# The moment problem for a Sobolev inner product


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**Abstract**

The concepts of definite and determinate Sobolev moment problem are introduced. The study of these questions is reduced to the definiteness or determinacy, respectively, of a system of classical moment problems by means of a canonical decomposition of the moment matrix associated with a Sobolev inner product in terms of Hankel matrices.



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## 1. Introduction

In the past two decades, there has been a growing interest in the study of the so called Sobolev inner products and the orthogonal polynomials defined by them. Such inner products are of the form

$$\langle f, g \rangle_S = \sum_{k=0}^{d} \langle f^{(k)}, g^{(k)} \rangle_k \,, \tag{1}$$

where $d \in I\!N$ is a fixed nonnegative integer,

$$\langle f, g \rangle_k = \int_{\Sigma_k} f(x) g(x) d\mu_k(x) \,, \qquad k = 0, 1, \ldots, d,$$

and $(\mu_0, \ldots, \mu_d), \mu_d \neq 0$, is a system of positive measures whose supports satisfy

$$supp(\mu_k) \subset \Sigma_k \subset I\!R \,, \qquad k = 0, 1 \ldots, d \,.$$

For a survey on recent advances of the algebraic aspect of the theory, inner products defined with classical weights, and the so called coherent pairs of measures see [1] and [5]. Recently, some important steps have been taken in the study of the asymptotic properties of Sobolev orthogonal polynomials defined through general measures. An account on this


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matter may be found in [4]. Nevertheless, to our knowledge, the moment theory of Sobolev inner products has not been treated so far. The object of this paper is to fill this absence.

We consider the following question (in the sequel *S-moment problem*). Given

$$(M; \Sigma_0, \Sigma_1, \ldots, \Sigma_d)\,,$$

where $M = (c_{i,j})_{i,j=0}^{\infty}$ is an infinite matrix of real numbers, and $\Sigma_k$ , $k = 0, 1, \ldots, d$ , are subsets of the real line, find a system of $d+1$ positive measures $(\mu_0, \mu_1, \ldots, \mu_d)$ with $supp(\mu_k) \subset \Sigma_k$ , $k = 0, 1, \ldots, d, \mu_d \neq 0$, such that

$$c_{i,j} = \langle x^i, x^j \rangle_S \in I\!R\,, \qquad i, j = 0, 1, \ldots\,, \tag{2}$$

takes place. In the sequel, the values $c_{i,j}$ are called *S-moments*. When $\Sigma_k = I\!R$, $k = 0, 1, \ldots, d$ , we drop $\Sigma_k$ from the notation and refer to the SH-moment problem (Sobolev-Hamburger moment problem) for $M$.

**Definition 1** *We say that the S-moment problem for* $(M; \Sigma_0, \Sigma_1, \ldots, \Sigma_d)$ , *is definite if it has at least one solution. This problem is said to be determinate if the solution is unique.*

It is easy to see that when $d = 0$ a necessary and sufficient condition for the corresponding S-moment problem to be determinate is that: 1) $M$ be a Hankel matrix, and 2) that the ordinary moment problem for the sequence $\{c_i\}$ of numbers given by the first row of $M$ be determinate. In this sense, the S-moment problem extends the classical moment problem. Thus, in the sequel, we refer indistinctly to the $(M; \Sigma)$ or the $(\{c_i\}; \Sigma)$ moment problem. We reduce the study of the S-moment problem to a system of $d + 1$ classical moment problems.

Before stating the main result, let us introduce some necessary notation. Let us assume that the S-moment problem $(M; \Sigma_0, \ldots, \Sigma_d)$ has for solution the system of measures $(\mu_0, \ldots, \mu_d)$. Denote by $M = (c_{i,j})_{i,j=0}^{\infty}$ the infinite matrix whose entry at the position $(i, j)$ is the S-moment $c_{i,j}$ given by (2). Due to (1) and (2), we have

$$c_{i,j} = \sum_{k=0}^{\delta} \frac{i!}{(i-k)!} \frac{j!}{(j-k)!} \int_{\Sigma_k} x^{i+j-2k} d\mu_k(x)\,, \quad \delta = \min\{i, j, d\}\,, \quad i, j = 0, 1, \ldots\,. \tag{3}$$

Let $M^{(k)} = \left(c_i^k\right)_{i=0}^{\infty}$ be the moment matrix associated to the measure $\mu_k$ , $k = 0, 1, \ldots, d$, where

$$c_i^k = \int_{\Sigma_k} x^m d\mu_k\,.$$

$M^{(k)}$ , $k = 0, 1, \ldots,$ is a Hankel matrix. From (3), it follows that

$$c_{i,j} = \sum_{k=0}^{\delta} \frac{i!}{(i-k)!} \frac{j!}{(j-k)!} c^k_{i+j-2k}\,, \qquad i, j = 0, 1, \ldots, \tag{4}$$

with $\delta = \min\{i, j, d\}$. Set $S^k = \left(s^k_{i,j}\right)_{i,j=0}^{\infty}$, with

$$s^k_{i,j} = \begin{cases} \dfrac{i!}{(i-k)!}\,, & i - j = k\ ,\ k = 0, \ldots, d\,, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Notice that $S^k$ equals the $k$-th power of the matrix $S^1$ and, in particular, $S^0$ is the infinite identity matrix. In the sequel, $A^t$ denotes the transpose of (the finite or infinite) matrix $A$. Considering the formal product between infinite matrices (which is well defined because $S^k$ and $(S^k)^t$ have in each row and column at most one element different from zero), from (4) we have

$$M = \sum_{k=0}^{d} S^k M^{(k)} (S^k)^t \,. \tag{6}$$

This equality is the matrix form of relations (4).

