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COALITION-PROOF SUPPLY FUNCTION EQUILIBRIA IN OLIGOPOLY *

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Abstract

In an industry where firms compete via supply functions the set of market outcomes that can arise is large. If decreasing supply functions are ruled out, the set of equilibrium outcomes reduces somewhat, but it remains large: any price between the competitive price and the Cournot price can be sustained by a supply function equilibrium. In sharp contrast, this multiplicity disappears when firms take into account the gains they can attain by coordinating their actions: if the number of firms is above a threshold we identify (e.g., three if demand is linear), then the Cournot equilibrium is the unique outcome that can be sustained by a coalition-proof supply function equilibrium.

Keywords: Oligopoly, Cournot, Supply Function, Coalition-proofness, Organized Markets, Electricity.

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1 Introduction

We study an oligopoly where firms compete via supply functions. The model of competition via supply functions describes many markets, like the spot electricity markets in Spain and in the UK or the markets for government procurement contracts, more realistically than the Cournot or the Bertrand models. Moreover, modelling competition via supply functions implicitly introduces firms' reactions (to exogenous shocks or to changes in the competitors' choices) while maintaining a static framework.

A shortcoming of this model is that there is a great multiplicity of equilibria; that is, the set of market outcomes that can be sustained by supply function equilibria is very large–see, e.g., Grossman (1981), Hart (1985), Klemperer and Meyer (1989).[1] Interestingly, we find that this multiplicity disappears when the firms in the industry take into account the gains they can attain by coordinating their actions.

In order to account for the coordination opportunities present in an industry we identify the equilibria that are self-enforcing in the strong sense that no coalition of firms has a (self-enforcing) improving deviation. We refer to these equilibria as coalition-proof supply function equilibria–an adaptation to our framework of the notion of coalition-proof Nash equilibrium introduced by Bernheim, Peleg and Whinston (1987). We identify conditions on the curvature of the demand and cost functions that guarantee that if the number of firms is above a threshold, then the Cournot equilibrium is the *unique* market outcome that can be sustained by a coalition-proof supply function equilibrium; e.g., if the market demand is a linear function, this result arises when there are three or more firms in the industry.

We study an industry in which firms have access to a technology that exhibits non-increasing returns to scale, and in which the market demand, known with certainty, is a decreasing and concave function. Firms in the industry compete by simultaneously choosing a (non-decreasing) supply function. Firms' strategy choices together with the market demand determine the market price and firms' profits.

A supply function equilibrium (SFE henceforth) of an industry is just a Nash equilibrium of the game where firms strategies are supply functions and payoffs are

[1] Klemperer and Meyer (1989) have shown, however, that with unbounded uncertainty equilibrium is unique.

profits. We show that every price between the competitive equilibrium price and the Cournot equilibrium price can be sustained by a supply function equilibrium. We then study which market outcomes can be sustained by coalition-proof supply function equilibria (CPSFE henceforth).

A CPSFE of an industry is a coalition-proof Nash equilibrium of the game described; i.e., a CPSFE is a profile of supply functions which is immune to improving and self-enforcing deviations by any coalition of firms. A deviation is self-enforcing if there is no further self-enforcing and improving deviation available to a proper subcoalition of the deviating coalition.

Since there are no general results on the existence of coalition-proof Nash equilibrium, we begin by studying whether a CPSFE exists in our framework.[2] In an industry with only two firms, a CPSFE is just a SFE which is not Pareto dominated by any other SFE; hence a CPSFE exists. Further, for an industry with an arbitrary number of firms, we find that firms' joint profits are maximized, within the set of market outcomes that can be sustained by SFE, at the Cournot equilibrium. Hence in an industry with only two firms the Cournot equilibrium can be sustained by a CPSFE. Using induction we show that in industries with more than two firms the Cournot equilibrium can be sustained by a CPSFE, and hence a CPSFE exists.

Next we establish conditions under which the Cournot equilibrium is the unique outcome that can be sustained by a CPSFE. In the search of these conditions, the class of SFE in which all but perhaps one firm supply inelastically the same fixed amount plays an important role. It turns out that for every SFE there is an equilibrium in this class for which the profits of the firms using an inelastic supply are greater than or equal to those of the firm with the greatest profits in the original equilibrium. Moreover, if the number of firms in the industry is above a threshold we identify, the Cournot equilibrium Pareto dominates every other market outcome that can be sustained by SFE in this class.[3] Thus, if the number of firms is above this threshold,

[2]There are games for which a coalition-proof Nash equilibrium does not exist; see Bernheim, Peleg and Whinston (1987). Moreno and Wooders (1996) provide conditions on the set of iteratively undominated strategies that guarantee existence of a coalition-proof Nash equilibrium; these conditions are difficult to check in our context.

[3]This threshold is always greater than two. In fact, if there are only two firms in the industry

the Cournot equilibrium Pareto dominates every other market outcome that can be sustained by a SFE, and is therefore the unique market outcome that can be sustained by a CPSFE.

An interesting interpretation of the present result, parallel to Kreps and Scheinkman (1983),[4] is that the Cournot model provides a "reduced form" of a structural model where firms choose their supply functions (and therefore their reactions to the competitors' choices) taking into account the coordination possibilities present in the industry. As we show, the Cournot model yields predictions (regarding market outcomes) identical to those obtained using the more cumbersome model of competition via supply functions. Thus, introducing firms' reactions into the model may add descriptive realism to the analysis, but it may be otherwise inconsequential.

2 Supply Function Equilibria

Consider an oligopolistic industry where n firms ($n \geq 2$) compete in the production of a homogeneous good. Demand is known to all firms with certainty;[5] throughout it is assumed that the demand function $D : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable, strictly decreasing, and concave on $(0, \rho)$, where $\rho > 0$ satisfies $D(p) > 0$ for $p < \rho$, and $D(p) = 0$ for $p \geq \rho$. All firms have access to the same technology, and therefore have identical cost function, $C : \mathbb{R} \to \mathbb{R}$; we assume that C is twice continuously differentiable, non-decreasing, and convex on $\mathbb{R}_+$, and satisfies $C'(0) < \rho$, and $C(q) = C(0)$ for $q < 0$. (Extending the domain of the cost function to include negative quantities is inconsequential and simplifies our analysis.) An industry is therefore described by triple (D, C, n), indicating the market demand, the firms' cost function, and the number of firms. In what follows, let us be given an industry (D, C, n).

the Stackelberg equilibria can be sustained as CPSFE.

[4] See also Klemperer and Meyer (1986) on this debate.

[5] In many markets the demand can be anticipated with a great accuracy, and therefore uncertainty plays a small role. This is the case, for example, in the spot electricity markets in Spain or in the UK. However, whereas in Spain each firm submits a bid for each of the (24 one-hour) periods in which the market is divided, in the UK firms submit a single bid for all (the 48 half-hour) periods. Green and Newbery's (1992) study of the UK market introduces demand uncertainty as a modeling strategy in order to account for the variability of demand along the different periods.

Firms compete by simultaneously choosing a supply function; that is, a real-valued function on $[0, \rho]$. Firms' supply functions are restricted to be twice continuously differentiable on $(0, \rho)$ and non-decreasing.[6] For a profile of supply functions $s = (s_1, \ldots, s_n)$, a *market clearing price* is a solution to the equation

$$\sum_{i=1}^{n} s_i(p) = D(p). \tag{MC}$$

Our assumptions on the market demand and the firms' supply functions guarantee that if a market clearing price exists, then it is unique. For each profile of supply functions s, let $p(s)$ be the market clearing price if it exists, and let $p(s)$ be zero if a market clearing price does not exist. Firm i's profits (payoff) is given by $\pi_i(s) = p(s)s_i(p(s)) - C(s_i(p(s)))$. (This construction implicitly assumes that firms' revenues are zero when a market clearing price does not exist.)

