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SECOND-PRICE COMMON-VALUE AUCTIONS UNDER MULTIDIMENSIONAL UNCERTAINTY

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Abstract

The literature has demonstrated that second-price common-value auctions are sensitive to the presence of asymmetries among bidders. Bikhchandani (1988) has shown that if it is common knowledge that a bidder has a disadvantage compared to her opponent, this bidder (almost surely) never wins the auction. This paper is the first to show that this result does not carry through when one allows for two-sided uncertainty. We show that even if the probabilities that one of the bidders is advantaged while the other one is disadvantaged are arbitrarily large, in every equilibrium, the disadvantaged bidder needs to win the auction with strictly positive probability. We then solve for the equilibria in two cases (one with two types and another with a continuum of types) and we characterize their expected revenues properties. We find that although they underperform relative to "comparable" symmetric auctions, they perform much better than what it is usually "assumed" in the literature.

Keywords: Common-value auctions, asymmetric bidders, mergers and acquisitions, privatization, liquidity constraints.

JEL Classification D44, D82, G34.

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1 Introduction

Auction participants are not unlike all other economic agents: they do not all come at the same size, and they do not all share the same aspirations. They do not necessarily intend to (or are not able to) use the object that is auctioned off in the same way, or they do not have the same financial backing to ensure that funding considerations do not affect them differently. These considerations come immediately into one's mind when, for example, the object being auctioned off is a failed firm or a state-owned firm about to be privatized.

Auction theory, for reasons of technical convenience, has mainly focused on symmetric environments. Nonetheless, attempts to extend the analysis to asymmetric environments have shown that most major insights derived in the symmetric case do not continue to hold. Perhaps, the most celebrated, violent reversal of fortune has been delivered in the analysis of single-object, second-price, common-value auctions. Bikhchandani (1988) has demonstrated that if in such an auction it is common knowledge that one of the bidders has a disadvantage compared to her opponent, even a minor one, this bidder (almost surely) never wins the auction. As a result, second-price, common-value auctions may be considerably less desirable in asymmetric environments compared to symmetric ones; e.g., in terms of expected revenues to the seller.¹

This paper is an attempt to further our understanding on the properties of asymmetric second-price, common-value auctions. It departs from the existing literature by introducing two-sided uncertainty, i.e., it analyzes the case in which both bidders' types are private information. This departure affects considerably the outcomes derived in Bikhchandani (1988). Even when one bidder is with a large probability (always bounded away from 1) disadvantaged while the other one is with an equally large probability advantaged, in equilibrium, the disadvantaged bidder has to win the auction with strictly positive probability. This is true even in the states of nature that her opponent is actually advantaged while she is disadvantaged.

This result restores the "common sense" intuition that a small advantage should not have an enormous impact on outcomes. It is actually shown in two examples (one with two types and another with a continuum of types) that the expected revenues generated in these

¹Milgrom and Weber (1982) have shown that in symmetric common-value environments second-price auctions, by reducing the winner's curse, generate larger expected revenues than first-price auctions do. Bikhchandani (1988) and Bulow and Klemperer (1997) argue that this result does not carry over when asymmetries are introduced.

asymmetric environments are not much smaller than the ones generated in “comparable” symmetric environments. Hence, one should put into a more rigorous examination the widely held belief that first-price auctions should be preferred in the presence of asymmetries (e.g., Klemperer (1997)).

Examples in which the informational structure of asymmetries among bidders fits into our description abound. Rival firms competing for the acquisition of a target firm in a takeover contest, of a failed firm under liquidation, or of a state-owned firm about to be privatized are obvious examples. In all those cases, the market value of the assets of the target firm is common to all bidders, but at the same time, each of them may have an additional private source of gains due to synergies between them and the target. It is highly unlikely that all these additional interests the contestants have are public information. Asymmetries among bidders can also be generated by the presence of liquidity constraints.² Firms that operate within imperfect capital markets face different costs of raising the amount of cash needed for their bids. Differences in retained earnings, in values of assets appropriate for collateral, or, more generally, in access to external finance may easily cause asymmetries among bidders. This last one is the source of asymmetries we will concentrate on.

