



The multicomponent 2D Toda hierarchy: generalized matrix orthogonal polynomials, multiple orthogonal polynomials and Riemann–Hilbert problems

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Abstract

We consider the relation of the multi-component 2D Toda hierarchy with matrix orthogonal and biorthogonal polynomials. The multi-graded Hänkel reduction of this hierarchy is considered and the corresponding generalized matrix orthogonal polynomials are studied. In particular for these polynomials we consider the recursion relations, and for rank one weights its relation with multiple orthogonal polynomials of mixed type with a type II normalization and the corresponding link with a Riemann–Hilbert problem.

1 Introduction

Recently, multiple orthogonal polynomials, the related Riemann–Hilbert problems and its application to different areas, for example Brownian motions have deserved much attention [1]. The field of matrix orthogonal polynomials has been also a growing area of research with some similarities with the scalar case but much more richness, see [2]. The relation of multiple orthogonal polynomials with multicomponent KP hierarchies has been noticed in [3] and the string equation formalism of integrable systems has been applied in [4]. Ten years ago Adler and van Moerbeke [5] –in the context of the so called discrete KP hierarchy– introduced what they named as generalized orthogonal polynomials, and what they claimed to be the corresponding Riemann–Hilbert problem, while later on they studied the related Darboux transformations [6]. Recently, and for the Toeplitz case Cafasso [7] extended this work in order to consider block matrices within the non-Abelian Ablowitz–Ladick lattice.

Following Ueno and Takasaki [8] and the seminal paper of Mulase [9] in [10] we gave a description of the infinite multicomponent 2D Toda lattice hierarchy in terms of a Gaussian factorization (also known as Borel factorization) in an infinite dimensional Lie group, while later on [11] we analyzed the dispersionless limit of the hierarchy with the aid of the factorization problem. (See [12] for a discussion on different cases where this factorization makes sense.) Following [5] we could argue as follows: i) in the one hand the multicomponent Toda hierarchy

may be viewed as an LU factorization of certain deformed infinite-dimensional matrix and ii) on the other hand, the same matrix could be thought as a moment matrix and the corresponding LU factorization should give us the corresponding generalized matrix orthogonal polynomials. In this manner, we would be able to build a bridge between multicomponent Toda hierarchy and matrix orthogonal polynomials. This is the main idea developed in this paper. First, we connect matrix orthogonal and biorthogonal polynomials with the multi-component 2DToda lattice hierarchy, focusing in particular on the Hankel reduction. Second, we generalize the band condition of [5] to the multicomponent case and consider what we refer as multi-graded Hankel. This leads to a multi-component extension of generalized orthogonal polynomials which in some cases can be described in terms of multiple orthogonal polynomials of mixed type with a type II normalization. This connection allows us to give an appropriate Riemann–Hilbert problem for this generalized orthogonal polynomials (notice that the one discussed in [5] is not correct).

The layout of the paper is as follows. In this introduction we give a overview of the Gaussian factorization and the semi-infinite multi-component 2D Toda lattice hierarchy. Then, in §2 we discuss how matrix orthogonal polynomials and the multi-component 2D Toda lattice hierarchy are connected. Finally, in §3 we consider multi-graded Hankel reduction, extended generalized orthogonal polynomials, mixed multiple orthogonal polynomials and corresponding Riemann–Hilbert problems.

1.1 Gaussian factorization and the semi-infinite multi-component 2D Toda lattice hierarchy

For the construction of a Lie group theoretical setting we denote by Λ the shift operator for matrix valued sequences. The associative algebra of linear operators on these sequences can be identified with the associative algebra of semi-infinite matrices with entries taking values in $\mathbb{C}^{N \times N}$, the set of $N \times N$ complex matrices. With the usual commutator for linear operators this algebra is also a Lie algebra denoted by \mathfrak{g} whose Lie group G is the group of invertible linear operators in \mathfrak{g} . Let's take $g \in G$ and consider the following Gaussian factorization problem $g = S^{-1}\bar{S}$ where S is a block lower triangular matrix, with $S_{ii} = \mathbb{I}_N$, being $\mathbb{I}_N \in \mathbb{C}^{N \times N}$ the identity matrix, and \bar{S} is an block upper triangular matrix. In [12] it is proven that the Borel decomposition holds if all the principal minors do not vanish. Thus, the factorization holds under “small” continuous deformations and we can consider the factorization $g(t) = S(t)^{-1}\bar{S}(t)$ where t stands for a set of complex variables. As was discussed in [10] this factorization problem leads to an integrable hierarchy of nonlinear PDE known as multicomponent 2D Toda lattice hierarchy. Let us discuss these issues in more depth. Observe that the matrix associated with the shift operator is the block matrix $(\mathbb{I}_N \delta_{i,i+1})$ and Λ^t is the operator associated with the transposed matrix $(\mathbb{I}_N \delta_{i+1,i})$. If E_{ab} , $a, b = 1, \dots, N$ is the canonical basis of $\mathbb{C}^{N \times N}$ and $t = (\{t_{ja}\}, \{\bar{t}_{ja}\})$, $j = 1, 2, \dots$ and $a = 1, \dots, N$, is a collection of complex parameters we introduce

$$W_0 := \sum_{a=1}^N E_{aa} \exp\left(\sum_{j=1}^{\infty} t_{ja} \Lambda^j\right),$$

$$\bar{W}_0 := \sum_{a=1}^N E_{aa} \exp\left(\sum_{j=1}^{\infty} \bar{t}_{ja} (\Lambda^t)^j\right),$$

and consider Gaussian factorization of $g(t) := W_0(t)g\bar{W}_0(t)^{-1}$.

Following [10] we define the Lax operators

$$\begin{aligned}
L &:= SAS^{-1} = \Lambda + u_0 + u_1\Lambda^t + u_2(\Lambda^t)^2 + \dots, \\
C_a &:= SE_{aa}S^{-1} = E_{aa} + c_{a1}\Lambda^t + \dots, \\
\bar{L} &:= \bar{S}\Lambda^t\bar{S}^{-1} = e^\phi\Lambda^t + \bar{u}_0 + \bar{u}_1\Lambda + \dots, \\
\bar{C}_a &:= \bar{S}E_{aa}\bar{S}^{-1} = \bar{c}_{a0} + \bar{c}_{a1}\Lambda + \dots,
\end{aligned} \tag{1}$$

where all the coefficients in the Λ -expansions belong to $\mathbb{C}^{N \times N}$. The multi-component 2D Toda hierarchy has the following Lax representation

$$\begin{aligned}
\frac{\partial L}{\partial t_{ja}} &= [(L^j C_a)_+, L], & \frac{\partial C_b}{\partial t_{ja}} &= [(L^j C_a)_+, C_b], & \frac{\partial \bar{L}}{\partial t_{ja}} &= [(\bar{L}^j \bar{C}_a)_+, \bar{L}], & \frac{\partial \bar{C}_b}{\partial t_{ja}} &= [(\bar{L}^j \bar{C}_a)_+, \bar{C}_b], \\
\frac{\partial L}{\partial \bar{t}_{ja}} &= [(\bar{L}^j \bar{C}_a)_-, L], & \frac{\partial C_b}{\partial \bar{t}_{ja}} &= [(\bar{L}^j \bar{C}_a)_-, C_b], & \frac{\partial \bar{L}}{\partial \bar{t}_{ja}} &= [(\bar{L}^j \bar{C}_a)_-, \bar{L}], & \frac{\partial \bar{C}_b}{\partial \bar{t}_{ja}} &= [(\bar{L}^j \bar{C}_a)_-, \bar{C}_b],
\end{aligned}$$

where the sub-indices $+$ and $-$ denote the block upper triangular, strictly block lower triangular projections, respectively.

