# WEAK CONVERGENCE OF VARYING MEASURES AND HERMITE-PADÉ ORTHOGONAL POLYNOMIALS

B. de la Calle Ysern
E. T. S. de Ingenieros Industriales
Universidad Politécnica de Madrid
Spain

G. López Lagomasino*
Escuela Politécnica Superior
Universidad Carlos III de Madrid
Spain

Key words and phrases: Orthogonal polynomials, varying measures, weak convergence, Nikishin system
AMS (MOS) subject classification: **42C05, 30E10.**

**Abstract**

It is known that the common denominator of the Hermite-Padé approximants of a mixed Angelesco-Nikishin system shares orthogonality relations with respect to each function in the system. It is less known that they also satisfy full orthogonality with respect to a varying measure. This problem motivates our interest in extending the class of varying measures with respect to which weak asymptotics of orthogonal polynomials takes place. In particular, for the case of a Nikishin system, we prove weak asymptotics of the corresponding varying measures.

## 1 Introduction

1. The denominators of interpolating rational functions satisfy orthogonality relations with respect to a measure which depends on the set of interpolation points. This has been the main cause of the increasing interest paid in the past two decades to the asymptotic properties of sequences of polynomials orthogonal with respect to so called varying measures.

Let $\{Q_n\}, n \in \mathbb{N}$, be a sequence of monic polynomials, $\deg Q_n = n$, and $\{\mu_n\}$ a sequence of finite positive Borel measures each of which has its support $S(\mu_n)$ contained in the real line $\mathbb{R}$. We say that the sequence of polynomials is orthogonal with respect to the (sequence of) varying measures if

$$0 = \int x^\nu Q_n(x) d\mu_n(x) \ , \ \ \nu = 0, \ldots, n-1 \ .$$

Notice that the $n-$th polynomial only satisfies orthogonality relations with respect to the $n-$th measure.

*The research of this author was supported by Dirección General de Enseñanza Superior under grant PB 96-0120-C03-01 and by INTAS under grant 93-0219 EXT.

So far, most of the applications are connected with the case when $d\mu_n(x) = \frac{d\mu(x)}{w_n(x)}$, where $\mu$ is a fixed measure and $w_n(x)$ is a polynomial. This situation appears in problems of rational interpolation. The zeros of $w_n(x)$ are the interpolation points by which the rational function is constructed. See, for example, [2], [10], and [11]. But, there are other questions in approximation theory where the varying part has a different expression (see, for example, [1], [3], and [9]).

The most general results have been obtained in connection with the $n-$th root asymptotic behavior of orthogonal polynomials with respect to varying measures. Here, $d\mu_n(x) = \phi_n(x)d\mu(x)$. Essentially, it is only required that the measure $\mu$ be regular ($\mu \in \mathbf{Reg}$) and $\lim_{n\to\infty} \phi_n(x)^{1/n}$ exists uniformly on $S(\mu)$. See [16], also [8]. Thus, the results for $n-$th root asymptotics parallel to a great extent those known for the case when the measure remains fixed ($\phi_n \equiv 1$).

This is by far not the situation for other types of asymptotic relations where most results only deal with the case when $d\mu_n(x) = \frac{d\mu(x)}{w_n(x)}$. The main objective of the present paper is to extend the class of varying orthogonality for which weak asymptotics takes place, derive from it other types of asymptotic properties, and apply these results in the study of the asymptotic behavior of the common denominator of Hermite-Padé approximants of a Nikishin system of functions.

2. In the introduction, we limit ourselves to a brief description of the application which we will consider. More details will be found in section 6.

Let us consider two finite positive Borel measure $\mu$ and $\sigma$ such that $Co(S(\mu)) \cap Co(S(\sigma)) = \emptyset$, where $Co(\cdot)$ denotes the convex hull of the set $(\cdot)$. Set

$$f_1(z) = \int \frac{d\sigma(t)}{z-t} \ , \ f_2(z) = \int \frac{\hat{\mu}(t)d\sigma(t)}{z-t} \ .$$

In the sequel, $\hat{\mu}$ denotes the Cauchy transform of the measure $\mu$; that is, the Markov function for the measure. The pair of functions $(f_1, f_2)$ forms what is called a Nikishin system of two functions. Fix two natural numbers $n_1, n_2 \in \mathbb{N}, n = n_1 + n_2$. We say that $(R_{n,1}, R_{n,2})$ is the $n-$th simultaneous Padé approximant of $(f_1, f_2)$ relative to $(n_1, n_2)$ if $R_{n,1} = \frac{P_{n,1}}{Q_n}, R_{n,2} = \frac{P_{n,2}}{Q_n}$, where

- $\deg P_{n,1} \leq n, \deg P_{n,2} \leq n, \deg Q_n \leq n, Q_n \not\equiv 0$ ,
- $(Q_n f_1 - P_{n,1})(z) = O\left(\frac{1}{z^{n_1+1}}\right) \ , \ z \to \infty$ ,
- $(Q_n f_2 - P_{n,2})(z) = O\left(\frac{1}{z^{n_2+1}}\right) \ , \ z \to \infty$ .

It is easy to see that the common denominator $Q_n$ shares orthogonality relations with the two measures $d\sigma(t)$ and $\hat{\mu}(t)d\sigma(t)$. More precisely, one has

$$0 = \int t^\nu Q_n(t) d\sigma(t) \ , \ \nu = 0, \ldots, n_1 - 1 \ ,$$

and

$$0 = \int t^\nu Q_n(t) \hat{\mu}(t) d\sigma(t) \ , \ \nu = 0, \ldots, n_2 - 1 \ .$$

Assume that $n_2 \leq n_1 + 1$. A non-trivial fact is that $Q_n$ also satisfies full orthogonality relation with respect to a varying measure. There exists a polynomial $w_n, \deg w_n = n_2$, whose zeros lie in $Co(S(\mu))$ such that

$$0 = \int t^\nu Q_n(t) \frac{d\sigma(t)}{w_n(t)} \ , \ \nu = 0, \ldots, n - 1 \ .$$

It is also known that the polynomials $w_n$ satisfy complete orthogonality relations with respect to a rather complicated type of varying measure

$$0 = \int x^{\nu} \frac{w_n(x)}{Q_n(x)} \int \frac{Q_n^2(t)}{t-x} \frac{d\sigma(t)}{w_n(t)} d\mu(x) \ , \ \nu = 0, \ldots, n_2 - 1$$

(for details see [2], [4] and [9]). We study the asymptotic behavior of the sequences $\{Q_n\}$ and $\{w_n\}$ as well as of the varying measures with respect to which these polynomials are orthogonal when $n_1 = n_2 \to \infty$.

In order to obtain the corresponding results on the real line, we start out with the unit circle.

3. Let $\rho_n$ and $\rho$ be finite positive Borel measures on $[0, 2\pi]$. By $\rho_n \xrightarrow{*} \rho$, we denote the weak * convergence of $\rho_n$ to $\rho$ as $n$ tends to infinity. This means that for every continuous real $2\pi$-periodic function $f$ on $[0, 2\pi]$

$$\lim_{n\to\infty} \int_0^{2\pi} f(\theta)\, d\rho_n(\theta) \ = \ \int_0^{2\pi} f(\theta)\, d\rho(\theta). \tag{1}$$

Unless otherwise stated, the limits of integration with respect to $\theta$ will always be 0 and $2\pi$, thus they will not be indicated in the following.

Let $\rho$ be a complex regular Borel measure on $[0, 2\pi]$. Set

$$\|\rho\| = |\rho|([0, 2\pi]) \ ,$$

where $|\rho|$ denotes the positive measure given by the total variation of $\rho$. This defines a norm on the space of all complex regular Borel measures. A sequence of complex regular Borel measures $\{\rho_n\}$ is said to converge in norm to $\rho$ if

$$\lim_{n\to\infty} \parallel \rho_n - \rho \parallel = 0 \ .$$

With this norm, the space of complex regular Borel measures on $[0, 2\pi]$ is the dual space of the space of all complex valued $2\pi$-periodic continuous functions on $[0, 2\pi]$. Thus, weak * convergence of a sequence of complex regular Borel measures means that (1) takes place for every complex valued $2\pi$-periodic continuous function on $[0, 2\pi]$. It is well known that any sequence of complex Borel measures uniformly bounded in norm is weak * relatively compact. All finite positive Borel measures are regular; therefore, their difference (usually called real or signed measure) is a complex regular measure. More details on these concepts and results may be found, for example, in chapters 2 and 6 of [15].

Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of finite positive Borel measures on the interval $[0, 2\pi]$ such that for each $n \in \mathbb{N}$ the support of $\rho_n$ contains an infinite set of points. By $d\theta$, we denote Lebesgue's measure on $[0, 2\pi]$, and $\rho_n' = d\rho_n/d\theta$, the Radon-Nikodym derivative of $\rho_n$ with respect to $d\theta$. By $\mathbb{N}$ (respectively $\mathbb{Z}, \mathbb{R}, \mathbb{C}$), we denote the set of natural (respectively integer, real, complex) numbers.

Let $\{W_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials such that, for each $n \in \mathbb{N}$, $W_n$ has degree $n$ (deg $W_n = n$) and all its zeros $\{w_{n,i}\}, 1 \leq i \leq n$, lie in the closed unit disk. We assume that the indices are taken so that if $w = 0$ is a zero of $W_n$ of multiplicity $m$ then $w_{n,1} = w_{n,2} = \ldots = w_{n,m} = 0$. Set

$$d\sigma_n(\theta) = \frac{d\rho_n(\theta)}{|W_n(z)|^2} \ , \ \ z = e^{i\theta}.$$

A certain link is needed between the measures $\rho_n$ and the polynomials $W_n$.

