STABILITY OF GROMOV HYPERBOLICITY

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ABSTRACT. A main problem when studying any mathematical property is to determine its stability, i.e., under what type of perturbations it is preserved. With this aim, here we study the stability of Gromov hyperbolicity, a property which has been proved to be fruitful in many fields. First of all we analyze the stability under appropriate limits, in the context of general metric spaces. We also prove the stability under some transformations in Riemann surfaces, even though the original surface and the modified one are not quasi-isometric.

1. INTRODUCTION.

The theory of Gromov hyperbolic spaces is a useful tool in order to understand the connections between graphs and Potential Theory (see e.g. [4], [11], [14], [27], [28], [29], [30], [37], [38], [43]). Besides, the concept of Gromov hyperbolicity grasps the essence of negatively curved spaces, and has been successfully used in the theory of groups (see e.g. [16], [18], [19] and the references therein).

A geodesic metric space is called hyperbolic (in the Gromov sense) if there exists an upper bound of the distance of every point in a side of any geodesic triangle to the union of the two other sides (see Definition 2.1). The latter condition is known as Rips condition.

But, it is not easy to determine whether a given space is Gromov hyperbolic or not. Recently, there has been some research aimed to show that metrics used in geometric function theory are Gromov hyperbolic. Some specific examples are showing that the Klein-Hilbert metric ([8], [31]) is Gromov hyperbolic (under particular conditions on the domain of Definition), that the Gehring-Osgood metric ([21]) is Gromov hyperbolic, and that the Vuorinen metric ([21]) is not Gromov hyperbolic (except for a particular case). Recently, some interesting results by Balogh and Buckley [5] about the hyperbolicity of Euclidean bounded domains with their quasihyperbolic metric have made significant progress in this direction (see also [10], [44] and the references therein). Another interesting instance is that of a Riemann surface endowed with the Poincaré metric. With such metric structure a Riemann surface is always negatively curved, but not every Riemann surface is Gromov hyperbolic, since topological obstacles may impede it: for instance, the two-dimensional jungle-gym (a $\mathbb{Z}^2$-covering of a torus with genus two) is not hyperbolic. We are interested in studying when Riemann surfaces equipped with their Poincaré metric are Gromov hyperbolic (see e.g. [22], [23], [24], [25], [40], [41], [42], [32], [33], [34], [3], [36]).

One of the important problems when studying any property is to obtain its stability under appropriate deformations, i.e., under what type of perturbations it is preserved. With this aim, here we study the stability of Gromov hyperbolicity.

First of all we analyze the stability under appropriate limits, in the context of general metric spaces (see Theorem 3.1), and we apply this result to plane domains with their Poincaré metrics and Euclidean domains

with their quasihyperbolic metrics (see respectively Theorems 3.5 and 3.7). We also have complementary results on stability of non-hyperbolicity for the Poincaré and quasihyperbolic metrics (see respectively Theorems 3.11 and 3.9).

We also prove the stability of Gromov hyperbolicity under some transformations in plane domains (endowed with their Poincaré metrics), even though the original domain and the modified one are not quasi-isometric. In particular, Theorem 5.3 gives some answers to the following question: how do some geometric perturbations affect on the hyperbolicity of a flute surface?

Notations. We denote by X a geodesic metric space. By d_X and L_X we shall denote, respectively, the distance and the length in the metric of X. From now on, when there is no possible confusion, we will not write the subindex X.

Finally, we denote by C, c and c_i, positive constants which can assume different values in different theorems.

2. Background in Gromov spaces and Riemann surfaces.

In our study of hyperbolic Gromov spaces we use the notations of [16]. We give now the basic facts about these spaces. We refer to [16] for more background and further results.

Definition 2.1. *If X is a geodesic metric space and $J = \{J_1, J_2, \ldots, J_n\}$, with $J_j \subseteq X$, we say that J is δ-thin if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_j) \leq \delta$. If $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of three geodesics $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_1]$. The space X is δ-hyperbolic (or satisfies the Rips condition with constant δ) if every geodesic triangle in X is δ-thin.*

We would like to point out that deciding whether or not a space is hyperbolic is usually extraordinarily difficult: Notice that, first of all, we have to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to. And then we have to take supremum over all the possible choices for P and then over all the possible choices for T. It means that if our space is, for instance, an n-dimensional manifold and we select two points P and Q on different sides of a triangle T, the function F that measures the distance between P and Q is a $(3n + 2)$-variable function. In order to prove that our space is hyperbolic we would have to take the minimum of F over the variable that describes Q, and then the supremum over the remaining $3n + 1$ variables, or at least prove that it is finite. Without disregarding the difficulty of solving a $(3n + 2)$-variable minimax problem, notice that the main obstacle is that we do not even know in an approximate way the location of geodesics in the space.

Examples:

(1) Every bounded metric space X is $(diam X)$-hyperbolic (see e.g. [16, p. 29]).
(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k$, with $k > 0$, is hyperbolic (see e.g. [16, p. 52]).
(3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [16, p. 29]).

Definition 2.2. *If $\gamma : [a, b] \longrightarrow X$ is a continuous curve in a metric space (X, d), the* length *of γ is*

$$L(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \cdots < t_n = b \right\}.$$

We say that γ is a geodesic *if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$. We say that X is a* geodesic metric space *if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[x, y]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but convenient as well).*

If E is a relatively closed subset of a geodesic metric space (X, d), we always consider in E the *inner metric* obtained by the metric in X, that is

$$d_{X|E}(z, w) := \inf \left\{ L_{X,d}(\gamma) : \gamma \subset E \text{ is a rectifiable} \right.$$
$$\left. \text{curve joining } z \text{ and } w \right\} \geq d(z, w).$$

Definition 2.3. *A function between two metric spaces $f : X \longrightarrow Y$ is an (a,b)-quasi-isometry, $a \geq 1$, $b \geq 0$, if*

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a\, d_X(x_1, x_2) + b\,, \qquad for\ every\ x_1, x_2 \in X.$$

We say that f is ε-full if for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq \varepsilon$. If f is ε-full for some $\varepsilon \geq 0$, we say that X and Y are quasi-isometric.

An (a,b)-quasigeodesic in X is an (a,b)-quasi-isometry between an interval of $\mathbb{R}$ and X.

Quasi-isometries are important since they are the maps which preserve hyperbolicity:

Theorem 2.4. *([16, p.88]) Let us consider an (a,b)-quasi-isometry between two geodesic metric spaces $f : X \longrightarrow Y$. If Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which only depends on δ, a and b. Besides, if the image of f is ε-full for some $\varepsilon \geq 0$, then X is hyperbolic if and only if Y is hyperbolic.*

It is well-known that if f is not ε-full, the hyperbolicity of X does not imply the hyperbolicity of Y: it is enough to consider the inclusion of $\mathbb{R}$ in $\mathbb{R}^2$ (which is indeed an isometry).

Definition 2.5. *Let us consider $H > 0$, a metric space (X,d), and subsets $Y, Z \subseteq X$. The set $V_H(Y) := \{x \in X : d(x, Y) \leq H\}$ is called the H-neighborhood of Y in X. The Hausdorff distance of Y to Z is defined by $\mathcal{H}_d(Y, Z) := \inf\{H > 0 : Y \subseteq V_H(Z),\ Z \subseteq V_H(Y)\}$.*

The following is a beautiful and useful result:

Theorem 2.6. *([16, p. 87]) For each $\delta \geq 0$, $a \geq 1$ and $b \geq 0$, there exists a constant $H = H(\delta, a, b)$ with the following property:*

Let (X,d) be a δ-hyperbolic geodesic metric space and let g be a (a,b)-quasigeodesic joining x and y. If γ is a geodesic joining x and y, then $\mathcal{H}_d(g, \gamma) \leq H$.