2 Matrix orthogonal polynomials and the multi-component 2DToda lattice hierarchy

Following Adler and van Moerbeke [5] we construct families of matrix orthogonal and bi-orthogonal polynomials associated with the 2D Toda lattice hierarchy.

In the first place, we define the following families of (time-dependent) matrix polynomials

$$p(z) \equiv \{p_i(z)\}_{i \geq 0} := S\chi(z), \quad \bar{p}(z) \equiv \{\bar{p}_i(z)\}_{i \geq 0} := (\bar{S}^{-1})^\dagger \chi(z),$$

where $\chi(z) := (\mathbb{I}_N, z\mathbb{I}_N, z^2\mathbb{I}_N, \dots)^t$ and the symbol \dagger denotes Hermitian conjugation. Next we consider a matrix-valued bilinear pairing between matrix polynomials. Given matrix polynomials $P(z) = \sum_{k=0}^i P_k z^k$ and $Q(z) = \sum_{l=0}^j Q_l z^l$ (of degrees i, j , respectively) we have

$$\langle P(z), Q(z) \rangle = \sum_{\substack{k=1, \dots, i \\ l=1, \dots, j}} P_k \langle z^k \mathbb{I}_N, z^l \mathbb{I}_N \rangle Q_l^\dagger,$$

where $\langle z^k \mathbb{I}_N, z^l \mathbb{I}_N \rangle$ denotes the matrix for the bilinear pairing in the canonical basis and for each (k, l) is an $N \times N$ complex matrix.

This pairing has the following properties

1. Is linear in the first component:

$$\langle c_1 P_1(z) + c_2 P_2(z), Q(z) \rangle = c_1 \langle P_1(z), Q(z) \rangle + c_2 \langle P_2(z), Q(z) \rangle, \quad \forall c_1, c_2 \in \mathbb{C}^{N \times N}$$

2. Is skew-linear in the second component:

$$\langle P(z), c_1 Q_1(z) + c_2 Q_2(z) \rangle = \langle P(z), Q_1(z) \rangle c_1^\dagger + \langle P(z), Q_2(z) \rangle c_2^\dagger, \quad \forall c_1, c_2 \in \mathbb{C}^{N \times N}$$

Proposition 1. 1. If $\langle z^i \mathbb{I}_N, z^j \mathbb{I}_N \rangle = g(t)_{ij}$ where g_{ij} is the $\mathbb{C}^{N \times N}$ block in the position (i, j) , then the families $p(z)$ and $\bar{p}(z)$ are biorthogonal matrix polynomials for the linear pairing, i.e.

$$\langle p_i(z), \bar{p}_j(z) \rangle = \delta_{ij} \mathbb{I}_N.$$

Moreover,

$$\langle p_i(z), z^l \mathbb{I}_N \rangle = 0, \quad l = 0, \dots, i-1, \quad \langle z^l \mathbb{I}_N, \bar{p}_j(z) \rangle = 0, \quad l = 0, \dots, j-1. \quad (2)$$

2. In addition, if the time-dependent initial condition $g(t)$ is Hermitian for all t then $p(z)$ and $\bar{p}(z)$ are two families of matrix orthogonal polynomials, moreover, the two families are proportional.

Proof. 1. With the previous definitions for $p(z)$ and $\bar{p}(z)$ we have:

$$p_i(z) = \sum_{k=0}^i S_{ik} z^k, \quad \bar{p}_j(z) = \sum_{l=0}^j (\bar{S}_{lj}^{-1})^\dagger z^l,$$

where S_{ik} and \bar{S}_{lj}^{-1} are the blocks (i, k) and (l, j) for S and \bar{S}^{-1} , respectively. Hence

$$\begin{aligned} \langle p_i(z), \bar{p}_j(z) \rangle &= \sum_{k,l=0}^{i,j} S_{ik} \langle z^k \mathbb{I}_N, z^l \mathbb{I}_N \rangle \bar{S}_{lj}^{-1} = \sum_{k,l \geq 0} S_{ik} \langle z^k \mathbb{I}_N, z^l \mathbb{I}_N \rangle \bar{S}_{lj}^{-1} \\ &= (Sg(t)\bar{S}^{-1})_{ij} = (S\bar{S}^{-1})_{ij} = \delta_{ij} \mathbb{I}_N, \end{aligned}$$

as desired. Finally, (2) is proven by induction. First we have that $\langle p_i(z), \bar{p}_0(z) \rangle = 0$, but $p_0(z) = (\bar{S}_{00}^{-1})^\dagger$ is invertible and therefore we conclude $\langle p_i(z), \mathbb{I}_N \rangle = 0$. Now, $\langle p_i(z), \bar{p}_1(z) \rangle = 0$, but $\bar{p}_1(z) = (\bar{S}_{11}^{-1})^\dagger z + (\bar{S}_{10}^{-1})^\dagger$, and using the skew-linearity, the previous result and the fact that $(\bar{S}_{11}^{-1})^\dagger$ is invertible we deduce that $\langle p_i(z), z \mathbb{I}_N \rangle = 0$, and so forth an so on.

2. Let's study the conditions under which $p(z)$ is a family of matrix orthogonal polynomials. If we take two polynomials in the family, such as $p_i(z) = \sum_{k=1}^i S_{ik} z^k$ and $p_j(z) = \sum_{l=1}^j S_{jl} z^l$ we have

$$\begin{aligned} \langle p_i(z), p_j(z) \rangle &= \sum_{k,l=1}^{i,j} S_{ik} \langle z^k I, z^l I \rangle (S_{jl})^\dagger = \sum_{k,l=0}^{\infty} S_{ik} \langle z^k \mathbb{I}_N, z^l \mathbb{I}_N \rangle (S^\dagger)_{lj} \\ &= (Sg(t)S^\dagger)_{ij} = (\bar{S}S^\dagger)_{ij}. \end{aligned}$$

Observe that $\bar{S}S^\dagger$ is clearly block upper-diagonal with its Hermitian conjugate given by

$$(\bar{S}S^\dagger)^\dagger = S\bar{S}^\dagger = Sg(t)(g(t)^{-1})^\dagger \bar{S}^\dagger = \bar{S}(\bar{S}g(t)^{-1})^\dagger = \bar{S}S^\dagger.$$

Therefore, $\bar{S}S^\dagger$ is Hermitian and block upper-diagonal, which implies that $\bar{S}S^\dagger$ is a block diagonal matrix and the blocks in the diagonal are $\mathbb{C}^{N \times N}$ Hermitian matrices.