**Definition 1** *Let $k \in \mathbb{Z}$ be a fixed integer. We say that $(\{\rho_n\}, \{W_n\}, k)$ is admissible on $[0, 2\pi]$ if :*

*(i) There exists a finite positive Borel measure $\rho$ on $[0, 2\pi]$ such that $\rho_n \xrightarrow{*} \rho\ , n \to \infty$ .*

*(ii) $||\sigma_n|| = \int d\sigma_n(\theta) < +\infty$ , $\forall n \in \mathbb{N}$ .*

*(iii) $\int \prod_{i=1}^{-k} |z - w_{n,i}|^{-2}\, d\rho_n(\theta) \le M < +\infty$ , $z = e^{i\theta}$ , $n \in \mathbb{N}$ (this condition applies only to the case when $k$ is a negative integer).*

*(iv) $\lim_{n\to\infty} \sum_{i=1}^{n} (1 - |w_{n,i}|) = +\infty$.*

A stronger connection is established by

**Definition 2** *Let $k \in \mathbb{Z}$ be a fixed integer. We say that $(\{\rho_n\}, \{W_n\}, k)$ is strongly admissible on $[0, 2\pi]$ if it is admissible and additionally*

*(i)*

$$\lim_{n\to\infty} \int |\rho_n'(\theta) - \rho'(\theta)|\, d\theta = 0$$

*(ii) $\rho' > 0$ almost everywhere on $[0, 2\pi]$.*

In [12] the definition of admissibility is introduced in analogous fashion but the component $\rho_n$ of the varying measures is taken to be constant. Obviously, in that situation, the conditions (i) in both definitions above are trivially satisfied. It is not difficult to prove that $\lim_{n\to\infty} \|\rho_n - \rho\| = 0$ implies (i) of definitions 1 and 2. Condition (ii) of admissibility guarantees that for each pair $(n, m)$ of natural numbers we can construct a polynomial $\varphi_{n,m}(z) = \alpha_{n,m} z^m + \cdots$ that is uniquely determined by the relations of orthogonality

$$\frac{1}{2\pi} \int \bar{z}^j \varphi_{n,m}(z)\, d\sigma_n(\theta) = 0\ , \quad j = 0, 1, \ldots, m-1\ , \quad z = e^{i\theta}\ ,$$

$$\frac{1}{2\pi} \int |\varphi_{n,m}(z)|^2\, d\sigma_n(\theta) = 1\ , \quad \deg \varphi_{n,m} = m\ , \quad \alpha_{n,m} > 0\ .$$

The following results are the key to all further arguments. In their proof (see sections 2 and 3 below), special difficulties arise when the degree of the polynomial $W_n$ exceeds that of of $\varphi_{n,n+k}$ (that is, for negative $k$). In order to handle these problems, condition (iii) of admissibility was introduced.

**Theorem 1** *Let $(\{\rho_n\}, \{W_n\}, k)$ be admissible on $[0, 2\pi]$, then*

$$\frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2}\, d\theta \xrightarrow{*} d\rho(\theta)\ , \quad z = e^{i\theta}\ . \tag{2}$$

**Theorem 2** *Let $(\{\rho_n\}, \{W_n\}, k)$ be strongly admissible on $[0, 2\pi]$, then*

$$\lim_{n\to\infty} \int \left| \frac{|\varphi_{n,n+k}(z)|^2}{|\varphi_{n,n+k+m}(z)|^2} - 1 \right| d\theta = 0\ , \tag{3}$$

*uniformly in $m \in \mathbb{N}$ .*

Let $\Phi_{n,m}(z) = z^m + \cdots = (\alpha_{n,m})^{-1}\varphi_{n,m}(z)$ and set $\Phi^*_{n,m}(z) = z^m\overline{\Phi_{n,m}}(1/z)$. For any sequence of positive Borel measures $\{\sigma_n\}$ such that $\int d\sigma_n(\theta) < +\infty, n \in \mathbb{N}$, the following relations hold. They are simple reformulations of known results (notice that $n$ is fixed).

$$\Phi_{n,m+1}(w) = w\Phi_{n,m}(w) + \Phi_{n,m+1}(0)\Phi^*_{n,m}(w) \ , \tag{4}$$

$$\Phi^*_{n,m+1}(w) = \Phi^*_{n,m}(w) + \overline{\Phi_{n,m+1}(0)}w\Phi_{n,m}(w) \ , \tag{5}$$

$$(\alpha_{n,m+1})^2 = (\alpha_{n,m})^2 + |\varphi_{n,m+1}(0)|^2 \ , \tag{6}$$

$$|\Phi_{n,m+1}(0)| \leq C \int \left| \frac{|\varphi_{n,m}(z)|^2}{|\varphi_{n,m+1}(z)|^2} - 1 \right| d\theta \ , \ z = e^{i\theta} \ , \tag{7}$$

where $C$ is an absolute constant independent of $n$ and $m$. For the proof of (4)-(6) see Chapter 1 of [7], for that of (7) see Theorem 2 in [13]. Combining (4)-(7) and using Theorem 2, we get

**Theorem 3** *Let $(\{\rho_n\}, \{W_n\}, k)$ be strongly admissible on $[0, 2\pi]$, then*

$$\lim_{n\to\infty} \Phi_{n,n+k+1}(0) = 0 \ , \tag{8}$$

$$\lim_{n\to\infty} \frac{\alpha_{n,n+k+1}}{\alpha_{n,n+k}} = 1 \ , \tag{9}$$

$$\lim_{n\to\infty} \frac{\Phi_{n,n+k+1}(w)}{\Phi_{n,n+k}(w)} = \lim_{n\to\infty} \frac{\varphi_{n,n+k+1}(w)}{\varphi_{n,n+k}(w)} = w, \quad |w| \geq 1 \ , \tag{10}$$

$$\lim_{n\to\infty} \frac{\Phi^*_{n,n+k+1}(w)}{\Phi^*_{n,n+k}(w)} = \lim_{n\to\infty} \frac{\varphi^*_{n,n+k+1}(w)}{\varphi^*_{n,n+k}(w)} = 1, \quad |w| \leq 1 \ , \tag{11}$$

$$\lim_{n\to\infty} \frac{\Phi^*_{n,n+k}(w)}{\Phi_{n,n+k}(w)} = \lim_{n\to\infty} \frac{\varphi^*_{n,n+k}(w)}{\varphi_{n,n+k}(w)} = 0 \quad |w| > 1 \tag{12}$$

*where in $(10) - (12)$ the convergence is uniform on each compact subset of the prescribed regions.*

## 2 Proof of Theorem 1

In all that follows $z = e^{i\theta}$, $\theta \in [0, 2\pi]$. We make use of the known relations

$$\frac{1}{2\pi} \int z^j \frac{1}{|\varphi_{n,m}(z)|^2} \, d\theta = \frac{1}{2\pi} \int z^j \, d\sigma_n(\theta), \quad |j| \leq m$$

(see Chapter 1 of [7]). From this, it follows that for each trigonometric polynomial $T_m$ of degree $\leq m$, we have

$$\frac{1}{2\pi} \int \frac{T_m(\theta)}{|\varphi_{n,m}(z)|^2} \, d\theta = \frac{1}{2\pi} \int T_m(\theta) \, d\sigma_n(\theta). \tag{13}$$

Let $A_n(z) = \prod_{i=1}^{-k}(z - w_{n,i})$ if $k = -1, -2, \ldots$; for $k = 0, 1, 2, \ldots$, we take $A_n(z) \equiv 1$. From (13) and (iii) of admissibility, it follows that

$$\int \frac{1}{|A_n(z)|^2} \frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2}\, d\theta \;=\; \int \frac{d\rho_n(\theta)}{|A_n(z)|^2} \;\leq\; M_1 \;<\; +\infty \;, \tag{14}$$

where $M_1 = \max(M, \sigma[0, 2\pi])$. From (14), we have that the sequence of signed measures $\{h_n\}$ defined by

$$dh_n(\theta) = \frac{1}{|A_n(z)|^2} \left( \frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2} d\theta - d\rho_n(\theta) \right)$$

is uniformly bounded in norm with respect to $n$ by $2M_1$. Therefore, in order to prove that

$$h_n \stackrel{*}{\longrightarrow} 0 \;,$$

it is sufficient to show that any convergent subsequence $\{h_n\}$, $n \in \Gamma$, $\Gamma \subset \mathbb{N}$, of such measures tends to zero.

Let us consider the function

$$e_n(w) = \int \frac{z}{z - w}\, dh_n(\theta), \quad |w| \,<\, 1 \;.$$

For each fixed $n \in \mathbb{N}$, $e_n(w)$ belongs to Nevalinna's class. In fact, denote

$$\hat{\beta}_n(w) \;=\; \int \frac{z}{z - w}\; \frac{1}{|A_n(z)|^2}\; \frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2}\; d\theta \;\;,$$

$$\hat{\rho}_n(w) \;=\; \int \frac{z}{z - w}\; \frac{d\rho_n(\theta)}{|A_n(z)|^2} \;\;.$$

It is easy to see that

$$-\frac{\pi}{2} < \arg \hat{\beta}_n(w) < \frac{\pi}{2} \;,$$

$$-\frac{\pi}{2} < \arg \hat{\rho}_n(w) < \frac{\pi}{2} \;,$$

for all $n \in \mathbb{N}$. But

$$\ln^+ |a| \leq \frac{|a|^\delta}{\delta} \leq \frac{Re(a^\delta)}{\delta \cos(\pi\delta/2)},$$

whenever $-\frac{\pi}{2} < \arg a < \frac{\pi}{2}$, $a \in \mathbb{C}$, where $\ln^+ x = \max(\ln x, 0)$, $x > 0$, $\delta \in (0,1)$. So, for $r \in (0,1)$ and $\delta$ fixed,

$$\begin{aligned} \int \ln^+ |\hat{\beta}_n(re^{i\theta'})|\, d\theta' \;\leq\; \int \frac{Re(\hat{\beta}_n(re^{i\theta'})^\delta)}{\delta \cos(\pi\delta/2)}\, d\theta' \;&=\; \frac{Re(\hat{\beta}_n(0)^\delta)}{\delta \cos(\pi\delta/2)} \\ =\; \frac{\left( \int \frac{1}{|A_n(z)|^2} \frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2}\, d\theta \right)^\delta}{\delta \cos(\pi\delta/2)} \;&\leq\; \frac{(M_1)^\delta}{\delta \cos(\pi\delta/2)} \;. \end{aligned} \tag{15}$$

