ON PERFECT NIKISHIN SYSTEMS

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Abstract. We prove perfectness for Nikishin systems made up of three functions and apply this to the convergence of the associated Hermite-Padé approximants.

1. Introduction

Let $S = (s_1, \dots, s_m)$ be a system of finite Borel measures. All the measures considered in this paper have constant sign and compact support $\operatorname{supp}(\cdot)$ contained in the real line $\mathbb{R}$ with infinitely many points. Fix a multi-index $n = (n_1, \dots, n_m) \in \mathbb{Z}_+^m$ and set $|n| = n_1 + \cdots + n_m$. We say that $Q_n, \deg Q_n \leq |n|, Q_n \not\equiv 0$, is a multiple orthogonal polynomial of S relative to the multi-index n if

$$0 = \int x^k Q_n(x) ds_j(x)\,, \quad k = 0, \dots, n_j - 1\,, \quad j = 1, \dots, m\,. \tag{1}$$

It is well known (see e.g. [2, 3, 5, 7] and Section 3 below) that orthogonality relations of this type arise in a natural way in the study of Hermite-Padé (or simultaneous Padé) approximation.

Basic questions are: if (1) determines Q_n uniquely (up to a constant factor); is $\deg Q_n = |n|$ for all non trivial solution of (1); are the zeros of Q_n simple and do they lie in the interior (with the euclidean topology of $\mathbb{R}$) of the smallest interval containing the support of all the measures s_j. In general, it is easy to construct examples where the answer to all these questions is negative (taking, for example, $s_1 = \cdots = s_m$).

Definition 1. We say that a multi-index n is weakly normal for the system S if Q_n is determined uniquely. A multi-index n is said to be normal if any non trivial solution Q_n of (1) satisfies $\deg Q_n = |n|$. If Q_n has exactly $|n|$ simple zeros and they all lie in the interior of the smallest interval containing $\cup_{j=1}^m \operatorname{supp}(s_j)$ the index is called strongly normal. When all the indices are weakly normal, normal, or strongly normal the system S is said to be weakly perfect, perfect, or strongly perfect respectively.

Normality of indices plays a crucial role in applications to number theory and Hermite-Padé approximation. Obviously, strong normality implies normality, and it is not hard to prove that normality implies weak normality (see Lemma 1 below).

Nikishin systems of measures were introduced in [7]. For them a large class of indices are known to be strongly normal. Such systems are defined as follows. We adopt the notation introduced in [5] which is clarifying.

Let σ_1 and σ_2 be two measures supported on $\mathbb{R}$ and let F_1, F_2 denote the smallest intervals containing $\operatorname{supp}(\sigma_1)$ and $\operatorname{supp}(\sigma_2)$ respectively. We write $F_i = \operatorname{Co}(\operatorname{supp}(\sigma_i))$. Assume that $F_1 \cap F_2 = \emptyset$. We define

$$\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{d\sigma_2(t)}{x - t} d\sigma_1(x) = \widehat{\sigma}_2(x) d\sigma_1(x)\,.$$

Therefore, $\langle \sigma_1, \sigma_2 \rangle$ is a measure with constant sign and support equal to that of σ_1.

For a system of closed intervals $F_1, \dots, F_m$ satisfying $F_{j-1} \cap F_j = \emptyset, j = 2, \dots, m$, and finite Borel measures $\sigma_1, \dots, \sigma_m$ with constant sign and $\operatorname{Co}(\operatorname{supp}(\sigma_j)) = F_j$, we define by induction

$$\langle \sigma_1, \sigma_2, \dots, \sigma_j \rangle = \langle \sigma_1, \langle \sigma_2, \dots, \sigma_j \rangle \rangle, \quad j = 2, \dots, m\,.$$

We say that $S = (s_1, \dots, s_m)$, where

$$s_1 = \langle \sigma_1 \rangle = \sigma_1, \quad s_2 = \langle \sigma_1, \sigma_2 \rangle, \dots, s_m = \langle \sigma_1, \dots, \sigma_m \rangle\,,$$

1991 *Mathematics Subject Classification.* Primary 42C05.

The work of both authors was partially supported by Dirección General de Enseñanza Superior under grant BFM2000-0206-C04-01 and the second author by grants PRAXIS XXI BCC-22201/99 and INTAS 00-272.

is the Nikishin system of measures associated with $(\sigma_1, \ldots, \sigma_m)$. Notice that all the measures in a Nikishin system have the same support.

For Nikishin systems of measures all multi-indices n satisfying $1 \leq i < j \leq m \Rightarrow n_j \leq n_i + 1$ are known to be strongly normal. This result was originally proved in [3]. More recently, an extension for so called generalized Nikishin systems was given in [5]. When $m = 2$, from the results in [2] it follows that the system is strongly perfect (a detailed proof may be found in [3]). In [1], the authors were able to include in the set of strongly normal indices all those for which there do not exist $1 \leq i < j < k \leq m$ such that $n_i < n_j < n_k$. In particular, for $m = 3$ all indices turn out to be strongly normal except (possibly) when $n_1 < n_2 < n_3$.