Let us return to the general problem. We have

**Theorem 1** *Given $(M; \Sigma_0, \dots, \Sigma_d)$, the S-moment problem is definite (determinate) if and only if $M$ admits a decomposition of the form* (6)*, where $M^{(k)}, k = 0, \dots, d$, are Hankel matrices, $S^k$ is defined by* (5)*, and the moment problems $(M^{(k)}; \Sigma_k)$ are definite (determinate) for each $k$.*

The paper is divided as follows. In section 2, we obtain some auxiliary results. We show that a matrix $M$ admits only one decomposition of type (6) if any. Then we give some criteria which allows to determine when $M$ can be decomposed in such form. Theorem 1 is proved in section 3 and some corollaries are given which follow from the classical moment theory. Section 4 is devoted to the connection between the moment matrix $M$ and the recurrence relation which satisfy Sobolev orthogonal polynomials.

## 2. Auxiliary results

In this section, our main object is to study when an infinite matrix $M$ is decomposable in the form (6) for some set $\{M^{(k)} : k = 0, 1, \dots, d\}, M^{(d)} \neq 0$, of Hankel matrices and $S^k$ as defined above. This question has an algebraic character. We will assume that $M$ is a real symmetric matrix because obviously this is a necessary condition for (6) to take place since Hankel matrices are symmetric. The following lemma is the key to all further considerations.

**Lemma 1** *Let $M = (c_{i,j})_{i,j=0}^{\infty}$ be an infinite symmetric matrix and $M^{(k)}$, $k = 0, 1, \dots, d$, infinite Hankel matrices such that* (6) *takes place and $M^{(d)} \neq 0$. Then this is the unique decomposition of this form of $M$.*

**Proof.** Assume that $\widetilde{M}^{(0)}$, $\widetilde{M}^{(1)}$, ..., $\widetilde{M}^{(d')}$ are also infinite Hankel matrices with respect to which (6) takes place, $\widetilde{M}^{(d')} \neq 0$. We must prove that $d = d'$ and $M^{(k)} = \widetilde{M}^{(k)}$, $k = 0, \dots, d$.

For definiteness, we can assume that $d' \leq d$ (if $d \leq d'$ the proof follows the same arguments). If $d' < d$, we complete the matrices $\widetilde{M}^{(k)}$ with zero matrices for $k = d' + 1, \dots, d$.

Notice that for any matrix $A$, the first $k$ rows (and columns) of $S^k A (S^k)^t$ are identically equal to zero. Since

$$M = S^0 M^{(0)} (S^0)^t + \sum_{k=1}^{d} S^k M^{(k)} (S^k)^t = S^0 \widetilde{M}^{(0)} 0 (S^0)^t + \sum_{k=1}^{d} S^k \widetilde{M}^{(k)} (S^k)^t \tag{7}$$

and

$$M^{(0)} = S^0 M^{(0)} (S^0)^t \,, \qquad \widetilde{M}^{(0)} = S^0 \widetilde{M}^{(0)} (S^0)^t$$

it follows that the first row of $M^{(0)}$ and $\widetilde{M}^{(0)}$ coincide with the first row of $M$. Since $M^{(0)}$ and $\widetilde{M}^{(0)}$ are Hankel matrices, we obtain that $M^{(0)} = \widetilde{M}^{(0)}$. From this and (7),

$$M - S^0 M^{(0)} (S^0)^t = S^1 M^{(1)} (S^1)^t + \sum_{k=2}^{d} S^k M^{(k)} (S^k)^t = S^1 \widetilde{M}^{(1)} (S^1)^t + \sum_{k=2}^{d} S^k \widetilde{M}^{(k)} (S^k)^t \,.$$

The second row of $\sum_{k=2}^{d} S^k M^{(k)} (S^k)^t$ and $\sum_{k=2}^{d} S^k \widetilde{M}^{(k)} (S^k)^t$ are identically zero; therefore, the second row of $S^1 M^{(1)} (S^1)^t$ and $S^1 \widetilde{M}^{(1)} (S^1)^t$ coincide with the second row of $M - S^0 M^{(0)} (S^0)^t$. From this, it immediately follows that the first row of $M^{(1)}$ is identical to that of $\widetilde{M}^{(1)}$. Since these are Hankel matrices they are equal. Repeating the same arguments, we obtain that $M^{(k)} = \widetilde{M}^{(k)}$, $k = 0, \ldots, d$. Therefore, $d' < d$ is not possible because then $\widetilde{M}^{(d)} = 0 \neq M^{(d)}$. Thus $d = d'$ and consequently $M^{(k)} = \widetilde{M}^{(k)}$, $k = 0, \ldots, d$. ■

The proof of the previous lemma gives a practical method for finding the matrices $M^{(k)}$ if one knows in advance that $M$ is decomposable in the form (6). One needs some necessary and sufficient condition in terms of the elements of $M$ to determine if such a decomposition is possible. Before proving the corresponding result (Theorem 2 below), we need some auxiliary relations.