We conclude that $\langle p_i(z), p_j(z) \rangle = \delta_{ij} (h_i)^{-1}$, where h_i is a Hermitian matrix. Notice also that as a consequence $\bar{p}(z) = hp(z)$ where $h = \text{diag}(h_1, h_2, \dots)$. □

2.1 The Hankel case

We choose g to be a block Hankel matrix so that $\Lambda g = g\Lambda^t$ or

$$(\Lambda g)_{ij} = g_{i+1,j} = g_{i,j+1} = (g\Lambda^t)_{ij},$$

In this case we have for the blocks of the moment matrix g

$$g_{ij} = \gamma^{(i+j)},$$

for some matrices $\gamma^{(j)} \in \mathbb{C}^{N \times N}$. From

$$\begin{aligned} \Lambda g(t) &= \Lambda W_0 g \bar{W}_0^{-1} = W_0 \Lambda g \bar{W}_0^{-1} \\ &= W_0 g \Lambda^t \bar{W}_0^{-1} = W_0 g \bar{W}_0^{-1} \Lambda^t = g(t) \Lambda^t, \end{aligned}$$

we easily deduce that $g(t)$ is block Hankel if g is. We have

Proposition 2. *Assume that g is block Hankel then with $\gamma^{(j)}$ a block moment matrix; i.e. $\gamma^{(j)} = \int_{\mathbb{R}} x^j \rho(x) dx$ then the pairing can be viewed as a scalar product in the real line whose matrix moment is g ; i.e.,*

$$\langle P(x), Q(x) \rangle = \int_{\mathbb{R}} P(x) \rho(x) Q(x)^\dagger dx.$$

Proof. On one hand we have $\langle P(x), Q(x) \rangle = \sum_{ij} P_i \gamma^{(i+j)} Q_j^\dagger$. Using the previous definition $\int_{\mathbb{R}} x^{j+k} \rho(x) dx = \langle x^j \mathbb{1}_N, x^k \mathbb{1}_N \rangle = \gamma^{(j+k)}$ (the Hankel symmetry ensures that there is only dependence in $j+k$) we have:

$$\begin{aligned} \langle P(x), Q(x) \rangle &= \sum_{ij} P_i \gamma^{(i+j)} Q_j^\dagger = \sum_{ij} P_i \int_{\mathbb{R}} x^{j+k} \rho(x) dx Q_j^\dagger \\ &= \int_{\mathbb{R}} P(x) \rho(x) Q(x)^\dagger dx. \end{aligned}$$

□

In general arbitrary continuous deformations do not preserve the Hermitian character of g . If we look for families of matrix orthogonal polynomials on the real line we should make restricted deformations. Let's make this point clear. In the following z^* denotes the complex conjugate of $z \in \mathbb{C}$.

Proposition 3. *If the matrix g is block Hankel and the matrices $\gamma^{(j)}$ are Hermitian then*

1. *The families $p(z)$ and $\bar{p}(z)$ are proportional and the two of them are matrix orthogonal polynomials in the real line.*
2. *Moreover, if the continuous deformation parameters satisfy one of the two following conditions*

- (a) $t_{ja}, \bar{t}_{ja} \in \mathbb{R}$ and satisfy $t_{ja} = t_j, \bar{t}_{ja} = \bar{t}_j, a = 1, \dots, N$.
- (b) t_{ja}, \bar{t}_{ja} satisfy $t_{ja} + \bar{t}_{ja}^* = 0$.

the result holds for the time dependent moment matrix.

Proof. 1. If the matrix g is block Hänkel and the blocks are Hermitian the matrix g is itself Hermitian. Given i, j , pair of indices for an element of g , there exist four integer indices $(k, l), (m, n)$ with $k, l \geq 0$ y $m, n = 1, \dots, N$ that satisfy $a_{ij} = (A_{kl})_{mn} = (A_{kl})_{nm}^* = (A_{lk})_{nm}^* = a_{ji}^*$ (we use $A \in \mathbb{C}^{N \times N}$ for a block of g) as a consequence $g = g^\dagger$ and by Proposition 1 the first part of the result holds.

2. Let us compute the time evolution for g . We have

$$W_0 = \sum_{a=1}^N E_{aa} \exp\left(\sum_{j=1}^{\infty} t_{ja} \Lambda^j\right) = \sum_{a=1}^N E_{aa} \sum_{j=0}^{\infty} s_j^{(a)} \Lambda^j,$$

where $s_j^{(a)}$ is the j -th Schur¹ polynomial for the component a . Consequently and taking $s_j := \sum_{a=1}^N E_{aa} s_j^{(a)}$

$$W_0 = \sum_{j=0}^{\infty} \sum_{k=1}^N E_{aa} s_j^{(a)} \Lambda^j = \sum_{j=0}^{\infty} s_j \Lambda^j,$$

so it is straight forward to see the block structure of W_0 , whose blocks are given by $(W_0)_{ij} = s_{j-i}$ if $j - i \geq 0$ and 0_N otherwise.

A similar argument for \bar{W}_0^{-1} leads to $(\bar{W}_0^{-1})_{ij} = \bar{s}_{i-j}$ if $i - j \geq 0$ and 0_N otherwise (here $\bar{s}_j := \sum_{a=1}^N E_{aa} \bar{s}_j^{(a)}$ and $\bar{s}_j^{(a)}$ is the j -th Schur polynomial for the component a but now in the variables $-t_{ja}$).