A *supply function equilibrium* (SFE henceforth) is a (pure strategy) Nash equilibrium of the game described. Denote by N the set $\{1, \ldots, n\}$, and write $SFE(D, C, n)$ for the set of supply function equilibria. In a SFE each firm maximizes profits on its "residual demand"; that is, if $\hat{s} \in SFE(D, C, n)$, then $p(\hat{s}) = \hat{p}$ solves

$$\max_{p \in [0,\rho]} \quad p\left(D(p) - \sum_{j \neq i} \hat{s}_j(p)\right) - C\left(D(p) - \sum_{j \neq i} \hat{s}_j(p)\right),$$

for each $i \in N$. If $\hat{p} \in (0, \rho)$, then $\hat{s}$ satisfies

$$D(\hat{p}) - \sum_{j \neq i} \hat{s}_j(\hat{p}) + \left(D'(\hat{p}) - \sum_{j \neq i} \hat{s}'_j(\hat{p})\right)\left(\hat{p} - C'\left(D(\hat{p}) - \sum_{j \neq i} \hat{s}_j(\hat{p})\right)\right) = 0,$$

for each $i \in N$. Writing $\hat{q}_i = \hat{s}_i(\hat{p})$ and using the market clearing condition (MC), this condition can be written as

$$\hat{q}_i + \left(D'(\hat{p}) - \sum_{j \neq i} \hat{s}'_j(\hat{p})\right)(\hat{p} - C'(\hat{q}_i)) = 0, \tag{E_i}$$

for each $i \in N$. If in addition each $\hat{s}_i$ is a convex function, then satisfying E_i for each $i \in N$ is a sufficient condition for a strategy profile $\hat{s}$ to be a SFE. For $s \in SFE(D, C, n)$, we denote by $(p(s), q_1(s), \ldots q_n(s))$ the associated market outcome.

[6] Other works, e.g., Kemplerer and Meyer (1989), do not restrict the strategy sets to include only non-decreasing supply functions. It may be questionable, however, whether in general firms can commit to decreasing supply functions. Moreover, in many markets, such as the spot electricity markets in Spain or in the UK, decreasing supply functions are explicitly ruled out.

In determining the market prices that can be sustained by SFE two prices play an important role: the competitive equilibrium price, characterized by marginal cost pricing, and the price associated to the Cournot equilibrium. In our framework each of these prices is a solution to a system of equations.

A *Cournot equilibrium* $(\bar{p}, \bar{q}_1, \ldots, \bar{q}_n)$ is characterized by the system of equations

$$\bar{q}_i + (\bar{p} - C'(\bar{q}_i)) D'(\bar{p}) = 0, \tag{$\bar{C}1$}$$

for $i \in N$, and

$$\sum_{i=1}^{n} \bar{q}_i = D(\bar{p}). \tag{$\bar{C}2$}$$

Equation $\bar{C}1$ ensures that each firm maximizes profits, whereas $\bar{C}2$ ensures that the market clears. Our assumptions on demand and cost functions imply the existence of a unique Cournot equilibrium, which is symmetric (i.e., satisfies $\bar{q}_1 = \ldots = \bar{q}_n$). Given an industry (D, C, n), we denote by $\bar{p}(D, C, n)$ and $\bar{q}(D, C, n)$ the price and each firm's output, respectively, at the Cournot equilibrium, and we refer to the Cournot equilibrium price of the industry as the *Cournot price.*

A *competitive equilibrium* $(\underline{p}, \underline{q}_1, \ldots, \underline{q}_n)$ satisfies the system of equations

$$C'\left(\underline{q}_i\right) = \underline{p}, \tag{$\underline{C}1$}$$

for $i \in N$, and

$$\sum_{i=1}^{n} \underline{q}_i = D(\underline{p}). \tag{$\underline{C}2$}$$

Our assumptions on demand and cost functions imply the existence of a unique *competitive equilibrium price*, which we denote by $\underline{p}(D, C, n)$. Clearly

$$0 \leq \underline{p}(D, C, n) < \bar{p}(D, C, n) < \rho.$$

Proposition 2.1 shows that if s is a SFE that leads to the Cournot price, then the market outcome associated with s is the Cournot equilibrium. Moreover, in every SFE that sustains the Cournot equilibrium the derivative of each firm's supply function vanishes at the Cournot price.

Proposition 2.1. *Let (D, C, n) be an industry, and let $s \in SFE(D, C, n)$. If $p(s) = \bar{p}(D, C, n)$, then $s_i(\bar{p}) = \bar{q}(D, C, n)$, and $s_i'(\bar{p}) = 0$ for $i \in N$.*

Proof: Let $s \in SFE(D,C,n)$ be such that $p(s) = \bar{p}(D,C,n) = \bar{p}$. Note that since $\bar{p} \in (0,\rho)$, s satisfies Condition E_i for $i \in N$. We show that $s_i(\bar{p}) = \bar{q}(D,C,n) = \bar{q}$ for $i \in N$. Assume by way of contradiction that there is a firm j such that $s_j(\bar{p}) \neq \bar{q}$. Since $\sum_{i=1}^{n} s_i(\bar{p}) = D(\bar{p}) = n\bar{q}$, assume without loss of generality that $s_j(\bar{p}) = q_j < \bar{q}$. Then

$$\begin{aligned}\bar{q} + (\bar{p} - C'(\bar{q}))\, D'(\bar{p}) &> q_j + (\bar{p} - C'(q_j))\, D'(\bar{p}) \\ &\geq q_j + (\bar{p} - C'(q_j)) \left(D'(\bar{p}) - \sum_{i \neq j} s_i'(\bar{p}) \right).\end{aligned}$$

The left hand side of this expression equals zero by $\bar{C}1$. Thus, Condition E_j is not satisfied, which is a contradiction.

We show that $s_i'(\bar{p}) = 0$ for $i \in N$. Let $i \in N$ arbitrary, and let $j \in N \backslash i$. From E_j and $\bar{C}1$ we have

$$0 = \bar{q} + D'(\bar{p})\,(\bar{p} - C'(\bar{q})) = \bar{q} + \left(D'(\bar{p}) - \sum_{k \neq j} \hat{s}_k'(\bar{p}) \right) (\bar{p} - C'(\bar{q})).$$

Therefore $\sum_{k \neq j} s_k'(\bar{p}) = 0$, and since $s_k'(\bar{p}) \geq 0$ for $i = 1, \ldots, n$, we have $s_k'(\bar{p}) = 0$ for $k \neq j$. Hence $s_i'(\bar{p}) = 0$. □

We now study the set of market outcomes that can be sustained by SFE. Klemperer and Meyer (1989) have shown that when supply functions are not restricted to be non-decreasing, then any market outcome $(p, q_1, \ldots, q_n) \in \mathbb{R}_+^n$ such that $p = D(\sum_{i=1}^{n} q_i)$ and $C'(q_i) < p$ for each $i \in N$ can be sustained by SFE. Our assumption that firms supply functions are non-decreasing reduces somewhat this set, although it remains large. Theorem 2.2 establishes that the set of prices that can be sustained by SFE, denoted by $SFE_p(D,C,n)$ (i.e., $p \in SFE_p(D,C,n)$ if $p = p(s)$ for some $s \in SFE(D,C,n)$), is the half open interval containing the prices between the Cournot price and the competitive equilibrium price.

