GENERALIZED HERMITE-PADÉ APPROXIMATION FOR NIKISHIN SYSTEMS OF THREE FUNCTIONS

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Abstract. Nikishin systems of three functions are considered. For such systems, the rate of convergence of simultaneous interpolating rational approximations with partially prescribed poles is studied. The solution is described in terms of the solution of a vector equilibrium problem in the presence of a vector external field.

1. Introduction.

Given a bounded interval Δ of the real line $\mathbb{R}$, by $\mathcal{M}(\Delta)$ we denote the set of all finite Borel measures σ with constant sign whose support $\mathrm{supp}(\sigma)$ is contained in Δ, and has infinitely many mass points. The associated Markov function is

$$\widehat{\sigma}(z) = \int \frac{d\sigma(x)}{z-x}. \tag{1}$$

Consider three bounded intervals Δ_1, Δ_2 and Δ_3 such that $\Delta_2 \cap (\Delta_1 \cup \Delta_3) = \emptyset$. Let $(\sigma_1, \sigma_2, \sigma_3)$ be a system of three measures, where $\sigma_j \in \mathcal{M}(\Delta_j), j = 1, 2, 3$. The associated Nikishin system of measures $S = (s_1, s_2, s_3) = \mathcal{N}(\sigma_1, \sigma_2, \sigma_3)$ is defined by

$$ds_1(x) = d\sigma_1(x), \quad ds_2(x) = \widehat{\sigma}_2(x) d\sigma_1(x), \quad ds_3(x) = \int \widehat{\sigma}_3(t) \frac{d\sigma_2(t)}{x-t} d\sigma_1(x). \tag{2}$$

For the general definition of a Nikishin system see [8]. All the measures in S have the same support Δ_1. By $\widehat{S} = (\widehat{s}_1, \widehat{s}_2, \widehat{s}_3)$, we denote the vector whose components are the Markov functions corresponding to each one of the measures s_i, $i = 1, 2, 3$. The functions $\widehat{s}_i$, $i = 1, 2, 3$, are holomorphic on $D = \overline{\mathbb{C}} \setminus \Delta_1$.

Let $\kappa_{\mathbf{n}}$ be an even integer, $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}_+^3$, and $|\mathbf{n}| = n_1 + n_2 + n_3$. Let $\alpha_{\mathbf{n}}$ and $\beta_{\mathbf{n}}$ be two monic polynomials with real coefficients such that $\deg \beta_{\mathbf{n}} = \kappa_{\mathbf{n}}$ and $\deg \alpha_{\mathbf{n}} \leq |\mathbf{n}| + \kappa_{\mathbf{n}} + \min\{n_i\}$. The zeros of $\alpha_{\mathbf{n}}$ belong to D and the zeros of $\beta_{\mathbf{n}}$ have even multiplicity and lie on Δ_1. We call generalized Hermite Padé approximant (GHPA) of the system $\widehat{S}$ relative to $(\mathbf{n}, \alpha_{\mathbf{n}}, \beta_{\mathbf{n}})$ the vector rational functions

$$R_{\mathbf{n}} = (R_{\mathbf{n},1}, R_{\mathbf{n},2}, R_{\mathbf{n},3}) = (P_{\mathbf{n},1}/\beta_{\mathbf{n}} Q_{\mathbf{n}}, P_{\mathbf{n},2}/\beta_{\mathbf{n}} Q_{\mathbf{n}}, P_{\mathbf{n},3}/\beta_{\mathbf{n}} Q_{\mathbf{n}}),$$

where $\deg Q_{\mathbf{n}} \leq |\mathbf{n}|, Q_{\mathbf{n}} \not\equiv 0$, $\deg P_{\mathbf{n},j} \leq |\mathbf{n}| + \kappa_{\mathbf{n}} - 1, j = 1, 2, 3,$, and

$$\left[\frac{\beta_{\mathbf{n}} Q_{\mathbf{n}} \widehat{s}_j - P_{\mathbf{n},j}}{\alpha_{\mathbf{n}}}\right](z) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right) \in \mathcal{H}(D). \tag{3}$$

Finding $R_{\mathbf{n}}$ reduces to solving a system of $4|n|+3\kappa_{\mathbf{n}}$ homogeneous linear equations on $4|n|+3\kappa_{\mathbf{n}}+1$ unknowns corresponding to the coefficients of $Q_{\mathbf{n}}$ and $P_{\mathbf{n},j}$, $j = 1, 2, 3$. We know (see [1]) that for any Nikishin system of three functions $\widehat{S} = (\widehat{s}_1, \widehat{s}_2, \widehat{s}_3)$, each multi-index $\mathbf{n} \in \mathbb{Z}^3$ determines $Q_{\mathbf{n}}$ uniquely except for a constant factor. Moreover, $Q_{\mathbf{n}}$ has exactly $|\mathbf{n}|$ simple zeros all lying in the interior of the interval Δ_1. (In reference to the interior of intervals of the real line we consider the usual Euclidean topology of $\mathbb{R}$.) In the sequel we take $Q_{\mathbf{n}}$ monic. When $\beta_{\mathbf{n}} \equiv 1$, GHPA reduce to the multipoint Hermite-Padé approximation (MHPA). If, additionally, $\alpha_{\mathbf{n}} \equiv 1$, we obtain classical Hermite-Padé approximants (CHPA).