**Lemma 2** *For $i, j \in \mathbb{N}$ and $d \in \{0, 1, \ldots\}$ fixed, we have*

$$\sum_{k=\nu}^{i} \frac{(-1)^{k-\nu}}{(k-\nu)!} \frac{(i+j-\nu-k-1)!}{(i-k)!(j-k)!} = 0 \,, \qquad \nu = 0, 1, \ldots \,, \qquad j \geq i \,. \tag{8}$$

*and*

$$\sum_{k=\nu}^{d} \frac{(-1)^{d-k}}{(k-\nu)!(d-k)!} \frac{i+j-2k}{(i-k)(j-k)} \frac{(i+j-k-d-1)!}{(i+j-k-\nu)!} = \frac{(i-d-1)!(j-d-1)!}{(i-\nu)!(j-\nu)!} \,, \qquad \nu = 0, 1, \ldots, d \,, \qquad i, j > d \,. \tag{9}$$

**Proof.** Let us prove (8). Consider the polynomial

$$p(z) := (1+z)^{i-\nu-1} = (1+z)^{i+j-2\nu-1} (1 - \frac{z}{1+z})^{j-\nu} \,.$$

Taking into account the binomial formula,

$$p(z) = \sum_{n=0}^{j-\nu} (-1)^n \binom{j-\nu}{n} z^n (1+z)^{i+j-2\nu-n-1} =$$

$$\sum_{n=0}^{j-\nu} (-1)^n \binom{j-\nu}{n} \sum_{m=0}^{i+j-2\nu-n-1} \binom{i+j-2\nu-n-1}{m} z^{m+n} \,. \tag{10}$$

Since $\deg p = i - \nu - 1$, the coefficient corresponding to $z^{i-\nu}$ in (10) must be equal to zero. That is,

$$0 = \sum_{n=0}^{i-\nu} (-1)^n \binom{j-\nu}{n} \binom{i+j-2\nu-n-1}{i-\nu-n} \,.$$

Taking $n = k - \nu$, we obtain (8).

In order to prove (9), first let us consider the case when $\nu = 0$; that is,

$$\sum_{k=0}^{d} \frac{(-1)^{d-k}}{k!(d-k)!} \left( \frac{1}{i-k} + \frac{1}{j-k} \right) \frac{(i+j-k-d-1)!}{(i+j-k)!} = \frac{(i-d-1)!(j-d-1)!}{i!j!} . \tag{11}$$

Notice that

$$s_i(i+j,d) := \sum_{k=0}^{d} \frac{(-1)^{d-k}}{k!(d-k)!} \frac{1}{i-k} \frac{(i+j-k-d-1)!}{(i+j-k)!} =$$

$$\frac{1}{\Gamma^2(d+1)} \sum_{k=0}^{d} \frac{(-1)^{d-k}}{i-k} \binom{d}{k} \frac{\Gamma(i-k+j-d)\Gamma(d+1)}{\Gamma(i-k+j+1)} =$$

$$\frac{1}{\Gamma^2(d+1)} \sum_{k=0}^{d} (-1)^{d-k} \binom{d}{k} \frac{1}{i-k} \beta(i-k+j-d, d+1) , \tag{12}$$

where $\Gamma(m) = (m-1)!$, $m \in \mathbb{Z}$, and $\beta(m,n) = \Gamma(m)\Gamma(n)/\Gamma(m+n)$ are, respectively, the usual Gamma and Beta functions. Using in (12) the integral representation of the Beta function, we obtain

$$s_i(i+j,d) = \frac{1}{\Gamma^2(d+1)} \sum_{k=0}^{d} (-1)^{d-k} \binom{d}{k} \int_0^1 \frac{x^{i-k}}{i-k} x^{j-d-1} (1-x)^d dx =$$

$$\frac{1}{\Gamma^2(d+1)} \sum_{k=0}^{d} (-1)^{d-k} \binom{d}{k} \int_0^1 x^{j-d-1} (1-x)^d \int_0^x t^{i-k-1} dt dx =$$

$$\frac{1}{\Gamma^2(d+1)} \int_0^1 x^{j-d-1} (1-x)^d \int_0^x t^{i-d-1} \left( \sum_{k=0}^{d} (-1)^{d-k} \binom{d}{k} t^{d-k} \right) dt dx =$$

$$\frac{1}{\Gamma^2(d+1)} \int_0^1 x^{j-d-1} (1-x)^d \int_0^x t^{i-d-1} (1-t)^d dt dx . \tag{13}$$

Analogously, one proves that

$$s_j(i+j,d) = \frac{1}{\Gamma^2(d+1)} \int_0^1 t^{i-d-1} (1-t)^d \left[ \int_0^t x^{j-d-1} (1-x)^d dx \right] dt . \tag{14}$$

Formulas (13) and (14) (after interchanging integrals in (14)) allow us to rewrite the left hand of (11) as

$$s_i(i+j,d) + s_j(i+j,d) =$$

$$\frac{1}{\Gamma^2(d+1)} \left[ \int_0^1 x^{j-d-1} (1-x)^d dx \right] \left[ \int_0^1 t^{i-d-1} (1-t)^d dt \right] =$$

$$\frac{1}{\Gamma^2(d+1)} \beta(j-d, d+1) \beta(i-d, d+1) = \frac{\Gamma(j-d)\Gamma(i-d)}{\Gamma(j+1)\Gamma(i+1)} ,$$

which proves (9) for $\nu = 0$.