The paper is organized as follows. The next section sets up the model. Section 3 delivers the general results in the two-type case. Moreover, it explicitly solves for a type-asymmetric equilibrium (the only class of admissible candidate equilibria) in an example. Section 4 discusses the continuum-of-types case and shows the existence of pure type-symmetric equilibria. Finally, Section 5 concludes.

2 The Model

An auctioneer, A , organizes an auction to sell an object, say a failed firm. There are 2 risk-neutral bidders, $i \in \{I, II\}$, competing to acquire it.

We assume that the value of the object is common to both bidders, and equals to $v \in [\underline{v}, \bar{v}] \subset \mathbb{R}_+$. This value is unknown. Each bidder though receives a private signal x_i , such that $x_I + x_{II} = v$.³ The signals are assumed to be independently and identically distributed (i.i.d.) conditional on v according to a cumulative distribution function F . The associated

²See, for instance, Che and Gale (1998).

³The normalization that the value of the object is the sum of the signals is only made for convenience, both notational and computational. No results rely on it.

density function, f , is assumed to be absolutely continuous and strictly positive over its closed and bounded support $[0, \bar{x}] \equiv X \subset \mathbb{R}_+$.⁴

Moreover, each bidder receives another private signal $\theta_i \in [\theta_A, \theta_D] \equiv \Theta \subset \mathbb{R}_{++}$, i.i.d. across bidders, distributed according to G , with absolutely continuous and strictly positive over its support density function, g . This signal contains private information regarding bidder i 's marginal utility of income. Finally, there is no correlation between the two signals.

Bidders' utility is $\hat{U}_i = v - \theta_i p_i$. Nonetheless, instead of understanding the auction as one of pure common-values in which bidders differ in their marginal utility of income, it is convenient to analyze it as one in which the bidders have identical marginal utilities of income, but they value the object differently. In this case, bidders' utility can be rewritten as

$$U_i = \frac{v}{\theta_i} - p_i.$$

From now on, we will refer to θ_i as the type of bidder i and to x_i as the signal she has received (i.e., a bidder who has observed the pair (x, θ) will be referred to hereafter as a θ -bidder with signal x).

The equilibrium notion we employ is stronger than that of the standard Bayesian-Nash equilibrium. In addition, we eliminate weakly dominated strategies. This restriction implies that we impose on bidders to bid at least their individually rational bid, i.e., $b_\theta(x) \geq \frac{x}{\theta}$.⁵

3 Disadvantaged Bidders Are Not Doomed

In this section, we consider that θ is distributed according to a two-point distribution $\Theta \equiv \{\theta_A, \theta_D\}$. Bidder i receives signal θ_A with probability μ_i and θ_D with the complementary probability. There exists a $\delta > 0$, such that $\mu_i > \delta$, $\forall i$; i.e., there is always uncertainty about both bidders' θ . It is important to notice that our framework encompasses the case in which one bidder receives signal θ_A with probability close to 1, while the other one receives signal θ_D with a similarly large probability (always bounded away from 1 by δ).

⁴The lower bound of the support is chosen to be zero to relieve notational burden. The extent to which this normalization affects the results will be explained as we proceed.

⁵Bikhchandani (1988) employs the same restriction to rule out unreasonable equilibria.

3.1 General Results

The most striking result in the literature on asymmetric, second-price, common-value auctions is that a bidder who has a disadvantage compared to her opponent, even a minor one, (almost surely) never wins the auction (e.g., Bikhchandani (1988) and Bulow and Klemperer (1997)). Bikhchandani actually shows that this result holds even if the probability that one bidder is more advantaged than the other is vanishingly small. In this section, we show that this result is not robust to the introduction of two sided-uncertainty about the type of the two bidders. Specifically, we show that even if the probabilities that one of the bidders is advantaged and the other disadvantaged are arbitrarily large (always bounded away from 1), it is possible that the disadvantaged bidder wins the auction. It should be noted that she wins the auction with strictly positive probability even in the state of nature in which her opponent is advantaged. The following proposition states formally the result.⁶

Proposition 1 *In every equilibrium in non-decreasing, pure strategies of this auction a D-bidder must win over an A-bidder with strictly positive probability.*