Analogously, one proves that

$$\int \ln^+ |\hat{\rho}_n(re^{i\theta'})| \, d\theta' \leq \frac{Re(\hat{\rho}_n(0)^\delta)}{\delta \cos(\pi\delta/2)} \leq \frac{(M_1)^\delta}{\delta \cos(\pi\delta/2)} \ . \tag{16}$$

From (15) and (16), it follows that

$$\sup_{n\to\infty} \lim_{r\to 1} \int \ln^+ |(\hat{\rho}_n - \hat{\beta}_n)(re^{i\theta'})| \, d\theta' \leq \frac{2(M_1)^\delta}{\delta \cos(\pi\delta/2)} < +\infty \ . \tag{17}$$

Therefore, the functions $e_n(w)$ are in Nevanlinna's class (uniformly with respect to $n$). Thus (see p. 16 of [5]),

$$e_n(w) = \frac{B_n(w)}{C_n(w)} \ , \quad n \in \mathbb{N} \ ,$$

where

$$\lim_{r\to 1} \max_{|w|=r} |B_n(w)| \leq 1 \ , \qquad \lim_{r\to 1} \max_{|w|=r} |C_n(w)| \leq 1 \ , \tag{18}$$

and $C_n(w) \neq 0 \ , \ |w| < 1$ . Also

$$C_n(w) = \lim_{r\to 1} \exp \left\{ -\frac{1}{2\pi} \int \ln^+ |(\hat{\rho}_n - \hat{\beta}_n)(re^{i\theta'})| \ \frac{re^{i\theta'} + w}{re^{i\theta'} - w} \ d\theta' \right\} . \tag{19}$$

Let $\Gamma' \subset \Gamma \subset \mathbb{N}$ be such that

$$\lim_{n\in\Gamma'} B_n(w) = B(w) \quad \text{and} \quad \lim_{n\in\Gamma'} C_n(w) = C(w)$$

uniformly on each compact subset of the open unit disk. Since, for each $n \in \mathbb{N}$, $C_n(w)$ is never zero in $\{|w| < 1\}$, then either $C(w) \equiv 0$ or $C(w)$ is never zero in that set. But (see (17) and (19))

$$\begin{aligned} \inf_{n\in\mathbb{N}} |C_n(0)| &= \exp \left\{ -\frac{1}{2\pi} \sup_{n\in\mathbb{N}} \lim_{r\to 1} \int \ln^+ |(\hat{\rho}_n - \hat{\beta}_n)(re^{i\theta'})| \ d\theta' \right\} \\ &\geq \exp \left\{ -\frac{(M_1)^\delta}{\pi\delta \cos(\pi\delta/2)} \right\} > 0 \ . \end{aligned}$$

Therefore, $C(w)$ never equals zero in the unit disk.

Let us prove that $B(w) \equiv 0$ in $\{|w| < 1\}$. First, we show that $\hat{\beta}_n$ interpolates $\hat{\rho}_n$ at all the zeros of $W_n/A_n$ inside $\{|w| < 1\}$ according to multiplicity. Let $w'$ be an arbitrary zero of $W_n/A_n$ inside $\{|w| < 1\}$ of multiplicity $m > 0$. Assume that $w' \neq 0$. From (13) it follows that

$$\int \frac{1}{|A_n(z)|^2} \ \frac{z}{(z-w)^j} \ \frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2} \, d\theta = \int \frac{z}{(z-w)^j} \ \frac{d\rho_n(\theta)}{|A_n(z)|^2} \ , \quad j = 1, \ldots, m.$$

Since

$$\frac{1}{z-w} = \sum_{j=1}^{m} \frac{(w-w')^{j-1}}{(z-w')^j} + \frac{(w-w')^m}{(z-w')^m} \ \frac{1}{z-w} \ ,$$

then, obviously, $e_n(w)$ has a zero of order $m$ at $w = w'$. On the other hand, if $w' = 0$ is a zero of $W_n/A_n$ of order $m$, then

$$\int \frac{1}{z^j} \frac{1}{|A_n(z)|^2} \frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2} d\theta = \int \frac{1}{z^j} \frac{d\rho_n(\theta)}{|A_n(z)|^2} , \quad j = 0, 1, \ldots, m.$$

Since

$$\frac{1}{z-w} = \sum_{j=1}^{m} \frac{w^{j-1}}{z^j} + \frac{w^{m+1}}{z^{m+1}} \frac{1}{z-w} ,$$

then $e_n(w)$ has a zero of order $m+1$ at $w = 0$.

Consider the sequence of functions

$$H_n(w) = B_n(w) \widetilde{\prod} \frac{1 - \bar{w}_{n,i} w}{w - w_{n,i}} , \quad n \in \Gamma',$$

where $\widetilde{\prod}$ denotes the product taken only over those i's such that $w_{n,i}$ is a zero of $W_n/A_n$ of module less than 1. For all $n \in \Gamma'$, $H_n(w)$ is analytic in $\{|w| < 1\}$, and using the maximum principle for analytic functions considering (18), we have that $|H_n(w)| \leq 1$, $n \in \Gamma'$, in $\{|w| < 1\}$. Thus

$$|B_n(w)| \leq \widetilde{\prod} \frac{|w - w_{n,i}|}{|1 - \bar{w}_{n,i} w|}, \quad n \in \Gamma', \quad |w| < 1.$$

The right hand member of this inequality tends to zero because of (iv) of the admissibility condition (see p. 281 of [18]). Since $A_n$ contains no more than $|k|$ zeros of $W_n$, they have no influence on the divergence of the limit in (iv). We have shown that $B(w) \equiv 0$ and $C(w)$ is never zero in $\{|w| < 1\}$, hence, $\lim_{n \in \Gamma'} e_n(w) = 0$ uniformly on each compact subset of $\{|w| < 1\}$.

Now,

$$e_n(w) = \int \frac{z}{z-w} \; dh_n(\theta) = \sum_{i=0}^{\infty} \left( \int \frac{1}{z^j} \; dh_n(\theta) \right) w^i .$$

Thus, for each fixed $i = 0, 1, \ldots$,

$$\lim_{n \in \Gamma'} \int \frac{1}{z^j} \; dh_n(\theta) = 0 ,$$

because $\lim_{n \in \Gamma'} e_n(w) = 0$. The measures $h_n$ are real; therefore, the same holds for positive powers of $z$. Since any complex valued $2\pi$ periodic continuous function on $[0, 2\pi]$ can be uniformly approximated by powers of $z$ and $z^{-1}$, we have proved that $h_n \stackrel{*}{\longrightarrow} 0$, $n \in \Gamma$. Hence

$$h_n \stackrel{*}{\longrightarrow} 0 , \quad n \in \mathbb{N}.$$

If $k \in \mathbb{N}$, then $A_n \equiv 1$ and the proof would be over. Suppose that $k$ is a negative integer. In this case, since the zeros of $A_n(z)$ are in $\{|w| \leq 1\}$ and $A_n$ is monic, the coefficients of the trigonometric polynomial

$$|A_n(z)|^2 = \sum_{i=k}^{-k} c_{n,i} \, z^i$$

are uniformly bounded,

$$|c_{n,i}| \leq C_0, \quad |i| \leq |k|, \quad n \in \mathbb{N}.$$

For any $m \in \mathbb{Z}$, we have

$$\int z^m \ |A_n(z)|^2 \ dh_n(\theta) = \sum_{i=k}^{-k} c_{n,i} \ \int z^{m+i} \ dh_n(\theta) \to 0 \quad \text{as} \quad n \to \infty \ .$$

Using this and the weak * convergence of $\rho_n$ to $\rho$, we obtain

$$\lim_{n\to\infty} \int z^m \ \frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2} \, d\theta = \lim_{n\to\infty} \int z^m \ d\rho_n(\theta) = \int z^m \ d\rho(\theta)$$

for each $m \in \mathbb{Z}$. Thus

$$\frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2} \, d\theta \xrightarrow{*} d\rho(\theta) \ , \quad n \in \mathbb{N} \ ,$$

as we wanted to prove.

**Corollary 1** *If* $|w_{n,j}| \leq C_1 < 1$ , $j = 1, 2, \ldots, n$ , *for all* $n \in \mathbb{N}$ , *and* $\rho_n \xrightarrow{*} \rho$ *then*

$$\frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2} \, d\theta \xrightarrow{*} d\rho(\theta) \ .$$

**Remark 1.** Condition (iv) of admissibility expresses that the zeros of the sequence of polynomials $\{W_n\}$, $n \in \mathbb{N}$, cannot tend globally very rapidly to the unit circle. If they do tend rapidly, there is still hope for (2) if $\rho$ is sufficiently weak near the accumulation points of zeros of $\{W_n\}$, $n \in \mathbb{N}$, on $\{|w| < 1\}$. In this more delicate situation, in order to prove Theorem 1 one must use the scheme of [11] instead of the scheme of [12] which was employed here.

