Strong asymptotics of orthogonal polynomials with varying measures and Hermite–Padé approximants

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Abstract

The strong asymptotic behaviour of orthogonal polynomials with respect to a general class of varying measures is given for the case of the unit circle and the real line. These results are used to obtain certain asymptotic relations for the polynomials involved in the construction of Hermite–Padé approximants of a Nikishin system of functions.

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1. Introduction

1. Let $\{Q_n\}$, $n \in \mathbb{N}$, be a sequence of monic polynomials, $\deg Q_n = n$, and $\{\mu_n\}$ a sequence of finite positive Borel measures each of which has its support $S(\mu_n)$ contained in the real line $\mathbb{R}$. We say that the sequence of polynomials is orthogonal with respect to the (sequence of) varying measures if

$$0 = \int x^{\nu} Q_n(x)\, \mathrm{d}\mu_n(x), \quad \nu = 0, \ldots, n-1.$$

Notice that the nth polynomial only satisfies orthogonality relations with respect to the nth measure.

Under general assumptions on the measures $\{\mu_n\}$, we aim to find the strong asymptotics of the polynomials $\{Q_n\}$. In [6] one of the authors obtained similar results but for a narrower class of

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varying measures. The object of the present extension is to cover classes of varying measures which appear in Hermite–Padé approximation and could not be treated with the previous results. We use the present results to study asymptotic relations between the polynomials involved in the construction of the common denominators of the Hermite–Padé approximants corresponding to a Nikishin system of functions. Similar relations for the more general mixed Angelesco–Nikishin systems remain unknown.

2. Let us see what these strong asymptotics results look like for measures on the unit circle. Let ρ_n and ρ finite positive Borel measures on $[0, 2\pi]$. By $\rho_n \xrightarrow{*} \rho$, we denote the weak convergence of ρ_n to ρ as n tends to infinity. This means that for every continuous 2π-periodic function f on $[0, 2\pi]$

$$\lim_{n\to\infty} \int_0^{2\pi} f(\theta)\,\mathrm{d}\rho_n(\theta) = \int_0^{2\pi} f(\theta)\,\mathrm{d}\rho(\theta).$$

Unless otherwise stated, the limits of integration with respect to θ will always be 0 and 2π, thus they will not be indicated in the following.

Let $\{\mathrm{d}\rho_n\}_{n\in\mathbb{N}}$ be a sequence of finite positive Borel measures on the interval $[0, 2\pi]$ such that for each $n \in \mathbb{N}$ the support of $\mathrm{d}\rho_n$ contains an infinite set of points. By $\mathrm{d}\theta$, we denote Lebesgue's measure on $[0, 2\pi]$, and $\rho_n' = \mathrm{d}\rho_n/\mathrm{d}\theta$, the Radon–Nykodym derivative of $\mathrm{d}\rho_n$ with respect to $\mathrm{d}\theta$. By $\mathbb{N}$ (resp. $\mathbb{Z}, \mathbb{R}, \mathbb{C}$) we denote the set of natural (respectively, integer, real, complex) numbers.

Let $\{W_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials such that, for each $n \in \mathbb{N}$, W_n has degree n ($\deg W_n = n$) and all its zeros $\{w_{n,i}\}$, $1 \leqslant i \leqslant n$, lie in the closed unit disk. We assume that the indexes are taken so that if $w = 0$ is a zero of W_n of degree m then $w_{n,1} = w_{n,2} = \cdots = w_{n,m} = 0$. Set

$$\mathrm{d}\sigma_n(\theta) = \frac{\mathrm{d}\rho_n(\theta)}{|W_n(z)|^2}, \quad z = \mathrm{e}^{\mathrm{i}\theta}.$$

A certain link is needed between the measures ρ_n and the polynomials W_n.

Definition 1. Let $k \in \mathbb{Z}$ be a fixed integer. We say that $(\{\mathrm{d}\rho_n\}, \{W_n\}, k)$ is admissible on $[0, 2\pi]$ if:

(i) There exists a finite positive Borel measure ρ on $[0, 2\pi]$ such that $\rho_n \xrightarrow{*} \rho$, $n \to \infty$.

(ii) $\|\mathrm{d}\sigma_n\| = \int \mathrm{d}\sigma_n(\theta) < +\infty$, $\forall n \in \mathbb{N}$.

(iii) $\int \prod_{i=1}^{-k} |z - w_{n,i}|^{-2}\,\mathrm{d}\rho_n(\theta) \leqslant M < +\infty$, $z = \mathrm{e}^{\mathrm{i}\theta}$, $n \in \mathbb{N}$ (this condition applies only to the case when k is a negative integer).

(iv) $\lim_{n\to\infty} \sum_{i=1}^{n} (1 - |w_{n,i}|) = +\infty$.

Condition (ii) of admissibility guarantees that for each pair (n, m) of natural numbers we can construct a polynomial $\varphi_{n,m}(z) = \alpha_{n,m} z^m + \cdots$ that is uniquely determined by the relations of orthogonality

$$\frac{1}{2\pi} \int \bar{z}^j \varphi_{n,m}(z)\,\mathrm{d}\sigma_n(\theta) = 0, \quad j = 0, 1, \ldots, m-1, \; z = \mathrm{e}^{\mathrm{i}\theta},$$

$$\frac{1}{2\pi} \int |\varphi_{n,m}(z)|^2\,\mathrm{d}\sigma_n(\theta) = 1, \quad \deg \varphi_{n,m} = m, \; \alpha_{n,m} > 0.$$

Under this admissibility condition, we will obtain the next result which is an extension of Theorem 2 in [6].