Proof: Assume not. In such case, in equilibrium, a D-bidder wins with zero probability over an A-bidder, and therefore an A-bidder should assign zero probability to the event that she ties with a D-bidder. Concentrate first on type-symmetric strategies. In such case, an A-bidder must bid in equilibrium $b_A(x) = \frac{2x}{\theta_A}$; i.e., the strategies derived in Milgrom (1981). This is due to the fact that an A-bidder assigns probability equal to 1 to the event that her opponent is an A-bidder conditional on tying. But then, in such equilibrium, for a D-bidder to never win, she must bid $b_D(x) = 0$. Consider a D-bidder with signal \bar{x} . The bid coming out of her individually rational strategy is $\frac{\bar{x}}{\theta_D}$. By bidding it, she wins whenever $\frac{2x}{\theta_A} < \frac{\bar{x}}{\theta_D}$. Take $y = \frac{\bar{x}\theta_A}{2\theta_D} > 0$. By bidding her individually rational bid, this D-bidder wins against all A-bidders with signals $x \in [0, y)$, while making positive profits conditional on winning. (The true value to her is $\frac{\bar{x}+x}{\theta_D}$, whereas she pays no more than $\frac{\bar{x}}{\theta_D}$.) Since this deviation is profitable it contradicts equilibrium strategies.

⁶The normalization that the lower bound in the support of x is zero allows us to make the proposition statement unconditionally. Otherwise, it would have been necessary to assume that $\frac{\bar{x}+x}{\theta_D} > \frac{2x}{\theta_A}$. That is, that the individually rational bid of the least advantaged bidder when receiving the highest possible signal is larger than the individually rational signal of the most advantaged bidder when receiving the lowest possible signal. This is nothing more than the weakest condition needed in an auction to ensure the presence of real competition among bidders. This condition is trivially satisfied when $\underline{x} = 0$.

Consider now type-asymmetric strategies. The proof is split into two cases: (a) We first show that there exists no equilibrium strategy in which one of the A-bidders, say A-I-bidder, always wins. For an equilibrium to have this property, there must be an equilibrium strategy for the A-II-bidder such that (i) A-I-bidder always wins against A-II, and (ii) D-I-bidder never wins against A-II. If strategies are strictly increasing, then because of (i) we have that $b_{A-II}(x) \leq \frac{\bar{x}}{\theta_A}$, $\forall x$. Otherwise, an A-I-bidder with signal 0 would regret winning. Because of (ii), $b_{A-II}(x) \geq \frac{\bar{x}}{\theta_D}$, $\forall x$. Otherwise, a D-I-bidder with signal \bar{x} would deviate upwards and win profitably. Since equilibrium strategies are assumed to be strictly increasing, (i) and (ii) contradict each other. Hence, A-II-bidder equilibrium strategy needs to have a flat interval. Now, assume that A-II-bidder's strategy has a flat interval, and hence takes the following form: $b_{A-II}(x) = \frac{\bar{x}}{\theta_D} + T$, $\forall x \in [m, M]$, where m and M are two scalars and $T \geq 0$. Clearly, $T > 0$, otherwise a D-I-bidder with signal \bar{x} would deviate and win profitably. A necessary condition for an A-I-bidder with signal 0 to want to outbid all A-II-bidders who bid according to this flat strategy is that the expected value when winning is larger than the payment; i.e., $\frac{m+M}{2\theta_A} \geq \frac{\bar{x}}{\theta_D} + T$. Nonetheless, the individually rational bid of an A-II-bidder with signal M is $b_{A-II}(M) \geq \frac{M}{\theta_A}$. Hence, a necessary condition for the flat equilibrium strategy to be individually rational is $\frac{\bar{x}}{\theta_D} + T \geq \frac{M}{\theta_A}$. Combining the two necessary conditions we get $\frac{m+M}{2\theta_A} \geq \frac{M}{\theta_A}$; a contradiction. Therefore, equilibrium strategies for an A-II-bidder cannot have a flat interval either.

