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Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (341) 624-98-75

A FIXED POINT THEOREM WITHOUT CONVEXITY

H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano and J.V. Llinares *

Abstract

The purpose of this paper is to extend Himmelberg's fixed point theorem replacing the usual convexity in topological vector spaces by an abstract topological notion of convexity which generalizes classical convexity as well as several metric convexity structures found in the literature. We prove the existence, under weak hypotheses , of a fixed point for a compact approachable map and we provide sufficient conditions under which this result applies to maps whose values are convex in the abstract sense mentionned above.

Keywords: Fixed point theorems; Generalized convexity; Compact correspondences; Approachable correspondences.

* Ben-El-Mechaiekh, Brock University, Canada; Chebbi, CERMSEM, University de Paris I; Florenzano, CNRS-CEPREMAP, E-mail: florenza@cepremap.msh.paris.fr; Llinares, CEPREMAP, the third anther would like to thank the Universidad Carlos III de Madrid for it hospitality while part of this work was completed.

A Fixed Point Theorem Without Convexity

H. BEN-EL-MECHAIEKH

Departement of Mathematics, Brock University, St. Catharines,
Ontario, L2S 3A1, Canada.

S. CHEBBI

CERMSEM, Université de Paris I, 90 rue de Tolbiac, 75013 Paris, France.

M. FLORENZANO

CNRS-CEPREMAP, 140 rue du Chevaleret, 75013 Paris, France.

and

J.V. LLINARES

CEPREMAP, 140 rue du Chevaleret, 75013 Paris, France.

The purpose of this paper is to extend Himmelberg's fixed point theorem replacing the usual convexity in topological vector spaces by an abstract topological notion of convexity which generalizes classical convexity as well as several metric convexity structures found in the literature. We prove the existence, under weak hypotheses, of a fixed point for a compact approachable map and we provide sufficient conditions under which this result applies to maps whose values are convex in the abstract sense mentionned above.

1. INTRODUCTION

The Himmelberg's theorem ([11], Theorem 2) generalizes to correspondences the Schauder fixed point theorem. It asserts that every compact upper semicontinuous correspondence Φ with nonempty closed convex values from a nonempty convex subset X of a locally convex topological vector space E into itself has a fixed point. The first important step in proving this theorem when E is a normed space, is to show that the correspondence Φ can be approximated (in the sense of the graph) by continuous single-valued functions. A straightforward use of the Brouwer fixed point theorem (applied to those approximations) together with compactness arguments conclude the proof (see for example, the proof of Theorem 5.11.3 in Dugundji and Granas [9]). In Ben-El-Mechaiekh and Deguire [5] and in Ben-El-Mecchaiekh [4], a thorough study of correspondences that can be approximated in this way - called approachable correspondences - was presented with emphasis on non-convex correspondences. Among other things, a version of Himmelberg's theorem for approachable upper semicontinuous compact correspondences defined on a convex subset of a locally convex topological vector space was proved (Lemma 4.1 below).

The first concern of this paper is to extend this theorem (for the class of approachable correspondences) by replacing the usual convexity in topological vector spaces with a quite general abstract convexity concept defined in topological spaces. Second, and in order to state a "topological analogue" of the Himmelberg's theorem, we provide sufficient conditions for a map whose values are convex in an abstract topological sense to be approachable.

Our paper is organized as follows:

In Section 2, we recall some definitions and extend to a uniform spaces setting some definitions previously given in the context of topological vector spaces.

In Section 3, we introduce a convexity concept which encompasses most of the convexity structures previously defined in the literature in order to extend the Brouwer or the Kakutani theorem. We then give some analogues of Cellina's approximation theorems for upper semicontinuous correspondences with nonempty (generalized) convex values.

In Section 4, a fixed point theorem for compact, upper semicontinuous with nonempty closed values, approachable self-correspondences is proved for a large class of uniform convex spaces and for the most general convexity structure. Taking into account the conditions of application of this result given in Section 3, this theorem allows for a generalization of most of the Himmelberg type fixed point theorems.

2. PRELIMINARIES

Let X, Y be two sets and $\Phi : X \to Y$ be a set-valued map (simply called correspondence). The *graph* of Φ is the set:

$$graph(\Phi) = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}.$$

If $X \subseteq Y$, a *fixed point* for a correspondence $\Phi : X \to Y$ is an element $x \in X$ with

$x \in \Phi(x)$.

If X and Y are topological spaces, a correspondence $\Phi : X \to Y$ is said to be *upper semicontinuous* (*u.s.c*) on X if for any given $x \in X$ and any open subset V of Y, the set $\{x \in X \mid \Phi(x) \subset V\}$ is open in X.

