Asymptotic behavior of solutions of general three term recurrence relations

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Dedicated to Mariano Gasca on his 60'th anniversary

Abstract

We consider the solutions of general three term recurrence relations whose coefficients are analytic functions in a prescribed region. We study the ratio asymptotic of such solutions under the assumption that the coefficients are asymptotically periodic and their strong asymptotic under more restrictive conditions.

1 Introduction

Let $a_n(z) \neq 0$ and $b_n(z)$, $n \in \mathbb{N}$, be analytic functions in a certain domain $\Omega \subseteq \mathbb{C}$. For the study of the solutions of the recurrence relation

$$w_n = b_n(z)w_{n-1} - a_n^2(z)w_{n-2}\,, \quad z \in \Omega\,, \tag{1}$$

it is convenient to consider the infinite tridiagonal matrix

$$\mathcal{D}(z) \quad = \quad \begin{pmatrix} -b_1(z) & a_2(z) & & & \\ a_2(z) & -b_2(z) & a_3(z) & & \\ & a_3(z) & -b_3(z) & \ddots & \\ & & \ddots & \ddots & \end{pmatrix}. \tag{2}$$

For each $j \in \mathbb{N}$, by $\mathcal{D}^{(j)}(z)$ we denote the infinite matrix which is obtained eliminating from $\mathcal{D}(z)$ its first j columns and rows ($\mathcal{D}^{(0)}(z) = \mathcal{D}(z)$). By $\mathcal{D}_n^{(j)}(z)$ we denote the principal section of order n of $\mathcal{D}^{(j)}(z)$. It is easy to check that $(-1)^n \det \mathcal{D}_n(z)$, $n \in \mathbb{N}$, is the solution to the recurrence relation (1) taking as initial conditions $w_{-1} \equiv 0$ and $w_0 \equiv 1$.

If we define the functions

$$w_n^{(j)}(z) = (-1)^n \det \mathcal{D}_n^{(j)}(z)\,, \quad n \in \mathbb{N}\,, \quad z \in \Omega\,, \quad w_0^{(j)} \equiv 1\,, \quad w_{-1}^{(j)} \equiv 0\,, \tag{3}$$

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expanding $\det \mathcal{D}_n^{(j)}(z)$ by its first row, we have

$$w_n^{(j)}(z) = b_{j+1}(z) w_{n-1}^{(j+1)}(z) - a_{j+2}^2(z) w_{n-2}^{(j+2)}(z)\,, \quad n \geq 1\,, \tag{4}$$

and if this is done by the last column one sees that $\{w_n^{(j)}\}$, $n \in \mathbb{N}$, is the solution to

$$w_n^{(j)}(z) = b_{n+j}(z) w_{n-1}^{(j)}(z) - a_{n+j}^2(z) w_{n-2}^{(j)}(z)\,, \quad n \geq 1\,, \tag{5}$$

with initial conditions $w_0^{(j)} \equiv 1$, $w_{-1}^{(j)} \equiv 0$. These solutions satisfy the following three-term recurrence relation (see Lemma 4, page 7, in [2])

$$\begin{aligned} 0 &= w_{n+s+1}^{(j)}(z) - w_s^{(n+j+1)}(z) w_{n+1}^{(j)}(z) + a_{n+j+2}^2(z) w_{s-1}^{(n+j+2)}(z) w_n^{(j)}(z)\,, \\ & \quad j, n \in \mathbb{N}\,, \quad s = 0, 1, \ldots\,, \end{aligned} \tag{6}$$

and

$$\begin{gathered} 0 = w_{s-1}^{(n+j-s+1)}(z) w_{n+s}^{(j)}(z) - \\ \left[w_{s-1}^{(n+j+1)}(z) w_s^{(n+j-s+1)}(z) - a_{n+j+2}^2(z) w_{s-2}^{(n+j+2)}(z) w_{s-1}^{(n+j-s+1)}(z) \right] w_n^{(j)}(z) + \\ + a_{n+j-s+2}^2(z) \ldots a_{n+j+1}^2(z) w_{s-1}^{(n+j+1)}(z) w_{n-s}^{(j)}(z)\,, \quad n \geq s \geq 2\,,\ j \geq 0\,. \end{gathered} \tag{7}$$

In this paper, we study the asymptotic behavior of the solution (3) of (1) when the coefficients of the recurrence relation are analytic functions in Ω and satisfy the following asymptotically periodic relations

$$\lim_{m\to\infty} a_{mN+j}(z) = a^{(j)}(z)\,, \quad \lim_{m\to\infty} b_{mN+j}(z) = b^{(j)}(z)\,, \quad j = 0, 1, \ldots, N-1\,, \tag{8}$$

where $N \in \mathbb{N}$ is fixed, the convergence in (8) is locally uniform in Ω, and

$$a^{(j)}(z) \neq 0\,, \quad z \in \Omega\,, \quad j = 0, 1, \ldots, N-1\,.$$

In the sequel, we assume that these conditions take place.

This general setting covers several cases treated in the literature:

i) Orthogonal polynomials with asymptotically periodic recurrence coefficients when $b_n(z) = z - b_n$ and $a_n(z) = a_n$, where b_n and a_n are independent of z. This case which is studied in detail by Geronimo and Van Assche [8] when $a_n > 0$, $b_n \in \mathbb{R}$ and in [1] when $a_n \neq 0$, $b_n \in \mathbb{C}$.

ii) Orthogonal polynomials $\{\Phi_n\}$ on the unit circle with $b_n(z) = z + \frac{\Phi_n(0)}{\Phi_{n-1}(0)}$, $a_n(z) = z\left(1 - |\Phi_{n-1}(0)|^2\right)$. In [11] and [12] is studied the case when the reflection coefficients $\{\Phi_n(0)\}$ are asymptotically periodic. In [2] is treated the situation in which the sequences $\{\Phi_n(0)/\Phi_{n+1}(0)\}$ and $\{|\Phi_n(0)|\}$ are asymptotically periodic.

iii) Laurent orthogonal polynomials and orthogonal rational functions as considered in [14] and [6].

2 Ratio asymptotic

In [4] V. I. Buslaev studies the convergence of the continued fraction

$$\cfrac{a_1^2(z)}{b_1(z) - \cfrac{a_2^2(z)}{b_2(z) - \cfrac{a_3^2(z)}{b_3(z) - \ddots}}} \tag{9}$$

under assumptions of type (8) and proves a result of considerable generality from which convergence is deduced. We make use of his result in proving ratio asymptotic for the sequence of functions which solves (1).

We state Buslaev's result in the form of a lemma for convenience of the reader. First let us introduce some notation. Set

$$\begin{pmatrix} \alpha^{(l)}(z) & \beta^{(l)}(z) \\ \gamma^{(l)}(z) & \delta^{(l)}(z) \end{pmatrix} = \begin{pmatrix} 0 & -\left(a^{(l)}\right)^2(z) \\ 1 & b^{(l)}(z) \end{pmatrix} \times \begin{pmatrix} 0 & -\left(a^{(l+1)}\right)^2(z) \\ 1 & b^{(l+1)}(z) \end{pmatrix} \times \tag{10}$$

$$\times \cdots \times \begin{pmatrix} 0 & -\left(a^{(N-1)}\right)^2(z) \\ 1 & b^{(N-1)}(z) \end{pmatrix} \times \begin{pmatrix} 0 & -\left(a^{(0)}\right)^2(z) \\ 1 & b^{(0)}(z) \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & -\left(a^{(l-1)}\right)^2(z) \\ 1 & b^{(l-1)}(z) \end{pmatrix},$$

$$I(z) = \alpha^{(l)}(z) + \delta^{(l)}(z), \quad \Delta(z) = \alpha^{(l)}(z)\delta^{(l)}(z) - \beta^{(l)}(z)\gamma^{(l)}(z) = \left(a^{(0)}(z)\cdots a^{(N-1)}(z)\right)^2,$$

$$\Gamma = \left\{ z \in \Omega \,:\, \left| I(z) + \sqrt{I^2(z) - 4\Delta(z)} \right| = \left| I(z) - \sqrt{I^2(z) - 4\Delta(z)} \right| \right\}.$$

For $z \in \Omega$, assuming that the root is taken so that $\left| I(z) + \sqrt{I^2(z) - 4\Delta(z)} \right| < \left| I(z) - \sqrt{I^2(z) - 4\Delta(z)} \right|$, we define

$$p^{(l)}(z) = \frac{\alpha^{(l)}(z) - \delta^{(l)}(z) + \sqrt{I^2(z) - 4\Delta(z)}}{2\gamma^{(l)}(z)}, \quad E = \bigcup_{l=0}^{N-1} \left\{ z \in \Omega \,:\, p^{(l)}(z) = 0 \right\}.$$

Lemma 1 ([4], Th. 1, page 675) *Assume that* (8) *takes place uniformly on compact subsets of* Ω. *Then, the continued fraction* (9) *converges uniformly on compact subsets of* $\Omega \setminus (\Gamma \cup E)$ *in the spherical metric to a meromorphic function in* $\Omega \setminus \Gamma$.

This result applied to the continued fraction

$$\cfrac{-a_{j+1}^2(z)}{b_{j+1}(z) - \cfrac{a_{j+2}^2(z)}{b_{j+2}(z) - \cfrac{a_{j+3}^2(z)}{b_{j+3}(z) - \ddots}}} \tag{11}$$

also guarantees the uniform convergence on compact subsets of $\Omega \setminus (\Gamma \cup E)$ to a meromorphic function $f^{(j)}$ in $\Omega \setminus \Gamma$. Define $v_n^{(j)}(z)$ as the solution of

$$w_n(z) = b_{n+j}(z)w_{n-1}(z) - a_{n+j}^2(z)w_{n-2}(z)$$

under the initial conditions $v_{-1}^{(j)}(z) \equiv 1$, $v_0^{(j)}(z) \equiv 0$. From the general theory of continued fractions, it is well known that the nth partial fraction of (11) is equal to $v_n^{(j)}(z)/w_n^{(j)}(z)$. On the other hand, from the recurrence relation it is easy to check that $v_n^{(j)}(z) = -a_{j+1}^2(z)w_{n-1}^{(j+1)}(z)$. Therefore, the nth partial fraction of (11) is $-a_{j+1}^2(z)w_{n-1}^{(j+1)}(z)/w_n^{(j)}(z)$. Hence, the convergence of (11) is equivalent to

$$f^{(j)}(z) = -a_{j+1}^2(z) \lim_{n\to\infty} \frac{w_{n-1}^{(j+1)}(z)}{w_n^{(j)}(z)} \tag{12}$$

for $z \in \Omega \setminus (\Gamma \cup E)$ (for details on the general theory of continued fractions see [15]).

One can use Lemma 1 to detect the limit behavior of the zeros of the solution $w_n^{(j)}(z)$ of (1). Let us define the set

$$E^{(j)} = \left\{ \text{ poles of } f^{(j)} \text{ in } \Omega \setminus \Gamma \right\} .$$

We have

Lemma 2 *For each fixed $j \in \mathbb{Z}_+$, if $\mathcal{K}$ is a compact set, $\mathcal{K} \subset \Omega \setminus \left(\Gamma \cup E \cup E^{(j)}\right)$, there exists $n_0 = n_0(j, \mathcal{K}) \in \mathbb{N}$ such that $w_n^{(j)}(z) \neq 0$, for all $n \geq n_0$ and $z \in \mathcal{K}$.*

Proof.- Let j and $\mathcal{K}$ be as above. Since $f^{(j)}$ has no poles on $\mathcal{K}$ from (12) it follows that $\left\{w_{n-1}^{(j+1)}(z)/w_n^{(j)}(z)\right\}$, $n \geq n_0$, is uniformly bounded on $\mathcal{K}$. Therefore, if $z_0 \in \mathcal{K}$ is a zero of $w_n^{(j)}(z)$ then $w_{n-1}^{(j+1)}(z_0) = 0$. Iterating (4) backwards on the index n, we would have that $w_0^{(j+n)}(z_0) = 0$ (recall that we have assumed that for all n, $a_n(z) \neq 0$). On the other hand, $w_0^{(j+n)}(z) \equiv 1$. This contradiction implies that for all $n \geq n_0$ and $z \in \mathcal{K}$, $w_n^{(j)}(z) \neq 0$ as we needed to prove. □

The previous result implies that for each $j \in \mathbb{N}$, the sequence of functions $\{w_n^{(j)}\}$, $n \in \mathbb{N}$, can only have accumulation points of zeros on the set $\Gamma \cup E \cup E^{(j)}$. Moreover, we have

Lemma 3 *For each fixed $j \in \mathbb{Z}_+$, $(E^{(j)} \cap E^{(j+1)}) \setminus E = \emptyset$. Let $z_0 \in E^{(j)} \setminus E$, then there exist $\varepsilon > 0$ and $n_2 \in \mathbb{N}$ such that for $n \geq n_2$ the number of zeros of $w_n^{(j)}$ in $\{z : |z - z_0| < \varepsilon\}$ is equal to the order of the pole which $f^{(j)}$ has at z_0 .*

Proof.- Fix $z_0 \in E^{(j)} \setminus E$. Since z_0 is a pole of $f^{(j)}$, there exist $M > 0$ and $\varepsilon > 0$ such that $\{z : 0 < |z - z_0| \leq \varepsilon\} \subset \Omega \setminus \left(\Gamma \cup E^{(j)} \cup E\right)$ and $\left|f^{(j)}(z)/a_{j+1}^2(z)\right| > M$ for $|z - z_0| \leq \varepsilon$. From (12) it follows that $|w_{n-1}^{(j+1)}(z)/w_n^{(j)}(z)| > M/2$ if $|z - z_0| \leq \varepsilon$ and $n \geq n_0$. This implies that either $w_{n-1}^{(j+1)}(z) \neq 0$ if $|z - z_0| \leq \varepsilon$ or $w_{n-1}^{(j+1)}$ and $w_n^{(j)}$ have a common zero in $\{z : |z - z_0| \leq \varepsilon\}$. As it was seen in the proof of Lemma 2, the second case is not possible; therefore, the first one holds. This means that $E^{(j)}$ cannot contain accumulation points of zeros of $\{w_{n-1}^{(j+1)}\}$. On the other hand, from (12) (applied to $f^{(j+1)}$) it also follows that every point in $E^{(j+1)}$ must be a limit point of zeros of $\{w_{n-1}^{(j+1)}\}$. Therefore, $(E^{(j)} \cap E^{(j+1)}) \setminus E = \emptyset$ as asserted.