(b) We now show that there does not exist an equilibrium in which A-I- and A-II-bidders tie among themselves while D-bidders always lose. Assume it does. If an A-I-bidder with signal ϵ ties with some A-II-bidder with signal M while D-bidders always lose, Bertrand competition among them implies that $b_{A-I}(\epsilon) = \frac{\epsilon+M}{\theta_A} \geq \frac{\bar{x}}{\theta_D}$. As $\epsilon \rightarrow 0$, $\frac{M}{\theta_A} \geq \frac{\bar{x}}{\theta_D}$ which implies that $M \geq \frac{\theta_A}{\theta_D} \bar{x} > 0$. Therefore, $M \not\rightarrow 0$. Take an A-II-bidder with signal $m < M$ who ties with an A-I-bidder with signal r . In a tie, $\frac{m+r}{\theta_A} \geq \frac{\bar{x}}{\theta_D}$. As $m \rightarrow 0$, $r \geq \frac{\theta_D}{\theta_A} \bar{x} > \epsilon$; which contradicts increasing strategies. So if an equilibrium exists, the bidding strategies must have flat intervals. Employing similar arguments as in the previous case one can show that this is not possible. Q.E.D.

The intuition behind the result is the following: An advantaged bidder cannot bid very aggressively because she knows that with positive probability she may be playing against an equally advantaged bidder. If both are aggressive, they end up regretting when winning. Their reluctance, on the other hand, makes the disadvantaged bidders less cautious, and as a result the extreme situation analyzed by Bikhchandani (1988) is not obtained. Disadvantaged bidders need to win with strictly positive probability in any equilibria of the auction.

The next proposition further characterizes the set of admissible equilibria in this auction. Namely, it shows that there are no type-symmetric equilibria.

Proposition 2 *There exist no type-symmetric equilibria in increasing pure strategies.*

Proof: We first show that if type-symmetric equilibria exist, they must be in continuous increasing pure strategies. Suppose not. Remember that by Proposition 1, a D-bidder cannot always lose. Assume w.l.o.g. that an A-bidder's equilibrium strategy has a gap at M . Then, this would imply that an A-bidder with signal $M - \epsilon$ would be willing to lose over a positive mass of D-bidders whom the A-bidder with signal M is willing to beat. (It is obvious that such discontinuity could not have occurred at the range of signals in which the A-bidders win with probability 1.) But the expected values of the object conditional of winning for the two A-bidders converge to each other as $\epsilon \rightarrow 0$; a contradiction.

In any equilibrium, an A-bidder with signal \bar{x} must win with probability 1. Since by Proposition 1, a D-bidder cannot always lose, and since strategies are assumed to be increasing, there exists a signal $M < \bar{x}$ such that $b_A(M) = b_D(\bar{x})$. Now, the unique equilibrium strategy for an A-bidder with signal greater than M is $b_A(x) = \frac{2x}{\theta_A}$, $\forall x > M$. Since strategies are assumed to be increasing and pure, and we have shown that they must be continuous, it must be that $b_A(M) = b_D(\bar{x}) = \frac{2M}{\theta_A}$.

Consider an A-bidder with signal $M - \epsilon$. Given that strategies are increasing, by following the equilibrium strategy she is losing against an A-bidder with signal M and a D-bidder with signal \bar{x} . By deviating and bidding $\frac{2M}{\theta_A}$, she pays at most her bid, while the true value is $\frac{1}{\theta_A} \{(M - \epsilon) + (\mu_i \bar{x} + (1 - \mu_i)M)\}$. Since μ_i is bounded by $\delta > 0$ and $M < \bar{x}$, as $\epsilon \rightarrow 0$, such deviation is profitable. Q.E.D.

Nonetheless, there exist type-asymmetric equilibria in such auction. In the next subsection, we provide an example in which we fully characterize a pure-strategy, type-asymmetric equilibrium, and we discuss its properties.

3.2 An Example

Consider the following example: x_i 's are uniformly drawn from $[0, 1]$, $\theta_A = 1$, $\theta_D = 2$, and $\mu_i = 1/2$. The following bidding strategies constitute an equilibrium in which a D-bidder has a strictly positive probability of winning over an A-bidder.

$$I) \quad b_{A-I}(x) = \begin{cases} x + 1/2 & x \in [0, 3/4) \\ 2x & x \in [3/4, 1] \end{cases} \quad b_{D-I}(x) = \begin{cases} x & x \in [0, 1/2) \\ x + 1/2 & x \in [1/2, 1] \end{cases}$$

$$II) \quad b_{A-II}(x) = \begin{cases} 1/2 & x \in [0, 1/2) \\ 3/2 & x \in [1/2, 3/4) \\ 2x & x \in [3/4, 1] \end{cases} \quad b_{D-II}(x) = \begin{cases} x & x \in [0, 1/2) \\ 1/2 & x \in [1/2, 1] \end{cases}$$

Before showing that these strategies constitute an equilibrium, it is worth noting some interesting properties of the equilibrium. First, a D-bidder has a strictly positive probability of winning the auction over an A-bidder. Just notice that a D-I-bidder with signal $x \in [1/2, 1]$ beats an A-II-bidder with signal $x \in [0, 1/2)$.

