + l_2 - l_1$, and $l_2 - l_1 \le 2c$. Consequently we have

$$d(\gamma_1(t),\gamma_2(l_2t/l_1)) \le d(\gamma_1(t),\gamma_2(t)) + d(\gamma_2(t),\gamma_2(l_2t/l_1)) \le 2\delta + 5c + l_2 - l_1 \le 2\delta + 7c\,. \qquad \square$$

Lemma 13. *Each side of T_5 is a (d_1,d_2)-quasigeodesic with its canonical parametrization, where $d_0 := (1+k_2)(1+2k_2)k_4$, $d_1 := d_0(1+k_2)(1+2k_2)$ and*

$$d_2 := \max\left\{k_1 + (1+k_2)k_5,\, k_4(15k_1 + 32k_6) + k_5,\, d_0^{-1}(17k_1 + 32k_6) + 2(k_1 + k_5) + (1+2k_2)^{-1}k_5\right\}.$$

Proof. Let us consider a side $g_4 : I \longrightarrow X$ in T_4 with its arc-length parametrization, and its corresponding side g_5 in T_5 with its canonical parametrization.

Given $s, t \in I$, let us choose a geodesic γ in Y between $g_5(s)$ an $g_5(t)$.

By hypothesis, γ meets at most a finite number of σ^i_{nm}'s. Let us assume first that γ does not meet $\cup_{n,m,i}\sigma^i_{nm}$. Then $\gamma \subseteq Y_n$, for some $n \in \Lambda$, and we have by $(c3)$ and $(c4)$

$$d_Y(g_5(t),g_5(s)) = d_{Y_n}(g_5(t),g_5(s)) \ge k_4^{-1} d_{X_n}(g_4(t),g_4(s)) - k_5 \ge k_4^{-1} d_X(g_4(t),g_4(s)) - k_5\,.$$

Let us assume now that γ meets $\cup_{n,m,i}\sigma^i_{nm}$. Our goal is to split γ into some curves joining two closed sets σ^i_{nm} and σ^j_{nk} in Y_n, so that we can relate them with the geodesics $\gamma^{ij}_{mnk} \subseteq X_n$ joining η^i_{nm} with η^j_{nk} mentioned in $(c3)$ for $n \in F_1$; if $n \in F_2$ we can take as γ^{ij}_{mnk} any geodesic joining η^i_{nm} with η^j_{nk}. If $\gamma : [\alpha,\beta] \longrightarrow Y$, let us define

$$v^1_1 := \min\{\alpha \le v \le \beta \,:\, \gamma(v) \in \cup_{n,m,i}\sigma^i_{nm}\}\,.$$

There exists this minimum since γ is a continuous function in a compact interval and $\gamma \cap (\cup_{n,m,i}\sigma^i_{nm})$ is a compact set: each σ^i_{nm} is a closed set and γ meets at most a finite number of σ^i_{nm}'s.

Then $\gamma(v^1_1) \in \sigma^{i_1}_{n_1m_1}$, for some n_1, m_1, i_1, and we define

$$v^2_1 := \max\{\alpha \le v \le \beta \,:\, \gamma(v) \in \sigma^{i_1}_{n_1m_1}\}\,.$$

In a similar way, we define recursively

$$v^1_j := \min\{v^2_{j-1} < v \le \beta \,:\, \gamma(v) \in \cup_{n,m,i}\sigma^i_{nm}\}\,;$$

if $\gamma(v_j^1) \in \sigma_{n_j m_j}^{i_j}$, for some n_j, m_j, i_j, we take

$$v_j^2 := \max\{v_{j-1}^2 < v \le \beta : \gamma(v) \in \sigma_{n_j m_j}^{i_j}\} .$$

We can continue this choice for $1 \le j \le r$. We have that

$$d_Y(g_5(t), g_5(s)) = L_Y(\gamma) = \beta - \alpha \ge v_1^1 - \alpha + \sum_{k=2}^{r} (v_k^1 - v_{k-1}^2) + \beta - v_r^2 .$$