Theorem 2.2. $SFE_p(D,C,n) = \left(\underline{p}(D,C,n), \bar{p}(D,C,n)\right].$

Proof: Write $\underline{p}(D,C,n) = \underline{p}$ and $\bar{p}(D,C,n) = \bar{p}$. Let $\hat{p} \in \left(\underline{p}, \bar{p}\right]$; we show that there is $\hat{s} \in SFE(D,C,n)$ such that $p(\hat{s}) = \hat{p}$. Write $\hat{q} = \frac{D(\hat{p})}{n}$, and for $i \in N$ let $\hat{s}_i(p) = \hat{a} + \hat{\alpha} p$, where $\hat{a}$ and $\hat{\alpha}$ satisfy

$$\hat{a} + \hat{\alpha}\hat{p} = \hat{q}, \tag{2.1}$$

and

$$\hat{q} + (\hat{p} - C'(\hat{q}))(D'(\hat{p}) - (n-1)\hat{\alpha}) = 0. \tag{2.2}$$

Note that $\hat{s}_i'(p) = \hat{\alpha}$ for $i \in N$. We show that $\hat{\alpha} \geq 0$, and therefore that each $\hat{s}_i$ is non-decreasing. Since $\hat{p} \leq \bar{p}$ we have

$$\hat{q} + (\hat{p} - C'(\hat{q}))\, D'(\hat{p}) \geq \bar{q} + (\bar{p} - C'(\bar{q}))\, D'(\bar{p}) = 0,$$

and therefore

$$\hat{s}_i'(p) = \hat{\alpha} = \frac{1}{n-1}\left(D'(\hat{p}) + \frac{\hat{q}}{\hat{p} - C'(\hat{q})}\right) \geq 0.$$

Equation 2.1 and the definition of $\hat{q}$ guarantees that $\hat{p}$ is the market clearing price for $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_n)$. Equation 2.2 ensures that $\hat{s}$ satisfies E_i for $i \in N$. Since each $\hat{s}_i$ is convex, we have $\hat{s} \in SFE(D, C, n)$.

Let $\hat{s}$ be such that $p(\hat{s}) = \hat{p} \leq \underline{p} < \rho$. We show that $\hat{s} \notin SFE(D, C, n)$. Since D is decreasing, we have $D(\hat{p}) \geq D(\underline{p}) > D(\rho) = 0$; hence there is one firm $i \in N$ producing $\hat{s}_i(\hat{p}) = \hat{q}_i \geq \frac{D(\underline{p})}{n} > 0$, which implies by $\underline{C}1$ that $C'(\hat{q}_i) \geq C'(\frac{D(\underline{p})}{n}) = \underline{p} \geq \hat{p}$. Since $s_j'(\hat{p}) \geq 0$ for $j \in N$, we have

$$\hat{q}_i + \left(D'(\hat{p}) - \sum_{j \neq i} \hat{s}_j'(\hat{p})\right)(\hat{p} - C'(\hat{q}_i)) > 0.$$

Thus, Condition E_i is not satisfied. Moreover, if $\hat{p} = 0$ then we have $\pi_i(\hat{s}) = -C(\hat{q}_i) < -C(0)$. Hence $\hat{s} \notin SFE(D, C, n)$.

Finally, let $\hat{s}$ be such that $\hat{p} > \bar{p} > 0$. As before we show that $\hat{s} \notin SFE(D, C, n)$. Since D is strictly decreasing, we have $D(\hat{p}) < D(\bar{p})$. Consequently, at least one firm $i \in N$ is producing $\hat{s}_i(\hat{p}) = \hat{q}_i < \bar{q} = \bar{q}(D, C, n)$. Thus, since $\hat{s}_i'(\hat{p}) \geq 0$, for $i \in N$, and $D'(\hat{p}) \leq D'(\bar{p})$ (recall that D is concave), $\bar{C}1$ implies

$$0 = \bar{q} + D'(\bar{p})(\bar{p} - C'(\bar{q})) > \hat{q}_i + \left(D'(\hat{p}) - \sum_{j \neq i} \hat{s}_j'(\hat{p})\right)(\hat{p} - C'(\hat{q}_i)).$$

Hence Condition E_i is not satisfied, and therefore $\hat{s} \notin SFE(D, C, n)$. □

Remark 2.3. Note that in the above construction $\hat{\alpha} = 0$ for $\hat{p} = \bar{p}$, whereas $\hat{\alpha}$ becomes arbitrarily large as $\hat{p}$ approaches $\underline{p}$ (because $\hat{p} - C'(\hat{q})$ approaches zero). Also note that for each $s \in SFE(D, C, n)$, we have $p(s) \in (0, \rho)$; hence in every SFE, E_i is satisfied for each $i \in N$.

The construction in the proof of Theorem 2.2 establishes that every price between the competitive equilibrium price and the Cournot price can be sustained by a symmetric SFE, which associated market outcome is therefore symmetric (i.e., all firms produce the same quantity). Nevertheless, there are asymmetric market outcomes that can be sustained by SFE (see Example 3.6.).

We finish this section by showing in Proposition 2.4 that the profits of an industry (D, C, n) are maximized, on the set of market outcomes that can be sustained by SFE, at the Cournot equilibrium. This result plays an important role in establishing in Section 3 that the Cournot equilibrium can be sustained by a CPSFE.

Proposition 2.4. *Let (D, C, n) be an industry, and let $\bar{s}, \hat{s} \in SFE(D, C, n)$ be such that $p(\bar{s}) = \bar{p}(D, C, n) > p(\hat{s})$. Then $\sum_{i=1}^{n} \pi_i(\bar{s}) > \sum_{i=1}^{n} \pi_i(\hat{s})$.*

Proof: For $p \in (0, \rho)$, define

$$\Pi(p) = pD(p) - nC\left(\frac{D(p)}{n}\right).$$

Then

$$\Pi'(p) = D(p) + D'(p)\left(p - C'\left(\frac{D(p)}{n}\right)\right),$$

and

$$\Pi''(p) = D'(p) + D''(p)\left(p - C'\left(\frac{D(p)}{n}\right)\right) + D'(p)\left(1 - \frac{D'(p)}{n}C''\left(\frac{D(p)}{n}\right)\right).$$

Write $\bar{p}(D, C, n) = \bar{p}$ and $\underline{p}(D, C, n) = \underline{p}$. Note that $\Pi''(p) < 0$ for $p \geq \underline{p}$. Also note that $\bar{C}1$ and $\bar{C}2$ imply

$$\begin{aligned} \Pi'(\bar{p}) &= D(\bar{p}) + D'(\bar{p})\left(\bar{p} - C'\left(\frac{D(\bar{p})}{n}\right)\right) \\ &= D(\bar{p}) - \frac{D(\bar{p})}{n} = \frac{(n-1)\, D(\bar{p})}{n} > 0. \end{aligned}$$

Hence $\Pi'(p) > \Pi'(\bar{p}) > 0$, and therefore $\Pi(\bar{p}) > \Pi(p)$, for $p \in (\underline{p}, \bar{p})$.

Let $\bar{s}, \hat{s} \in SFE(D, C, n)$ be such that $p(\bar{s}) = \bar{p} > \hat{p} = p(\hat{s})$. For $i \in N$ denote $q_i(\bar{s}) = \bar{q}$ and $q_i(\hat{s}) = \hat{q}_i$. Note that $\bar{q} = \frac{D(\bar{p})}{n}$ by Proposition 2.2, and therefore

$$\sum_{i=1}^{n} \pi_i(\bar{s}) = \Pi(\bar{p}).$$

Since C is convex we have

$$\sum_{i=1}^{n} \pi_i(\hat{s}) = \sum_{i=1}^{n} (\hat{p}\hat{q}_i - C(\hat{q}_i)) \leq \Pi(\hat{p}).$$

Hence

$$\sum_{i=1}^{n} \pi_i(\bar{s}) = \Pi(\bar{p}) > \Pi(\hat{p}) \geq \sum_{i=1}^{n} \pi_i(\hat{s}),$$

which establishes Proposition 2.4. □

3 Coalition-Proof Supply Function Equilibria

Consider an industry where firms are free to discuss their strategies, although they cannot make binding agreements. In this setting, one would expect that firms will agree to coordinate their actions on a strategy profile that realizes the highest possible profits. Since the agreements the firms may reach are non-binding, they must be *self-enforcing,* i.e., immune to profitable deviations. In particular, a firm will implement an agreement only if it is in its own interest to do so; hence agreements must be SFE. Not all SFE are self-enforcing, however, as they may be vulnerable to deviations by coalitions of firms. We introduce the notion of *coalition-proof supply function equilibrium* (CPSFE), which identifies the agreements that are self-enforcing in the strong sense that neither individual firms nor coalitions of firms have (self-enforcing) improving deviations. As we shall see, the possibility of coalitional deviations reduces drastically the set of equilibria.

