Some properties of zeros of Sobolev-type orthogonal polynomials

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Abstract

For polynomials orthogonal with respect to a discrete Sobolev product, we prove that, for each n, Q_n has at least $n-m$ zeros on the convex hull of the support of the measure, where m denotes the number of terms in the discrete part. Interlacing properties of zeros are also described.

Keywords: Orthogonal and quasi-orthogonal polynomials; Sobolev-type products; Gauss–Jacobi quadrature formula; Zeros

AMS classification: 33C45

1. Introduction

(1) During the last years several authors studied polynomials orthogonal with respect to the so-called Sobolev-type (or discrete Sobolev) inner products, that is, inner products of the form

$$\langle f,g\rangle = \int_I fg\,\mathrm{d}\mu + \sum_{i=0}^{r} M_i f^{(i)}(c)g^{(i)}(c), \tag{1}$$

where μ is a finite positive Borel measure supported on an interval $I \subset \mathbb{R}$, $c \notin \mathring{I}$ (the interior of I), $r \geqslant 1$, $M_i \geqslant 0$ for $i = 0, \dots, r-1$ and $M_r > 0$ (see for instance [1, 2, 5, 6, 8, 10]). The location of zeros of the polynomials Q_n orthogonal with respect to the product (1) has been considered in them, among other questions.

It is known that Q_n has at least $n-(r+1)$ zeros with odd multiplicity in $\mathring{I}$, whenever $n \geqslant r+1$. Moreover in the following particular situations, we have:

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[1] This research was partially supported by Diputación General de Aragón P CB-12/91 and by Comisión Interministerial de Ciencia y Tecnologia (CICYT-Spain) PB93-0228-C02-02.

[2] This research was carried out while visiting the Departamento de Matemáticas, Universidad de Zaragoza, Spain.

(a) Suppose that $M_i = 0$ for $i = 1, \ldots, r-1$ and $M_0 > 0$, then whenever $n \geqslant r+1$, Q_n has at least $n-2$ zeros with odd multiplicity in $\mathring{I}$. Moreover if $c \in \partial I$ (the boundary of I), Q_n has at least $n-1$ zeros with odd multiplicity in $\mathring{I}$ (see [2, 11, 13]).

(b) When the inner product (1) is

$$\langle f, g \rangle = \int_I fg \, \mathrm{d}\mu + M_r f^{(r)}(c) g^{(r)}(c) + M_s f^{(s)}(c) g^{(s)}(c),$$

where $1 \leqslant r < s$ and $M_r, M_s > 0$, then for every $n \geqslant s+1$, Q_n has at least $n-2$ zeros with odd multiplicity in $\mathring{I}$ (see [3]).

These last two results seem to suggest that the number of zeros of Q_n in $\mathring{I}$ does not depend on the order of the derivatives in (1) but on the number of terms in the discrete part of the inner product.

In Section 2, we prove that this conjecture is true. Furthermore, we shall prove that the coefficients M_i may well be negative numbers, although in this case the product ceases to be positive definite.

In what follows we shall be concerned with the discrete Sobolev product

$$\langle f, g \rangle = \int_{S_\mu} fg \, \mathrm{d}\mu + \sum_{i=1}^{m} M_i f^{(\nu_i)}(c) g^{(\nu_i)}(c), \tag{2}$$

with μ a finite positive Borel measure whose support, S_μ, contains an infinite set of points, $S_\mu \subset \mathbb{R}$, $0 \leqslant \nu_1 < \cdots < \nu_m$ and $M_i \in \mathbb{R} \setminus \{0\}$. We will denote by Δ the convex hull of S_μ and by $\mathring{\Delta}$ the interior of Δ. We will suppose that $\Delta \neq \mathbb{R}$ and $c \in \mathbb{R} \setminus \mathring{\Delta}$.

Let $\mathbb{Z}_+$ be the set of positive integers. By Q_n, $n \in \mathbb{Z}_+$, we will denote the nth monic polynomial of least degree, not identically equal to zero, such that

$$\langle p, Q_n \rangle = 0, \quad p \in \mathscr{P}_{n-1},$$

where $\mathscr{P}_{n-1}$ denotes the linear space of all polynomials of degree $\leqslant n-1$.