Given $\sigma_{n_{k-1} m_{k-1}}^{i_{k-1}}$ and $\sigma_{n_k m_k}^{i_k}$, we have $n_{k-1} = n_k$, $n_{k-1} = m_k$, $m_{k-1} = n_k$ or $m_{k-1} = m_k$. Since $\sigma_{nm}^i = \sigma_{mn}^i$, by simplicity in the notation we can assume that $m_{k-1} = n_k$ and that the curve $f_{n_{k-1} n_k m_k}^{i_{k-1} i_k} \circ \gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}$ joining $\sigma_{n_{k-1} m_{k-1}}^{i_{k-1}}$ and $\sigma_{n_k m_k}^{i_k}$, is contained in Y_{n_k}. If $\gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k} : [\alpha_k, \beta_k] \longrightarrow X_{n_k}$ $(k = 2, \dots, r)$, then $(c3)$ and $(c4)$ give that

$$\begin{aligned} k_4^{-1} d_{X_{n_k}} &\big(\gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\beta_k), \gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\alpha_k)\big) - b_{n_{k-1} n_k m_k}^{i_{k-1} i_k} \\ &\le d_{Y_{n_k}} \big(f_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\beta_k)), f_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\alpha_k))\big) . \end{aligned}$$

By $(c2)$,

$$\begin{aligned} d_{Y_{n_k}} &\big(f_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\beta_k)), f_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\alpha_k))\big) \\ &\le \operatorname{diam}_{Y_{n_k}} \big(\sigma_{n_{k-1} m_{k-1}}^{i_{k-1}}\big) + d_{Y_{n_k}} \big(\sigma_{n_{k-1} m_{k-1}}^{i_{k-1}}, \sigma_{n_k m_k}^{i_k}\big) + \operatorname{diam}_{Y_{n_k}} \big(\sigma_{n_k m_k}^{i_k}\big) \\ &\le (1 + 2k_2)\, d_{Y_{n_k}} \big(\sigma_{n_{k-1} m_{k-1}}^{i_{k-1}}, \sigma_{n_k m_k}^{i_k}\big) . \end{aligned}$$

Consequently we have

$$\begin{aligned} v_k^1 - v_{k-1}^2 &\ge d_{Y_{n_k}} \big(\sigma_{n_{k-1} m_{k-1}}^{i_{k-1}}, \sigma_{n_k m_k}^{i_k}\big) \\ &\ge (1 + 2k_2)^{-1} d_{Y_{n_k}} \big(f_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\beta_k)), f_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\alpha_k))\big) \\ &\ge (1 + 2k_2)^{-1} \Big(k_4^{-1} d_{X_{n_k}} \big(\gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\beta_k), \gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\alpha_k)\big) - b_{n_{k-1} n_k m_k}^{i_{k-1} i_k}\Big). \end{aligned}$$

We have that $\gamma([\alpha, v_1^1]) \subseteq Y_{n_1}$ or $\gamma([\alpha, v_1^1]) \subseteq Y_{m_1}$, and $\gamma([v_r^2, \beta]) \subseteq Y_{n_r}$ or $\gamma([v_r^2, \beta]) \subseteq Y_{m_r}$. By simplicity in the notation, we can assume that $\gamma([\alpha, v_1^1]) \subseteq Y_{n_1}$ and $\gamma([v_r^2, \beta]) \subseteq Y_{n_r}$. Then Remark 6 before Theorem 2 gives

$$\begin{aligned} v_1^1 - \alpha &\ge d_{Y_{n_1}} \big(g_5(s), \sigma_{n_1 m_1}^{i_1}\big) \ge k_4^{-1} d_{X_{n_1}} \big(g_4(s), \eta_{n_1 m_1}^{i_1}\big) - k_1 - k_5 , \\ \beta - v_r^2 &\ge d_{Y_{n_r}} \big(g_5(t), \sigma_{n_r m_r}^{i_r}\big) \ge k_4^{-1} d_{X_{n_r}} \big(g_4(t), \eta_{n_r m_r}^{i_r}\big) - k_1 - k_5 . \end{aligned}$$