When a bidder ties with her opponent, and according to the equilibrium she assigns zero probability to the event that she tied with an opposite type, her strategy is part of what would have been a symmetric equilibrium in an auction where (a) it is common knowledge that both bidders are of her type, and (b) the signal space is truncated to contain only the signals in which by following the above specified strategies they would have never tied with a bidder of the opposite type (i.e., when $x \in [3/4, 1]$ for the A-bidders, and when $x \in [0, 1/2)$ for the D-bidders).⁷

We now sketch the arguments used to solve for the equilibrium. As it is already noted, the bidders play their "symmetric equilibrium strategies" when they assign zero probability to the event that they have tied with an opposite type bidder. Although it is quite obvious that there exist values of x for which such situation arises for the A-bidders, this is not the case for the D-bidders. For this to be true, it is necessary that both A-bidders bid aggressively enough so that a D-bidder would never tie with them. But if both of them bid aggressively, they may end up regretting winning. If an A-II-bidder bids aggressively, as in the above specified strategies, to win over the D-I-bidder with signal less than one-half, an A-I-bidder with signal 0 may regret winning. This is not the case because although she regrets winning over all A-II-bidders with signals less than one-half she is compensated for these losses by the gains she makes by winning over all D-II-bidders with signals greater than one-half. Given that both A-II- and D-II-bidders submit the same bid (1/2), an A-I-bidder cannot find an alternative bid that only beats the D-II-bidders. The same is true for a D-I-bidder with signal one-half. She just breaks even by beating the same set of A-II- and D-II-bidders. Finally, all A-II-bidders with signals between one-half and three-fourths

⁷In this sense, this equilibrium can be understood as the type-asymmetric equilibrium that is the "closest possible" to a type-symmetric one.

bid very aggressively and make sure that a D-I-bidder with signal 1 does not want to outbid them.

The seller's expected revenue can be readily computed, and is found to be equal to 0.4661. It is interesting to make revenue comparisons between this situation and the ones arising were the bidders symmetric, or were there complete information about their type (i.e., as in Klemperer (1997)). To be able to make meaningful comparisons, we first need to define the appropriate alternative situations. We accomplish this by ensuring that under all three situations the expected value of the object being auctioned off is the same. The expected value of the object in the example is $\frac{3}{4}$. An "expected value-equivalent" symmetric auction can be constructed by considering an auction in which both bidders have the "fictitious type" $\theta = \frac{32}{27}$. The expected price of such auction is 0.56.⁸ The expected price of the auction in which it is common knowledge that there is an A-bidder and a D-bidder (as in Klemperer (1997)) is 0.25.⁹ By comparing the expected revenues, we see that the difference in the performance of second-price auctions in asymmetric and in symmetric environments is not too large. The negative reputation second-price auctions have acquired in asymmetric environments is due to the fact that the literature has focused on the certainty or the one-sided uncertainty cases in which second-price auctions seriously underperform compared to our framework.

4 Continuum-of-types case

4.1 Existence and Characterization of the Equilibrium

We now characterize the equilibrium of the auction when the θ_i 's are continuously distributed in the interval $[\theta_A, \theta_D]$. We show that in the continuous case there exists a type-symmetric equilibrium in pure, continuous, and strictly increasing strategies.

⁸This is the expected price arising from the symmetric equilibrium of this auction. Milgrom (1981) has shown that the expected prices of all other asymmetric equilibria of this auction are lower.

⁹There is actually a continuum of expected prices ranging from 0 to 0.5. Nonetheless, the lower bound can be reached only if the D-bidder bids less than her individual rational bid. On the other hand, the upper bound can be reached only if a D-bidder with signal less than one-half bids "unnaturally" aggressively. More aggressively than in the symmetric equilibrium had her opponent been of her own type.