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In [5] the authors study the exact rate of convergence of CHPA for a large class of Nikishin systems corresponding to sequences multi-indices $\mathbf{n} = (n_1, \ldots, n_m)$ such that $n_1 \geq \cdots \geq n_m$. Inspired in that paper, in [2] we introduced GHPA for these systems and extended the results of [5] considering also more general sequences of multi-indices. For the case $m = 3$, multi-indices such that $n_1 < n_2 < n_m$ were excluded. The results we present here are precisely for such sequences of multi-indices so this paper complements [2]. In number theory applications of Hermite-Padé approximation, Markov systems of 2 and 3 functions have great importance, thus the interest of the present results.

In the rest of the paper, $\mathbf{n} \in \mathbb{Z}_+^3$ and $n_1 < n_2 < n_3, \Lambda \subset \mathbb{Z}_+^3$ is a sequence of distinct multi-indices such that

$$\lim_{\mathbf{n}\in\Lambda} n_i/|\mathbf{n}| = p_i > 0, \qquad i = 1,2,3, \tag{4}$$

($p_1 \leq p_2 \leq p_3$.) Finally, $\{\alpha_{\mathbf{n}}\}, \{\beta_{\mathbf{n}}\}, \mathbf{n} \in \Lambda$, will denote two sequences of polynomials with real coefficients such that $\deg \beta_{\mathbf{n}} = \kappa_{\mathbf{n}}, \deg \alpha_{\mathbf{n}} \leq |\mathbf{n}| + \kappa_{\mathbf{n}} + n_1$. The zeros of $\beta_{\mathbf{n}}$ have even multiplicity and lie on Δ_1 and the zeros of $\alpha_{\mathbf{n}}$ belong to a compact subset E of $D = \overline{\mathbb{C}} \setminus \Delta_1$. We assume that there exist measures α, β with support contained in $E \subset D$ and Δ_1 respectively such that

$$\lim_{\mathbf{n}\in\Lambda} \chi(\alpha_{\mathbf{n}})/|\mathbf{n}| = \alpha\,, \qquad \lim_{\mathbf{n}\in\Lambda} \chi(\beta_{\mathbf{n}})/|\mathbf{n}| = \beta\,, \tag{5}$$

where $\chi(q) = \sum_{q(\zeta)=0} \delta_\zeta$ denotes the zero counting measure associated with the polynomial q which assigns mass 1 to each zero of q (counting multiplicities) and measure zero to all Borel sets not containing zeros of q. The convergence of measures is in the weak star sense.

The main result of the paper states

THEOREM 1. *Assume that* $|\sigma_1'| > 0$ *almost everywhere on* $\Delta_1, \sigma_j \in \mathbf{Reg}, j = 2,3,$ *and* (4) – (5) *take place. Then, for each* $j = 1,2,3$,

$$\lim_{n\in\Lambda'} |(\widehat{s}_j - R_{\mathbf{n},j})(z)|^{1/|\mathbf{n}|} = F_j(z)\,, \qquad \mathcal{K} \in \mathcal{G}_j\,,$$

(uniformly on each compact subsets $\mathcal{K}$ *of the region* $\mathcal{G}_j$*).*

Expressions for F_j and $\mathcal{G}_j$ as well as complementary results are given in the last section. For example, conditions are given so that $\mathcal{G}_j = \overline{\mathbb{C}} \setminus (\Delta_1 \cup E)$. Section 2 is dedicated to the presentation of concepts and auxiliary results needed for the proofs contained in Section 3.

For upper bounds on the rate of convergence of MHPA for general sequences of multi-indices and generating measures see [3].

2. Auxiliary results.

It easy to verify that the relations (3) (see (4) in [2]) are equivalent to

$$\int x^\nu Q_{\mathbf{n}}(x) \frac{\beta_{\mathbf{n}}(x) ds_j(x)}{\alpha_{\mathbf{n}}(x)} = 0, \qquad \nu = 0, \ldots, n_j - 1, \qquad j = 1,2,3. \tag{6}$$

In (50) of [2] it is shown that

$$\widehat{s}_j(z) - R_{\mathbf{n},j}(z) = \frac{\alpha_{\mathbf{n}}(z)}{(\beta_{\mathbf{n}} Q_{\mathbf{n}})(z)} \int \frac{(\beta_{\mathbf{n}} Q_{\mathbf{n}})(x)}{\alpha_{\mathbf{n}}(x)} \frac{ds_j(x)}{z - x} = \frac{\alpha_{\mathbf{n}}(z)}{(\beta_{\mathbf{n}} Q_{\mathbf{n}})(z)} \Phi_{\mathbf{n},j}(z), \quad j = 1,2,3. \tag{7}$$

From here an integral expression for the polynomials $P_{\mathbf{n},j}$ readily follows. Formula (7) is the main tool for finding the limit behavior of the remainder of the approximation.

Many of the ingredients we use in the proofs, are contained in previous papers. We will state them in the form of lemmas pointing out references where their proofs may be found.

For each pair $(j,k) \in \{(2,3),(3,2)\}$ we denote

$$ds_{j,k}(x) = \widehat{\sigma}_k(x) d\sigma_j(x),$$

It is well known (see the Appendix in [6]) that there exists a first degree polynomial $\ell_{j,k}$ and a measure with constant sign $\tau_{j,k}$ such that

$$\frac{1}{\widehat{s}_{j,k}(z)} = \ell_{j,k}(z) + \widehat{\tau}_{j,k}(z). \tag{8}$$

Consider the following functions which we call of second type

$$
\begin{aligned}
\Psi_{\mathbf{n},0}(z) &= \tfrac{Q_{\mathbf{n}}(z)\beta_{\mathbf{n}}(z)}{\alpha_{\mathbf{n}}(z)}, & \Psi_{\mathbf{n},1}(z) &= \int \tfrac{\Psi_{\mathbf{n},0}(x)}{z-x} ds_3(x), \\
\Psi_{\mathbf{n},2}(z) &= \int \tfrac{\Psi_{\mathbf{n},1}(x)}{z-x}\tfrac{\widehat{s}_{3,2}(x)}{\widehat{\sigma}_3(x)} d\tau_{2,3}(x), & \Psi_{\mathbf{n},3}(z) &= \int \tfrac{\Psi_{\mathbf{n},2}(x)}{z-x}\tfrac{\widehat{s}_{2,3}(x)}{\widehat{\sigma}_2(x)} d\tau_{3,2}(x).
\end{aligned} \tag{9}
$$

These functions satisfy (the proof is contained in that of (14)-(16) in [7], just substitute what is there denoted as $ds_{1,3}$ by $\beta_{\mathbf{n}} ds_3/\alpha_{\mathbf{n}}$).

