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THE ASYMPTOTIC NUCLEOLUS OF LARGE MONOPOLISTIC GAMES*

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Abstract

We study the asymptotic nucleolus of large differentiable monopolistic games. We show that if v is a monopolistic game which is a composition of a non-decreasing concave and differentiable function with a vector of measures, then v has an asymptotic nucleolus. We also provide an explicit formula for the asymptotic nucleolus of v and show that it coincides with the center of symmetry of the subset of the core of v in which all the monopolists obtain the same payoff. We apply this result to large monopolistic market games to obtain a relationship between the asymptotic nucleolus of the game and the competitive payoff distributions of the market.

Keywords: Monopolistic market games; Asymptotic nucleolus; Core; Competitive payoffs.

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§1 - Introduction

Monopolistic coalitional games (or more generally, mixed games) describe situations in which some of the players are "small," i.e., individually insignificant, whereas others are "large," i.e., individually significant. The main purpose of this work is to study the asymptotic nucleolus in such games. Since Shitovitz's (1973) seminal paper (which analyzed the core of large oligopolistic markets), many works on mixed markets have been written (for a comprehensive survey see Gabszewicz and Shitovitz (1992)). Guesnerie (1977) and Gardner (1977) investigated the asymptotic behavior of the Shapley value in such markets. Legros (1989) deals with the nucleolus of a bilateral market with two complementary commodities. In this work we study the asymptotic nucleolus of large differentiable monopolistic coalitional games.

Mathematically, we shall present the set of players by a measure space in which the small players form a non-atomic part and in which the large players are atoms. We assume that any atom has a monopolistic power, that is, the worth of a coalition which does not contain all the atoms is zero. In the asymptotic approach, a game with an infinite set of players is regarded as a limit of games with a finite set of players.

We first prove (see Section 3) that if v is a monopolistic game of the form $v = f \circ \mu$, where $\mu = (\mu_1, \dots, \mu_m)$ is a vector of measures and $f: \Re^m_+ \to \Re_+$ is a non-decreasing concave function which is continuously differentiable in the interior of $\Re^m_+$, then the game v has an asymptotic nucleolus. We also provide an explicit formula for the asymptotic nucleolus. This formula implies that it coincides with the center of symmetry of the subset of the core of v in which all the atoms receive the same payoff. Actually, we prove a stronger result, namely that every sequence of payoff vectors which belongs to the kernels of any admissible sequence of finite partition games which approximate the game v converges to the center of symmetry of the above mentioned subset of the core of v (see Theorem 3.1 and Corollary 3.3).

We note that any game of the above-mentioned form can be viewed as a large production game, where μ is the distribution of the production factors among the owners and f is the production function.

In Section 4 we apply the above-mentioned result to large monopolistic market games. We prove that under some mild conditions (on the untility funcitons of the traders) the asymptotic nucleolus of the transferable utility monopolstic market game which is associated with an economy with a finite number of types exists and coincides on the atomless part of the players' space with half of a competitive payoff distribution of the economy (see Proposition 4.1 and Theorem 4.3).

§2 - Preliminaries

In this section we define the basic notions which are relevant to our work. Let (T,Σ) be a measurable space, i.e., T is a set and Σ is a σ-field of subsets of T. We refer to the member of T as *players* and to those of Σ as *coalitions*. A *coalitional game*, or simply a *game* on (T,Σ), is a function $v: \Sigma \to \Re$ with $v(\emptyset) = 0$. If T is finite and $\Sigma = 2^T$ is the set of all subsets of T, the game v will be called a *finite game*. A game v is *superadditive* if $v(S_1 \cup S_2) \geq v(S_1) + v(S_2)$ whenever S_1 and S_2 are disjoint coalitions. A *payoff measure* in a game v on (T,Σ) is a bounded finitely additive measure $\lambda: \Sigma \to \Re$ which satisfies $\lambda(T) \leq v(T)$.

We denote by $ba = ba(T,\Sigma)$ the Banach space of all bounded finitely additive measures on (T,Σ) with the variation norm. The subspace of ba which consists of all bounded countably additive measures on (T,Σ) is denoted by $ca = ca(T,\Sigma)$. If λ is a measure in ca then $ca(\lambda) = ca(T,\Sigma,\lambda)$ denotes the set of all members of ca which are absolutely continuous with respect to λ. If A is a subset of an ordered vector space we denote by A_+ the set of all non-negative members of A.

Let K be a convex subset of an Euclidean space and let $f: K \to \Re$ be a concave function. A vector p is a *supergradient* of f at $x \in K$ if $f(y) - f(x) \leq p \cdot (y - x)$ for all $y \in K$. The set of all supergradients of f at x will be denoted by $\partial f(x)$. It is well known that if x is an interior point of K then $\partial f(x) \neq \emptyset$ and f is differentiable at x iff it has a unique supergradient at x which, in this case, coincides with the *gradient* vector.

For two vectors x,y in $\Re^m$ we write $x \geq y$ to mean $x_i \geq y_i$ for all $1 \leq i \leq m$, $x > y$ to mean $x \geq y$ and $x \neq y$, and $x >> y$ to mean $x_i > y_i$ for all $1 \leq i \leq m$. A function f defined on a set $A \subset \Re^m$ is called *non-decreasing* if for every $x, y \in A$ we have $x \geq y$ implies $f(x) \geq f(y)$. It is called *increasing* if, in addition, $x > y$ implies $f(x) > f(y)$.

§3 - The Asymptotic Behavior of the Kernel and the Nucleolus in Mixed Games

In this section we investigate the asymptotic behavior of the kernel and the nucleolus in a class of mixed games.

Let v be a finite game (that is, T is finite and $\Sigma = 2^T$). If $x \in \Re^{|T|}$ and $S \subset T$ we define $x(S) = \sum_{i \in S} x_i$ if $S \neq \emptyset$, and $x(\emptyset) = 0$. Denote

$$I(v) = \left\{ x \in \Re^{|T|} \mid x_i \geq v(\{i\}) \text{ for every } i \in T \text{ and } x(T) = v(T) \right\}$$

and

$$I^*(v) = \left\{ x \in \Re^{|T|} \mid x(T) = v(T) \right\}.$$

For every $i, j \in T$, $i \neq j$ and $x \in \Re^{|T|}$ define

$$s_{ij}(x) = max\{v(S) - x(S) \mid S \subset T,\ i \in S \text{ and } j \notin S\}$$

The *prekernel* of the game v is the set

$$PK(v) = \{x \in I^*(v) \mid s_{ij}(x) = s_{ji}(x)\ \forall i, j \in T,\ i \neq j\}.$$

The *kernel* of the game v is the set

$$K(v) = \{x \in I(v) \mid (s_{ij}(x) - s_{ji}(x))\,(x_j - v(\{j\})) \leq 0 \quad \forall i, j \in T,\ i \neq j\}.$$

It is well known that if v is a finite game which is zero monotonic (that is, $v(S \cup \{i\}) \geq v(S) + v(\{i\})$ for every $S \subset T$ and $i \in T \setminus S$), then $PK(v)$ and $K(v)$ coincide (see Theorem 2.7 in Maschler, Peleg and Shapley (1972)). For a further discussion of the kernel the reader is referred to Maschler (1992).

Let v be a finite game. For every $x \in I(v)$, let $\theta(x)$ be a $2^{|T|}$-tuple whose components are the numbers $v(S) - x(S),\ S \subset T$, arranged in non-increasing order, i.e., $\theta_i(x) \geq \theta_j(x)$ for $1 \leq i \leq j \leq n$. The *nucleolus* of the game v, denoted by Nv, is the payoff vector which is "closest" to v in the sense that $\theta(Nv)$ is the minimum in the lexicographic order of the set $\{\theta(x) \mid x \in I(v)\}$. It is well known that the nucleolus of a finite game v always exists when $I(v) \neq \emptyset$ and it consists of a unique point which belongs to the kernel of v (e.g., Schmeidler (1969)).

In the rest of the paper we assume that a fixed measure $\lambda \in ca_+(T, \Sigma)$ is given. We interpret λ as a *population measure*, that is, if S is a coalition, then $\lambda(S)$ is the proportion of the total population which is contained in S. We also assume that T can be represented in the form $T = T_0 \cup T_1$, where T_0 and T_1 are non-empty disjoint coalitions, the restriction of λ to (T_0, Σ_{T_0}) is non-atomic (where, here and in the sequel, if S is a coalition $\Sigma_S = \{Q \in \Sigma | Q \subset S\}$) and T_1 is a finite set of atoms of λ such that every subset of T_1 is in Σ.

Let v be a game on (T, Σ) and let π be a finite subfield of Σ. The set of all atoms of π is denoted by A_π. The set of all subsets of A_π is identified naturally with π, and thus a finite game with a set of players A_π is identified with a function $w: \pi \to \Re$ with $w(\varnothing) = 0$. The restriction of the game v to π is denoted by v_π. An *admissible sequence* of finite fields is an increasing sequence $(\pi_n)_{n=1}^{\infty}$ of finite subfields of Σ such that every subset of T_1 is in π_1 and $\bigcup_{n=1}^{\infty} \pi_n$ generates Σ.

Let v be a superadditive game on (T, Σ). It is said that v has an asymptotic nucleolus if there exists a game ψv such that, for every admissible sequence of finite fields $(\pi_n)_{n=1}^{\infty}$ and every S in π_1, $\lim_{n \to \infty} Nv_{\pi_n}(S)$ exists and equals $\psi v(S)$. It follows that $\psi v \in ba$, and it is called the *asymptotic nucleolus* of the game v.

The asymptotic approach was introduced in Kannai (1966) in the context of the Shapley value of non-atomic games (see also chapter III of Aumann and Shapley (1974)).

We are now ready to state and prove the main result of this section.

Theorem 3.1

Let $\mu = (\mu_1, \ldots, \mu_m)$ be a vector of non-trivial measures in $ca_+(\lambda)$. Assume that $f: \Re_+^m \to \Re_+$ is a non-decreasing concave function which is continuously differentiable in int $\Re_+^m$ and satisfies, $\nabla f(\mu(T)) >> 0$ and $f(\mu(T \setminus \{a\})) = 0$ for every $a \in T_1$. Then the game $v = f \circ \mu$ has an asymptotic nucleolus. Moreover, if

$\left(\pi_n\right)_{n=1}^{\infty}$ *is an admissible sequence of finite fields and* $x_n \in K\left(v_{\pi_n}\right)$ *for every* n,

then for every $S \in \pi_1$ *we have*

$$\lim_{n\to\infty} x_n(S) = \tfrac{1}{2}\nabla f(\mu(T))\cdot\mu(S\cap T_o) + \frac{f(\mu(T)) - \frac{1}{2}\nabla f(\mu(T))\cdot\mu(T_o)}{|T_1|}|S\cap T_1|$$

Proof

Let $\left(\pi_n\right)_{n=1}^{\infty}$ be an admissible sequence of finite fields. We first show that if $S \in \pi_1 \cap \Sigma_{T_o}$ and $x_n \in K\left(v_{\pi_n}\right)$ for every n, then $\lim_{n\to\infty} x_n(S) = \frac{1}{2}\nabla f(\mu(T))\cdot\mu(S)$.