Such a polynomial does exist. In fact it is deduced solving a homogeneous linear system with n equations and $n+1$ unknowns. Uniqueness follows from the minimality of the degree for the polynomial solution. If the product is positive definite then $\deg Q_n = n$ and thus all the Q_n's are distinct. In general this is not so and for different values of n the same polynomial Q_n can appear.

It is easy to see that the sequence (Q_n) is quasi-orthogonal of order $d = \nu_m + 1$ on S_μ with respect to the measure $(x-c)^d \, \mathrm{d}\mu$, that is $\int_{S_\mu} PQ_n (x-c)^d \, \mathrm{d}\mu = 0$ for every polynomial P with $\deg P \leqslant n-d-1$.

In the sequel, for every $n \in \mathbb{Z}_+$, $\bar{n}$ denotes the number of terms in the discrete part of the product (2) whose order of derivative is less than n.

The main results of this paper are given in the next two theorems, which will be proved in Sections 2 and 3 (see Theorems 2.2 and 3.3, respectively).

Theorem. *For every $n \in \mathbb{Z}_+$, Q_n has at least $n - \bar{n}$ changes of sign in the interior of the convex hull of the support of the measure μ.*

(2) Another interesting question is that connected with the interlacing property of the zeros of such orthogonal polynomials. When there is no discrete part we have the classical definition of orthogonality and all the zeros of Q_{n+1} interlace with those of Q_n. For Sobolev-type inner products with $M_i \geqslant 0$, the polynomials Q_n and Q_{n+1} can have common zeros (see [1]). If the coefficients M_i are allowed to be real numbers, it is easy to see that it may occur that $Q_n \equiv Q_{n+1}$.

In Section 3, we give an estimate of the number of consecutive zeros of Q_n which have in between a zero of Q_{n+1} (for a particular product (2), a partial result appears in [14]).

Let $(x_{nh})_{h=1}^{N_n}$ be the points in $\mathring{\Delta}$ where Q_n changes sign. Let κ_n be the number of intervals $I_{nh} = (x_{nh}, x_{n,h+1})$, $h = 1, \ldots, N_n - 1$, containing at least one point where Q_{n+1} changes sign, then

Theorem. *For n such that $2\nu_{\bar{n}} + 3 \leqslant n < \nu_{1+\bar{n}}$, then one of two cases occurs:*
(a) $\kappa_n \geqslant n - 2\nu_{\bar{n}} - 3$, *or*
(b) Q_n *and* Q_{n+1} *have at least* $[\frac{1}{2}(n + 1 - \nu_{\bar{n}} + N_{n+1})]$ *common zeros in* $\mathring{\Delta}$.

2. Location of zeros

We assume the conditions imposed above and we are going to obtain a lower bound for the number of zeros of Q_n with odd multiplicity located in $\mathring{\Delta}$.

Lemma 2.1. *Let Q be a polynomial whose zeros are located in an interval $I \subset \mathbb{R}$ $(I \neq \mathbb{R})$ and $c \in \mathbb{R}\backslash I$. Given $(\nu_i)_{i=1}^{k} \subset \mathbb{Z}_+ \cup \{0\}$ such that $0 \leqslant \nu_1 < \nu_2 < \cdots < \nu_k$, if $\deg Q > \nu_k - k$ there exists a polynomial φ with $\deg \varphi = k$ such that*

$$(Q\varphi)^{(\nu_i)}(c) = 0, \quad i = 1, \ldots, k, \tag{3}$$

holds. Moreover, all the zeros of φ are out of $\mathring{I}$ (the interior of I).

Proof. First of all, note that such a polynomial φ exists; it is the solution of a system of k homogeneous linear equations with $k + 1$ unknowns (the coefficients of φ). Furthermore, φ is not identically zero and $\deg \varphi \leqslant k$.