Now, let $\nu \in \{1, 2, \ldots, d\}$. The change of parameters $k = \overline{k} + \nu$, $d = \overline{d} + \nu$, $i = \overline{i} + \nu$, $j = \overline{j} + \nu$ allow to reduce the left hand of (9) to the case just studied. That is, the left hand of (9) equals

$$\sum_{\overline{k}=0}^{\overline{d}} \frac{(-1)^{\overline{d}-\overline{k}}}{\overline{k}!(\overline{d}-\overline{k})!} \frac{\overline{i}+\overline{j}-2\overline{k}}{(\overline{i}-\overline{k})(\overline{j}-\overline{k})} \frac{(\overline{i}+\overline{j}-\overline{k}-\overline{d}-1)!}{(\overline{i}+\overline{j}-\overline{k})!} =$$

$$\frac{(\overline{i}-\overline{d}-1)!(\overline{j}-\overline{d}-1)!}{\overline{i}!\overline{j}!} \quad , \quad \overline{i}, \overline{j} > \overline{d} \, ,$$

which is equivalent to the expression on the right hand of (9). With this we conclude the proof. ∎

The next result gives us conditions under which an infinite matrix $M$ verifies (6).

**Theorem 2** *Let $M = (c_{i,j})_{i,j=0}^{\infty}$ be an infinite real matrix. A necessary and sufficient condition in order that there exist infinite Hankel matrices $M^{(0)}, M^{(1)}, \ldots, M^{(d)}$ satisfying (6) is*

$$\left.\begin{array}{rcll} c_{i,j} & = & \displaystyle\sum_{k=0}^{d} \alpha_k(i,j) c_{k,i+j-k} & , \quad \forall i,j \in I\!N, \, i,j > d, \\ c_{i,j} & = & c_{j,i} & , \quad \forall i,j = 0,1,\ldots \, , \end{array}\right\} \tag{15}$$

*where*

$$\alpha_k(i,j) = \frac{(-1)^{d-k}}{k!(d-k)!} \frac{i+j-2k}{(i-k)(j-k)} \frac{(i+j-k-d-1)!}{(i+j-k)!} \frac{i!j!}{(i-d-1)!(j-d-1)!}. \tag{16}$$

**Proof.** First we prove that condition (15) is necessary. Since the Hankel matrices $M^{(0)}, M^{(1)}, \ldots, M^{(d)}$ are symmetric, from (6) we have that $M$ is a symmetric matrix and the second part of (15) holds. Moreover, if (6) holds with $M^{(k)} = \left(c_{i,j}^{k}\right)_{i,j=0}^{\infty}$, for each $k = 0, 1, \ldots, d$, then since $M^{(k)}$ is a Hankel matrix there exists a sequence of numbers $\{c_p^k\}$, $p = 0, 1, \ldots$, such that $c_{i,j}^k = c_p^k$ whenever $i + j = p$. With this definition of the numbers $c_p^k$, it follows from (6) that (4) takes place for all $c_{i,j}$. In particular, for each $k = 0, 1, \ldots, d$ fixed and $i, j \geq d$,

$$c_{k,i+j-k} = \sum_{\nu=0}^{k} \frac{k!}{(k-\nu)!} \frac{(i+j-k)!}{(i+j-k-\nu)!} c_{i+j-2\nu}^{\nu} \, . \tag{17}$$

Moreover, from (4), (9) and (16), for $i, j > d$ we obtain

$$c_{i,j} = \sum_{\nu=0}^{d} \frac{i!}{(i-\nu)!} \frac{j!}{(j-\nu)!} c_{i+j-2\nu}^{\nu} =$$

$$\sum_{\nu=0}^{d} \frac{i!}{(i-d-1)!} \frac{j!}{(j-d-1)!} c_{i+j-2\nu}^{\nu} \left[ \frac{(i-d-1)!(j-d-1)!}{(i-\nu)!(j-\nu)!} \right] =$$

$$\sum_{\nu=0}^{d} \frac{i!}{(i-d-1)!}\frac{j!}{(j-d-1)!} c^{\nu}_{i+j-2\nu} \times$$
$$\left[\sum_{k=\nu}^{d} \frac{(-1)^{d-k}}{(k-\nu)!(d-k)!}\frac{i+j-2k}{(i-k)(j-k)}\frac{(i+j-k-d-1)!}{(i+j-k-\nu)!}\right] =$$
$$\sum_{k=0}^{d} \alpha_k(i,j) \sum_{\nu=0}^{k} \frac{k!}{(k-\nu)!}\frac{(i+j-k)!}{(i+j-k-\nu)!} c^{\nu}_{i+j-2\nu}\,, \qquad i,j > d\,.$$

Then, using (17), we arrive to (15).

Now, we prove that condition (15) is sufficient. We may define the Hankel matrices $M^{(k)} = \left(c^k_{m,n}\right)_{m,n=0}^{\infty}$ for $k = 0, 1, \dots, d$, where $c^k_{m,n} = c^k_j$ if $m+n = j$, and $c^k_j$ is defined as

$$c^k_j := \sum_{\nu=0}^{k} \frac{(-1)^{k-\nu}}{\nu!(k-\nu)!}\frac{(j+2k-2\nu)(j+k-\nu-1)!}{(j+2k-\nu)!} c_{\nu,j+2k-\nu}\,. \tag{18}$$