We are now ready to compute

$$g(t)_{ij} = \sum_{k, l \geq 0} (W_0)_{ik} g_{kl} (\bar{W}_0^{-1})_{lj} = \sum_{k \geq i, l \geq j} s_{k-i} \gamma^{(k+l)} \bar{s}_{l-j} = \sum_{k, l \geq 0} s_k \gamma^{(i+j+k+l)} \bar{s}_l.$$

Then

(a) If the first condition holds then all matrices s_j and \bar{s}_j are real and scalar so

$$\begin{aligned} g(t)_{ij}^\dagger &= \sum_{k, l \geq 0} \bar{s}_l^\dagger \gamma^{(i+j+k+l)} s_k^\dagger = \sum_{k, l \geq 0} s_k^\dagger \gamma^{(i+j+k+l)} \bar{s}_l^\dagger = \sum_{k, l \geq 0} s_k \gamma^{(i+j+k+l)} \bar{s}_l \\ &= g(t)_{ij}. \end{aligned}$$

(b) If the second condition holds $s_j = \bar{s}_j^\dagger$ for all j so

$$\begin{aligned} g(t)_{ij}^\dagger &= \sum_{k, l \geq 0} \bar{s}_l^\dagger \gamma^{(i+j+k+l)} s_k^\dagger = \sum_{l, k \geq 0} \bar{s}_k^\dagger \gamma^{(i+j+l+k)} s_l^\dagger = \sum_{k, l \geq 0} s_k \gamma^{(i+j+k+l)} \bar{s}_l \\ &= g(t)_{ij}. \end{aligned}$$

¹We remind the reader that the Schur polynomials are given by $\exp(\sum_{j=1}^{\infty} t_{ja} \Lambda^j) = \sum_{j=0}^{\infty} s_j^{(a)} \Lambda^j$ so $s_0^{(a)} = 1$ and $s_j^{(a)} = \sum_{p=1}^j \sum_{j_1+\dots+j_p=j} t_{j_1 a} \cdots t_{j_p a} = t_{ja} + \cdots$ where the missing term is a polynomial of utmost degree j in the variables $t_{1a}, t_{2a}, \dots, t_{j-1, a}$.

Under any of the two conditions $g(t)$ is block-Hänkel with Hermitian blocks, so is itself Hermitian. That proves the second part of the proposition. \square

We now discuss the recursion formulae. Using the Hänkel condition $\Lambda g = g\Lambda^t$ and taking the usual definition for the Lax operator we conclude $L = \bar{L}$ and hence we have $L = \Lambda + u(n) + v(n)\Lambda^t$.

Proposition 4. *The polynomials $p(z), \bar{p}(z)$ satisfy a three term recurrence law given by $p_{n+1}(z) = zp_n(z) - u(n)p_n(z) - v(n)p_{n-1}(z)$*

Proof. From the definition of L we have $Lp(z) = LS\chi(z) = zp(z) = (\Lambda + u(n) + v(n)\Lambda^t)p(z)$. If we take the sequence terms $\{p_n(z)\}_n$ we conclude $zp_n(z) = p_{n+1}(z) + u(n)p_n(z) + v(n)p_{n-1}(z)$. For the polynomials $\bar{p}(z)$ we have $\bar{L}^\dagger \bar{p}(z) = z\bar{p}(z) = (\Lambda v^\dagger(n) + u^\dagger(n) + \Lambda^t)\bar{p}(z)$, and using the sequence $\{\bar{p}_n(z)\}_n$ we obtain another recurrence law. \square

3 Multi-graded Hänkel reduction, generalized orthogonality, multiple orthogonal polynomials and Riemann–Hilbert problems

In this section we will study generalized Hänkel type conditions. Given a multi-index $\bar{n} = (n_1, \dots, n_N)$ with n_a non-negative integers we define for $A \in \mathfrak{g}$ the power $A^{\bar{n}} = \sum_{a=1}^N A^{n_a} E_{aa}$. For two multi-indices \bar{n} and \bar{m} a matrix g is said to be a (\bar{n}, \bar{m}) multi-graded Hänkel if

$$\Lambda^{\bar{n}} g = g(\Lambda^t)^{\bar{m}}. \quad (3)$$

If as before $g_{ij} \in \mathbb{C}^{N \times N}$ denotes a block in g then we can write $g_{ij} = (g_{ij,ab})_{1 \leq a, b \leq N}$ and the multi-graded Hänkel condition reads $g_{i+n_a, j, ab} = g_{i, j+m_b, ab}$. An ample family of multi-graded Hänkel matrices can be constructed in terms of weights $\rho_{j,ab}$ as the moments

$$g_{ij,ab} = \int_{\mathbb{R}} x^i \rho_{j,ab}(x) dx, \quad (4)$$

where the weights satisfy a generalized periodicity condition of the form

$$\rho_{j+m_b, ab}(x) = x^{n_a} \rho_{j,ab}(x). \quad (5)$$

Thus, given the weights $\rho_{0,ab}, \dots, \rho_{m_b-1,ab}$, all the others are fixed by (5). From now on, we concentrate only in these cases of multigraded Hänkel matrices. We notice that for the 1-component case and for $n_1 = m_1$ case these moments matrices were studied in [5]-[6].

Proposition 5. *For multi-graded Hänkel matrices the matrix polynomials p_i satisfy the following generalized orthogonality conditions*

$$\int_{\mathbb{R}} p_i(x) \rho_j(x) dx = 0, \quad j = 0, \dots, i-1, \quad \rho_j := (\rho_{j,ab}) \in \mathbb{C}^{N \times N}. \quad (6)$$

Proof. From $Sg(t)\bar{S}^{-1} = 1_G$ we get $\sum_{\substack{j=1, \dots, i \\ b=1, \dots, N}} p_{ij,ab} g_{jl,bc} = 0$ for $a, c = 1, \dots, N$ and $l = 0, \dots, i-1$. Now recalling (4) we get the result. \square

Using the Euclidean division $i = \theta_c m_c + \sigma_c$, with $\theta_c \geq 0$, $0 \leq \sigma_c < m_c$ we get a better insight of the orthogonality relations (6), for $a, c = 1, \dots, N$,

$$\begin{aligned} \sum_{b=1}^N \int_{\mathbb{R}} p_{i,ab}(x) \rho_{j,bc}(x) (x^{n_b})^l dx &= 0, \quad j = 0, \dots, m_c - 1, \quad l = 0, \dots, \theta_c - 1, \\ \sum_{b=1}^N \int_{\mathbb{R}} p_{i,ab}(x) \rho_{j,bc}(x) (x^{n_b})^{\theta_c} dx &= 0, \quad j = 0, \dots, \sigma_c - 1. \end{aligned} \quad (7)$$