Suppose that $\deg \varphi = r \leqslant k - 1$. If we denote $n = \deg Q$, the polynomial $Q\varphi$ has at least n zeros in I, then by Rolle's theorem $(Q\varphi)^{(\nu_1)}$ has at least $n - \nu_1$ zeros in I and one extra zero in c, because of (3). Therefore $(Q\varphi)^{(\nu_1)}$ has at least $n - \nu_1 + 1$ zeros in the convex hull of $I \cup \{c\}$, $\operatorname{co}(I \cup \{c\})$.

Now we proceed by induction. As $(Q\varphi)^{(\nu_{r+1})}(x) = [(Q\varphi)^{(\nu_r)}]^{(\nu_{r+1} - \nu_r)}(x)$, again by Rolle's theorem we have that $(Q\varphi)^{(\nu_{r+1})}$ has at least $n + r - \nu_r - (\nu_{r+1} - \nu_r) = n - \nu_{r+1} + r$ zeros in $\operatorname{co}(I \cup \{c\})$ and one extra zero in c because of (3); that is, $(Q\varphi)^{(\nu_{r+1})}$ has at least $n - \nu_{r+1} + (r + 1)$ zeros in $\mathbb{R}$, which contradicts the fact that $\deg(Q\varphi)^{(\nu_{r+1})} = n + r - \nu_{r+1}$. (Notice that, since $Q\varphi \not\equiv 0$ and $n \geqslant \nu_k - k + 1 \geqslant \nu_{r+1} - r$, we have $(Q\varphi)^{(\nu_{r+1})} \not\equiv 0$.) Therefore we deduce that $\deg \varphi = k$.

If φ has at least one zero in $\mathring{I}$, $(Q\varphi)^{(\nu_1)}$ has at least $n + 2 - \nu_1$ zeros in $\operatorname{co}(I \cup \{c\})$. Repeating the same argument as above it follows that $(Q\varphi)^{(\nu_k)}$ is a polynomial not identically zero with degree $n + k - \nu_k$ and at least $n + k + 1 - \nu_k$ zeros in $\mathbb{R}$; hence all the zeros of φ are out of $\mathring{I}$. $\square$

Remark. The same conclusion as in the preceding lemma is true if c belongs to the boundary of I, $\nu_1 > 0$ and c is at most a simple zero of Q.

Theorem 2.2. *Let (Q_n) be the sequence of monic orthogonal polynomials with respect to the product (2). Then the polynomial Q_n, for each $n \in \mathbb{Z}_+$, has at least $n - \bar{n}$ changes of sign in $\mathring{\Delta}$, where $\bar{n}$ is the number of terms in the discrete part of the product whose order of derivative is less than n.*

Proof. The result of the theorem is derived from the following:

Claim. *If $\nu_j + 1 \leqslant n \leqslant \nu_{j+1}$, $j = 1, \dots, m$, then Q_n has at least $n - j$ changes of sign in $\mathring{\Delta}$.*

Proof of the Claim. For $n = j$ the result is trivial, so we assume $n \geqslant j + 1$. Suppose that Q_n has L changes of sign in $\mathring{\Delta}$ with $L \leqslant n - j - 1$.

If $c \in \mathbb{R}\backslash\Delta$, we can define a polynomial Q such that

$$\begin{cases} \deg Q = n - j - 1 \text{ and all the zeros of } Q \text{ belong to } \Delta, \\ QQ_n \text{ does not change sign in } \mathring{\Delta}. \end{cases} \tag{4}$$

Indeed, we take Q the polynomial with a simple root at each one of the L points where Q_n changes its sign in Δ and one zero of multiplicity $n - j - 1 - L$ at one of the (finite) endpoints of the interval Δ.

For $\nu_j + 1 \leqslant n \leqslant \nu_{j+1}$, from (2), we get that

$$0 = \int_{S_\mu} pQ_n \,\mathrm{d}\mu + \sum_{i=1}^{j} M_i \, p^{(\nu_i)}(c) \, Q_n^{(\nu_i)}(c) \tag{5}$$

holds for every $p \in \mathscr{P}_{n-1}$.