The notion of CPSFE is an adaptation to our setting of the concept of coalition-proof Nash equilibrium–see Bernheim, Peleg and Whinston (1987). A CPSFE is a strategy profile which is not subject to credible (i.e., self-enforcing) improving deviations by any coalition of players. Self-enforcing deviations are those that are not subject to further self-enforcing and improving deviations by a proper subcoalition of the deviating coalition. A formal definition of the concept of CPSFE follows.

Let (D, C, n) be an industry. We denote by 2^N the set of all possible coalitions. For a strategy profile s and a coalition $M \in 2^N$, write s_M for the profile of supply functions of the members of M, and write m for the cardinality of the set M. Let s be a strategy profile and let $M \in 2^N$, $2 \leq m < n$, be a coalition of firms (recall that

$n \geq 2$). Holding fixed the strategies of the members of the complementary coalition, $s_{N \setminus M}$, the situation the group of firms in M faces can be modeled as that of an "industry" $(D_{s,M}, C, m)$, where $D_{s,M}$ is given for $p \in \mathbb{R}_+$ by

$$D_{s,M}(p) = \begin{cases} D(p) - \sum_{i \in N \setminus M} s_i(p) & \text{if } D(p) - \sum_{i \in N \setminus M} s_i(p) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

This recursive structure allows us to formalize the notion of CPSFE.

Coalition-Proof Supply Function Equilibrium: *Let (D, C, n) be an industry.*

(1) If $n = 2$, a coalition-proof supply function equilibrium is a strategy profile $s \in SFE(D, C, n)$ such that there is no $\tilde{s} \in SFE(D, C, n)$ satisfying $\pi_i(\tilde{s}) > \pi_i(s)$ for every $i \in N$.

(2) Assume that $CPSFE(D, C, k)$ has been defined for $2 \leq k \leq n - 1$, and let (D, C, n) be an industry.

(i) A strategy profile s is self-enforcing if $s \in SFE(D, C, n)$, and if for all $M \in 2^N$, $2 \leq m < n$, $s_M \in CPSFE(D_{s,M}, C, m)$.

(ii) A strategy profile s is a coalition-proof supply function equilibrium if it is self-enforcing and if there is no self-enforcing strategy profile $\tilde{s}$ such that $\pi_i(\tilde{s}) > \pi_i(s)$ for every $i \in N$.

The definition of CPSFE applies to industries with no fewer than two firms. Note that a CPSFE is a SFE, and therefore it is invulnerable to deviations by an individual firm.

Theorem 3.1 establishes that every industry has a CPSFE. In fact, it is shown that the Cournot equilibrium always can be sustained by a CPSFE.

Theorem 3.1. *Every industry (D, C, n) has a coalition-proof supply function equilibrium. Moreover, there is $s \in CPSFE(D, C, n)$ such that $p(s) = \bar{p}(D, C, n)$.*

Proof: Let (D, C, n) be an industry. For $i \in N$, let $\bar{s}_i$ be given for $p \in [0, \rho]$ by $\bar{s}_i(p) = \bar{q}(D, C, n) = \bar{q}$. Clearly $p(\bar{s}) = \bar{p}(D, C, n) = \bar{p}$. Moreover, $\bar{s} \in SFE(D, C, n)$ as Condition E_i is satisfied for each $i \in N$, and each $\bar{s}_i$ is a convex function. We prove that $\bar{s} \in CPSFE(D, C, n)$ by induction on the number of firms.

If $n = 2$, then $\bar{s} \in CPSFE(D, C, n)$ follows from Proposition 2.4. Assume that it has been shown that $\bar{s} \in CPSFE(D, C, n)$ whenever $2 \leq n \leq k - 1$. We show that $\bar{s} \in CPSFE(D, C, k)$.

We show that $\bar{s}$ is self-enforcing. Since $\bar{s} \in SFE(D, C, k)$ we must show that $\bar{s}_M \in CPSFE(D_{\bar{s},M}, C, m)$ for $M \in 2^N$ such that $2 \leq m < k$. Let $\hat{M} \in 2^N$ be a coalition such that $2 \leq \hat{m} < k$, and consider the industry $(D_{\bar{s},\hat{M}}, C, \hat{m})$. Note that $D_{\bar{s},\hat{M}}(p) = D(p) - (k - \hat{m})\bar{q}$, and $\rho_{\bar{s},\hat{M}} = \rho$. Therefore the unique Cournot equilibrium of the industry $(D_{\bar{s},\hat{M}}, C, \hat{m})$ is characterized by the equations

$$\bar{q}_{\bar{s},\hat{M}} + \left(\bar{p}_{\bar{s},\hat{M}} - C'(\bar{q}_{\bar{s},\hat{M}})\right) D'_{\bar{s},\hat{M}}(\bar{p}_{\bar{s},\hat{M}}) = 0, \tag{3.1}$$

and

$$\hat{m}\bar{q}_{\bar{s},\hat{M}} = D_{\bar{s},\hat{M}}(\bar{p}_{\bar{s},\hat{M}}). \tag{3.2}$$

Since $D'_{\bar{s},\hat{M}}(p) = D'(p)$, the system of equations (3.1) $-$ (3.2) is equivalent to $(\bar{C}1) - (\bar{C}2)$. Hence its unique solution is $(\bar{p}_{\bar{s},\hat{M}}, \bar{q}_{\bar{s},M}) = (\bar{p}, \bar{q})$. Thus, the induction hypothesis implies that $\bar{s}_{\hat{M}} \in CPSFE(D_{\bar{s},\hat{M}}, C, \hat{m})$.

In order to show that $\bar{s} \in CPSFE(D, C, n)$, we must prove that no other self-enforcing strategy s yields higher profits than $\bar{s}$ to all the firms. Assume by way of contradiction that there is a self-enforcing strategy s such that $\pi_i(s) > \pi_i(\bar{s})$ for $i \in N$. Then

$$\sum_{i \in N} \pi_i(s) > \sum_{i \in N} \pi_i(\bar{s}).$$

Note, however, that $s \in SFE(D, C, n)$ since it is self-enforcing, which contradicts Proposition 2.4. □

We now study conditions under which the Cournot equilibrium is the unique outcome that can be sustained by a CPSFE. Lemma 3.2 below plays an important role in the search for these conditions. It establishes that for every SFE one can find another SFE in which all but (perhaps) one firm supply inelastically the output of the firm with the greatest profits (and output) at the original equilibrium. Moreover, this new SFE leads to a price that is greater than or equal to the price associated to the original equilibrium. Consequently the profits of the firms using an inelastic supply are greater or equal to those of the firm with the greatest profits at the original equilibrium.

Let (D, C, n) be an industry. Consider the market outcomes that can be sustained by $s \in SFE(D, C, n)$ for which there are $j \in N$ and $v \in \mathbb{R}_+$, such that for each $i \in N \backslash \{j\}$, $s_i(p) = v$ for all $p \in [0, \rho]$. Writing $p(s) = u$, $s_j(u) = w$, and $s_j'(u) = \alpha$, this set is characterized by the vectors (u, v, w, α) in $\mathbb{R}_+^4$ satisfying the system of equations

$$w = -D'(u)\left(u - C'(w)\right), \tag{E.1}$$

$$v = \left(\alpha - D'(u)\right)\left(u - C'(v)\right), \tag{E.2}$$

$$(n-1)v + w = D(u). \tag{E.3}$$

In Lemma 3.2 below it is shown that fixing v (to be any real number corresponding to the production of a firm with the greatest production in an arbitrary SFE) the system $(E.1)$-$(E.3)$ has a unique solution, $(u(v), w(v), \alpha(v))$. Moreover, $\alpha' > 0$, and therefore that the function α is invertible. Hence the system $(E.1)$-$(E.3)$ implicitly defines a function $(u(\alpha), v(\alpha), w(\alpha))$ on $[0, \infty)$. This function is continuously differentiable on $(0, \infty)$.