LEMMA 1. *Let $n_1 < n_2 < n_3$. We have*

$$
\begin{aligned}
i)\quad 0 &= \int x^\nu \Psi_{\mathbf{n},1}(x) d\tau_{2,3}(x), & \nu &= 0,\ldots,n_1-1, \\
ii)\quad 0 &= \int x^\nu \Psi_{\mathbf{n},1}(x) \tfrac{\widehat{s}_{3,2}(x)}{\widehat{\sigma}_3(x)} d\tau_{2,3}(x), & \nu &= 0,\ldots,n_2-1, \\
iii)\quad 0 &= \int x^\nu \Psi_{\mathbf{n},2}(x) \tfrac{\widehat{s}_{2,3}(x)}{\widehat{\sigma}_2(x)} d\tau_{3,2}(x), & \nu &= 0,\ldots,n_1-1.
\end{aligned} \tag{10}
$$

From relations (10) immediately follows that $\Psi_{\mathbf{n},1}$ has at least n_1+n_2 zeros on Δ_2 and $\Psi_{\mathbf{n},2}$ has at least n_1 zeros on Δ_3. It turns out that these zeros are all simple and these functions have no other zeros on $\overline{\mathbb{C}} \setminus \Delta_1$ and $\overline{\mathbb{C}} \setminus \Delta_2$, respectively. These assertions are contained and proved in Lemma 3.2 of [7]. Let $Q_{\mathbf{n},j+1}, j = 1, 2,$ be the monic polynomial whose zeros are those of $\Psi_{\mathbf{n},j}$ in $\overline{\mathbb{C}} \setminus \Delta_j$, $Q_{\mathbf{n},1} = Q_{\mathbf{n}}$, and $Q_{\mathbf{n},4} \equiv 1$.

Set

$$
d\widetilde{\sigma}_1(x) = ds_3(x), \quad d\widetilde{\sigma}_2(x) = \frac{\widehat{s}_{2,3}(x)}{\widehat{\sigma}_2(x)} d\tau_{2,3}(x), \quad \text{and} \quad d\widetilde{\sigma}_3(x) = \frac{\widehat{s}_{3,2}(x)}{\widehat{\sigma}_3(x)} d\tau_{3,2}(x).
$$

We have (see the proof of (21) in [7] and (42) in [2])

LEMMA 2. *Let $n_1 < n_2 < n_3$. Then*

$$
0 = \int x^\nu \Psi_{\mathbf{n},j}(x) \frac{d\widetilde{\sigma}_{j+1}(x)}{Q_{\mathbf{n},j+2}(x)}, \quad \nu = 0,\ldots,n_1+\cdots+n_{3-j}-1, \quad j = 0,1,2, \tag{11}
$$

and

$$
\frac{\Psi_{\mathbf{n},j+1}(z)}{Q_{\mathbf{n},j+2}(z)} = \frac{1}{Q_{\mathbf{n},j+1}(z)} \int \frac{Q^2_{\mathbf{n},j+1}(x)}{z-x} \frac{\Psi_{\mathbf{n},j}(x) d\widetilde{\sigma}_{j+1}(x)}{Q_{\mathbf{n},j+1}(x) Q_{\mathbf{n},j+2}(x)}, \qquad j = 0,1,2. \tag{12}
$$

The functions $\Phi_{\mathbf{n},j}$ and $\Psi_{\mathbf{n},j}$ are connected as indicated in the next result. The proof is analogous to that of Lemma 5 in [2], taking into consideration Lemma 2.1 of [7].

LEMMA 3. *Let $n_1 < n_2 < n_3$. We have that*

$$
\begin{aligned}
\Phi_{\mathbf{n},1}(z) &= \tfrac{1}{\widehat{s}_{2,3}(z)} \Psi_{\mathbf{n},1}(z) - \tfrac{\widehat{\sigma}_3(z)}{\widehat{s}_{2,3}(z)} \Psi_{\mathbf{n},2}(z) + \Psi_{\mathbf{n},3}(z), && z \in \mathbb{C} \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3), \\
\Phi_{\mathbf{n},2}(z) &= \tfrac{\widehat{\sigma}_2(z)}{\widehat{s}_{2,3}(z)} \Psi_{\mathbf{n},1}(z) - \Psi_{\mathbf{n},2}(z) && z \in \mathbb{C} \setminus (\Delta_1 \cup \Delta_2) \\
\Phi_{\mathbf{n},3}(z) &= \Psi_{\mathbf{n},1}(z), && z \in \mathbb{C} \setminus \Delta_1.
\end{aligned} \tag{13}
$$

If we find the asymptotic behavior of the polynomials $Q_{\mathbf{n},j}$, (12) allows us to obtain that of the second type functions $\Psi_{\mathbf{n},j}$. Then, formulas (13) will help us find that of the functions $\Phi_{\mathbf{n},j}$. The asymptotic behavior of the polynomials $Q_{\mathbf{n},j}$ is obtained using results from vector potential theory in the presence of an external field. For this purpose, all the assumptions of Theorem 1 are essential. In particular, recall that (4) takes place.