Now, we have to consider several cases according to whether the orders of derivatives are consecutive or not.

(a) *Case* $n = \nu_j + 1$. (i) Let $n = \nu_j + 1 = \nu_{j-1} + 2 = \cdots = \nu_1 + j$. Since S_μ contains an infinite set, putting in (5) $p = Q$ and taking into account (4) we have a contradiction and the result follows. Notice that this is the only situation for $j = 1$. If $j > 1$, then $j - 1$ more situations can occur.

(ii) Let $n = \nu_j + 1 = \nu_{j-1} + 2 = \cdots = \nu_l + j + 1 - l > \nu_{l-1} + j + 2 - l \geqslant \cdots \geqslant \nu_1 + j$, with $l = 2, \dots, j$.

By applying Lemma 2.1 with $k = l - 1$, as $\deg Q = n - j - 1 > \nu_{l-1} - (l - 1)$, there exists a polynomial φ, with $\deg \varphi = l - 1$, which satisfies $(\varphi Q)^{(\nu_i)}(c) = 0$, $i = 1, \dots, l - 1$, and with no zeros in $\mathring{\Delta}$. Then, $\deg \varphi Q = n - j - 1 + l - 1 = \nu_l - 1$, hence taking $p = \varphi Q$, (5) leads to a contradiction, because $\varphi Q Q_n$ has constant sign in S_μ.

Notice that the claim has already been proved whenever $\nu_j + 1 = \nu_{j+1}$. It remains to consider:

(b) *Case* $n > \nu_j + 1$. Since $\deg Q = n - j - 1 > \nu_j - j$, again by applying the lemma with $k = j$ and taking $p = \varphi Q$ in (5), the claim follows.

Eventually if $c \in \partial\Delta$, we construct a polynomial Q satisfying (4) and such that it has at most a simple zero at c. If $\nu_1 \neq 0$, we proceed as before, by using the remark instead of the lemma. When $\nu_1 = 0$, one has to distinguish the case $Q(c) \neq 0$, which is deduced in the same way as when $c \in \mathbb{R}\backslash\Delta$, from the case $Q(c) = 0$, where the remark should be applied only for $i = 2, \dots, k$. (Observe that if $\nu_1 = 0$ and $M_1 > 0$, it can be deduced that Q_n has at least $n - j + 1$ changes of sign in $\mathring{\Delta}$.)

Thus the claim is proved and we are ready to deduce the theorem. □

Denote $\nu_0 = 0$ and $\nu_{m+1} = +\infty$. If $\nu_0 + 1 \leqslant n \leqslant \nu_1$, then Q_n coincides with the nth monic orthogonal polynomial with respect to the product

$$\langle f, g \rangle_0 = \int_{S_\mu} fg \,\mathrm{d}\mu.$$

If $j = 1, \dots, m$ and $v_j + 1 \leqslant n \leqslant v_{j+1}$, Q_n coincides with the nth monic orthogonal polynomial with respect to the product

$$\langle f, g \rangle_j = \int_{S_\mu} fg \, d\mu + \sum_{i=1}^{j} M_i f^{(v_i)}(c) g^{(v_i)}(c). \tag{6}$$

Now it suffices to apply the claim and the theorem follows. $\square$

Remark. From the preceding theorem and Theorem 4 in [9], it follows that, for n large enough and special types of measures for which ratio asymptotics of the sequence (Q_n) can be obtained (for instance, measures such that $\mu' > 0$ a.e. in Δ), there are precisely $n - m$ simple zeros of Q_n in $\mathring{\Delta}$ while the m remaining zeros are attracted by the point c.

Now, using that (Q_n) is quasi-orthogonal of order d with respect to the measure $(x - c)^d \, d\mu$, we give another result about the location of the zeros in $\mathring{\Delta}$.

Let C_α denote the open connected components of $\mathring{\Delta} \setminus S_\mu$ and we write $[x]$ for the integer part of x.