Consequently we have

$$\begin{aligned} d_Y(g_5(t), g_5(s)) &\ge v_1^1 - \alpha + \sum_{k=2}^{r} (v_k^1 - v_{k-1}^2) + \beta - v_r^2 \\ &\ge k_4^{-1} d_{X_{n_1}} \big(g_4(s), \eta_{n_1 m_1}^{i_1}\big) - k_1 - k_5 + k_4^{-1} d_{X_{n_r}} \big(g_4(t), \eta_{n_r m_r}^{i_r}\big) - k_1 - k_5 \\ &\quad + (1 + 2k_2)^{-1} \sum_{k=2}^{r} \Big(k_4^{-1} d_{X_{n_k}} \big(\gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\beta_k), \gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\alpha_k)\big) - b_{n_{k-1} n_k m_k}^{i_{k-1} i_k}\Big) \\ &\ge k_4^{-1} d_{X_{n_1}} \big(g_4(s), \eta_{n_1 m_1}^{i_1}\big) + k_4^{-1} d_{X_{n_r}} \big(g_4(t), \eta_{n_r m_r}^{i_r}\big) - 2(k_1 + k_5) - (1 + 2k_2)^{-1} k_5 \\ &\quad + (1 + 2k_2)^{-1} k_4^{-1} \sum_{k=2}^{r} d_{X_{n_k}} \big(\gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\beta_k), \gamma_{n_{k-1} n_k m_k}^{i_{k-1} i_k}(\alpha_k)\big). \end{aligned}$$

Now we want to obtain a continuous curve γ' in X joining $g_4(s)$ with $g_4(t)$.

By $(c2)$ we can choose geodesics γ_k in $X_{n_{k+1}}$ $(2 \le k \le r-1)$ joining $\gamma^{i_{k-1}i_k}_{n_{k-1}n_k m_k}(\beta_k)$ with $\gamma^{i_k i_{k+1}}_{n_k n_{k+1} m_{k+1}}(\alpha_{k+1})$, such that

$$L_{X_{n_{k+1}}}(\gamma_k) = d_{X_{n_{k+1}}}\big(\gamma^{i_k i_{k+1}}_{n_k n_{k+1} m_{k+1}}(\alpha_{k+1}), \gamma^{i_{k-1}i_k}_{n_{k-1}n_k m_k}(\beta_k)\big)$$
$$\le k_2\, d_{X_{n_{k+1}}}\big(\gamma^{i_k i_{k+1}}_{n_k n_{k+1} m_{k+1}}(\beta_{k+1}), \gamma^{i_k i_{k+1}}_{n_k n_{k+1} m_{k+1}}(\alpha_{k+1})\big).$$

By (b) we can choose a geodesic γ_1 in X_{n_1} joining $g_4(s)$ with $\gamma^{i_1 i_2}_{n_1 n_2 m_2}(\alpha_2)$, such that

$$L_{X_{n_1}}(\gamma_1) = d_{X_{n_1}}\big(g_4(s), \gamma^{i_1 i_2}_{n_1 n_2 m_2}(\alpha_2)\big) \le d_{X_{n_1}}\big(g_4(s), \eta^{i_1}_{n_1 m_1}\big) + k_1\,,$$

and a geodesic γ_r in X_{n_r} joining $\gamma^{i_{r-1}i_r}_{n_{r-1}n_r m_r}(\beta_r)$ with $g_4(t)$, such that $L_{X_{n_r}}(\gamma_r) \le d_{X_{n_r}}\big(g_4(t), \eta^{i_r}_{n_r m_r}\big) + k_1$.

We consider now the continuous curve γ' in X joining $g_4(s)$ with $g_4(t)$ obtained by the juxtaposition of the geodesics $\{\gamma_k\}_{k=1}^r$ and $\{\gamma^{i_{k-1}i_k}_{n_{k-1}n_k m_k}\}_{k=2}^r$.

On the one hand, these facts give

$$\begin{aligned}
d_X(g_4(t), g_4(s)) &\le L_X(\gamma') \le d_{X_{n_1}}\big(g_4(s), \eta^{i_1}_{n_1 m_1}\big) + k_1 + d_{X_{n_r}}\big(g_4(t), \eta^{i_r}_{n_r m_r}\big) + k_1 \\
&\quad + (1+k_2)\sum_{k=2}^{r} d_{X_{n_k}}\big(\gamma^{i_{k-1}i_k}_{n_{k-1}n_k m_k}(\beta_k), \gamma^{i_{k-1}i_k}_{n_{k-1}n_k m_k}(\alpha_k)\big) \\
&\le 2k_1 + d_0(2(k_1+k_5) + (1+2k_2)^{-1}k_5) + d_0\Big(k_4^{-1} d_{X_{n_1}}\big(g_4(s), \eta^{i_1}_{n_1 m_1}\big) \\
&\quad + k_4^{-1} d_{X_{n_r}}\big(g_4(t), \eta^{i_r}_{n_r m_r}\big) + (1+2k_2)^{-1}k_4^{-1}\sum_{k=2}^{r} d_{X_{n_k}}\big(\gamma^{i_{k-1}i_k}_{n_{k-1}n_k m_k}(\beta_k), \gamma^{i_{k-1}i_k}_{n_{k-1}n_k m_k}(\alpha_k)\big) \\
&\quad - 2(k_1+k_5) - (1+2k_2)^{-1}k_5\Big) \\
&\le 2k_1 + d_0(2(k_1+k_5) + (1+2k_2)^{-1}k_5) + d_0\, d_Y(g_5(t), g_5(s))
\end{aligned}$$

(recall that $d_0 := (1+k_2)(1+2k_2)k_4$); then we have Corollary 4, since so far we have not used that $g_4(s)$ and $g_4(t)$ belong to the same side of T_4.