Set

$$
\theta_1 = p_1 + p_2 + p_3 = 1, \; \theta_2 = p_1 + p_2, \; \theta_3 = p_1, \qquad \theta = (\theta_1, \theta_2, \theta_3),
$$

$$
A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}
$$

and

$$\mathbf{f} = (f_1, f_2, f_3)^t = (V^{\beta} - V^{\alpha}, 0, 0)^t$$

where

$$(V^{\beta} - V^{\alpha})(z) = \int \log \frac{1}{|z-x|} d(\beta - \alpha)(x), \qquad z \in \mathbb{C}.$$

is the difference of the logarithmic potentials of the measures β and α.

For each $i = 1, 2, 3$, $\mathcal{M}_{\theta_i}(\Delta_i)$ is the set of all finite positive Borel measures in $\mathcal{M}(\Delta_i)$ whose total mass equals θ_i. Denote $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ and

$$\mathcal{M}_{\theta}(\Delta) = \mathcal{M}_{\theta_1}(\Delta_1) \times \mathcal{M}_{\theta_2}(\Delta_2) \times \mathcal{M}_{\theta_3}(\Delta_3).$$

For each column vector of measures $\mu = (\mu_1, \mu_2, \mu_3)^t \in \mathcal{M}_{\theta}(\Delta)$, we define the vector function

$$W^{\mu}(z) = (V^{A\mu} + f)(z) = \int \ln \frac{1}{|z-x|} dA\mu(x) + f(z), \qquad z \in \mathbb{C}.$$

The i-th component of W^{μ} is given by

$$W_i^{\mu} = \sum_{j=1}^{3} a_{i,j} V^{\mu_j} + f_i, \qquad i = 1, 2, 3,$$

where $a_{i,j}$, $i, j = 1, 2, 3$, represents the entry in row i and column j of A and

$$V^{\mu_j}(z) = \int \ln \frac{1}{|z-x|} d\mu_j(x), \qquad z \in \mathbb{C}.$$

Finally, for each $\mu \in \mathcal{M}_{\theta}(\Delta)$, we define

$$w_i^{\mu} = \min_{x \in \Delta_i} W_i^{\mu}(x), \qquad i = 1, 2, 3.$$

The following result is a particular case of Theorem 4 in [2].

Lemma 4. *There exists a unique vector measure $\bar{\mu} \in \mathcal{M}_{\theta}(\Delta)$, such that*

$$W_i^{\bar{\mu}}(x) = w_i^{\bar{\mu}}, \qquad x \in \operatorname{supp}(\bar{\mu}_i), \qquad i = 1, 2, 3\,.$$

The measure $\bar{\mu} = \bar{\mu}(\Delta, \theta, A, \mathbf{f})$ is called extremal or equilibrium measure with respect to the initial data (Δ, θ, A, f).

Set

$$f_{n,1}(x) = \frac{1}{|n|}(V^{\chi(\beta_n)}(x) - V^{\chi(\alpha_n)}(x))\,.$$

By (5) and known properties of the potential (see [4]), it follows that:

- $f_1(x) = \lim_{n \in \Lambda} f_{n,1}(x) = V^{\beta}(x) - V^{\alpha}(x)\,,\ x \in \Delta_1\,,$ in measure on Δ_1.
- Each $f_{n,1}$ as well as f_1 is lower semi-continuous on Δ_1.
- Each $f_{n,1}$ and f_1 is weakly approximatively continuous on Δ_1. g is weakly approximatively continuous at $x_0 \in \Delta_1$, if there exists a set $e(x_0) \subset \Delta_1$ of positive measure such that

$$\liminf_{x \to x_0, x \in \Delta_1} g(x) = \lim_{x \to x_0, x \in e(x_0)} g(x) = g(x_0)\,.$$

- $\lim_{n \to \infty} \min_{\Delta_1} f_{n,1}(x) = \min_{\Delta_1} f_1(x)\,.$

This type of convergence will be denoted $\mathcal{F} - \lim_{n \to \infty} f_{n,1} = f_1$. f_1 is the first component of $\mathbf{f}$.

A positive measure σ with compact support is said to belong to class **Reg** if

$$\lim_{l} \kappa_l^{1/l} = 1/\operatorname{cap}(\operatorname{supp}(\sigma))\,,$$

where $\kappa_l > 0$ denotes the leading coefficient of the orthonormal polynomial of degree l with respect to σ and $\operatorname{cap}(\operatorname{supp}(\sigma))$ is the logarithmic capacity of the indicated set. For details on this definition and properties of the so called class of regular measures see Chapter 2 in [10]. In particular, it is well known that $\sigma' > 0$ almost everywhere (on its support contained in $\mathbb{R}$) implies that $\sigma \in$ **Reg**. A negative measure measure σ is said to belong to **Reg** if $-\sigma \in$ **Reg**

The final auxiliary result is a combination of Theorem 1 in [4] and Theorem 3.3.3 in [10].