The main result of this paper states the following.

Theorem 1. *An arbitrary Nikishin system of three measures is strongly perfect.*

This result is proved in section 2. Section 3 is devoted to some applications.

2. Proofs

Let $S = (s_1, \ldots, s_m)$ be a system of measures in $\mathbb{R}$ (not necessarily of Nikishin type) and $n = (n_1, \ldots, n_m)$ a multi-index. The moment matrix of the system S relative to the multi-index n is the square matrix M_n of order $|n|$ obtained placing the submatrices

$$\begin{pmatrix} \int ds_j(x) & \cdots & \int x^{|n|-1} ds_j(x) \\ \vdots & \ddots & \vdots \\ \int x^{n_j-1} ds_j(x) & \cdots & \int x^{|n|+n_j-2} ds_j(x) \end{pmatrix}, \quad j = 1, \ldots, m,$$

consecutively one on top of the other. If $n_j = 0$, this index is skipped in the construction of M_n. By M'_n we denote the matrix obtained adding to M_n at the end the column vector

$$\left(\int x^{|n|} ds_1(x), \ldots, \int x^{|n|+n_1-1} ds_1(x), \int x^{|n|} ds_2(x), \ldots, \int x^{|n|+n_m-1} ds_m(x)\right)^t.$$

Let $Q_n(x) = a_{|n|} x^{|n|} + a_{|n|-1} x^{|n|-1} + \cdots + a_0$ be a solution of (1) and $A = (a_0, \ldots, a_{|n|})^t$ the vector of coefficients corresponding to Q_n. In matrix form, the system of equations defined by (1) may be expressed as follows

$$M'_n A = \overline{0}, \tag{2}$$

where $\overline{0}$ denotes the $|n|$-dimensional zero vector. In algebraic terms it is easy to answer the first two questions posed in the previous section. By $\mathrm{rk}(\cdot)$ we denote the rank of the indicated matrix.

Lemma 1. *Let $(s_1, \ldots, s_m)$ be a system of measures and $n = (n_1, \ldots, n_m)$ a multi-index. A necessary and sufficient condition in order that n be weakly normal is that $\mathrm{rk}(M'_n) = |n|$. In turn, a necessary and sufficient condition in order that n be normal is that $\mathrm{rk}(M_n) = |n|$. In particular, normality implies weak normality.*

Proof. In fact, according to the Rouche-Frobenius Theorem the solution space of the homogeneous system of equations (2) is one dimensional if and only if $\mathrm{rk}(M'_n) = |n|$. Of course, this is equivalent to the fact that Q_n be determined uniquely up to a constant factor. On the other hand, $\mathrm{rk}(M_n) = |n|$ and $a_n = 0$ imply that all the other entries of A must equal zero, whereas if $\mathrm{rk}(M) < |n|$ we can find a non trivial solution of (2) with $a_n = 0$. □

Related to (1) there is the so called dual problem. Let σ_1 be a Borel measure on $\mathbb{R}$ with infinitely many points in its support and $(w_1, \ldots, w_m)$ a system of continuous functions on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$ with constant sign. Consider the system of measures $(s_1, \ldots, s_m) = (w_1 d\sigma_1, \ldots, w_m d\sigma_m)$. Whenever it is convenient, we adopt the differential notation for a measure.

Definition 2. We say that $n \in \mathbb{Z}_+^m$ is normal with respect to the dual problem if there do not exist polynomials $P_{n_1}, \ldots, P_{n_m}$, not all identically equal to zero, such that $\deg P_{n_j} \leq n_j - 1$ and

$$\int x^\nu (P_{n_1}(x) w_1(x) + \cdots + P_{n_m}(x) w_m(x)) d\sigma_1(x) = 0, \quad \nu = 0, \ldots, |n| - 1, \tag{3}$$

($\deg P_{n_j} \leq -1$ means that $P_{n_j} \equiv 0$).

LEMMA 2. *Let σ_1 and $(w_1, \dots, w_m)$ be as above. Set $S = (w_1 d\sigma_1, \dots, w_m d\sigma_m)$. The index $n \in \mathbb{Z}_+^m$ is normal with respect to S if and only if it is normal with respect to the dual problem.*

Proof. According to Lemma 1, n is normal with respect to S if and only if the rows of M_n are linearly independent. Taking arbitrary linear combinations of the rows of M_n one sees that this is equivalent to saying that n is normal for the dual problem. □

DEFINITION 3. It is said that $(w_1, \dots, w_m)$ forms an AT system for the index $n = (n_1, \dots, n_m)$ on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$ if no matter what polynomials $P_{n_1}, \dots, P_{n_m}$ one chooses with $\deg P_{n_j} \leq n_j - 1$, not all identically equal to zero, the function

$$\mathcal{P}_n(x) = \mathcal{P}_n(P_{n_1}, \dots, P_{n_m}; x) = P_{n_1}(x) w_1(x) + \cdots + P_{n_m}(x) w_m(x) .$$

has at most $|n| - 1$ zeros on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. The system $(w_1, \dots, w_m)$ forms an AT system on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$ if it is an AT system on that interval for all $n \in \mathbb{Z}_+^m$.