For $\alpha \geq 0$ denote by $SFE_\alpha(D, C, n)$ the set of SFE of the form described above. Note that the market outcomes sustained by $s \in SFE_\alpha(D, C, n)$ are given by $p(s) = u(\alpha)$, $q_i(s) = v(\alpha)$ for $i \in N \backslash \{j\}$, and $q_j(s) = w(\alpha)$. Also note that for $\alpha = 0$ the system $(E.1)$-$(E.3)$ reduces to equations $\bar{C}1$ and $\bar{C}2$, and therefore $s \in SFE_0(D, C, n)$ if and only if the associated market outcome is the Cournot equilibrium; i.e., $(p(s), q_1(s), \ldots, q_n(s)) = (\bar{p}(D, C, n), \bar{q}(D, C, n), \ldots, \bar{q}(D, C, n))$.

Lemma 3.2. *Let (D, C, n) be an industry such that $C'''(q) \geq 0$ for $q > 0$, and let $\hat{s} \in SFE(D, C, n)$ and $i^* \in N$ be such that $\pi_{i*}(\hat{s}) \geq \pi_i(\hat{s})$ for all $i \in N$. Then there is $\tilde{\alpha} \geq 0$ and $\tilde{s} \in SFE_{\tilde{\alpha}}(D, C, n)$ such that $p(\tilde{s}) \geq p(\hat{s})$, and $\pi_{i^*}(\tilde{s}) \geq \pi_{i^*}(\hat{s})$.*

Proof: Let $\hat{s} \in SFE(D, C, n)$ and $i^* \in N$ be such that $\pi_{i^*}(\hat{s}) \geq \pi_i(\hat{s}) = \hat{\pi}_i$ for all $i \in N$. Without loss of generality assume that $i^* = 1$. Write $p(\hat{s}) = \hat{p}$, and for $i \in N$, write $q_i(\hat{s}) = \hat{q}_i$, $\hat{\alpha}_i = \hat{s}_i'(\hat{p})$, and $\hat{A}_i = \sum_{j \in N \backslash \{i\}} \hat{\alpha}_j$. Since $C'(\hat{q}_i) < \hat{p}$ and $\hat{\pi}_1 \geq \hat{\pi}_i$ for $i \in N$, then $\hat{q}_1 \geq \hat{q}_i$ for $i \in N$.

First we show that $D(\hat{p}) - (n-1)\hat{q}_1 > 0$. Since $\hat{s} \in SFE(D, C, n)$, for $i \in N$ we have

$$\hat{q}_i = \left(\hat{A}_i - D'(\hat{p})\right)(\hat{p} - C'(\hat{q}_i)),$$

by condition E_i. Hence for $i \in N$ we have

$$\begin{aligned}
\hat{q}_i &= \hat{q}_1 - \left(\hat{A}_1 - D'(\hat{p})\right)(\hat{p} - C'(\hat{q}_1)) + \left(\hat{A}_i - D'(\hat{p})\right)(\hat{p} - C'(\hat{q}_i)) \\
&= \hat{q}_1 + (\hat{\alpha}_1 - \hat{\alpha}_i)(\hat{p} - C'(\hat{q}_1)) + \left(\hat{A}_i - D'(\hat{p})\right)(C'(\hat{q}_1) - C'(\hat{q}_i)).
\end{aligned}$$

Thus, since $\hat{p}$ is a maximizes the profits of Firm 1 on the residual demand, we have $\hat{p} \geq C'(\hat{q}_1)$, and therefore

$$\begin{aligned}
D(\hat{p}) - (n-1)\hat{q}_1 &= \left(\hat{q}_1 + \sum_{i>1} \hat{q}_i\right) - (n-1)\hat{q}_1 \\
&= \hat{q}_1 + \sum_{i>1}\left((\hat{\alpha}_1 - \hat{\alpha}_i)(\hat{p} - C'(\hat{q}_1)) + \left(\hat{A}_i - D'(\hat{p})\right)(C'(\hat{q}_1) - C'(\hat{q}_i))\right) \\
&= \left(\hat{A}_1 - D'(\hat{p}) + (n-1)\hat{\alpha}_1 - \hat{A}_1\right)(\hat{p} - C'(\hat{q}_1)) \\
&\quad + \sum_{i>1}\left(\hat{A}_i - D'(\hat{p})\right)(C'(\hat{q}_1) - C'(\hat{q}_i)) \\
&= ((n-1)\hat{\alpha}_1 - D'(\hat{p}))(\hat{p} - C'(\hat{q}_1)) \\
&\quad + \sum_{i>1}\left(\hat{A}_i - D'(\hat{p})\right)(C'(\hat{q}_1) - C'(\hat{q}_i)) > 0.
\end{aligned}$$

Let $\check{p}$ be such that $D(\check{p}) - (n-1)\hat{q}_1 = 0$, and for $u \in [\hat{p}, \check{p}]$ define

$$\phi(u) = D(u) - (n-1)\hat{q}_1 + D'(u)(u - C'(D(u) - (n-1)\hat{q}_1)).$$

We establish below that the equation

$$\phi(u) = 0$$

has a unique solution on $[\hat{p}, \check{p}]$, $\tilde{u}$. Hence $(\tilde{u}, \tilde{w}, \tilde{\alpha})$, where

$$\tilde{w} = D(\tilde{u}) - (n-1)\hat{q}_1,$$

and

$$\tilde{\alpha} = D'(\tilde{u}) + \frac{\hat{q}_1}{\tilde{u} - C'(\hat{q}_1)}$$

is the unique solution to the system $(E.1)$-$(E.3)$ for $v^* = \hat{q}_1$. Moreover, $\tilde{u} \geq \hat{p}$ (because $\phi(\hat{p}) \geq 0$ and $\phi' < 0$). Further $\tilde{w} > 0$, and $\tilde{\alpha} \geq 0$. To see this note that $\tilde{u} \geq \hat{p}$ and $\hat{q}_1 \geq \hat{q}_i$ for $i > 1$ imply

$$\begin{aligned}\hat{q}_n &= D(\hat{p}) - \hat{q}_1 - \sum_{i=2}^{n-1} \hat{q}_i \\ &\geq D(\tilde{u}) - (n-1)\hat{q}_1 = \tilde{w}.\end{aligned}$$

And since C is increasing, we have $\tilde{u} - C'(\tilde{w}) \geq \hat{p} - C'(\hat{q}_n) > 0$, and therefore

$$\tilde{w} = -D'(\tilde{u})(\tilde{u} - C'(\tilde{w})) > 0.$$

Now, because C' is increasing, $\hat{q}_1 \geq \hat{q}_n \geq \tilde{w}$ implies

$$\tilde{\alpha} = D'(\tilde{u}) + \frac{\hat{q}_1}{\tilde{u} - C'(\hat{q}_1)} = \frac{\hat{q}_1}{\tilde{u} - C'(\hat{q}_1)} - \frac{\tilde{w}}{\tilde{u} - C'(\tilde{w})} \geq \frac{\hat{q}_1 - \tilde{w}}{\tilde{u} - C'(\tilde{w})} \geq 0.$$

Let s be a profile of supply functions satisfying for $i \in N \backslash \{n\}$, $\tilde{s}_i(p) = \hat{q}_1$ for $p \in [0, \rho]$, and $\tilde{s}_n(\tilde{u}) = \tilde{w}$, $\tilde{s}_n'(\tilde{u}) = \tilde{\alpha}$. The above construction implies $\tilde{s} \in SFE_{\tilde{\alpha}}(D, C, n)$. Moreover, $p(\tilde{s}) = \tilde{u} = u(\tilde{\alpha}) \geq \hat{p} = p(\hat{s})$, and

$$\pi_1(\tilde{s}) = p(\tilde{s})\tilde{s}_1(p(\tilde{s})) - C(\tilde{s}_1(p(\tilde{s}))) = p(\tilde{s})\hat{q}_1 - C(\hat{q}_1) \geq \hat{p}\hat{q}_1 - C(\hat{q}_1) = \pi_1(\hat{s}),$$

which establishes Lemma 3.2.