3.1 Evolution

From (3) we conclude that $g(t)$ is of (\bar{n}, \bar{m}) multi-graded Hänkel type if g is. In general, the evolution of g is given in terms of Schur polynomials by

$$g_{ij}(t) = \sum_{k,l \geq 0} s_k g_{i+k, j+l} \bar{s}_l$$

which recalling (4) leads to the following evolution of the weights

$$\rho_j(t) = \sum_{k,l \geq 0} x^k s_k \rho_{j+l} \bar{s}_l = \exp(t(x)) \sum_{l \geq 0} \rho_{j+l} \bar{s}_l, \quad (8)$$

where $t(x) = \sum_{a=1}^N t_a(x) E_{aa}$ and $t_a(x) := \sum_{j \geq 1} t_{ja} x^j$. It can be easily checked that $\rho_j(t)$ satisfies the periodicity condition (5) if ρ_j does. From (8) we infer that

$$\begin{pmatrix} \rho_{0,ab}(t) \\ \vdots \\ \rho_{m_b-1,ab}(t) \end{pmatrix} = \exp(t_a(x)) \begin{pmatrix} \mathcal{S}_{0,ab} & \mathcal{S}_{1,ab} & \mathcal{S}_{2,ab} & \cdots & \mathcal{S}_{m_b-1,ab} \\ x^{n_a} \mathcal{S}_{m_b-1,ab} & \mathcal{S}_{0,ab} & \mathcal{S}_{1,ab} & \cdots & \mathcal{S}_{m_b-2,ab} \\ x^{n_a} \mathcal{S}_{m_b-2,ab} & x^{n_a} \mathcal{S}_{m_b-1,ab} & \mathcal{S}_{0,ab} & \cdots & \mathcal{S}_{m_b-2,ab} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x^{n_a} \mathcal{S}_{1,ab} & x^{n_a} \mathcal{S}_{2,ab} & x^{n_a} \mathcal{S}_{3,ab} & \cdots & \mathcal{S}_{0,ab} \end{pmatrix} \begin{pmatrix} \rho_{0,ab} \\ \vdots \\ \rho_{m_b-1,ab} \end{pmatrix} \quad (9)$$

where

$$\mathcal{S}_{i,ab} = \sum_{j \geq 0} \bar{s}_{i+m_b j}^{(b)} (x^{n_a})^j = \frac{1}{m_b x^{i n_a / m_b}} \sum_{k=0}^{m_b-1} \varepsilon_b^{ik} \exp(-\bar{t}_b(\varepsilon_b^k x^{n_a / m_b})), \quad \varepsilon_b^{m_b} = 1.$$

If we denote

$$\bar{t}_a^{(l)}(x) = \sum_{j \geq 0} \bar{t}_{j m_b + l, a} x^{j m_b + l}, \quad l = 0, 1, \dots, m_b - 1,$$

we have

$$\bar{t}_a(\varepsilon_b^k x) = \bar{t}_a^{(0)}(x) + \bar{t}_a^{[k]}(x), \quad \bar{t}_a^{[k]}(x) = \sum_{\substack{j \geq 0 \\ l=1, \dots, m_b-1}} \bar{t}_{j m_b + l, a} \varepsilon_b^{kl} x^{j m_b + l},$$

and therefore

$$\mathcal{L}_{i,ab} = \frac{1}{m_b x^{in_a/m_b}} \exp(-\bar{t}_b^{(0)}(x^{n_a/m_b})) \sum_{k=0}^{m_b-1} \varepsilon_b^{ik} \exp(-\bar{t}_b^{[k]}(x^{n_a/m_b})). \quad (10)$$

Finally, we deduce

$$\rho_{j,ab}(t) = \exp(t_a(x) - \bar{t}_b^{(0)}(x^{n_a/m_b})) \sum_{k=0}^{m_b-1} \hat{\rho}_{j,ab}^{(k)} \exp(-\bar{t}_b^{[k]}(x^{n_a/m_b})), \quad (11)$$

where we have used the discrete Fourier transform of the weights

$$\hat{\rho}_{j,ab}^{(k)} := \frac{1}{m_b} \sum_{i=1}^{m_b-1} \varepsilon_b^{ik} x^{-in_a/m_b} \rho_{j+i,ab}.$$

3.2 Recursion relations and symmetries

In terms of Lax operators the multi-graded H\"ankel reduction reads [10]

$$\mathcal{L} := \sum_{a=1}^N L^{n_a} C_a = \sum_{b=1}^N \bar{L}^{m_b} \bar{C}_b. \quad (12)$$

Within this subsection we assume that

$$n_1 \geq \dots \geq n_N \geq 1, \quad m_1 \geq \dots \geq m_N \geq 1,$$

and suppose that $n_1 = \dots = n_r$ and $n_r > n_{r+1}$. Given (1) from (12) we deduce that

$$\mathcal{L} = (E_{11} + \dots + E_{rr})\Lambda^{n_1} + \mathcal{L}_{n_1-1}\Lambda^{n_1-1} + \dots + \mathcal{L}_0 + \mathcal{L}_{-1}\Lambda^t + \mathcal{L}_{-2}(\Lambda^t)^2 + \dots,$$

while we also have $\mathcal{L} = \sum_{b=1}^N \bar{L}^{m_b} \bar{C}_b$ and therefore

$$\mathcal{L} = \mathcal{L}_{-m_1}(\Lambda^t)^{m_1} + \mathcal{L}_{-m_1+1}(\Lambda^t)^{m_1-1} + \dots + \mathcal{L}_0 + \mathcal{L}_1\Lambda + \mathcal{L}_2\Lambda^2 + \dots.$$

We conclude the block band structure

$$\mathcal{L} = (E_{11} + \dots + E_{rr})\Lambda^{n_1} + \mathcal{L}_{n_1-1}\Lambda^{n_1-1} + \dots + \mathcal{L}_{-m_1+1}(\Lambda^t)^{m_1-1} + \mathcal{L}_{-m_1}(\Lambda^t)^{m_1}. \quad (13)$$

Proposition 6. *The polynomials $p_i(z)$ are subject to*

$$(E_{11} + \dots + E_{rr})p_{i+n_1}(z) + \dots + \mathcal{L}_{-m_1}p_{i-m_1}(z) = p_i(z) \left(\sum_{a=1}^n z^{n_a} E_{aa} \right)$$

Proof. We only have to show that $\mathcal{L}p = p(\sum_{a=1}^n z^{n_a} E_{aa})$. But this follows from $\mathcal{L}p = S(\sum_{a=1}^N \Lambda^{n_a} E_{aa})S^{-1}S\chi = S\chi(\sum_{a=1}^N z^{n_a} E_{aa}) = p(z)(\sum_{a=1}^n z^{n_a} E_{aa})$. \square

Similarly, from $\mathcal{L}^\dagger \bar{p}(z) = \bar{p}(z)(\sum_{b=1}^N z^{m_b} E_{bb})$ a recursion relation follows for the \bar{p}_i .