We wish to show that (6) holds. For this purpose, take

$$\widetilde{M} = \sum_{k=0}^{d} S^k M^{(k)} (S^k)^t\,. \tag{19}$$

If $\widetilde{M} = (\widetilde{c}_{i,j})$, from (19) we obtain

$$\widetilde{c}_{i,j} = \sum_{k=0}^{\delta} \frac{i!}{(i-k)!}\frac{j!}{(j-k)!} c^k_{i+j-2k}\ ,\ i,j = 0, 1, \dots\,, \tag{20}$$

where $\delta = \min\{i,j,d\}$. Using the expression for $c^k_{i+j-2k}$ given by (18) and substituting in (20), we find that

$$\widetilde{c}_{i,j} = \sum_{k=0}^{\delta} \frac{i!}{(i-k)!}\frac{j!}{(j-k)!} \sum_{\nu=0}^{k} \frac{(-1)^{k-\nu}}{\nu!(k-\nu)!}\frac{(i+j-2\nu)(i+j-k-\nu-1)!}{(i+j-\nu)!} c_{\nu,i+j-\nu} =$$
$$\sum_{\nu=0}^{\delta} \frac{i!j!(i+j-2\nu)}{\nu!(i+j-\nu)!} \sum_{k=\nu}^{\delta} \frac{(-1)^{k-\nu}}{(k-\nu)!}\frac{(i+j-k-\nu-1)!}{(i-k)!(j-k)!} c_{\nu,i+j-\nu}\,. \tag{21}$$

We must prove that $\widetilde{c}_{i,j} = c_{i,j}\ ,\ i,j = 0, 1, \dots$ . Since $M$ and $\widetilde{M}$ are symmetric matrices (see (15) and (20)) it is sufficient to consider that $i \le j$. If $i \le d$, we have that $\delta = i$ and (21) can be expressed as

$$\widetilde{c}_{i,j} = c_{i,j} + \sum_{\nu=0}^{i-1} \frac{i!j!(i+j-2\nu)}{\nu!(i+j-\nu)!} \sum_{k=\nu}^{i} \frac{(-1)^{k-\nu}}{(k-\nu)!}\frac{(i+j-k-\nu-1)!}{(i-k)!(j-k)!} c_{\nu,i+j-\nu}\,.$$

Because of (8) in Lemma 2, we obtain

$$\widetilde{c}_{i,j} = c_{i,j}\,. \tag{22}$$

If $d < i$, since $\widetilde{c}_{k,i+j-k} = c_{k,i+j-k}$ for $k \le d$ (see (22)), from (15) and (20) we have

$$c_{i,j} = \sum_{k=0}^{d} \alpha_k(i,j)\widetilde{c}_{k,i+j-k} = \sum_{k=0}^{d} \alpha_k(i,j) \sum_{\nu=0}^{k} \frac{k!(i+j-k)!}{(k-\nu)!(i+j-k-\nu)!} c^{\nu}_{i+j-2\nu} =$$

$$\sum_{\nu=0}^{d}\sum_{k=\nu}^{d}\alpha_k(i,j)\frac{k!(i+j-k)!}{(k-\nu)!(i+j-k-\nu)!}c^{\nu}_{i+j-2\nu}=$$

$$\sum_{\nu=0}^{d}\frac{i!}{(i-d-1)!}\frac{j!}{(j-d-1)!}\sum_{k=\nu}^{d}\frac{(-1)^{d-k}}{(d-k)!}\frac{i+j-2k}{(i-k)(j-k)}\frac{(i+j-k-d-1)!}{(i+j-k-\nu)!(k-\nu)!}c^{\nu}_{i+j-2\nu}.$$

Thus, from (9)

$$c_{i,j}=\sum_{\nu=0}^{d}\frac{i!}{(i-\nu)!}\frac{j!}{(j-\nu)!}c^{\nu}_{i+j-2\nu}\,,$$

which is $\widetilde{c}_{i,j}$ according to (20). ■

A consequence of the sufficiency proof of Theorem 2 and Lemma 1 is

**Corollary 1** *Assume that $M=(c_{i,j})_{i,j=0}^{\infty}$ admits a decomposition of form (6), then for each $k=0,\dots,d$ the sequence $\{c^k_j\}$, $j=0,1,\dots$, which determines the Hankel matrix $M^{(k)}=\left(c^k_{m,n}\right)_{m,n=0}^{\infty}$, where $c^k_{m,n}=c^k_j$, if $m+n=j$, is given by formula (18).*

**Proof.** In fact, if $M$ satisfies (6) for some $(M^{(0)},\dots,M^{(d)})$ then (15) holds. In this case, in the proof of Theorem 2, it was shown that $M$ also satisfies (6) for $\widetilde{M}^{(0)},\dots,\widetilde{M}^{(d)}$, where $\widetilde{M}^{(k)}=\left(c^k_{m,n}\right)_{m,n=0}^{\infty}$ and

$$c^k_{m,n}=c^k_j=\sum_{\nu=0}^{k}\frac{(-1)^{k-\nu}}{\nu!(k-\nu)!}\frac{(j+2k-2\nu)(j+k-\nu-1)!}{(j+2k-\nu)!}c_{\nu,j+2k-\nu}\ ,\quad \text{if } m+n=j\,.$$

But according to Lemma 1 the decomposition of $M$ in form (6) is unique, thus $M^{(k)}=\widetilde{M}^{(k)}$ and the elements of $M^{(k)}$ are given by formula (18). ■

**Remark 1** *Formula (9) for $\nu=0$ indicates that*

$$\sum_{k=0}^{d}\alpha_k(i,j)=1$$

*for each pair $(i,j)$ of indices. That is, from Theorem 2, we know that it is possible to write each entry $c_{i,j}$ of an anti-diagonal of $M$ as a linear combination (15) of the first $d+1$ elements in this anti-diagonal whose coeffients sum one, $c_{0,i+j},c_{1,i+j-1},\dots,c_{d,i+j-d}$. In this sense these matrices generalize Hankel matrices.*

## 3. The S-moment problem

We are ready for the study of the definiteness and determinacy of the S-moment problem $(M;\Sigma_0,\dots,\Sigma_d)$.