LEMMA 5. *Let $\Lambda \subset \mathbb{N}$. Suppose that a sequence of monic polynomials $\{q_l\}_{l\in\Lambda}$ satisfies*

$$\int x^k q_l(x) d\sigma_l(x) = 0, \qquad k = 0, \dots, \deg q_l - 1, \quad l \in \Lambda\,, \tag{14}$$

where $d\sigma_l = \exp(-g_l)d\sigma$, and σ is a positive measure supported on a finite interval Δ. Assume that either $\sigma' > 0$ a.e. on Δ, the functions in $\{g_l\}_{l\in\Lambda}$ and g are lower semi-continuous on Δ and $\mathcal{F} - \lim_{l\in\Lambda} \frac{1}{l} g_l(x) = g(x), x \in \Delta$, or $\sigma \in \mathbf{Reg}$, the functions in $\{g_l\}_{l\in\Lambda}$ and g are continuous on Δ and $\lim_{l\in\Lambda} \frac{1}{l} g_l(x) = g(x)$, uniformly on Δ. Suppose that $\lim_{l\in\Lambda} \deg q_l / l = \theta$. Then

$$\lim_{l\in\Lambda} \frac{1}{l}\chi(q_l) = \overline{\mu}\,, \qquad \lim_{l\in\Lambda} \left(\int q_l^2 d\sigma_l(x)\right)^{\frac{1}{l}} = e^{-v}, \tag{15}$$

where $\overline{\mu} = \overline{\mu}(\Delta, \theta, 2, g)$ is the unique solution of the scalar equilibrium problem

$$2V^{\overline{\mu}}(x) + g(x) = v, \qquad x \in \operatorname{supp}(\overline{\mu})$$

and $v = \min_{x\in\Delta} 2V^{\overline{\mu}}(x) + g(x)$ is the associated equilibrium constant.

3. PROOF OF THE MAIN RESULTS.

We are ready to prove

THEOREM 2. *Assume that $|\sigma_1'| > 0$ almost everywhere on $\Delta_1, \sigma_j \in \mathbf{Reg}, j = 2, 3,$ and $(4) - (5)$ take place. Then, for each $j = 1, 2, 3,$*

$$\lim_{\mathbf{n}\in\Lambda} \chi(Q_{\mathbf{n},j})/|\mathbf{n}| = \bar{\mu}_j\,, \tag{16}$$

and

$$\lim_{\mathbf{n}\in\Lambda} |Q_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = \exp(-V^{\bar{\mu}_j}(z)), \qquad \mathcal{K} \subset \mathbb{C} \setminus \Delta_j\,, \tag{17}$$

Moreover, for $j = 1, 2, 3,$

$$\lim_{\mathbf{n}\in\Lambda} \left(\int Q_{\mathbf{n},j}^2 \frac{|\Psi_{\mathbf{n},j-1}| d|\widetilde{\sigma}_j|}{|Q_{\mathbf{n},j} Q_{\mathbf{n},j+1}|}\right)^{1/|\mathbf{n}|} = e^{-v_j}, \tag{18}$$

where $v_j = w_1^{\bar{\mu}} + \cdots + w_j^{\bar{\mu}}$, and

$$\lim_{n\in\Lambda} |\Psi_{n,j}|^{1/|\mathbf{n}|} = \exp(V^{\bar{\mu}_j} - V^{\bar{\mu}_{j+1}} - v_j)\,, \qquad \mathcal{K} \subset \mathbb{C} \setminus (\Delta_j \cup \Delta_{j+1}) \tag{19}$$

$(\Delta_4 = \emptyset, Q_{\mathbf{n},4} \equiv 1, V^{\bar{\mu}_4} \equiv 0)$.

Proof. The proof is similar to that of Theorem 6 in [2] so we sketch the main ingredients. The sequences of measures $\{\chi(Q_{\mathbf{n},j})/|\mathbf{n}|\}, \mathbf{n} \in \Lambda, j = 1, 2, 3,$ are weakly compact. Therefore, in order to prove (16), it is sufficient to prove that for any sequence of multi-indices $\Lambda' \subset \Lambda$ such that

$$\lim_{\mathbf{n}\in\Lambda'} \chi(Q_{\mathbf{n},j})/|\mathbf{n}| = \mu_j, \qquad j = 1, 2, 3, \tag{20}$$

we have that $\mu_j = \bar{\mu}_j, j = 1, 2, 3$. If (20) takes place. then

$$\lim_{\mathbf{n}\in\Lambda'} \frac{1}{|\mathbf{n}|} \log |Q_{\mathbf{n},j}| = -V^{\mu_j}, \qquad \mathcal{K} \subset \mathbb{C} \setminus \Delta_j. \tag{21}$$

Relation (11) with $j = 0$, can be written as

$$0 = \int x^\nu Q_{\mathbf{n},1}(x) \frac{|\beta_{\mathbf{n}}(x)| d|\widetilde{\sigma}_1|(x)}{|\alpha_{\mathbf{n}}(x) Q_{\mathbf{n},2}(x)|}, \qquad \nu = 0, \dots, \deg Q_{\mathbf{n},1} - 1.$$

From (4), (5), and (21), we find that

$$\mathcal{F} - \lim_{\mathbf{n}\in\Lambda'} \frac{1}{|\mathbf{n}|} \log \frac{|\alpha_{\mathbf{n}}(x) Q_{\mathbf{n},2}(x)|}{|\beta_{\mathbf{n}}(x)|} = V^{\beta}(x) - V^{\alpha}(x) - V^{\mu_2}(x), \qquad x \in \Delta_1.$$

Using Lemma 5 it follows that $\mu_1 \in \mathcal{M}_{\theta_1}(\Delta_1)$ is the unique measure that satisfies the scalar boundary value equilibrium problem

$$2V^{\mu_1} - V^{\mu_2} + f_1 = v_1, \qquad x \in \operatorname{supp}(\mu_1)$$

where

$$v_1 = \min_{x\in\Delta_1}(2V^{\mu_1} - V^{\mu_2} + f_1)(x),$$

and

$$\lim_{\mathbf{n}\in\Lambda'}\left(\int |Q_{\mathbf{n},1}(x)|^2 \frac{|\beta_{\mathbf{n}}(x)||d|\widetilde{\sigma}_1|(x)}{|Q_{\mathbf{n},2}(x)\alpha_{\mathbf{n}}(x)|}\right)^{1/|\mathbf{n}|} = e^{-v_1}. \tag{22}$$