Proposition 2.3. (a) *The number of zeros of the polynomial Q_n ($n > d$) located in each component C_α is less than or equal to either $d + 1$ or d, whenever d is even or odd, respectively. Moreover, if a component C_α has the maximum number of zeros, the remaining zeros are simple.*

(b) *Let j be a positive integer, $j > 1$. If j is even (respectively odd), there are at most $[(d + 1)/j]$ components C_α (respectively $[(d + 1)/(j - 1)]$), each one containing at least j zeros. (Notice that there are at most $[\frac{1}{2}(d + 1)]$ components C_α, each one containing more than one zero of Q_n.)*

Proof. (a) Suppose d is even and let $C_\alpha = (a_\alpha, b_\alpha)$ be a component with r zeros of Q_n ($r \geqslant d + 2$). We can construct a polynomial Q such that QQ_n does not change sign on S_μ; indeed, if r is even, it suffices to take Q the polynomial with the rest of the roots of Q_n and if r is odd, we add to Q one more zero taken among the rest of the zeros of Q_n in C_α. So $\int_{S_\mu} QQ_n(x - c)^d \, d\mu \neq 0$ and, by quasi-orthogonality, we have $\deg Q > n - d - 1$ which leads to a contradiction. The case d odd can be proved in a similar way.

Besides, if a component C_α has the maximum number of zeros, then by an argument of quasi-orthogonality, it follows that Q_n has at least $n - (d + 1)$ changes of sign in $\mathring{\Delta} \setminus C_\alpha$. Since Q_n has at most $n - d$ zeros in $\mathbb{R} \setminus C_\alpha$, the remaining zeros are simple.

(b) Let C_α ($\alpha = 1, \dots, k$) be the components each one containing precisely j zeros of Q_n ($j > 1$). There is a polynomial Q such that QQ_n does not change sign on S_μ with

$$\deg Q \leqslant \begin{cases} n - kj & \text{if } j \text{ even,} \\ n - k(j - 1) & \text{if } j \text{ odd.} \end{cases}$$

Now, using again the quasi-orthogonality of the sequence (Q_n) we have $k < (d + 1)/j$ for j even and $k < (d + 1)/(j - 1)$ for j odd and the result follows. $\square$

3. Interlacing properties of the zeros

The separation property of the zeros of standard orthogonal polynomials can be deduced from the Gauss–Jacobi quadrature formula (for example, see [4, Theorem 6.2, p. 34]). We will use this technique to study this property for Sobolev-type orthogonal polynomials.

Let $x_{n1}, \dots, x_{nN_n}$ be the points where Q_n changes sign in $\mathring{\Delta}$; so because of Theorem 2.2, we have $N_n \geqslant n - \bar{n} \geqslant n - d$ $(d = v_m + 1)$. The polynomial Q_n can be represented in the form $Q_n = Q_{n1} Q_{n2}$ where Q_{n1} has simple zeros at $(x_{nk})_{k=1}^{N_n}$ with $\deg Q_{n1} = N_n$ and the sign of Q_{n2} is constant in $\mathring{\Delta}$; hence $\deg Q_{n2} \leqslant n - N_n \leqslant d$. We can suppose, without loss of generality, that $Q_{n2}(x)(x-c)^d \,\mathrm{d}\mu$ is a positive measure. Next, we study the separation of the zeros for the polynomials Q_{n1}.

By using the quasi-orthogonality of (Q_n) with respect to the measure $(x-c)^d \,\mathrm{d}\mu$ it is easy to obtain the following analog of the Gauss–Jacobi quadrature formula (see [7, Lemma 3]).

Lemma 3.1. *For every $n > d$ and every polynomial P with $\deg P \leqslant n - d + N_n - 1$ the formula*

$$\int_{S_\mu} P Q_{n2}(x-c)^d \,\mathrm{d}\mu = \sum_{k=1}^{N_n} \lambda_{nk} P(x_{nk}) \tag{7}$$

with

$$\lambda_{nk} = \int_{S_\mu} \frac{Q_n(x)}{Q'_{n1}(x_{nk})(x - x_{nk})}(x-c)^d \,\mathrm{d}\mu$$

holds.