## 3 Proof of Theorem 2

We start by proving the stronger statement

$$\lim_{n\to\infty} \int \left( \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)} \right| - 1 \right)^2 d\theta = 0. \tag{20}$$

The arguments follow closely those in the proof of Theorem 2.1 and Corollary 2.2 in [14]. We have (see (13))

$$0 \leq \int \left| \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)} \right| - 1 \right|^2 d\theta = \int \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)} \right|^2 d\theta + 2\pi$$

$$-2 \int \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)} \right| d\theta = 4\pi - 2 \int \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)} \right| d\theta \ .$$

In order to obtain (20), the inequalities above show that it is sufficient to prove that

$$\liminf_{n\to\infty} \frac{1}{2\pi} \int \left| \frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)} \right| d\theta \geq 1 \tag{21}$$

Let $f$ be a $2\pi$-periodic nonnegative continuous function, and let $m$ be a nonnegative integer. Applying Cauchy-Schwarz's inequality twice, we obtain

$$\int (f(\theta)\rho_n'(\theta))^{\frac{1}{4}}\, d\theta = \int \left|\frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)}\right|^{\frac{1}{2}} \left|\frac{\varphi_{n,n+k+m}(z)}{W_n(z)}\right|^{\frac{1}{2}} \rho_n'(\theta)^{\frac{1}{4}} \left|\frac{W_n(z)}{\varphi_{n,n+k}(z)}\right|^{\frac{1}{2}} f(\theta)^{\frac{1}{4}}\, d\theta$$

$$\leq \left(\int \left|\frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)}\right| d\theta\right)^{\frac{1}{2}} \left(\int \left|\frac{\varphi_{n,n+k+m}(z)}{W_n(z)}\right|^2 \rho_n'(\theta)\, d\theta\right)^{\frac{1}{4}} \left(\int \left|\frac{W_n(z)}{\varphi_{n,n+k}(z)}\right|^2 f(\theta)\, d\theta\right)^{\frac{1}{4}}$$

$$\leq (2\pi)^{\frac{1}{4}} \left(\int \left|\frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)}\right| d\theta\right)^{\frac{1}{2}} \left(\int \left|\frac{W_n(z)}{\varphi_{n,n+k}(z)}\right|^2 f(\theta)\ d\theta\right)^{\frac{1}{4}} .$$

Thus

$$\left(\frac{1}{2\pi}\int (f(\theta)\rho_n'(\theta))^{\frac{1}{4}}\, d\theta\right)^4 \leq$$
$$\left(\frac{1}{2\pi}\int \left|\frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)}\right| d\theta\right)^2 \left(\frac{1}{2\pi}\int \left|\frac{W_n(z)}{\varphi_{n,n+k}(z)}\right|^2 f(\theta)\ d\theta\right) . \tag{22}$$

On the other hand,

$$\left(\frac{1}{2\pi}\int \left|(f\rho_n')^{\frac{1}{4}} - (f\rho')^{\frac{1}{4}}\right| d\theta\right)^4 \leq$$
$$\left(\frac{1}{2\pi}\int |f\rho_n' - f\rho'|^{\frac{1}{4}}\, d\theta\right)^4 \leq \left(\frac{1}{2\pi}\int |f\rho_n' - f\rho'|\, d\theta\right) .$$

This inequality and condition (i) of strong admissibility give

$$\lim_{n\to\infty} \left(\frac{1}{2\pi}\int (f(\theta)\rho_n'(\theta))^{\frac{1}{4}}\, d\theta\right)^4 = \left(\frac{1}{2\pi}\int (f(\theta)\rho'(\theta))^{\frac{1}{4}}\, d\theta\right)^4 \tag{23}$$

Taking limits in (22), by use of (2) and (23), it follows that

$$\left(\frac{1}{2\pi}\int (f(\theta)\rho'(\theta))^{\frac{1}{4}}\, d\theta\right)^4 \leq \liminf_{n\to\infty} \left(\frac{1}{2\pi}\int \left|\frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)}\right| d\theta\right)^2 \left(\frac{1}{2\pi}\int f(\theta)\ d\rho(\theta)\right)$$

By Corollary 3.3 in [14], from this inequality we get (21) and so (20) is satisfied. Notice that this relation holds uniformly in $m \geq 0$, the key reason for this is that $m$ occurs neither in the second factor of the right-hand side of (22) nor in the left-hand side.

Finally, we have that

$$\left(\int \left|\left|\frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)}\right|^2 - 1\right| d\theta\right)^2 \leq$$
$$\int \left(\left|\frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)}\right| - 1\right)^2 d\theta \int \left(\left|\frac{\varphi_{n,n+k}(z)}{\varphi_{n,n+k+m}(z)}\right| + 1\right)^2 d\theta.$$

Because of (20), the first integral on the right-hand side tends to zero and the second remains bounded, hence the proof is complete.

## 4 Weak Convergence on the Unit Circle

**Theorem 4** *Let* $(\{\rho_n\},\{W_n\},k)$ *be strongly admissible on* $[0,2\pi]$*, then*

$$\lim_{n\to\infty}\int\left|\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2\rho_n'(\theta)\,-\,1\right|\,d\theta=0\,. \tag{24}$$

*Moreover, for each* $m\in\mathbb{N}$*,*

$$\lim_{n\to\infty}\int\left|\frac{\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+m}(z)}z^m}{|W_n(z)|^2}\,\rho_n'(\theta)-1\right|\,d\theta=0\,. \tag{25}$$

**Proof.** The proof of this result follows the same ideas as in Theorem 2. Here, the main step is to show that

$$\liminf_{n\to\infty}\frac{1}{2\pi}\int\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|\sqrt{\rho_n'(\theta)}\,d\theta\geq 1$$

which is done in a similar fashion as in the proof of (21). (25) follows from (24) and (10). For more details see Theorem 4 in [12].

From Theorem 4, we get

**Theorem 5** *Let* $(\{\rho_n\},\{W_n\},k)$ *be strongly admissible on* $[0,2\pi]$*, then for every bounded Borel-measurable function* $f$ *on* $[0,2\pi]$ *and* $m\in\mathbb{N}$*, we have*

$$\lim_{n\to\infty}\int f(\theta)\,\frac{\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+m}(z)}z^m}{|W_n(z)|^2}\,\rho_n'(\theta)\,d\theta=\int f(\theta)\,d\theta \tag{26}$$

*and*

$$\lim_{n\to\infty}\int f(\theta)\,\frac{\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+m}(z)}z^m}{|W_n(z)|^2}\,d\rho_n(\theta)=\int f(\theta)\,d\theta. \tag{27}$$

**Proof.** From (25), equation (26) is immediate. To prove (27) notice that

$$\lim_{n\to\infty}\int\frac{|\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+m}(z)}z^m|}{|W_n(z)|^2}\,(d\rho_n)_s(\theta)=0\,, \tag{28}$$

where $(d\rho_n)_s$ represents the singular part of $d\rho_n$ with respect to Lebesgue's measure; that is, $(d\rho_n)_s(\theta)=d\rho_n(\theta)-\rho_n'(\theta)d(\theta)$. To see this, take $f\equiv 1$ in (26) with $m=0$ and use the fact that

$$\int\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2\rho_n'(\theta)\,d\theta\leq\int\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2d\rho_n(\theta)\;=2\pi\,.$$

We arrive at

$$\lim_{n\to\infty}\int\left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2(d\rho_n)_s(\theta)\;=0\,.$$

By Cauchy-Schwarz's inequality

$$\int \frac{|\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+m}(z)}z^m|}{|W_n(z)|^2} \ (d\rho_n)_s(\theta)$$

$$\leq \left(\int \left|\frac{\varphi_{n,n+k}(z)}{W_n(z)}\right|^2 \ (d\rho_n)_s(\theta)\right)^{\frac{1}{2}} \left(\int \left|\frac{\varphi_{n,n+k+m}(z)}{W_n(z)}z^m\right|^2 \ (d\rho_n)_s(\theta)\right)^{\frac{1}{2}} ,$$

and both integrals on the right-hand side tend to zero, thus (28) holds. Equations (28) and (26) give (27) and so we are done.

# 5 Ratio and Weak Convergence on the Real Line

In this section, we provide similar results to those above for sequences $\{\mu_n\}_{n\in\mathbb{N}}$ of finite positive Borel measures on $[-1,1]$ whose supports contain infinitely many points.

Let $\{w_{2n}\}_{n\in\mathbb{N}}$ be a sequence of polynomials with real coefficients such that, for each $n \in \mathbb{N}$: $\deg w_{2n} = i_n, 0 \leq i_n \leq 2n$; and $w_{2n} \geq 0$ on $[-1,1]$. If $i_n < 2n$, let $x_{2n,i} = \infty$ for $1 \leq i \leq 2n - i_n$; if, additionally, $i_n > 0$, then $\{x_{2n,i}\}_{2n-i_n+1\leq i\leq 2n}$, denotes the set of zeros of $w_{2n}$. When $i_n = 2n$, then $\{x_{2n,i}\}_{1\leq i\leq 2n}$, is the set of zeros of $w_{2n}$.

Set $d\tau_n = \dfrac{d\mu_n}{w_{2n}}$. If, for each $n \in \mathbb{N}$,

$$\int_{-1}^{1} \frac{d\mu_n(x)}{w_{2n}(x)} < +\infty ,$$

we can construct the table of polynomials $\{l_{n,m}\}_{n,m\in\mathbb{N}}$, where $l_{n,m}(x) = \beta_{n,m}x^m + \cdots$, $\beta_{n,m} > 0$, is the $m$-th orthonormal polynomial with respect to $\tau_n$; that is, these polynomials are uniquely determined by having positive leading coefficients and satisfying the relations

$$\int_{-1}^{1} l_{n,k}(x)\, l_{n,m}(x)\, d\tau_n(x) = \delta_{k,m} .$$

The limits of integration with respect to $x$ will always be $-1$ and 1, thus they will not be indicated.