Note that since f is non-decreasing the game v is superadditive. Therefore, for every n, the game v_{π_n} is zero-monotonic, and thus $K\left(v_{\pi_n}\right) = PK\left(v_{\pi_n}\right)$ for every n. Let n be a fixed natural number and let $j \in \pi_n \cap \Sigma_{T_o}$. Assume that $x_n \in K\left(v_{\pi_n}\right)$. Then for every $i \in T_1$ we have

$$s_{ji}(x_n) = \max\{v(Q) - x_n(Q)\,|\,Q \subset \pi_n,\ j \in Q,\ \{i\} \notin Q\} = -x_n(j)$$

and

$$s_{ij}(x_n) \geq v(T\setminus j) - x_n(T) + x_n(j) = f(\mu(T\setminus j)) - f(\mu(T)) + x_n(j)$$

Since $x_n \in PK\left(v_{\pi_n}\right)$, we have

$$s_{ij}(x_n) = s_{ji}(x_n).$$

Therefore

$$x_n(j) \leq \tfrac{1}{2}\left(f(\mu(T)) - f(\mu(T\setminus j))\right)$$

Since f is concave and differentiable,

$$f(\mu(T)) \le f(\mu(T \setminus j)) + \nabla f(\mu(T \setminus j)) \cdot \mu(j)$$

Thus,

$$(3.1) \quad x_n(j) \le \tfrac{1}{2} \nabla f(\mu(T \setminus j)) \cdot \mu(j)$$

Let $\varepsilon > 0$. As f is continuously differentiable on $int\, \Re^m_+$, there exists $\delta > 0$ such that for every $x \in \Re^m_+$ we have

$$(3.2) \quad \|x - \mu(T)\| < \delta \Rightarrow \nabla f(x) \le \nabla f(\mu(T)) + \varepsilon e$$

where $e = (1,1,\dots,1)$. Since $\mu_1,\dots,\mu_m$ are absolutely continuous with respect to λ and the restriction of λ to (T_o, Σ_{T_o}) is non-atomic, there exists a natural number n_o such that $\|\mu(j)\| < \delta$ for every $j \in \pi_{n_o} \cap \Sigma_{T_o}$. Therefore by (3.1) and (3.2), for every $n \ge n_o$ and $j \in \pi_n \cap \Sigma_{T_o}$ we have

$$x_n(j) \le \tfrac{1}{2}\left(\nabla f(\mu(T)) + \varepsilon e\right) \cdot \mu(j)$$

Let $S \in \pi_1 \cap \Sigma_{T_o}$. Then S is the union of members of π_n for every n. Therefore for every $n \ge n_o$

$$x_n(S) \le \tfrac{1}{2}\left(\nabla f(\mu(T)) + \varepsilon e\right) \cdot \mu(S)$$

Since ε is arbitrary, we have

$$\overline{lim}\, x_n(S) \le \tfrac{1}{2} \nabla f(\mu(T)) \cdot \mu(S)$$

We now show that $\underline{lim}\, x_n(S) \ge \frac{1}{2} \nabla f(\mu(T)) \cdot \mu(S)$. Since f is continuously differentiable on $int\, \Re^m_+$ and $\nabla f(\mu(T)) >> 0$, there exists $\hat{\delta} > 0$ such that for every $x \in \Re^m_+$ we have

$$\|x-\mu(T)\|<\hat{\delta} \Rightarrow \nabla f(x) \le \frac{3}{2}\nabla f(\mu(T)).$$

Let n_1 be a natural number such that $\|\mu(j)\|<\hat{\delta}$ for every $j \in \pi_{n_1} \cap \Sigma_{T_o}$.

Then

$$\nabla f(\mu(T \setminus j)) \le \frac{3}{2}\nabla f(\mu(T)).$$

Therefore by (3.1), for every $n \ge n_1$ and $j \in \pi_n \cap \Sigma_{T_o}$ we have

$$x_n(j) \le \frac{3}{4}\nabla f(\mu(T)) \cdot \mu(j).$$

Hence,

$$(3.3) \quad x_n(S) \le \frac{3}{4}\nabla f(\mu(T)) \cdot \mu(S)$$

Now there exists a natural number $n_2 \ge n_1$ such that for every $n \ge n_2$ and $j \in \pi_n \cap \Sigma_{T_o}$ we have

$$(3.4) \quad x_n(j) < \frac{1}{|T_1|} f(\frac{1}{4}\mu(T))$$

(note that since f is concave, $f(\frac{1}{4}\mu(T)) \ge \frac{1}{4}f(\mu(T)) > 0$).

Let $n \ge n_2$ be fixed and let $i \in T_1$ and $j \in \pi_n \cap \Sigma_{T_o}$. Choose $Q_n \subset \pi_n$ such that $\{i\} \in Q_n$, $j \notin Q_n$ and

$$v_{\pi_n}(Q_n) - x_n(Q_n) = max\{v_{\pi_n}(Q) - x_n(Q) \mid Q \subset \pi_n, \{i\} \in Q, j \notin Q\}$$

As $x_n \in K(v_{\pi_n})$, then $v_{\pi_n}(Q_n) - x_n(Q_n) = -x_n(j)$.

Let $S_n = \bigcup_{I \in Q_n} I$. We show that $S_n \supset T_1$. Assume not. Then $v(S_n) = 0$, and thus $x_n(j) = x_n(S_n) \ge x_n(\{i\})$. Since all the players in T_1 are interchangeable in the game v_{π_n} (two players in a finite game are interchangeable if they have the same marginal contribution to every coalition which does not contain them), they get the same payoff in every member of $K\left(v_{\pi_n}\right)$. Hence,

$$f(\mu(T)) = x_n(T) = |T_1|\, x_n(\{i\}) + x_n(T_o).$$

By (3.3), $x_n(T_o) \le \frac{3}{4}\nabla f(\mu(T))\cdot\mu(T_o)$. Therefore,

$$x_n(\{i\}) \ge \frac{f(\mu(T)) - \frac{3}{4}\nabla f(\mu(T))\cdot\mu(T_o)}{|T_1|}$$

Since f is concave and differentiable,

$$f(\mu(T)) - \frac{3}{4}\nabla f(\mu(T))\cdot\mu(T_o) \ge f(\mu(T) - \frac{3}{4}\mu(T_o)) \ge f(\frac{1}{4}\mu(T))$$

Thus, $x_n(\{i\}) \ge \frac{1}{|T_1|} f(\frac{1}{4}\mu(T))$. Since $x_n(j) \ge x_n(\{i\})$, this contradicts (3.4).

Therefore $S_n \supset T_1$, and thus there exists $\hat{S}_n \in \Sigma_{T_o}$ such that $S_n = (T \setminus j)\setminus \hat{S}_n$. Hence,

$$-x_n(j) = v(S_n) - x_n(S_n) = f(\mu(T) - \mu(j) - \mu(\hat{S}_n)) - f(\mu(T)) + x_n(j) + x_n(\hat{S}_n)$$

Thus

$$x_n(j) = \frac{1}{2}\left[f(\mu(T)) - f(\mu(T) - \mu(j) - \mu(\hat{S}_n)) - x_n(\hat{S}_n)\right]$$

By (3.3)

$$x_n(\hat{S}_n) \le \frac{3}{4}\nabla f(\mu(T))\cdot\mu(\hat{S}_n).$$

Since f is concave,

$$f(\mu(T)) - f(\mu(T) - \mu(j) - \mu(\hat{S}_n)) \ge \nabla f(\mu(T))\cdot(\mu(j) + \mu(\hat{S}_n))$$

Therefore,

$$x_n(j) \ge \frac{1}{2}\left[\nabla f(\mu(T))\cdot\mu(j) + \frac{1}{4}\nabla f(\mu(T))\cdot\mu(\hat{S}_n)\right] \ge \frac{1}{2}\nabla f(\mu(T))\cdot\mu(j).$$

Hence,

$$x_n(S) \geq \frac{1}{2}\nabla f(\mu(T)) \cdot \mu(S) \text{ for every } n \geq n_2.$$

This implies that $\underline{lim}\, x_n(S) \geq \frac{1}{2}\nabla f(\mu(T)) \cdot \mu(S)$.

Assume now that $S \in \pi_1$ is any coalition. Then for every natural number n we have

$$x_n(S) = x_n(S \cap T_o) + x_n(S \cap T_1)$$

Let t_n be the payoff which is assigned by x_n to a player in T_1. Then

$$v(T) = x_n(T) = |T_1|\, t_n + x_n(T_o) \Rightarrow \lim_{n\to\infty} t_n = \frac{v(T) - \frac{1}{2}\nabla f(\mu(T))\cdot\mu(T_o)}{|T_1|}$$

Therefore,

$$\lim_{n\to\infty} x_n(S) = \frac{1}{2}\nabla f(\mu(T)) \cdot \mu(S \cap T_o) + \frac{v(T) - \frac{1}{2}\nabla f(\mu(T))\cdot\mu(T_o)}{|T_1|}\,|\, S \cap T_1 \,| \quad \text{Q.E.D.}$$

Let v be a game on (T, Σ). The core of v, denoted by *Core(v)*, is the set of all payoff measures $\mu \in ba$ such that $\mu(S) \geq v(S)$ for every $S \in \Sigma$.

We want to determine the location in the core of the asymptotic nucleolus of a game which satisfies the conditions of Theorem 3.1. We first state and prove a representation theorem for the core of such games.

<u>Theorem 3.2</u>

Let $\mu = (\mu_1, \ldots, \mu_m)$ be a vector of non-trivial measures in $ca_+(\lambda)$. Assume that $f: \Re^m_+ \to \Re_+$ is a concave function which is differentiable at $\mu(T)$ and satisfies $f(\mu(T \setminus \{a\})) = 0$ for every $a \in T_1$. Then the core of the game $v = f \circ \mu$ is given by

$$Core(v) = \left\{ \xi \in ca_+(\lambda) \;\middle|\; \xi(T) = f(\mu(T)) \text{ and } \forall S \in \Sigma_{T_o},\ \xi(S) \leq \nabla f(\mu(T)) \cdot \mu(S) \right\}$$

Proof

Let

$$M(v) = \left\{\xi \in ca_+(\lambda) \;\middle|\; \xi(T) = f\big(\mu(T)\big) \text{ and } \forall S \in \Sigma_{T_0},\ \xi(S) \le \nabla f\big(\mu(T)\big)\cdot \mu(S)\right\}.$$

We will show that $M(v) = Core(v)$. We first show that $M(v) \subset Core(v)$. Let $\xi \in M(v)$ and $S \in \Sigma$. Now if S does not include T_1 then $v(S) = 0$ and clearly, $\xi(S) \ge v(S)$. If $S \supset T_1$ then $T \setminus S \subset T_0$. As $\xi \in M(v)$,

$$\xi(T \setminus S) \le \nabla f\big(\mu(T)\big)\cdot \mu(T \setminus S).$$

Therefore

$$\xi(S) = \xi(T) - \xi(T \setminus S) \ge \xi(T) - \nabla f\big(\mu(T)\big)\cdot \mu(T \setminus S) = f\big(\mu(T)\big) - \nabla f\big(\mu(T)\big)\cdot \mu(T \setminus S).$$

As f is concave,

$$v(S) = f\big(\mu(S)\big) \le f\big(\mu(T)\big) - \nabla f\big(\mu(T)\big)\cdot \mu(T \setminus S).$$

Hence, $\xi(S) \ge v(S)$, and thus $\xi \in Core(v)$.