Now take $0 < \varepsilon_1 \leq \varepsilon$ so that $|f^{(j)}(z)| \neq 0$ if $|z - z_0| = \varepsilon_1$. From (12) we have that $w_n^{(j)}(z)$ has no zeros on $\{z : |z - z_0| = \varepsilon_1\}$ for all $n \geq n_1 \geq n_0$. By (12) and the argument principle, we have that

$$\lim_n \frac{-1}{2\pi i} \int_{[|z-z_0|=\varepsilon]} \frac{\left(w_{n-1}^{(j+1)}(z)/w_n^{(j)}(z)\right)'}{\left(w_{n-1}^{(j+1)}(z)/w_n^{(j)}(z)\right)} dz = \frac{-1}{2\pi i} \int_{[|z-z_0|=\varepsilon]} \frac{\left(f^{(j)}(z)\right)'}{f^{(j)}(z)} dz = \nu(z_0)\,,$$

where $\nu(z_0)$ is the order of the pole which $f^{(j)}$ has at z_0. For $n \geq n_1$ the integral on the left is equal to the number of zeros of $w_n^{(j)}(z)$ inside $\{z : |z - z_0| < \varepsilon\}$. The existence of limit implies that for all $n \geq n_2 \geq n_1$ the number of zeros of $w_n^{(j)}(z)$ inside $\{z : |z - z_0| < \varepsilon\}$ must be equal to $\nu(z_0)$. With this we conclude the proof. □

Let $i \in \{0, \ldots, N-1\}$, $s = N$, $n = mN + i$, and $m \in \mathbb{Z}_+$. Then (7) is a three term recurrence relation with solution $\{w_{mN+i}^{(j)}(z)\}$ for each $z \in \Omega$. The coefficients of this recurrence have limit which are holomorphic functions that can be expressed in terms of $a^{(k)}(z)$, $b^{(k)}(z)$, $k = 0, \ldots, N-1$. For $z \in \Omega$ such that $w_{N-1}^{((m-1)N+i+j+1)}(z) \neq 0$, let

$$\delta_{m,i+j} := \frac{w_{N-1}^{(mN+i+j+1)} w_N^{((m-1)N+i+j+1)} - a_{mN+i+j+2}^2 w_{N-2}^{(mN+i+j+2)} w_{N-1}^{((m-1)N+i+j+1)}}{w_{N-1}^{((m-1)N+i+j+1)}}\,,$$

$$\gamma_{m,i+j} := \frac{a_{(m-1)N+i+j+2}^2 \cdots a_{mN+i+j+1}^2 w_{N-1}^{(mN+i+j+1)}}{w_{N-1}^{((m-1)N+i+j+1)}}\,.$$

We can rewrite (7) as

$$w_{(m+1)N+i}^{(j)}(z) - \delta_{m,i+j}(z) w_{mN+i}^{(j)}(z) + \gamma_{m,i+j}(z) w_{(m-1)N+i}^{(j)}(z) = 0\,, \quad m \geq 1\,. \tag{13}$$

We have

$$\lim_{m\to\infty} w_{N-1}^{((m-1)N+i+j+1)}(z) = (-1)^{N-1} \det \widetilde{\mathcal{D}}_{N-1}^{(i+j+1)}(z)\,, \tag{14}$$

$$\lim_{m\to\infty} \delta_{m,i+j}(z) = (-1)^N \left[\det \widetilde{\mathcal{D}}_N^{(i+j+1)}(z) - \left(a^{(i+j+2)}(z)\right)^2 \det \widetilde{\mathcal{D}}_{N-2}^{(i+j+2)}(z)\right]\,, \tag{15}$$

$$\lim_{m\to\infty} \gamma_{m,i+j}(z) = \Delta(z)\,, \tag{16}$$

where the limits are locally uniform in Ω and

$$\widetilde{\mathcal{D}}^{(k)}(z) \;=\; \begin{pmatrix} -b^{(k+1)}(z) & a^{(k+2)}(z) & & \\ a^{(k+2)}(z) & -b^{(k+2)}(z) & a^{(k+3)}(z) & \\ & a^{(k+3)}(z) & -b^{(k+3)}(z) & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad k = 0, 1, \ldots\,, \tag{17}$$

($\widetilde{\mathcal{D}}(z) \equiv \widetilde{\mathcal{D}}^{(0)}(z)$) and $\widetilde{\mathcal{D}}_n^{(k)}(z)$ is the n-th principal section of $\widetilde{\mathcal{D}}^{(k)}(z)$. This matrix is a particular case of (2). Taking (14), (15), and (16) into consideration, the characteristic equation associated with (13) is

$$
\begin{aligned}
0 = \det \widetilde{\mathcal{D}}_{N-1}^{(i+j+1)}(z)\lambda^2 \quad &+ \quad (-1)^{N-1} P_N(z) \det \widetilde{\mathcal{D}}_{N-1}^{(i+j+1)}(z)\lambda + \\
&+ \quad \left(a^{(0)}(z) \cdots a^{(N-1)}(z)\right)^2 \det \widetilde{\mathcal{D}}_{N-1}^{(i+j+1)}(z)\,, \qquad (18)
\end{aligned}
$$

where

$$
\begin{aligned}
P_N(z) &= \det \widetilde{\mathcal{D}}_{N}^{(i+j+1)}(z) - \left(a^{(i+j+2)}(z)\right)^2 \det \widetilde{\mathcal{D}}_{N-2}^{(i+j+2)}(z) \\
&= \det \widetilde{\mathcal{D}}_{N}^{(N-1)}(z) - \left(a^{(0)}(z)\right)^2 \det \widetilde{\mathcal{D}}_{N-2}(z)\,.
\end{aligned}
$$

For the proof of the last equality see Lemma 5, page 8 in [2]. This means that P_N does not depend on $i+j$.

For $z \in \Omega$ such that $\det \widetilde{\mathcal{D}}_{N-1}^{(i+j+1)}(z) \neq 0$, the solutions of (18) are

$$
\lambda_k(z) = \frac{(-1)^N P_N(z) + (-1)^k \sqrt{(P_N(z))^2 - 4\left(a^{(0)}(z) \cdots a^{(N-1)}(z)\right)^2}}{2}\,, \quad k = 1, 2\,, \qquad (19)
$$

and (18) may be expressed as

$$
0 = \lambda^2 + (-1)^{N-1} P_N(z)\lambda + \Delta(z). \qquad (20)
$$

Lemma 4 *With the notation used in Lemma 1, for each $z \in \Omega$ we have*

$$
\begin{aligned}
E \quad &= \quad \{z \in \Omega \,:\, \exists l \in \{0, \ldots, N-1\}\,, \det \widetilde{\mathcal{D}}_{N-1}^{(l)}(z) = 0\}\,, \qquad (21) \\
I(z) \quad &= \quad (-1)^N P_N(z)\,. \qquad (22)
\end{aligned}
$$

Consequently,

$$
\Gamma = \{z \in \Omega \,:\, |\lambda_1(z)| = |\lambda_2(z)|\}\,.
$$

Proof.- Let $z \in \Omega$. $I(z)$ is independent of $l \in \{0, \ldots, N-1\}$ because for different values of l it represents the trace of different similar matrices. For $m = 1, \ldots, N$, set

$$
\begin{pmatrix} \alpha_z(m) & \beta_z(m) \\ \gamma_z(m) & \delta_z(m) \end{pmatrix} = \begin{pmatrix} 0 & -\left(a^{(0)}(z)\right)^2 \\ 1 & b^{(0)}(z) \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & -\left(a^{(m-1)}(z)\right)^2 \\ 1 & b^{(m-1)}(z) \end{pmatrix},
$$

where $I(z) = \alpha_z(N) + \delta_z(N) = \alpha^{(0)}(z) + \delta^{(0)}(z)$. Let us prove

$$
\left.
\begin{aligned}
\alpha_z(m) \quad &= \quad \left(a^{(0)}\right)^2 (-1)^{m-1} \det \widetilde{\mathcal{D}}_{m-2}(z) \\
\beta_z(m) \quad &= \quad \left(a^{(0)}\right)^2 (-1)^{m} \det \widetilde{\mathcal{D}}_{m-1}(z) \\
\gamma_z(m) \quad &= \quad (-1)^{m-1} \det \widetilde{\mathcal{D}}_{m-1}^{(N-1)}(z) \\
\delta_z(m) \quad &= \quad (-1)^{m} \det \widetilde{\mathcal{D}}_{m}^{(N-1)}(z)
\end{aligned}
\right\}\,, \quad m = 1, \ldots, N\,. \qquad (23)
$$

We proceed by induction. For $m = 1, 2$, (23) is trivial (considering that $\widetilde{\mathcal{D}}_{-1}^{(q)}(z) = 0$, $\widetilde{\mathcal{D}}_0^{(q)}(z) = 1$). Let us assume that (23) holds for $m = s$, where $1 \le s < N$. Taking into consideration that

$$\begin{aligned} \begin{pmatrix} \alpha_z(s+1) & \beta_z(s+1) \\ \gamma_z(s+1) & \delta_z(s+1) \end{pmatrix} &= \begin{pmatrix} \alpha_z(s) & \beta_z(s) \\ \gamma_z(s) & \delta_z(s) \end{pmatrix} \times \begin{pmatrix} 0 & -\left(a^{(s)}(z)\right)^2 \\ 1 & b^{(s)}(z) \end{pmatrix} \\ &= \begin{pmatrix} \beta_z(s) & -\left(a^{(s)}(z)\right)^2 \alpha_z(s) + b^{(s)}(z)\beta_z(s) \\ \delta_z(s) & -\left(a^{(s)}(z)\right)^2 \gamma_z(s) + b^{(s)}(z)\delta_z(s) \end{pmatrix}, \end{aligned}$$

we obtain (23) for $m = s + 1$.

In particular, (23) is true for $m = N$. Since $I(z) = \alpha_z(N) + \delta_z(N)$ we obtain (22). On the other hand, for each $l \in \{0, \ldots, N-1\}$ it immediately follows that $p^{(l)}(z) = 0$ if and only if $\beta^{(l)}(z)\gamma^{(l)}(z) = 0$. Repeating the arguments used in proving (23), we can obtain the entries of the matrix given by (10). We get

$$\left.\begin{aligned} \alpha^{(l)}(z) &= \left(a^{(l)}\right)^2 (-1)^{N-1} \det \widetilde{\mathcal{D}}_{N-2}^{(l)}(z) \\ \beta^{(l)}(z) &= \left(a^{(l)}\right)^2 (-1)^{N} \det \widetilde{\mathcal{D}}_{N-1}^{(l)}(z) \\ \gamma^{(l)}(z) &= (-1)^{N-1} \det \widetilde{\mathcal{D}}_{N-1}^{(l-1)}(z) \\ \delta^{(l)}(z) &= (-1)^{N} \det \widetilde{\mathcal{D}}_{N}^{(l-1)}(z) \end{aligned}\right\} . \tag{24}$$

Using the formulas for $\beta^{(l)}(z)$ and $\gamma^{(l)}(z)$ it follows that (21) takes place. □

The set E is given by the zeros of N analytic functions in Ω which are not identically equal to zero in Ω; therefore, its points can only accumulate on $\partial\Omega$.

For each $j \in \mathbb{Z}_+$ fixed, the sequence $\{\widetilde{w}_n^{(j)}(z)\}$, $n \in \mathbb{N}$, given by (3) associated with the matrix (17) satisfies the following three term recurrence relation for $z \in \Omega \setminus E$,

$$0 = \widetilde{w}_{(m+1)N+i}^{(j)}(z) + (-1)^{N-1} P_N(z) \widetilde{w}_{mN+i}^{(j)}(z) + \Delta(z) \widetilde{w}_{(m-1)N+i}^{(j)}(z)\,,\ m \in \mathbb{N}, \tag{25}$$

which is (13) in this case. Thus, for $z \in \Omega \setminus E$ the sequence $\{\widetilde{w}_{mN+i}^{(j)}(z)\}$, $m \in \mathbb{N}$, verifies

$$0 = c_{m+1} + (-1)^{N-1} P_N(z) c_m + \Delta(z) c_{m-1}\,,\ m \ge 0\,. \tag{26}$$

The associated characteristic equation is also (20), and its solutions λ_1, λ_2, verify $\lambda_1(z) \neq \lambda_2(z)$ for $z \in \Omega \setminus \Gamma$. Therefore, for $z \in \Omega \setminus \Gamma$, any solution of (26) can be expressed as

$$c_m = A(z)(\lambda_1(z))^m + B(z)(\lambda_2(z))^m\,.$$

In particular, taking $c_m = \widetilde{w}_{mN+i}^{(j)}(z)$ with initial conditions $\widetilde{w}_i^{(j)}(z)$, $\widetilde{w}_{N+i}^{(j)}(z)$ (for $m = 0, 1$, respectively), we can determine $A(z)$ and $B(z)$ obtaining

$$\begin{aligned} \widetilde{w}_{mN+i}^{(j)}(z) &= \frac{(\lambda_1(z))^m - (\lambda_2(z))^m}{\lambda_1(z) - \lambda_2(z)} \widetilde{w}_{N+i}^{(j)}(z) - \Delta(z) \frac{(\lambda_1(z))^{m-1} - (\lambda_2(z))^{m-1}}{\lambda_1(z) - \lambda_2(z)} \widetilde{w}_i^{(j)}(z) \\ &= \frac{\widetilde{w}_{N+i}^{(j)}(z) - \lambda_2(z)\widetilde{w}_i^{(j)}(z)}{\lambda_1(z) - \lambda_2(z)} \lambda_1(z)^m - \frac{\widetilde{w}_{N+i}^{(j)}(z) - \lambda_1(z)\widetilde{w}_i^{(j)}(z)}{\lambda_1(z) - \lambda_2(z)} \lambda_2(z)^m\,. \end{aligned} \tag{27}$$

In the sequel we assume that the root in (19) is taken so that $|\lambda_2(z)| < |\lambda_1(z)|$ for $z \in \Omega \setminus \Gamma$.

¿From (27), we obtain the strong asymptotic behavior of the functions $\widetilde{w}_{mN+i}^{(j)}(z)$.

Theorem 1 *For each i, $j \in \{0,1,\ldots,N-1\}$ fixed, we have*

$$\lim_{m\to\infty} \frac{\widetilde{w}^{(j)}_{mN+i}(z)}{(\lambda_1(z))^m} = \frac{\widetilde{w}^{(j)}_{N+i}(z) - \lambda_2(z)\widetilde{w}^{(j)}_i(z)}{\lambda_1(z)-\lambda_2(z)} \tag{28}$$

uniformly on compact subsets of $\Omega \setminus (\Gamma \cup E)$.