Since σ_1 has infinitely many points in its support, (3) forces $\mathcal{P}_n(x)$ to have at least $|n|$ changes of sign in the interior of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. Therefore, a sufficient condition in order that an index n be normal for the dual problem is that $(w_1, \dots, w_m)$ form an AT system for the index n on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. In fact, the AT property has more substantial consequences. The following result appears as Theorem 1 in [7] where more on AT systems may be found. For convenience of the reader we include a proof with the additional assumption that the functions w_j are analytic on a neighborhood of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$ which is sufficient for our further considerations.

LEMMA 3. *Let σ_1 and $(w_1, \dots, w_m)$ be as above. Set $S = (w_1 d\sigma_1, \dots, w_m d\sigma_m)$. Assume that $(w_1, \dots, w_m)$ is an AT system for the multi-index $n = (n_1, \dots, n_m)$. Then n is strongly normal for S.*

Proof. From (1) it follows that

$$0 = \int Q_n(x) \mathcal{P}_n(x) d\sigma_1(x) \tag{4}$$

for all $\mathcal{P}_n(x) = \mathcal{P}_n(P_{n_1}, \dots, P_{n_m}; x)$. Assume that Q_n has at most $N < |n|$ changes of sign in the interior of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. Choose the polynomials $P_{n_j}, j = 1, \dots, m$, so that $\mathcal{P}_n$ has a simple zero at each of the points where Q_n changes sign in the interior of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$ and a zero of multiplicity $|n| - N - 1$ at one of the extreme points of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$ (recall that in the proof we are assuming additionally that the functions w_j are analytic on a neighborhood of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$). Finding such polynomials $P_{n_j}, j = 1, \dots, m$, reduces to solving a homogeneous system of $|n| - 1$ equations on $|n|$ unknowns formed by the coefficients of these polynomials, thus a non trivial solution exists. Since $\mathcal{P}_n(x)$ can have no more zeros on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$ than the $|n| - 1$ already assigned, we have that $Q_n(x)\mathcal{P}_n(x)$ does not change sign on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. Therefore, (4) cannot take place for this $\mathcal{P}_n$ arriving to a contradiction. □

From Lemma 3 it follows that Theorem 1 is an immediate consequence of the following.

THEOREM 2. *Let $S = (s_1, s_2, s_m)$ be the Nikishin system associated with $(\sigma_1, \sigma_2, \sigma_3)$. Then (w_1, w_2, w_3) forms an AT system on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$, where $w_1 \equiv 1, w_2 = \widehat{\sigma}_2$, and $w_3 = \widehat{s}_{2,3}$ with $s_{2,3} = \langle \sigma_2, \sigma_3 \rangle$.*

To prove this theorem we use

LEMMA 4. *Let σ_2, σ_3 be two measures on $\mathbb{R}$ such that $\mathrm{Co}(\mathrm{supp}(\sigma_2)) \cap \mathrm{Co}(\mathrm{supp}(\sigma_3)) = \emptyset$. Then*

$$\widehat{\sigma}_2(z)\widehat{\sigma}_3(z) = \widehat{s}_{2,3}(z) + \widehat{s}_{3,2}(z) , \quad z \in \mathbb{C} \setminus (\mathrm{supp}(\sigma_2) \cup \mathrm{supp}(\sigma_3)) , \tag{5}$$

where $s_{2,3} = \langle \sigma_2, \sigma_3 \rangle$ and $s_{3,2} = \langle \sigma_3, \sigma_2 \rangle$.

Proof. In fact,

$$\widehat{\sigma}_2(z)\widehat{\sigma}_3(z) = \int\int \frac{d\sigma_2(x) d\sigma_3(t)}{(z-x)(z-t)} = \int\int \left(\frac{1}{z-x} - \frac{1}{z-t} \right) \frac{d\sigma_2(x) d\sigma_3(t)}{x-t}$$
$$= \int \frac{\widehat{\sigma}_3(x) d\sigma_2(x)}{z-x} + \int \frac{\widehat{\sigma}_2(t) d\sigma_3(t)}{z-t}$$

which is what we needed to prove. □

Let σ be a measure supported on $\mathbb{R}$ with constant sign. Notice that the statement of Theorem 2 implies that a system of the form $(1, \widehat{\sigma})$ forms an AT system on any closed interval contained in $\mathbb{R}$ disjoint from $\mathrm{Co}(\mathrm{supp}(\sigma))$. This is the reason why Nikishin systems of two measures in strongly perfect. Let us prove this particular case separately. Before doing so let us recall the well known property (see [6, Appendix]) that there exists a finite measure τ with constant sign such that $\mathrm{Co}(\mathrm{supp}(\tau)) \subset \mathrm{Co}(\mathrm{supp}(\sigma))$ and

$$\frac{1}{\widehat{\sigma}(z)} = l(z) + \widehat{\tau}(z), \quad z \in \mathbb{C} \setminus \mathrm{supp}(\sigma)\,, \tag{6}$$

where $l(z)$ is a polynomial of degree one. This will be used frequently in the sequel.