It remains to show that $Core(v) \subset M(v)$. Let $\xi \in Core(v)$. Then for every $S \in \Sigma$ we have

(3.4) $\quad 0 \le \xi(S) \le \xi(T) - v(T \setminus S)$.

As f is continuous at $\mu(T)$ and $\mu_1, \ldots, \mu_m$ are in $ca_+(\lambda)$, the inequality in (3.4) implies that $\xi \in ca_+(\lambda)$. Since the restriction of λ to $\big(T_0, \Sigma_{T_0}\big)$ is non-atomic, the restrictions of $\mu_1, \ldots, \mu_m$ and ξ to $\big(T_0, \Sigma_{T_0}\big)$ are also non-atomic. Let $S \in \Sigma_{T_0}$. We will show that $\xi(S) \le \nabla f\big(\mu(T)\big)\cdot \mu(S)$. By Lyapunov's theorem, for every $0 < \alpha < 1$ there exists a coalition $S_\alpha \in \Sigma_{T_0}$ such that $\mu(S_\alpha) = \alpha\mu(S)$ and $\xi(S_\alpha) = \alpha\xi(S)$. As f is differentiable at $\mu(T)$, for every $0 < \alpha < 1$ we have

$$f\left(\mu(T \setminus S_\alpha)\right) = f\left(\mu(T)\right) - \alpha \nabla f\left(\mu(T)\right) \cdot \mu(S) + o(\alpha).$$

As $\xi \in Core(v)$, we have

$$\xi\left(S_\alpha\right) = \xi(T) - \xi\left(T \setminus S_\alpha\right) \leq f\left(\mu(T)\right) - f\left(\mu(T \setminus S_\alpha)\right).$$

Hence,

$$\xi(S) \leq \nabla f\left(\mu(T)\right) \cdot \mu(S) + g(\alpha),$$

where $\lim_{\alpha \to o} g(\alpha) = 0$. Therefore $\xi(S) \leq \nabla f\left(\mu(T)\right) \cdot \mu(S)$, and the proof is complete. Q.E.D.

Let A be a subset of a linear space. A point $x_o \in A$ is called a *center of symmetry* of A if for every $x \in A$, the point $2x_o - x$ also belongs to A. Note that if A is bounded, there may be at most one center of symmetry.

The following corollary is a direct consequence of Theorems 3.1 and 3.2.

Corollary 3.3

Let $\mu = \left(\mu_1, \ldots, \mu_m\right)$ be a vector of non-trivial measures in $ca_+(\lambda)$. Assume that $f: \Re^m_+ \to \Re_+$ is a non-decreasing concave function which is differentiable in int $\Re^m_+$ and satisfies, $\nabla f\left(\mu(T)\right) >> 0$ and $f\left(\mu(T \setminus \{a\})\right) = 0$ for every $a \in T_1$. Then the asymptotic nucleolus of the game $v = f \circ \mu$ coincides with the center of symmetry of the subset of the core of v in which all the members of T_1 receive the same payoff.

§4 - Market Games

In this section we apply Theorem 3.1 to games which arise in economic applications.

We consider a pure exchange economy E in which the commodity space is $\Re^m_+$. The traders' space is represented by the measure space (T, Σ, λ). We assume again that $T = T_o \cup T_1$, where T_o and T_1 are non-empty and disjoint coalitions, T_1 is a finite set of atoms of λ such that every subset of T_1 is in Σ, and the restriction of λ to $\left(T_o, \Sigma_{T_o}\right)$ is non-atomic. We will interpret the members of T_1 as monopolists. Every trader $t \in T$ has a *utility function* $u_t : \Re_+^m \to \Re_+$. An *assignment* in E is an integrable function $x: T \to \Re_+^m$. There is a fixed *initial assignment* ω ($\omega(t)$ represents the *initial bundle density* of trader t). An *allocation* is an assignment x such that $\int_T x d\lambda \le \int_T \omega \, d\lambda$. A *transferable utility competitive equilibrium* (t.u.c.e.) of the economy E is a pair (x, p), where x is an allocation and $p \in \Re_+^m$, such that for all $t \in T$, $u_t(x) - p \cdot (x - \omega(t))$ attains its maximum (over $\Re_+^m$) at $x = x(t)$. The measure $\varphi(S) = \int_S \left[u_t(x(t)) - p \cdot (x(t) - \omega(t))\right] d\lambda$ (when the function $u_t(x(t))$ is integrable) is called the *competitive payoff distribution*; and p is the *vector competitive prices*. We assume the following

(4.1) $\int_T \omega \, d\lambda >> 0$

(4.2) For every trader $a \in T_1$ there exists a commodity $1 \le k_a \le m$ such that $\omega_{k_a}(t) = 0$ for every $t \in T \setminus \{a\}$ (where ω_{k_a} denotes the k_a-component of ω).

The meaning of (4.2) is that every atom of λ has a corner on one of the commodities in the economy.

We restrict our analysis to two cases: (1) when every trader in E has the same utility function and (2) when E has a finite number of types.

Denote by U the set of all functions $u: \Re_+^m \to \Re_+$ which are continuous and concave on $\Re_+^m$, continuously differentiable and increasing on the interior of $\Re_+^m$ and vanish on the boundary of $\Re_+^m$. Note that any differentiable neoclassical utility function is in U (see Definition 1.4.2 in Aliprantis, Brown and Burkinshaw (1989)).

We first study the case in which all the traders in the economy E has the same utility function $u: \Re_+^m \to R_+$. We assume that $u \in U$ and that u is homogeneous of degree one on $\Re_+^m$ (note that, for example, any Cobb-Douglas utility function satisfies these assumptions). The *Aumann-Shapley Shubik market game* which is associated with the economy E (see Shapley and Shubik (1969) and Section 30 of Aumann and Shapley (1974)) in this special case is defined by

(4.3) $$v(S) = sup\left\{\int_S u(x(t))d\lambda \mid x \text{ is an assignment such that } \int_S x\, d\lambda = \int_S \omega\, d\lambda \right\}$$

Proposition 4.1

Assume that the economy E satisfies (4.1) and (4.2) and that every trader in E has the same utility function $u \in U$ which is also homogeneous of degree one. Then the market game v which is defined in (4.3) has an asymptotic nucleolus ψv which is given by

(4.4) $$\psi v(S) = \frac{1}{2}\nabla u(\int_T \omega\, d\lambda)\cdot \int_{S\cap T_o} \omega\, d\lambda + \frac{u\left(\int_T \omega\, d\lambda\right) - \frac{1}{2}\nabla u\left(\int_T \omega\, d\lambda\right)\cdot \int_{T_o} \omega\, d\lambda}{|T_1|}|S \cap T_1|$$

Moreover, there exists a competitive payoff distribution φ which corresponds to a t.u.c.e. of E such that $\psi v(S) = \frac{1}{2}\varphi(S)$ for every $S \in \Sigma_{T_o}$.

Proof

We first note that for every $S \in \Sigma$, $v(S) = u(\int_S \omega\, d\lambda)$. Indeed, let $S \in \Sigma$. Then by the definition of v, we have $v(S) \geq u(\int_S \omega\, d\lambda)$. Since u is concave and homogeneous of degree one, by Jensen's inequality, for every assignment x such that $\int_S x\, d\lambda = \int_S \omega\, d\lambda$ we have $\int_S u(x(t))\, d\lambda \leq u(\int_S \omega\, d\lambda)$. Therefore $v(S) = u(\int_S \omega\, d\lambda)$. Now, since u vanishes on the boundary of $\Re_+^m$, by (4.2), for every $a \in T_1$ we have $v(T\setminus\{a\}) = u(\int_{T\setminus\{a\}} \omega\, d\lambda) = 0$.

Also the assumption that u is increasing in the interior of $\Re_+^m$ implies that $\nabla u(\int_T \omega \, d\lambda) >> 0$.

Thus the game v satisfies the requirements of Theorem 3.1 and therefore (4.4) is satisfied. Let $b = \int_T \omega \, d\lambda$. Since u is homogeneous of degree one, by Euler's theorem $\nabla u(b) \cdot b = u(b)$. As u is concave, for every $x \in \Re_+^m$ we have

$$u(x) \leq u(b) + \nabla u(b) \cdot (x - b) = \nabla u(b) \cdot x .$$

Therefore $\max_{x \in \Re_+^m} \left(u(x) - \nabla u(b) \cdot x\right) = 0$. Consequently, for every $t \in T$ we have

$$\max_{x \in \Re_+^m} \left(u(x) - \nabla u(b) \cdot \left(x - \omega\,(t)\right)\right) = \nabla u(b) \cdot \omega\,(t) .$$

Let $\varphi = \nabla u(b) \cdot \int \omega \, d\lambda$. Then φ is a competitive payoff distribution in E and $\psi v(S) = \frac{1}{2}\varphi(S)$ for every $S \in \Sigma_{T_0}$. Q.E.D.

We now analyze the case when there is a finite number of traders' types in the economy E. Two traders are of the *same type* if they have identical initial bundles and identical utility functions. We assume that the number of different types of traders in T_0 is n. For every $1 \leq i \leq n$, we denote by S_i the set of traders in T_0 which are of type i. We assume that S_i is measurable (i.e., $S_i \in \Sigma$) and $\lambda(S_i) > 0$. The utility function of the traders of type i $(1 \leq i \leq n)$ is denoted by u_i, and their initial bundle by ω_i. We assume that for every $1 \leq i \leq n$, $u_i \in U$ and in addition u_i is homogeneous of degree one. We also assume that for every $a \in T_1$ the utility function u_a of the trader a is in U (but not necessarily homogeneous of degree one). The Aumann-Shapley-Shubik market game which is associated with the economy E in this case of finite number of types is

$$(4.5)\ v(S) = sup\left\{ \sum_{a \in S \cap T_1} \lambda(\{a\}) u_a(x(a)) + \sum_{i=1}^{n} \int_{S \cap S_i} u_i(x(t))\, d\lambda \ \middle|\ x \in X(S) \right\}$$

where, $X(S) = \left\{ x \,\middle|\, x \text{ is an assignment such that } \int_S x d\lambda = \int_S \omega\, d\lambda \right\}$.

Define a function $f: \Re_+^m \to \Re_+$ by

$$(4.6)\ f(y) = max\left\{ \sum_{a \in T_1} \lambda(\{a\}) u_a(x_a) + \sum_{i=1}^{n} u_i(x_i) \,\middle|\, x_a, x_i \in \Re_+^m,\ \sum_{a \in T_1} \lambda(\{a\}) x_a + \sum_{i=1}^{n} x_i \leq y \right\}$$

Since the utility functions of the traders are continuous and concave, it is easy to see that f is well defined and concave on $\Re_+^m$.