Theorem 1. *Let* $(\{d\rho_n\}, \{W_n\}, k)$ *be admissible on* $[0, 2\pi]$. *Suppose that* $\log \rho' \in L^1[0, 2\pi]$ *and* $\liminf_{n\to\infty} \int \log \rho_n'(\theta)\, d\theta \geqslant \int \log \rho'(\theta)\, d\theta$. *Then*

$$\lim_{n\to\infty} \frac{\varphi_{n,n+k}(z)}{z^k W_n(z)} = S(\rho, z), \tag{1}$$

where the limit is uniform on each compact subset of $\{z\colon |z| > 1\}$ *and*

$$S(\rho, z) = \exp\left\{\frac{1}{4\pi} \int \frac{w+z}{w-z} \log \rho'(\theta)\, d(\theta)\right\}, \quad w = e^{i\theta}.$$

The paper is divided as follows. In Section 2 we prove Theorem 1 and we state some results needed for the proof of the corresponding strong asymptotics on the real line. The case of the real line is treated in Section 3. The final Section 4 is dedicated to the study of Hermite–Padé orthogonal polynomial corresponding to a Nikishin system of functions.

2. Szegő type theorem on the unit circle

The next result is an extension of a similar one which appears in [7] suitable for the type of varying weights which we are considering. In [7] the varying weight is of the form $d\rho/|W_n(z)|^2$ and (i) of Definition 1 is immediate. The proof in this more general setting may be found in [3].

Theorem 2. *Let* $(\{d\rho_n\}, \{W_n\}, k)$ *be admissible on* $[0, 2\pi]$, *then*

$$\frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2}\, d\theta \xrightarrow{\ *\ } d\rho(\theta), \quad z = e^{i\theta}. \tag{2}$$

Proof of Theorem 1. First, we notice that if we multiply W_n by a certain constant the corresponding orthonormal polynomials are multiplied by the same constant and the ratio on the left-hand side of (1) remains unaltered. Thus without loss of generality we may suppose in the following that W_n is monic. If P_n is a polynomial of degree exactly equal to n, we denote as usual $P_n^*(z) = z^n \overline{P_n}(1/z)$. It will more convenient for us to prove the equivalent relation

$$\lim_{n\to\infty} \frac{W_n^*(z)}{\varphi_{n,n+k}^*(z)} = S(\rho, z)$$

uniformly on each compact subset of $\{z\colon |z| < 1\}$. In the following, we will denote $\{z\colon |z| < 1\}$ by D.

First, let us show that

$$\left\{\frac{W_n^*(z)}{\varphi_{n,n+k}^*(z)}\right\}_{n\in\mathbb{N}}$$

is uniformly bounded on each compact subset of D. In fact, since $\varphi^*_{n,n+k}(z)$ has no zeros in D, using Cauchy's formula and Cauchy–Schwarz's inequality we obtain for $z \in D$

$$\begin{aligned}\left|\frac{W_n^*(z)}{\varphi^*_{n,n+k}(z)}\right| &\leqslant \frac{1}{2\pi}\int\left|\frac{wW_n^*(w)}{(w-z)\varphi^*_{n,n+k}(w)}\right|\,\mathrm{d}\theta \\ &\leqslant \sqrt{\frac{1}{2\pi}\int\left|\frac{w}{w-z}\right|^2\mathrm{d}\theta}\times\sqrt{\frac{1}{2\pi}\int\left|\frac{W_n^*(w)}{\varphi^*_{n,n+k}(w)}\right|^2\mathrm{d}\theta} \\ &\leqslant \frac{1}{\inf_{|w|=1}|z-w|}\sqrt{\frac{1}{2\pi}\int\left|\frac{W_n^*(w)}{\varphi^*_{n,n+k}(w)}\right|^2\mathrm{d}\theta}.\end{aligned}$$

From Theorem 2 we have that the last integral tends to $\sqrt{\rho[0,2\pi]/2\pi}$ as n goes to ∞, since

$$\left|\frac{W_n^*(w)}{\varphi^*_{n,n+k}(w)}\right| = \left|\frac{W_n(w)}{\varphi_{n,n+k}(w)}\right|, \quad |w|=1.$$

Thus, if K is a compact subset of D, for each $\varepsilon>0$ and all sufficiently large $n\in\mathbb{N}$

$$\sup_{z\in K}\left|\frac{W_n^*(z)}{\varphi^*_{n,n+k}(z)}\right| \leqslant \frac{1}{\inf_{|w|=1,z\in K}|z-w|}\sqrt{\frac{\rho[0,2\pi]}{2\pi}}+\varepsilon$$

as we needed to prove.