Proof. Let P be an arbitrary polynomial with $\deg P \leqslant n - d + N_n - 1$ and denote by L the Lagrange polynomial interpolating P at the points $x_{n1}, \dots, x_{nN_n}$ $(\deg L < N_n)$. Then, $P - L = Q_{n1} q$ where $\deg q \leqslant n - d - 1$. Integrating with respect to the measure $Q_{n2}(x)(x-c)^d \,\mathrm{d}\mu$, because of the quasi-orthogonality of the sequence (Q_n), the result follows. $\square$

Note that formula (7) is true whenever $\deg P \leqslant 2(n-d) - 1$.

Lemma 3.2. *For every $n > d$, the number of positive coefficients in formula (7) is greater than or equal to $[\frac{1}{2}(n - d + N_n + 1)]$.*

Proof. Suppose that the number of positive coefficients λ_{nk}, $k = 1, \dots, N_n$, is less than or equal to $[\frac{1}{2}(n - d + N_n + 1)] - 1$. Let $P(x) = \prod^+ (x - x_{nk})^2$, where $\prod^+$ denotes the product over all indices k for which $\lambda_{nk} > 0$. Since $\deg P \leqslant n - d + N_n - 1$, formula (7) applied to P leads to a contradiction. $\square$

Remark. Note that the number of nonpositive coefficients in (7) is less than or equal to $[\frac{1}{2}(N_n + d - n)] \leqslant \frac{1}{2}d$.

Concerning the number of positive coefficients in a mechanical quadrature formula, see also [7, 15]. More recent references related to this subject are [12, 16].

Now, we use the above results to deduce:

Theorem 3.3. *Let (Q_n) be a sequence of monic orthogonal polynomials with respect to the product (2). Let $(x_{nh})_{h=1}^{N_n}$ be the points in the interior of the convex hull of the support of μ where Q_n changes sign. Assume that they are indexed so that $x_{n1} < x_{n2} < \cdots < x_{n,N_n}$. By κ_n denote the number of intervals $I_{nh} = (x_{nh}, x_{n,h+1})$, $h = 1, \ldots, N_n - 1$, containing at least one point where Q_{n+1} changes sign. For n such that $2v_j + 3 \leqslant n < v_{j+1}$ $(j = 0, \ldots, m)$, one of two cases occurs:*

(a) $\kappa_n \geqslant n - 2v_j - 3$, *or*

(b) Q_n *and* Q_{n+1} *have at least* $[\frac{1}{2}(n + 1 - v_j + N_{n+1})]$ *common zeros in* $\mathring{\Delta}$.

Proof. As above, $v_0 = 0$ and $v_{m+1} = \infty$. To begin with let $v_j \leqslant n < v_{j+1}, j = 0, \ldots, m$. Obviously, for such n's, Q_{n+1} coincides with the $(n+1)$th monic orthogonal polynomial with respect to the product (6). Therefore, in regards to those indices, $d = v_j + 1$.

For $n + 1$, formula (7) adopts the form

$$\int_{S_\mu} PQ_{n+1,2}(x-c)^d \, d\mu = \sum_{k=1}^{N_{n+1}} \lambda_{n+1,k} P(x_{n+1,k}), \tag{8}$$

where P is any polynomial with

$$\deg P \leqslant n + N_{n+1} - d. \tag{9}$$

Take $P(x) = \prod^-(x - x_{n+1,h})^2 Q_n(x) q(x)$, where $\prod^-$ denotes the product over those indices h such that $\lambda_{n+1,h} \leqslant 0$ and q is a polynomial. We wish to place this P in (8).

We know that Q_n is orthogonal to all polynomials of degree $\leqslant n - d - 1$ with respect to the measure $(x-c)^d \, d\mu$. If q is a polynomial of degree $\leqslant n - 2d - 1$, then from the remark of Lemma 3.2 we have $\deg(\prod^-(x - x_{n+1,h})^2 q Q_{n+1,2}) \leqslant n - d - 1$ and hence the left-hand side of (8) is equal to 0.