This property is known as geodesic stability. M. Bonk has proved that, in fact, geodesic stability is equivalent to Gromov hyperbolicity [9].

Definition 2.7. *Let (X,d) be a metric space, and let $\{X_n\}_n \subseteq X$ be a family of geodesic metric spaces such that $\eta_{nm} := X_n \cap X_m$ are compact sets. Further, assume that for any n and m the set $X \setminus \eta_{nm}$ is not connected, and that a and b are in different components of $X \setminus \eta_{nm}$ for any $a \in X_n \setminus \eta_{nm}$, $b \in X_m \setminus \eta_{nm}$, with $m \neq n$. If there exists positive constants c_1 and c_2 such that $\mathrm{diam}_{X_n}(\eta_{nm}) \leq c_1$ for every n, m, and $d_{X_n}(\eta_{nm}, \eta_{nk}) \geq c_2$ for every n and $m \neq k$, we say that $\{X_n\}_n$ is a (c_1, c_2)-tree decomposition of X.*

Theorem 2.8. *([40, Theorem 2.4] and [32, Theorem 2.9]) Let us consider a metric space X and a family of geodesic metric spaces $\{X_n\}_n \subseteq X$ which is a (c_1, c_2)-tree decomposition of X. Then X is hyperbolic if and only if there exists a constant δ_0 such that X_n is δ_0-hyperbolic for every n.*

A *non-exceptional* Riemann surface S is a Riemann surface whose universal covering space is the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, endowed with its Poincaré metric, i.e. the metric obtained by projecting the Poincaré metric of the unit disk $ds = 2|dz|/(1 - |z|^2)$. Therefore, any simply connected subset of S is isometric to a subset of $\mathbb{D}$. With this metric, S is a geodesically complete Riemannian manifold with constant curvature -1, and therefore S is a geodesic metric space. The only Riemann surfaces which are left out are the *exceptional* Riemann surfaces, that is to say, the sphere, the plane, the punctured plane and the tori. It is easy to study the hyperbolicity of these particular cases. The Poincaré metric is natural and useful in Complex Analysis: for instance, any holomorphic function between two domains is Lipschitz with constant 1, when we consider the respective Poincaré metrics.

For $x \in D \subsetneq \mathbb{R}^k$ we denote by $\delta_D(x)$ the distance of x to the boundary of D, $\min_{a \in \partial D} |x - a|$. The quasihyperbolic metric in a domain $D \subsetneq \mathbb{R}^k$ is the distance induced by the density $1/\delta_D(x)$. We will denote by d_D the quasihyperbolic or the Poincaré distance in D, indistinctly; the context will determine without a doubt which metric we are using each time. The subscript Eucl will be used to denote the distance or length with respect to the Euclidean metric.

It is well known that for every non-exceptional domain $\Omega \subset \mathbb{C}$

$$\lambda_\Omega(z) \leq \frac{2}{\delta_\Omega(z)} \qquad \forall\, z \in \Omega,$$

and that for all domains $\Omega_1 \subset \Omega_2$ we have $\lambda_{\Omega_1}(z) \geq \lambda_{\Omega_2}(z)$ for every $z \in \Omega_1$.

We will need the following result.

Theorem 2.9. ([22, Theorem 1.1]) *Let Ω be a plane domain with $\partial\Omega \subset \mathbb{R}$, $\Omega \cap \mathbb{R} = (-\infty, 0) \cup \bigcup_{k=1}^{\infty}(a_k, b_k)$, $b_k \leq a_{k+1}$ for every k, and $\lim_{k \to \infty} a_k = \infty$.*

(1) The metric spaces Ω, with either the Poincaré or the quasihyperbolic metric, are Gromov hyperbolic if

$$\liminf_{k \to \infty} \frac{b_k - a_k}{a_k} > 0.$$

(2) The metric spaces Ω, with either the Poincaré or the quasihyperbolic metric, are not Gromov hyperbolic if

$$\lim_{k \to \infty} \frac{b_k - a_k}{a_k} = 0.$$

3. Limits of Gromov hyperbolic spaces.

First of all we analyze the stability of Gromov hyperbolicity under appropriate limits, in the context of general metric spaces.

Theorem 3.1. *Let us consider geodesic metric spaces X and $\{X_n\}_n$, with $X, X_n \subseteq Y$, where Y is a space with a measure μ and functions λ, λ_n, such that $L_X(\gamma) = \int_\gamma \lambda \, d\mu$ and $L_{X_n}(\gamma) = \int_\gamma \lambda_n \, d\mu$, for every curve γ in X and X_n respectively. We assume also that for each ball $\mathcal{B}$ of X there is a positive constant $c_\mathcal{B}$ with $\lambda \geq c_\mathcal{B}$ in $\mathcal{B}$. Let us assume that for every closed ball $\mathcal{B} \subset X$ there exists N with $\mathcal{B} \subseteq X_n$ for every $n \geq N$ and λ_n converges to λ uniformly in $\mathcal{B}$. If there exists a constant δ_0 such that X_n is δ_0-hyperbolic for every n, then X is δ-hyperbolic, with δ is a constant which just depends on δ_0.*

Proof. Given $\varepsilon > 0$ and a closed ball $\mathcal{B} \subseteq X$, there exists N with $\mathcal{B} \subseteq X_n$ and $|\lambda_n - \lambda| \leq \varepsilon c_\mathcal{B}$ in $\mathcal{B}$ for $n \geq N$. Observe that in $\mathcal{B}$ we have that

$$\left| \frac{\lambda_n}{\lambda} - 1 \right| \leq \frac{\varepsilon c_\mathcal{B}}{\lambda} \leq \varepsilon.$$

Therefore we obtain

(3.1) $$1 - \varepsilon \leq \frac{\lambda_n}{\lambda} \leq 1 + \varepsilon, \qquad \text{and} \qquad 1 - \varepsilon \leq \frac{L_{X_n}(\gamma)}{L_X(\gamma)} \leq 1 + \varepsilon,$$

for $n \geq N$ and for every curve $\gamma \subseteq \mathcal{B}$.

First of all we prove that for each $p \in X$ and $0 < r_1 < r_2 < r_3$, there exists N with $\overline{B_{X_n}(p, r_1)} \subseteq \overline{B_X(p, r_2)} \subseteq \overline{B_{X_n}(p, r_3)}$, for $n \geq N$.

We show now the first content. Let us consider $0 < \varepsilon \leq (r_2 - r_1)/r_2$ and $\mathcal{B} := \overline{B_X(p, r_2)}$. For any $x \in \partial B_X(p, r_2)$ we take any curve η joining p and x. By (3.1) there exists N_1 such that for $n \geq N_1$ we have

$$L_{X_n}(\eta) \geq L_{X_n}(\eta \cap B_X(p, r_2)) > (1 - \varepsilon)L_X(\eta \cap B_X(p, r_2)) \geq (1 - \varepsilon)r_2 \geq r_1.$$

Then $d_{X_n}(p, x) \geq r_1$ for any $x \in \partial B_X(p, r_2)$, and consequently $d_{X_n}(p, \partial B_X(p, r_2)) \geq r_1$. Therefore $\overline{B_{X_n}(p, r_1)} \subseteq \overline{B_X(p, r_2)}$, for $n \geq N_1$.