Proof.- The result follows dividing (27) by $(\lambda_1(z))^m$ and taking into consideration that $\lim_m (\lambda_2(z)/\lambda_1(z))^m = 0$ uniformly on compact subsets of $\Omega \setminus \Gamma$. □

Corollary 1 *a) Let $i_1, i_2 \in \{0,1,\ldots,N-1\}$ and $j_1, j_2 \in \mathbb{N}$ be fixed. We have*

$$\lim_{m\to\infty} \frac{\widetilde{w}^{(j_1)}_{(m+1)N+i_1}(z)}{\widetilde{w}^{(j_2)}_{mN+i_2}(z)} = \frac{\widetilde{w}^{(j_1)}_{N+i_1}(z) - \lambda_2(z)\widetilde{w}^{(j_1)}_{i_1}(z)}{\widetilde{w}^{(j_2)}_{N+i_2}(z) - \lambda_2(z)\widetilde{w}^{(j_2)}_{i_2}(z)}\lambda_1(z) \tag{29}$$

uniformly on each compact subset of $\Omega \setminus \left(\Gamma \cup E \cup \widetilde{E}\right)$, where

$$\widetilde{E} := \{\zeta \in \mathbb{C} : \widetilde{w}^{(j)}_{N+i}(\zeta) - \lambda_2(\zeta)\widetilde{w}^{(j)}_i(\zeta) = 0 \text{ for some } i,\, j \in \{0,1,\ldots,N-1\}\}.$$

In particular, for each $j \in \{0,1,\ldots,N-1\}$ fixed, we have

$$\lim_{n\to\infty} \frac{\widetilde{w}^{(j)}_{n+N}(z)}{\widetilde{w}^{(j)}_n(z)} = \lambda_1(z) \tag{30}$$

uniformly on each compact subset of $\Omega \setminus \left(\Gamma \cup E \cup \widetilde{E}_j\right)$, where

$$\widetilde{E}_j = \{\zeta \in \mathbb{C} : \widetilde{w}^{(j)}_{N+i}(\zeta) - \lambda_2(\zeta)\widetilde{w}^{(j)}_i(\zeta) = 0 \text{ for some } i \in \{0,1,\ldots,N-1\}\}.$$

b) Let $j \in \mathbb{Z}_+$ be fixed. For each fixed $z \in \widetilde{E}_j$, if $i \in \{0,\ldots,N-1\}$ is such that $\widetilde{w}^{(j)}_{N+i}(z) - \lambda_2(z)\widetilde{w}^{(j)}_i(z) = 0$, then we have

$$\frac{\widetilde{w}^{(j)}_{(m+1)N+i}(z)}{\widetilde{w}^{(j)}_{mN+i}(z)} = \lambda_2(z). \tag{31}$$

Proof.- To obtain (29), take into consideration that in (28) the limit is different from zero for $z \in \Omega \setminus \left(\Gamma \cup E \cup \widetilde{E}\right)$.

Given $j \in \{0,1,\ldots,N-1\}$, if $i \in \{0,1,\ldots,N-1\}$ and $z \in \Omega \setminus \left(\Gamma \cup E \cup \widetilde{E}_j\right)$ the limit in (28) is not zero. Therefore, for $n = mN+i$ it is sufficient to take limit as $m \to \infty$ in

$$\frac{\widetilde{w}^{(j)}_{n+N}(z)}{\widetilde{w}^{(j)}_n(z)} = \frac{\dfrac{\widetilde{w}^{(j)}_{n+N}(z)}{(\lambda_1(z))^{m+1}}}{\dfrac{\widetilde{w}^{(j)}_n(z)}{(\lambda_1(z))^m}}\lambda_1(z)$$

in order to obtain (30).

If $i \in \{0, \ldots, N-1\}$ is such that $\widetilde{w}_{N+i}^{(j)}(z) - \lambda_2(z)\widetilde{w}_i^{(j)}(z) = 0$, from (27) we obtain $\widetilde{w}_{(m+1)N+i}^{(j)}(z) - \lambda_2(z)\widetilde{w}_{mN+i}^{(j)}(z) = 0$ for each $m \in \mathbb{N}$. From here we deduce (31). $\square$

In the sequel, $j \in \mathbb{Z}_+$ is fixed and $\widetilde{f}^{(j)}(z)$ is the limit of the continued fraction (11) for the purely periodic case.

Remark 1 *Given $i \in \{1, 2, \ldots, N-1\}$ and $z \in \Omega \setminus (\Gamma \cup E)$, according to* (12),

$$\widetilde{f}^{(j)}(z) = -\left(a^{(j+1)}(z)\right)^2 \lim_{m\to\infty} \frac{\widetilde{w}_{mN+i-1}^{(j+1)}(z)}{\widetilde{w}_{mN+i}^{(j)}(z)}\,. \tag{32}$$

Then, by Theorem 1*, if there exists some $i \in \{0, \ldots, N-1\}$ such that $\widetilde{w}_{N+i}^{(j)}(z) - \lambda_2(z)\widetilde{w}_i^{(j)}(z) \neq 0$ we have the following explicit expression for $\widetilde{f}^{(j)}$,*

$$\begin{aligned} \widetilde{f}^{(j)}(z) &= -\left(a^{(j+1)}(z)\right)^2 \frac{\widetilde{w}_{N+i-1}^{(j+1)}(z) - \lambda_2(z)\widetilde{w}_{i-1}^{(j+1)}(z)}{\widetilde{w}_{N+i}^{(j)}(z) - \lambda_2(z)\widetilde{w}_i^{(j)}(z)} \\ &= -\left(a^{(j+1)}(z)\right)^2 \frac{\widetilde{w}_{kN+i-1}^{(j+1)}(z) - \lambda_2(z)\widetilde{w}_{(k-1)N+i-1}^{(j+1)}(z)}{\widetilde{w}_{kN+i}^{(j)}(z) - \lambda_2(z)\widetilde{w}_{(k-1)N+i}^{(j)}(z)}\,, \quad k \in \mathbb{N}\,. \end{aligned} \tag{33}$$

Moreover, with the condition above, (33) *is independent of i.*

We can extend (33) *to the case when $\widetilde{w}_{N+i}^{(j)}(z) - \lambda_2(z)\widetilde{w}_i^{(j)}(z) = 0$ for all $i = 0, \ldots, N-1$ but $\widetilde{w}_{N+i_1}^{(j+1)}(z) - \lambda_2(z)\widetilde{w}_{i_1}^{(j+1)}(z) \neq 0$ for some $i_1 \in \{0, \ldots N-1\}$. In this case, from* (32) *and Theorem* 1 *(applied to j and $j+1$) we know that $z \in \widetilde{E}^{(j)}$ and $\widetilde{f}^{(j)}(z) = \infty$ in* (33).

Corollary 2 *If $\widetilde{E}^{(j)}$ is the set of poles of $\widetilde{f}^{(j)}(z)$ on $\Omega \setminus (\Gamma \cup E)$, then*

$$[\Omega \setminus (\Gamma \cup E)] \cap \widetilde{E}_j \subseteq \widetilde{E}^{(j)}\,.$$

Proof.- Let $z \in [\Omega \setminus (\Gamma \cup E)] \cap \widetilde{E}_j$ be fixed. Since $z \in \widetilde{E}_j$, we have that there exists $i_0 \in \{0, 1, \ldots, N-1\}$ such that

$$\widetilde{w}_{mN+i_0}^{(j)}(z) = (\lambda_2(z))^m\, \widetilde{w}_{i_0}^{(j)}(z)\,, \quad m \in \mathbb{N}\,, \tag{34}$$

(see (31)). If

$$\widetilde{w}_{N+i}^{(j)}(z) - \lambda_2(z)\widetilde{w}_i^{(j)}(z) \neq 0 \text{ for some } i \in \{0, \ldots, N-1\}\,, \tag{35}$$

from (28) we obtain $\lim_n \dfrac{\widetilde{w}_{(m+1)N+i}^{(j)}(z)}{\widetilde{w}_{mN+i}^{(j)}(z)} = \lambda_1(z)$. From Poincaré's Theorem (see [7]) applied to (25), if $z \notin \widetilde{E}^{(j)}$ we know that there exists the limit of the ratio $\widetilde{w}_{n+N}^{(j)}(z)/\widetilde{w}_n^{(j)}(z)$. Thus, this limit must be $\lambda_1(z)$, which is false (see (34)). In other words, if (35) holds we arrive to $z \in \widetilde{E}^{(j)}$ and the proof is finished.

Now assume that $\widetilde{w}^{(j)}_{N+i}(z) - \lambda_2(z)\widetilde{w}^{(j)}_i(z) = 0$ for all $i \in \{0, \ldots, N-1\}$; that is,

$$\widetilde{w}^{(j)}_{mN+i}(z) = (\lambda_2(z))^m\, \widetilde{w}^{(j)}_i(z)\,, \quad m \in \mathbb{N}\,, \quad i = 0, \ldots, N-1\,. \tag{36}$$

If $\widetilde{w}^{(j)}_i(z) = 0$ for some $i \in \{0, \ldots, N-1\}$ then from (36) and Lemma 2 we have $z \in \widetilde{E}^{(j)}$. Suppose $\widetilde{w}^{(j)}_i(z) \neq 0$ for all $i \in \{0, \ldots, N-1\}$. We want to show that

$$\widetilde{w}^{(j+1)}_{N+i_1}(z) - \lambda_2(z)\widetilde{w}^{(j+1)}_{i_1}(z) \neq 0 \tag{37}$$

for some $i_1 \in \{0, \ldots, N-1\}$ because, in this case, also $z \in \widetilde{E}^{(j)}$ (see (33) and (36)).

In fact, if $\widetilde{w}^{(j+1)}_{N+i-1}(z) - \lambda_2(z)\widetilde{w}^{(j+1)}_{i-1}(z) = 0$ for all $i = 0, \ldots, N-1$ we have, again by (31),

$$\widetilde{w}^{(j+1)}_{mN+i}(z) = (\lambda_2(z))^m\, \widetilde{w}^{(j+1)}_i(z)\,, \quad m \in \mathbb{N}\,.$$

From this and (36),

$$\widetilde{w}^{(j+1)}_{mN+i-1}(z) = \frac{\widetilde{w}^{(j+1)}_{i-1}(z)}{\widetilde{w}^{(j)}_i(z)}\widetilde{w}^{(j)}_{mN+i}(z)\,, \quad m \in \mathbb{N}\,, \quad i = 0, \ldots, N-1\,. \tag{38}$$

But $z \in \Omega \setminus (\Gamma \cup E)$ and from Lemma 1 we know that the continued fraction (11) (corresponding to the periodic case) converges. Therefore,

$$\frac{\widetilde{w}^{(j+1)}_{mN+i-1}(z)}{\widetilde{w}^{(j)}_{mN+i}(z)} = \frac{\widetilde{w}^{(j+1)}_{i-1}(z)}{\widetilde{w}^{(j)}_i(z)} = -\frac{\widetilde{f}^{(j)}(z)}{\left(a^{(j+1)}(z)\right)^2}$$

(see (12)). That is, the ratio $\widetilde{w}^{(j+1)}_{i-1}(z)/\widetilde{w}^{(j)}_i(z)$ is independent of i, and (38) means that $\{\widetilde{w}^{(j+1)}_{n-1}(z)\}_n$ and $\{\widetilde{w}^{(j)}_n(z)\}_n$ are two linearly independent solutions of recurrence (5) (for the periodic case). However, this is not possible because of the convergence in (11) (see [10, Th. 1, pag. 192]). Thus, (37) holds as we needed to prove. □

For the study of the quotient $w^{(j)}_{n+N}(z)/w^{(j)}_n(z)$ the following result is useful

Lemma 5 ([5], Th. 6, page 1754) *Let the continued fraction*

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \tag{39}$$

be such that $\lim_n \delta^1_n = \delta'$ *and* $\limsup_n |\delta^2_n| < |\delta'|$*, where* δ^1_n , δ^2_n *are the zeros of the polynomial* $\delta^2 - b_n\delta - a_n = 0$*. We have that if* (39) *converges to a finite value then* $\lim_n H_n = \delta'$ *where*

$$H_n = b_n + \cfrac{a_n}{b_{n-1} + \cfrac{a_{n-1}}{b_{n-2} + \ddots + \cfrac{a_2}{b_1}}}\,.$$

Theorem 2 *The ratio* $w_{n+N}^{(j)}(z)/w_n^{(j)}(z)$ *converges uniformly on each compact subset of* $\Omega \setminus \left(\Gamma \cup E \cup E^{(j)}\right)$ *to the root of greater absolute value of* (20).

Proof.- According to Lemma 2 there are no accumulation points of the zeros of the functions $w_n^{(j)}(z)$ in $\Omega \setminus \left(\Gamma \cup E \cup E^{(j)}\right)$. Therefore, for $z \in \Omega \setminus \left(\Gamma \cup E \cup E^{(j)}\right)$ and $i \in \{0, 1, \ldots, N-1\}$ fixed, Poincaré's Theorem applied to the recurrence relation (13) guarantees pointwise limit of the ratio $w_{(m+1)N+i}^{(j)}(z)/w_{mN+i}^{(j)}(z)$, $m \to \infty$, this limit being one of the roots $\lambda_k(z)$ of the characteristic equation (20). Moreover, also from Lemma 2 we know that there exists $m_0 \in \mathbb{Z}_+$ such that

$$w_{(m-1)N+i}^{(j)}(z) \neq 0, \quad m \geq m_0. \tag{40}$$

On the other hand, since $\lim_n w_{N-1}^{(mN+i+j+1)}(z) = \widetilde{w}_{N-1}^{(i+j+1)}(z)$ and $z \notin E$ we can also suppose that

$$w_{N-1}^{((m-1)N+i+j+1)}(z) \neq 0, \quad m \geq m_0. \tag{41}$$

We assume that m_0 is chosen such that $m_0 \geq 2$ and

$$\gamma_{m,i+j}(z) \neq 0 \neq \delta_{m,i+j}(z), \quad m \geq m_0$$

(see (15) and (16)). Under these conditions, from (13), we have

$$\frac{w_{(m+1)N+i}^{(j)}(z)}{w_{mN+i}^{(j)}(z)} = \delta_{m,i+j}(z) - \cfrac{\gamma_{m,i+j}(z)}{\left(\cfrac{w_{mN+i}^{(j)}(z)}{w_{(m-1)N+i}^{(j)}(z)}\right)}$$

$$= \delta_{m,i+j}(z) - \cfrac{\gamma_{m,i+j}(z)}{\delta_{m-1,i+j}(z) - \ddots - \cfrac{\gamma_{m_0,i+j}(z)}{\left(\cfrac{w_{m_0N+i}^{(j)}(z)}{w_{(m_0-1)N+i}^{(j)}(z)}\right)}}. \tag{42}$$

Let us show that the limit of the quotient is at each point of $\Omega \setminus \left(\Gamma \cup E \cup E^{(j)}\right)$ the root of greater absolute value. For this, we apply Lemma 5 to the continued fraction

$$\cfrac{c}{\left(\cfrac{w_{m_0N+i}^{(j)}(z)}{w_{(m_0-1)N+i}^{(j)}(z)}\right) - \cfrac{\gamma_{m_0,i+j}(z)}{\delta_{m_0,i+j}(z) - \cfrac{\gamma_{m_0+1,i+j}(z)}{\delta_{m_0+1,i+j}(z) - \ddots}}} \tag{43}$$

(where $c \neq 0$ is arbitrary) for $z \in \Omega \setminus \left(\Gamma \cup E \cup E^{(j)}\right)$ fixed. By Lemma 1 and the convergence of $\gamma_{m,i+j}$ and $\delta_{m,i+j}$ this fraction converges uniformly on compact subsets of $\Omega \setminus \Gamma$ to a meromorphic function. With the notation used in Lemma 5, let $\delta_m^k = \delta_m^k(z), k = 1, 2,$ be the roots of $\delta^2 - \delta_{m,i+j}(z)\delta + \gamma_{m,i+j}(z) = 0$. Clearly, $\delta_m^k \to \lambda_k(z), m \to \infty$, $k = 1, 2$.