Finally we notice that in this case the following symmetry conditions hold [10]

$$\begin{aligned} \left(\sum_{a=1}^N \frac{\partial}{\partial t_{in_a a}} + \sum_{a=1}^N \frac{\partial}{\partial \bar{t}_{im_a a}} \right) L &= \left(\sum_{a=1}^N \frac{\partial}{\partial t_{in_a a}} + \sum_{a=1}^N \frac{\partial}{\partial \bar{t}_{im_a a}} \right) \bar{L} = 0, \\ \left(\sum_{a=1}^N \frac{\partial}{\partial t_{in_a a}} + \sum_{a=1}^N \frac{\partial}{\partial \bar{t}_{im_a a}} \right) C_l &= \left(\sum_{a=1}^N \frac{\partial}{\partial t_{in_a a}} + \sum_{a=1}^N \frac{\partial}{\partial \bar{t}_{im_a a}} \right) \bar{C}_b = 0, \end{aligned}$$

for $i \geq 1$ and $b = 1, \dots, N$.

3.3 Relation with multiple orthogonal polynomials

We will see that when the weights ρ_j are particular rank one matrices there is a nice correspondence with multiple orthogonal polynomial of mixed type with a normalization of type II. Following [1] we take two sets of non negative multi-indices $\bar{\nu} = (\nu_1, \dots, \nu_p)$, $\bar{\mu} = (\mu_1, \dots, \mu_q)$ and write $|\nu| = \sum_{J=1}^p \nu_J$ and $|\mu| = \sum_{K=1}^q \mu_K$. We also take weights $\{w_{1J}\}_{J=1}^p$ and $\{w_{2K}\}_{K=1}^q$ which are assumed to be non-negative functions on the real line. For a fixed pair $\bar{\nu}, \bar{\mu}$ we say that $\{A_{\bar{\nu}, \bar{\mu}, J}\}_{J=1, \dots, p}$ is a set of multiple orthogonal polynomials of mixed type if $\deg A_{\bar{\nu}, \bar{\mu}, J} \leq \nu_J - 1$ and the following orthogonality relations

$$\int_{\mathbb{R}} \sum_{J=1}^p A_{\bar{\nu}, \bar{\mu}, J}(x) w_{1J}(x) w_{2K}(x) x^\alpha dx = 0, \quad \alpha = 0, \dots, \mu_K - 1, \quad K = 1, \dots, q, \quad (14)$$

are satisfied. Alternatively defining the following linear forms $Q_{\bar{\nu}, \bar{\mu}}(x) := \sum_{J=1}^p A_{\bar{\nu}, \bar{\mu}, J}(x) w_{1J}(x)$ the orthogonality relations can be written in the following way

$$\int_{\mathbb{R}} Q_{\bar{\nu}, \bar{\mu}}(x) w_{2K}(x) x^\alpha dx = 0, \quad \alpha = 0, \dots, \mu_K - 1, \quad K = 1, \dots, q.$$

For multiple orthogonal polynomials as described above we will take $|\nu| = |\mu|$ and assume that the following conditions hold

1. For each $K = 1, \dots, p$ the orthogonality relations for the multi-indices $\bar{\nu} + e_K, \bar{\mu}$ have a unique solution with $A_{\bar{\nu}, \bar{\mu}, K}$ monic and $\deg A_{\bar{\nu}, \bar{\mu}, K} = \nu_K - 1$, with $\deg A_{\bar{\nu}, \bar{\mu}, J} < \nu_J - 1$ if $J \neq K$. We will call it type II normalization to the K -th component and write that normalized solution as $\{A_{\bar{\nu}, \bar{\mu}, J}^{(II, K)}\}_{J=1, \dots, p}$.
2. For each $K = 1, \dots, q$ the orthogonality relations for the multi-indices $\bar{\nu}, \bar{\mu} - e_K$ have a unique solution with the following normalization: $\deg A_{\bar{\nu}, \bar{\mu}, J} = \nu_J - 1$ and $\int_{\mathbb{R}} Q_{\bar{\nu}, \bar{\mu}}(x) w_{2K}(x) x^{\mu_K} dx = 1$. We will call it normalization of type I to the K -th component and write that normalized solution as $\{A_{\bar{\nu}, \bar{\mu}, J}^{(I, K)}\}_{J=1, \dots, p}$.

In order to connect (7) with multiple orthogonal polynomials we consider the Euclidean division $i = q_a n_a + r_a$, with $q_a \geq 0$ and $0 \leq r_a < n_a$, and write

$$p_{i, ab}(z) = \sum_{j=1}^{n_b} z^{j-1} \Pi_{ij, ab}(z^{n_b}), \quad (15)$$

where $\Pi_{ij, ab}(z^{n_b})$ are polynomials in z^{n_b} such that

$$\deg \Pi_{ij, ab} \leq \begin{cases} q_b, & j \leq r_b \text{ or } j = r_b + 1 \text{ and } b = a, \\ q_b - 1, & j > r_b + 1 \text{ or } j = r_b + 1 \text{ and } b \neq a. \end{cases}$$

Notice that the monic character of p_i gives the normalization of $\Pi_{i r_a + 1, aa}$ which happens to be a monic polynomial with $\deg \Pi_{i r_a + 1, aa} = q_a$.

The inversion formula for (15) can be deduced as follows. If we denote by $\epsilon_b := \exp(2\pi i/n_b)$, a primitive n_b -th root of the unity, and evaluate at $\epsilon_b^k z$, $k = 0, \dots, n_b - 1$ we get the following system of equations

$$p_{i, ab}(\epsilon_b^k z) = \sum_{j=1}^{n_b} (\epsilon_b^k z)^{j-1} \Pi_{ij, ab}(z^{n_b}) \quad k = 0, \dots, n_b - 1,$$

that we solve in order to obtain the polynomials $\Pi_{ik,ab}$ in terms of a discrete Fourier transform of the polynomial p_i through the formula

$$\Pi_{i,j+1,ab}(z^{n_b}) = \frac{1}{n_b z^j} \sum_{k=0}^{n_b-1} \epsilon_b^{-jk} p_{i,ab}(\epsilon_b^k z).$$

Then, (7) can be written as

$$\begin{aligned} \int_{\mathbb{R}} \sum_{b=1}^N \sum_{j=1}^{n_b} \Pi_{ij,ab}(x^{n_b}) x^{j-1} \rho_{k,bc}(x) (x^{n_b})^l dx &= 0, \quad k = 0, \dots, m_c - 1, \quad l = 0, \dots, \theta_c - 1, \\ \int_{\mathbb{R}} \sum_{b=1}^N \sum_{j=1}^{n_b} \Pi_{ij,ab}(x^{n_b}) x^{j-1} \rho_{k,bc}(x) (x^{n_b})^{\theta_c} dx &= 0, \quad k = 0, \dots, \sigma_c - 1. \end{aligned} \quad (16)$$