We show now the second content. As we have just proved, there exists $N_2 \geq N_1$ such that $\overline{B_{X_n}(p, r_3)} \subseteq \overline{B_X(p, 2r_3)}$, for $n \geq N_2$. Let us consider $0 < \varepsilon \leq (r_3 - r_2)/r_2$ and $\mathcal{B} := \overline{B_X(p, 2r_3)}$. For any $x \in \partial B_{X_n}(p, r_3)$ we take any curve η joining p and x. By (3.1) there exists $N \geq N_2$ such that for $n \geq N$ we have

$$L_X(\eta) \geq L_X(\eta \cap B_{X_n}(p, r_3)) \geq \frac{L_{X_n}(\eta \cap B_{X_n}(p, r_3))}{1 + \varepsilon} \geq \frac{r_3}{1 + \varepsilon} \geq r_2.$$

Then $d_X(p, x) \geq r_2$ for any $x \in \partial B_{X_n}(p, r_3)$, and consequently $d_X(p, \partial B_{X_n}(p, r_3)) \geq r_2$. Therefore $\overline{B_X(p, r_2)} \subseteq \overline{B_{X_n}(p, r_3)}$, for $n \geq N$.

Let us consider $p \in X$, $r > 0$ and $x, y \in \overline{B_X(p,r)}$. Then there exists N with $\overline{B_X(p,r)} \subseteq \overline{B_{X_n}(p,2r)}$ and $\overline{B_{X_n}(p,4r)} \subseteq \overline{B_X(p,5r)}$, for $n \geq N$. We consider $\varepsilon > 0$ and $\mathcal{B} := \overline{B_X(p,5r)}$; by (3.1) there exists $N_3 \geq N$ with $\mathcal{B} \subseteq X_n$ and

$$1 - \varepsilon \leq \frac{L_{X_n}(\gamma)}{L_X(\gamma)} \leq 1 + \varepsilon,$$

for $n \geq N_3$ and for every curve $\gamma \subseteq \mathcal{B}$ joining x and y.

If η is a curve joining x and y in X_n and η is not contained $\overline{B_{X_n}(p,4r)}$, then $L_{X_n}(\eta) > 4r$, since $x, y \in \overline{B_{X_n}(p,2r)}$. We also have that $d_{X_n}(x,y) \leq 4r < L_{X_n}(\eta)$. Hence, every geodesic in X_n joining x and y is contained in $\mathcal{B} = \overline{B_X(p,5r)}$. Consequently,

$$1 - \varepsilon \leq \frac{d_{X_n}(x,y)}{d_X(x,y)} \leq 1 + \varepsilon,$$

for $n \geq N_0$ and for every $x, y \in \overline{B_X(p,r)}$. This implies that d_{X_n} converges uniformly on closed balls of X to d_X.

We consider now a geodesic triangle T in X, a point $p \in T$ and $0 < \varepsilon < 1/2$. Let us define $a_\varepsilon := (1+\varepsilon)/(1-\varepsilon)$, $M := H(\delta_0, a_{1/2}, 0)$, where H is the constant in Theorem 2.6, and $r := \mathrm{diam}_X(T)$. Then there exists N_4 such that for any $n \geq N_4$, $x, y \in \overline{B_X(p,r)}$ and $\gamma \subset \overline{B_X(p,r)}$,

$$1 - \varepsilon \leq \frac{L_{X_n}(\gamma)}{L_X(\gamma)} \leq 1 + \varepsilon, \qquad 1 - \varepsilon \leq \frac{d_{X_n}(x,y)}{d_X(x,y)} \leq 1 + \varepsilon,$$

and $\overline{B_X(p,r)} \subseteq \overline{B_{X_n}(p,2r)}$.

Let us consider $g : [a,b] \longrightarrow X$ a side of T. We check now that for any $n \geq N_4$, g is an $(\alpha_\varepsilon, 0)$-quasigeodesic in X_n: if $a \leq c < d \leq b$, we obtain

$$L_{X_n}(g([c,d])) \leq (1+\varepsilon)L_X(g([c,d])) = (1+\varepsilon)d_X(g(c),g(d)) \leq \frac{1+\varepsilon}{1-\varepsilon} d_{X_n}(g(c),g(d)),$$

$$L_{X_n}(g([c,d])) \geq (1-\varepsilon)L_X(g([c,d])) = (1-\varepsilon)d_X(g(c),g(d)) \geq \frac{1-\varepsilon}{1+\varepsilon} d_{X_n}(g(c),g(d)).$$

By Theorem 2.6, if $n \geq N_4$, we have that the Hausdorff distance of g and any geodesic segment $[g(a),g(b)]$ in X_n is less or equal than M. Then, T is $(\delta_0 + 2M)$-thin in X_n for every $n \geq N_4$, and consequently, T is δ-thin in X, with $\delta = (\delta_0 + 2M)/(1-\varepsilon)$. Since $0 < \varepsilon < 1/2$, we have that T is $(\delta_0 + 2M)$-thin in X. $\square$

We want to apply this result to the context of Euclidean domains. In order to do it, we will need some definitions.

Definition 3.2. *Let $\{D_n\}_{n=1}^\infty$ be a sequence of domains in the Riemann sphere such that $z_0 \in D_n$, for some fixed point z_0 in the Riemann sphere. The* kernel *of the sequence $\{D_n\}_{n=1}^\infty$ with respect to z_0, denoted by $\ker_{z_0}\{D_n\}$, is defined as the largest domain D such that: (a) $z_0 \in D$; (b) for each compact subset K of D, $K \subset D_n$ for all n sufficiently large. It is simple to check that the definition makes sense and that $\ker_{z_0}\{D_n\}$ is nonvoid if and only if $\cap_{n=1}^\infty D_n$ contains some neighborhood of z_0. We say that $D_n \longrightarrow D$ if and only if $\ker_{z_0}\{D_{n_j}\} = \ker_{z_0}\{D_n\}$ for every subsequence $\{n_j\}$ of $\{n\}$.*

Definition 3.3. *Let Ω be any domain in the Riemann sphere such that $\infty \in \Omega$ and $\mathrm{card}(\partial\Omega) \geq 3$. The* normalized universal covering map $\pi : \mathbb{D} \longrightarrow \Omega$ *is the unique universal covering map with $\pi(z) \approx c\,z^{-1}$, $c > 0$, as $z \to 0$.*

The following is a result of Hejhal (see [26]); in fact, the result in [26] is better, but this version is good enough for the application that we need.

Theorem 3.4. *([26, Theorem 1]) Let $\{D_n\}_{n=1}^\infty$ be a sequence of domains such that $\infty \in D_n$ and $\mathrm{card}(\partial D_n) \geq 3$. In addition, let π_n be the normalized universal covering map for D_n. If $D = \ker_\infty\{D_n\}$ is nonvoid, $\mathrm{card}(\partial D) \geq 3$ and $D_n \longrightarrow D$, then π_n converge uniformly on compact sets to the normalized covering map for D.*

Theorems 3.1 and 3.4 give the following result for plane domains endowed with their Poincaré metrics.

Theorem 3.5. *Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of domains such that $z_0 \in D_n$, $\mathrm{card}(\partial D_n) \geq 3$ and D_n endowed with its Poincaré metric is δ_0-hyperbolic for every n. If $D = \ker_{z_0}\{D_n\}$ is nonvoid, $\mathrm{card}(\partial D) \geq 3$ and $D_n \longrightarrow D$, then D endowed with its Poincaré metric is δ-hyperbolic, where δ is a constant which just depends on δ_0.*

Proof. Applying a Möbius map if it is necessary (which is an isometry for the Poincaré metric), we can assume that $z_0 = \infty$. The Poincaré density λ_n of D_n only depends on the universal covering map and its derivative. Then, Theorem 3.4 gives that λ_n converges to λ uniformly on closed balls of D. Since λ is a positive continuous function, for each ball of D there is a positive constant c with $\lambda \geq c$. Consequently, the result holds by Theorem 3.1. $\qquad \square$

We need the following standard result.