Proposition 3 *The type-symmetric equilibrium bidding strategies are the following:*

$$b_i^*(x_i, \theta_i) = \frac{x_i + x^E(b_i(x_i, \theta_i))}{\theta_i},$$

where

$$x^E(b_i(x_i, \theta_i)) = \int_{m(x_i)}^{M(x_i)} z d\hat{F}(z),$$

and where $m(x_i) \equiv \{\min x \geq 0 : \exists \theta' \in [\theta_A, \theta_D] \text{ such that } b_{-i}(m, \theta') = b_i(x_i, \theta_i)\}$, $M(x_i) \equiv \{\max x \leq \bar{x} : \exists \theta'' \in [\theta_A, \theta_D] \text{ such that } b_{-i}(M, \theta'') = b_i(x_i, \theta_i)\}$, and $\hat{F}(z)$ is the truncation of F such that $z \in [m(x_i), M(x_i)]$.

Proof: It is clear that the purported equilibrium strategies have to be decreasing in θ . Notice that for any price bidder (x, θ) is willing to pay for the object, bidder $(x, \theta - \epsilon)$ makes strictly more profits when outbidding the same set of opponents. Let us assume that they are increasing in x . If this were the case, then $m(x_i)$ is such that $b_i(x_i, \theta_i) = b_j(m, \theta_A)$, and $M(x_i)$ is such that $b_i(x_i, \theta_i) = b_j(M, \theta_D)$ if $b_i < U$, where $U \equiv b_j(\bar{x}, \theta_D)$. Thus if bidder (\bar{x}, θ_D) follows these strategies then straightforward calculations yield:

$$U^* \equiv b^*(\bar{x}, \theta_D) = \frac{\bar{x} + \int_{m(\bar{x})}^{\bar{x}} z d\hat{F}(z)}{\theta_D} = b^*(m(\bar{x}), \theta_A) = \frac{m(\bar{x}) + \int_{m(\bar{x})}^{\bar{x}} z d\hat{F}(z)}{\theta_A}.$$

Moreover, $b^*(\bar{x}, \theta_A) = \frac{2\bar{x}}{\theta_A}$.

We claim that for all $b \in [U^*, \frac{2\bar{x}}{\theta_A}]$ there exists $\tilde{\theta} \in [\theta_A, \theta_D]$ such that $b^*(\bar{x}, \tilde{\theta}) = b$. We prove this claim by contradiction. Assume that there exists $b^*(\bar{x}, \theta) = b'$ and $b^*(\bar{x}, \theta - \epsilon) = b''$ such that $\lim_{\epsilon \rightarrow 0} b^*(\bar{x}, \theta - \epsilon) \neq b'$. Let y be the signal of the $\theta - \epsilon$ -bidder who bids b' . Now, due to the assumed gap, a $\theta - \epsilon$ -bidder with signal slightly greater than y should bid less than b' (for this bidder the expected signal of her opponent is smaller than it is for the $\theta - \epsilon$ -bidder with signal y). This contradicts increasing bidding strategies. Consequently, for all (x_i, θ_i) such that $x_i \geq m(\bar{x})$ it is deduced that $M(x_i) = \bar{x}$, and $m(x_i)$ is such that $b(x_i, \theta_i) = b(m(x_i), \theta_A)$. Hence, since the expectation $x^E(b_i(x_i, \theta_i))$ is a continuous and increasing function, $b^*(x_i, \theta_i)$ is also increasing and continuous in x_i , for all $x_i \geq U^*$. As for the case in which $x_i < U^*$, since $x^E(b_i(x_i, \theta_i))$ does not depend on x_i , it is also the case that $b^*(x_i, \theta_i)$ is increasing and continuous in x_i .¹⁰

¹⁰The normalization that the lower bound in the support of x is zero ensures that the bid strategies of all types start at the same point, i.e., from 0. Had we not used this normalization, it would have been necessary to define in a similar manner $u^* \equiv b_j^*(\bar{x}, \theta_A)$. Then for the range of the bids $b \in [\frac{2\bar{x}}{\theta_D}, u^*]$ the analysis would have been along the same lines as when $b \in [U^*, \frac{2\bar{x}}{\theta_A}]$.