Take $\{W_n^*/\varphi^*_{n,n+k}\}$, $n\in\Gamma$, $\Gamma\subset\mathbb{N}$, a convergent subsequence whose limit is S_Γ. We will prove that $S_\Gamma\in H_2(D)$, $S_\Gamma\neq 0$, and

$$S_\Gamma(0)\geqslant \exp\left\{\frac{1}{4\pi}\int\log\rho'(\theta)\,\mathrm{d}\theta\right\}. \tag{3}$$

In fact,

$$\lim_{n\in\Gamma}\left|\frac{W_n^*(z)}{\varphi^*_{n,n+k}(z)}\right|^2 = |S_\Gamma(z)|^2$$

uniformly on each compact subset of D. On the other hand, for each $\varepsilon>0$ and all sufficiently large n, $0<r<1$,

$$\int_{|w|=r}\left|\frac{W_n^*(w)}{\varphi^*_{n,n+k}(w)}\right|^2\mathrm{d}\theta\leqslant\int_{|w|=1}\left|\frac{W_n^*(w)}{\varphi^*_{n,n+k}(w)}\right|^2\mathrm{d}\theta\leqslant\rho[0,2\pi]+\varepsilon.$$

Thus, taking limits, we obtain that

$$\int_{|w|=r}|S_\Gamma(w)|^2\,\mathrm{d}\theta\leqslant\rho[0,2\pi]+\varepsilon.$$

Since this inequality holds for each $0<r<1$, we have that $S_\Gamma\in H_2(D)$.

It is obvious that $W_n^* \neq 0$ in D. Hence, either $S_\Gamma \equiv 0$ or $S_\Gamma(z) \neq 0$ for all $z \in D$. But, denoting $\Phi_{n,n+k} = \alpha_{n,n+k}^{-1} \varphi_{n,n+k}$ and using Jensen's inequality, we obtain

$$\left(\frac{W_n^*(0)}{\varphi_{n,n+k}^*(0)}\right)^2 = \frac{1}{\alpha_{n,n+k}^2} = \frac{1}{2\pi}\int \left|\frac{\Phi_{n,n+k}(w)}{W_n(w)}\right|^2 \mathrm{d}\rho_n(\theta) \geqslant \frac{1}{2\pi}\int \left|\frac{\Phi_{n,n+k}^*(w)}{W_n^*(w)}\right|^2 \rho_n'(\theta)\,\mathrm{d}\theta$$
$$\geqslant \exp\left\{\frac{1}{2\pi}\int \log\left|\frac{\Phi_{n,n+k}^*(w)}{W_n^*(w)}\right|^2 \mathrm{d}\theta + \frac{1}{2\pi}\int \log \rho_n'(\theta)\,\mathrm{d}\theta\right\}$$
$$= \exp\left\{\frac{1}{2\pi}\int \log\, \rho_n'(\theta)\,\mathrm{d}\theta\right\}.$$

Therefore

$$S_\Gamma(0) = \liminf_{n\in\Gamma} \frac{W_n^*(0)}{\varphi_{n,n+k}^*(0)} \geqslant \liminf_{n\to\infty} S(\rho_n, 0) \geqslant S(\rho, 0) = \exp\left\{\frac{1}{4\pi}\int \log \rho'(\theta)\,\mathrm{d}\theta\right\} > 0,$$

where the second inequality holds by hypothesis.

Since the family is uniformly bounded on each compact subset, it is sufficient to prove that any convergent subsequence tends to $S(\rho, z)$. To this end, it is sufficient to show that

$$\mathrm{Re}(\log S_\Gamma(z)) = \log|S_\Gamma(z)| = \mathrm{Re}(\log S(\rho, z))$$
$$= \frac{1}{4\pi}\int P(z,w)\,\log\rho'(\theta)\,\mathrm{d}\theta = \mathrm{Re}(\log S(\rho,z)),$$

where $P(z, w)$ denotes Poisson's kernel.

However, using once more Jensen's inequality and Theorem 2, we get

$$|S_\Gamma(z)|^2 = \lim_{n\in\Gamma}\left|\frac{W_n^*(z)}{\varphi_{n,n+k}^*(z)}\right|^2 = \lim_{n\in\Gamma}\exp\left\{\frac{1}{2\pi}\int P(z,w)\,\log\left|\frac{W_n^*(w)}{\Phi_{n,n+k}^*(w)}\right|^2 \mathrm{d}\theta\right\}$$
$$\leqslant \lim_{n\in\Gamma}\frac{1}{2\pi}\int P(z,w)\left|\frac{W_n^*(w)}{\Phi_{n,n+k}^*(w)}\right|^2 \mathrm{d}\theta = \frac{1}{2\pi}\int P(z,w)\,\mathrm{d}\rho(\theta).$$

Taking limits in the above inequality as r tends to 1, $r<1$, $z = r\mathrm{e}^{\mathrm{i}\theta'}$, using Fatou's Theorem (see Chapters 11 and 17 in [8]) and the fact that $S_\Gamma \in H_2(D)$ we obtain

$$|S_\Gamma(\mathrm{e}^{\mathrm{i}}\theta')|^2 \leqslant \rho'(\theta'),$$

almost everywhere with respect to Lebesgue's measure on $[0, 2\pi]$.

Hence

$$\mathrm{Re}(\log S_\Gamma(z)) = \log|S_\Gamma(z)| = \frac{1}{4\pi}\int P(z,w)\log|S_\Gamma(w)|^2\mathrm{d}\theta$$
$$\leqslant \frac{1}{4\pi}\int P(z,w)\log\rho'(\theta)\,\mathrm{d}\theta = \mathrm{Re}(\log S(\rho,z)).$$

But, according to (3), we have that

$$\mathrm{Re}(\log S_\Gamma(0)) = \log|S_\Gamma(0)| \geqslant \frac{1}{4\pi}\int \log\rho'(\theta)\,\mathrm{d}\theta = \frac{1}{4\pi}\int P(0,w)\log\rho'(\theta)\,\mathrm{d}\theta.$$