According to the prescribed conditions $w_{2n}(\cos\theta)$ is nonnegative for $\theta \in \mathbb{R}$, thus (see p. 3 of [17]) there exists an algebraic polynomial $W'_{2n}(w)$ of degree $i_n$ whose zeros lie in $\{|w| \leq 1\}$ such that

$$w_{2n}(\cos\theta) = |W'_{2n}(e^{i\theta})|^2 , \quad \theta \in [0, 2\pi] .$$

It is easy to see that the zeros of $W'_{2n}$ are the points

$$\left\{\frac{1}{\Psi(x_{2n,i})}\right\}_{2n-i_n+1\leq i\leq 2n} ,$$

where $\Psi(x) = x + \sqrt{x^2-1}$ is conformal mapping of $\overline{\mathbf{C}} \setminus [-1,1]$ onto $\{|w| > 1\}$ such that $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$ (on $[-1,1]$ we extend $\Psi$ continuously, considering the interval to have two sides as it is usually done). Take $W_{2n}(w) = w^{2n-i_n}W'_{2n}(w)$ ; then, $\deg W_{2n} = 2n$ and

$$w_{2n}(\cos\theta) = |W_{2n}(e^{i\theta})|^2 , \quad \theta \in [0, 2\pi] .$$

The polynomials $l_{n,m}$ are closely related to the polynomials $\varphi_{2n,2m}$ orthonormal with respect to the measure $\sigma_{2n}$ defined by

$$d\sigma_{2n}(\theta) = d\tau_n(\cos\theta) = \frac{d\mu_n(\cos\theta)}{|W_{2n}(z)|^2} \ , \quad z = e^{i\theta} \ .$$

That is, $\sigma_{2n}(E) = \tau_n(\{\cos\theta;\ \theta \in E\})$ whenever $E \subset [0,\pi]$ or $E \subset [\pi, 2\pi]$. Thus, writing $\sigma_{2n}(\theta) = \sigma_{2n}(\{0 \le t \le \theta\})$, we have

$$\sigma_{2n}(\theta) = \begin{cases} G_n(\cos\theta) \ , & 0 \le \theta \le \pi \ , \\ -G_n(\cos\theta) \ , & \pi \le \theta \le 2\pi \ , \end{cases}$$

where $G_n(x) = \int_{-1}^{x} d\tau_n(t)$, $x \in [-1,1]$, at every point $\theta$ where $\sigma_{2n}$ is continuous; and so, almost everywhere in $[0, 2\pi]$. Furthermore,

$$\sigma'_{2n}(\theta) = |\sin\theta|\, G_n(\cos\theta) = |\sin\theta| \frac{\mu'_n(\cos\theta)}{|W_{2n}(e^{i\theta})|^2} = |\sin\theta|\, \tau'_n(\cos\theta),$$

whenever either side exists (thus almost everywhere). Notice that $\sigma' > 0$ almost everywhere on $[0, 2\pi]$ if and only if $\tau' > 0$ almost everywhere on $[-1, 1]$, where

$$\sigma(\theta) = \begin{cases} \tau(\cos\theta) \ , & 0 \le \theta \le \pi \ , \\ -\tau(\cos\theta) \ , & \pi \le \theta \le 2\pi \ . \end{cases}$$

For $n$ fixed, we can use the well-known formula (see Theorem $V$.1.4 of [6])

$$l_{n,m}(x) = \frac{\varphi_{2n,2m}(w) + \varphi^*_{2n,2m}(w)}{w^m\sqrt{2\pi(1+\Phi_{2n,2m}(0))}} \ , \tag{29}$$

where $\Phi_{2n,2m} = \dfrac{\varphi_{2n,2m}}{\alpha_{2n,2m}}$ and $x = \dfrac{1}{2}(w + 1/w)$. If $L_{2n,2m} = \dfrac{l_{2n,2m}}{\beta_{2n,2m}}$, the previous relation can be written as follows

$$L_{n,m}(x) = \frac{\Phi_{2n,2m}(w) + \Phi^*_{2n,2m}(w)}{(2w)^m(1+\Phi_{2n,2m}(0))}. \tag{30}$$

Consequently, using (29) and (30), for $k \in \mathbb{Z}$ fixed and $n + k \ge 0$, we have

$$\frac{l_{n,n+k+1}(x)}{l_{n,n+k}(x)} = \frac{1}{w} \frac{\sqrt{1+\Phi_{2n,2n+2k}(0)}}{\sqrt{1+\Phi_{2n,2n+2k+2}(0)}} \frac{\varphi_{2n,2n+2k+2}(w)}{\varphi_{2n,2n+2k}(w)} \frac{1 + \dfrac{\varphi^*_{2n,2n+2k+2}(w)}{\varphi_{2n,2n+2k+2}(w)}}{1 + \dfrac{\varphi^*_{2n,2n+2k}(w)}{\varphi_{2n,2n+2k}(w)}}, \tag{31}$$

$$\frac{L_{n,n+k+1}(x)}{L_{n,n+k}(x)} = \frac{1}{2w} \frac{1+\Phi_{2n,2n+2k}(0)}{1+\Phi_{2n,2n+2k+2}(0)} \frac{\Phi_{2n,2n+2k+2}(w)}{\Phi_{2n,2n+2k}(w)} \frac{1 + \dfrac{\Phi^*_{2n,2n+2k+2}(w)}{\Phi_{2n,2n+2k+2}(w)}}{1 + \dfrac{\Phi^*_{2n,2n+2k}(w)}{\Phi_{2n,2n+2k}(w)}}. \tag{32}$$

We define $d\rho_n(\theta) = d\mu_n(\cos\theta)$ as above.
**Definition 3** *Let $k \in \mathbb{Z}$ be fixed, we say that $(\{\mu_n\}, \{w_{2n}\}, k)$ is strongly admissible on the interval $[-1,1]$ if $(\{\rho_n\}, \{W_{2n}\}, k)$ is strongly admissible on $[0, 2\pi]$.*

From the construction above, it is easy to see that this reduces to $(\{\mu_n\}, \{w_{2n}\}, k)$ satisfying the following conditions

(I) There exists a finite positive Borel measure $\mu$ on $[-1,1]$ such that $\mu_n \xrightarrow{*} \mu$ , $n \to \infty$, and

$$\lim_{n\to\infty} \int |\mu_n'(x) - \mu'(x)|\, dx = 0 \; .$$

(II) $\mu' > 0$ almost everywhere.

(III) $\|\tau_n\| = \int d\tau_n(x) < +\infty, \quad n \in \mathbb{N}$ .

(IV) $\int \prod_{i=1}^{-k} |1 - \dfrac{x}{x_{2n,i}}|^{-1}\, d\mu_n(x) \leq M < +\infty$ , $n \in \mathbb{N}$ , where $\dfrac{x}{x_{2n,i}} \equiv 0$ if $x_{2n,i} = \infty$ (this condition applies only to the case when $k$ is a negative integer).

(V) $\lim_{n\to\infty} \sum_{i=1}^{2n} \left( 1 - \dfrac{1}{|\Psi(x_{2n,i})|} \right) = +\infty$ .

Under these conditions one gets

**Theorem 6** *Let* $(\{\mu_n\}, \{w_{2n}\}, 2k)$ *be strongly admissible on the interval* $[-1,1]$*, then*

$$\lim_{n\to\infty} \frac{L_{n,n+k+1}(x)}{L_{n,n+k}(x)} = \frac{1}{2} \lim_{n\to\infty} \frac{l_{n,n+k+1}(x)}{l_{n,n+k}(x)} = \frac{1}{2}\, \Psi(x) \; , \quad x \in \mathbb{C} \setminus [-1,1] \; , \tag{33}$$

*where the limit holds uniformly on each compact subset of* $\mathbb{C} \setminus [-1,1]$*, and*

$$\lim_{n\to\infty} \frac{\beta_{n,n+k+1}}{\beta_{n,n+k}} = 2 \; . \tag{34}$$

**Proof.** The proof of (33) is immediate using (31), (32), (8), (10) and (12), while (34) is a direct consequence of (33).

The next two results are proved exactly the same way as Theorems 8 and 9 in [12] respectively. At certain points one must substitute the result used from [12] by the corresponding one proved in the sections above. Therefore, we simply state them.

Given $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $n+k-1 \geq 0$, as in the general theory of orthogonal polynomials, one proves that the polynomial $l_{n,n+k}$ satisfies the three-term relation

$$x\, l_{n,n+k}(x) = a_{n,k,1}\, l_{n,n+k+1}(x) + a_{n,k,0}\, l_{n,n+k}(x) + a_{n,k,-1}\, l_{n,n+k-1}(x) \; .$$

With this notation, we have

**Theorem 7** *Let* $(\{\mu_n\}, \{w_{2n}\}, 2k-2)$ *be strongly admissible on the interval* $[-1,1]$*, then*

$$\lim_{n\to\infty} a_{n,k,1} = \lim_{n\to\infty} a_{n,k,-1} = \frac{1}{2} \quad \textit{and} \quad \lim_{n\to\infty} a_{n,k,0} = 0 \; . \tag{35}$$

**Theorem 8** *Let* $(\{\mu_n\}, \{w_{2n}\}, 2k)$ *be strongly admissible on the interval* $[-1,1]$ *for each* $k \in \mathbb{Z}$ *and let* $T_n$ *denote the* $n$*-th Chebyshev polynomial, i.e.,* $T_n(\cos\theta) = \cos n\theta$*. Then, for every* $m \in \mathbb{N}$ *and every bounded Borel-measurable function* $f$ *on* $[-1,1]$*, we have*

$$\lim_{n\to\infty} \int f(x) \frac{l_{n,n+k}(x) l_{n,n+k+m}(x)}{w_{2n}(x)} \mu_n'(x)\, dx = \frac{1}{\pi} \int f(x)\, T_m(x) \frac{dx}{\sqrt{1-x^2}} \tag{36}$$

*and*

$$\lim_{n\to\infty} \int f(x) \frac{l_{n,n+k}(x) l_{n,n+k+m}(x)}{w_{2n}(x)} d\mu_n(x) = \frac{1}{\pi} \int f(x)\, T_m(x) \frac{dx}{\sqrt{1-x^2}} . \tag{37}$$

The only condition of strong admissibility in which $k$ is involved is $(IV)$. If the zeros of $w_{2n}$ are bounded away from $[-1,1]$ then for all $2k$ that property holds.