Relation (11) with $j = 1$, can be expressed as

$$0 = \int x^\nu Q_{\mathbf{n},2}(x)\frac{|\Psi_{\mathbf{n},1}(x)|d|\widetilde{\sigma}_2|(x)}{|Q_{\mathbf{n},2}(x)Q_{\mathbf{n},3}(x)|}, \qquad \nu = 0,\dots,\deg Q_{\mathbf{n},2} - 1.$$

From (4), (12) (with $j = 0$), (21), and (22), we obtain

$$\lim_{\mathbf{n}\in\Lambda'}\frac{1}{|\mathbf{n}|}\log\frac{|Q_{\mathbf{n},2}(x)Q_{\mathbf{n},3}(x)|}{|\Psi_{\mathbf{n},1}(x)|} = -V^{\mu_1}(x) - V^{\mu_3}(x) + v_1,$$

uniformly on Δ_2. From Lemma 5 it follows that $\mu_2 \in \mathcal{M}_{\theta_2}(\Delta_2)$ is the unique measure that satisfies the scalar the boundary value equilibrium problem

$$2V^{\mu_2} - V^{\mu_1} - V^{\mu_3} + v_1 = v_2, \qquad x \in \operatorname{supp}(\mu_2)$$

where

$$v_2 = \min_{x\in\Delta_2}(2V^{\mu_2} - V^{\mu_2} - V^{\mu_3})(x) + v_1,$$

and

$$\lim_{\mathbf{n}\in\Lambda'}\left(\int |Q_{\mathbf{n},2}(x)|^2 \frac{|\Psi_{\mathbf{n},1}(x)|d|\widetilde{\sigma}_2|(x)}{|Q_{\mathbf{n},2}(x)Q_{\mathbf{n},3}(x)|}\right)^{1/|\mathbf{n}|} = e^{-v_2}. \tag{23}$$

Finally, relation (11) with $j = 2$, can be expressed as

$$0 = \int x^\nu Q_{\mathbf{n},3}(x)\frac{|\Psi_{\mathbf{n},2}(x)|d|\widetilde{\sigma}_3|(x)}{|Q_{\mathbf{n},3}(x)|}, \qquad \nu = 0,\dots,\deg Q_{\mathbf{n},3} - 1.$$

From (4), (12) (with $j = 1$), (21), and (23), we obtain

$$\lim_{\mathbf{n}\in\Lambda'}\frac{1}{|\mathbf{n}|}\log\frac{|Q_{\mathbf{n},3}(x)|}{|\Psi_{\mathbf{n},2}(x)|} = -V^{\mu_2}(x) + v_2,$$

uniformly on Δ_3. From Lemma 5 it follows that $\mu_3 \in \mathcal{M}_{\theta_3}(\Delta_3)$ is the unique measure that satisfies the scalar the boundary value equilibrium problem

$$2V^{\mu_3} - V^{\mu_2} + v_2 = v_3, \qquad x \in \operatorname{supp}(\mu_3)$$

where

$$v_3 = \min_{x\in\Delta_3}(2V^{\mu_3} - V^{\mu_2})(x) + v_2,$$

and

$$\lim_{\mathbf{n}\in\Lambda'}\left(\int |Q_{\mathbf{n},3}(x)|^2 \frac{|\Psi_{\mathbf{n},2}(x)|d|\widetilde{\sigma}_3|(x)}{|Q_{\mathbf{n},3}(x)|}\right)^{1/|\mathbf{n}|} = e^{-v_3}. \tag{24}$$

Putting together the three scalar equilibrium problems, we obtain a vector equilibrium problem that according to Lemma 4 has the unique solution $\bar{\mu} = \bar{\mu}(\Delta,\theta,A,\mathbf{f})$. We immediately obtain that $\mu = \bar{\mu}$ for any such Λ' as we needed to prove and $v_j = w_1^{\bar{\mu}} + \cdots + w_j^{\bar{\mu}}$ as indicated. Now, (17) follows directly (17), and (19) is a consequence of (12), (17), and (18). □

We are ready for the the proof of Theorem 1

Proof of Theorem 1. By Theorem 2 (recall that $Q_{\mathbf{n},1} = Q_{\mathbf{n}}$) and (5), we know that

$$\lim_{\mathbf{n}\in\Lambda}\left|\frac{\alpha_{\mathbf{n}}}{\beta_{\mathbf{n}}Q_{\mathbf{n}}}\right|^{1/|\mathbf{n}|} = \exp(V^{\bar{\mu}_1} + f_1), \qquad \mathcal{K} \subset D \setminus E. \tag{25}$$

On compact subsets of D the same holds taking upper limit instead of limit. On account of (7), the proof of Theorem 1 reduces to finding the limit of $\{\Phi_{\mathbf{n},j}\}, \mathbf{n} \in \Lambda, j = 1,2,3$.