**Proof of Theorem 1.** Assume that the S-moment problem is definite. As we saw in the introduction (6) takes place with $M^{(k)}$ the moment matrix associated with the measure $\mu_k$. Therefore, the ordinary moment problems $(M^{(k)};\Sigma_k)$ are definite for each $k=0,\dots,d$.

If for some $k$ the moment problem $(M^{(k)};\Sigma_k)$ is indeterminate, then obviously the S-moment problem is indeterminate because we would have two measures $\mu_k^1$ , $\mu_k^2$ giving the same moment matrix $M^{(k)}$ without affecting relations (4) (equivalent to (6)) and thus (2) would take place for the sets of measures $(\mu_0,\ldots,\mu_k^1,\ldots,\mu_d)$ and $(\mu_0,\ldots,\mu_k^2,\ldots,\mu_d)$. This settles the necessity.

Conversely, if for each $k=0,\ldots,d$ the moment problem $(M^{(k)};\Sigma_k)$ is definite then there exist measures $\mu_0,\ldots,\mu_d$ whose moment matrices are $M^{(0)},\ldots,M^{(d)}$ respectively. From (6), we have that the elements $c_{i,j}$ of $M$ are related with the elements of the Hankel matrices $M^{(0)},\ldots,M^{(d)}$ through relations (4) which imply (2). That is, the S-moment problem is definite. Suppose that the S-moment problem is not determinate. Then there exist two distinct systems of measures $(\mu_0^1,\ldots,\mu_d^1)$ , $(\mu_0^2,\ldots,\mu_d^2)$ , $supp(\mu_k^i)\subset\Sigma_k$ , $i\in\{1,2\}$ , $k=0,\ldots,d$, whose Hankel matrices $(M_1^{(0)},\ldots,M_1^{(d)})$ , $(M_2^{(0)},\ldots,M_2^{(d)})$ are related with $M$ through (6). According to Lemma 1, $M_1^{(k)}=M_2^{(k)}$ , $k=0,\ldots,d$. But at least for some $k$ , $\mu_k^1\neq\mu_k^2$ and $supp(\mu_k^i)\subset\Sigma_k$ , $i=1,2$. This means that the moment problem $(M^{(k)};\Sigma_k)$ is indeterminate. With this we conclude the proof. ∎

From Theorems 1 and 2, we obtain

**Corollary 2** *Given $(M;\Sigma_0,\ldots,\Sigma_d)$ the S-moment problem is definite (determinate) if and only if*

*i)* (15) *holds*

*ii)* *for each $k=0,1,\ldots,d$, the ordinary moment problem $(\left\{c_j^k\right\};\Sigma_k)$ is definite (determinate), where $c_j^k$ is given by* (18).

**Proof.** It is sufficient to point out that according to Theorem 2, (6) and (15) are equivalent. On the other hand, from Corollary 1 we have that under (6) the elements $c_i^k$ of the first row of the Hankel matrix $M^{(k)}$ are given by formula (18). The rest of the proof follows directly from Theorem 1. ∎

Theorem 1 is the link needed in order to translate results from the classical theory of moments into the context of the Sobolev moment problem. Before stating some of these consequences, let us introduce some new notation. For all $n\in I\!N$, set

$$\Delta_n^k := \begin{vmatrix} c_0^k & c_1^k & \cdots & c_{n-1}^k \\ c_1^k & c_2^k & \cdots & c_n^k \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1}^k & c_n^k & \cdots & c_{2n-2}^k \end{vmatrix}, \qquad \left(\Delta_n^k\right)^{(1)} := \begin{vmatrix} c_1^k & c_2^k & \cdots & c_n^k \\ c_2^k & c_3^k & \cdots & c_{n+1}^k \\ \vdots & \vdots & \ddots & \vdots \\ c_n^k & c_{n+1}^k & \cdots & c_{2n-1}^k \end{vmatrix}$$

where $c_j^k$ , $j=0,1,\ldots$ , is given by (18).

**Corollary 3** *The SH-moment problem is definite if and only if*

*i)* (15) *holds*

*ii)* *for each* $k \in \{0, 1, \ldots, d\}$ , *either*

$$\Delta_n^k > 0 \qquad \textit{for all} \qquad n = 1, 2, \ldots$$

*or for some* $m \in I\!N$ *we have*

$$\Delta_1^k, \Delta_2^k, \ldots, \Delta_m^k > 0 \qquad \textit{and} \qquad \Delta_j^k = 0 \; j = m+1, m+2, \ldots.$$

**Proof.** It is immediate from Theorem 1 and the classical condition for determinacy of the Hamburger moment problem (see [6], p. 5). ■

The following two corollaries are also consequences of Theorem 1 and known results for definiteness in moment theory (see [6] pp. 5-9 and [2] p. 64).