Lemma 4.2

Let v be the market game in (4.5), then $v(S) = f(\int_S \omega\, d\lambda)$ for every $S \in \Sigma$, where f is given by (4.6).

Proof

Let $S \in \Sigma$. Assume first that S does not include T_1. Then by (4.2), $\int_S \omega\, d\lambda$ belongs to the boundary of $\Re_+^m$. Since the utility functions of the traders in T vanish on the boundary of $\Re_+^m$, we have $v(S) = 0$ and $f(\int_S \omega\, d\lambda) = 0$. So assume that $S \supset T_1$.

We first show that $v(S) \geq f(\int_S \omega\, d\lambda)$. Let $(x_a)_{a \in T_1}$ and $(x_i)_{i=1}^n$ such that

$$f(\int_S \omega\, d\lambda) = \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a) + \sum_{i=1}^{n} u_i(x_i).$$

Define an assignment x by $x(t) = x_t$ if $t \in T_1$ and for every $t \in S_i$ $(1 \leq i \leq n)$

$$x(t) = \begin{cases} \frac{1}{\lambda(S \cap S_i)} x_i & \text{if } \lambda(S \cap S_i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\int_S x \, d\lambda = \sum_{a \in T_1} \lambda(\{a\}) x_a + \sum_{i=1}^{n} x_i \leq \int_S \omega \, d\lambda$$

Therefore $v(S) \geq \int_S u_t(x(t)) \, d\lambda$. Since the u_i are homogeneous of degree one,

$$\int_S u_t(x(t)) \, d\lambda = \sum_{a \in T_1} \lambda(\{a\}) u_a(x_a) + \sum_{i=1}^{n} u_i(x_i) = f(\int_S \omega \, d\lambda)$$

It remains to show that $v(S) \leq f(\int_S \omega \, d\lambda)$. Let x be an assignment such that $\int_S x \, d\lambda = \int_S \omega \, d\lambda$. For every $a \in T_1$ let $x_a = x(a)$ and for every $1 \leq i \leq n$ let $x_i = \int_{S \cap S_i} x \, d\lambda$. Then

$$\sum_{a \in T_1} \lambda(\{a\}) x_a + \sum_{i=1}^{n} x_i = \int_S x \, d\lambda = \int_S \omega \, d\lambda$$

Therefore by the definition of f, we have

$$f(\int_S \omega \, d\lambda) \geq \sum_{a \in T_1} \lambda(\{a\}) u_a(x_a) + \sum_{i=1}^{n} u_i(x_i)$$

Since the u_i are concave and homogeneous of degree one,

$$\sum_{a \in T_1} \lambda(\{a\}) u_a(x_a) + \sum_{i=1}^{n} \int_{S \cap S_i} u_i(x(t)) \, d\lambda \leq \sum_{a \in T_1} \lambda(\{a\}) u_a(x_a) + \sum_{i=1}^{n} u_i(x_i)$$

As x was an arbitrary assignment which satisfies $\int_S x \, d\lambda = \int_S \omega \, d\lambda$, we obtain that

$v(S) \leq f(\int_S \omega \, d\lambda)$.

Lemma 4.3

The function f which is defined in (4.6) is continuously differentiable on

int $\Re^m_+$ *and* $\nabla f(\int_T \omega \, d\lambda) >> 0$.

Proof

We first show that f is differentiable at every point in the interior of $\Re^m_+$. Let $y^* \in int\ \Re^m_+$. Then from the definition of f it is clear that $f(y^*) > 0$. Since f is concave on $\Re^m_+$, it is sufficient to show that $\partial f(y^*)$ consists of a unique point. Let $(x_a^*)_{a \in T_1}$ and $(x_i^*)_{i=1}^n$ be such that

$$f(y^*) = \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a^*) + \sum_{i=1}^{n} u_i(x_i^*)$$

Since the utility functions of the traders are non-decreasing, we have

$$\sum_{a \in T_1} \lambda(\{a\}) x_a^* + \sum_{i=1}^{n} x_i^* = y^*$$

Since $f(y^*) > 0$, the assumption that the utility functions of the traders vanish on the boundary of $\Re^m_+$ implies that there exists $j \in T_1 \cup \{1,\ldots,n\}$ such that $x_j^* \in int\ \Re^m_+$. Assume first that $1 \le j \le n$. We will show that $\partial f(y^*) \subset \partial u_j(x_j^*)$. Let $p \in \partial f(y^*)$. Then for every $x \in \Re^m_+$ we have

$$u_j(x) - u_j\left(x_j^*\right) = u_j(x) + \sum_{a \in T_I} \lambda\,(\{a\}) u_a\left(x_a^*\right) + \sum_{i \neq j} u_i\left(x_i^*\right)$$

$$- u_j\left(x_j^*\right) - \sum_{a \in T_I} \lambda\,(\{a\}) u_a\left(x_a^*\right) - \sum_{i \neq j} u_i\left(x_i^*\right) \leq$$

$$f(x + \sum_{a \in T_I} \lambda\,(\{a\}) x_a^* + \sum_{i \neq j} x_i^*) - f\left(y^*\right) \leq p \cdot \left(x - x_j^*\right).$$

Thus $p \in \partial u_j\left(x_j^*\right)$ and $\partial f\left(y^*\right) \subset \partial u_j\left(x_j^*\right)$. Since u_j is differentiable at x_j^*, we have $\partial u_j\left(x_j^*\right) = \left\{\nabla u_j\left(x_j^*\right)\right\}$. As $\partial f\left(y^*\right) \neq \emptyset$, we have $\partial f\left(y^*\right) = \left\{\nabla u_j\left(x_j^*\right)\right\}$. If $j \in T_I$, for every $x \in \Re_+^m$ we define $\bar{u}_j(x) = \lambda\,(\{j\}) u_j(x)$. Then the above argument implies that $\partial f\left(y^*\right) = \left\{\nabla \bar{u}_j\left(x_j^*\right)\right\}$. Thus, in any case $\partial f\left(y^*\right)$ consists of a unique point, and therefore f is differentiable at y^*. The assumption that the utility funcitons of the traders are increasing in *int* $\Re_+^m$ implies that $\nabla f(\int_T \omega\, d\lambda) >> 0$. Now since f is concave on $\Re_+^m$, it is continuous on *int* $\Re_+^m$. Moreover, since the utility functions of the traders vanish on the boundary of $\Re_+^m$ it is easy to see that f is also continuous on the boundary of $\Re_+^m$. Now Proposition 39.1 of Aumann and Shapley (1974) asserts that any continuous concave function on $\Re_+^m$ which is differentiable on *int* $\Re_+^m$ is continuously differentiable in *int* $\Re_+^m$. Therefore f is continuously differentiable on *int* $\Re_+^m$. Q.E.D.

We are now ready to state and prove the main result of this section.

Theorem 4.4

Assume that the economy E satisfies (4.1), (4.2) and also

(1) There is a finite number n of traders' types in T_0.

(2) The utility functions $u_1,\dots,u_n$ *of the traders in* T_0 *are in U and in addition they are homogeneous of degree one on* $\Re_+^m$.

(3) The utility functions $\{u_a\}_{a\in T_1}$ *of the traders in* T_1 *are in U.*

Let f be the function which is given by (4.6). Then the market game v which is defined in (4.5) has an asymptotic nucleolus ψv *which is given by*

$$(4.7)\quad \psi v(S) = \frac{1}{2}\nabla f\left(\int_T \omega\, d\lambda\right)\cdot \int_{S\cap T_0} \omega\, d\lambda + \frac{f\left(\int_T \omega\, d\lambda\right) - \frac{1}{2}\nabla f\left(\int_T \omega\, d\lambda\right)\cdot \int_{T_0} \omega\, d\lambda}{|T_1|}\,|S\cap T_1|.$$

Moreover, there exists a competitive payoff distribution φ *which corresponds to a t.u.c.e. in the economy* E *such that* $\psi v(S) = \frac{1}{2}\varphi(S)$ *for every* $S\in\Sigma_{T_0}$.

Proof

(4.7) follows from Theorem 3.1 and Lemmata 4.2 and 4.3. Denote $b=\int_T \omega\, d\lambda$. Let $\left(x_a^*\right)_{a\in T_1}$ and $\left(x_i^*\right)_{i=1}^n$ be such that

$$f(b) = \sum_{a\in T_1}\lambda(\{a\})u_a\left(x_a^*\right) + \sum_{i=1}^n u_i\left(x_i^*\right).$$

For every $t\in T$, let $x^*(t) = \begin{cases} x_t^* & t\in T_1 \\ x_i^* & t\in S_i \end{cases}$.

Then by a similar argument to that which was used in the proof of Lemma 4.3, we obtain that for every $t\in T$ and $x\in\Re_+^m$

$$(4.8)\quad u_t(x) \le u_t\left(x^*(t)\right) + \nabla f(b)\cdot\left(x - x^*(t)\right).$$

Since f is non-decreasing on $\Re^m_+$, $\nabla f(b) \geq 0$. Let $1 \leq i \leq m$. Now if x_i^* is on the boundary of $\Re^m_+$, then $u_i\left(x_i^*\right) = 0$, and thus by (4.8), $u_i(x) - \nabla f(b) \cdot x \leq 0$ for every $x \in \Re^m_+$. If $x_i^* \in int\ \Re^m_+$, then $\nabla f(b) = \nabla u_i\left(x_i^*\right)$. Since u_i is homogeneous of degree one, $\nabla u_i\left(x_i^*\right) \cdot x_i^* = u_i\left(x_i^*\right)$. Therefore we again have by (4.8),

$u_i(x) - \nabla f(b) \cdot x \leq 0$ for every $x \in \Re^m_+$ and thus

$$\max_{x \in \Re^m_+} \left(u_i(x) - \nabla f(b) \cdot x\right) = 0$$

This implies that for every $t \in T$

$$\max_{x \in \Re^m_+} \left(u_i(x) - \nabla f(b) \cdot (x - \omega(t))\right) = \nabla f(b) \cdot \omega(t).$$

Now by (4.8), for every $a \in T_1$ and $t \in T$ we have

$$\max_{x \in \Re^m_+} \left(u_a(x) - \nabla f(b) \cdot (x - \omega(t))\right) = u_a\left(x_a^*\right) - \nabla f(b) \cdot \left(x_a^* - \omega(t)\right).$$

For every $t \in T$ let

$$g(t) = \begin{cases} u_t(x^*(t)) - \nabla f(b) \cdot (x^*(t) - \omega(t)) & t \in T_1 \\ \nabla f(b) \cdot \omega(t) & t \in T_0 \end{cases}$$

For every $S \in \Sigma$ define $\varphi(S) = \int_S g d\lambda$. Then φ is a competitive payoff distribution in the economy E and for every $S \in \Sigma_{T_0}$ we have $\psi v(S) = \frac{1}{2}\varphi(S)$. Q.E.D.