Set

$$U_0^{\bar{\mu}} = -V^{\bar{\mu}_1}, \qquad U_j^{\bar{\mu}} = V^{\bar{\mu}_j} - V^{\bar{\mu}_{j+1}} - v_j, \quad j = 1,2,3,$$

$(V^{\bar\mu_4} \equiv 0)$. Fix $j \in \{1,2,3\}$. For each integer $k, 1 \le k \le 4-j$, define the regions

$$D_k^j = \{z \in D = \overline{\mathbb{C}} \setminus \Delta_1 : U_k^{\bar\mu}(z) > U_i^{\bar\mu}(z), i = 1, \ldots, 4-j, i \neq k\}.$$

Some D_k^j could be empty ($D_1^3 = D$). By (13) and (19) we have that

$$\lim_{\mathbf{n}\in\Lambda} |\Phi_{\mathbf{n},j}|^{1/|\mathbf{n}|} = \exp U_k^{\bar\mu}, \qquad \mathcal{K} \subset D_k^j.$$

Denote

$$\xi_j(z) = \max\{U_k^{\bar\mu}(z) : k = 1, \ldots, 4-j\};$$

then (notice that the functions multiplying the $\Psi's$ in (13) are different form zero for the specified values of z for which those relations hold)

$$\lim_{n\in\Lambda} |\Phi_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = \exp \xi_j(z)\,, \qquad \mathcal{K} \subset \bigcup_{k=1}^{4-j} D_k^j\,, \tag{26}$$

and

$$\limsup_{\mathbf{n}\in\Lambda} |\Phi_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} \le \exp \xi_j(z)\,, \qquad \mathcal{K} \subset D\,. \tag{27}$$

Using (7) and (25)-(27), it follows that

$$\lim_{\mathbf{n}\in\Lambda} |\widehat{s}_j(z) - R_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = \exp(V^{\bar\mu_1} + f_1 + \xi_j)(z)\,, \qquad \mathcal{K} \subset \mathcal{G}_j = \bigcup_{k=1}^{4-j} D_k^j \setminus E\,, \tag{28}$$

and

$$\limsup_{\mathbf{n}\in\Lambda} |\widehat{s}_j(z) - R_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} \le F_j(z) := \exp(V^{\bar\mu_1} + f_1 + \xi_j)(z)\,, \qquad \mathcal{K} \subset D\,. \tag{29}$$

(28) and (29) complement and make precise the statement of Theorem 1. □

Along the way, we have also obtained

Theorem 3. *Assume that $|\sigma_1'| > 0$ almost everywhere on $\Delta_1, \sigma_j \in \mathbf{Reg}, j = 2,3,$ and $(4)-(5)$ take place. Then, for each $j = 1,2,3$, (26) and (27) are satisfied.*

The set

$$\Omega_j^c = \{z \in D : (V^{\bar\mu_1} + f_1 + \xi_j)(z) < 0\}$$

is the domain of convergence of the approximants $R_{\mathbf{n},j}$ to $\widehat{s}_j$. This set contains a neighborhood of infinity whenever $|\alpha| < 1 + |\beta| + p_j$ because then

$$(V^{\bar\mu_1} + f_1 + \xi_j)(z) = \mathcal{O}\left((1 + |\beta| + p_j - |\alpha|)\log\frac{1}{|z|}\right) \to -\infty \quad \text{as} \quad z \to \infty\,. \tag{30}$$

(By assumption $|\alpha| \le 1 + |\beta| + p_1$ since for all $\mathbf{n} \in \Lambda, \deg \alpha_{\mathbf{n}} \le |n| + \deg \beta_{\mathbf{n}} + n_1$.) The convergence is uniform on compact subsets of Ω_j^c and the rate is geometric. There can also be a non-empty domain of divergence given by

$$\Omega_j^d = \{z \in D : (V^{\bar\mu_1} + f_1 + \xi_j)(z) > 0\}\,.$$

Let us see under what conditions we can guarantee that $\Omega_j^c = D$. For MHPA, in [3] we proved convergence in D for more general sequences of multi-indices and general Nikishin systems, but here (28)-(29) give an exact expression for the rate of convergence.

Corollary 1. *Assume that $|\sigma_1'| > 0$ almost everywhere on $\Delta_1, \sigma_j \in \mathbf{Reg}, j = 2,3,$ $(4)-(5)$ hold, $\mathrm{supp}(\bar\mu_1) = \Delta_1$, and $|\alpha| < 1 + |\beta| + p_1$. Then, $\Omega_j^c = D, j = 1,2,3.$*

Proof. For $j = 3$,

$$V^{\bar\mu_1} + f_1 + \xi_3 = V^{\bar\mu_1} + f_1 + U_1^{\bar\mu} = W_1^{\bar\mu} - w_1^{\bar\mu}.$$

This function is subharmonic in $\overline{\mathbb{C}} \setminus \Delta_1$. Since $\mathrm{supp}(\bar\mu_1) = \Delta_1$, from the equilibrium condition it is constantly equal to zero on Δ_1. Then, using that $|\alpha| < 1 + |\beta| + p_1$ and (30) for $j = 3$, it follows that $\Omega_3^c = D$.

Let us consider the case when $j = 2$. We have

$$U_2^{\bar\mu} - U_1^{\bar\mu} = W_2^{\bar\mu} - w_2^{\bar\mu} = \mathcal{O}((p_2 - p_3)\log(1/|z|)), \quad z \to \infty. \tag{31}$$

From the equilibrium condition, this function equals zero on $\mathrm{supp}(\bar{\mu}_2)$. If $p_2 = p_3$, it is subharmonic in $\overline{\mathbb{C}} \setminus \mathrm{supp}(\bar{\mu}_2)$. Using the maximum principle, it follows that $U_2^{\bar{\mu}}(z) \leq U_1^{\bar{\mu}}(z), z \in \overline{\mathbb{C}}$, and $\xi_2(z) = \max\{U_1^{\bar{\mu}}(z), U_2^{\bar{\mu}}(z)\} = U_1^{\bar{\mu}}(z) = \xi_3(z)$. Therefore $(V^{\bar{\mu}_1} + f_1 + \xi_2)(z) = (V^{\bar{\mu}_1} + f_1 + \xi_3)(z), z \in D$. Consequently, if $p_2 = p_3$, $\Omega_2^c = D$.