Therefore, using the maximum principle for harmonic functions it follows that

$$\mathrm{Re}(\log S_\Gamma(z)) = \frac{1}{4\pi}\int P(z,w)\log\rho'(\theta)\,\mathrm{d}\theta$$

as we wanted to prove. From this, it follows that $\mathrm{Im}(\log S_\Gamma(z))$ and $\mathrm{Im}(\log S(\rho,z))$ differ in a constant value. Since $\mathrm{Im}(\log S_\Gamma(0)) = 0 = \mathrm{Im}(\log S(\rho,0))$ this constant is zero. With this we conclude the proof of Theorem 1. □

For the analogue of Theorem 1 on the real line, we need an additional result on the unit circle. When the varying measure is $\mathrm{d}\rho/|W_n(z)|^2$ its proof is given in [7]. For the type of varying measures considered here the proof appears in [3]. Therefore, we limit ourselves to its statement. First, let us introduce another definition.

Definition 2. Let $k \in \mathbb{Z}$ be a fixed integer. We say that $(\{\mathrm{d}\rho_n\},\{W_n\},k)$ is strongly admissible on $[0,2\pi]$ if it is admissible and additionally

(i) $\lim_{n\to\infty}\int|\rho_n'(\theta)-\rho'(\theta)|\,\mathrm{d}\theta = 0$,

(ii) $\rho' > 0$ almost everywhere on $[0,2\pi]$.

Theorem 3. *Let* $(\{\mathrm{d}\rho_n\},\{W_n\},k)$ *be strongly admissible on* $[0,2\pi]$, *then*

$$\lim_{n\to\infty}\Phi_{n,n+k+1}(0) = 0, \tag{4}$$

$$\lim_{n\to\infty}\frac{\Phi^*_{n,n+k}(w)}{\Phi_{n,n+k}(w)} = \lim_{n\to\infty}\frac{\varphi^*_{n,n+k}(w)}{\varphi_{n,n+k}(w)} = 0, \quad |w|>1, \tag{5}$$

where in (5) *the convergence is uniform on each compact subset of the prescribed regions.*

3. Szegő type theorem on the real line

In this section, we provide a similar result to Theorem 1 for sequences $\{\mu_n\}_{n\in\mathbb{N}}$ of finite positive Borel measures on $[-1,1]$ whose supports contain infinitely many points.

Let $\{w_{2n}\}_{n\in\mathbb{N}}$ be a sequence of polynomials with real coefficients such that, for each $n\in\mathbb{N}$: $\deg w_{2n} = i_n$, $0\leqslant i_n\leqslant 2n$; and $w_{2n}\geqslant 0$ on $[-1,1]$. If $i_n<2n$, let $x_{2n,i}=\infty$ for $1\leqslant i\leqslant 2n-i_n$; if, additionally, $i_n>0$, then $\{x_{2n,i}\}_{2n-i_n+1\leqslant i\leqslant 2n}$, denotes the set of zeros of w_{2n}. When $i_n=2n$, then $\{x_{2n,i}\}_{1\leqslant i\leqslant 2n}$, is the set of zeros of w_{2n}.

Set $\mathrm{d}\tau_n = \mathrm{d}\mu_n/w_{2n}$. If, for each $n\in\mathbb{N}$,

$$\int_{-1}^{1}\frac{\mathrm{d}\mu_n(x)}{w_{2n}(x)} < +\infty,$$

we can construct the table of polynomials $\{l_{n,m}\}_{n,m\in\mathbb{N}}$, such that $l_{n,m} = \beta_{n,m}x^m+\cdots$, $\beta_{n,m}>0$, is the mth orthonormal polynomial with respect to τ_n; that is, these polynomials are uniquely determined by having positive leading coefficients and satisfying the relations

$$\int_{-1}^{1} l_{n,k}\,l_{n,m}\,\mathrm{d}\tau_n(x) = \delta_{k,m}.$$

The limits of integration with respect to x will always be -1 and 1, thus they will not be indicated.

According to the prescribed conditions $w_{2n}(\cos\theta)$ is nonnegative for $\theta\in\mathbb{R}$, thus (see p. 3 of [9]) there exists an algebraic polynomial $W'_{2n}(w)$ of degree i_n whose zeros lie in $\{|z|\leqslant 1\}$ such that

$$w_{2n}(\cos\theta)=|W'_{2n}(e^{i\theta})|^2, \quad \theta\in[0,2\pi].$$

It is easy to see that the zeros of W'_{2n} are the points $\{1/\Psi(x_{2n,i})\}_{2n-i_n+1\leqslant i\leqslant 2n}$, where Ψ denotes the conformal mapping of $\overline{\mathbf{C}}\setminus[-1,1]$ onto $\{|z|>1\}$ such that $\Psi(\infty)=\infty$ and $\Psi'(\infty)>0$ (on $[-1,1]$ we extend Ψ continuously, considering the interval to have two sides as it is usually done). Take $W_{2n}=z_n^{2n-i_n}W'_{2n}$; then, $\deg W_{2n}=2n$ and

$$w_{2n}(\cos\theta)=|W_{2n}(e^{i\theta})|^2, \quad \theta\in[0,2\pi].$$

The polynomials $l_{n,m}$ are closely related to the polynomials $\varphi_{2n,2m}$ orthonormal with respect to the measure σ_{2n} defined by

$$d\sigma_{2n}(\theta)=d\tau_n(\cos\theta)=\frac{d\mu_n(\cos\theta)}{|W_{2n}(z)|^2}, \quad z=e^{i\theta}.$$