Proposition 3.6. *Let X, Y be metric spaces with X compact, and let $\{f_n\}_n$ be a sequence of functions $f_n : X \longrightarrow Y$ verifying $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in X$, and $d_Y(f_n(x), f_n(y)) \leq M d_X(x, y)$ for every $x, y \in X$ and every n. Then $\{f_n\}_n$ converges uniformly to f on X.*

Proof. Note that $d_Y(f(x), f(y)) \leq M d_X(x, y)$ for every $x, y \in X$. Assume that $\{f_n\}$ does not converge uniformly to f. Then there exist $\varepsilon > 0$, $\{x_j\} \subset X$ and $\{n_j\}$ with $d_Y(f_{n_j}(x_j), f(x_j)) \geq \varepsilon$ for every j. Since X is compact, without loss of generality we can assume that $\{x_j\}$ converges to $x \in X$. Hence,

$$d_Y(f_{n_j}(x), f(x)) \geq \varepsilon - d_Y(f_{n_j}(x_j), f_{n_j}(x)) - d_Y(f(x_j), f(x))$$
$$\geq \varepsilon - 2M d_X(x_j, x).$$

Therefore, $\liminf_{j \to \infty} d_Y(f_{n_j}(x), f(x)) \geq \varepsilon$, which contradicts $\lim_{n \to \infty} f_n(x) = f(x)$. $\qquad \square$

Now we can obtain a consequence of Theorem 3.1 for the quasihyperbolic metric.

Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of domains in $\mathbb{R}^k$ such that $x_0 \in D_n$ for some fixed point $x_0 \in \mathbb{R}^k$, and $D_n \neq \mathbb{R}^k$. We can define the *kernel* of the sequence $\{D_n\}_{n=1}^{\infty}$ with respect to x_0, denoted by $\ker_{x_0}\{D_n\}$, in a similar way than in the case of the Riemann sphere.

Theorem 3.7. *Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of domains such that $x_0 \in D_n$, $D_n \neq \mathbb{R}^k$ and D_n endowed with its quasihyperbolic metric is δ_0-hyperbolic for every n. If $D = \ker_{x_0}\{D_n\} \neq \varnothing, \mathbb{R}^k$, and $D_n \longrightarrow D$, then D endowed with its quasihyperbolic metric is δ-hyperbolic, where δ is a constant which just depends on δ_0.*

Proof. Note that we just need to show that $\lim_{n \to \infty} \delta_{D_n}(x) = \delta_D(x)$ for every $x \in D$, since if this holds, then Proposition 3.6 gives the uniform convergence on compact subsets of D (recall that $|\delta_{D_n}(x) - \delta_{D_n}(y)| \leq |x - y|$ for every $x, y \in D_n$ and every n), and therefore the result is a direct consequence of Theorem 3.1.

Let us show the pointwise convergence of δ_{D_n}.

For any $x \in D$ and $\varepsilon > 0$ there exists N with $B_{Eucl}\big(x, (1 - \varepsilon)\delta_D(x)\big) \subset D_n$ for every $n \geq N$. Hence, $\delta_{D_n}(x) \geq (1 - \varepsilon)\delta_D(x)$ for every $n \geq N$, and we deduce that $\liminf_{n \to \infty} \delta_{D_n}(x) \geq \delta_D(x)$.

Seeking for a contradiction, let us assume that $\lim_{n \to \infty} \delta_{D_n}(x) \neq \delta_D(x)$ for some $x \in D$. Therefore, there exists a subsequence $\{n_j\}$ of $\{n\}$ with $\limsup_{j \to \infty} \delta_{D_{n_j}}(x) := R > \delta_D(x)$. Hence, there exists J with $\delta_{D_{n_j}}(x) \geq (R + \delta_D(x))/2 > \delta_D(x)$ for every $j \geq J$. Then $B_{Eucl}\big(x, (R + \delta_D(x))/2\big) \subseteq D_{n_j}$ for every $j \geq J$. Since $D = \ker_{x_0}\{D_{n_j}\}$, we deduce that $B_{Eucl}\big(x, (R + \delta_D(x))/2\big) \subseteq D$, which is a contradiction. Therefore $\lim_{n \to \infty} \delta_{D_n}(x) = \delta_D(x)$ for every $x \in D$, and this finishes the proof. $\qquad \square$

After these results on stability of Gromov hyperbolicity under limits, we are interested in similar results on stability of non-hyperbolicity under limits.

First of all, we show with the following example that the limit of non-hyperbolic spaces can be hyperbolic.

Example 3.8. *Let us consider an increasing sequence $\{a_k\}_k \subset (1, \infty)$ with $\lim_{k \to \infty} a_k = \infty$ and $\lim_{k \to \infty} a_{k+1}/a_k = 1$. If we define $D_n := \mathbb{C} \setminus \big(\{0\} \cup \{1\} \cup_{k=n}^{\infty} \{a_k\}\big)$, then D_n endowed either with the Poincaré or the quasihyperbolic metric, are not Gromov hyperbolic by Theorem 2.9. However $D_n \longrightarrow D := \mathbb{C} \setminus \big(\{0\} \cup \{1\}\big)$, and D is hyperbolic with both metrics (see e.g. [22, Proposition 3.5]).*

The limit in this case does not preserve the non-hyperbolicity since "the obstacles for the hyperbolicity escape to infinity". One can think that if the topological obstacles of D_n grow (i.e. $\Pi_1(D_n)$ is a subgroup of $\Pi_1(D_{n+1})$ for every n) and D_n are not hyperbolic for every n, then D will not be hyperbolic.

In fact, we have a stronger result which can be stated in an informal way as follows: in order to get the non-hyperbolicity of D, it is sufficient to require the non-hyperbolicity for just one D_n, for both the quasihyperbolic and the Poincaré metric.

Theorem 3.9. *Let $D \neq \varnothing, \mathbb{R}^k$ and let W be a bounded domain in $\mathbb{R}^k$ with ∂W a hypersurface contained in D. If $D_0 := D \cup (\mathbb{R}^k \setminus W)$ endowed with its quasihyperbolic metric is not hyperbolic, then D endowed with its quasihyperbolic metric is not hyperbolic either.*

Proof. Since ∂W is a compact set that does not intersect ∂D, we have $d_{Eucl}(\partial W, \partial D \setminus \partial D_0) =: c_0 > 0$.

We always have $\delta_D(x) \leq \delta_{D_0}(x)$ for every $x \in D$.

If $x \in D \cap \overline{W}$ and $\delta_D(x) < \delta_{D_0}(x)$, then $\delta_D(x) = d_{Eucl}(x, \partial D \setminus \partial D_0)$ and

$$\delta_{D_0}(x) \leq \operatorname{diam}_{Eucl}(W) \leq \frac{\operatorname{diam}_{Eucl}(W)}{c_0} d_{Eucl}(x, \partial D \setminus \partial D_0)$$
$$= \frac{\operatorname{diam}_{Eucl}(W)}{c_0} \delta_D(x) =: C \delta_D(x).$$

Therefore,

$$\delta_D(x) \leq \delta_{D_0}(x) \leq C \delta_D(x),$$

for every $x \in D \cap \overline{W}$.