(Recall that $|\lambda_1(z)| \geq |\lambda_2(z)|$.) Thus, to apply Lemma 5 it is sufficient to prove that z is not a pole of the limit function in (43). In other words, we want to show that the continued fraction

$$\cfrac{-\gamma_{m_0,i+j}(z)}{\delta_{m_0,i+j}(z) - \cfrac{\gamma_{m_0+1,i+j}(z)}{\delta_{m_0+1,i+j}(z) - \ddots}} \tag{44}$$

does not converge to $-w^{(j)}_{m_0N+i}(z)/w^{(j)}_{(m_0-1)N+i}(z)$.

The numerators $\{v_n^{(i+j+1)}(z)\}$ and denominators $\{v_n^{(i+j)}(z)\}$ of the partial fractions of (44) are analytic functions which are solution of the recurrence relation

$$w_n = \delta_{m_0+n-1,i+j}(z)w_{n-1} - \gamma_{m_0+n-1,i+j}(z)w_{n-2}\,, \quad n \geq 1\,, \tag{45}$$

with initial conditions given by $v_{-1}^{(i+j+1)}(z) = 1$, $v_0^{(i+j+1)}(z) = 0$ and $v_{-1}^{(i+j)}(z) = 0$, $v_0^{(i+j)}(z) = 1$, respectively.

Suppose that

$$w^{(j)}_{(m_0-1)N+i}(z)w^{(j+1)}_{m_0N+i-1}(z) - w^{(j)}_{m_0N+i}(z)w^{(j+1)}_{(m_0-1)N+i-1}(z) \neq 0\,. \tag{46}$$

This condition means that the solutions $\{w^{(j)}_{(m_0+m)N+i}(z)\}_m$ and $\{w^{(j+1)}_{(m_0+m)N+i-1}(z)\}_m$ of (45) are linearly independent (see [10, pag. 196]). Therefore, each solution of (45) can be expressed as a linear combination of these solutions. In particular, using the initial conditions we obtain

$$v_n^{(i+j+1)} = \frac{w^{(j)}_{(m_0+n)N+i}(z)w^{(j+1)}_{m_0N+i-1}(z) - w^{(j)}_{m_0N+i}(z)w^{(j+1)}_{(m_0+n)N+i-1}(z)}{w^{(j)}_{(m_0-1)N+i}(z)w^{(j+1)}_{m_0N+i-1}(z) - w^{(j)}_{m_0N+i}(z)w^{(j+1)}_{(m_0-1)N+i-1}(z)}\,, \quad n \geq 0\,,$$

and

$$v_n^{(i+j)} = -\frac{w^{(j)}_{(m_0+n)N+i}(z)w^{(j+1)}_{(m_0-1)N+i-1}(z) - w^{(j)}_{(m_0-1)N+i}(z)w^{(j+1)}_{(m_0+n)N+i-1}(z)}{w^{(j)}_{(m_0-1)N+i}(z)w^{(j+1)}_{m_0N+i-1}(z) - w^{(j)}_{m_0N+i}(z)w^{(j+1)}_{(m_0-1)N+i-1}(z)}\,, \quad n \geq 0\,.$$

Thus, the partial fractions for (44) are

$$\frac{v_n^{(i+j+1)}}{v_n^{(i+j)}} = -\frac{w^{(j+1)}_{m_0N+i-1}(z) - w^{(j)}_{m_0N+i}(z)\dfrac{w^{(j+1)}_{(m_0+n)N+i-1}(z)}{w^{(j)}_{(m_0+n)N+i}(z)}}{w^{(j+1)}_{(m_0-1)N+i-1}(z) - w^{(j)}_{(m_0-1)N+i}(z)\dfrac{w^{(j+1)}_{(m_0+n)N+i-1}(z)}{w^{(j)}_{(m_0+n)N+i}(z)}}\,, \quad n \geq 0\,, \tag{47}$$

(see (40)). If the limit function in (44) takes the value $-w^{(j)}_{m_0N+i}(z)/w^{(j)}_{(m_0-1)N+i}(z)$, taking limit in (47) as $n \to \infty$ we have

$$\frac{w^{(j+1)}_{m_0N+i-1}(z) + w^{(j)}_{m_0N+i}(z)\dfrac{f^{(j)}(z)}{a^2_{j+1}(z)}}{w^{(j+1)}_{(m_0-1)N+i-1}(z) + w^{(j)}_{(m_0-1)N+i}(z)\dfrac{f^{(j)}(z)}{a^2_{j+1}(z)}} = \frac{w^{(j)}_{m_0N+i}(z)}{w^{(j)}_{(m_0-1)N+i}(z)}$$

(see (12)). But $z \notin E^{(j)}$ and $f^{(j)}(z) \neq \infty$. Then we can simplify the expression above and we arrive to a contradiction with (46). This fact proofs that, under this condition, the limit in (44) is not $-w^{(j)}_{m_0N+i}(z)/w^{(j)}_{(m_0-1)N+i}(z)$ for $z \in \Omega \setminus \left(\Gamma \cup E \cup E^{(j)}\right)$ and, therefore, for these points the limit of $w^{(j)}_{(m+1)N+i}(z)/w^{(j)}_{mN+i}(z)$ is $\lambda_1(z)$.

Suppose now that (46) is false; that is,

$$w^{(j)}_{(m_0-1)N+i}(z)w^{(j+1)}_{m_0N+i-1}(z) - w^{(j)}_{m_0N+i}(z)w^{(j+1)}_{(m_0-1)N+i-1}(z) = 0\,.$$

Then we have

$$w^{(j)}_{mN+i}(z)w^{(j+1)}_{(m+1)N+i-1}(z) - w^{(j)}_{(m+1)N+i}(z)w^{(j+1)}_{mN+i-1}(z) = 0\,, \quad m \geq m_0 - 1\,, \tag{48}$$

(see again [10, pag. 196]). From this, if $w^{(j+1)}_{mN+i-1}(z) = 0$ for some $m \geq m_0 - 1$ we have $w^{(j+1)}_{mN+i-1}(z) = 0$ for all $m \geq m_0-1$. Change j to $j+1$ in (6). For $s = N$ and $n = mN+i-2$ we obtain there $w^{(j+1)}_{mN+i-2}(z) = 0$ (see (41)). But $w^{(j+1)}_{mN+i-1}(z) = w^{(j+1)}_{mN+i-2}(z) = 0$ implies $w^{(j+1)}_0(z) = 0$ (see (5)), which is false. Hence,

$$w^{(j+1)}_{mN+i-1}(z) \neq 0\,, \quad m \geq m_0 - 1\,. \tag{49}$$

Because of (48) and (49) we can write

$$\frac{w^{(j+1)}_{(m+1)N+i-1}(z)}{w^{(j+1)}_{mN+i-1}(z)} = \frac{w^{(j)}_{(m+1)N+i}(z)}{w^{(j)}_{mN+i}(z)}\,, \quad m \geq m_0 - 1\,.$$

Proceeding by induction, suppose

$$w^{(j+k)}_{mN+i-k}(z) \neq 0\,, \quad m \geq m_0 - 1\,, \quad k = 1, 2, \ldots, \widetilde{k}\,, \tag{50}$$

and

$$\frac{w^{(j+k)}_{(m+1)N+i-k}(z)}{w^{(j+k)}_{mN+i-k}(z)} = \frac{w^{(j)}_{(m+1)N+i}(z)}{w^{(j)}_{mN+i}(z)}\,, \quad m \geq m_0 - 1\,, \quad k = 1, 2, \ldots, \widetilde{k}\,, \tag{51}$$

where $\widetilde{k} \leq mN + i$. We want to proof that (50) and (51) also hold for $\widetilde{k} + 1 \leq mN + i$.

Replace j by $j + \widetilde{k} - 1$ and n by $nN + i - \widetilde{k} + 1$ in (4). Taking into account (50) we have

$$\frac{1}{\left(\dfrac{w^{(j+\widetilde{k})}_{nN+i-\widetilde{k}}(z)}{w^{(j+\widetilde{k}-1)}_{nN+i-\widetilde{k}+1}(z)}\right)} = b_{j+\widetilde{k}}(z) - a^2_{j+\widetilde{k}+1}\frac{w^{(j+\widetilde{k}+1)}_{nN+i-\widetilde{k}-1}(z)}{w^{(j+\widetilde{k})}_{nN+i-\widetilde{k}}(z)}\,, \quad n \geq m_0 - 1\,. \tag{52}$$

¿From (51) and (50),

$$\frac{w^{(j+\widetilde{k})}_{(m+1)N+i-\widetilde{k}}(z)}{w^{(j+\widetilde{k}-1)}_{(m+1)N+i-\widetilde{k}+1}(z)} = \frac{w^{(j+\widetilde{k})}_{mN+i-\widetilde{k}}(z)}{w^{(j+\widetilde{k}-1)}_{mN+i-\widetilde{k}+1}(z)}\,, \quad m \geq m_0 - 1\,. \tag{53}$$

Write (52) for $n = m$ and $n = m+1$ respectively. Comparing both expressions and taking into account (53) we arrive to

$$\frac{w^{(j+\widetilde{k}+1)}_{(m+1)N+i-\widetilde{k}-1}(z)}{w^{(j+\widetilde{k})}_{(m+1)N+i-\widetilde{k}}(z)} = \frac{w^{(j+\widetilde{k}+1)}_{mN+i-\widetilde{k}-1}(z)}{w^{(j+\widetilde{k})}_{mN+i-\widetilde{k}}(z)}\,, \quad m \geq m_0 - 1\,. \tag{54}$$

Hence, if $w^{(j+\widetilde{k}+1)}_{mN+i-\widetilde{k}-1}(z) = 0$ for some $m \geq m_0 - 1$ we have that the same is true for all $m \geq m_0 - 1$. Replace j by $j + \widetilde{k} + 1$ in (6). Taking there $s = N$ and $n = mN + i - \widetilde{k} - 2$ we obtain $w^{(j+\widetilde{k}+1)}_{mN+i-\widetilde{k}-2}(z) = 0$. This is,

$$w^{(j+\widetilde{k}+1)}_{mN+i-\widetilde{k}-1}(z) = w^{(j+\widetilde{k}+1)}_{mN+i-\widetilde{k}-2}(z) = 0\,, \quad m \geq m_0 - 1\,. \tag{55}$$

If $\widetilde{k} = mN + i - 1$ then $w^{(j+\widetilde{k}+1)}_0(z) = 0$ which is false. If $\widetilde{k} \leq mN + i - 2$, from (55) we arrive to

$$w^{(j+\widetilde{k}+1)}_{mN+i-\widetilde{k}-3}(z) = w^{(j+\widetilde{k}+1)}_{mN+i-\widetilde{k}-4}(z) = \ldots = w^{(j+\widetilde{k}+1)}_0(z) = 0\,,$$

which also is false because of (5). Therefore, we deduce that (50) holds for $k = \widetilde{k} + 1$.

Now, we can rewrite (54) in the form

$$\frac{w^{(j+\widetilde{k}+1)}_{(m+1)N+i-\widetilde{k}-1}(z)}{w^{(j+\widetilde{k}+1)}_{mN+i-\widetilde{k}-1}(z)} = \frac{w^{(j+\widetilde{k})}_{(m+1)N+i-\widetilde{k}}(z)}{w^{(j+\widetilde{k})}_{mN+i-\widetilde{k}}(z)}\,, \quad m \geq m_0 - 1\,;$$

that is, we also have (51) for $k = \widetilde{k} + 1$.

We just proved that (50) and (51) hold for $m \geq m_0 - 1$ and $k \leq mN + i$. In particular, take $k = (m - \widetilde{m} + 1)N$, being $\widetilde{m}$ fixed such that $\widetilde{m} \in \mathbb{Z}_+$, $\widetilde{m} \geq m_0$. Then

$$\frac{w^{(j+(m-\widetilde{m}+1)N)}_{\widetilde{m}N+i}(z)}{w^{(j+(m-\widetilde{m}+1)N)}_{(\widetilde{m}-1)N+i}(z)} = \frac{w^{(j)}_{(m+1)N+i}(z)}{w^{(j)}_{mN+i}(z)}\,, \quad m \geq m_0 - 1\,.$$

Taking limits as $m \to \infty$,

$$\frac{\widetilde{w}^{(j)}_{\widetilde{m}N+i}(z)}{\widetilde{w}^{(j)}_{(\widetilde{m}-1)N+i}(z)} = \lim_m \frac{w^{(j)}_{(m+1)N+i}(z)}{w^{(j)}_{mN+i}(z)} \in \{\lambda_1(z), \lambda_2(z)\}\,. \tag{56}$$

Therefore, taking limits in (56) as $\widetilde{m} \to \infty$,

$$\frac{\widetilde{w}^{(j)}_{m_0N+i}(z)}{\widetilde{w}^{(j)}_{(m_0-1)N+i}(z)} = \lim_{\widetilde{m}} \frac{\widetilde{w}^{(j)}_{\widetilde{m}N+i}(z)}{\widetilde{w}^{(j)}_{(\widetilde{m}-1)N+i}(z)} = \lim_m \frac{w^{(j)}_{(m+1)N+i}(z)}{w^{(j)}_{mN+i}(z)}$$

(the sequence on the left hand of (56) is a constant sequence). If the above limit is $\lambda_2(z)$, because of Corollary 1 (see (30)) and Corollary 2 we have

$$z \in \widetilde{E}_j \cap [\Omega \setminus (\Gamma \cup E)] \subset \widetilde{E}^{(j)}\,. \tag{57}$$

On the other hand, taking $k = (m - \widetilde{m} + 1)N + 1$, $\widetilde{m}$ as above, and proceeding in the same form that before, we obtain from (51)

$$\lim_{\widetilde{m}} \frac{\widetilde{w}^{(j+1)}_{\widetilde{m}N+i-1}(z)}{\widetilde{w}^{(j+1)}_{(\widetilde{m}-1)N+i-1}(z)} = \lim_{m} \frac{w^{(j)}_{(m+1)N+i}(z)}{w^{(j)}_{mN+i}(z)} = \lambda_2(z)\,.$$

Hence, as in (57), $z \in \widetilde{E}^{(j+1)}$. In other words, $z \in \widetilde{E}^{(j)} \cap \widetilde{E}^{(j+1)}$, which is not possible because of Lemma 3. Therefore, the limit in (56) is $\lambda_1(z)$, as we wanted to prove.