These equations strongly suggest to perform a change of variables in each integrand of the type $y = x^{n_b}$. For that aim, it is relevant that when n_b is an even number then $\text{supp}(\rho_{j,bc}) \subset \mathbb{R}^+$, otherwise the change of variables is ill defined. In fact, for these even cases, one easily see that the weights must be supported on the positive axis or uniqueness of the orthogonal polynomials is not ensured. Moreover, for n_b odd it is also necessary to assume that the weight is supported either only in the positive real numbers or only in the negative real line, this requirement comes from the positivity condition on the weights and the use of (19). Hereon we will assume that the all weights are supported on the positive real semiline. When the mentioned change of variable is performed in (16) we get

$$\begin{aligned} \int_{\mathbb{R}} \sum_{b=1}^N \sum_{j=1}^{n_b} \Pi_{ij,ab}(y) \tilde{\rho}_{jk,bc}(y) y^l dy &= 0, \quad k = 0, \dots, m_c - 1, \quad l = 0, \dots, \theta_c - 1, \\ \int_{\mathbb{R}} \sum_{b=1}^N \sum_{j=1}^{n_b} \Pi_{ij,ab}(y) \tilde{\rho}_{jk,bc}(y) y^{\theta_c} dy &= 0, \quad k = 0, \dots, \sigma_c - 1, \end{aligned} \quad (17)$$

with

$$\tilde{\rho}_{jk,bc}(y) = \frac{1}{n_b} y^{\frac{j}{n_b}-1} \rho_{k,bc}(y^{\frac{1}{n_b}}).$$

Now, if the matrix weights ρ_k are rank-1 matrices of the following particular form

$$\rho_{k,bc}(x) = v_{1,b}(x) w_{2,kc}(x^{n_b}),$$

we get

$$\begin{aligned} \int_{\mathbb{R}} \sum_{b=1}^N \sum_{j=1}^{n_b} \Pi_{ij,ab}(y) w_{1,jb}(y) w_{2,kc}(y) y^l dy &= 0, \quad k = 0, \dots, m_c - 1, \quad l = 0, \dots, \theta_c - 1, \\ \int_{\mathbb{R}} \sum_{b=1}^N \sum_{j=1}^{n_b} \Pi_{ij,ab}(y) w_{1,jb}(y) w_{2,kc}(y) y^{\theta_c} dy &= 0, \quad k = 0, \dots, \sigma_c - 1, \end{aligned} \quad (18)$$

where

$$w_{1,jb}(y) = \frac{1}{n_b} y^{\frac{j}{n_b}-1} v_{1,b}(y^{\frac{1}{n_b}}). \quad (19)$$

We are ready to describe the relation among multiple orthogonal polynomials of type II and generalized matrix polynomials. First, given (a, j) with $a = 1, \dots, N$ and $j \in \mathbb{N}$ we make the definitions $\mathcal{N}(a, j) := n_1 + \dots + n_{a-1} + j$ and $\mathcal{M}(a, j) := m_1 + \dots + n_{a-1} + j + 1$.

Proposition 7. *Relations (18) are particular cases of (14) with:*

1. $J = \mathcal{N}(b, j)$ for $b = 1, \dots, N$ and $j = 1, \dots, n_b$; and $K = \mathcal{M}(c, k)$ for $c = 1, \dots, N$ and $k = 0, \dots, m_c - 1$. We have therefore the identification $J = n_1 + \dots + n_{b-1} + j$ and $K = m_1 + \dots + m_{c-1} + k + 1$.
2. $p = |\bar{n}| = n_1 + \dots + n_N$ and $q = |\bar{m}| = m_1 + \dots + m_N$.
- 3.

$$\nu_{\mathcal{N}(b,j)} = \begin{cases} q_b + 1, & j \leq r_b \text{ or } j = r_b + 1 \text{ and } b = a, \\ q_b, & j > r_b + 1 \text{ or } j = r_b + 1 \text{ and } b \neq a, \end{cases}$$

$$\mu_{\mathcal{M}(c,k)} = \begin{cases} \theta_c + 1, & 0 \leq k \leq \sigma_c - 1, \\ \theta_c, & \sigma_c \leq k \leq m_c - 1. \end{cases}$$

$$4. |\bar{\nu}| = \sum_J \nu_J = Ni + 1 \text{ and } |\bar{\mu}| = \sum_K \mu_K = Ni.$$

$$5. A_{\bar{\nu}, \bar{\mu}, J} = \Pi_{ij, ab} \text{ and } p_{i, ab}(z) = \sum_{j=1}^{n_b} z^{j-1} A_{\bar{\nu}, \bar{\mu}, \mathcal{N}(b,j)}(z^{n_b}).$$

$$6. \deg A_{\bar{\nu}, \bar{\mu}, J} = \nu_J \text{ if } J = \mathcal{N}(a, r_a + 1) \text{ so that } A_{\bar{\nu}, \bar{\mu}, J} = A_{\bar{\nu}, \bar{\mu}, J}^{(II, \mathcal{N}(a, r_a + 1))}, \text{ and we have a type II normalization w.r.t } \mathcal{N}(a, r_a + 1). \text{ Thus, } p_{i, ab}(z) = \sum_{j=1}^{n_b} z^{j-1} A_{\bar{\nu}, \bar{\mu}, \mathcal{N}(b,j)}^{(II, \mathcal{N}(a, r_a + 1))}(z^{n_b}).$$

The reader should notice that evolution of the weights given trough (8) give the following evolution of $w_{1,jb}$ and $w_{2,kc}$

$$w_{1,jb}(t) = \exp\left(\sum_{k \geq 1} y^{k/n_b} t_{kb}\right) w_{1,jb}, \quad w_{2,kc}(t) = \sum_{l \geq 0} w_{2,k+l} c \bar{s}_l^{(c)}.$$

Using (11) we may write

$$w_{1,jb}(t) = \exp\left(\sum_{k \geq 1} y^{k/n_b} t_{kb}\right) w_{1,jb},$$

$$w_{2,kc}(t) = \exp(-\bar{t}_b^{(0)}(y^{1/m_c})) \sum_{l=0}^{m_b-1} \tilde{w}_{2k,c}^{(l)} \exp(-\bar{t}_b^{(l)}(y^{1/m_c})), \quad \tilde{w}_{2,kc}^{(l)} := \frac{1}{m_c} \sum_{i=1}^{m_b-1} \varepsilon_c^{il} y^{-i/m_c} w_{2,k+i,c}.$$

These evolved weights fulfill the positivity condition (recall that their support is included in the positive real line) when we have $t_{jb}, \bar{t}_{jm_c} \in \mathbb{R}$ and $\bar{t}_{jc} \leq 0$ for $j \neq 0 \pmod{m_c}$.

3.4 Riemann–Hilbert problems

As we have just shown the generalized matrix polynomials are connected with a family of multiple orthogonal polynomials for a particular rank one moment matrix. In [1] the Riemann–Hilbert problem for multiple orthogonal polynomials of mixed type was presented, see also [13]. We will discuss its relation with the generalized orthogonal polynomials $p_i(z)$.