**Corollary 4** *A definite S-moment problem* $(M; \Sigma_0, \ldots, \Sigma_d)$ *has a solution with*

$$\mathrm{supp}(\mu_k) \subset [0, +\infty)$$

*for some* $k \in \{0, 1, \ldots, d\}$ *if and only if either*

$$\Delta_n^k > 0 \ , \ \left(\Delta_n^k\right)^{(1)} > 0 \qquad \textit{for all} \qquad n = 0, 1, \ldots,$$

*or for some* $m \in I\!N$ *we have*

$$\Delta_n^k > 0 \ , \ \left(\Delta_n^k\right)^{(1)} > 0 \ , \ n = 0, 1, \ldots, m \qquad \textit{and} \qquad \Delta_n^k = \left(\Delta_n^k\right)^{(1)} = 0 \ , \ n = m+1, \ldots.$$

**Corollary 5** *A definite S-moment problem* $(M; \Sigma_0, \ldots, \Sigma_d)$ *has a solution with*

$$\mathrm{supp}(\mu_k) \subset [a, b]$$

*for some* $k \in \{0, 1, \ldots, d\}$ *if and only if*

$$\sum_{i,j=0}^{\infty} c_{i+j}^k x_i x_j \ , \ \sum_{i,j=0}^{\infty} \left[(a+b)c_{i+j-1}^k - abc_{i+j}^k - c_{i+j-2}^k\right] x_i x_j$$

*are two non-negative infinite quadratic forms.*

Regarding determinacy, we have (see [6], p. 19)

**Corollary 6** *A definite S-moment problem* $(M; \Sigma_0, \ldots, \Sigma_d)$ *such that each set* $\Sigma_k, k = 0, \ldots, d$, *is a bounded interval of the real line is determinate.*

**Corollary 7** *A definite SH-moment problem is determinate if for each* $k = 0, \ldots, d$

$$\sum_{n=0}^{\infty} |c_{2n}^k|^{-1/2n} = \infty. \tag{23}$$

*In particular, this is true if*

$$\sum_{n=d}^{\infty} |c_{d,2n-d}|^{-1/2n} = \infty. \tag{24}$$

**Proof.** From (23), we have that Carleman's sufficient condition for determinacy of the moment problem is satisfied for each $(M^{(k)}, \Sigma_k)$, $k = 0, \ldots, d$. Hence, according to Theorem 1, the Sobolev moment problem is determinate.

From (4), we obtain

$$c_{d,2n-d} = \sum_{k=0}^{d} \frac{d!}{(d-k)!} \frac{(2n-d)!}{(2n-d-k)!} c^k_{2n-2k} \,, \qquad n = d, d+1, \ldots \,.$$

Since the terms on the right hand of this equality are all positive, it follows that

$$c^k_{2n-2k} \le c_{d,2n-d} \,, \qquad k = 0, \ldots, d.$$

Therefore, using (24), we have that for each $k = 0, \ldots, d$

$$\infty = \sum_{n=d}^{\infty} |c_{d,2n-d}|^{-1/2n} \le \sum_{n=d}^{\infty} |c^k_{2n-2k}|^{-1/2n} \,,$$

which is equivalent to what we needed to prove. ∎

Notice that analogous sufficient conditions for the determinacy of the SH-moment problem may be stated in terms of any of the rows of matrix $M$ below row $d+1$.

## 4. S-moment matrices and recurrence relations of Sobolev orthogonal polynomials

In the following, we assume that the S-moment problem $(M; \Sigma_0, \ldots, \Sigma_d)$ is definite. Let $(\mu_0, \mu_1, \ldots, \mu_d)$ be a solution. We also assume that $supp(\mu_k)$ is an infinite set for some $k \in \{0, 1, \ldots, d\}$. The $n$-th principal section $M_n$ of $M$ is formed by the S-moments

$$c_{i,j} = \langle z^i, z^j \rangle_S \,, \qquad i, j = 0, \ldots, n-1 \,.$$

From definition (1) and the fact that $\cup_{i=0}^{d} supp(\mu_i)$ contains infinitely many points, it is obvious that $\langle \cdot, \cdot \rangle_S$ defines a real positive definite quadratic form on the space of polynomials of degree $\le n-1$ and thus its matrix representation $M_n$ in the basis $\{1, z, \ldots, z^{n-1}\}$ is a real positive definite matrix. In particular, from Sylvester's Theorem, for each $n \in \mathbb{N}$, we have that the determinant of $M_n$ is greater than zero for all $n$. Thus, for each $n$, the following system of equations has a unique solution $(a_{n,0}, a_{n,1}, \ldots, a_{n,n-1})$ ,

$$\sum_{k=0}^{n-1} a_{n,k} \langle x^k, x^i \rangle_S = -\langle x^n, x^i \rangle_S \,, \qquad i = 0, 1, \ldots, n-1 \,.$$

In other words, there exists a unique monic polynomial $P_n$ of degree $n$,

$$P_n(x) = x^n + \sum_{k=0}^{n-1} a_{n,k} x^k \,,$$

such that

$$\langle P_n, x^i \rangle_S = 0 \,, \qquad i = 0, 1, \ldots, n-1 \,.$$

Then $\{P_n\}$ , $n = 0, 1, \dots$ , is the sequence of (monic) orthogonal polynomials with respect to the inner product given by (1).

Let $\|P_j\|_S^2 = \langle P_j, P_j \rangle_S$ , $j = 0, 1, \dots$. We introduce the sequence $\{p_n\}$ , $n = 0, 1, \dots$ , of orthonormal polynomials

$$p_n(z) = \frac{P_n(z)}{\|P_n\|_S}, \qquad n = 0, 1, \dots .$$

Obviously, the polynomial $zp_{n-1}(z)$ can be expressed in the form

$$zp_{n-1}(z) = \sum_{j=0}^{n} \langle zp_{n-1}, p_j \rangle_S p_j(z), \qquad n \geq 1. \tag{25}$$