It remains to show that $\phi(u) = 0$ has a unique solution on $[\hat{p}, \check{p}]$. Since $\check{p} > \hat{p}$, $0 < \hat{p} - C'(\hat{q}_1) < \check{p} - C'(0)$, and therefore

$$\begin{aligned}\phi(\check{p}) &= D(\check{p}) - (n-1)\hat{q}_1 + D'(\check{p})\left(\check{p} - C'\left(D(\check{p}) - (n-1)\hat{q}_1\right)\right) \\ &= D'(\check{p})\left(\check{p} - C'(0)\right) < 0.\end{aligned}$$

We show that $\phi(\hat{p}) \geq 0$. Denote $\hat{q} = D(\hat{p}) - (n-1)\hat{q}_1$. Then we have

$$\begin{aligned}\phi(\hat{p}) &= D(\hat{p}) - (n-1)\hat{q}_1 + D'(\hat{p})\left(\hat{p} - C'(\hat{q})\right) \\ &= (n-1)\hat{\alpha}_1\left(\hat{p} - C'(\hat{q}_1)\right) + \sum_{i>1} \hat{A}_i\left(C'(\hat{q}_1) - C'(\hat{q}_i)\right) \\ &\quad -D'(\hat{p})\left(C'(\hat{q}) - C'(\hat{q}_1) + (n-1)C'(\hat{q}_1) - \sum_{i>1} C'(\hat{q}_i)\right).\end{aligned}$$

As noted above $\hat{p} \geq C'(\hat{q}_1)$; moreover, because C is a convex function and $\hat{q}_1 \geq \hat{q}_i$, we have $C'(\hat{q}_1) \geq C'(\hat{q}_i)$ for $i > 1$. Hence the first term in the above expression is non-negative; i.e.,

$$(n-1)\hat{\alpha}_1 (\hat{p} - C'(\hat{q}_1)) + \sum_{i>1} \hat{A}_i (C'(\hat{q}_1) - C'(\hat{q}_i)) \geq 0.$$

We show that the second term is also non-negative. Since $\hat{q}_1 \geq \hat{q}_i$ for $i > 1$, we have $\hat{q}_i = D(\hat{p}) - \sum_{j \neq i} \hat{q}_j \geq D(\hat{p}) - (n-1)\hat{q}_1 = \check{q}$ for $i > 1$. Thus, for $i > 1$ there is $\lambda_i \in [0,1]$ such that $\hat{q}_i = \lambda_i \hat{q}_1 + (1-\lambda_i)\check{q}$. Hence

$$\check{q} + (n-1)\hat{q}_1 = D(\hat{p}) = \hat{q}_1 + \sum_{i>1} \hat{q}_i = \hat{q}_1 + \sum_{i>1} (\lambda_i \hat{q}_1 + (1-\lambda_i)\check{q}).$$

Thus,

$$\hat{q}_1 - \check{q} = (\hat{q}_1 - \check{q}) \sum_{i>1} (1-\lambda_i),$$

and therefore either $\sum_{i>1}(1-\lambda_i) = 1$ or $\hat{q}_1 = \check{q}$. This equation together with our assumption that $C'''(q) \geq 0$ for $q > 0$, yields the inequality

$$\begin{aligned}
(n-1)C'(\hat{q}_1) - \sum_{i>1} C'(\hat{q}_i) &= (n-1)C'(\hat{q}_1) - \sum_{i>1} C'(\lambda_i \hat{q}_1 + (1-\lambda_i)\check{q}) \\
&\geq (n-1)C'(\hat{q}_1) - \sum_{i>1} \lambda_i C'(\hat{q}_1) + \sum_{i>1} (1-\lambda_i) C'(\check{q}) \\
&= (C'(\hat{q}_1) - C'(\check{q})) \sum_{i>1} (1-\lambda_i) \\
&= C'(\hat{q}_1) - C'(\check{q}).
\end{aligned}$$

Hence

$$-D'(\hat{p}) \left(C'(\check{q}) - C'(\hat{q}_1) + (n-1)C'(\hat{q}_1) - \sum_{i>1} C'(\hat{q}_i) \right) \geq 0.$$

Therefore $\phi(\hat{p}) \geq 0$.

The Intermediate Value Theorem implies that the equation $\phi(u) = 0$ has a solution on $[\hat{p}, \check{p}]$. Moreover, the solution is unique since $\phi' < 0$. □

Given an industry (D, C, n), define the function π on $[0, \infty)$ by

$$\pi(\alpha) = u(\alpha)v(\alpha) - C(v(\alpha)).$$

For $\alpha \geq 0$ and $s \in SFE_\alpha(D, C, n)$, the function $\pi(\alpha)$ provides the profits of the firms using an inelastic supply. Clearly the function π is well defined, and is continuously differentiable on $(0, \infty)$.

Lemma 3.3 establishes that if the function π of an industry (D, C, n) reaches a maximum at $\alpha = 0$, then the Cournot equilibrium is the unique outcome that can be sustained by a CPSFE. This fact is a straightforward implication of Theorem 3.1 and Lemma 3.2.

Lemma 3.3. *Let (D, C, n) be an industry such that $C'''(q) \geq 0$ for $q > 0$. If $\pi(0) > \pi(\alpha)$ for all $\alpha \in (0, \infty)$, then*

$$(p(s), q_1(s), \ldots, q_n(s)) = (\bar{p}(D, C, n), \bar{q}(D, C, n), \ldots, \bar{q}(D, C, n))$$

for all $s \in CPSFE(D, C, n)$.

Proof: Let $\hat{s} \in SFE(D, C, n)$ be such that $p(\hat{s}) \neq \bar{p}(D, C, n) = \bar{p}$. We show that $\hat{s} \notin CPSFE(D, C, n)$. Assume, without loss of generality, that $\pi_1(\hat{s}) \geq \pi_i(\hat{s})$ for $i \in N$. By Lemma 3.2 there is $\tilde{\alpha} \geq 0$ and $\tilde{s} \in SFE_{\tilde{\alpha}}(D, C, n)$ such that $\pi_1(\tilde{s}) \geq \pi_1(\hat{s})$. Moreover, $p(\hat{s}) \neq \bar{p}$ implies $\tilde{\alpha} > 0$. Let $\bar{s} \in CPSFE(D, C, n)$ be such that $(p(\bar{s}), q_1(\bar{s}), \ldots, q_n(\bar{s})) = (\bar{p}, \bar{q}, \ldots, \bar{q})$. (The strategy profile $\bar{s}$ exists by Theorem 3.1.) Since $\bar{s} \in SFE_0(D, C, n)$, for $i \in N$ we have

$$\pi_i(\bar{s}) = \pi(0) > \pi(\tilde{\alpha}) = \pi_1(\tilde{s}) \geq \pi_1(\hat{s}) \geq \pi_i(\hat{s}).$$

Hence all firms benefit by jointly deviating to the strategy profile $\bar{s}$. Since $\bar{s} \in CPSFE(D, C, n)$, then $\bar{s}$ is a self-enforcing and improving deviation, and therefore $\hat{s} \notin CPSFE(D, C, n)$. □

In what follows we study conditions under which the assumption on the function π of Lemma 3.3 holds. Interestingly, in some cases it is possible to directly calculate the function π and to check whether this assumption holds. Following this approach we are able to establish in Proposition 3.4 that in a *linear industry*, the assumptions of Lemma 3.3 hold whenever there are three or more firms in the industry. A *linear industry* is described by a linear demand function, (i.e., $D(p) = a - bp$, for $p \in [0, \frac{a}{b}]$, where $a, b \in \mathbb{R}_+$), and a linear cost function (i.e., $C(q) = cq$, for $q \geq 0$, where $c \in \mathbb{R}_+$).