Now, with $j = 2$ suppose that $p_2 < p_3$. Taking (31) into consideration, we know that in a neighborhood of infinity $U_2^{\bar{\mu}} > U_1^{\bar{\mu}}$. Let Γ_2 be the set of all points in D where $U_2^{\bar{\mu}} - U_1^{\bar{\mu}} = 0$. According to what we have just said, this set is bounded, and from the equilibrium condition on Δ_2 it contains $\mathrm{supp}(\bar{\mu}_2)$. Γ_2 divides D into several regions. In any bounded connected component of the complement of Γ_2, $U_2^{\bar{\mu}} - U_1^{\bar{\mu}}$ is subharmonic and on its boundary equals zero. Thus in these regions and on Γ_2, using the maximum principle, $\xi_2(z) = \xi_3(z)$ and we conclude that these sets are contained in Ω_2^c.

Let's see what happens in the unbounded connected component of the complement of Γ_2 in D. Here,

$$V^{\bar{\mu}_1} + f_1 + \xi_2 = V^{\bar{\mu}_1} + f_1 + U_2^{\bar{\mu}} = (2V^{\bar{\mu}_1} - V^{\bar{\mu}_2} + f_1 - w_1^{\bar{\mu}_1}) + (-V^{\bar{\mu}_1} + 2V^{\bar{\mu}_2} - V^{\bar{\mu}_3} - w_2^{\bar{\mu}}) \tag{32}$$

is subharmonic (use again (30) with $j = 2$). The boundary of this unbounded region is formed by Δ_1 and a subset of Γ_2. On the subset of Γ_2, $V^{\bar{\mu}_1} + f_1 + \xi_2 = V^{\bar{\mu}_1} + f_1 + U_1^{\bar{\mu}} < 0$. On Δ_1, from the equilibrium condition on Δ_1 the first parenthesis is (32) equals zero. The second parenthesis is a subharmonic function on $\overline{\mathbb{C}} \setminus \mathrm{supp}(\bar{\mu}_2)$. Due to the equilibrium condition on Δ_2, using the maximum principle the second parenthesis in (32) is less than zero on Δ_1. Consequently, by the maximum principle on the unbounded connected component of the complement of Γ_2 in D the subharmonic function $V^{\bar{\mu}_1} + f_1 + U_2^{\bar{\mu}}$ is less than zero. Thus we have proved that when $j = 2$, we have that $\Omega_2^c = D$.

The case $j = 1$ is treated analogously and we sketch the proof. Obviously $\xi_1 = \max(\xi_2, U_3^{\bar{\mu}})$. When $p_1 = p_2$, due to the equilibrium condition on Δ_3, and that $U_3^{\bar{\mu}} - U_2^{\bar{\mu}}$ is subharmonic on $\overline{\mathbb{C}} \setminus \mathrm{supp}(\bar{\mu}_3)$, we conclude that $U_3^{\bar{\mu}} \leq U_2^{\bar{\mu}}$ on all of $\overline{\mathbb{C}}$ and the proof reduces to $j = 2$. If $p_1 < p_2$, in a neighborhood of infinity $U_3^{\bar{\mu}} > U_2^{\bar{\mu}}$. Let Γ_3 be the set of all points in D where $U_3^{\bar{\mu}} - U_2^{\bar{\mu}} = 0$. This set is bounded, and from the equilibrium condition on Δ_3 it contains $\mathrm{supp}(\bar{\mu}_3)$. Γ_3 divides D into several regions. In any bounded connected component of the complement of Γ_3, $U_3^{\bar{\mu}} - U_2^{\bar{\mu}}$ is subharmonic and on its boundary equals zero. Thus in these regions and on Γ_3, $\xi_1(z) = \xi_2(z)$ and we conclude that these sets are contained in Ω_1^c. On the unbounded region, $\{z \in D : U_3^{\bar{\mu}} > U_2^{\bar{\mu}}\}$ the function $V^{\bar{\mu}_1} + f_1 + U_3^{\bar{\mu}}$ is subharmonic and on its boundary it is negative; therefore, this region is also contained in Ω_1^c and we conclude the proof. □

Remark . Suppose that

$$\beta \leq \lambda := (\bar{\mu}_2 + \alpha)' + (2\theta_1 + |\beta| - \theta_2 - |\alpha|)\omega_{\Delta_1} , \tag{33}$$

where $(\bar{\mu}_2 + \alpha)'$ denotes the balayage of $\bar{\mu}_2 + \alpha$ onto Δ_1 and ω_{Δ_1} denotes the equilibrium measure on Δ_1 in the absence of an external field. The inequality in (33) means that the measure on the right hand dominates the one on the left on any Borel set. In this case

$$\bar{\mu}_1 = (\lambda - \beta)/2$$

and it is easy to verify that $\mathrm{supp}(\bar{\mu}_1) = \Delta_1$ (see Lemma 6 in [2].) For example this condition is satisfied in the case of MHPA.

Remark . If $\beta_n \equiv 1, \mathbf{n} \in \Lambda$ (that is, for MHPA), all the previous results hold true if $\sigma_j \in \mathbf{Reg}, j = 1, 2, 3$, since in the proof of Theorem 2 we can use Lemma 5 with $\sigma_1 \in \mathbf{Reg}$.

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