To conclude the proof, we show that the sequence $\left\{w^{(j)}_{(m+1)N+i}/w^{(j)}_{mN+i}\right\}$, $m \in \mathbb{N}$, is uniformly bounded on each compact subset of $\Omega \setminus \left(\Gamma \cup E \cup E^{(j)}\right)$. Let $z_1 \in \Omega \setminus \left(\Gamma \cup E \cup E^{(j)}\right)$ be fixed. It will be sufficient to prove that there exists a closed disk centered at z_1, entirely contained in $\Omega \setminus \left(\Gamma \cup E \cup E^{(j)}\right)$, on which $\left\{w^{(j)}_{(m+1)N+i}/w^{(j)}_{mN+i}\right\}$, $n \in \mathbb{N}$, is uniformly bounded.

Let $\varepsilon_1 > 0$ be such that $\overline{D_{\varepsilon_1}(z_1)} \subset \Omega \setminus \left(\Gamma \cup E \cup E^{(j)}\right)$, where $D_{\varepsilon_1}(z_1) := \{z \,:\, |z - z_1| < \varepsilon_1\}$. Using Lemma 2, there exists $n_1 = n_1(\varepsilon_1)$ such that $w^{(j)}_{mN+i}(z) \neq 0$ for $z \in \overline{D_{\varepsilon_1}(z_1)}$ and $m \geq n_1$. Let $\widehat{w}^{(j)}_p(z) = w^{(j)}_p(z)/\left(a_j(z)a_{j+1}(z)\dots a_p(z)\right)$. From (13) we have

$$\widehat{w}^{(j)}_{(m+1)N+i} - \frac{\delta_{m,i+j}}{a_{mN+i+1}\cdots a_{(m+1)N+i}}\widehat{w}^{(j)}_{mN+i} + \frac{\gamma_{m,i+j}}{a_{(m-1)N+i+1}\cdots a_{(m+1)N+i}}\widehat{w}^{(j)}_{(m-1)N+i} = 0 \qquad (58)$$

for $m \geq 1$, and uniformly on compact subsets of Ω we have that

$$\lim_{m} \frac{\delta_{m,i+j}}{a_{mN+i+1}\cdots a_{(m+1)N+i}} = \frac{(-1)^N P_N}{\Delta^{1/2}}\,, \quad \lim_{m} \frac{\gamma_{m,i+j}}{a_{(m-1)N+i+1}\cdots a_{(m+1)N+i}} = 1\,. \qquad (59)$$

Here and in the sequel we will refrain from writing the variable z.

The characteristic equation associated with (58) is

$$\delta^2 + \frac{(-1)^{N-1}P_N}{\Delta^{1/2}}\delta + 1 = 0\,,$$

and its roots are

$$\widehat{\lambda}_i = \lambda_i/\Delta^{1/2}\,,\ i = 1, 2\,,$$

with $\widehat{\lambda}_1\widehat{\lambda}_2 = 1$. Recall that λ_1, and hence $\widehat{\lambda}_1$, is the root of greater absolute value at each z. Therefore, $|\widehat{\lambda}_1| \geq 1 \geq |\widehat{\lambda}_2|$. Let $\rho := |\widehat{\lambda}_1(z_1)| - 1$. Obviously, $\rho > 0$. Since

$$\lim_{m\to\infty} \frac{\widehat{w}^{(j)}_{(m+1)N+i}(z_1)}{\widehat{w}^{(j)}_{mN+i}(z_1)} = \widehat{\lambda}_1(z_1)$$

there exists $n_2 \geq n_1$ such that

$$\left| \frac{\widehat{w}^{(j)}_{(m+1)N+i}(z_1)}{\widehat{w}^{(j)}_{mN+i}(z_1)} - \widehat{\lambda}_1(z_1) \right| < \rho\,, \quad m \geq n_2\,. \qquad (60)$$

Let us rewrite (58) as

$$\frac{\widehat{w}^{(j)}_{(m+1)N+i}}{\widehat{w}^{(j)}_{mN+i}} = \frac{\delta_{m,i+j}}{a_{mN+i+1}\cdots a_{(m+1)N+i}} - \frac{\gamma_{m,i+j}}{a_{(m-1)N+i+1}\cdots a_{(m+1)N+i}} \frac{1}{\left(\dfrac{\widehat{w}^{(j)}_{mN+i}}{\widehat{w}^{(j)}_{(m-1)N+i}}\right)} \tag{61}$$

for $m \geq n_2$ and $z \in \overline{D_{\varepsilon_1}(z_1)}$. Taking (59) into consideration and the continuity of the function $(-1)^N P_N/\Delta^{1/2}$, there exists $n_3 \geq n_2$ such that

$$\begin{aligned} \left|\frac{\delta_{m,i+j}}{a_{mN+i+1}\cdots a_{(m+1)N+i}} - \frac{(-1)^N P_N(z_1)}{\Delta(z_1)^{1/2}}\right| &\leq \left|\frac{\delta_{m,i+j}}{a_{mN+i+1}\cdots a_{(m+1)N+i}} - \frac{(-1)^N P_N}{\Delta^{1/2}}\right| \\ + \left|\frac{(-1)^N P_N}{\Delta^{1/2}} - \frac{(-1)^N P_N(z_1)}{\Delta(z_1)^{1/2}}\right| &\leq \frac{\rho^2}{3|\widehat{\lambda}_1(z_1)|}\,, \quad m \geq n_3\,, \quad z \in D_{\varepsilon_2}(z_1)\,, \end{aligned} \tag{62}$$

where $0 < \varepsilon_2 \leq \varepsilon_1$ depends on z_1 and the function $P_N/\Delta^{1/2}$. Moreover, there exists $n_4 > n_3$ such that

$$\left|\frac{\gamma_{m,i+j}}{a_{(m-1)N+i+1}\cdots a_{(m+1)N+i}} - 1\right| < \frac{\rho^2}{3}\,, \quad m \geq n_4\,, \quad z \in D_{\varepsilon_2}(z_1) \tag{63}$$

and

$$\left|\frac{\gamma_{m,i+j}}{a_{(m-1)N+i+1}\cdots a_{(m+1)N+i}}\right| < 1 + \frac{\rho}{3}\,, \quad m \geq n_4\,, \quad z \in D_{\varepsilon_2}(z_1)\,. \tag{64}$$

The function $\widehat{w}^{(j)}_{n_4N+i}/\widehat{w}^{(j)}_{(n_4-1)N+i}$ is continuous at $z = z_1$. From this and (60), we have

$$\left|\frac{\widehat{w}^{(j)}_{n_4N+i}}{\widehat{w}^{(j)}_{(n_4-1)N+i}} - \widehat{\lambda}_1(z_1)\right| < \rho\,, \quad z \in \overline{D_{\varepsilon_3}(z_1)} \tag{65}$$

(where $0 < \varepsilon_3 \leq \varepsilon_2$ depends on z_1 and n_4). Therefore,

$$\begin{aligned} \left|\left(\frac{\widehat{w}^{(j)}_{n_4N+i}}{\widehat{w}^{(j)}_{(n_4-1)N+i}}\right)^{-1} - \left(\widehat{\lambda}_1(z_1)\right)^{-1}\right| &= \\ = \left|\frac{\dfrac{\widehat{w}^{(j)}_{n_4N+i}}{\widehat{w}^{(j)}_{(n_4-1)N+i}} - \widehat{\lambda}_1(z_1)}{\dfrac{\widehat{w}^{(j)}_{n_4N+i}}{\widehat{w}^{(j)}_{(n_4-1)N+i}}\widehat{\lambda}_1(z_1)}\right| &< \frac{\rho}{\left|\dfrac{\widehat{w}^{(j)}_{n_4N+i}}{\widehat{w}^{(j)}_{(n_4-1)N+i}}\right|\left|\widehat{\lambda}_1(z_1)\right|}\,, \quad z \in \overline{D_{\varepsilon_3}(z_1)}\,. \end{aligned}$$

Also, from (65) and the way in which ρ was selected

$$1 = \left|\widehat{\lambda}_1(z_1)\right| - \rho < \left|\frac{\widehat{w}^{(j)}_{n_4N+i}}{\widehat{w}^{(j)}_{(n_4-1)N+i}}\right|\,, \quad z \in \overline{D_{\varepsilon_3}(z_1)}\,.$$

Hence

$$\left|\left(\frac{\widehat{w}^{(j)}_{n_4N+i}}{\widehat{w}^{(j)}_{(n_4-1)N+i}}\right)^{-1} - \left(\widehat{\lambda}_1(z_1)\right)^{-1}\right| < \frac{\rho}{\left|\widehat{\lambda}_1(z_1)\right|}\,, \quad z \in \overline{D_{\varepsilon_3}(z_1)}\,. \tag{66}$$

Returning to (61), for $m = n_4$ we have (notice that $\widehat{\lambda}_1 + \widehat{\lambda}_2 = (-1)^N P_N/\Delta^{1/2}$)

$$\frac{\widehat{w}^{(j)}_{(n_4+1)N+i}}{\widehat{w}^{(j)}_{n_4N+i}} - \widehat{\lambda}_1(z_1) = \left[\frac{\delta_{n_4,i+j}}{a_{n_4N+i+1}\cdots a_{(n_4+1)N+i}} - \frac{(-1)^N P_N(z_1)}{\Delta(z_1)^{1/2}} - \right]$$

$$\frac{\gamma_{n_4,i+j}}{a_{(n_4-1)N+i+1}\cdots a_{(n_4+1)N+i}}\left[\left(\frac{\widehat{w}^{(j)}_{n_4N+i}}{\widehat{w}^{(j)}_{(n_4-1)N+i}}\right)^{-1} - \widehat{\lambda}_2(z_1)\right] +$$

$$\widehat{\lambda}_2(z_1)\left[1 - \frac{\gamma_{n_4,i+j}}{a_{(n_4-1)N+i+1}\cdots a_{(n_4+1)N+i}}\right].$$

Therefore, taking absolute values in the previous equality and taking into consideration (62), (63), (64), and (66), we obtain for $z \in \overline{D_{\varepsilon_3}(z_1)}$

$$\left|\frac{\widehat{w}^{(j)}_{(n_4+1)N+i+j}}{\widehat{w}^{(j)}_{n_4N+i+j}} - \widehat{\lambda}_1(z_1)\right| < \frac{\rho^2}{3\left|\widehat{\lambda}_1(z_1)\right|} + (1+\frac{\rho}{3})\frac{\rho}{\left|\widehat{\lambda}_1(z_1)\right|} + \frac{1}{\left|\widehat{\lambda}_1(z_1)\right|}\frac{\rho^2}{3} = \rho\,, \tag{67}$$

which is (65) with n_4 replaced by n_4+1. In turn, this gives (66) and, consequently, (67) with n_4 replaced by n_4+1. Repeating this process we conclude that

$$\left|\frac{\widehat{w}^{(j)}_{(m+1)N+i}}{\widehat{w}^{(j)}_{mN+i}} - \widehat{\lambda}_1(z_1)\right| < \rho\,, \quad m \geq n_4\,, \quad z \in \overline{D_{\varepsilon_3}(z_1)}\,,$$

as we needed to prove. □

For $N = 1$, Theorem 2 indicates that for each $j \in \mathbb{N}$ fixed there is uniform convergence of the quotient $w^{(j)}_{n+1}/w^{(j)}_n, n \to \infty$, on each compact subset of $\Omega \setminus \left(\Gamma \cup E^{(j)}\right)$ since in this case $E = \emptyset$. Now, (13) and (5) coincide, with $n = m + i + 1$. Its characteristic equation (20) given by $\lambda^2 - b(z)\lambda + a(z) = 0$ has for roots

$$\lambda_k = \frac{b(z) \pm \sqrt{b(z)^2 - 4a(z)^2}}{2}\,, \quad k = 1, 2\,.$$

In the limiting case, when the difference equation (5) has constant coefficients, we have

$$\widetilde{w}^{(j)}_n(z) = \frac{(\lambda_1(z))^{n+1} - (\lambda_2(z))^{n+1}}{\lambda_1(z) - \lambda_2(z)}$$

when $\lambda_1(z) \neq \lambda_2(z)$. Therefore, the roots of $\widetilde{w}^{(j)}_n(z)$ lie on $\Gamma = \{z : |\lambda_1(z)| = |\lambda_2(z)|\}$. We can write

$$\lambda_i(z) = a(z)\left(\frac{b(z)}{2a(z)} \pm \sqrt{\left(\frac{b(z)}{2a(z)}\right)^2 - 1}\right) = a(z)\varphi_\pm(\xi_z)\,, \, \xi_z = \frac{b(z)}{2a(z)}\,, \, \varphi_\pm(\xi) = \xi \pm \sqrt{\xi^2 - 1}\,.$$

Since $|\varphi_+(\xi)| = \dfrac{1}{|\varphi_-(\xi)|}$, it follows that $|\lambda_1(z)| = |\lambda_2(z)|$ if and only if $|\varphi_\pm(\xi_z)| = 1$. This occurs if and only if $\xi_z \in [-1, 1]$.