Let us recall the reader that the Cauchy transform is defined by

$$\hat{Q}_{\bar{\nu}', \bar{\mu}', K}(z) := -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{Q_{\bar{\nu}', \bar{\mu}'}(x)}{z-x} w_{2K}(x) dx.$$

Now we can make the following definition for the $(p+q) \times (p+q)$ complex valued matrix $Y(z)$

$$\begin{aligned} Y_{K,J} &:= A_{\bar{\nu}'+e_K, \bar{\mu}', J}^{(II,K)} & J=1, \dots, p \quad K=1, \dots, p \\ Y_{K, J+p} &:= \hat{Q}_{\bar{\nu}'+e_K, \bar{\mu}, J}^{(II,K)} & J=1, \dots, q \quad K=1, \dots, p \\ Y_{K+p, J} &:= -2\pi i A_{\bar{\nu}', \bar{\mu}'-e_K, J}^{(I,K)} & J=1, \dots, p \quad K=1, \dots, q \\ Y_{K+p, J+p} &:= -2\pi i \hat{Q}_{\bar{\nu}', \bar{\mu}'-e_K, J}^{(I,K)} & J=1, \dots, q \quad K=1, \dots, q \end{aligned}$$

we will also use the following real valued $(p+q) \times (p+q)$ matrix $D(x)$ defined by blocks

$$D(x) := \begin{pmatrix} \mathbb{I}_p & W(x) \\ 0_{q \times p} & \mathbb{I}_q \end{pmatrix}$$

where $W_{JK}(x) = w_{1J}(x)w_{2K}(x)$.

We adapt to the present situation a result of [1] taking into account the support of the weights

Theorem 1. *Let be $\bar{\nu}', \bar{\mu}'$ two multi-indices such that $|\bar{\nu}'| = |\bar{\mu}'|$ and suppose that the normality conditions hold. Let also be two sets of weight functions $\{w_{1J}\}_{J=1, \dots, p}$ and $\{w_{2K}\}_{K=1, \dots, q}$ such that for every J, K $w_{1J}, (x)w_{2K}(x)$ are differentiable a.e. in \mathbb{R}_+ and $x^j w_{1J}, x^j w_{2K} \in H^1(\mathbb{R}_+)$, $j = 0, \dots, \nu'_K - 1$. In $x = 0$ we require the weight functions to be bounded. Then, the matrix $Y(z)$ is the only solution of the following Riemann–Hilbert problem.*

1. $Y(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}_+$.
2. $Y(x)_+ = Y(x)_- D(x)$ for all $x > 0$.
3. $Y(z) \text{diag}(z^{-\nu'_1}, \dots, z^{-\nu'_p}, z^{\mu'_1}, \dots, z^{\mu'_p}) = \mathbb{I}_{(p+q)} + O(z^{-1})$ for $z \rightarrow \infty$.

$$4. Y(z) = O \left(\begin{pmatrix} 1 & 1 & \cdots & 1 & \log |z| & \log |z| & \cdots & \log |z| \\ 1 & 1 & \cdots & 1 & \log |z| & \log |z| & \cdots & \log |z| \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & \log |z| & \log |z| & \cdots & \log |z| \end{pmatrix} \right) \text{ when } z \rightarrow 0.$$

Proof. First we see uniqueness of the solutions. First we see that as $\det D(x) = 1 \det Y(z)$ is analytical across the integration contour, then the only possible singularity of $\det Y(z)$ is in $z = 0$. As $z \det Y(z) \rightarrow 0$ when $z \rightarrow 0$ the isolated singularity must be removable and $\det Y(z)$ is an entire function. The asymptotic behavior and Liouville's Theorem gives $\det Y(z) = 1$. Consequently $Y(z)^{-1}$ exists. Given two possible solutions for the problem $Y(z)$ and $\tilde{Y}(z)$, the matrix $Y(z)\tilde{Y}(z)^{-1}$ can be singular only in $z = 0$. As before the singularity must be removable so $Y(z)\tilde{Y}(z)^{-1}$ is entire and bounded, and hence $Y(z)\tilde{Y}(z)^{-1} = \mathbb{I}_{p+q}$.

Now we prove that the matrix $Y(z)$ is a solution of the R–H problem. Condition 1 follows from the fact that polynomials are entire functions and general theory of Cauchy integrals [14] gives analytic behavior outside the integration contour. Now, observe that condition 2 is a jumping condition on the positive real axis and is a consequence of Plemelj formulae. For condition 3 we notice that $Y(z)_{ii}$ is a monic polynomial with degree ν'_i for $i = 1, \dots, p$. We also see that the leading term of each $Y(z)_{ii}$ is $z^{-\mu'_i}$ for $i = p, \dots, (p+q)$ due to the orthogonality relations and the type I normalization. Consequently the diagonal elements of $Y(z) \text{diag}(z^{-\nu'_1}, \dots, z^{-\nu'_p}, z^{\mu'_1}, \dots, z^{\mu'_p})$ are equal to 1 and the rest of them vanish like $\frac{1}{z}$ asymptotically. Finally, condition 4 is a consequence of the behavior of the polynomials and the Cauchy transforms. The boundness condition of the weights at $z = 0$ makes the Cauchy integrals to have at much log-type singularities at $z = 0$. \square

The Theorem applies to our situation giving

Proposition 8. *If $\bar{\nu}' = \bar{\nu} - e_{\mathcal{N}(a, r_a+1)}$ and $\bar{\mu}' = \bar{\mu}$, with $\bar{\nu}$ and $\bar{\mu}$ as in Proposition 7, we have $\Pi_{ij, ab} = Y_{\mathcal{N}(a, r_a+1), \mathcal{N}(b, j)}$ and*

$$p_{i, ab}(z) = \sum_{j=1}^{n_b} z^{j-1} Y_{\mathcal{N}(a, r_a+1), \mathcal{N}(b, k)}(z^{n_b}),$$

$$Y_{\mathcal{N}(a, r_a+1), \mathcal{N}(b, j)}(z^{n_b}) = \frac{1}{n_b z^{j-1}} \sum_{l=0}^{n_b-1} \epsilon_b^{-l(j-1)} p_{i, ab}(\epsilon_b^j z).$$

We observe that in [5] and [6] these generalized polynomials are considered for the one component case $N = 1$ and for $n = m$ and therefore fit as a particular example of our polynomials and consequently are related with multiple orthogonal polynomials of mixed type. Apparently they might noticed this fact later, as in [3] they claim that they considered multiple orthogonal polynomials in [5]. However, we must say that the Riemann–Hilbert problem derived there is different from the Daems–Kuijlaars problem considered here for multiple orthogonal polynomials. In fact, the matrix Y considered in [5] is not analytic in the upper-half plane and hence fails to satisfy the Riemann–Hilbert problem posed there.

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