Our assumption that $C'(0) < \rho$ implies $c < \frac{a}{b}$. Thus, a linear industry is described by the parameters a, b, c, and n.

Proposition 3.4. *Let (D, C, n) be a linear industry. If $n \geq 3$, then*

$$(p(s), q_1(s), \ldots, q_n(s)) = (\bar{p}(D, C, n), \bar{q}(D, C, n), \ldots, \bar{q}(D, C, n))$$

for all $s \in CPSFE(D, C, n)$.

Proof: The solution to the system $(E.1)$-$(E.3)$ yields

$$\pi(\alpha) = \frac{(\alpha + b)(a - bc)^2}{((n-1)\alpha + (n+1)b)^2}.$$

It is easy to see that whenever $n \geq 3$ we have $\pi'(\alpha) < 0$ on $(0, \infty)$. Hence $\pi(0) > \pi(\alpha)$ for $\alpha \in (0, \infty)$, whenever $n \geq 3$, and therefore Proposition 3.4 follows from Lemma 3.3. □

The conclusion of Proposition 3.4 does not hold when there are only two firms in the industry; in this case the Stackelberg equilibria also can be sustained by CPSFE. In turns out that in a linear industry, the presence of three firms reduces the set of market outcomes that can be sustained by CPSFE to just the Cournot Equilibrium. This conclusion extends to every industry (D, C, n) with a quadratic cost function and linear demand (see Proposition 3.7 below). Nevertheless, as Example 3.6 shows, there are industries with three firms for which market outcomes other than Cournot can be sustained by CPSFE. Moreover, under appropriate conditions on the functions D and C, the number

$$n^* = 2 + \frac{D'(\bar{p}) + D''(\bar{p})(\bar{p} - C'(\bar{q}))}{D'(\bar{p})(1 - D'(\bar{p})C''(\bar{q}))}$$

provides a threshold on the number of firms that guarantees that if $n \geq n^*$, then the Cournot equilibrium is the unique outcome that can be sustained by a CPSFE.

Remark 3.5. Note that $n^* > 2$. Moreover, if the industry's demand is linear, then $n^* \leq 3$, and since n is an integer $n \geq n^*$ if and only if $n \geq 3$.

Example 3.6: Consider the industry $(D, C, 3)$ where $C(q) = 0$ for $q \in \mathbb{R}$, and $D(p) = 1 - p^2$ for $p \in [0, 1]$. From the solution to system $(E.1)$-$(E.3)$ one can compute the function π to obtain

$$\pi(\alpha) = \frac{1}{49}(\sqrt{(\alpha^2+7)} - \alpha)\left(2 + \frac{3\alpha}{7}(\sqrt{(\alpha^2+7)} - \alpha)\right)$$

This function has a maximum at $\alpha^* = \frac{1}{3}$. Let $s \in SFE_{\frac{1}{3}}(D,C,3)$. In this SFE, two of the firms produce $\frac{1}{3}$ and the remaining firm produces $\frac{2}{9}$. We show $s \in CPSFE(D,C,3)$. Since $\pi(\frac{1}{3}) \geq \pi(\alpha)$ for $\alpha \geq 0$, by Lemma 3.2 s is not Pareto dominated by any other market equilibrium. Hence the coalition of all three firms does not have an improving deviation. In addition, the two firms producing $\frac{1}{3}$ are in fact producing the Cournot equilibrium quantity of the (two-firm) "residual industry," and therefore by Theorem 3.1 this coalition does not have an improving deviation either. Nor does have an improving deviation a coalition formed by a firm producing $\frac{1}{3}$ and the firm producing $\frac{2}{9}$, since these productions correspond to a Stackelberg equilibrium of their residual industry. Hence s is a CPSFE. Note that for this industry

$$n^* = 2 + \frac{2\bar{p} + 2\bar{p}}{2\bar{p}} = 4.$$

Indeed if $n \geq 4$ then the Cournot equilibrium becomes the unique outcome that can be sustained by a CPSFE.

Proposition 3.7 establishes conditions more general than those of Proposition 3.4 under which the Cournot equilibrium is the unique coalition-proof supply function equilibrium of an industry.

Proposition 3.7. *Let (D,C,n) be an industry such that $n \geq n^*$. If*

(3.7.1) $C''(q) = 0$ for $q > 0$ and $D'''(p) \leq 0$ for $p \in (0,\rho)$, or

(3.7.2) $C'''(q) = 0$ for $q > 0$ and $D''(p) = 0$ for $p \in (0,\rho)$,

then

$$(p(s), q_1(s), \ldots, q_n(s)) = (\bar{p}(D,C,n), \bar{q}(D,C,n), \ldots, \bar{q}(D,C,n))$$

for all $s \in CPSFE(D,C,n)$.

Proof: It is shown that under the assumptions of the proposition we have $\frac{\partial^2 w}{\partial u^2} \geq 0$. Hence the condition $(A.1)$ of Lemma A (see the Appendix) is satisfied and therefore $\pi'(\alpha) < 0$ for $\alpha \in (0,\infty)$. This in turns implies that $\pi(0) > \pi(\alpha)$ for $\alpha \in (0,\infty)$. The conclusion of Proposition 3.7 then follows from Lemma 3.3.

Suppose that (3.7.1) holds. Implicit differentiation of $E.1$ yields

$$\frac{\partial w}{\partial u} = -D'(u) - D''(u)\,(u - C'(w))\,,$$

and therefore

$$\frac{\partial^2 w}{\partial u^2} = -2D''(u) - D'''(u)\,(u - C'(w)) \geq 0.$$

Suppose that (3.7.2) holds. Then again differentiation of $E.1$ we have

$$\frac{\partial w}{\partial u} = \frac{-D'(u)}{1 - C''(w)D'(u)},$$

and therefore

$$\frac{\partial^2 w}{\partial u^2} = 0. \ \square$$

Lemma A of the Appendix provides alternative conditions implying that the function π is decreasing $(0, \infty)$. Checking these conditions directly allows us to establish the conclusion of Proposition 3.7 for a quadratic industry (Example 3.8), and for an industry where the cost function is quadratic and the demand is a polynomial of third order (Example 3.9). These examples are outside the scope of Proposition 3.7.

Example 3.8. Let $(D, C, 3)$ be an industry where $C(q) = \frac{1}{2}q^2$ for $q \geq 0$, and $D(p) = 1 - p^2$ for $p \in [0, 1]$. For this industry, the Cournot equilibrium is the unique outcome that can be sustained by a CPSFE. We show that $(A.1)$ of Lemma A holds; hence that $\pi'(\alpha) < 0$ for $\alpha > 0$, and therefore this result follows from Lemma 3.3.