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THE ASYMPTOTIC NUCLEOLUS OF LARGE MONOPOLISTIC GAMES*

Ezra Einy, Diego Moreno and Benyamin Shitovitz†

Abstract

We study the asymptotic nucleolus of large differentiable monopolistic games. We show that if v is a monopolistic game which is a composition of a non-decreasing concave and differentiable function with a vector of measures, then v has an asymptotic nucleolus. We also provide an explicit formula for the asymptotic nucleolus of v and show that it coincides with the center of symmetry of the subset of the core of v in which all the monopolists obtain the same payoff. We apply this result to large monopolistic market games to obtain a relationship between the asymptotic nucleolus of the game and the competitive payoff distributions of the market.

Keywords: Monopolistic market games; Asymptotic nucleolus; Core; Competitive payoffs.

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§1 - Introduction

Monopolistic coalitional games (or more generally, mixed games) describe situations in which some of the players are "small," i.e., individually insignificant, whereas others are "large," i.e., individually significant. The main purpose of this work is to study the asymptotic nucleolus in such games. Since Shitovitz's (1973) seminal paper (which analyzed the core of large oligopolistic markets), many works on mixed markets have been written (for a comprehensive survey see Gabszewicz and Shitovitz (1992)). Guesnerie (1977) and Gardner (1977) investigated the asymptotic behavior of the Shapley value in such markets. Legros (1989) deals with the nucleolus of a bilateral market with two complementary commodities. In this work we study the asymptotic nucleolus of large differentiable monopolistic coalitional games.

Mathematically, we shall present the set of players by a measure space in which the small players form a non-atomic part and in which the large players are atoms. We assume that any atom has a monopolistic power, that is, the worth of a coalition which does not contain all the atoms is zero. In the asymptotic approach, a game with an infinite set of players is regarded as a limit of games with a finite set of players.

We first prove (see Section 3) that if v is a monopolistic game of the form $v = f \circ \mu$, where $\mu = (\mu_1, \ldots, \mu_m)$ is a vector of measures and $f: \Re_+^m \to \Re_+$ is a non-decreasing concave function which is continuously differentiable in the interior of $\Re_+^m$, then the game v has an asymptotic nucleolus. We also provide an explicit formula for the asymptotic nucleolus. This formula implies that it coincides with the center of symmetry of the subset of the core of v in which all the atoms receive the same payoff. Actually, we prove a stronger result, namely that every sequence of payoff vectors which belongs to the kernels of any admissible sequence of finite partition games which approximate the game v converges to the center of symmetry of the above mentioned subset of the core of v (see Theorem 3.1 and Corollary 3.3).

We note that any game of the above-mentioned form can be viewed as a large production game, where μ is the distribution of the production factors among the owners and f is the production function.

In Section 4 we apply the above-mentioned result to large monopolistic market games. We prove that under some mild conditions (on the untility funcitons of the traders) the asymptotic nucleolus of the transferable utility monopolstic market game which is associated with an economy with a finite number of types exists and coincides on the atomless part of the players' space with half of a competitive payoff distribution of the economy (see Proposition 4.1 and Theorem 4.3).

§2 - Preliminaries

In this section we define the basic notions which are relevant to our work. Let (T, Σ) be a measurable space, i.e., T is a set and Σ is a σ-field of subsets of T. We refer to the member of T as *players* and to those of Σ as *coalitions*. A *coalitional game*, or simply a *game* on (T, Σ), is a function $v: \Sigma \to \Re$ with $v(\emptyset) = 0$. If T is finite and $\Sigma = 2^T$ is the set of all subsets of T, the game v will be called a *finite game*. A game v is *superadditive* if $v(S_1 \cup S_2) \geq v(S_1) + v(S_2)$ whenever S_1 and S_2 are disjoint coalitions. A *payoff measure* in a game v on (T, Σ) is a bounded finitely additive measure $\lambda: \Sigma \to \Re$ which satisfies $\lambda(T) \leq v(T)$.

We denote by $ba = ba(T, \Sigma)$ the Banach space of all bounded finitely additive measures on (T, Σ) with the variation norm. The subspace of ba which consists of all bounded countably additive measures on (T, Σ) is denoted by $ca = ca(T, \Sigma)$. If λ is a measure in ca then $ca(\lambda) = ca(T, \Sigma, \lambda)$ denotes the set of all members of ca which are absolutely continuous with respect to λ. If A is a subset of an ordered vector space we denote by A_+ the set of all non-negative members of A.

Let K be a convex subset of an Euclidean space and let $f: K \to \Re$ be a concave function. A vector p is a *supergradient* of f at $x \in K$ if $f(y) - f(x) \leq p \cdot (y - x)$ for all $y \in K$. The set of all supergradients of f at x will be denoted by $\partial f(x)$. It is well known that if x is an interior point of K then $\partial f(x) \neq \emptyset$ and f is differentiable at x iff it has a unique supergradient at x which, in this case, coincides with the *gradient* vector.

For two vectors x,y in $\Re^m$ we write $x \geq y$ to mean $x_i \geq y_i$ for all $1 \leq i \leq m$, $x > y$ to mean $x \geq y$ and $x \neq y$, and $x >> y$ to mean $x_i > y_i$ for all $1 \leq i \leq m$. A function f defined on a set $A \subset \Re^m$ is called *non-decreasing* if for every $x, y \in A$ we have $x \geq y$ implies $f(x) \geq f(y)$. It is called *increasing* if, in addition, $x > y$ implies $f(x) > f(y)$.

§3 - The Asymptotic Behavior of the Kernel and the Nucleolus in Mixed Games

In this section we investigate the asymptotic behavior of the kernel and the nucleolus in a class of mixed games.

Let v be a finite game (that is, T is finite and $\Sigma = 2^T$). If $x \in \Re^{|T|}$ and $S \subset T$ we define $x(S) = \sum_{i \in S} x_i$ if $S \neq \emptyset$, and $x(\emptyset) = 0$. Denote

$$I(v) = \left\{ x \in \Re^{|T|} \,\middle|\, x_i \geq v(\{i\}) \text{ for every } i \in T \text{ and } x(T) = v(T) \right\}$$

and

$$I^*(v) = \left\{ x \in \Re^{|T|} \mid x(T) = v(T) \right\}.$$

For every $i, j \in T$, $i \neq j$ and $x \in \Re^{|T|}$ define

$$s_{ij}(x) = max\{v(S) - x(S) \mid S \subset T,\ i \in S \text{ and } j \notin S\}$$

The *prekernel* of the game v is the set

$$PK(v) = \{x \in I^*(v) \mid s_{ij}(x) = s_{ji}(x)\ \forall i, j \in T,\ i \neq j\}.$$

The *kernel* of the game v is the set

$$K(v) = \{x \in I(v) \mid (s_{ij}(x) - s_{ji}(x))\,(x_j - v(\{j\})) \leq 0\ \ \forall i, j \in T,\ i \neq j\}.$$

It is well known that if v is a finite game which is zero monotonic (that is, $v(S \cup \{i\}) \geq v(S) + v(\{i\})$ for every $S \subset T$ and $i \in T \setminus S$), then $PK(v)$ and $K(v)$ coincide (see Theorem 2.7 in Maschler, Peleg and Shapley (1972)). For a further discussion of the kernel the reader is referred to Maschler (1992).

Let v be a finite game. For every $x \in I(v)$, let $\theta(x)$ be a $2^{|T|}$-tuple whose components are the numbers $v(S) - x(S),\ S \subset T$, arranged in non-increasing order, i.e., $\theta_i(x) \geq \theta_j(x)$ for $1 \leq i \leq j \leq n$. The *nucleolus* of the game v, denoted by Nv, is the payoff vector which is "closest" to v in the sense that $\theta(Nv)$ is the minimum in the lexicographic order of the set $\{\theta(x) \mid x \in I(v)\}$. It is well known that the nucleolus of a finite game v always exists when $I(v) \neq \varnothing$ and it consists of a unique point which belongs to the kernel of v (e.g., Schmeidler (1969)).

In the rest of the paper we assume that a fixed measure $\lambda \in ca_+(T, \Sigma)$ is given. We interpret λ as a *population measure*, that is, if S is a coalition, then $\lambda(S)$ is the proportion of the total population which is contained in S. We also assume that T can be represented in the form $T = T_0 \cup T_1$, where T_0 and T_1 are non-empty disjoint coalitions, the restriction of λ to (T_0, Σ_{T_0}) is non-atomic (where, here and in the sequel, if S is a coalition $\Sigma_S = \{Q \in \Sigma | Q \subset S\}$) and T_1 is a finite set of atoms of λ such that every subset of T_1 is in Σ.

Let v be a game on (T, Σ) and let π be a finite subfield of Σ. The set of all atoms of π is denoted by A_π. The set of all subsets of A_π is identified naturally with π, and thus a finite game with a set of players A_π is identified with a function $w: \pi \to \Re$ with $w(\varnothing) = 0$. The restriction of the game v to π is denoted by v_π. An *admissible sequence* of finite fields is an increasing sequence $(\pi_n)_{n=1}^{\infty}$ of finite subfields of Σ such that every subset of T_1 is in π_1 and $\bigcup_{n=1}^{\infty} \pi_n$ generates Σ.

Let v be a superadditive game on (T, Σ). It is said that v has an asymptotic nucleolus if there exists a game ψv such that, for every admissible sequence of finite fields $(\pi_n)_{n=1}^{\infty}$ and every S in π_1, $\lim_{n\to\infty} Nv_{\pi_n}(S)$ exists and equals $\psi v(S)$. It follows that $\psi v \in ba$, and it is called the *asymptotic nucleolus* of the game v.

The asymptotic approach was introduced in Kannai (1966) in the context of the Shapley value of non-atomic games (see also chapter III of Aumann and Shapley (1974)).

We are now ready to state and prove the main result of this section.

<u>Theorem 3.1</u>

Let $\mu = (\mu_1, \ldots, \mu_m)$ be a vector of non-trivial measures in $ca_+(\lambda)$. Assume that $f: \Re_+^m \to \Re_+$ is a non-decreasing concave function which is continuously differentiable in int $\Re_+^m$ and satisfies, $\nabla f(\mu(T)) >> 0$ and $f(\mu(T \setminus \{a\})) = 0$ for every $a \in T_1$. Then the game $v = f \circ \mu$ has an asymptotic nucleolus. Moreover, if

$(\pi_n)_{n=1}^{\infty}$ is an admissible sequence of finite fields and $x_n \in K(v_{\pi_n})$ for every n,

then for every $S \in \pi_1$ we have

$$\lim_{n\to\infty} x_n(S) = \frac{1}{2}\nabla f(\mu(T))\cdot\mu(S\cap T_o) + \frac{f(\mu(T)) - \frac{1}{2}\nabla f(\mu(T))\cdot\mu(T_o)}{|T_1|}|S\cap T_1|$$

<u>Proof</u>

Let $(\pi_n)_{n=1}^{\infty}$ be an admissible sequence of finite fields. We first show that if $S \in \pi_1 \cap \Sigma_{T_o}$ and $x_n \in K(v_{\pi_n})$ for every n, then $\lim_{n\to\infty} x_n(S) = \frac{1}{2}\nabla f(\mu(T))\cdot\mu(S)$.