# 6 Applications

In order to fully understand the interest of this application, we must first explain the connection between the systems of orthogonal polynomials with respect to varying measures which we are about to consider and Hermite-Padé approximation of Nikishin systems of functions. In [9] (see also [1]), the more general construction of mixed Angelesco-Nikishin systems (or generalized Nikishin systems) is studied and the $n$-th root asymptotic behavior of their Hermite-Padé approximants is given. We adopt the notation introduced in [9] and use some of the new orthogonality relations revealed in that paper. Nevertheless, we restrict our attention to purely Nikishin systems because our present methods do not cover the generalized case. Theorem 9 below is useful for obtaining strong (or Szegő-type) asymptotics of Nikishin systems.

Let $F_1$ and $F_2$ be two nonintersecting segments of the real line, $\sigma_1$ and $\sigma_2$ two finite positive Borel measures such that $S(\sigma_1) \subset F_1$ , $S(\sigma_2) \subset F_2$. We define a new measure $\langle \sigma_1, \sigma_2 \rangle$

$$d\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{d\,\sigma_2(t)}{x-t}\, d\,\sigma_1(x) = \hat{\sigma}_2(x)\, d\sigma_1(x)$$

This measure $\langle \sigma_1, \sigma_2 \rangle$, obviously has constant sign on its support $F_1$.

For a system of segments $F_1, F_2, \ldots, F_m$, such that $F_k \cap F_{k+1} = \emptyset$, $k = 1, 2, \ldots, m-1$, and finite, positive Borel measures $\sigma_1, \sigma_2, \ldots, \sigma_m$, $S(\sigma_k) \subset F_k$, $k = 1, 2, \ldots, m$, we define inductively the measures

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_{k+1} \rangle = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_{k+1} \rangle \rangle , \quad k = 2, \ldots, m-1 .$$

Thus, on $F_1$, we have defined $m$ finite Borel measures each one with constant sign. Set

$$s_1 = \langle \sigma_1 \rangle = \sigma_1 \ , \ s_2 = \langle \sigma_1, \sigma_2 \rangle \ , \ \ldots \ , \ s_m = \langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle \ .$$

The system of functions $(f_1, f_2, \ldots, f_m)$, where

$$f_k(z) = \hat{s}_k(z) = \int \frac{ds_k(x)}{z-x} , \quad k = 1, \ldots, m$$

defines what is called a Nikishin system.

Consider a multi-index $n = (n_1, n_2, \ldots, n_m) \in \mathbb{N}^m$. There exists a polynomials $Q_n$ that satisfies the conditions

$$\begin{gathered} Q_n \not\equiv 0 \ , \quad \deg Q_n \le |n| = n_1 + \ldots + n_m \\ (Q_n f_k - P_{n,k})(z) = O\left(\frac{1}{z^{n_k+1}}\right) \ , \quad z \to \infty \ , \quad k = 1, \ldots, m \ , \end{gathered} \tag{38}$$

where $P_{n,k}$ is a polynomial. The rational functions

$$R_{n,k} = \frac{P_{n,k}}{Q_n} \ , \quad k = 1, \ldots, m \ ,$$

are the Hermite-Padé approximants (or simultaneous Padé approximants) of the system $(f_1, \ldots, f_m)$ relative to the multi-index $n$.

From (38), it follows that

$$0 = \int x^\nu Q_n(x)\, ds_k(x) \ , \quad \nu = 0, \ldots, n_k - 1 \ , \quad k = 1, \ldots, m \ . \tag{39}$$

Define the functions of second kind

$$\Phi_{n,k}(z) = \int \frac{Q_n(x)}{z - x}\, ds_k(x) \ , \quad k = 1, \ldots, m \ .$$

From (39), it follows that

$$\Phi_{n,k}(z) = \frac{1}{q(z)} \int \frac{Q_n(x) q(x)}{z - x}\, ds_k(x) \ , \quad k = 1, \ldots, m \ , \tag{40}$$

where $q$ is any polynomial of degree $\le n_k$. Therefore,

$$\Phi_{n,k}(z) = O\left(\frac{1}{z^{n_k+1}}\right) \ , \quad z \to \infty \ . \tag{41}$$

Using (41) (see (38)) and the obvious identity

$$Q_n(z) \int \frac{ds_k(x)}{z - x} - \int \frac{Q_n(z) - Q_n(x)}{z - x}\, ds_k(x) = \int \frac{Q_n(x)}{z - x}\, ds_k(x) \ ,$$

we obtain formulas for the numerators of the Hermite-Padé approximants

$$P_{n,k}(z) = \int \frac{Q_n(z) - Q_n(x)}{z - x}\, ds_k(x) \ , \quad k = 1, \ldots, m \ , \tag{42}$$

and for the remainder term

$$f_k(z) - R_{n,k}(z) = \frac{1}{(Q_n q)(z)} \int \frac{(Q_n q)(x)}{z - x}\, ds_k(x) \ , \quad k = 1, \ldots, m \ , \tag{43}$$

where $q$ is any polynomial of degree $\le n_k$.

In the sequel, we assume that the multi-index $n = (n_1, \ldots, n_m)$ satisfies

$$j \le k \ \Rightarrow \ n_k \le n_j + 1 \ . \tag{44}$$

For each multi-index $n$ (with property (44)), we define inductively the following functions

$$\Psi_{n,0}(z) = Q_n(z) \ , \quad \Psi_{n,k}(z) = \int \frac{\Psi_{n,k-1}(x)}{z - x}\, d\sigma_k(x) \ , \quad k = 1, \ldots, m \ . \tag{45}$$

For each $j = 0, \ldots, m-1$ and $k = j+1, \ldots, m$ , we define the measure

$$s_k^j = \langle \sigma_{j+1}, \ldots, \sigma_k \rangle \ .$$

We have

**Lemma 1** *For each $j = 0, \ldots, m-1$ , the functions $\Psi_{n,j}$ satisfy*

$$0 = \int x^\nu \Psi_{n,j} \, ds_k^j(x) \ , \quad \nu = 0, \ldots, n_k - 1 \ , \quad k = j+1, \ldots, m \ . \tag{46}$$

For $j = 0$ , (46) coincides with (39). The proof may be carried out by induction showing that if the statement is true for $j \in \{0, \ldots, m-2\}$ then it also holds for $j+1$ (for details see Proposition 1 in [9]).

Taking $k = j+1$, $s_{j+1}^j = \sigma_{j+1}$ and (46) indicates that

$$0 = \int x^\nu \Psi_{n,j}(x) \, d\sigma_{j+1}(x) \ , \quad \nu = 0, \ldots, n_{j+1} - 1 \ , \quad j = 0, \ldots, m-1 \ . \tag{47}$$

From (47), it follows that $\Psi_{n,j}(z)$ has at least $n_{j+1}$ changes of sign on $F_{j+1}$. Denote by $Q_{n,k}$ the monic polynomial whose zeros are the zeros of $\Psi_{n,k-1}$ on $F_k$ (counting their order). According to (47), $\deg Q_{n,k} \geq n_k$ . Denote $Q_{n,m+1} \equiv 1$. Set

$$N_{n,k} = \sum_{j=k}^{m} n_j \ , \quad k = 1, \ldots, m \ .$$

**Lemma 2** *For $k = 1, \ldots, m$*

$$0 = \int x^\nu \Psi_{n,k-1}(x) \frac{d\sigma_k(x)}{Q_{n,k+1}(x)} \ , \quad \nu = 0, \ldots, N_{n,k} - 1 \ . \tag{48}$$

For $k = m$, (48) reduces to (47) with $j = m-1$. For the rest of the indicated values of $k$, the formula may be proved by induction for decreasing values of the index $k$ (for details, see Proposition 2 in [9]).

Using (48), we have that $\deg Q_{n,k} \geq N_{n,k}$. Using (45), (48), and Cauchy's integral formula, it is easy to deduce that if for some $k$, $\deg Q_{n,k} > N_{n,k}$, then $\deg Q_{n,k-1} > N_{n,k-1}$. Since $\Psi_{n,0} = Q_n$, and $\deg Q_n \leq |n| = N_{n,1}$, we obtain (for details, see Proposition 3 in [9])

**Lemma 3** *For each $k = 1, \ldots, m$, the polynomial $Q_{n,k}$ has exactly $N_{n,k}$ simple zeros on the interval $F_k$ and $\deg Q_{n,k} = N_{n,k}$. In particular, $Q_{n,1} = Q_n$.*

Set $Q_{n,0} \equiv 1$. For each $k = 1, \ldots, m$, (48) may be rewritten as follows

$$0 = \int x^\nu Q_{n,k}(x) \left| \frac{Q_{n,k-1}(x)\Psi_{n,k-1}(x)}{Q_{n,k}(x)} \right| \frac{d\sigma_k(x)}{|Q_{n,k-1}(x) Q_{n,k+1}(x)|} \ , \quad \nu = 0, \ldots, N_{n,k} - 1 \ . \tag{49}$$

On the other hand, for $k = 1, \ldots, m$, $\frac{\Psi_{n,k}}{Q_{n,k+1}}$ is holomorphic in $\overline{\mathbb{C}} \setminus F_k$, and (recall that $Q_{n,m+1} \equiv 1$)

$$\frac{\Psi_{n,k}(z)}{Q_{n,k+1}(z)} = O\left( \frac{1}{z^{N_k+1}} \right) \ , \quad z \to \infty \ .$$