In order that such a polynomial q exists ($\deg q \geqslant 0$), we must restrict our attention to those indices n such that ($d = v_j + 1$)

$$2v_j + 3 \leqslant n < v_{j+1}. \tag{10}$$

Given j, if there is no n for which such inequalities hold, we have nothing to prove and we consider a different j. Obviously, at least for $j = m$, such n's are possible. Moreover, because of (9) and the remark to Lemma 3.2, formula (8) holds for the above polynomial P whenever q is of degree $\leqslant n - 2d + 1$.

Therefore, if n satisfies (10), from (8) we obtain that

$$0 = \sum{}^+ \lambda_{n+1,k} (\prod{}^-(x_{n+1,k} - x_{n+1,h}))^2 (qQ_n)(x_{n+1,k}), \tag{11}$$

where q is any polynomial with degree $\leqslant n - 2d - 1$ and $\sum^+$ denotes the sum over those indices k such that $\lambda_{n+1,k} > 0$.

If $\kappa_n \geqslant n - 2d - 1 = n - 2v_j - 3$ we have nothing to prove. Therefore, let us assume that $\kappa_n \leqslant n - 2d - 2$. We shall construct a polynomial q, whose zeros are contained in the set of zeros of Q_{n1}, such that

$$(qQ_{n1})(x_{n+1,k}) \geqslant 0 \quad \text{for all } k. \tag{12}$$

In order to construct q, we follow the following rule. We analyze the intervals I_{nh} from $h = N_n - 1$ down to $h = 0$ where $I_{n0} = (a, x_{n1})$ and a is the left endpoint of Δ (possibly $-\infty$). If I_{nh} contains a zero of $Q_{n+1,1}$ we assign to q one zero at $x_{n,h+1}$ and move to the next interval $I_{n,h-1}$; if I_{nh} has no zero of $Q_{n+1,1}$, then neither x_{nh} nor $x_{n,h+1}$ are to be zeros of q, we skip the interval $I_{n,h-1}$ and consider next the interval $I_{n,h-2}$.

Notice that each time that I_{nh}, $h = 0, \ldots, N_n - 1$, has no zero of $Q_{n+1,1}$ we save at least one degree for q. The worst situation occurs when all the intervals I_{nh} that do not contain zeros of $Q_{n+1,1}$ are consecutive.

It is not hard to see that q satisfies $\deg q \leqslant \kappa_n + 1$ and (12). Since q divides Q_{n1} (and thus Q_n), $\lambda_{n+1,k} > 0$ and $(\prod^{-}(x_{n+1,k} - x_{n+1,h}))^2 > 0$, from (11), we conclude that $Q_n(x_{n+1,k}) = 0$ for all k.

A lower bound for the number of terms in (11) is given by Lemma 3.2 from which follows (b). With this we conclude the proof. $\square$

Notice that in (b), Q_{n+1} may be substituted by $Q_{n+1,1}$.

We wish to underline that the proof of Theorem 3.3 is based only on properties of quasi-orthogonality. Therefore, for quasi-orthogonal polynomials a version of this result is immediate.

Remark. An interesting case arises when

$$\langle f, g \rangle = \int_{S_\mu} fg \, d\mu + \sum_{i=0}^{m-1} M_i f^{(i)}(c) g^{(i)}(c),$$

where $c \in \mathbb{R} \setminus \Delta$, $M_i \in \mathbb{R} \setminus \{0\}$ and $\mu' > 0$ a.e. in Δ. From the remark to Theorem 2.2, we know that for all sufficiently large n, Q_n has exactly $n - m$ simple zeros in $\mathring{\Delta}$ and the rest are outside of Δ. Therefore, (b) in Theorem 3.3 cannot occur (notice that $d = m$ and $N_{n+1} = n + 1 - m$) and from (a) we obtain that $\kappa_n \geqslant n - 2m - 1$ for all large n.

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