Note that since f is non-decreasing the game v is superadditive. Therefore, for every n, the game v_{π_n} is zero-monotonic, and thus $K(v_{\pi_n}) = PK(v_{\pi_n})$ for every n. Let n be a fixed natural number and let $j \in \pi_n \cap \Sigma_{T_o}$. Assume that $x_n \in K(v_{\pi_n})$. Then for every $i \in T_1$ we have

$$s_{ji}(x_n) = max\{v(Q) - x_n(Q)\,|Q \subset \pi_n, j \in Q, \{i\} \notin Q\} = -x_n(j)$$

and

$$s_{ij}(x_n) \geq v(T\setminus j) - x_n(T) + x_n(j) = f(\mu(T\setminus j)) - f(\mu(T)) + x_n(j)$$

Since $x_n \in PK(v_{\pi_n})$, we have

$$s_{ij}(x_n) = s_{ji}(x_n).$$

Therefore

$$x_n(j) \leq \frac{1}{2}\left(f(\mu(T)) - f(\mu(T\setminus j))\right)$$

Since f is concave and differentiable,

$$f(\mu(T)) \le f(\mu(T \setminus j)) + \nabla f(\mu(T \setminus j)) \cdot \mu(j)$$

Thus,

$$(3.1) \quad x_n(j) \le \tfrac{1}{2} \nabla f(\mu(T \setminus j)) \cdot \mu(j)$$

Let $\varepsilon > 0$. As f is continuously differentiable on $int\, \Re^m_+$, there exists $\delta > 0$ such that for every $x \in \Re^m_+$ we have

$$(3.2) \quad \|x - \mu(T)\| < \delta \Rightarrow \nabla f(x) \le \nabla f(\mu(T)) + \varepsilon e$$

where $e = (1,1,\ldots,1)$. Since $\mu_1,\ldots,\mu_m$ are absolutely continuous with respect to λ and the restriction of λ to (T_o, Σ_{T_o}) is non-atomic, there exists a natural number n_o such that $\|\mu(j)\| < \delta$ for every $j \in \pi_{n_o} \cap \Sigma_{T_o}$. Therefore by (3.1) and (3.2), for every $n \ge n_o$ and $j \in \pi_n \cap \Sigma_{T_o}$ we have

$$x_n(j) \le \tfrac{1}{2}\left(\nabla f(\mu(T)) + \varepsilon e\right) \cdot \mu(j)$$

Let $S \in \pi_1 \cap \Sigma_{T_o}$. Then S is the union of members of π_n for every n.

Therefore for every $n \ge n_o$

$$x_n(S) \le \tfrac{1}{2}\left(\nabla f(\mu(T)) + \varepsilon e\right) \cdot \mu(S)$$

Since ε is arbitrary, we have

$$\overline{lim}\; x_n(S) \le \tfrac{1}{2} \nabla f(\mu(T)) \cdot \mu(S)$$

We now show that $\underline{lim}\, x_n(S) \ge \tfrac{1}{2} \nabla f(\mu(T)) \cdot \mu(S)$. Since f is continuously differentiable on $int\, \Re^m_+$ and $\nabla f(\mu(T)) >> 0$, there exists $\hat{\delta} > 0$ such that for every $x \in \Re^m_+$ we have

$$\|x - \mu(T)\| < \hat{\delta} \Rightarrow \nabla f(x) \leq \frac{3}{2} \nabla f(\mu(T)).$$

Let n_1 be a natural number such that $\|\mu(j)\| < \hat{\delta}$ for every $j \in \pi_{n_1} \cap \Sigma_{T_o}$.

Then

$$\nabla f(\mu(T \setminus j)) \leq \frac{3}{2} \nabla f(\mu(T)).$$

Therefore by (3.1), for every $n \geq n_1$ and $j \in \pi_n \cap \Sigma_{T_o}$ we have

$$x_n(j) \leq \frac{3}{4} \nabla f(\mu(T)) \cdot \mu(j).$$

Hence,

$$\text{(3.3)} \quad x_n(S) \leq \frac{3}{4} \nabla f(\mu(T)) \cdot \mu(S)$$

Now there exists a natural number $n_2 \geq n_1$ such that for every $n \geq n_2$ and $j \in \pi_n \cap \Sigma_{T_o}$ we have

$$\text{(3.4)} \quad x_n(j) < \frac{1}{|T_1|} f(\frac{1}{4} \mu(T))$$

(note that since f is concave, $f(\frac{1}{4}\mu(T)) \geq \frac{1}{4} f(\mu(T)) > 0$).

Let $n \geq n_2$ be fixed and let $i \in T_1$ and $j \in \pi_n \cap \Sigma_{T_o}$. Choose $Q_n \subset \pi_n$ such that $\{i\} \in Q_n$, $j \notin Q_n$ and

$$v_{\pi_n}(Q_n) - x_n(Q_n) = max\{v_{\pi_n}(Q) - x_n(Q) \mid Q \subset \pi_n,\ \{i\} \in Q, j \notin Q\}$$

As $x_n \in K(v_{\pi_n})$, then $v_{\pi_n}(Q_n) - x_n(Q_n) = -x_n(j)$.

Let $S_n = \bigcup_{l \in Q_n} l$. We show that $S_n \supset T_1$. Assume not. Then $v(S_n) = 0$, and thus $x_n(j) = x_n(S_n) \geq x_n(\{i\})$. Since all the players in T_1 are interchangeable in the game v_{π_n} (two players in a finite game are interchangeable if they have the same marginal contribution to every coalition which does not contain them), they get the same payoff in every member of $K\left(v_{\pi_n}\right)$. Hence,

$$f(\mu(T)) = x_n(T) = |T_1|\, x_n(\{i\}) + x_n(T_o).$$

By (3.3), $x_n(T_o) \leq \frac{3}{4}\nabla f(\mu(T))\cdot\mu(T_o)$. Therefore,

$$x_n(\{i\}) \geq \frac{f(\mu(T)) - \frac{3}{4}\nabla f(\mu(T))\cdot\mu(T_o)}{|T_1|}$$

Since f is concave and differentiable,

$$f(\mu(T)) - \frac{3}{4}\nabla f(\mu(T))\cdot\mu(T_o) \geq f(\mu(T) - \frac{3}{4}\mu(T_o)) \geq f(\frac{1}{4}\mu(T))$$

Thus, $x_n(\{i\}) \geq \frac{1}{|T_1|} f(\frac{1}{4}\mu(T))$. Since $x_n(j) \geq x_n(\{i\})$, this contradicts (3.4).

Therefore $S_n \supset T_1$, and thus there exists $\hat{S}_n \in \Sigma_{T_o}$ such that $S_n = (T \setminus j) \setminus \hat{S}_n$. Hence,

$$-x_n(j) = v(S_n) - x_n(S_n) = f(\mu(T) - \mu(j) - \mu(\hat{S}_n)) - f(\mu(T)) + x_n(j) + x_n(\hat{S}_n)$$

Thus

$$x_n(j) = \frac{1}{2}\left[f(\mu(T)) - f(\mu(T) - \mu(j) - \mu(\hat{S}_n)) - x_n(\hat{S}_n)\right]$$

By (3.3)

$$x_n(\hat{S}_n) \leq \frac{3}{4}\nabla f(\mu(T))\cdot\mu(\hat{S}_n)\ .$$

Since f is concave,

$$f(\mu(T)) - f(\mu(T) - \mu(j) - \mu(\hat{S}_n)) \geq \nabla f(\mu(T))\cdot(\mu(j) + \mu(\hat{S}_n))$$

Therefore,

$$x_n(j) \geq \frac{1}{2}\left[\nabla f(\mu(T))\cdot\mu(j) + \frac{1}{4}\nabla f(\mu(T))\cdot\mu(\hat{S}_n)\right] \geq \frac{1}{2}\nabla f(\mu(T))\cdot\mu(j).$$

Hence,

$$x_n(S) \geq \frac{1}{2}\nabla f(\mu(T)) \cdot \mu(S) \text{ for every } n \geq n_2.$$

This implies that $\underline{lim}\, x_n(S) \geq \frac{1}{2}\nabla f(\mu(T)) \cdot \mu(S)$.

Assume now that $S \in \pi_1$ is any coalition. Then for every natural number n we have

$$x_n(S) = x_n(S \cap T_o) + x_n(S \cap T_1)$$

Let t_n be the payoff which is assigned by x_n to a player in T_1. Then

$$v(T) = x_n(T) = |T_1|\, t_n + x_n(T_o) \Rightarrow \lim_{n\to\infty} t_n = \frac{v(T) - \frac{1}{2}\nabla f(\mu(T)) \cdot \mu(T_o)}{|T_1|}$$

Therefore,

$$\lim_{n\to\infty} x_n(S) = \frac{1}{2}\nabla f(\mu(T)) \cdot \mu(S \cap T_o) + \frac{v(T) - \frac{1}{2}\nabla f(\mu(T)) \cdot \mu(T_o)}{|T_1|} |\, S \cap T_1 \,| \quad \text{Q.E.D.}$$

Let v be a game on (T, Σ). The core of v, denoted by *Core(v)*, is the set of all payoff measures $\mu \in ba$ such that $\mu(S) \geq v(S)$ for every $S \in \Sigma$.

We want to determine the location in the core of the asymptotic nucleolus of a game which satisfies the conditions of Theorem 3.1. We first state and prove a representation theorem for the core of such games.

<u>Theorem 3.2</u>

Let $\mu = (\mu_1, \ldots, \mu_m)$ be a vector of non-trivial measures in $ca_+(\lambda)$. Assume that $f : \Re_+^m \to \Re_+$ is a concave function which is differentiable at $\mu(T)$ and satisfies $f(\mu(T \setminus \{a\})) = 0$ for every $a \in T_1$. Then the core of the game $v = f \circ \mu$ is given by

$$Core(v) = \left\{ \xi \in ca_+(\lambda) \;\middle|\; \xi(T) = f(\mu(T)) \text{ and } \forall S \in \Sigma_{T_o},\ \xi(S) \leq \nabla f(\mu(T)) \cdot \mu(S) \right\}$$

Proof

Let

$$M(v) = \left\{ \xi \in ca_+(\lambda) \;\middle|\; \xi(T) = f(\mu(T)) \text{ and } \forall S \in \Sigma_{T_0},\ \xi(S) \le \nabla f(\mu(T)) \cdot \mu(S) \right\}.$$

We will show that $M(v) = Core(v)$. We first show that $M(v) \subset Core(v)$. Let $\xi \in M(v)$ and $S \in \Sigma$. Now if S does not include T_1 then $v(S) = 0$ and clearly, $\xi(S) \ge v(S)$. If $S \supset T_1$ then $T \setminus S \subset T_0$. As $\xi \in M(v)$,

$$\xi(T \setminus S) \le \nabla f(\mu(T)) \cdot \mu(T \setminus S).$$

Therefore

$$\xi(S) = \xi(T) - \xi(T \setminus S) \ge \xi(T) - \nabla f(\mu(T)) \cdot \mu(T \setminus S) = f(\mu(T)) - \nabla f(\mu(T)) \cdot \mu(T \setminus S).$$

As f is concave,

$$v(S) = f(\mu(S)) \le f(\mu(T)) - \nabla f(\mu(T)) \cdot \mu(T \setminus S).$$

Hence, $\xi(S) \ge v(S)$, and thus $\xi \in Core(v)$.