Lemma 5. *The system* $(1, \widehat{\sigma})$ *forms an AT system on any closed interval contained in* $\mathbb{R}$ *disjoint from* $Co(\mathrm{supp}(\sigma))$.

Proof. Let us assume that $(1, \widehat{\sigma})$ is not an AT system on some interval $[a, b]$ disjoint from $\mathrm{Co}(\mathrm{supp}(\sigma))$. Then there exists a multi-index $n = (n_1, n_2) \in \mathbb{Z}_+^2$ and polynomials $P_{n_i}, \deg P_{n_i} \leq n_i - 1, i = 1, 2$, not both identically equal to zero, such that $\mathcal{P}_n = P_{n_1} + P_{n_2}\widehat{\sigma}$ has exactly $N \geq |n| = n_1 + n_2$ zeros on $[a, b]$ counting multiplicities. Obviously, $N < \infty$ since $\mathcal{P}_n$ is analytic on a neighborhood of $[a, b]$ and $N = \infty$ would imply that $\mathcal{P}_n \equiv 0$ and by the same token $P_{n_i} \equiv 0, i = 1, 2$.

Let W_n be the monic polynomial whose zeros are the zeros of $\mathcal{P}_n$ on $[a, b]$ (counting multiplicities). Therefore,

$$\frac{\mathcal{P}_n(z)}{W_n(z)} = O\left(\frac{1}{z^{N-M}}\right) \in H(\mathbb{C} \setminus \mathrm{Co}(\mathrm{supp}(\sigma)))\,, \tag{7}$$

where $M = \max\{n_1 - 1, n_2 - 2\}$.

Assume that $M = n_1 - 1$. From (7) we have that

$$\frac{z^\nu \mathcal{P}_n(z)}{W_n(z)} = O\left(\frac{1}{z^2}\right) \in H(\mathbb{C} \setminus \mathrm{Co}(\mathrm{supp}(\sigma)))\,, \quad \nu = 0, \ldots, n_2 - 1\,.$$

Let Γ be a closed integration path with winding number 1 for all its interior points. Denote $\mathrm{Ext}(\Gamma)$ and $\mathrm{Int}(\Gamma)$ the unbounded and bounded connected components respectively of the complement of Γ. Take Γ so that $\mathrm{Co}(\mathrm{supp}(\sigma)) \subset \mathrm{Int}(\Gamma)$ and $[a, b] \subset \mathrm{Ext}(\Gamma)$. From Cauchy's Theorem, it follows that

$$0 = \frac{1}{2\pi i} \int_\Gamma \frac{z^\nu \mathcal{P}_n(z)}{W_n(z)} dz = \frac{1}{2\pi i} \int_\Gamma \frac{z^\nu (P_{n_2}\widehat{\sigma})(z)}{W_n(z)} dz\,, \quad \nu = 0, \ldots, n_2 - 1\,.$$

Using Fubini's Theorem and Cauchy's integral formula, we obtain

$$0 = \int \frac{x^\nu P_{n_2}(x)}{W_n(x)} d\sigma(x)\,, \quad \nu = 0, \ldots, n_2 - 1\,.$$

Since $d\sigma(x)/W_n(x)$ is a measure with constant sign on $\mathrm{Co}(\mathrm{supp}(\sigma))$ it follows that P_{n_2} has at least n_2 zeros on $\mathrm{Co}(\mathrm{supp}(\sigma))$. But this is impossible unless $P_{n_2} \equiv 0$ which in turn would imply that P_{n_1}, having $N > n_1 - 1$ zeros on $[a, b]$, would also be identically equal to zero against our initial assumption on these polynomials.

If $M = n_2 - 2$ the proof is the same except for one additional ingredient. From (7) it follows that

$$\frac{z^\nu \mathcal{P}_n(z)}{\widehat{\sigma}(z) W_n(z)} = O\left(\frac{1}{z^2}\right) \in H(\mathbb{C} \setminus \mathrm{Co}(\mathrm{supp}(\sigma)))\,, \quad \nu = 0, \ldots, n_1 - 1\,.$$

Take Γ as before. From Cauchy's Theorem and (6), it follows that

$$0 = \frac{1}{2\pi i} \int_\Gamma \frac{z^\nu \mathcal{P}_n(z)}{(\widehat{\sigma} W_n)(z)} dz = \frac{1}{2\pi i} \int_\Gamma \frac{z^\nu (P_{n_1}\widehat{\tau})(z)}{W_n(z)} dz\,, \quad \nu = 0, \ldots, n_1 - 1\,.$$

Using Fubini's Theorem and Cauchy's integral formula, we obtain

$$0 = \int \frac{x^\nu P_{n_1}(x)}{W_n(x)} d\tau(x)\,, \quad \nu = 0, \ldots, n_1 - 1\,.$$