In particular, for $b_k(z) = z + b_k$ and $a_k(z)^2 = zb_k(1 - a_k^2)$, where $a_k, b_k \in \mathbb{C}\, k \in \mathbb{N}$, and $b_k \to b$, $a_k \to a$ as $k \to \infty$, the recurrence (5) is

$$w_n^{(j)}(z) - (z + b_{n+j})w_{n-1}^{(j)}(z) + zb_{n+j}(1 - a_{n+j}^2)w_{n-2}^{(j)}(z) = 0\,.$$

If $|a_n| < 1$, $n \in \mathbb{N}$, the polynomials $\{w_n^{(j)}\}$, $n \in \mathbb{N}$, are orthogonal on the unit circle with respect to a measure whose continuous part has Γ for its support. In [2], we obtained that the set Γ when $a \in (0, 1]$ is the arc of the unit circle

$$\Gamma = \{z \,:\, |z| = 1\,,\, \alpha + \arg b \le \arg z \le 2\pi - \alpha + \arg b\}$$

where $a = \sin\alpha/2$. When $a = 1$ the arc reduces to the point $\Gamma = \{-b\}$.

If $a > 1$, we have $\xi_z = \dfrac{z + b}{2i\sqrt{zb(a^2 - 1)}} \in [-1, 1]$ if and only if

$$\sqrt{\frac{z}{b}} + \sqrt{\frac{b}{z}} \in [-2i\sqrt{a^2 - 1}, 2i\sqrt{a^2 - 1}]\,,$$

where the right hand represents the segment whose end points are $\pm 2i\sqrt{a^2 - 1}$. Set $\sqrt{\dfrac{z}{b}} = re^{i\alpha}$, $r > 0$, $\alpha \in (-\pi, \pi]$, then ξ_z can be written as

$$\xi_z = \frac{1}{2i\sqrt{a^2 - 1}}(re^{i\alpha} + \frac{1}{r}e^{-i\alpha}) = \frac{1}{2i\sqrt{a^2 - 1}}\left((r + \frac{1}{r})\cos\alpha + i(r - \frac{1}{r})\sin\alpha\right)\,.$$

In order that $\xi_z \in [-1, 1]$ it is necessary, in the first place, that $\cos\alpha = 0$; that is, $2\alpha = \pi$. Therefore, $\arg z = \pi + \arg b$. In other words, Γ is contained in the half line beginning at the origin which passes through $-b$. Secondly, in order that $\xi_z = \dfrac{1}{2\sqrt{a^2 - 1}}(r - \dfrac{1}{r}) \in [-1, 1]$, we have that $-2r\sqrt{a^2 - 1} \le r^2 - 1 \le 2r\sqrt{a^2 - 1}$. Considering both inequalities, we obtain that $r \in [a - \sqrt{a^2 - 1}, a + \sqrt{a^2 - 1}]$. Therefore, in passing from $a \le 1$ to $a > 1$ the set Γ transforms from an arc of the unit circle to a segment perpendicular to the unit circle each one passing through $z = -b$ and symmetric with respect to this point.

In [14, sections 3 and 4] the authors study the asymptotic properties of the so called Laurent polynomials. These polynomials $\{B_n\}$ satisfy the recurrence relation

$$B_n(z) = (z - \beta_n)B_{n-1}(z) - \alpha_n z B_{n-2}(z)\,,$$

where $\beta_n, \alpha_n > 0$ and $\beta_n \to \beta, \alpha_n \to \alpha$ (as $n \to \infty$). Our previous example generalizes the study to the case when $\{\alpha_n\}$ and $\{\beta_n\}$ are arbitrary sequences of complex numbers such that $\alpha\beta > 0$ and extends Theorem 4.1 which is one of the main results of that paper.

3 Strong asymptotic

Under conditions more restrictive than those imposed in the previous section it is possible to deduce a result similar to Theorem 1 for the functions $\{w_n^{(j)}\}$, $n \in \mathbb{N}$. More precisely, this result depends on the proximity of the families of functions $\{\widetilde{w}_n^{(j)}\}$ and $\{w_n^{(j)}\}$ expressed in terms of the coefficients of the recurrence (1). This fact has been used by other authors to relate the properties of both families of functions in special situations (see [3, Theorem 3.8] and [13, section 5.3]).

Lemma 6 *The family of functions* $\{\widetilde{Q}_n^{(j)}(z)\}$, $n \in \mathbb{N}$, *solution to the recurrence*

$$a^{(j+n+1)}(z)s_n = b^{(j+n)}(z)s_{n-1} - a^{(j+n)}(z)s_{n-2}\,, \quad n \geq 1\,, \tag{68}$$

with initial conditions

$$\widetilde{Q}_0^{(j)}(z) = -\frac{\widetilde{f}^{(j)}(z)}{\left(a^{(j+1)}(z)\right)^2}\,, \quad \widetilde{Q}_{-1}^{(j)}(z) = 1/a^{(j+1)}(z)\,, \tag{69}$$

analytic in $\Omega \setminus \left(\Gamma \cup E \cup \widetilde{E}^{(j)}\right)$, *verify*

$$\widetilde{Q}_n^{(j)}(z) = \frac{-\widetilde{w}_n^{(j)}(z)\dfrac{\widetilde{f}^{(j)}(z)}{\left(a^{(j+1)}(z)\right)^2} - \widetilde{w}_{n-1}^{(j+1)}(z)}{a^{(j+2)}(z)a^{(j+3)}(z)\dots a^{(j+n+1)}(z)}\,, \quad n \in \mathbb{N}\,, \quad z \in \Omega \setminus (\Gamma \cup E)\;. \tag{70}$$

Proof.- We proceed by induction on n. For $n = 1$ and $n = 2$, (70) is a direct consequence of (68) and (69). Suppose that (70) holds for $n = 1, 2, \dots, m$. Then, it is sufficient to write (68) for $n = m + 1$ and take into consideration that the functions $\widetilde{w}_n^{(j)}(z)$ satisfy the recurrence relation (5) (with periodic coefficients). $\square$

Lemma 7 *For* $i \in \{0, 1, \dots, N-1\}$ *and* $m \in \mathbb{N}$, *we have*

$$\frac{\widetilde{Q}_{mN+i}^{(j)}(z)}{\left(\widehat{\lambda}_2(z)\right)^m} = \widetilde{Q}_i^{(j)}(z)\,, \quad z \in \Omega \setminus \left(\Gamma \cup E \cup \widetilde{E}^{(j)}\right)\;. \tag{71}$$

Proof.- Let $z \in \Omega \setminus \left(\Gamma \cup E \cup \widetilde{E}^{(j)}\right)$ and let $i \in \{0, 1, \dots, N-1\}$ be fixed. Since the sequences $\{\widetilde{w}_{mN+i}^{(j)}\}$ and $\{\widetilde{w}_{mN+i-1}^{(j+1)}\}$, $m \in \mathbb{N}$, are solutions of the recurrence relation (25), then the sequence $\{Q_{mN+i}^{(j)}\}$, $m \in \mathbb{N}$, given by $Q_n^{(j)}(z) = -\widetilde{w}_n^{(j)}(z)\dfrac{\widetilde{f}^{(j)}(z)}{\left(a^{(j+1)}(z)\right)^2} - \widetilde{w}_{n-1}^{(j+1)}(z)$, is also a solution of (25). Since

$$Q_n^{(j)} = a^{(j+2)}a^{(j+3)}\dots a^{(j+n+1)}\widetilde{Q}_n^{(j)}$$

we deduce that $\{\widetilde{Q}_{mN+i}^{(j)}\}$, $m \in \mathbb{N}$, is solution of the difference equation with constant coefficients

$$\Delta(z)^{1/2}s_{m+1} + (-1)^{N-1}P_N(z)s_m + \Delta(z)^{1/2}s_{m-1} = 0$$

whose characteristic equation

$$\Delta(z)^{1/2}s^2 + (-1)^{N-1}P_N(z)s + \Delta(z)^{1/2} = 0$$

has for solutions $\widehat{\lambda}_1(z)$ and $\widehat{\lambda}_2(z)$. Therefore,

$$\widetilde{Q}^{(j)}_{mN+i}(z) = A(z)\left(\widehat{\lambda}_1(z)\right)^m + B(z)\left(\widehat{\lambda}_2(z)\right)^m\,, \quad m \geq 1\,. \tag{72}$$

¿From the initial conditions $\widetilde{Q}^{(j)}_{N+i}(z)$ and $\widetilde{Q}^{(j)}_{i}(z)$, for $m = 1, 0$, respectively, we obtain

$$A(z) = \frac{\widetilde{Q}^{(j)}_{N+i}(z) - \widehat{\lambda}_2(z)\widetilde{Q}^{(j)}_{i}(z)}{\widehat{\lambda}_1(z) - \widehat{\lambda}_2(z)}\,, \quad B(z) = -\frac{\widetilde{Q}^{(j)}_{N+i}(z) - \widehat{\lambda}_1(z)\widetilde{Q}^{(j)}_{i}(z)}{\widehat{\lambda}_1(z) - \widehat{\lambda}_2(z)}\,.$$

Let us show that $A(z) \equiv 0$.

In the rest of the proof, we delete the variable z from the notation. We have

$$\widetilde{Q}^{(j)}_{N+i} - \widehat{\lambda}_2\widetilde{Q}^{(j)}_{i} = -\frac{\widetilde{w}^{(j)}_{N+i}\dfrac{\widetilde{f}^{(j)}}{\left(a^{(j+1)}\right)^2} + \widetilde{w}^{(j+1)}_{N+i-1}}{a^{(j+2)}\ldots a^{(j+N+i+1)}} + \widehat{\lambda}_2\frac{\widetilde{w}^{(j)}_{i}\dfrac{\widetilde{f}^{(j)}}{\left(a^{(j+1)}\right)^2} + \widetilde{w}^{(j+1)}_{i-1}}{a^{(j+2)}\ldots a^{(j+i+1)}}$$

$$= \frac{1}{a^{(j+2)}\ldots a^{(j+i+1)}}\left[\frac{-\widetilde{w}^{(j)}_{N+i}\dfrac{\widetilde{f}^{(j)}}{\left(a^{(j+1)}\right)^2} - \widetilde{w}^{(j+1)}_{N+i-1}}{a^{(j+i+2)}\ldots a^{(j+i+N+1)}} + \widehat{\lambda}_2\left(\widetilde{w}^{(j)}_{i}\frac{\widetilde{f}^{(j)}}{\left(a^{(j+1)}\right)^2} + \widetilde{w}^{(j+1)}_{i-1}\right)\right]\,.$$

Since $a^{(j+i+2)}\ldots a^{(j+i+N+1)} = \Delta^{1/2}$ and $\widehat{\lambda}_2 = \lambda_2/\Delta^{1/2}$, rearranging the terms we arrive at

$$\widetilde{Q}^{(j)}_{N+i} - \widehat{\lambda}_2\widetilde{Q}^{(j)}_{i} = \frac{-\left(\widetilde{w}^{(j)}_{N+i} - \lambda_2\widetilde{w}^{(j)}_{i}\right)\dfrac{\widetilde{f}^{(j)}}{\left(a^{(j+1)}\right)^2} - \left(\widetilde{w}^{(j+1)}_{N+i-1} - \lambda_2\widetilde{w}^{(j+1)}_{i-1}\right)}{\Delta^{1/2}a^{(j+2)}\ldots a^{(j+i+1)}}\,.$$

But, according to (33),

$$-\frac{\widetilde{f}^{(j)}}{\left(a^{(j+1)}\right)^2} = \frac{\widetilde{w}^{(j+1)}_{N+i-1} - \lambda_2\widetilde{w}^{(j+1)}_{i-1}}{\widetilde{w}^{(j)}_{N+i} - \lambda_2\widetilde{w}^{(j)}_{i}}\,.$$

Hence $\widetilde{Q}^{(j)}_{N+i} - \widehat{\lambda}_2\widetilde{Q}^{(j)}_{i} = 0$. Thus $\widetilde{Q}^{(j)}_{N+i} - \widehat{\lambda}_1\widetilde{Q}^{(j)}_{i} = (\widehat{\lambda}_2 - \widehat{\lambda}_1)\widetilde{Q}^{(j)}_{i}$ and (71) follows from (72). □

The following lemma provides relations between different functions $\widetilde{Q}^{(j+s)}_{n}(z)$ which we will use later.

Lemma 8 *For each $k = 0, 1, \ldots$ and for each $n \in \mathbb{N}$, $n \geq k-1$, we have*

$$\widetilde{Q}^{(j)}_{n}(z) = a^{(j+k+1)}(z)\widetilde{Q}^{(j+k)}_{n-k}(z)\widetilde{Q}^{(j)}_{k-1}(z)\,. \tag{73}$$

Proof.- For k fixed, the families of functions $\{a^{(j+k+1)}(z)\widetilde{Q}^{(j+k)}_{n-k}(z)\widetilde{Q}^{(j)}_{k-1}(z)\}$, $n \geq k-1$, and $\{\widetilde{Q}^{(j)}_{n}(z)\}$ are both solutions of (68). Using the uniqueness of such solutions, in order to conclude that they coincide it is sufficient to verify that both satisfy the same initial conditions.

In fact, for $n = k-1$ the right hand of (73) is $a^{(j+k+1)}(z)\widetilde{Q}_{-1}^{(j+k)}(z)\widetilde{Q}_{k-1}^{(j)}(z) = \widetilde{Q}_{k-1}^{(j)}(z)$ and we have the desired equality (see (69)).