It remains to show that $Core(v) \subset M(v)$. Let $\xi \in Core(v)$. Then for every $S \in \Sigma$ we have

(3.4) $\quad 0 \le \xi(S) \le \xi(T) - v(T \setminus S).$

As f is continuous at $\mu(T)$ and $\mu_1, \ldots, \mu_m$ are in $ca_+(\lambda)$, the inequality in (3.4) implies that $\xi \in ca_+(\lambda)$. Since the restriction of λ to (T_0, Σ_{T_0}) is non-atomic, the restrictions of $\mu_1, \ldots, \mu_m$ and ξ to (T_0, Σ_{T_0}) are also non-atomic. Let $S \in \Sigma_{T_0}$. We will show that $\xi(S) \le \nabla f(\mu(T)) \cdot \mu(S)$. By Lyapunov's theorem, for every $0 < \alpha < 1$ there exists a coalition $S_\alpha \in \Sigma_{T_0}$ such that $\mu(S_\alpha) = \alpha\mu(S)$ and $\xi(S_\alpha) = \alpha\xi(S)$. As f is differentiable at $\mu(T)$, for every $0 < \alpha < 1$ we have

$$f\left(\mu\left(T \setminus S_{\alpha}\right)\right) = f\left(\mu(T)\right) - \alpha \nabla f\left(\mu(T)\right) \cdot \mu\,(S) + o(\alpha).$$

As $\xi \in Core(v)$, we have

$$\xi\left(S_{\alpha}\right) = \xi\,(T) - \xi\left(T \setminus S_{\alpha}\right) \leq f\left(\mu(T)\right) - f\left(\mu\left(T \setminus S_{\alpha}\right)\right).$$

Hence,

$$\xi\,(S) \leq \nabla f\left(\mu(T)\right) \cdot \mu\,(S) + g(\alpha),$$

where $\lim_{\alpha \to o} g(\alpha) = 0$. Therefore $\xi\,(S) \leq \nabla f\left(\mu(T)\right) \cdot \mu\,(S)$, and the proof is complete. Q.E.D.

Let A be a subset of a linear space. A point $x_o \in A$ is called a *center of symmetry* of A if for every $x \in A$, the point $2x_o - x$ also belongs to A. Note that if A is bounded, there may be at most one center of symmetry.

The following corollary is a direct consequence of Theorems 3.1 and 3.2.

Corollary 3.3

Let $\mu = \left(\mu_1, \ldots, \mu_m\right)$ be a vector of non-trivial measures in $ca_+(\lambda)$. Assume that $f: \Re_+^m \to \Re_+$ is a non-decreasing concave function which is differentiable in $int\,\Re_+^m$ and satisfies, $\nabla f\left(\mu(T)\right) >> 0$ and $f\left(\mu\left(T \setminus \{a\}\right)\right) = 0$ for every $a \in T_1$. Then the asymptotic nucleolus of the game $v = f \circ \mu$ coincides with the center of symmetry of the subset of the core of v in which all the members of T_1 receive the same payoff.

§4 - Market Games

In this section we apply Theorem 3.1 to games which arise in economic applications.

We consider a pure exchange economy E in which the commodity space is $\Re_+^m$. The traders' space is represented by the measure space (T, Σ, λ). We assume again that $T = T_o \cup T_1$, where T_o and T_1 are non-empty and disjoint coalitions, T_1 is a finite set of atoms of λ such that every subset of T_I is in Σ, and the restriction of λ to $\left(T_o, \Sigma_{T_o}\right)$ is non-atomic. We will interpret the members of T_I as monopolists. Every trader $t \in T$ has a *utility function* $u_t : \Re^m_+ \rightarrow \Re_+$. An *assignment* in E is an integrable function $x: T \rightarrow \Re^m_+$. There is a fixed *initial assignment* ω ($\omega(t)$ represents the *initial bundle density* of trader t). An *allocation* is an assignment x such that $\int_T x d\lambda \leq \int_T \omega\, d\lambda$. A *transferable utility competitive equilibrium* (t.u.c.e.) of the economy E is a pair (x,p), where x is an allocation and $p \in \Re^m_+$, such that for all $t \in T$, $u_t(x) - p \cdot (x - \omega(t))$ attains its maximum (over $\Re^m_+$) at $x = x(t)$. The measure $\varphi(S) = \int_S \left[u_t(x(t)) - p \cdot (x(t) - \omega(t))\right] d\lambda$ (when the function $u_t(x(t))$ is integrable) is called the *competitive payoff distribution*; and p is the *vector competitive prices*. We assume the following

(4.1) $\int_T \omega\, d\lambda >> 0$

(4.2) For every trader $a \in T_I$ there exists a commodity $1 \leq k_a \leq m$ such that $\omega_{k_a}(t) = 0$ for every $t \in T \setminus \{a\}$ (where ω_{k_a} denotes the k_a-component of ω).

The meaning of (4.2) is that every atom of λ has a corner on one of the commodities in the economy.

We restrict our analysis to two cases: (1) when every trader in E has the same utility function and (2) when E has a finite number of types.

Denote by U the set of all functions $u: \Re^m_+ \rightarrow \Re_+$ which are continuous and concave on $\Re^m_+$, continuously differentiable and increasing on the interior of $\Re^m_+$ and vanish on the boundary of $\Re^m_+$. Note that any differentiable neoclassical utility function is in U (see Definition 1.4.2 in Aliprantis, Brown and Burkinshaw (1989)).

We first study the case in which all the traders in the economy E has the same utility function $u: \Re_+^m \to R_+$. We assume that $u \in U$ and that u is homogeneous of degree one on $\Re_+^m$ (note that, for example, any Cobb-Douglas utility function satisfies these assumptions). The *Aumann-Shapley Shubik market game* which is associated with the economy E (see Shapley and Shubik (1969) and Section 30 of Aumann and Shapley (1974)) in this special case is defined by

$$(4.3) \quad v(S) = sup\left\{\int_S u(x(t))d\lambda \mid x \text{ is an assignment such that } \int_S x\, d\lambda = \int_S \omega\, d\lambda \right\}$$

Proposition 4.1

Assume that the economy E satisfies (4.1) and (4.2) and that every trader in E has the same utility function $u \in U$ which is also homogeneous of degree one. Then the market game v which is defined in (4.3) has an asymptotic nucleolus ψv which is given by

$$(4.4) \quad \psi v(S) = \frac{1}{2}\nabla u(\int_T \omega\, d\lambda) \cdot \int_{S \cap T_o} \omega\, d\lambda + \frac{u\left(\int_T \omega\, d\lambda\right) - \frac{1}{2}\nabla u\left(\int_T \omega\, d\lambda\right) \cdot \int_{T_o} \omega\, d\lambda}{|T_1|} |S \cap T_1|$$

Moreover, there exists a competitive payoff distribution φ which corresponds to a t.u.c.e. of E such that $\psi v(S) = \frac{1}{2}\varphi(S)$ for every $S \in \Sigma_{T_o}$.

Proof

We first note that for every $S \in \Sigma$, $v(S) = u(\int_S \omega\, d\lambda)$. Indeed, let $S \in \Sigma$. Then by the definition of v, we have $v(S) \geq u(\int_S \omega\, d\lambda)$. Since u is concave and homogeneous of degree one, by Jensen's inequality, for every assignment x such that $\int_S x\, d\lambda = \int_S \omega\, d\lambda$ we have $\int_S u(x(t))\, d\lambda \leq u(\int_S \omega\, d\lambda)$. Therefore $v(S) = u(\int_S \omega\, d\lambda)$. Now, since u vanishes on the boundary of $\Re_+^m$, by (4.2), for every $a \in T_1$ we have $v(T \setminus \{a\}) = u(\int_{T \setminus \{a\}} \omega\, d\lambda) = 0$.

Also the assumption that u is increasing in the interior of $\Re_+^m$ implies that $\nabla u(\int_T \omega\, d\lambda) >> 0$. Thus the game v satisfies the requirements of Theorem 3.1 and therefore (4.4) is satisfied. Let $b = \int_T \omega\, d\lambda$. Since u is homogeneous of degree one, by Euler's theorem $\nabla u(b) \cdot b = u(b)$. As u is concave, for every $x \in \Re_+^m$ we have

$$u(x) \leq u(b) + \nabla u(b) \cdot (x - b) = \nabla u(b) \cdot x .$$

Therefore $\max_{x \in \Re_+^m} \left(u(x) - \nabla u(b) \cdot x\right) = 0$. Consequently, for every $t \in T$ we have

$$\max_{x \in \Re_+^m} \left(u(x) - \nabla u(b) \cdot (x - \omega(t))\right) = \nabla u(b) \cdot \omega(t) .$$

Let $\varphi = \nabla u(b) \cdot \int \omega\, d\lambda$. Then φ is a competitive payoff distribution in E and $\psi v(S) = \frac{1}{2}\varphi(S)$ for every $S \in \Sigma_{T_o}$. Q.E.D.

We now analyze the case when there is a finite number of traders' types in the economy E. Two traders are of the *same type* if they have identical initial bundles and identical utility functions. We assume that the number of different types of traders in T_o is n. For every $1 \leq i \leq n$, we denote by S_i the set of traders in T_o which are of type i. We assume that S_i is measurable (i.e., $S_i \in \Sigma$) and $\lambda(S_i) > 0$. The utility function of the traders of type i $(1 \leq i \leq n)$ is denoted by u_i, and their initial bundle by ω_i. We assume that for every $1 \leq i \leq n$, $u_i \in U$ and in addition u_i is homogeneous of degree one. We also assume that for every $a \in T_1$ the utility function u_a of the trader a is in U (but not necessarily homogeneous of degree one). The Aumann-Shapley-Shubik market game which is associated with the economy E in this case of finite number of types is

$$(4.5)\ v(S) = sup\left\{ \sum_{a \in S \cap T_1} \lambda(\{a\}) u_a(x(a)) + \sum_{i=1}^{n} \int_{S \cap S_i} u_i(x(t))\, d\lambda \;\middle|\; x \in X(S) \right\}$$

where, $X(S) = \left\{ x \,\middle|\, x \text{ is an assignment such that } \int_S x d\lambda = \int_S \omega\, d\lambda \right\}$.