Let $\alpha > 0$. From $E.1$ we have

$$w = 2u\,(u - w) = 2\frac{u^2}{1 + 2u}.$$

Hence

$$\frac{\partial^2 w}{\partial u^2} = \frac{4}{(1 + 2u)^3} > 0.$$

Example 3.9. Let $(D, C, 3)$ be an industry where $C(q) = \frac{1}{2}q^2$ for $q \geq 0$, and $D(p) = \frac{1}{3}(27 - p^3)$, for $p \in [0, 3]$. The Cournot equilibrium for this industry is $\bar{p} = 2.1977$, $\bar{q} = 1.8206$. Implicit differentiation of $E.1$ yields (see the Appendix)

$$\frac{\partial^2 w}{\partial u^2}(\bar{p}) = \frac{2\bar{p}(3 - \bar{p}^2)}{(1 + \bar{p}^2)^3} < 0;$$

hence $(A.1)$ of Lemma A does not hold. Nonetheless, it is easy to prove that $(A.2)$ holds. Write

$$\gamma(\alpha) = \frac{-1}{n-1}\left((n-1)\,\alpha - (n-2)\,D'(u) - \frac{\partial w}{\partial u}\right) = \frac{-1}{2}\left(2\alpha - D'(u) - \frac{\partial w}{\partial u}\right).$$

We show $\gamma(\alpha) \leq 0$. One can compute $\gamma\,(0)$ (see the Appendix) to obtain

$$\gamma\,(0) = D'(\bar{p})(n - n^*).$$

As

$$n = 3 > n^* = 2 + \frac{\bar{p}^2 + 2\bar{p}\,(\bar{p} - \bar{q})}{\bar{p}^2\,(1 + \bar{p}^2)} = 2.2304,$$

we have $\gamma\,(0) < 0$. Hence it suffices to show that $\gamma'(\alpha) < 0$. We have

$$\gamma'(\alpha) = -\,(1 - D''(u)u'(\alpha)) + \frac{1}{n-1}\left(-D''(u) + \frac{\partial^2 w}{\partial u^2}\right)u'(\alpha),$$

and since $u'(\alpha) < 0$, then $-D''(u) + \frac{\partial^2 w}{\partial u^2} > 0$ implies $\gamma'(\alpha) < 0$. Here we have

$$-D''(u) + \frac{\partial^2 w}{\partial u^2} = 2u\left(1 + \frac{3 - u^2}{(1 + u^2)^3}\right) > 0.$$

4 Appendix

The following lemma provides alternative conditions that ensure that the function π has a unique maximum on $\alpha = 0$.

Lemma A. *Let (D, C, n) be an industry. Assume that for each $\alpha \in (0, \infty)$ either*

(A.1) $n \geq n^*$ and $\frac{\partial^2 w}{\partial u^2} \geq 0$, or

(A.2) $(n-1)\,\alpha - (n-2)\,D'(u) - \frac{\partial w}{\partial u} > 0$.

Then $\pi'(\alpha) < 0$ for each $\alpha \in (0, \infty)$.

Proof: Let $\alpha > 0$. From the definition of π we have

$$\pi'(\alpha) = u'\,(\alpha)\,v(\alpha) + (u\,(\alpha) - C'(v))\left(\frac{\partial v}{\partial u}u'(\alpha) + \frac{\partial v}{\partial \alpha}\right).$$

From $E.1$ we have

$$\frac{\partial w}{\partial u} = \frac{-D'(u) - D''(u)\,(u - C'(w))}{1 - C''(w)D'(u)} > 0.$$

From $E.2$ we get

$$\frac{\partial v}{\partial \alpha} = \frac{(u - C'(v))}{1 + C''(v)(\alpha - D'(u))} > 0,$$

and

$$\frac{\partial v}{\partial u} = \frac{\alpha - D'(u) - D''(u)(u - C'(v))}{1 + C''(v)(\alpha - D'(u))} > 0.$$

From $E.3$ we have

$$u'(\alpha) = \frac{(n-1)\frac{\partial v}{\partial \alpha}}{D'(u) - (n-1)\frac{\partial v}{\partial u} - \frac{\partial w}{\partial u}} < 0.$$

Substituting in the expression of $\pi'(\alpha)$ we have

$$\begin{aligned}
\pi'(\alpha) &= u'(\alpha)\left(v(\alpha) + (u(\alpha) - C'(v(\alpha))\frac{1}{n-1}\left(D'(u(\alpha)) - \frac{\partial w}{\partial u}\right)\right) \\
&= u'(\alpha)(u(\alpha) - C'(v(\alpha)))\left(\alpha - D'(u(\alpha)) + \frac{1}{n-1}\left(D'(u(\alpha)) - \frac{\partial w}{\partial u}\right)\right) \\
&= \beta(\alpha)\gamma(\alpha),
\end{aligned}$$

where

$$\beta(\alpha) = -u'(\alpha)(u(\alpha) - C'(v(\alpha))) > 0, \qquad (3.\beta)$$

and

$$\begin{aligned}
\gamma(\alpha) &= -\left(\alpha - D'(u(\alpha)) + \frac{1}{n-1}\left(D'(u(\alpha)) - \frac{\partial w}{\partial u}\right)\right) \qquad (3.\gamma) \\
&= \frac{-1}{n-1}\left((n-1)\alpha - (n-2)D'(u(\alpha)) - \frac{\partial w}{\partial u}\right).
\end{aligned}$$

This establishes the conclusion of Lemma A under $(A.2)$.

Note that

$$\begin{aligned}
(n-1)\gamma(0) &= (n-2)D'(\bar{p}) - \frac{D'(\bar{p}) + D''(\bar{p})(\bar{p} - C'(\bar{q}))}{1 - C''(\bar{q})D'(\bar{p})} \\
&= D'(\bar{p})\left(n - 2 - \frac{D'(\bar{p}) + D''(\bar{p})(\bar{p} - C'(\bar{q}))}{D'(\bar{p})(1 - C''(\bar{q})D'(\bar{p}))}\right) \\
&= D'(\bar{p})(n - n^*) \le 0.
\end{aligned}$$

We establish Lemma A under $(A.1)$ by showing that $\gamma'(\alpha) < 0$ for $\alpha \in (0, \infty)$. We have

$$\begin{aligned}
\gamma'(\alpha) &= \frac{1}{n-1}\left(-(n-1) + (n-2)D''(u)u'(\alpha) + \frac{\partial^2 w}{\partial u^2}u'(\alpha)\right) \\
&= -(1 - D''(u)u'(\alpha)) + \frac{1}{n-1}\left(-D''(u) + \frac{\partial^2 w}{\partial u^2}\right)u'(\alpha).
\end{aligned}$$

Since $\frac{\partial^2 w}{\partial u^2} \geq 0$ and $u'(\alpha) < 0$, we have (recall that D is a concave function)

$$\left(-D''(u) + \frac{\partial^2 w}{\partial u^2}\right) u'(\alpha) \leq 0.$$

Hence in order to prove that $\gamma'(\alpha) < 0$ it suffices to show that $1 - D''(u)u'(\alpha) > 0$. As $u'(\alpha) < 0$, $E.3$ implies

$$0 < D'(u)u'(\alpha) = (n-1)\left(\frac{\partial v}{\partial u}u'(\alpha) + v'(\alpha)\right) + \frac{\partial w}{\partial u}u'(\alpha).$$

Since

$$\frac{\partial w}{\partial u}u'(\alpha) < 0,$$

the above inequality implies

$$\frac{\partial v}{\partial u}u'(\alpha) + v'(\alpha) > 0.$$

Since

$$v'(\alpha) = \frac{(u - C'(v))}{1 + C''(v)(\alpha - D'(u))} > 0,$$

we have

$$\begin{aligned}
\frac{\partial v}{\partial u}u'(\alpha) + v'(\alpha) &= \left(\frac{\alpha - D'(u) - D''(u)(u - C'(v))}{1 + C''(v)(\alpha - D'(u))}\right) u'(\alpha) + v'(\alpha) \\
&= \left(\frac{\alpha - D'(u)}{1 + C''(v)(\alpha - D'(u))}\right) u'(\alpha) + (1 - D''(u)u'(\alpha))\, v'(\alpha) > 0.
\end{aligned}$$

The first term in the right hand side of the above expression is negative. Hence the second term must be positive. Since $v'(\alpha) > 0$, we must have $1 - D''(u)u'(\alpha) > 0$. Therefore $\gamma'(\alpha) < 0$, and since $\gamma(0) \leq 0$, we have $\gamma(\alpha) < 0$ for $\alpha > 0$. □

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