That is, $d\sigma_{2n}(E)=d\tau_n(\{\cos\theta;\ \theta\in E\})$ whenever $E\subset[0,\pi]$ or $E\subset[\pi,2\pi]$. Thus, writting $\sigma_{2n}(\theta)=d\sigma_{2n}(\{0\leqslant t\leqslant\theta\})$, we have

$$\sigma_{2n}(\theta)=\begin{cases} G_n(\cos\theta), & 0\leqslant\theta\leqslant\pi,\\ -G_n(\cos\theta), & \pi\leqslant\theta\leqslant 2\pi,\end{cases}$$

where $G_n(x)=\int_{-1}^{x}d\tau_n(t)$, $x\in[-1,1]$, at every point θ where σ_{2n} is continuous; and so, almost everywhere in $[0,2\pi]$. Furthermore,

$$\sigma'_{2n}(\theta)=|\sin\theta|\,G_n(\cos\theta)=|\sin\theta|\,\frac{\mu'_n(\cos\theta)}{|W_{2n}(e^{i\theta})|^2}=|\sin\theta|\,\tau'_n(\cos\theta),$$

whenever either side exists (thus almost everywhere). Notice that $\sigma'>0$ almost everywhere if $\tau'>0$ almost everywhere, where

$$\sigma(\theta)=\begin{cases} \tau(\cos\theta), & 0\leqslant\theta\leqslant\pi\,,\\ -\tau(\cos\theta), & \pi\leqslant\theta\leqslant 2\pi\,.\end{cases}$$

For n fixed, we can use the well-known formula (see Theorem V.1.4 of [4])

$$l_{n,m}(x)=\frac{\varphi_{2n,2m}(z)+\varphi^*_{2n,2m}(z)}{z^m\sqrt{2\pi(1+\Phi_{2n,2m}(0))}}, \tag{6}$$

where $\Phi_{2n,2m}=\varphi_{2n,2m}/\alpha_{2n,2m}$ and $x=\frac{1}{2}(z+1/z)$.

We denote $d\rho_n(\theta)=d\mu_n(\cos\theta)$ and $d\rho(\theta)=d\mu(\cos\theta)$ as above. Notice that, since $\log|\sin(\theta)|\in L^1[0,2\pi]$, $\log(\rho'(\theta))$ belongs to $L^1[0,2\pi]$ whenever $\log(\mu'(x))$ belongs to $L^1[-1,1]$

Definition 3. Let $k\in\mathbb{Z}$ be fixed, we say that $(\{\mu_n\},\{w_{2n}\},\ k)$ is strongly admissible on the interval $[-1,1]$ if $(\{\rho_n\},\ \{W_{2n}\},k)$ is strongly admissible on $[0,2\pi]$.

From the construction above, it is easy to see that this reduces to $(\{\mu_n\}, \{w_{2n}\}, k)$ satisfying

(I) There exists a finite positive Borel measure μ on $[-1,1]$ such that $\mu_n \xrightarrow{*} \mu$, $n \to \infty$, and

$$\lim_{n\to\infty} \int |\mu_n' - \mu'| \, dx = 0.$$

(II) $\mu' > 0$ almost everywhere.

(III) $\|d\tau_n\| = \int d\tau_n(x) < +\infty$, $n \in \mathbb{N}$.

(IV) $\int \prod_{i=1}^{-k} |1 - (x/x_{2n,i})|^{-1} \, d\mu_n(x) \leqslant M < +\infty$, $n \in \mathbb{N}$, where $x/x_{2n,i} \equiv 0$ if $x_{2n,i} = \infty$ (this condition applies only to the case when k is a negative integer).

(V) $\lim_{n\to\infty} \sum_{i=1}^{2n} \left(1 - \dfrac{1}{|\Psi(x_{2n,i})|}\right) = +\infty$.

Under these conditions one gets

Theorem 4. *Let $(\{\mu_n\}, \{w_{2n}\}, 2k-1)$ be strongly admissible on the interval $[-1,1]$. Suppose that $\log \mu' \in L^1[-1,1]$ and $\liminf_{n\to\infty} \int \log \mu_n'(x)\, dx \geqslant \int \log \mu'(x)\, dx$. Then*

$$\lim_{n\to\infty} \frac{l_{n,n+k}^2(x)}{[\Psi(x)]^{2k}\, w_{2n}(x)} B_{2n}(x) = \frac{1}{2\pi}[S(\mu(\cos\theta), \Psi(x))]^2, \tag{7}$$

where the limit is uniform on each compact subset of $\mathbb{C} \setminus [-1,1]$ and

$$B_{2n}(x) = \prod_{i=1}^{2n} \frac{\Psi(x) - \Psi(x_{2n,i})}{1 - \overline{\Psi(x_{2n,i})}\Psi(x)}.$$

Proof. We have the equality $w_{2n}(x) = W_{2n}(z)\overline{W_{2n}(z)}$, where $x = \cos\theta$ and $z = e^{i\theta}$. This is equivalent to $w_{2n}(x) = W_{2n}(z)\overline{W_{2n}}(1/z)$. Therefore $z^{2n} w_{2n}(x) = W_{2n}(z) W_{2n}^*(z)$, $|z| = 1$, $x = \frac{1}{2}(z + (1/z))$. By analytic continuation, we have

$$z^{2n} w_{2n}(x) = W_{2n}(z) W_{2n}^*(z), \qquad x = \tfrac{1}{2}(z + (1/z)), \quad z \in \mathbb{C}.$$