Reasoning as in the previous case we obtain a contradiction. □

Proof of Theorem 2. We use the notation introduced in the statement of the Theorem. Let us assume that (w_1, w_2, w_3) is not an AT system on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. Then there exists a multi-index

$n=(n_1,n_2,n_3)\in\mathbb{Z}_+^3$ and polynomials $P_{n_i},\deg P_{n_i}\le n_i-1, i=1,2,3$, not all identically equal to zero, such that $\mathcal{P}_n=P_{n_1}+P_{n_2}w_2+P_{n_3}w_3$ has exactly $N\ge|n|$ zeros on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$ counting multiplicities. Obviously, $N<\infty$ since $\mathcal{P}_n$ is analytic on a neighborhood of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$ and $N=\infty$ would imply that $\mathcal{P}_n\equiv 0$ and by the same token $P_{n_i}\equiv 0, i=1,2,3$.

Let W_n be the monic polynomial whose zeros are the zeros of $\mathcal{P}_n$ on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$ (counting multiplicities). Therefore,

$$\frac{\mathcal{P}_n(z)}{W_n(z)}=O\left(\frac{1}{z^{N-M}}\right)\in H(\mathbb{C}\setminus\mathrm{Co}(\mathrm{supp}(\sigma_2)))\,, \tag{8}$$

where $M=\max\{n_1-1,n_2-2,n_3-2\}$.

Assume that $M=n_1-1$. From (8) we have that

$$\frac{z^\nu\mathcal{P}_n(z)}{W_n(z)}=O\left(\frac{1}{z^2}\right)\in H(\mathbb{C}\setminus\mathrm{Co}(\mathrm{supp}(\sigma_2)))\,,\quad \nu=0,\dots,n_2+n_3-1\,.$$

Let Γ be a closed integration path with winding number 1 for all its interior points such that $\mathrm{Co}(\mathrm{supp}(\sigma_2))\subset\mathrm{Int}(\Gamma)$ and $\mathrm{Co}(\mathrm{supp}(\sigma_1))\subset\mathrm{Ext}(\Gamma)$. From Cauchy's Theorem, it follows that

$$0=\frac{1}{2\pi i}\int_\Gamma\frac{z^\nu\mathcal{P}_n(z)}{W_n(z)}dz=\frac{1}{2\pi i}\int_\Gamma\frac{z^\nu(P_{n_2}w_2+P_{n_3}w_3)(z)}{W_n(z)}dz\,,\quad \nu=0,\dots,n_2+n_3-1\,.$$

Substituting w_2 and w_3 by their expressions, using Fubini's Theorem and Cauchy's integral formula, we obtain

$$0=\int\frac{x^\nu(P_{n_2}+P_{n_3}\widehat{\sigma}_3)(x)}{W_n(x)}d\sigma_2(x)\,,\quad \nu=0,\dots,n_2+n_3-1\,.$$

Since $d\sigma_2(x)/W_n(x)$ is a measure with constant sign on $\mathrm{supp}\,\sigma_2$, it follows that $(P_{n_2}+P_{n_3}\widehat{\sigma}_3)(x)$ must have at least n_2+n_3 changes of sign on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$. According to Lemma 5 this is not possible unless $P_{n_2}\equiv 0$ and $P_{n_3}\equiv 0$. But this is not possible either because then P_{n_1} would have $N>n_1-1$ zeros on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$ and would also be identically equal to zero contrary to our assumption that these polynomials are not all identically equal to zero.

Let us consider the case when $M=n_2-2$. From (8) it follows that

$$\frac{z^\nu\mathcal{P}_n(z)}{\widehat{\sigma}_2(z)W_n(z)}=O\left(\frac{1}{z^2}\right)\in H(\mathbb{C}\setminus\mathrm{Co}(\mathrm{supp}(\sigma)))\,,\quad \nu=0,\dots,n_1+n_3-1\,.$$

Take Γ as before. From Cauchy's Theorem we obtain

$$0=\frac{1}{2\pi i}\int_\Gamma\frac{z^\nu\mathcal{P}_n(z)}{(\widehat{\sigma}_2W_n)(z)}dz=$$

$$\frac{1}{2\pi i}\int_\Gamma\frac{z^\nu P_{n_1}(z)}{(\widehat{\sigma}_2W_n)(z)}dz+\frac{1}{2\pi i}\int_\Gamma\frac{z^\nu(P_{n_3}\widehat{s}_{2,3})(z)}{(\widehat{\sigma}_2W_n)(z)}dz\,,\quad \nu=0,\dots,n_1+n_3-1\,.$$

According to (6), there exists a finite measure τ_2 with constant sign such that

$$\frac{1}{\widehat{\sigma}_2(z)}=l_2(z)+\widehat{\tau}_2(z),\quad z\in\mathbb{C}\setminus\mathrm{supp}(\sigma_2)\,, \tag{9}$$