Let us define

$$X_1 := D \cap \overline{W}, \qquad X_2 := D \cap (\mathbb{R}^k \setminus W),$$
$$Y_1 := D_0 \cap \overline{W} = X_1, \qquad Y_2 := D_0 \cap (\mathbb{R}^k \setminus W) = \mathbb{R}^k \setminus W,$$
$$c_1 := \max\left\{ \operatorname{diam}_{D|X_1}(\partial W), \operatorname{diam}_{D|X_2}(\partial W) \right\},$$
$$k_1 := \max\left\{ \operatorname{diam}_{D_0|X_1}(\partial W), \operatorname{diam}_{D_0|X_2}(\partial W) \right\}.$$

Note that ∂W is a compact hypersurface contained in D and that every hypersurface is connected; therefore, X_2 and Y_2 are path-connected sets and geodesic metric spaces. It is clear that $\{X_1, X_2\}$ is a (c_1, c_2)-tree decomposition of D for any c_2, and that $\{Y_1, Y_2\}$ is a (k_1, k_2)-tree decomposition of D_0 for any k_2.

Since D_0 is not hyperbolic, Theorem 2.8 gives that $(Y_1, d_{D_0|Y_1})$ is not hyperbolic.

Since $(X_1, d_{D|X_1})$ and $(X_1, d_{D_0|X_1}) = (Y_1, d_{D_0|Y_1})$ are $(C, 0)$-quasi-isometric, we deduce that $(X_1, d_{D|X_1})$ is not hyperbolic by Theorem 2.4. Hence, D is not hyperbolic by Theorem 2.8. $\qquad\square$

We have a similar result for the Poincaré metric. We need a previous result from [2].

Lemma 3.10. ([2, Lemma 3.1]) *Let D_0 be a plane domain, let E be a closed non-empty subset of D_0, $D := D_0 \setminus E$ and ε a positive constant. Then we have that*

$$\lambda_{D_0}(z) \leq \lambda_D(z) \leq \operatorname{cotanh}(\varepsilon/2)\lambda_{D_0}(z),$$

for every $z \in D_0$ with $d_{D_0}(z, E) \geq \varepsilon$.

Theorem 3.11. *Let $D \subset \mathbb{C}$ and let W be a bounded Jordan domain in $\mathbb{C}$ with $\partial W \subset D$. If $D_0 := D \cup (\mathbb{C} \setminus W)$ endowed with its Poincaré metric is not hyperbolic, then D endowed with its Poincaré metric is not hyperbolic either.*

Proof. Note that $D = D_0 \setminus E$, with $E := \mathbb{C} \setminus (D \cup W)$. Let us define $\varepsilon := \min\{d_{D_0}(z, E) : z \in D_0 \cap \overline{W}\}$. Then Lemma 3.10 gives

$$\lambda_{D_0}(z) \leq \lambda_D(z) \leq \operatorname{cotanh}(\varepsilon/2)\lambda_{D_0}(z),$$

for every $z \in D_0 \cap \overline{W}$. Now, a similar argument to the one in the proof of Theorem 3.9, using Theorem 2.8, finishes the proof. $\qquad\square$

4. Background in Denjoy domains.

From now on, we will make extensive use of a specially interesting kind of Riemann surfaces, endowed with their Poincaré metrics: the *Denjoy domains*, that is to say, plane domains Ω with $\partial\Omega \subset \mathbb{R}$. This kind of surfaces are becoming more and more important in Geometric Theory of Functions, since, on the one hand, they are a very general type of Riemann surfaces, and, on the other hand, they are more manageable due to its symmetry. For instance, Garnett and Jones have proved in [15] the Corona Theorem for Denjoy domains, and in [2] and [39] the authors have got characterizations of Denjoy domains which satisfy a linear isoperimetric inequality. See also [1], [3] and [17].

We will consider a particular type of Denjoy domain, which we will call *train*. A train can be defined as the complement of a sequence of ordered closed intervals (see Definition 4.1). Trains do include a especially important case of surfaces which are the flute surfaces (see, e.g. [6], [7]). These ones are the simplest examples of infinite ends, and besides, in a flute surface it is possible to give a fairly precise description of the ending geometry (see, e.g. [20]). In [3] there are some partial results on hyperbolicity of trains.

We need some definitions and background. So far we have used the word *geodesic* in the sense of Definition 2.2, that is to say, as a global geodesic or a minimizing geodesic; however, we need now to deal with a special type of local geodesics: simple closed geodesics, which obviously can not be minimizing geodesics. We will continue using the word geodesic with the meaning of Definition 2.2, unless we are dealing with closed geodesics.

Definition 4.1. *A* train *is a Denjoy domain* $\Omega \subset \mathbb{C}$ *with* $\Omega \cap \mathbb{R} = \cup_{n=0}^{\infty}(a_n, b_n)$, *such that* $-\infty \leq a_0$ *and* $b_n \leq a_{n+1}$ *for every* n. *A* flute surface *is a train with* $b_n = a_{n+1}$ *for every* n.

For each $n > 0$, *we denote by* γ_n *the simple closed geodesic which just intersects* $\mathbb{R}$ *in* (a_0, b_0) *and* (a_n, b_n), $2l_n := L_\Omega(\gamma_n)$.

For each $n > 0$, *we denote by* σ_n *the simple closed geodesic which just intersects* $\mathbb{R}$ *in* (a_n, b_n) *and* (a_{n+1}, b_{n+1}), *and* $2r_n := L_\Omega(\sigma_n)$ *(see figure below). If* $b_n = a_{n+1}$, *we define* σ_n *as the puncture at this point and* $r_n = 0$.
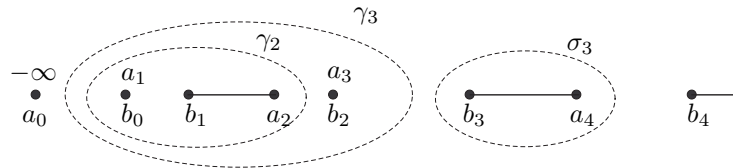
(a) Train seen as a subset of the complex plane.
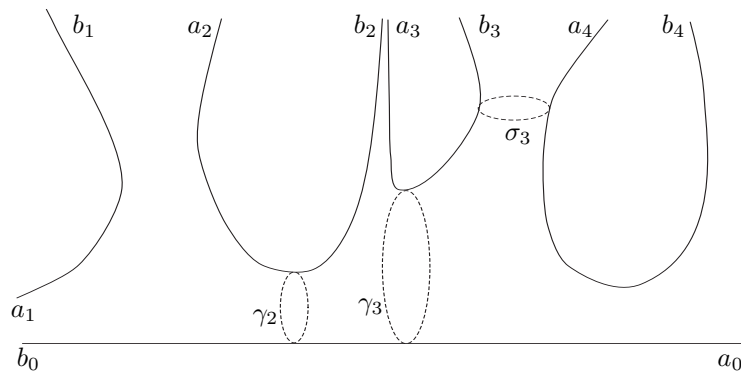
(b) The same train seen with "Euclidean eyes".

Remark. Recall that in every free homotopy class there exists a single simple closed geodesic, assuming that punctures are simple closed geodesics with length equal to zero. That is why both γ_n and σ_n are unique for every n.

A train is a flute surface if and only if every σ_n is a puncture, i.e., if $a_{n+1} = b_n$ for every $n \geq 0$.

It is not difficult to see that the values of $\{l_n\}$ and $\{r_n\}$ determine a train. Then, there must exist a characterization of hyperbolicity in terms of the lengths of these sequences (see [35, Theorem 3.2]). This theorem has several interesting consequences.