Since bids are such that the bidders maximize the probability of winning conditional on not regretting winning, then by standard arguments employed in the analysis of second-price auctions we conclude that these strategies constitute an equilibrium. Q.E.D.

In the next subsection we analyze an example with uniform distributions. It allows us to compute the expected revenues to the seller.

4.2 An Example with Uniform Distributions

The distributions of both θ and x are assumed to be uniform. Moreover, we make the following normalizations: $[\theta_A, \theta_D] \equiv [1, 2]$, and $[0, \bar{x}] \equiv [0, 1]$.

It is now straightforward to calculate the equilibrium bid functions:

$$b^*(x, \theta) = \begin{cases} \frac{4x}{4\theta-3} & \text{if } x \in [0, \frac{4\theta-3}{5}), \\ \frac{3x+1}{3\theta-1} & \text{if } x \in [\frac{4\theta-3}{5}, 1]. \end{cases}$$

It is worth noting that the bid functions are piecewise linear in x . They have a steeper slope in the first segment, and moreover this slope is decreasing in θ . The fact that the slope is decreasing in θ shows that the more efficient a bidder is (the smaller her θ is), the more aggressively she bids.

[Insert Figure]

Since the bid functions are piecewise linear, to compute the expected revenue to the seller we have to distinguish between two cases: bids between 0 and $4/5$ and bids between $4/5$ and 2.

a) Consider a bidder choosing a bid $b \leq 4/5$. This bidder faces an opponent with an expected θ equal to 1.5. Since $b(x, \theta^E) = b(x, 1.5) = \frac{4x}{3}$, the expected payment when choosing b will be

$$\int_0^{\frac{3b}{4}} \frac{4}{3} z dz = \frac{3}{8} b^2.$$

b) Consider now that the bidder chooses $b > 4/5$. If her opponent ties with her, then the expected θ of the opponent will be $\theta^E = \frac{2+2b}{3b}$. Notice that when $b = 2$, $\theta^E = 1$, and when $b = 4/5$, $\theta^E = 1.5$. Since $b(x, \theta^E) = (3x+1)\frac{b}{2+b}$, the expected payment as a function of b is:

$$\int_0^{\frac{8}{15b}-\frac{1}{15}} \frac{4z}{3} dz + \int_{\frac{8}{15b}-\frac{1}{15}}^{\frac{1+b}{3}} (3z+1)\frac{b}{2+b} dz = \frac{256 - 352b - 140b^2 + 450b^3 + 225b^4}{1350b^2}.$$

Adding up we get that the expected payment is:

$$R = \int_1^2 \int_0^{\frac{4\theta-3}{5}} \frac{3}{8} \left(\frac{4z}{4\theta-3} \right)^2 dzd\theta + \int_1^2 \int_{\frac{4\theta-3}{5}}^1 \frac{1}{1350 \left(\frac{3z+1}{3\theta-1} \right)^2} \times \left[256 - 352 \left(\frac{3z+1}{3\theta-1} \right) - 140 \left(\frac{3z+1}{3\theta-1} \right)^2 + 450 \left(\frac{3z+1}{3\theta-1} \right)^3 + 225 \left(\frac{3z+1}{3\theta-1} \right)^4 \right] dzd\theta.$$

Integrating the above expression we get $R = 0.211$. Hence, the expected revenues to the seller equals to 0.422.