Using this relation and (6), we obtain

$$\frac{l_{n,n+k}^2(x)}{w_{2n}(x)} = \frac{\varphi_{2n,2n+2k}^2(z)}{z^{2k} W_{2n}(z) W_{2n}^*(z)} \frac{\left(1 + \frac{\varphi_{2n,2n+2k}^*(z)}{\varphi_{2n,2n+2k}(z)}\right)^2}{2\pi(1 + \Phi_{2n,2n+2k}(0))}, \quad |z| > 1.$$

Let us rewrite this formula in the more convenient form.

$$\frac{l_{n,n+k}^2(x)}{[\Psi(x)]^{2k}\, w_{2n}(x)} \frac{W_{2n}^*(\Psi(x))}{W_{2n}(\Psi(x))} = \left(\frac{\varphi_{2n,2n+2k}(z)}{z^{2k} W_{2n}(z)}\right)^2 \frac{\left(1 + \frac{\varphi_{2n,2n+2k}^*(z)}{\varphi_{2n,2n+2k}(z)}\right)^2}{2\pi(1 + \Phi_{2n,2n+2k}(0))}, \quad |z| > 1. \tag{8}$$

Notice that

$$W_{2n}(\Psi(x)) = \prod_{i=1}^{2n} \left(\Psi(x) - \frac{1}{\Psi(x_{2n,i})}\right),$$

Therefore,

$$\frac{W_{2n}^*(\Psi(x))}{W_{2n}(\Psi(x))} = B_{2n}(x),$$

where $(\Psi(x) - \Psi(x_{2n,i}))/(1 - \overline{\Psi(x_{2n,i})}\Psi(x)) \equiv 1/\Psi(x)$ if $x_{2n,i} = \infty$. Taking limits in (8) as n tends to ∞, using (1), (4), and (5) we arrive to (7). The proof is complete. □

4. Application

Let us explain the connection between the systems of orthogonal polynomials with respect to varying measures and Hermite–Padé approximation of Nikishin systems of functions. In [1–5], the more general construction of mixed Angelesco–Nikishin systems (or generalized Nikishin systems) is studied and the nth root asymptotic behavior of their Hermite–Padé approximants is given. We adopt the notation introduced in [5] and use some of the new orthogonality relations which those authors have revealed.

Let F_1 and F_2 be two nonintersecting segments of the real line, σ_1 and σ_2 two finite positive Borel measures such that $S(\sigma_1) \subset F_1$, $S(\sigma_2) \subset F_2$. We define a new measure $\langle \sigma_1, \sigma_2 \rangle$

$$d\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{d\sigma_2(t)}{x - t} \, d\sigma_1(x) = \hat{\sigma}_2(x) \, d\sigma_1(x).$$

This measure $\langle \sigma_1, \sigma_2 \rangle$, obviously has constant sign on its support F_1.

For a system of segments $F_1, F_2, \ldots, F_m$, such that $F_k \cap F_{k+1} = \emptyset$, $k = 1, 2, \ldots, m-1$, and finite, positive Borel measures $\sigma_1, \sigma_2, \ldots, \sigma_m$, $S(\sigma_k) \subset F_k$, $k = 1, 2, \ldots, m$, we define inductively the measures

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_{k+1} \rangle = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_{k+1} \rangle \rangle, \quad k = 2, \ldots, m-1.$$

Thus, on F_1, we have defined m finite Borel measures each one with constant sign. Set

$$s_1 = \langle \sigma_1 \rangle = \sigma_1, \quad s_2 = \langle \sigma_1, \sigma_2 \rangle, \ldots, s_m = \langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle.$$

The system of functions $(f_1, f_2, \ldots, f_m)$, where

$$f_k(z) = \hat{s}_k(z) = \int \frac{ds_k(x)}{z - x}, \quad k = 1, \ldots, m$$

defines what is called a Nikishin system.

Consider a multi-index $n = (n_1, n_2, \ldots, n_m) \in \mathbb{N}^m$. There exists a polynomials Q_n that satisfies the conditions

$$\begin{aligned} &Q_n \not\equiv 0, \quad \deg Q_n \leqslant |n| = n_1 + \cdots + n_m \\ &(Q_n f_k - P_{n,k})(z) = O\left(\frac{1}{z^{n_k+1}}\right), \quad z \to \infty, \ k = 1, \ldots, m, \end{aligned} \tag{9}$$

where $P_{n,k}$ is a polynomial. The rational functions

$$R_{n,k} = \frac{P_{n,k}}{Q_n}, \quad k = 1, \ldots, m,$$

constitute the Hermite–Padé approximants (or simultaneous Padé approximants) of the system $(f_1, \ldots, f_m)$ relative to the multi-index n. From (9), it follows that

$$0 = \int x^v Q_n(x) \, ds_k(x), \quad v = 0, \ldots, n_{k-1}, \ k = 1, \ldots, m. \tag{10}$$

In the sequel, we assume that the multi-index $n=(n_1,\dots,n_m)$ satisfies

$$j\leqslant k \;\Rightarrow\; n_k\leqslant n_j+1. \tag{11}$$

For each multi-index n (with property (11)), we define inductively the following functions:

$$\Psi_{n,0}(z)=Q_n(z), \qquad \Psi_{n,k}(z)=\int\frac{\Psi_{n,k-1}(x)}{z-x}\,\mathrm{d}\sigma_k(x), \quad k=1,\dots,m. \tag{12}$$

For each $j=0,\dots,m-1$ and $k=j+1,\dots,m$, we define the measure

$$s_k^j=\langle\sigma_{j+1},\dots,\sigma_k\rangle.$$

We have

Lemma 1. *For each $j=0,\dots,m-1$, the functions $\Psi_{n,j}$ satisfy*

$$0=\int x^{\nu}\Psi_{n,j}\,\mathrm{d}s_k^j(x), \quad \nu=0,\dots,n_{k-1},\; k=j+1,\dots,m. \tag{13}$$

For $j=0$, (13) coincides with (10). The proof may be carried out by induction showing that if the statement is true for $j\in\{0,\dots,m-2\}$ then it also holds for $j+1$ (for details see Proposition 1 in [5]).