For $n = k$, from (69) we have $-\dfrac{\widetilde{f}^{(j+k)}(z)}{a^{(j+k+1)}(z)}\widetilde{Q}_{k-1}^{(j)}(z)$ on the right hand of (73). Taking $n = k-1$ and $s = m$ in (6) (adapting the formula to the purely periodic case), we have

$$\widetilde{w}_{k+m}^{(j)}(z) = \widetilde{w}_m^{(j+k)}(z)\widetilde{w}_k^{(j)}(z) - \left(a^{(j+k+1)}(z)\right)^2 \widetilde{w}_{m-1}^{(j+k+1)}(z)\widetilde{w}_{k-1}^{(j)}(z)\,. \tag{74}$$

Substituting j by $j+1$ and k by $k-1$ in (74), it follows that

$$\widetilde{w}_{m+k-1}^{(j+1)}(z) = \widetilde{w}_m^{(j+k)}(z)\widetilde{w}_{k-1}^{(j+1)}(z) - \left(a^{(j+k+1)}(z)\right)^2 \widetilde{w}_{m-1}^{(j+k+1)}(z)\widetilde{w}_{k-2}^{(j+1)}(z)\,. \tag{75}$$

¿From (74) and (75),

$$\frac{\widetilde{w}_{m+k-1}^{(j+1)}(z)}{\widetilde{w}_{k+m}^{(j)}(z)} = \frac{\widetilde{w}_{k-1}^{(j+1)}(z) - \left(a^{(j+k+1)}(z)\right)^2 \widetilde{w}_{k-2}^{(j+1)}(z)\dfrac{\widetilde{w}_{m-1}^{(j+k+1)}(z)}{\widetilde{w}_m^{(j+k)}(z)}}{\widetilde{w}_k^{(j)}(z) - \left(a^{(j+k+1)}(z)\right)^2 \widetilde{w}_{k-1}^{(j)}(z)\dfrac{\widetilde{w}_{m-1}^{(j+k+1)}(z)}{\widetilde{w}_m^{(j+k)}(z)}}\,. \quad m \in \mathbb{N}\,, \quad k = 0,1,\ldots \tag{76}$$

Taking limits in (76) as $m \to \infty$ (see (32)),

$$\begin{aligned}
-\frac{\widetilde{f}^{(j)}(z)}{\left(a^{(j+1)}(z)\right)^2} &= \frac{\widetilde{w}_{k-1}^{(j+1)}(z) - \left(a^{(j+k+1)}(z)\right)^2 \widetilde{w}_{k-2}^{(j+1)}(z)\dfrac{\widetilde{f}^{(j+k)}(z)}{-\left(a^{(j+1)}(z)\right)^2}}{\widetilde{w}_k^{(j)}(z) - \left(a^{(j+k+1)}(z)\right)^2 \widetilde{w}_{k-1}^{(j)}(z)\dfrac{\widetilde{f}^{(j+k)}(z)}{-\left(a^{(j+1)}(z)\right)^2}} \\
&= \frac{\widetilde{w}_{k-1}^{(j+1)}(z) + \widetilde{w}_{k-2}^{(j+1)}(z)\widetilde{f}^{(j+k)}(z)}{\widetilde{w}_k^{(j)}(z) + \widetilde{w}_{k-1}^{(j)}(z)\widetilde{f}^{(j+k)}(z)}
\end{aligned}$$

or, what is the same,

$$\widetilde{f}^{(j+k)}(z)\left[-\frac{\widetilde{f}^{(j)}(z)}{\left(a^{(j+1)}(z)\right)^2} - \widetilde{w}_{k-2}^{(j+1)}(z)\right] = \frac{\widetilde{f}^{(j)}(z)}{\left(a^{(j+1)}(z)\right)^2}\widetilde{w}_k^{(j)}(z) + \widetilde{w}_{k-1}^{(j+1)}(z)\,.$$

That is, from (70),

$$\widetilde{f}^{(j+k)}(z) = -a^{(j+k+1)}(z)\frac{\widetilde{Q}_k^{(j)}(z)}{\widetilde{Q}_{k-1}^{(j)}(z)}$$

and $-\dfrac{\widetilde{f}^{(j+k)}(z)}{a^{(j+k+1)}(z)}\widetilde{Q}_{k-1}^{(j)}(z) = \widetilde{Q}_k^{(j)}(z)$, as we needed to prove. □

Remark 2 *¿From Lemma 8 we have*

$$\widetilde{Q}_{mN+i+1}^{(j)}(z) = a^{(j+i+2)}(z)\widetilde{Q}_0^{(j+i+1)}(z)\widetilde{Q}_{mN+i}^{(j)}(z)\,, \quad m \in \mathbb{N}\,, \quad i = 0,\ldots,N-1\,.$$

From this and (71), if $\widetilde{Q}^{(j)}_{m_0N+i_0}(z)=0$ *for some* $m_0\in\mathbb{N}$ *and* $i_0\in\{0,\dots,N-1\}$ *we arrive to* $\widetilde{Q}^{(j)}_n(z)=0$ *for all* $n\in\mathbb{N}$*. In this case,*

$$\widetilde{w}^{(j+1)}_{n-1}(z)=-\frac{\widetilde{f}^{(j)}(z)}{\left(a^{(j+1)(z)}\right)^2}\widetilde{w}^{(j)}_n(z)\,,\quad n\in\mathbb{N} \tag{77}$$

(see (70)). But, as we saw in the proof of Corollary 2, (77) is not possible for $z\in\Omega\setminus\left(\Gamma\cup E\cup\widetilde{E}^{(j)}\right)$*. Consequently, we have* $\widetilde{Q}^{(j)}_n(z)\neq 0$ *for all* $n\in\mathbb{N}$*.*

Remark 3 *Remark 2 and Lemma 7 indicate that not only do we have strong asymptotic for the functions* $\widetilde{Q}^{(j)}_{mN+i}(z)$ *but also that the relation between* $\widetilde{Q}^{(j)}_{mN+i}(z)$ *and* $\widehat{\lambda}_2(z)^m$ *is exact and*

$$\frac{\widetilde{Q}^{(j)}_{n+N}(z)}{\widetilde{Q}^{(j)}_n(z)}=\widehat{\lambda}_2(z)\,,\quad n\geq 1\,,\quad z\in\Omega\setminus\left(\Gamma\cup E\cup\widetilde{E}^{(j)}\right).$$

We define the functions

$$\widetilde{p}^{(j)}_n(z)=\frac{\widetilde{w}^{(j)}_n(z)}{a^{(j+2)}(z)\dots a^{(j+n+1)}(z)}\,,\quad p^{(j)}_n(z)=\frac{w^{(j)}_n(z)}{a^{(j+2)}(z)\dots a^{(j+n+1)}(z)}\,,\quad n\in\mathbb{N}\,,$$

taking

$$\widetilde{p}^{(j)}_0(z)=p^{(j)}_0(z)=1\,,\quad \widetilde{p}^{(j)}_{-1}(z)=p^{(j)}_{-1}(z)=0. \tag{78}$$

Since the functions $\{\widetilde{w}^{(j)}_n(z)\}$ and $\{w^{(j)}_n(z)\}$ verify the recurrences (4) and (5) (with periodic coefficients in the first case), we obtain the following relations

$$a^{(j+n+1)}(z)\widetilde{p}^{(j+n-1)}_m(z)=b^{(j+n)}(z)\widetilde{p}^{(j+n)}_{m-1}(z)-\frac{\left(a^{(j+n+1)}(z)\right)^2}{a^{(j+n+2)}(z)}\widetilde{p}^{(j+n+1)}_{m-2}(z)\,, \tag{79}$$

$$a^{(j+n+1)}(z)p^{(j)}_n(z)=b_{j+n}(z)p^{(j)}_{n-1}(z)-\frac{\left(a_{j+n}(z)\right)^2}{a^{(j+n)}(z)}p^{(j)}_{n-2}(z)\,, \tag{80}$$

$$(m\,,\,n\geq 2)\,.$$

In the following, we study the connection between these two families of functions.

Lemma 9 *Let* $z\in\Omega$*. For each* $n\in\mathbb{N}$ *we have*

$$p^{(j)}_n(z)=\widetilde{p}^{(j)}_n(z)+ \tag{81}$$

$$+\sum_{s=1}^{n}\left[\frac{b_{j+s}(z)-b^{(j+s)}(z)}{a^{(j+s+1)}(z)}\widetilde{p}^{(j+s)}_{n-s}(z)+\frac{a^{(j+s+1)}(z)^2-a_{j+s+1}(z)^2}{a^{(j+s+1)}(z)a^{(j+s+2)}(z)}\widetilde{p}^{(j+s+1)}_{n-s-1}(z)\right]p^{(j)}_{s-1}(z)\,.$$

Proof.- Let $z\in\Omega$ be fixed. We follow the scheme of the proof of (5.13) in [13, page 108].

We delete z from the notation. Multiplying (80) by $\widetilde{p}^{(j+n)}_{m-1}$, (79) by $p^{(j)}_{n-1}$ and deleting one expression from the other, we obtain

$$a^{(j+n+1)}\left(\widetilde{p}^{(j+n)}_{m-1}p^{(j)}_n-\widetilde{p}^{(j+n-1)}_m p^{(j)}_{n-1}\right)=$$

$$\left(b_{j+n}-b^{(j+n)}\right)\widetilde{p}_{m-1}^{(j+n)}p_{n-1}^{(j)}-\frac{a_{j+n}^2}{a^{(j+n)}}\widetilde{p}_{m-1}^{(j+n)}p_{n-2}^{(j)}+\frac{\left(a^{(j+n+1)}\right)^2}{a^{(j+n+2)}}\widetilde{p}_{m-2}^{(j+n+1)}p_{n-1}^{(j)}\,.$$

Substituting n by k and m by $n-k+1$, it follows that

$$a^{(j+k+1)}\left(\widetilde{p}_{n-k}^{(j+k)}p_k^{(j)}-\widetilde{p}_{n-k+1}^{(j+k-1)}p_{k-1}^{(j)}\right)=$$

$$\left(b_{j+k}-b^{(j+k)}\right)\widetilde{p}_{n-k}^{(j+k)}p_{k-1}^{(j)}-\frac{\left(a_{j+k}\right)^2}{a^{(j+k)}}\widetilde{p}_{n-k}^{(j+k)}p_{k-2}^{(j)}+\frac{\left(a^{(j+k+1)}\right)^2}{a^{(j+k+2)}}\widetilde{p}_{n-k-1}^{(j+k+1)}p_{k-1}^{(j)}\,.$$

Taking $k=1,2,\ldots,n$ in the previous equality and adding the results, we obtain the desired expression. □

Relation (81) suggests the proximity between the functions $p_n^{(j)}(z)$ and $\widetilde{p}_n^{(j)}(z)$ when the coefficients of the recurrence formulas (1) corresponding to the purely periodic and asymptotically periodic cases are close enough. This will be measured in terms of the convergence of the series

$$\sum_{s=1}^{\infty}\left(\left|b_s(z)-b^{(s)}(z)\right|+\left|a^{(s+1)}(z)^2-a_{s+1}(z)^2\right|\right)\,. \tag{82}$$

The following theorem guarantees, under this restriction, a similar asymptotic behavior for the functions $w_n^{(j)}(z)$ and $\widetilde{w}_n^{(j)}(z)$.

Theorem 3 *Assume that the series* (82) *converges uniformly on compact subsets of* Ω. *For each* $i=0,1,\ldots,N-1$, *we have*

$$\lim_{m\to\infty}\frac{w_{mN+i}^{(j)}(z)}{(\lambda_1(z))^m}=\phi(z)\frac{\widetilde{w}_{N+i}^{(j)}(z)-\lambda_2(z)\widetilde{w}_i^{(j)}(z)}{\lambda_1(z)-\lambda_2(z)} \tag{83}$$

uniformly on compact subsets of $\Omega\setminus\left(\Gamma\cup E\cup E^{(j)}\cup\widetilde{E}^{(j)}\right)$ *and* ϕ *is analytic in this region.*

Proof.- As above, we will not write explicitly the variable z. To begin with, let us prove the uniform boundedness on compact subsets of $\Omega\setminus\left(\Gamma\cup E\cup\widetilde{E}^{(j)}\right)$ of the sequence $\{p_n^{(j)}\widetilde{Q}_n^{(j)}\}$. Multiplying both sides of (81) by $\widetilde{Q}_n^{(j)}$ and taking account of (73),

$$\begin{aligned}p_n^{(j)}\widetilde{Q}_n^{(j)} &= \widetilde{p}_n^{(j)}\widetilde{Q}_n^{(j)}+\sum_{s=1}^{n}\Big[(b_{j+s}-b^{(j+s)})\widetilde{p}_{n-s}^{(j+s)}\widetilde{Q}_{n-s}^{(j+s)}+\\ &+ \frac{\left(a^{(j+s+1)}\right)^2-\left(a_{j+s+1}\right)^2}{a^{(j+s+2)}}\widetilde{p}_{n-s-1}^{(j+s+1)}\widetilde{Q}_{n-s}^{(j+s)}\Big]p_{s-1}^{(j)}\widetilde{Q}_{s-1}^{(j)}\,.\end{aligned}$$

Taking absolute values,

$$\begin{aligned}\left|p_n^{(j)}\widetilde{Q}_n^{(j)}\right| &\leq \left|\widetilde{p}_n^{(j)}\widetilde{Q}_n^{(j)}\right|+\sum_{s=1}^{n}\Big[\left|b_{j+s}-b^{(j+s)}\right|\left|\widetilde{p}_{n-s}^{(j+s)}\widetilde{Q}_{n-s}^{(j+s)}\right|\\ &+ \left|\frac{\left(a^{(j+s+1)}\right)^2-\left(a_{j+s+1}\right)^2}{a^{(j+s+2)}}\right|\left|\frac{\widetilde{p}_{n-s-1}^{(j+s+1)}}{\widetilde{p}_{n-s}^{(j+s)}}\right|\left|\widetilde{p}_{n-s}^{(j+s)}\widetilde{Q}_{n-s}^{(j+s)}\right|\Big]\left|p_{s-1}^{(j)}\widetilde{Q}_{s-1}^{(j)}\right|\,.\end{aligned} \tag{84}$$

For each $m \in \mathbb{N}$, $i \in \{0,1,\ldots,N-1\}$, we have $\left|\widetilde{p}^{(j)}_{mN+i}\widetilde{Q}^{(j)}_{mN+i}\right| = \left|\dfrac{\widetilde{p}^{(j)}_{mN+i}}{\widehat{\lambda}_1^m}\right| \left|\widetilde{Q}^{(j)}_{mN+i}\widehat{\lambda}_1^m\right|$. Therefore, taking into consideration (28) and (71), for each compact subset $\mathcal{K} \subset \Omega \setminus (\Gamma \cup E \cup \widetilde{E}^{(j)})$ there exists a constant $C_1 = C_1(\mathcal{K})$ (depending on $\mathcal{K}$) such that

$$\left|\widetilde{p}^{(j)}_n \widetilde{Q}^{(j)}_n\right| \le C_1\,, \quad n = 0,1,\ldots\,. \tag{85}$$

Moreover, by (32), there exists $C_2 = C_2(\mathcal{K})$ such that

$$\left|\frac{\widetilde{p}^{(j+s+1)}_{n-s-1}}{\widetilde{p}^{(j+s)}_{n-s}}\right| \le C_2\,, \quad \forall s = 1,2,\ldots,n\,, \quad n \in \mathbb{N}\,. \tag{86}$$

Taking account of (85) and (86) in (84), we obtain that there exists $C_3 = C_3(\mathcal{K})$ such that

$$\left|p^{(j)}_n \widetilde{Q}^{(j)}_n\right| \le C_3 \left(1 + \sum_{s=1}^{n} \left[\left|b_{j+s} - b^{(j+s)}\right| + \left|\left(a^{(j+s+1)}\right)^2 - (a_{j+s+1})^2\right|\right] \left|p^{(j)}_{s-1}\widetilde{Q}^{(j)}_{s-1}\right|\right)$$

uniformly on compact subsets of $\Omega \setminus \left(\Gamma \cup E \cup \widetilde{E}^{(j)}\right)$. Using the Gronwall inequality, it follows that

$$\left|p^{(j)}_n \widetilde{Q}^{(j)}_n\right| \le C_3 \exp\left\{C_3 \sum_{s=1}^{n} \left(\left|b_{j+s} - b^{(j+s)}\right| + \left|\left(a^{(j+s+1)}\right)^2 - (a_{j+s+1})^2\right|\right)\right\}\,,$$

uniformly on compact subsets of $\Omega \setminus \left(\Gamma \cup E \cup \widetilde{E}^{(j)}\right)$. On account of the convergence of (82), it follows that there exists a constant $C = C(\mathcal{K})$ such that

$$\left|p^{(j)}_n \widetilde{Q}^{(j)}_n\right| \le C\,, \quad n = 0,1,\ldots\,, \tag{87}$$

uniformly on compact subsets $\mathcal{K}$ of $\Omega \setminus \left(\Gamma \cup E \cup \widetilde{E}^{(j)}\right)$.