Define a function $f: \Re_+^m \to \Re_+$ by

$$(4.6)\ f(y) = max\left\{ \sum_{a \in T_1} \lambda(\{a\}) u_a(x_a) + \sum_{i=1}^{n} u_i(x_i) \;\middle|\; x_a, x_i \in \Re_+^m,\ \sum_{a \in T_1} \lambda(\{a\}) x_a + \sum_{i=1}^{n} x_i \le y \right\}$$

Since the utility functions of the traders are continuous and concave, it is easy to see that f is well defined and concave on $\Re_+^m$.

<u>Lemma 4.2</u>

Let v be the market game in (4.5), then $v(S) = f(\int_S \omega\, d\lambda)$ for every $S \in \Sigma$, where f is given by (4.6).

<u>Proof</u>

Let $S \in \Sigma$. Assume first that S does not include T_1. Then by (4.2), $\int_S \omega\, d\lambda$ belongs to the boundary of $\Re_+^m$. Since the utility functions of the traders in T vanish on the boundary of $\Re_+^m$, we have $v(S) = 0$ and $f(\int_S \omega\, d\lambda) = 0$. So assume that $S \supset T_1$.

We first show that $v(S) \ge f(\int_S \omega\, d\lambda)$. Let $(x_a)_{a \in T_1}$ and $(x_i)_{i=1}^n$ such that

$$f(\int_S \omega\, d\lambda) = \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a) + \sum_{i=1}^{n} u_i(x_i).$$

Define an assignment x by $x(t) = x_t$ if $t \in T_1$ and for every $t \in S_i$ $(1 \le i \le n)$

$$x(t) = \begin{cases} \frac{1}{\lambda(S \cap S_i)} x_i & \text{if } \lambda(S \cap S_i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\int_S x\, d\lambda = \sum_{a \in T_1} \lambda(\{a\}) x_a + \sum_{i=1}^{n} x_i \leq \int_S \omega\, d\lambda$$

Therefore $v(S) \geq \int_S u_t(x(t))\, d\lambda$. Since the u_i are homogeneous of degree one,

$$\int_S u_t(x(t))\, d\lambda = \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a) + \sum_{i=1}^{n} u_i(x_i) = f(\int_S \omega\, d\lambda)$$

It remains to show that $v(S) \leq f(\int_S \omega\, d\lambda)$. Let x be an assignment such that $\int_S x\, d\lambda = \int_S \omega\, d\lambda$. For every $a \in T_1$ let $x_a = x(a)$ and for every $1 \leq i \leq n$ let $x_i = \int_{S \cap S_i} x\, d\lambda$. Then

$$\sum_{a \in T_1} \lambda(\{a\})\, x_a + \sum_{i=1}^{n} x_i = \int_S x\, d\lambda = \int_S \omega\, d\lambda$$

Therefore by the definition of f, we have

$$f(\int_S \omega\, d\lambda) \geq \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a) + \sum_{i=1}^{n} u_i(x_i)$$

Since the u_i are concave and homogeneous of degree one,

$$\sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a) + \sum_{i=1}^{n} \int_{S \cap S_i} u_i(x(t))\, d\lambda \leq \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a) + \sum_{i=1}^{n} u_i(x_i)$$

As x was an arbitrary assignment which satisfies $\int_S x\, d\lambda = \int_S \omega\, d\lambda$, we obtain that $v(S) \leq f(\int_S \omega\, d\lambda)$.

Lemma 4.3

The function f which is defined in (4.6) is continuously differentiable on *int* $\Re^m_+$ *and* $\nabla f(\int_T \omega\, d\lambda) >> 0$.

Proof

We first show that f is differentiable at every point in the interior of $\Re^m_+$. Let $y^* \in int\ \Re^m_+$. Then from the definition of f it is clear that $f(y^*) > 0$. Since f is concave on $\Re^m_+$, it is sufficient to show that $\partial f(y^*)$ consists of a unique point. Let $(x_a^*)_{a \in T_1}$ and $(x_i^*)_{i=1}^n$ be such that

$$f(y^*) = \sum_{a \in T_1} \lambda(\{a\})\, u_a(x_a^*) + \sum_{i=1}^n u_i(x_i^*)$$

Since the utility functions of the traders are non-decreasing, we have

$$\sum_{a \in T_1} \lambda(\{a\}) x_a^* + \sum_{i=1}^n x_i^* = y^*$$

Since $f(y^*) > 0$, the assumption that the utility functions of the traders vanish on the boundary of $\Re^m_+$ implies that there exists $j \in T_1 \cup \{1,\ldots,n\}$ such that $x_j^* \in int\ \Re^m_+$. Assume first that $1 \le j \le n$. We will show that $\partial f(y^*) \subset \partial u_j(x_j^*)$. Let $p \in \partial f(y^*)$. Then for every $x \in \Re^m_+$ we have

$$u_j(x) - u_j\left(x_j^*\right) = u_j(x) + \sum_{a \in T_l} \lambda\left(\{a\}\right) u_a\left(x_a^*\right) + \sum_{i \neq j} u_i\left(x_i^*\right)$$

$$- u_j\left(x_j^*\right) - \sum_{a \in T_l} \lambda\left(\{a\}\right) u_a\left(x_a^*\right) - \sum_{i \neq j} u_i\left(x_i^*\right) \leq$$

$$f(x + \sum_{a \in T_l} \lambda\left(\{a\}\right) x_a^* + \sum_{i \neq j} x_i^*) - f\left(y^*\right) \leq p \cdot \left(x - x_j^*\right).$$

Thus $p \in \partial u_j\left(x_j^*\right)$ and $\partial f\left(y^*\right) \subset \partial u_j\left(x_j^*\right)$. Since u_j is differentiable at x_j^*, we have $\partial u_j\left(x_j^*\right) = \left\{\nabla u_j\left(x_j^*\right)\right\}$. As $\partial f\left(y^*\right) \neq \emptyset$, we have $\partial f\left(y^*\right) = \left\{\nabla u_j\left(x_j^*\right)\right\}$. If $j \in T_l$, for every $x \in \Re_+^m$ we define $\bar{u}_j(x) = \lambda\left(\{j\}\right) u_j(x)$. Then the above argument implies that $\partial f\left(y^*\right) = \left\{\nabla \bar{u}_j\left(x_j^*\right)\right\}$. Thus, in any case $\partial f\left(y^*\right)$ consists of a unique point, and therefore f is differentiable at y^*. The assumption that the utility funcitons of the traders are increasing in *int* $\Re_+^m$ implies that $\nabla f(\int_T \omega \, d\lambda) >> 0$. Now since f is concave on $\Re_+^m$, it is continuous on *int* $\Re_+^m$. Moreover, since the utility functions of the traders vanish on the boundary of $\Re_+^m$ it is easy to see that f is also continuous on the boundary of $\Re_+^m$. Now Proposition 39.1 of Aumann and Shapley (1974) asserts that any continuous concave function on $\Re_+^m$ which is differentiable on *int* $\Re_+^m$ is continuously differentiable in *int* $\Re_+^m$. Therefore f is continuously differentiable on *int* $\Re_+^m$. Q.E.D.

We are now ready to state and prove the main result of this section.

Theorem 4.4

Assume that the economy E satisfies (4.1), (4.2) and also

(1) There is a finite number n of traders' types in T_o.

(2) The utility functions $u_1,\ldots,u_n$ *of the traders in* T_o *are in U and in addition they are homogeneous of degree one on* $\Re_+^m$.

(3) The utility functions $\{u_a\}_{a\in T_1}$ *of the traders in* T_1 *are in U.*

Let f be the function which is given by (4.6). Then the market game v which is defined in (4.5) has an asymptotic nucleolus ψv *which is given by*

$$(4.7)\quad \psi v(S) = \frac{1}{2}\nabla f\left(\int_T \omega\, d\lambda\right)\cdot \int_{S\cap T_o} \omega\, d\lambda + \frac{f\left(\int_T \omega\, d\lambda\right) - \frac{1}{2}\nabla f\left(\int_T \omega\, d\lambda\right)\cdot \int_{T_o} \omega\, d\lambda}{|T_1|}|S\cap T_1|.$$

Moreover, there exists a competitive payoff distribution φ *which corresponds to a t.u.c.e. in the economy* E *such that* $\psi v(S) = \frac{1}{2}\varphi(S)$ *for every* $S\in\Sigma_{T_o}$.

Proof

(4.7) follows from Theorem 3.1 and Lemmata 4.2 and 4.3. Denote $b = \int_T \omega\, d\lambda$. Let $\left(x_a^*\right)_{a\in T_1}$ and $\left(x_i^*\right)_{i=1}^n$ be such that

$$f(b) = \sum_{a\in T_1} \lambda(\{a\}) u_a\left(x_a^*\right) + \sum_{i=1}^n u_i\left(x_i^*\right).$$ For every $t\in T$, let $x^*(t) = \begin{cases} x_t^* & t\in T_1 \\ x_i^* & t\in S_i \end{cases}$.

Then by a similar argument to that which was used in the proof of Lemma 4.3, we obtain that for every $t\in T$ and $x\in\Re_+^m$

$$(4.8)\quad u_t(x)\le u_t\left(x^*(t)\right) + \nabla f(b)\cdot\left(x - x^*(t)\right).$$

Since f is non-decreasing on $\Re_+^m$, $\nabla f(b) \geq 0$. Let $1 \leq i \leq m$. Now if x_i^* is on the boundary of $\Re_+^m$, then $u_i\left(x_i^*\right) = 0$, and thus by (4.8), $u_i(x) - \nabla f(b) \cdot x \leq 0$ for every $x \in \Re_+^m$. If $x_i^* \in int\ \Re_+^m$, then $\nabla f(b) = \nabla u_i\left(x_i^*\right)$. Since u_i is homogeneous of degree one, $\nabla u_i\left(x_i^*\right) \cdot x_i^* = u_i\left(x_i^*\right)$. Therefore we again have by (4.8),

$u_i(x) - \nabla f(b) \cdot x \leq 0$ for every $x \in \Re_+^m$ and thus

$$\max_{x \in \Re_+^m} \left(u_i(x) - \nabla f(b) \cdot x\right) = 0$$

This implies that for every $t \in T$

$$\max_{x \in \Re_+^m} \left(u_i(x) - \nabla f(b) \cdot \left(x - \omega\ (t)\right)\right) = \nabla f(b) \cdot \omega\ (t).$$

Now by (4.8), for every $a \in T_1$ and $t \in T$ we have

$$\max_{x \in \Re_+^m} \left(u_a(x) - \nabla f(b) \cdot \left(x - \omega\ (t)\right)\right) = u_a\left(x_a^*\right) - \nabla f(b) \cdot \left(x_a^* - \omega\ (t)\right).$$

For every $t \in T$ let

$$g(t) = \begin{cases} u_t\left(x^*(t)\right) - \nabla f(b) \cdot \left(x^*(t) - \omega\ (t)\right) & t \in T_1 \\ \nabla f(b) \cdot \omega\ (t) & t \in T_0 \end{cases}$$

For every $S \in \Sigma$ define $\varphi\ (S) = \int_S g d\lambda$. Then φ is a competitive payoff distribution in the economy E and for every $S \in \Sigma_{T_o}$ we have $\psi v(S) = \frac{1}{2}\varphi(S)$. Q.E.D.

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