Proposition 4.2. ([35, Proposition 3.6]) *Let us consider a train Ω with $l_n \leq c$ for every n. Then Ω is δ-hyperbolic, where δ is a constant which only depends on c.*

The following result shows that hyperbolicity is stable under bounded perturbations of the lengths of $\{\gamma_n\}_n$ and $\{\sigma_n\}_n$.

Theorem 4.3. ([35, Theorem 3.8]) *Let us consider two trains Ω, Ω' and a constant c such that $|r_n' - r_n| \leq c$, and $|l_n' - l_n| \leq c$ for every $n \geq 1$. Then Ω is hyperbolic if and only if Ω' is hyperbolic.*
Furthermore, if Ω is δ-hyperbolic, then Ω' is δ'-hyperbolic, with δ' a constant which only depends on δ and c.

Theorem 4.5 below is a simpler version of [35, Theorem 3.2]. Next, we are going to define some functions that will appear in the statement of Theorem 4.5.

Definition 4.4. *Let us consider a sequence of positive numbers $\{l_n\}_{n=1}^{\infty}$ and a sequence of non-negative numbers $\{r_n\}_{n=1}^{\infty}$. Consider $n \geq 1$ and $0 \leq h \leq l_n$. We define*

$$
\Gamma_{nm}^0(h) := \begin{cases}
e^h \sum\limits_{k=m+1}^{n} e^{-l_k}\,, & \text{if } m < n \text{ and } l_m \leq h\,, \\[2ex]
l_m - h + e^h \sum\limits_{k=m}^{n} e^{-l_k}\,, & \text{if } m < n \text{ and } l_m > h\,, \\[2ex]
\min\left\{h,\, l_n - h\right\}\,, & \text{if } m = n\,, \\[2ex]
l_m - h + e^h \sum\limits_{k=n}^{m} e^{-l_k}\,, & \text{if } m > n \text{ and } l_m > h\,, \\[2ex]
e^h \sum\limits_{k=n}^{m-1} e^{-l_k}\,, & \text{if } m > n \text{ and } l_m \leq h\,,
\end{cases}
$$

The functions $\Gamma_{nm}^0(h)$ are naturally associated to trains by taking $\{l_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ as the half-lengths of $\{\gamma_n\}_{n=1}^{\infty}$ and $\{\sigma_n\}_{n=1}^{\infty}$.

Theorem 4.5. ([35, Theorem 3.12]) *Let us consider a train Ω such that there exists a constant $c > 0$ with $r_n \leq c$ for every $n \geq 1$. Then Ω is hyperbolic if and only if*

$$
K^0 := \sup_{n \geq 1} \sup_{h \in [0, l_n]} \min_{m \in [A_n(h), B_n(h)]} \Gamma_{nm}^0(h) < \infty\,.
$$

Furthermore, if Ω is δ-hyperbolic, then K^0 is bounded by a constant which only depends on δ and c; if $K^0 < \infty$, then Ω is δ-hyperbolic, with δ a constant which only depends on K^0 and c.

We also have the following facts.

Proposition 4.6. ([35, Proposition 3.13]) *In any train Ω we have*

$$
\min_{m \in [A_n(h), B_n(h)]} \Gamma_{nm}^0(h) = \min_{m \geq 1} \Gamma_{nm}^0(h)\,,
$$

for every $n \geq 1$ and $0 \leq h \leq l_n$.

Theorem 4.7. ([35, Corollary 3.14]) *Let us consider a train Ω with $l_1 \leq l^0$, $r_n \leq c_1$ for every n and*

$$(4.2) \qquad \sum_{k=n}^{\infty} e^{-l_k} \leq c_2\, e^{-l_n}, \qquad \text{for every } n > 1\,.$$

Then Ω is δ-hyperbolic, where δ is a constant which only depends on c_1, c_2 and l^0.

5. Transformations which preserve hyperbolicity.

In the current paper our main aim is to study the stability of Gromov hyperbolicity. Keeping that in mind, from now on we will adopt a different viewpoint: we will study under what kind of geometric perturbations Gromov hyperbolicity is preserved.

To be more precise, we want to study now the following problem: if we have an hyperbolic train with $\{r_n\} \in l^\infty$, what kind of perturbations are allowed on $\{l_n\}$ so that the train is still hyperbolic? Theorem 5.3 below answers this question, and furthermore provides methods to construct a great deal of hyperbolic flute surfaces.

We need the following definitions.

Definition 5.1. *We denote by H the following set of sequences:*
$$H := \big\{ \{x_n\} : \quad \text{the train with } l_n = x_n \text{ and } r_n = 0 \text{ for every } n \text{ is hyperbolic} \big\}$$
$$= \big\{ \{x_n\} : \quad \text{every train with } l_n = x_n \text{ for every } n \text{ and } \{r_n\} \in l^\infty \text{ is hyperbolic} \big\}\,.$$

The second equality is a direct consequence of Theorem 4.3.

Definition 5.2. *We say that the sequence $\{y_n\}$ is a union of the sequences $\{x_n^1\}, \ldots, \{x_n^N\}$, if $\{x_n^1\}, \ldots, \{x_n^N\}$ are subsequences of $\{y_n\}$, and $\{x_n^1\}, \ldots, \{x_n^N\}$ is a partition of $\{y_n\}$.*

Theorem 5.3. *Let us consider a sequence $\{l_n\} \in H$.*
  (1) *If $l'_n = l_n + x_n$ with $\{x_n\} \in l^\infty$, then $\{l'_n\} \in H$.*
  (2) *Fix a positive integer N. Let us assume that $\{l_n\}$ is a subsequence $\{l'_{n_k}\}$ of $\{l'_n\}$ such that $n_{k+1} - n_k \leq N$ for every k, and $\max\{l'_{n_k}, l'_{n_{k+1}}\} \leq l'_m + N$ for every $m \in (n_k, n_{k+1})$ and every k. Then $\{l'_n\} \in H$.*
  (3) *If $\{l'_n\}$ is any union of the sequences $\{l_n^1\}, \ldots, \{l_n^N\} \in H$, then $\{l'_n\} \in H$.*
  (4) *If $\{l'_n\}$ is a union of $\{l_n\}$ and a sequence $\{x_n\} \in l^\infty$, then $\{l'_n\} \in H$.*
  (5) *Let us assume that $\{l'_n\}$ is any union of the sequences $\{l_n^1\}, \ldots, \{l_n^N\}$ which verify*

$$\sum_{k=n}^{\infty} e^{-l_k^j} \leq c\, e^{-l_n^j}, \qquad \text{for every } n > 1 \text{ and } j = 1, \ldots, N\,.$$

  Then $\{l'_n\} \in H$.
  (6) *Fix a positive integer N. Let us assume that $\{x_n\}$ is a subsequence $\{l'_{n_k}\}$ of $\{l'_n\}$ such that $\max\{l'_{n_k}, l'_{n_{k+1}}\} \leq l'_m + N$ for every $m \in (n_k, n_{k+1})$ and every k. If $\{x_n\} \notin H$, then $\{l'_n\} \notin H$.*
  (7) *Fix a positive integer N. Let σ be a permutation of the positive integer numbers such that $|\sigma(n) - n| \leq N$ for every n, and consider $l'_n := l_{\sigma(n)}$. Then $\{l'_n\} \in H$.*

Remarks.
   (1) In fact, (7) gives the following stronger statement: If σ is a permutation of the positive integer numbers such that $|\sigma(n) - n| \leq N$ for every n, then $\{l_{\sigma(n)}\} \in H$ if and only if $\{l_n\} \in H$ (since σ^{-1} also satisfies $|\sigma^{-1}(n) - n| \leq N$ for every n).
   (2) We have examples showing that the conclusions of Theorem 5.3 do not hold if we remove any of the hypothesis.