From (9), Cauchy's Theorem, Fubini's Theorem, and Cauchy's Integral Formula, for the first integral on the right hand we have

$$\frac{1}{2\pi i}\int_\Gamma\frac{z^\nu P_{n_1}(z)}{(\widehat{\sigma}_2W_n)(z)}dz=\frac{1}{2\pi i}\int_\Gamma\frac{z^\nu(P_{n_1}\widehat{\tau}_2)(z)}{W_n(z)}dz=\int\frac{x^\nu P_{n_1}(x)}{W_n(x)}d\tau_2(x)\,.$$

For the second integral, using (5), (9), Cauchy's Theorem, Fubini's Theorem, and Cauchy's Integral formula, we obtain

$$\frac{1}{2\pi i}\int_\Gamma\frac{z^\nu(P_{n_3}\widehat{s}_{2,3})(z)}{(\widehat{\sigma}_2W_n)(z)}dz=\frac{1}{2\pi i}\int_\Gamma\frac{z^\nu P_{n_3}(z)}{W_n(z)}\left(\widehat{\sigma}_3(z)-\frac{\widehat{s}_{3,2}(z)}{\widehat{\sigma}_2(z)}\right)dz=$$

$$-\frac{1}{2\pi i}\int_\Gamma\frac{z^\nu(P_{n_3}\widehat{s}_{3,2}\widehat{\tau}_2)(z)}{W_n(z)}dz=-\int\frac{x^\nu(P_{n_3}\widehat{s}_{3,2})(x)}{W_n(x)}d\tau_2(x)\,.$$

Summing up the last three relations, we get

$$0=\int\frac{x^\nu(P_{n_1}-P_{n_3}\widehat{s}_{3,2})(x)}{W_n(x)}d\tau_2(x)\,,\quad \nu=0,\dots,n_1+n_3-1\,.$$

On account of Lemma 5, reasoning as in the previous case we obtain a contradiction.

If $M = n_3 - 2$, from (8) we have that

$$\frac{z^\nu \mathcal{P}_n(z)}{\widehat{s}_{2,3}(z) W_n(z)} = O\left(\frac{1}{z^2}\right) \in H(\mathbb{C} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_2)))\,, \quad \nu = 0, \ldots, n_1 + n_2 - 1\,.$$

Taking Γ as above and using Cauchy's Theorem it follows that

$$0 = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu \mathcal{P}_n(z)}{(\widehat{s}_{2,3} W_n)(z)} dz =$$

$$\frac{1}{2\pi i}\int_\Gamma \frac{z^\nu P_{n_1}(z)}{(\widehat{s}_{2,3} W_n)(z)} dz + \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (P_{n_2}\widehat{\sigma}_2)(z)}{(\widehat{s}_{2,3} W_n)(z)} dz\,, \quad \nu = 0, \ldots, n_1 + n_2 - 1\,.$$

According to (6), let $\tau_{2,3}$ be such that $\mathrm{Co}(\mathrm{supp}(\tau_{2,3})) \subset \mathrm{Co}(\mathrm{supp}(s_{2,3})) = \mathrm{Co}(\mathrm{supp}(\sigma_2))$ and

$$\frac{1}{\widehat{s}_{2,3}(z)} = l_{2,3}(z) + \widehat{\tau}_{2,3}(z), \quad z \in \mathbb{C} \setminus \mathrm{supp}(s_{2,3})\,, \tag{10}$$

From (10)

$$\frac{1}{2\pi i}\int_\Gamma \frac{z^\nu P_{n_1}(z)}{(\widehat{s}_{2,3} W_n)(z)} dz = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (P_{n_1}\widehat{\tau}_{2,3})(z)}{(W_n)(z)} dz = \int \frac{x^\nu P_{n_1}(x)}{W_n(x)} d\tau_{2,3}(x)\,.$$

For the second integral, using (5) and (10), we get

$$\frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (P_{n_2}\widehat{\sigma}_2)(z)}{(\widehat{s}_{2,3} W_n)(z)} dz = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu P_{n_2}(z)}{(\widehat{\sigma}_3 W_n)(z)} \frac{(\widehat{s}_{2,3} + \widehat{s}_{3,2})(z)}{\widehat{s}_{2,3}(z)} dz =$$

$$\frac{1}{2\pi i}\int_\Gamma \frac{z^\nu P_{n_2}(z)}{(\widehat{\sigma}_3 W_n)(z)} \frac{\widehat{s}_{3,2}(z)}{\widehat{s}_{2,3}(z)} dz = \int \frac{x^\nu (P_{n_2}\widehat{s}_{3,2})(x)}{(\widehat{\sigma}_3 W_n)(x)} d\tau_{2,3}(x)\,.$$

Putting together these relations, it follows that

$$0 = \int \frac{x^\nu (P_{n_1}\widehat{\sigma}_3 + P_{n_2}\widehat{s}_{3,2})(x)}{(\widehat{\sigma}_3 W_n)(x)} d\tau_{2,3}(x)\,, \quad \nu = 0, \ldots, n_1 + n_2 - 1\,. \tag{11}$$

We cannot apply directly Lemma 5 as we did before to conclude the proof. Instead, we must go one step further down.