Taking $k=j+1$, $s_{j+1}^j=\sigma_{j+1}$ and (13) indicates that

$$0=\int x^{\nu}\Psi_{n,j}(x)\,\mathrm{d}\sigma_{j+1}(x)\,, \quad \nu=0,\dots,n_{j+1}-1,\; j=0,\dots,m-1. \tag{14}$$

From (14), it follows that $\Psi_{n,j}(z)$ has at least n_{j+1} changes of sign on F_{j+1}. Denote by $Q_{n,k}$ the monic polynomial whose zeros are the zeros of $\Psi_{n,k-1}$ on F_k (counting their multiplicities). According to (14), $\deg Q_{n,k}\geqslant n_k$. Denote $Q_{n,m+1}\equiv 1$. Set

$$N_{n,k}=\sum_{j=k}^{m}n_j, \quad k=1,\dots,m.$$

Lemma 2. *For $k=1,\dots,m$*

$$0=\int x^{\nu}\Psi_{n,k-1}(x)\frac{\mathrm{d}\sigma_k(x)}{Q_{n,k+1}(x)}, \quad \nu=0,\dots,N_{n,k}-1. \tag{15}$$

For $k=m$, (15) reduces to (14) with $j=m-1$. For the rest of the indicated values of k, the formula may be proved by induction for decreasing values of the index k (for details, see Proposition 2 in [5]).

Using (15), we have that $\deg Q_{n,k}\geqslant N_{n,k}$. From (12), (15), and Cauchy's integral formula, it is easy to deduce that if for some k, $\deg Q_{n,k}>N_{n,k}$, then $\deg Q_{n,k-1}>N_{n,k-1}$. Since $\Psi_{n,0}=Q_n$, and $\deg Q_n\leqslant|n|=N_{n,1}$, we obtain (for details, see Proposition 3 in [5])

Lemma 3. *For each $k=1,\dots,m$, the polynomial $Q_{n,k}$ has exactly $N_{n,k}$ simple zeros on the interval F_k and $\deg Q_{n,k}=N_{n,k}$. In particular, $Q_{n,1}=Q_n$.*

Set $Q_{n,0} \equiv 1$. For each $k = 1, \ldots, m$, (15) may be rewritten as follows:

$$0 = \int x^{\nu} Q_{n,k}(x) \left| \frac{Q_{n,k-1}(x)\Psi_{n,k-1}(x)}{Q_{n,k}(x)} \right| \frac{d\sigma_k(x)}{|Q_{n,k-1}(x)Q_{n,k+1}(x)|}, \quad \nu = 0, \ldots, N_{n,k} - 1.$$

Denote

$$K_{n,k} = \left(\int Q_{n,k}^2(x) \left| \frac{Q_{n,k-1}(x)\Psi_{n,k-1}(x)}{Q_{n,k}(x)} \right| \frac{d\sigma_k(x)}{|Q_{n,k-1}(x)Q_{n,k+1}(x)|} \right)^{-1/2}, \quad k = 1, \ldots, m.$$

Take

$$K_{n,0} = 1, \quad \kappa_{n,k} = \frac{K_{n,k}}{K_{n,k-1}}, \quad k = 1, \ldots, m.$$

Define

$$q_{n,k} = \kappa_{n,k} Q_{n,k}, \quad F_{n,k}(z) = K_{n,k-1}^2 \left| \frac{Q_{n,k-1}(z)\Psi_{n,k-1}(z)}{Q_{n,k}(z)} \right|, \quad k = 1, \ldots, m.$$

With this notation, $q_{n,k}$ is orthonormal with respect to the varying measure

$$\frac{F_{n,k}(x)\, d\sigma_k(x)}{|Q_{n,k-1}(x)Q_{n,k+1}(x)|}. \tag{16}$$

In Theorem 5 there is no need that the index n cover the whole sequence of natural numbers. If the conditions of strong admissibility are satisfied for $n \in \Lambda \subset \mathbb{N}$ then the statement is valid taking limit as $n \to \infty$ for $n \in \Lambda$. Next, notice that $\deg w_{2n} \leqslant 2n$. Thus, the degree of the orthonormal polynomials can increase to infinity much faster than $\deg w_{2n}$. Moreover, one can even take $w_{2n} \equiv 1$, for all n, so that their degrees may not tend to infinity at all. On the other hand,

$$\deg w_{2n} - 2 \deg l_{n,n+k} \leqslant 2n - 2(n+k) \leqslant 2|k|,$$

where k is a fixed integer, so as n tends to infinity this difference of degrees must remain bounded above.