Proof. (1) is a direct consequence of Theorem 4.3.
   (2) Fix $n \geq 1$ and $h \in [0, l'_n]$.
   Let us consider the maximum integer k_0 such that $n_{k_0} \leq n < n_{k_0+1}$.

If $l'_s \leq h$ for some $s \in [n_{k_0}, n_{k_0+1}]$, by symmetry, without loss of generality we can assume that there exists some $s \in [n_{k_0}, n)$ with $l'_s \leq h$ (the case $s = n$ is trivial: if $l'_n \leq h$, then $h = l'_n$ and $\left(\Gamma^0_{nn}\right)'(h) = 0$). Hence $A'_n(h) \in [n_{k_0}, n)$ and then $l'_k \geq h$ for every $k \in (A'_n(h), n]$ and $n - A'_n(h) \leq n - n_{k_0} \leq N - 1$; consequently,

$$\left(\Gamma^0_{nA'_n(h)}\right)'(h) = \sum_{k=A'_n(h)+1}^{n} e^{h-l'_k} \leq \sum_{k=A'_n(h)+1}^{n} 1 = n - A'_n(h) \leq N - 1\,.$$

Let us assume now that $l'_s > h$ for every $s \in [n_{k_0}, n_{k_0+1}]$. There exists some integer m with $\Gamma^0_{k_0 m}(h) \leq K^0$. By symmetry, without loss of generality we can assume that $m \leq k_0$.

If $m = k_0$, then $\min\{h, l_{k_0} - h\} \leq K^0$. If $\min\{h, l_{k_0} - h\} = h$, then $h \leq K^0$ and we can deduce

$$\left(\Gamma^0_{nn}\right)'(h) = \min\{h, l'_n - h\} \leq h \leq K^0.$$

If $\min\{h, l_{k_0} - h\} = l_{k_0} - h$, then $l_{k_0} - h \leq K^0$ and

$$\left(\Gamma^0_{nn_{k_0}}\right)'(h) = l'_{n_{k_0}} - h + \sum_{k=n_{k_0}}^{n} e^{h-l'_k} \leq l_{k_0} - h + \sum_{k=n_{k_0}}^{n} 1 \leq K^0 + N.$$

If $m < k_0$ and $l_m > h$, then $\Gamma^0_{k_0 m}(h) = l_m - h + e^h \sum_{k=m}^{k_0} e^{-l_k} \leq K^0$. Hence

$$\left(\Gamma^0_{nn_m}\right)'(h) = l'_{n_m} - h + e^h \sum_{k=n_m}^{n_{k_0}} e^{-l'_k} + \sum_{k=n_{k_0}+1}^{n} e^{h-l'_k}$$

$$\leq l'_{n_m} - h + e^h\left(e^{-l'_{n_m}} + \sum_{j=m+1}^{k_0} \sum_{k=n_{j-1}+1}^{n_j} e^{-l'_k}\right) + \sum_{k=n_{k_0}+1}^{n} 1$$

$$\leq l'_{n_m} - h + e^h\left(e^{-l'_{n_m}} + \sum_{j=m+1}^{k_0} N\,e^{N-l'_{n_j}}\right) + N - 1$$

$$\leq N\,e^N\left(l_m - h + e^h \sum_{j=m}^{k_0} e^{-l_j}\right) + N - 1 \leq N\,e^N K^0 + N - 1\,.$$

If $m < k_0$ and $l_m \leq h$, a similar argument gives the same bound for $\left(\Gamma^0_{nn_m}\right)'(h)$.

Then, $\left(K^0\right)' \leq N\,e^N K^0 + N$ and Theorem 4.5 implies (2).

(3) Assume first that $N = 2$; then $\{l'_n\}$ is the union of $\{l^1_n\}$ and $\{l^2_n\}$. We denote by $\{l'_{n^i_k}\}$ the subsequence $\{l^i_n\}$ in $\{l'_n\}$, for $i = 1, 2$. Fix $n \geq 1$ and $h \in [0, l'_n]$. By symmetry, without loss of generality we can assume that there exist k_1 with $n^1_{k_1} = n$ and $m_1 \leq k_1$ with $\left(\Gamma^0_{k_1 m_1}\right)^1(h) \leq (K^0)^1$.

We can assume that $l'_s > h$ for every $s \in (n^1_{m_1}, n^1_{k_1})$, since the other case is similar.

If there is no k with $n^2_k \in [n^1_{m_1}, n^1_{k_1}]$, then $\left(\Gamma^0_{n^1_{k_1} n^1_{m_1}}\right)'(h) = \left(\Gamma^0_{k_1 m_1}\right)^1(h) \leq (K^0)^1$.

Assume now that there exists k with $n^2_k \in (n^1_{m_1}, n^1_{k_1})$. Let us define $k_2 := \max\{k : n^2_k \in (n^1_{m_1}, n^1_{k_1})\}$. If there exists $m_2 \leq k_2$ such that $\left(\Gamma^0_{k_2 m_2}\right)^2(h) \leq (K^0)^2$, then

$$\left(\Gamma^0_{n^1_{k_1}, \max\{n^1_{m_1}, n^2_{m_2}\}}\right)'(h) \leq \left(\Gamma^0_{k_1 m_1}\right)^1(h) + \left(\Gamma^0_{k_2 m_2}\right)^2(h) \leq (K^0)^1 + (K^0)^2.$$

If there exists k_3 verifying the next three conditions simultaneously:
(a) $n^2_{k_3} \in (n^1_{m_1}, n^1_{k_1})$,
(b) there exists $m_3 \leq k_3$ such that $\left(\Gamma^0_{k_3 m_3}\right)^2(h) \leq (K^0)^2$,
(c) for every $k \in (k_3, k_2]$ we have $\left(\Gamma^0_{km}\right)^2(h) > (K^0)^2$ for every $m \leq k$,
then there exists $m_0 > k_2$ such that $\left(\Gamma^0_{k_3+1, m_0}\right)^2(h) \leq (K^0)^2$: In fact, seeking for a contradiction, let us assume that there exists $m_0 \in (k_3+1, k_2]$ with $\left(\Gamma^0_{k_3+1, m_0}\right)^2(h) \leq (K^0)^2$; then $\left(\Gamma^0_{m_0 m_0}\right)^2(h) \leq \left(\Gamma^0_{k_3+1, m_0}\right)^2(h) \leq$

$(K^0)^2$ (recall that $l'_s > h$ for every $s \in (n^1_{m_1}, n^1_{k_1})$), which is actually a contradiction with (c). Hence,

$$\left(\Gamma^0_{n^1_{k_1}, \max\{n^1_{m_1}, n^2_{m_3}\}}\right)'(h) \le \left(\Gamma^0_{k_1 m_1}\right)^1(h) + \left(\Gamma^0_{k_3 m_3}\right)^2(h) + \left(\Gamma^0_{k_3+1, m_0}\right)^2(h) \le (K^0)^1 + 2(K^0)^2.$$

If for any k with $n^2_k \in (n^1_{m_1} n^1_{k_1})$ we have $\left(\Gamma^0_{km}\right)^2(h) > (K^0)^2$ for every $m \le k$, let us define $k_4 := \min\{k : n^2_k \in (n^1_{m_1}, n^1_{k_1})\}$. As in the last case, then there exists $m_4 > k_2$ such that $\left(\Gamma^0_{k_4 m_4}\right)^2(h) \le (K^0)^2$, and hence

$$\left(\Gamma^0_{n^1_{k_1} n^1_{m_1}}\right)'(h) \le \left(\Gamma^0_{k_1 m_1}\right)^1(h) + \left(\Gamma^0_{k_4 m_4}\right)^2(h) \le (K^0)^1 + (K^0)^2.$$

Consequently, $\left(K^0\right)' \le 2(K^0)^1 + 2(K^0)^2$ and Theorem 4.5 implies (3) with $N = 2$. The result for N sequences is obtained by applying $N - 1$ times this result for 2 sequences.