From (11) we know that $P_{n_1}\widehat{\sigma}_3 + P_{n_2}\widehat{s}_{3,2}$ must have $N_1 \geq n_1 + n_2$ zeros on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$ and P_{n_1}, P_{n_2} cannot be simultaneously identically equal to zero because by the way these polynomials were chosen it would turn out that all three would be identically equal to zero against our initial assumption. Let V be the monic polynomial whose zeros are the zeros of $P_{n_1}\widehat{\sigma}_3 + P_{n_2}\widehat{s}_{3,2}$ on $\mathrm{Co}(\mathrm{supp}(\sigma_2))$. Therefore,

$$\frac{(P_{n_1}\widehat{\sigma}_3 + P_{n_2}\widehat{s}_{3,2})(z)}{V(z)} = O\left(\frac{1}{z^{N_1 - M_1}}\right) \in H(\mathbb{C} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_3)))\,, \tag{12}$$

where $M_1 = \max\{n_1 - 2, n_2 - 2\}$.

Assume that $M_1 = n_1 - 2$. From (12) we have that

$$\frac{z^\nu (P_{n_1}\widehat{\sigma}_3 + P_{n_2}\widehat{s}_{3,2})(z)}{(\widehat{\sigma}_3 V)(z)} = O\left(\frac{1}{z^2}\right) \in H(\mathbb{C} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_3)))\,, \quad \nu = 0, \ldots, n_2 - 1\,.$$

Let Γ be a closed integration path with winding number 1 for all its interior points such that $\mathrm{Co}(\mathrm{supp}(\sigma_3)) \subset \mathrm{Int}(\Gamma)$ and $\mathrm{Co}(\mathrm{supp}(\sigma_2)) \subset \mathrm{Ext}(\Gamma)$. Using Lemma 5, it follows that

$$0 = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (P_{n_1}\widehat{\sigma}_3 + P_{n_2}\widehat{s}_{3,2})(z)}{(\widehat{\sigma}_3 V)(z)} dz = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (P_{n_2}\widehat{s}_{3,2})(z)}{(\widehat{\sigma}_3 V)(z)} dz =$$

$$-\frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (P_{n_2}\widehat{s}_{2,3})(z)}{(\widehat{\sigma}_3 V)(z)} dz = -\frac{1}{2\pi i}\int_\Gamma \frac{z^\nu (P_{n_2}\widehat{s}_{2,3}\widehat{\tau}_3)(z)}{V(z)} dz =$$

$$-\int \frac{x^\nu (P_{n_2}\widehat{s}_{2,3})(x)}{V(z)} d\tau_3(x)\,, \quad \nu = 0, \ldots, n_2 - 1\,,$$

where $1/\widehat{\sigma}_3(z) = l_3(z) + \widehat{\tau}_3(z), z \in \mathbb{C} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_3))$. This is not possible unless $P_{n_2} \equiv 0$ and, consequently, $P_{n_1} \equiv 0$ which contradicts the initial assumptions on these polynomials.

If $M_1 = n_2 - 2$, from (12) we have that

$$\frac{z^\nu(P_{n_1}\widehat{\sigma}_3 + P_{n_2}\widehat{s}_{3,2})(z)}{(\widehat{s}_{3,2}V)(z)} = O\left(\frac{1}{z^2}\right) \in H(\mathbb{C}\setminus \mathrm{Co}(\mathrm{supp}(\sigma_3)))\,, \quad \nu = 0,\ldots,n_1-1\,.$$

Taking Γ as before and $1/\widehat{s}_{3,2}(z) = l_{3,2}(z) + \widehat{\tau}_{3,2}(z)$, $z \in \mathbb{C}\setminus \mathrm{Co}(\mathrm{supp}(\sigma_3))$, it follows that

$$0 = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu(P_{n_1}\widehat{\sigma}_3 + P_{n_2}\widehat{s}_{3,2})(z)}{(\widehat{s}_{3,2}V)(z)}dz = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu(P_{n_1}\widehat{\sigma}_2\widehat{\sigma}_3)(z)}{(\widehat{\sigma}_2\widehat{s}_{3,2}V)(z)}dz =$$

$$\frac{1}{2\pi i}\int_\Gamma \frac{z^\nu(P_{n_1}\widehat{s}_{2,3})(z)}{(\widehat{\sigma}_2\widehat{s}_{3,2}V)(z)}dz = \frac{1}{2\pi i}\int_\Gamma \frac{z^\nu(P_{n_1}\widehat{s}_{2,3}\widehat{\tau}_{3,2})(z)}{(\widehat{\sigma}_2 V)(z)}dz =$$

$$\int \frac{x^\nu(P_{n_1}\widehat{s}_{2,3})(x)}{(\widehat{\sigma}_2 V)(x)}d\tau_{3,2}(x)\,, \quad \nu = 0,\ldots,n_1-1\,.$$

Reasoning as in the previous case, we obtain a contradiction. With this we conclude the proof of Theorem 2. □

Remark . The proof of Theorem 2 may be conveniently modified so as to allow the possibility that consecutive intervals $F_j, j = 1,\ldots,m$, appearing in the definition of a Nikishin system of measures have a common end point (as long as their interiors remain non intersecting). It suffices to require that the corresponding measures be sufficiently weak in a neighborhood of the point of contact. In the same direction, it is possible to allow that some of the measures have unbounded support requiring that the corresponding measures be sufficiently weak at infinity.