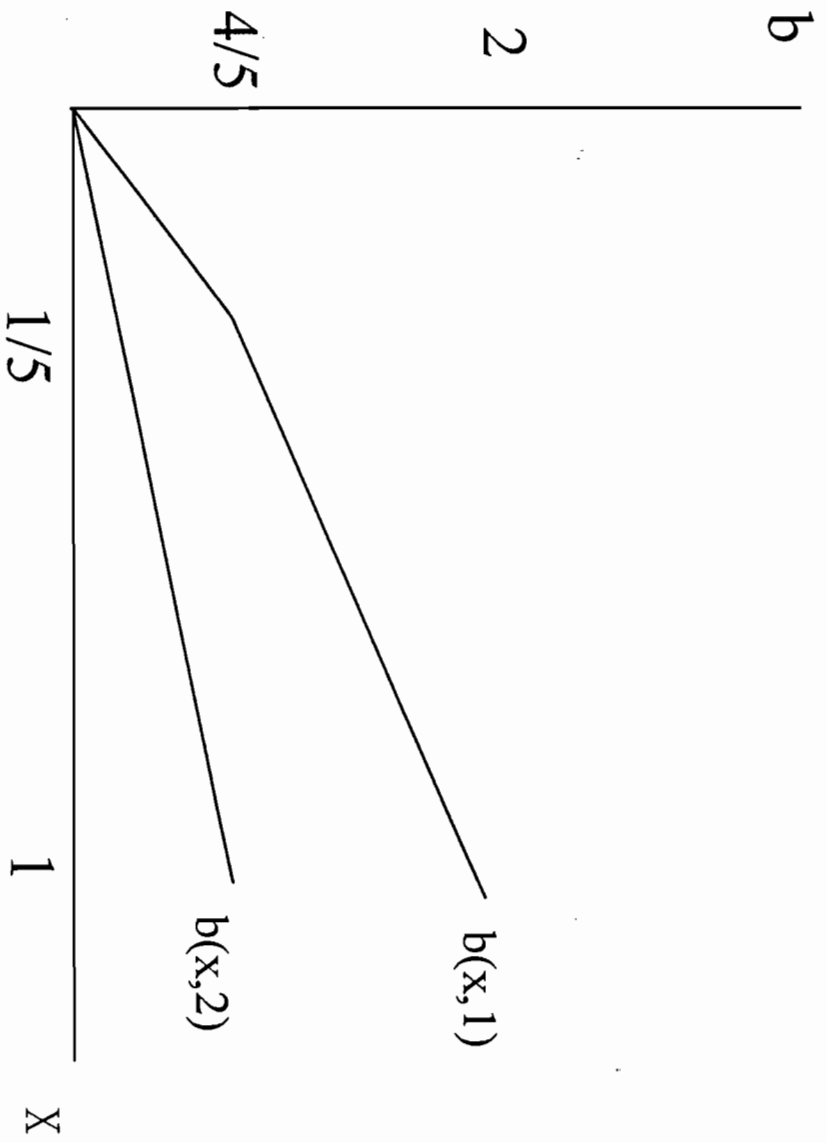
For comparison purposes, consider a symmetric common-value, second-price auction, where it is common knowledge that both bidders have the same efficiency parameter θ . Let $B^*(x, \theta)$ denote the symmetric equilibrium bid function. It is well known that $B^*(x, \theta) = \frac{2x}{\theta}$. Notice that $B^*(x, \theta) < b^*(x, \theta)$ for all $x \in [0, \frac{4\theta-3}{5}]$ when $\theta < 1.5$. Moreover, $B^*(x, 1.5) = b^*(x, 1.5)$ for all $x \in [0, 0.6)$ whereas $B^*(x, 1.5) > b^*(x, 1.5)$ for all $x \in (0.6, 1]$. Thus, a 1.5-bidder bids less aggressively when there is uncertainty about the θ of her opponent (while her expectation of her opponent's θ is equal to 1.5) than when she knows for sure that her opponent is like her. Nonetheless, we should note that the difference in the expected revenues to the seller is rather small. The expected price when $B^*(x, 1.5)$ is the symmetric equilibrium strategy is 0.444, while as we have shown above, the expected price when there is uncertainty about θ is 0.422.

5 Conclusions

The literature has demonstrated that second-price, common-value auctions are sensitive to the presence of asymmetries among bidders. For example, Bikhchandani (1988) has shown that if it is common knowledge that a bidder has a disadvantage compared to her opponent, this bidder (almost surely) never wins the auction. This paper is the first to show that this result does not carry through when one allows for two-sided uncertainty. We show that even if the probabilities that one of the bidders is advantaged while the other one is disadvantaged are arbitrarily large, in every equilibrium, the disadvantaged bidder needs to win the auction with strictly positive probability. We then solve for the equilibria in two cases (one with two types and another with a continuum of types) and we characterize their expected revenues properties. We find that although they underperform relative to "comparable" symmetric auctions, they perform much better than what is usually "assumed" in the literature.

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Equilibrium Bid Functions