In addition, it is easy to see that Theorem 4 remains valid if for each $n \in \Lambda$, the sign of w_{2n} on the interval of orthogonality $F = [a, b]$ is fixed (positive or negative depending on n). The only change in the expression (7) is that in the left hand one must add a new factor $\lambda_n = \pm 1$ depending on the sign of $w_{2n}(x)$. In order to consider this fact in the next theorem we define the numbers $\lambda_{n,k}$ as $+1$ if $Q_{n,k-1}(x)Q_{n,k+1}(x)$ is positive on F_k and -1 if negative.

In [3] it was proved.

Lemma 4. *Let Λ be a sequence of multi-indexes such that* (11) *takes place, $n_{k-1} - n_k \leqslant C$, $k = 2, \ldots, m$, where C is a constant independent of $n \in \Lambda$, and $n_1 \to \infty$ as n varies over Λ. Assume that for each $k = 1, \ldots, m$, $\sigma_k' > 0$ almost everywhere on F_k. Then, for each $k = 1, \ldots, m$*

$$\lim_{n \in \Lambda} F_{n,k}(x) = f_k(x),$$

uniformly on F_k, where

$$f_1 \equiv 1, \quad f_k(x) = \frac{1}{\sqrt{|(x - b_{k-1})(x - a_{k-1})|}}, \quad k = 2, \ldots, m.$$

For $k=1,\ldots,m$, we denote Ψ_k the conformal mapping of $\overline{\mathbb{C}}\setminus F_k$ onto $\{|z|>1\}$ such that $\Psi_k(\infty)=\infty$ and $\Psi_k'(\infty)>0$. Finally, $M_{n,k}$ denotes $\deg Q_{n,k-1}Q_{n,k+1}$ and

$$Q_{n,k-1}(x)\,Q_{n,k+1}(x)=\prod_{i=1}^{M_{n,k}}(x-x_{n,k,i}).$$

Now, we are ready for

Theorem 5. *Let Λ be a sequence of multi-indexes such that* (11) *takes place, $n_{k-1}-n_k\leqslant C$, $k=2,\ldots,m$, where C is a constant independent of $n\in\Lambda$, and $n_1\to\infty$ as n varies over Λ. Assume that for each $k=1,\ldots,m$, $\log(\sigma_k')\in L^1(F_k)$. Then, for each $k=1,\ldots,m$*

$$\lim_{n\in\Lambda}\frac{Q_{n,k}(x)Q_{n,m}(x)}{Q_{n,k-1}(x)}\,\frac{K_{n,m}^2}{K_{n,k-1}^2}\prod_{j=k}^{m}A_{n,j}(x)=\prod_{j=k}^{m}\frac{1}{2\pi}[S(f_j\sigma_j',\Psi_j(x))]^2, \tag{17}$$

where the limit is uniform on each compact subset of $\mathbb{C}\setminus(\bigcup_{j=k}^{m}F_j)$ and

$$A_{n,k}(x)=\lambda_{n,k}\,\Psi_k(x)^{n_{k-1}-n_k}\prod_{i=1}^{M_{n,k}}\frac{\Psi_k(x)-\Psi_k(x_{n,k,i})}{1-\overline{\Psi_k(x_{n,k,i})}\Psi_k(x)}.$$

Proof. From Lemma 4 it is easy to see that the varying measures (16) verify the strong admissibility condition because the functions $F_{n,k}$ converge to f_k uniformly on F_k. On the other hand, $\log(f_k\sigma_k')\in L^1(F_k)$ since f_k is a strictly positive continuous function on the interval F_k. From this and the uniform convergence of $F_{n,k}$ to f_k, it is obtained that all the functions $F_{n,k}$ are bounded from below by a positive constant independent of $n\in\Lambda$. Hence, $\log(F_{n,k}\sigma_k')\in L^1(F_k)$ and $\log(F_{n,k})$ tends to $\log(f_k)$ uniformly on F_k as $n\in\Lambda$. Then, the condition

$$\liminf_{n\in\Lambda}\int_{F_k}\log(F_{n,k}(x)\,\sigma_k'(x))\,\mathrm{d}x\geqslant\int_{F_k}\log(f_k(x)\,\sigma_k'(x))\,\mathrm{d}x$$

also takes place.

Notice that $\deg(Q_{n,k}^2)-\deg Q_{n,k-1}\deg Q_{n,k+1}=2N_{n,k}-N_{n,k-1}-N_{n,k+1}=n_k-n_{k-1}$, $n_0=0$. The numbers n_k-n_{k-1} depend on n but they remain bounded so, it has no relevance in order to calculate the limit. Therefore, using (7) for $j=1,\ldots,m$, we obtain

$$\lim_{n\in\Lambda}\frac{Q_{n,j}^2(x)}{Q_{n,j-1}(x)Q_{n,j+1}(x)}\,\frac{K_{n,j}^2}{K_{n,j-1}^2}\,\lambda_{n,j}\,\Psi_j(x)^{n_{j-1}-n_j}B_{n,j}=\frac{1}{2\pi}[S(f_j\sigma_j',\Psi_j(x))]^2, \tag{18}$$

where the limit is uniform on each compact subset of $\mathbb{C}\setminus F_j$ and

$$B_{n,j}=\prod_{i=1}^{M_{n,j}}\frac{\Psi_j(x)-\Psi_j(x_{n,j,i})}{1-\overline{\Psi_j(x_{n,j,i})}\Psi_j(x)}.$$

If we multiply the expressions (18) for $j=k,\ldots,m$, we obtain the result stated. □

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