3. Applications

Let $f = (f_1,\ldots,f_m)$ be a finite system of (formal) power series

$$f_j(z) = \sum_{k=0}^{\infty}\frac{c_{j,k}}{z^{k+1}}\,, \quad j = 1,\ldots,m\,.$$

Fix a multi–index $n = (n_1,\ldots,n_m) \in \mathbb{Z}_+^m$. It is easy to see that there exists a polynomial Q_n such that

$$\begin{array}{ll} \text{(i)} & Q_n(z) \not\equiv 0\,, \quad \deg Q_n \le |n|, \\ \text{(ii)} & (Q_n f_j - P_{n,j})(z) = \frac{A_{n,j}}{z^{n_j+1}} + \ldots\,, \quad j = 1,\ldots,m\,, \end{array} \tag{13}$$

where on the right hand of (ii) we have a series in increasing powers of $1/z$ and $P_{n,j}$ is the polynomial part of the power expansion of $Q_n f_j$ at $z = \infty$ (hence $\deg P_{n,j} \le |n| - 1$). The construction of Q_n, reduces to finding a non–trivial solution of a homogeneous linear system of $|n|$ equations on $|n| + 1$ unknowns (the coefficients of Q_n). Therefore, a non–trivial solution always exists. For each solution of (13), the vector $\left(\frac{P_{n,1}}{Q_n},\ldots,\frac{P_{n,m}}{Q_n}\right)$ is called the Hermite–Padé approximant (or simultaneous Padé approximant) of $(f_1,\ldots,f_m)$ relative to the multi–index $(n_1,\ldots,n_m)$. In the case of one function the definition reduces to that of a diagonal Padé approximant.

It is well known that Padé approximants and in particular diagonal Padé approximants are uniquely determined. This is not the case for Hermite–Padé approximants when $m \ge 2$. Different solutions to the homogeneous system mentioned above can give rise to different vector Hermite–Padé approximants. From (13) it follows that a sufficient condition in order that the multi–index $(n_1,\ldots,n_m)$ determines a unique Hermite–Padé approximant, is to be able to ensure that any Q_n which solves (13) has $\deg Q_n = |n|$. In fact, if Q_n and $\tilde{Q}_n$ satisfy (ii), we would have that $\tilde{Q}_n = \lambda Q_n$, $\lambda \ne 0$, since otherwise we can obtain a polynomial of degree less than $|n|$ that verifies (i) and (ii).

Let $S = (s_1,\ldots,s_m$ be the Nikishin system of measures associated with $(\sigma_1,\ldots,\sigma_m)$. We say that $\widehat{S} = (\widehat{s}_1,\ldots,\widehat{s}_m)$ is the Nikishin system of functions associated with $(\sigma_1,\ldots,\sigma_m)$. Consider the Hermite-Padé approximants relative to $\widehat{S}$. Using Cauchy's Theorem, from (13) immediately follows that the common denominator Q_n of the corresponding Hermite-Padé approximant verifies (1).

From Theorem 1 and what was said above, we obtain

Corollary 1. *For a Nikishin system of three functions the Hermite-Padé approximant of any muti–index is determined uniquely.*

Another consequence of Theorem 1 is the following.

Corollary 2. *Let* $\widehat{S} = (\widehat{s}_1, \widehat{s}_2, \widehat{s}_3)$ *be a Nikishin system. Let* $\{n(r)\}, r \in \mathbb{N}$, *be a sequence of multi–indices in* $\mathbb{Z}_+^3$ *such that* $\lim_{r\to\infty} |n(r)| = \infty$, *and there exists a constant* c *such that* $n_i(r) \geq (|n(r)|/3) - c, i = 1, 2, 3$. *Then*

$$\lim_{r\to\infty} \frac{P_{n(r),i}}{Q_{n(r)}} = f_i\,, \quad i = 1, 2, 3\,.$$

uniformly on each compact subset of $\mathbb{C} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1))$.

Proof. According to [2, Theorem 1], our assumptions imply that each component of the Hermite-Padé approximant converges to the corresponding component of $\widehat{S}$ in logarithmic capacity on each compact subset of $\mathbb{C} \setminus \mathrm{Co}(\mathrm{supp}(\sigma_1))$. On the other hand, for all $i = 1, 2, 3$, and $r \in \mathbb{N}$, all the poles of $P_{n(r),i}/Q_{n(r)}$ lie on $\mathrm{Co}(\mathrm{supp}(\sigma_1))$. According to [4, Lemma 1], this and the convergence in capacity imply our statement. □

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