On the other hand, using (87) and again the convergence of (82), we can define the function

$$\phi = 1 + \sum_{k=0}^{\infty} \left\{(b_{j+k} - b^{(j+k)})\widetilde{Q}^{(j)}_{k-1} + \frac{\left(a^{(j+k+1)}\right)^2 - (a_{j+k+1})^2}{a^{(j+k+1)}}\widetilde{Q}^{(j)}_k\right\} p^{(j)}_{k-1}\,,$$

where the series converges absolutely and uniformly on compact subsets of the prescribed region.

In the following, we shall prove that

$$\lim_{n\to\infty} a^{(j+n+2)} \left(p^{(j)}_{n+1}\widetilde{Q}^{(j)}_n - \frac{a^2_{j+n+2}}{\left(a^{(j+n+2)}\right)^2} p^{(j)}_n \widetilde{Q}^{(j)}_{n+1}\right) = \phi\,. \tag{88}$$

Multiplying (81) by $\widetilde{Q}^{(j)}_{n+1}$, substituting n by $n+1$ in (81), multiplying this last expression by $\widetilde{Q}^{(j)}_n$ and deleting one expression from the other, we obtain

$$\begin{aligned} p^{(j)}_n \widetilde{Q}^{(j)}_{n+1} - p^{(j)}_{n+1}\widetilde{Q}^{(j)}_n &= \\ \widetilde{p}^{(j)}_n \widetilde{Q}^{(j)}_{n+1} - \widetilde{p}^{(j)}_{n+1}\widetilde{Q}^{(j)}_n &+ \sum_{s=1}^{n} \left[\frac{b_{j+s} - b^{(j+s)}}{a^{(j+s+1)}} \left(\widetilde{p}^{(j+s)}_{n-s}\widetilde{Q}^{(j)}_{n+1} - \widetilde{p}^{(j+s)}_{n-s+1}\widetilde{Q}^{(j)}_n\right) + \right. \end{aligned} \tag{89}$$

$$+\frac{\left(a^{(j+s+1)}\right)^2-a_{j+s+1}^2}{a^{(j+s+1)}a^{(j+s+2)}}\left(\widetilde{p}_{n-s-1}^{(j+s+1)}\widetilde{Q}_{n+1}^{(j)}-\widetilde{p}_{n-s}^{(j+s+1)}\widetilde{Q}_{n}^{(j)}\right)\Bigg]\,p_{s-1}^{(j)}-\frac{b_{j+n+1}-b^{(j+n+1)}}{a^{(j+n+2)}}\widetilde{Q}_n^{(j)}p_n^{(j)}\,.$$

We know that $\{\widetilde{Q}_n^{(j)}\}$ and $\{\widetilde{p}_{n-s}^{(j+s)}\}$ are solutions of the difference equation given in (68). That is,

$$a^{(j+n+1)}\widetilde{Q}_n^{(j)}=b^{(j+n)}\widetilde{Q}_{n-1}^{(j)}-a^{(j+n)}\widetilde{Q}_{n-2}^{(j)} \tag{90}$$

and

$$a^{(j+n+1)}\widetilde{p}_{n-s}^{(j+s)}=b^{(j+n)}\widetilde{p}_{n-s-1}^{(j+s)}-a^{(j+n)}\widetilde{p}_{n-s-2}^{(j+s)}\,. \tag{91}$$

Thus, multiplying (90) by $\widetilde{p}_{n-s-1}^{(j+s)}$ and (91) by $\widetilde{Q}_{n-1}^{(j)}$, and subtracting both expressions, we obtain

$$a^{(j+n+1)}\left[\widetilde{p}_{n-s-1}^{(j+s)}\widetilde{Q}_n^{(j)}-\widetilde{p}_{n-s}^{(j+s)}\widetilde{Q}_{n-1}^{(j)}\right]=a^{(j+n)}\left[\widetilde{p}_{n-s-2}^{(j+s)}\widetilde{Q}_{n-1}^{(j)}-\widetilde{p}_{n-s-1}^{(j+s)}\widetilde{Q}_{n-2}^{(j)}\right]\,.$$

Iterating and taking into account the initial conditions for $\{\widetilde{p}_{n-s}^{(j+s)}\}$ (see (78)) we arrive to

$$\widetilde{p}_{n-s-1}^{(j+s)}\widetilde{Q}_n^{(j)}-\widetilde{p}_{n-s}^{(j+s)}\widetilde{Q}_{n-1}^{(j)}=-\frac{a^{(j+s+1)}}{a^{(j+n+1)}}\widetilde{Q}_{s-1}^{(j)}\,,\quad s=0,1,\ldots\,. \tag{92}$$

Considering (89) and (92), it follows that

$$p_n^{(j)}\widetilde{Q}_{n+1}^{(j)}-p_{n+1}^{(j)}\widetilde{Q}_n^{(j)}=-\frac{a^{(j+1)}}{a^{(j+n+2)}}\widetilde{Q}_{-1}^{(j)}-$$

$$-\sum_{s=1}^{n}\left[\frac{b_{j+s}-b^{(j+s)}}{a^{(j+n+2)}}\widetilde{Q}_{s-1}^{(j)}+\frac{\left(a^{(j+s+1)}\right)^2-a_{j+s+1}^2}{a^{(j+s+1)}a^{(j+n+2)}}\widetilde{Q}_s^{(j)}\right]p_{s-1}^{(j)}-\frac{b_{j+n+1}-b^{(j+n+1)}}{a^{(j+n+2)}}\widetilde{Q}_n^{(j)}p_n^{(j)}\,.$$

Multiplying by $-a^{(j+n+2)}$, it follows that

$$a^{(j+n+2)}\left(p_{n+1}^{(j)}\widetilde{Q}_n^{(j)}-p_n^{(j)}\widetilde{Q}_{n+1}^{(j)}\right)=$$

$$1+\sum_{s=1}^{n+1}\left[\left(b_{j+s}-b^{(j+s)}\right)\widetilde{Q}_{s-1}^{(j)}+\frac{\left(a^{(j+s+1)}\right)^2-a_{j+s+1}^2}{a^{(j+s+1)}}\widetilde{Q}_s^{(j)}\right]p_{s-1}^{(j)}-$$

$$\frac{\left(a^{(j+n+2)}\right)^2-a_{j+n+2}^2}{a^{(j+n+2)}}\widetilde{Q}_{n+1}^{(j)}p_n^{(j)}\,.$$

That is,

$$a^{(j+n+2)}\left(p_{n+1}^{(j)}\widetilde{Q}_n^{(j)}-\frac{a_{j+n+2}^2}{\left(a^{(j+n+2)}\right)^2}p_n^{(j)}\widetilde{Q}_{n+1}^{(j)}\right)=1\quad+ \tag{93}$$

$$\sum_{s=1}^{n+1}\left[\left(b_{j+s}-b^{(j+s)}\right)\widetilde{Q}_{s-1}^{(j)}\quad+\quad\frac{\left(a^{(j+s+1)}\right)^2-a_{j+s+1}^2}{a^{(j+s+1)}}\widetilde{Q}_s^{(j)}\right]p_{s-1}^{(j)}\,.$$

Taking (82) into consideration and taking limit in (93), we arrive at (88).

Finally, let us show that from this we deduce the result we are looking for. Substituting n by $n+1$ in (92) and taking $s=0$ we have

$$a^{(j+n+2)} = \frac{1}{\widetilde{p}_{n+1}^{(j)}\widetilde{Q}_n^{(j)} - \widetilde{p}_n^{(j)}\widetilde{Q}_{n+1}^{(j)}} \,. \tag{94}$$

Substituting this expression of $a^{(j+n+2)}$ into (88), we obtain

$$\lim_{n\to\infty} \frac{p_{n+1}^{(j)}\widetilde{Q}_n^{(j)} - \left(\frac{a_{j+n+2}}{a^{(j+n+2)}}\right)^2 p_n^{(j)}\widetilde{Q}_{n+1}^{(j)}}{\widetilde{p}_{n+1}^{(j)}\widetilde{Q}_n^{(j)} - \widetilde{p}_n^{(j)}\widetilde{Q}_{n+1}^{(j)}} =$$

$$\lim_{n\to\infty} \frac{p_n^{(j)}}{\widetilde{p}_n^{(j)}} \frac{\frac{p_{n+1}^{(j)}}{p_n^{(j)}}\widetilde{Q}_n^{(j)} - \left(\frac{a_{j+n+2}}{a^{(j+n+2)}}\right)^2 \widetilde{Q}_{n+1}^{(j)}}{\frac{\widetilde{p}_{n+1}^{(j)}}{\widetilde{p}_n^{(j)}}\widetilde{Q}_n^{(j)} - \widetilde{Q}_{n+1}^{(j)}} = \phi \,. \tag{95}$$

Dividing (6) by $w_{mN+i+1}^{(j)}$ (for $s=N$, $n=mN+i, i\in\{0,1,\ldots,N-1\}$) and taking limit as $m\to\infty$ (which is possible by Theorem 2), it follows that

$$\lim_{m\to\infty} \frac{w_{(m+1)N+i+1}^{(j)}}{w_{mN+i+1}^{(j)}} = \widetilde{w}_N^{(j+i+1)} - \frac{\left(a^{(j+i+2)}\right)^2 \widetilde{w}_{N-1}^{(j+i+2)}}{\lim_{m\to\infty} \frac{w_{mN+i+1}^{(j)}}{w_{mN+i}^{(j)}}} \,.$$

Therefore,

$$\lim_{m\to\infty} \frac{w_{mN+i+1}^{(j)}}{w_{mN+i}^{(j)}} = \frac{\left(a^{(j+i+2)}\right)^2 \widetilde{w}_{N-1}^{(j+i+2)}}{\widetilde{w}_N^{(j+i+1)} - \lambda_1} \tag{96}$$

with uniform convergence on compact subsets of $\Omega \setminus \left(\Gamma \cup E \cup E^{(j)}\right)$. From (96)

$$\lim_{m\to\infty} \frac{p_{mN+i+1}^{(j)}}{p_{mN+i}^{(j)}} = \lim_{m\to\infty} \frac{\widetilde{p}_{mN+i+1}^{(j)}}{\widetilde{p}_{mN+i}^{(j)}} = \frac{a^{(j+i+2)}\widetilde{w}_{N-1}^{(j+i+2)}}{\widetilde{w}_N^{(j+i+1)} - \lambda_1} \tag{97}$$

(also uniformly).

On the other hand, from (94), for $n=mN+i$ we can write

$$a^{(j+i+2)} \frac{\widetilde{p}_{mN+i}^{(j)}}{\widehat{\lambda}_1^m} = \frac{1}{\frac{\widetilde{p}_{mN+i+1}^{(j)}}{\widetilde{p}_{mN+i}^{(j)}} \frac{\widetilde{Q}_{mN+i}^{(j)}}{\widehat{\lambda}_2^m} - \frac{\widetilde{Q}_{mN+i+1}^{(j)}}{\widehat{\lambda}_2^m}} \,. \tag{98}$$

Moreover, because of the definition of $\widetilde{p}_n^{(j)}$,

$$\frac{\widetilde{p}_{mN+i}^{(j)}}{\widehat{\lambda}_1^m} = \begin{cases} \dfrac{\widetilde{w}_{mN+i}^{(j)}}{\lambda_1^m} \dfrac{1}{a^{(j+2)}\cdots a^{(j+i+1)}} & , \quad \text{if } i\in\{1,\ldots,N-1\}\,, \\[2ex] \dfrac{\widetilde{w}_{mN}^{(j)}}{\lambda_1^m} & , \quad \text{if } i=0\,. \end{cases}$$

Therefore, the limit of the denominator on the right hand of (98) is not zero (see Theorem 1)). From this fact and (97), dividing the numerator and denominator of the second fraction in (95) by $\widehat{\lambda}_2^m$ and taking limit as $m \to \infty$ in that fraction, we obtain

$$\lim_{m\to\infty} \frac{\dfrac{p_{mN+i+1}^{(j)}}{p_{mN+i}^{(j)}} \dfrac{\widetilde{Q}_{mN+i}^{(j)}}{\widehat{\lambda}_2^m} - \left(\dfrac{a_{j+mN+i+2}}{a^{(j+i+2)}}\right)^2 \dfrac{\widetilde{Q}_{mN+i+1}^{(j)}}{\widehat{\lambda}_2^m}}{\dfrac{\widetilde{p}_{mN+i+1}^{(j)}}{\widetilde{p}_{mN+i}^{(j)}} \dfrac{\widetilde{Q}_{mN+i}^{(j)}}{\widehat{\lambda}_2^m} - \dfrac{\widetilde{Q}_{mN+i+1}^{(j)}}{\widehat{\lambda}_2^m}} = 1\,.$$

That is, the limit of that fraction exists, independent of the value of i. Using this fact in (95) and taking limit there as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{p_n^{(j)}}{\widetilde{p}_n^{(j)}} = \lim_{n\to\infty} \frac{w_n^{(j)}}{\widetilde{w}_n^{(j)}} = \phi$$

uniformly on compact subsets of $\Omega \setminus \left(\Gamma \cup E \cup E^{(j)} \cup \widetilde{E}^{(j)}\right)$, from which, using Theorem 1, we have (83). □

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