In this paper, the pair $(X,\mathcal{U})$ denotes a uniform space with $\mathcal{U}$ being a basis of symmetric entourages for some uniformity on the space X. Given $U \in \mathcal{U}$, the U-ball around a given element $x \in X$ is the set $U[x] = \{x' \in X \mid (x,x') \in U\}$. The U-neighborhood around a given subset $A \subset X$ is the set $U[A] = \bigcup_{x \in A} U[x] = \{y \in X \mid U[y] \bigcap A \neq \emptyset\}$. It is well known that for every $x \in X$, the sets $\{U[x] \mid U \in \mathcal{U}\}$ form a basis of neighborhoods of x for the uniform topology on X. A topological space is said to be uniformizable if there exists $\mathcal{U}$ such that the topology given on X is the uniform topology associated with $(X,\mathcal{U})$.

If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are uniform spaces and if $\Phi : X \to Y$ is a correspondence, instead of upper semicontinuity of Φ, we will use the following slightly weaker continuity property (equivalent to upper semicontinuity for compact-valued maps) :

$$\forall V \in \mathcal{V},\ \forall x \in X,\ \exists U \in \mathcal{U},\ \Phi(U[x]) \subset V[\Phi(x)]. \tag{1}$$

In what follows, all topological spaces are supposed to be Hausdorff, which means for uniform spaces that $\bigcap\{U \mid U \in \mathcal{U}\}$ is the diagonal of $X \times X$.

The definitions and properties of the class $\mathcal{A}$ of approachable correspondences, given in Ben-El-Mechaiekh [4] and Ben-El-Mechaiek and Deguire [5] in the case where X and Y are topological vector spaces, easily extend to the uniform spaces setting.

Let us first recall that if $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are two uniform spaces and if

$$W = \{((x,y),(x',y')) \in (X \times Y) \times (X \times Y) \mid (x,x') \in U, (y,y') \in V\},$$

then the family $\left(\mathcal{W}\right)_{\substack{U \in \mathcal{U} \\ V \in \mathcal{V}}}$ is a basis of symmetric entourages for the product uniformity and that the associated uniform topology on $X \times Y$ is the product of the uniform topologies on X and Y.

DEFINITION (2.1). Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two uniform spaces and let $\Phi : X \to Y$ be a correspondence. Given a member W of $\mathcal{W}$, a function $s : X \to Y$ is said to be a *W-approximative selection* of Φ if and only if:

$$graph(s) \subset W[graph(\Phi)].$$

DEFINITION (2.2). Let X and Y be topological spaces. The correspondence $\Phi : X \to Y$ is said to be *approachable* if and only if:

(i) X and Y are uniformizable (with respective basis of symmetric entourages $\mathcal{U}$ and $\mathcal{V}$),

(ii) Φ admits a continuous W-approximative selection for each W in $\mathcal{W}$ (the basis of the product uniformity on $X \times Y$).

Let $c(X,Y)$ be the class of continuous functions $X \to Y$. Note that, in view of the symmetry of the entourages in $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$, the condition (ii) is equivalent to :

$$\begin{cases} \forall U \in \mathcal{U},\ \forall V \in \mathcal{V},\ \exists s \in c(X,Y) \text{ such that} \\ \forall x \in X,\ \exists x' \in U[x] \text{ with } s(x) \in V[\Phi(x')]. \end{cases}$$

The class $\mathcal{A}$ of correspondences from X into Y is defined by:

$$\mathcal{A}(X,Y) := \{\Phi : X \twoheadrightarrow Y \mid \Phi \text{ is approachable}\}.$$

We write $\mathcal{A}(X)$ for $\mathcal{A}(X,X)$.

It is clear from Definitions (2.1) and (2.2) that the problem of finding a graph-approximation for a given correspondence Φ reduces to that of finding a selection for an open neighborhood of the graph of Φ. With this remark in mind, let us first observe that with some compactness, every nonempty valued u.s.c. correspondence admits an open-graph majorant. More precisely:

LEMMA (2.3). *Let $(X,\mathcal{U}),(Y,\mathcal{V})$ be two uniform spaces with X paracompact, and assume that $\Phi : X \twoheadrightarrow Y$ satisfies (1) and has nonempty values. Then, for any pair of entourages $U \in \mathcal{U}, V \in \mathcal{V}$, there exists an open-graph map, depending on U and V, $\Psi_{U,V} : X \twoheadrightarrow Y$ such that:*