On the other hand, Lemma 12 gives

$$(1+k_2)^{-1}(1+2k_2)^{-1}|t-s| - 15k_1 - 32k_6 \le d_X(g_4(t), g_4(s))\,.$$

Consequently we have

$$d_Y(g_5(t), g_5(s)) \ge d_0^{-1}(1+k_2)^{-1}(1+2k_2)^{-1}|t-s| - d_0^{-1}(17k_1 + 32k_6) - 2(k_1+k_5) - (1+2k_2)^{-1}k_5\,.$$

In order to see the other inequality, we consider the domain I of g_5 and $s,t \in I$, with $s<t$.

If $g_5([s,t])$ does not intersect with any σ^i_{nm}, then $g_5([s,t]) \subseteq h^{ij}_{mnk}$, for some m,n,k,i,j. This fact, $(c3)$ and $(c4)$ give

$$d_Y(g_5(t), g_5(s)) \le d_{Y_n}(g_5(t), g_5(s)) \le k_4 d_{X_n}(g_4(t), g_4(s)) + k_5\,.$$

In other case, we can split the interval $[s,t]$ into a union of intervals $[u_0,u_1]\cup(u_1,u_2]\cup\cdots\cup(u_{l-1},u_l]$, with $l\ge 1$, such that $g_5((u_{r-1},u_r]) \subseteq h^{i_r j_r}_{m_r n_r k_r} \subseteq Y_{n_r}$ $(1\le r\le l)$, $u_0 = s$ and $u_l = t$. We have that $g_5(u_r)$ is an end point of $h^{i_r j_r}_{m_r n_r k_r}$; we denote by $g_5(u_{r-1}+)$ the other end point of $h^{i_r j_r}_{m_r n_r k_r}$.

By (b) and $(c2)$ we have that $d_{Y_{n_{r+1}}}\big(g_5(u_r+), g_5(u_r)\big) \le k_2\, d_{Y_{n_{r+1}}}\big(g_5(u_{r+1}), g_5(u_r+)\big)$ $(1 \le r \le l-2)$, and $d_{Y_{n_l}}\big(g_5(u_{l-1}+), g_5(u_{l-1})\big) \le k_1$. These facts, $(c3)$, $(c4)$ and Lemma 12 give

$$\begin{aligned} d_Y(g_5(t), g_5(s)) &\le \sum_{r=0}^{l-1} d_{Y_{n_{r+1}}}\big(g_5(u_{r+1}), g_5(u_r+)\big) + \sum_{r=1}^{l-2} d_{Y_{n_{r+1}}}\big(g_5(u_r+), g_5(u_r)\big) \\ &\quad + d_{Y_{n_l}}\big(g_5(u_{l-1}+), g_5(u_{l-1})\big) \\ &\le k_1 + (1+k_2) \sum_{r=0}^{l-1} d_{Y_{n_{r+1}}}\big(g_5(u_{r+1}), g_5(u_r+)\big) \\ &\le k_1 + (1+k_2) \sum_{r=0}^{l-1} \big(k_4\, d_{X_{n_{r+1}}}\big(g_4(u_{r+1}), g_4(u_r+)\big) + b_r\big) \\ &\le k_1 + (1+k_2)\, k_5 + (1+k_2)\, k_4 \sum_{r=0}^{l-1} |u_{r+1} - u_r| \le (1+k_2)\, k_4 |t-s| + k_1 + (1+k_2)\, k_5 \,. \end{aligned}$$

Consequently we have the result. □

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Ana Portilla, José M. Rodríguez and Eva Tourís
Departamento de Matemáticas
Universidad Carlos III de Madrid
Avenida de la Universidad, 30
28911 Leganés, Madrid, SPAIN
e-mails: apferrei@math.uc3m.es
jomaro@math.uc3m.es
etouris@math.uc3m.es