Let

$$D = \begin{pmatrix} \langle zp_0, p_0 \rangle_S & \langle zp_1, p_0 \rangle_S & \langle zp_2, p_0 \rangle_S & \cdots \\ \langle zp_0, p_1 \rangle_S & \langle zp_1, p_1 \rangle_S & \langle zp_2, p_1 \rangle_S & \cdots \\ & \langle zp_1, p_2 \rangle_S & \langle zp_2, p_2 \rangle_S & \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix} \tag{26}$$

and $D_m, m = 1, 2, \dots$, be the $m$-th principal section of $D$. By $D_m^t$ we denote the transpose of $D_m$. For each $z \in \mathbb{C}$ such that $p_m(z) = 0$, relations (25) can be expressed in matrix form as

$$(D_m^t - zI_m)v_m(z) = 0$$

where

$$v_m(z)^t = (p_0(z), p_1(z), \dots, p_{m-1}(z)) .$$

Thus, the zeros of $p_m$ are the eigenvalues of $D_m$. That is

$$\sigma(D_m) = \{z \, : \, p_m(z) = 0\} . \tag{27}$$

For each $n = 1, 2, \dots$ , $M_n$ is a symmetric positive definite matrix; therefore, there exists a unique Cholesky factorization

$$M_n = T_n T_n^t . \tag{28}$$

(see [3]), where $T_n$ is a lower triangular matrix of order $n$. It is well known and easy to verify that $T_n$ is the $n$-th principal section of $T_{n+1}$. Thus, $T_n$ is the $n$-th principal section of an infinite lower triangular matrix $T$ given by

$$T = \begin{pmatrix} \tau_{0,0} & & & & \\ \tau_{1,0} & \tau_{1,1} & & & \\ \vdots & \vdots & \ddots & & \\ \tau_{i,0} & \tau_{i,1} & \cdots & \tau_{i,i} & \\ \vdots & \vdots & & & \ddots \end{pmatrix}, \tag{29}$$

$\tau_{i,j} \in \mathbb{R}$ , $i, j = 0, 1, \dots$ and $\tau_{i,i} > 0$ , $i = 0, 1, \dots$ .

**Theorem 3** *For each $n = 1, 2, \ldots,$ we have*

$$T_n \begin{pmatrix} p_0(z) \\ p_1(z) \\ \vdots \\ p_{n-1}(z) \end{pmatrix} = \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{pmatrix}, \qquad \text{for all} \qquad z \in \mathbb{C}. \tag{30}$$

*and*

$$D_n = T_n^{-1} M_n' (T_n^{-1})^t \,,$$

*where $M_n'$ is the $n$-th principal section of the infinite matrix*

$$M' = \begin{pmatrix} c_{0,1} & c_{0,2} & \cdots \\ c_{1,1} & c_{1,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

*which is obtained eliminating the first column from $M$.*

**Proof.** Let $z^m = \sum_{k=0}^{m} \beta_{m,k} p_k(z)$ , $m = 0, 1, \ldots$ . Notice that $\beta_{m,m} > 0$, for all $m = 0, 1, \ldots$ . Then

$$c_{i,j} = \langle z^i, z^j \rangle_S = \sum_{k=0}^{i} \sum_{r=0}^{j} \beta_{i,k} \beta_{j,r} \langle p_k, p_r \rangle_S$$

where $\langle p_k, p_r \rangle_S = \delta_{k,r}$ , $k, r = 0, 1, \ldots$ . Thus

$$c_{i,j} = \sum_{k=0}^{\min\{i,j\}} \beta_{i,k} \beta_{j,k}.$$

On the other hand, from (28) and (29), we obtain

$$c_{i,j} = \sum_{k=0}^{\min\{i,j\}} \tau_{i,k} \tau_{j,k}.$$

Thus, because of the uniqueness of the factorization (28), we have that $\beta_{i,k} = \tau_{i,k}$ , $i, k = 0, 1, \ldots$ . That is,

$$z^i = \sum_{k=0}^{i} \tau_{i,k} p_k(z) \,, \qquad i = 0, 1, \ldots \,,$$

which is equivalent to (30).

Set

$$p_i(z) = \sum_{h=0}^{i} \gamma_{i,h} z^h \,, \qquad i = 0, 1, \ldots \,.$$

From (30), we have

$$T_n^{-1} = \begin{pmatrix} \gamma_{0,0} & & & \\ \gamma_{1,0} & \gamma_{1,1} & & \\ \vdots & \vdots & \ddots & \\ \gamma_{n-1,0} & \gamma_{n-1,1} & \cdots & \gamma_{n-1,n-1} \end{pmatrix}.$$

Therefore, for any fixed $j = 0, 1, \ldots,$

$$zp_j(z) = \sum_{r=0}^{j} \gamma_{j,r} z^{r+1} .$$

Let $n \geq i, j$. The entry $(i, j)$ of $D_n$ which is given by $\langle zp_j, p_i \rangle_S$ satisfies

$$\langle p_i, zp_j \rangle_S = \sum_{h=0}^{i} \sum_{r=0}^{j} \gamma_{i,h} \gamma_{j,r} \langle z^h, z^{r+1} \rangle_S = \sum_{h=0}^{i} \sum_{r=0}^{j} \gamma_{i,h} \gamma_{j,r} c_{r+1,h} .$$

Therefore, it is the $(i, j)$ entry of matrix $T_n^{-1} M_n' (T_n^{-1})^t$ for any $n \geq i, j$ . ■

An immediate consequence of Theorem 3 is

**Corollary 8** *Let $z_n \in \sigma(D_n)$ and $D_n w_n = z_n w_n$ with $\|w_n\| = 1$. Then*

$$z_n = \langle M_n' (T_n^{-1})^t w_n, (T_n^{-1})^t w_n \rangle .$$

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