This follows from (47) for $j = k-1$, formulas (45), and the fact that $\deg Q_{n,k+1} = N_{k+1}$ ($N_{m+1} = 0$). In particular,

$$\frac{Q_{n,k}(z)\Psi_{n,k}(z)}{Q_{n,k+1}(z)} = O\left(\frac{1}{z}\right) , \quad z \to \infty .$$

Let $\Gamma$ be any contour surrounding the interval $F_k$ such that $F_{k+1}$ and $z$ lie outside $\Gamma$. For $k = m$, $\Gamma$ surrounds $F_k$ and $z$ lies outside this curve. From Cauchy's integral formula and (45), we obtain

$$\begin{aligned}
&\frac{Q_{n,k}(z)\Psi_{n,k}(z)}{Q_{n,k+1}(z)} = \frac{1}{2\pi i}\int_\Gamma \frac{Q_{n,k}(\zeta)\Psi_{n,k}(\zeta)}{Q_{n,k+1}(\zeta)}\frac{d\zeta}{z-\zeta} \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{Q_{n,k}(\zeta)d\zeta}{Q_{n,k+1}(\zeta)(z-\zeta)}\int \frac{\Psi_{n,k-1}(t)}{\zeta - t}d\sigma_k(t) \\
&= \int \frac{\Psi_{n,k-1}(t)d\sigma_k(t)}{2\pi i}\int_\Gamma \frac{Q_{n,k}(\zeta)}{Q_{n,k+1}(\zeta)(z-\zeta)}\frac{d\zeta}{\zeta - t} \\
&= \int \frac{Q_{n,k}(t)\Psi_{n,k-1}(t)}{Q_{n,k+1}(t)}\frac{d\sigma_k(t)}{z-t} .
\end{aligned}$$

We can rewrite the equality above in the more symmetric form

$$\begin{aligned}
&\frac{Q_{n,k}(z)\Psi_{n,k}(z)}{Q_{n,k+1}(z)} = \\
&\int \frac{Q^2_{n,k}(t)}{z-t}\frac{Q_{n,k-1}(t)\Psi_{n,k-1}(t)}{Q_{n,k}(t)}\frac{d\sigma_k(t)}{Q_{n,k-1}(t)Q_{n,k+1}(t)} , \quad k = 1, \ldots, m .
\end{aligned} \tag{50}$$

Set

$$K_{n,k} = \left(\int Q^2_{n,k}(x)\left|\frac{Q_{n,k-1}(x)\Psi_{n,k-1}(x)}{Q_{n,k}(x)}\right|\frac{d\sigma_k(x)}{|Q_{n,k-1}(x)Q_{n,k+1}(x)|}\right)^{-1/2} , \quad k = 1, \ldots, m .$$

Take

$$K_{n,0} = 1 , \quad \kappa_{n,k} = \frac{K_{n,k}}{K_{n,k-1}} , \quad k = 1, \ldots, m .$$

Define

$$q_{n,k} = \kappa_{n,k}Q_{n,k} , \quad F_{n,k}(z) = K^2_{n,k-1}\left|\frac{Q_{n,k-1}(z)\Psi_{n,k-1}(z)}{Q_{n,k}(z)}\right| , \quad k = 1, \ldots, m .$$

With this notation, $q_{n,k}$ is orthonormal with respect to the varying measure

$$\frac{F_{n,k}(x)d\sigma_k(x)}{|Q_{n,k-1}(x)Q_{n,k+1}(x)|} .$$

From the statement of the following result, it is obvious that the main tool in its proof is Theorem 8. Because of the change in notation in this section, in order to avoid confusion, we wish to emphasize what the different indices in Theorem 8 mean in relation with their use in the present situation. In Theorem 8, $n \in \mathbb{N}$ is not a multi-index, it controls the connection between the orthonormal polynomial, the measure with respect to which it satisfies orthogonality relations, and together with $k$ the relation between the degree of $l_{n,n+k}$ and $w_{2n}$ (we take $m = 0$ in Theorem 8, which has nothing to do with the $m$ functions in the Nikishin system).

The first important observation is that in Theorem 8 there is no need that the index $n$ covers the whole sequence of natural numbers. If the conditions of strong admissibility are satisfied for $n \in \Lambda \subset \mathbb{N}$ then the statement is valid taking limit as $n \to \infty$ for $n \in \Lambda$. Next, notice that $\deg w_{2n} \leq 2n$. Thus, the degree of the orthonormal polynomials can increase to infinity much faster than $\deg w_{2n}$. Moreover, one can even take $w_{2n} \equiv 1$, for all $n$, so that their degrees may not tend to infinity at all. On the other hand,

$$\deg w_{2n} - 2 \deg l_{n,n+k} \leq 2n - 2(n+k) \leq 2|k| \ ,$$

where $k$ is a fixed integer, so as $n$ tends to infinity this difference of degrees must remain bounded above. This restriction is essential in the method by which we reached Theorem 8 (moreover, it is in the matter of things). In proving Theorem 1, which is the cornerstone of this paper, we pass over to the left hand of (13) as much of $W_n$ as relation (12) allows. Then, at the end of the proof, we are able to deal with what was left behind thanks to the fact that $\deg A_n$ remains bounded as $n$ tends to infinity.

Finally, it is easy to see that Theorem 8 remains valid if for each $n \in \Lambda$, the sign of $w_{2n}$ on the interval of orthogonality $F = [a, b]$ is fixed (positive or negative depending on $n$). The only changes in the expressions (35) and (36) is that on the left hand you must place $|w_{2n}(x)|$ instead of $w_{2n}(x)$, and on the right you substitute $\sqrt{1-x^2}$ by $\sqrt{(b-x)(x-a)}$.

We are ready for

**Theorem 9** *Let $\Lambda$ be a sequence of multi-indices such that (44) takes place, $n_{k-1} - n_k \leq C, k = 2, \ldots, m$, where $C$ is a constant independent of $n \in \Lambda$, and $n_1 \to \infty$ as $n$ varies over $\Lambda$. Assume that for each $k = 1, \ldots, m, \sigma_k' > 0$ almost everywhere on $F_k$. Then, for each $k = 1, \ldots, m$, and every bounded Borel measureable function $f$ on $F_k = [a_k, b_k]$*

$$\lim_{n \in \Lambda} \int f(x) \frac{q_{n,k}^2(x)}{|Q_{n,k-1}(x) Q_{n,k+1}(x)|} F_{n,k}(x) \sigma_k'(x) dx = \frac{1}{\pi} \int_{a_k}^{b_k} f(x) \frac{dx}{\sqrt{(b_k - x)(x - a_k)}} \qquad (51)$$

*and*

$$\lim_{n \in \Lambda} \int f(x) \frac{q_{n,k}^2(x)}{|Q_{n,k-1}(x) Q_{n,k+1}(x)|} F_{n,k}(x) d\sigma_k(x) = \frac{1}{\pi} \int_{a_k}^{b_k} f(x) \frac{dx}{\sqrt{(b_k - x)(x - a_k)}} \ . \qquad (52)$$

**Proof.** It will be carried out by induction on the index $k$ (according to its meaning in Theorem 9) and using Theorem 8. For each $k \in \{1, \ldots, m\}$ the role of $\mu_n$ is carried by $F_{n,k} d\sigma_k$, that of $w_{2n}$ is taken by $Q_{n,k-1} Q_{n,k+1}$, and $l_{n,n+k}$ is $q_{n,k}$. The multi-indices $n$ will run over the sequence $\Lambda$.

Taking into consideration that the zeros of $Q_{n,k-1} Q_{n,k}$ remain uniformly bounded away from $F_k$ (inside $F_{k-1} \cup F_{k+1}$ which does not intersect $F_k$) it is easy to check that all the conditions of strong admissibility from (III) to (V) are fulfilled. Moreover ($N_{m+1} = 0$), for $k = 2, \ldots, m$

$$\deg Q_{n,k-1} Q_{n,k+1} - 2 \deg Q_{n,k} = N_{n,k-1} + N_{n,k+1} - 2N_{n,k} = n_{k-1} - n_k \leq C \ ,$$

while

$$\deg Q_{n,0} Q_{n,2} - 2 \deg Q_{n,1} = N_{n,2} - 2N_{n,1} = -n_1 - N_{n,1} \leq 0 \ ,$$

Thus the difference between the degree of the denominator and the numerator of the rational function $\frac{q_{n,k}^2}{Q_{n,k-1} Q_{n,k+1}}$ in (51)-(52) remains uniformly bounded as required.

Conditions (I)-(II) will be checked step by step as we carry out the induction process. For $k = 1, F_{n,1}(x) \equiv 1$ and the conditions (I)-(II) are immediate, taking as limit the measure $\sigma_1$, because $F_{n,1}(x)d\sigma_1(x) = d\sigma_1(x)$ remains fixed. Since $q_{n,1}$ is orthonormal with respect to $d\sigma_1(x)/|Q_{n,0}(x)Q_{n,2}(x)|$ (recall that $Q_{n,0} \equiv 1$) then (51) and (52) follow directly from (35) and (36) respectively (taking into consideration the remarks made before the statement of this theorem). Assume that (51) and (52) take place for some $k-1$, where $k \in \{2, \ldots, m\}$, and let us prove that then they also hold for $k$.

To this end, it is sufficient to show that condition (I)-(II) of strong admissibility take place. More precisely, we must show that there exists a finite positive Borel measure $\sigma$ on $F_k$ such that $F_{n,k}(x)d\sigma_k(x) \xrightarrow{*} d\sigma(x)$ as $n$ runs over the sequence of multi-indices $\Lambda$ and

$$\lim_{n\in\Lambda} \int |F_{n,k}(x)\sigma_k'(x) - \sigma'(x)|dx \, .$$

Such a measure is easy to find if we prove that the sequence of functions $\{F_{n,k}\}$ converges uniformly on $F_k$ to a strictly positive continuous function as $n$ varies over $\Lambda$.