(4) is a direct consequence of (3) and Proposition 4.2.

(5) is a direct consequence of (3) and Theorem 4.7.

(6) Since $\{x_n\} \notin H$, by Theorem 4.5 and Proposition 4.6, for each $M > N$ there exist k_0 and $h \in (0, x_{k_0})$ with $\Gamma^0_{k_0 m}(h) \ge M$, for every $m \ge 1$.

Consider $m \ge 1$. By symmetry, without loss of generality we can assume that $m \le n_{k_0}$. If $m = n_{k_0}$, then

$$\left(\Gamma^0_{n_{k_0} n_{k_0}}\right)'(h) = \min\left\{h, l'_{n_{k_0}} - h\right\} = \min\left\{h, x_{k_0} - h\right\} = \Gamma^0_{k_0 k_0}(h) \ge M.$$

Notice that if $m \in (n_{k_0-1}, n_{k_0})$, then

$$l'_m - h \ge l'_{n_{k_0}} - h - N = x_{k_0} - h - N \ge \Gamma^0_{k_0 k_0}(h) - N \ge M - N > 0,$$

and $l'_m > h$. Hence $\left(\Gamma^0_{n_{k_0} m}\right)'(h) \ge l'_m - h \ge M - N$.

In the case $m \le n_{k_0-1}$, we have $n_{k_1-1} < m \le n_{k_1}$ for some $k_1 < k_0$.

If $x_{k_1} \le h$, then

$$\left(\Gamma^0_{n_{k_0} m}\right)'(h) \ge e^h \sum_{k=m+1}^{n_{k_0}} e^{-l'_k} \ge e^h \sum_{k=k_1+1}^{k_0} e^{-x_k} = \Gamma^0_{k_0 k_1}(h) \ge M.$$

If $x_{k_1} > h$ and $l'_m > h$, then

$$\left(\Gamma^0_{n_{k_0} m}\right)'(h) = l'_m - h + e^h \sum_{k=m}^{n_{k_0}} e^{-l'_k} \ge l'_{n_{k_1}} - h - N + e^h \sum_{k=m}^{n_{k_0}} e^{-l'_k}$$

$$\ge x_{k_1} - h - N + e^h \sum_{k=k_1}^{k_0} e^{-x_k} = \Gamma^0_{k_0 k_1}(h) - N \ge M - N.$$

If $x_{k_1} > h$ and $l'_m \le h$, then $x_{k_1} - N = l'_{n_{k_1}} - N \le l'_m \le h$ and $0 \ge x_{k_1} - h - N$; therefore

$$\left(\Gamma^0_{n_{k_0} m}\right)'(h) = e^h \sum_{k=m+1}^{n_{k_0}} e^{-l'_k} \ge x_{k_1} - h - N + e^h e^{-x_{k_1}} - 1 + e^h \sum_{k=k_1+1}^{k_0} e^{-x_k} = \Gamma^0_{k_0 k_1}(h) - N - 1 \ge M - N - 1.$$

Consequently, $\left(K^0\right)' \ge M - N - 1$ for every $M > N$, and hence $\left(K^0\right)' = \infty$. Then $\{l'_n\} \notin H$ by Theorem 4.5.

(7) First, we want to remark the following elementary fact: If $i < j$ and $\sigma(i) > \sigma(j)$, then $|i - j| < 2N$: $|i - j| = j - i < j - \sigma(j) + \sigma(i) - i \le 2N$.

Fix $n \ge 1$ and $h \in [0, l'_n]$. There exists $\sigma(m)$ with $\Gamma^0_{\sigma(n)\sigma(m)}(h) \le K^0$. By symmetry, without loss of generality we can assume that $\sigma(m) \le \sigma(n)$.

If $m = n$, then $\sigma(m) = \sigma(n)$ and $\left(\Gamma^0_{nn}\right)'(h) = \Gamma^0_{\sigma(n)\sigma(n)}(h) \le K^0$.

We consider now the case $\sigma(m) < \sigma(n)$.

If $m > n$, then $m - n < 2N$.

If $B'_n(h) > m$, then $l'_k > h$ for every $k \in (n, m]$ and

$$\left(\Gamma^0_{nm}\right)'(h) = l'_m - h + \sum_{k=n}^{m} e^{h-l'_k} \leq l_{\sigma(m)} - h + 2N \leq \Gamma^0_{\sigma(n)\sigma(m)}(h) + 2N \leq K^0 + 2N.$$

If $B'_n(h) \leq m$, then $l'_k > h$ for every $k \in (n, B'_n(h))$ and

$$\left(\Gamma^0_{nB'_n(h)}\right)'(h) = \sum_{k=n}^{B'_n(h)-1} e^{h-l'_k} \leq 2N.$$

We deal now with the case $m < n$. Notice first that $\sigma([m,n]) \subset [m-N, n+N]$ and $[m+N, n-N] \subset [\sigma(m), \sigma(n)]$; then, in $\sigma([m,n]) \setminus [\sigma(m), \sigma(n)]$ there are at most $4N$ integers.

If $A'_n(h) \geq m$, then $l'_k > h$ for every $k \in (A'_n(h), n)$, and

$$\left(\Gamma^0_{nA'_n(h)}\right)'(h) = e^h \sum_{k=A'_n(h)+1}^{n} e^{-l'_k} \leq e^h \sum_{\substack{k \in [m,n] \\ l_{\sigma(k)} \geq h}} e^{-l_{\sigma(k)}} = e^h \sum_{\substack{j \in \sigma([m,n]) \\ l_j \geq h}} e^{-l_j} \leq \sum_{\substack{j \in \sigma([m,n]) \setminus [\sigma(m),\sigma(n)] \\ l_j \geq h}} e^{h-l_j} + e^h \sum_{\substack{j=\sigma(m) \\ l_j \geq h}}^{\sigma(n)} e^{-l_j}$$

$$\leq 4N + 1 + e^h \sum_{j=\sigma(m)+1}^{\sigma(n)} e^{-l_j} \leq 4N + 1 + \Gamma^0_{\sigma(n)\sigma(m)}(h) \leq 4N + 1 + K^0.$$

If $A'_n(h) < m$, then $l'_k > h$ for every $k \in [m, n)$, and

$$\left(\Gamma^0_{nm}\right)'(h) = l'_m - h + e^h \sum_{k=m}^{n} e^{-l'_k} = l_{\sigma(m)} - h + e^h \sum_{k \in [m,n]} e^{-l_{\sigma(k)}} = l_{\sigma(m)} - h + e^h \sum_{j \in \sigma([m,n])} e^{-l_j}$$

$$\leq \sum_{j \in \sigma([m,n]) \setminus [\sigma(m),\sigma(n)]} e^{h-l_j} + l_{\sigma(m)} - h + e^h \sum_{j=\sigma(m)}^{\sigma(n)} e^{-l_j} \leq 4N + \Gamma^0_{\sigma(n)\sigma(m)}(h) \leq 4N + K^0.$$

Hence, $\left(K^0\right)' \leq 4N + 1 + K^0$, and Theorem 4.5 gives (7). $\qquad\square$

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