In (50) substitute $k$ by $k-1$, multiply either sides by $K^2_{n,k-1}$ and take the modulus. We obtain (notice that $K_{n,k-1} = \kappa_{n,k-1}K_{n,k-2}$)

$$F_{n,k}(z) = \left| \int \frac{1}{z-t} \frac{q^2_{n,k-1}(t)}{|Q_{n,k-2}(t)Q_{n,k}(t)|} F_{n,k-1}(t)d\sigma_{k-1}(t) \right| \, . \tag{53}$$

Using (52) for $k-1$, taking $f(t) = (z-t)^{-1}$, we obtain for each $z \in \overline{\mathbb{C}} \setminus F_{k-1}$ the pointwise limit

$$\lim_{n\in\Lambda} \int \frac{1}{z-t} \frac{q^2_{n,k-1}(t)}{|Q_{n,k-2}(t)Q_{n,k}(t)|} F_{n,k-1}(t)d\sigma_{k-1}(t) = \frac{1}{\pi} \int_{a_{k-1}}^{b_{k-1}} \frac{1}{z-t} \frac{dt}{\sqrt{(b_{k-1}-t)(t-a_{k-1})}} \, . \tag{54}$$

The integrals on the left hand of (54) define a sequence of analytic functions in $\overline{\mathbb{C}} \setminus F_{k-1}$ which is uniformly bounded on each compact subset $K$ of $\overline{\mathbb{C}} \setminus F_{k-1}$ by $\frac{1}{d(K,F_{k-1})}$, where $d(K, F_{k-1})$ denotes the distance between the non-intersecting compact sets $K$ and $F_{k-1}$. Therefore, the limit in (54) is uniform on $K$. In particular, it is uniform on $F_k$. It is well known and easy to verify using Cauchy's integral formula that

$$\frac{1}{\pi} \int_{a_{k-1}}^{b_{k-1}} \frac{1}{z-t} \frac{dt}{\sqrt{(b_{k-1}-t)(t-a_{k-1})}} = \frac{1}{\sqrt{(z-b_{k-1})(z-a_{k-1})}} \, , \tag{55}$$

where the square root is taken so that $\sqrt{(z-b_{k-1})(z-a_{k-1})} > 0$ for $z = x > b_{k-1}$.

From (53), (54), and (55) it follows that

$$\lim_{n\in\Lambda} F_{n,k}(x) = \frac{1}{\sqrt{|(x-b_{k-1})(x-a_{k-1})|}} \, ,$$

uniformly on $F_k$, which is what we wanted to prove. Conditions (I)-(II) follow taking $d\sigma(x) = \frac{1}{\sqrt{|(x-b_{k-1})(x-a_{k-1})|}} d\sigma_k(x)$. With this, the proof of Theorem 9 is complete.

From Lemma 3 and the fact that $F_{k-1} \cap F_k = \emptyset = F_k \cap F_{k+1}$, for each $n \in \Lambda$ and $k \in \{1, \ldots, m\}$, the function $\frac{Q_{n,k-1}\Psi_{n,k-1}}{Q_{n,k-1}Q_{n,k}Q_{n,k+1}}$ has a constant sign on $F_k$. If it is positive, from (50), we have

$$\frac{Q_{n,k}(z)\Psi_{n,k}(z)}{Q_{n,k+1}(z)} = \int \frac{Q^2_{n,k}(t)}{z-t} \frac{Q_{n,k-1}(t)\Psi_{n,k-1}(t)}{Q_{n,k}(t)} \frac{d\sigma_k(t)}{Q_{n,k-1}(t)Q_{n,k+1}(t)}$$

$$= \int \frac{Q^2_{n,k}(t)}{z-t} \left| \frac{Q_{n,k-1}(t)\Psi_{n,k-1}(t)}{Q_{n,k}(t)} \right| \frac{d\sigma_k(t)}{|Q_{n,k-1}(t)Q_{n,k+1}(t)|} \, .$$

If negative,

$$-\frac{Q_{n,k}(z)\Psi_{n,k}(z)}{Q_{n,k+1}(z)} = \int \frac{Q^2_{n,k}(t)}{z-t} \frac{-Q_{n,k-1}(t)\Psi_{n,k-1}(t)}{Q_{n,k}(t)} \frac{d\sigma_k(t)}{Q_{n,k-1}(t)Q_{n,k+1}(t)}$$

$$= \int \frac{Q^2_{n,k}(t)}{z-t} \left| \frac{Q_{n,k-1}(t)\Psi_{n,k-1}(t)}{Q_{n,k}(t)} \right| \frac{d\sigma_k(t)}{|Q_{n,k-1}(t)Q_{n,k+1}(t)|} \, .$$

Normalize $\Psi_{n,k}$ according to the following rule. In the first case above, take $\psi_{n,k} = K^2_{n,k}\Psi_{n,k}$, in the second $\psi_{n,k} = -K^2_{n,k}\Psi_{n,k}$. In either cases, multiplying either sides of the two previous formulas by $K^2_{n,k}$, we obtain

$$\frac{Q_{n,k}(z)\psi_{n,k}(z)}{Q_{n,k+1}(z)} = \int \frac{q^2_{n,k}(t)}{z-t} F_{n,k}(t) \frac{d\sigma_k(t)}{|Q_{n,k-1}(t)Q_{n,k+1}(t)|} \tag{56}$$

**Corollary 2** *Assume that the conditions of Theorem 9 take place, then for each* $k \in \{1, \ldots, m\}$

$$\lim_{n\in\Lambda} Q_{n,k}(z) \prod_{j=k}^{m} \psi_{n,j}(z) = \prod_{j=k}^{m} \frac{1}{\sqrt{(z-b_j)(z-a_j)}} \tag{57}$$

*where the limit is uniform on each compact subset of* $\overline{\mathbb{C}} \setminus \left(\cup_{j=k}^m F_j\right)$ *and for each* $k$ *the square root is taken so that* $\sqrt{(z-b_k)(z-a_k)} > 0$ *for* $z = x > b_k$.

**Proof.** From (56),

$$Q_{n,k}(z) \prod_{j=k}^{m} \psi_{n,j}(z) = \prod_{j=k}^{m} \frac{Q_{n,j}(z)\psi_{n,j}(z)}{Q_{n,j+1}(z)} = \prod_{j=k}^{m} \int \frac{q^2_{n,j}(t)}{z-t} F_{n,j}(t) \frac{d\sigma_j(t)}{|Q_{n,j-1}(t)Q_{n,j+1}(t)|} \, .$$

Using Theorem 9, for $f(t) = (z-t)^{-1}$, we obtain pointwise convergence for each $z \in \overline{\mathbb{C}} \setminus \left(\cup_{j=k}^m F_j\right)$ . In the proof of Theorem 9, we showed that for each $j \in \{1, \ldots, m-1\}$ fixed, this limit is uniform in $\overline{\mathbb{C}} \setminus F_j$. The same arguments are valid for $j = m$ (we didn't need to consider this case in the proof of that result). Therefore, (57) immediately follows and the proof is complete.

## References


[1] J. Bustamante, Asymptotics for Angelesco-Nikishin systems, *J. of Approx. Theory* **85** (1996), 43-68.

[2] J. Bustamante, G. López Lagomasino, Hermite-Padé approximation for Nikishin systems of analytic functions, *Russian Acad. Sci. Sb. Math.* **77** (1994), 367-384.

[3] F. Cala and G. López Lagomasino, Multipoint Padé-type approximants. Exact rate of convergence, *Const. Approx.* **14** (1998), 259-272.

[4] K. Driver and H. Stahl, Simultaneous rational approximants to Nikishin systems, I, II, *Acta Sci. Math. (Szeged)* **60** (1995), 245-263; **61** (1995), 261-284.

[5] P. L. Duren, "Theory of $H^P$ Spaces," Academic Press, New York, 1970.

[6] G. Freud, "Orthogonal Polynomials," Pergamon Press, New York, 1971.

[7] J. L. Geronimus, "Polynomials Orthogonal on the Circle and the Segment," Gosizdat Fiz-Mat, Moscow, 1958.

[8] A. A. Gonchar and E. A. Rakhmanov, Equilibrium measure and distribution of zeros of extremal polynomials, *Math. USSR-Sb.* **125(167)** (1984), 117-127.

[9] A. A. Gonchar, E. A. Rakhmanov, and V. N. Sorokin, On Hermite-Padé approximants for systems of Markov type functions, *Mat. Sb.* **188** (1997), 33-58.

[10] G. López Lagomasino, On the asymptotics of the ratio of orthogonal polynomials and the convergence of multipoint Padé approximants, *Math. USSR-Sb.* **128(170)** (1985), 216-229.

[11] G. López Lagomasino, Convergence of Padé approximants of Stieltjes type meromorphic functions and comparative asymptotic for orthogonal polynomials, *Math. USSR Sb.* **64** (1989), 207-227.

[12] G. López Lagomasino, Asymptotics of polynomials orthogonal with respect to varying measures, *Constr. Approx.* **5** (1989), 199-219.

[13] A. Maté, P. Nevai, and V. Totik, Asymptotics for the ratio of leading coefficients of orthogonal polynomials, *Constr. Approx.* **1** (1985), 63-69.

[14] A. Maté, P. Nevai, and V. Totik, Strong and weak convergence of orthogonal polynomials on the unit circle, *Amer. J. Math.* **109** (1987), 239-282.

[15] W. Rudin, "Real and Complex Analysis," Mc Graw Hill, New York, 1966.

[16] H. Stahl and V. Totik, "General Orthogonal Polynomials," Cambridge University Press, Cambridge, 1992.

[17] G. Szegő, "Orthogonal Polynomials," Coll. Publ. XXIII, Amer. Math. Soc., Providence, Rhode Island, 1975 (4th edition).

[18] J. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Domain," Coll. Publ. XX, Amer. Math. Soc., Providence, Rhode Island, 1969.