$$\Phi(x) \subseteq \Psi_{U,V}(x) \subseteq V[\Phi(U[x])], \forall x \in X.$$

Proof. Let $(U,V) \in (\mathcal{U} \times \mathcal{V})$ be given. Without loss of generality, we can assume that all $U \in \mathcal{U}$ and $V \in \mathcal{V}$ are open. By (1), for each $x \in X$, there exists $U_x \subset U$ such that $\Phi(U_x[x]) \subset V[\Phi(x)]$. Let $\{O_i\}_{i \in I}$ be a point-finite open refinement of the cover $\{U_x[x]\}_{x \in X}$, i.e. for every $i \in I$, $O_i \subset U_{x_i}[x_i]$ for some $x_i \in X$ and the set $I(x) := \{i \in I \mid x \in O_i\}$ is finite. Define the correspondence $\Psi_{U,V} : X \twoheadrightarrow Y$ by putting:

$$\Psi_{U,V}(x) = \bigcap_{i \in I(x)} V[\Phi(x_i)],\ x \in X.$$

Clearly, $\Phi(x) \subset \Psi_{U,V}(x)$ for all $x \in X$. Moreover, for every $x \in X$ and every $i \in I(x)$, we have $\Psi_{U,V}(x) \subset V[\Phi(x_i)]$. If $x' \in \bigcap_{i \in I(x)} O_i$, then $I(x) \subset I(x')$ and consequently,

$\Psi_{U,V}(x) \subset \Psi_{U,V}(\bigcap_{i \in I(x)} O_i) \subset \Psi_{U,V}(x)$, i.e. $\Psi_{U,V}$ is locally constant. Finally, for any given $x \in X$, the set $\left(\bigcap_{i \in I(x)} O_i\right) \times \Psi_{U,V}(x)$ is an open set around $\{x\} \times \Psi_{U,V}(x)$ contained in the graph of $\Psi_{U,V}$, i.e. $\Psi_{U,V}$ has an open graph. □

We will also make use of the following lemma; it rephrases, in the context of uniform spaces, Proposition 2.5 in Ben-El-Mechaiekh [4] and its proof is therefore omitted.

LEMMA (2.4). *Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ $(Z,\mathcal{W})$ be three uniform spaces, with Z compact and let $\Psi : Z \to X$, $\Phi : X \to Y$ be two u.s.c. closed valued approachable correspondences, then so is their composition product $\Phi \circ \Psi$.*

3. APPROACHABLE CORRESPONDENCES AND LOCALLY (GENERALIZED) CONVEX SPACES

We now precise the topological convexity notions that we will use in this paper. Generally speaking, a convexity structure on a set E is given by a family $\mathcal{C}$ of subsets of E, stable under finite or infinite intersections and containing E and the empty set. The elements of $\mathcal{C}$ are said to be $\mathcal{C}$-convex and, for any $X \subset E$, a natural definition of the $\mathcal{C}$-convex hull of E is:

$$\mathrm{co}_{\mathcal{C}} X = \bigcap \{Y \mid X \subset Y \text{ and } Y \text{is } \mathcal{C} - \text{convex}\}.$$

In what follows, E stands for a topological space. If $(E,\mathcal{U})$ is a uniform space and $\mathcal{C}$ is a convexity structure defined on E, then $(E,\mathcal{U})$ is said to be *locally $\mathcal{C}$-convex* if $\mathcal{U}$ is such that for every $U \in \mathcal{U}$, $U[X]$ is $\mathcal{C}$-convex whenever $X \subset E$ is $\mathcal{C}$-convex. It should be noticed that, with this definition, U-balls $U[x]$ are not necessarily convex.

A quite general convexity structure on the topological space E can be given as follows. Let $\langle E \rangle$ denote the family of all nonempty finite subsets of E. If n is any integer and if $J \subset \{0,1,...,n\}$, $\triangle_n$ will denote the unit-simplex of $\mathbb{R}^{n+1}$ and $\triangle_J$ the face of $\triangle_n$ corresponding to J, i.e. $\triangle_J =$ co $\{e_j \mid j \in J\}$ where $e_0, e_1, ..., e_n$ is the canonical basis of $\mathbb{R}^{n+1}$.

DEFINITION (3.1). An *L-structure* on E is given by a nonempty set-valued map $\Gamma : \langle E \rangle \to E$ verifying:

($*$) For every $A = \{x_0, x_1, ..., x_n\} \in \langle E \rangle$, there exists a continuous function $f^A : \triangle_n \to \Gamma(A)$ such that for all $J \subset \{0,1,...,n\}$, $f^A(\triangle_J) \subset \Gamma(\{x_j, j \in J\})$.

The pair (E, Γ) is then called an *L-space* and $X \subset E$ is said to be *L-convex* if $\forall A \in \langle X \rangle$, $\Gamma(A) \subset X$.

Clearly, the L-convex subsets of E form a convexity on E. If X is an L-convex subset of E, the pair $(X, \Gamma|_{\langle X \rangle})$ is an L-space on its own right. If $(E, \mathcal{U})$ is locally L-convex, then for the uniformity induced on X by $\mathcal{U}$, $(X, \Gamma|_{\langle X \rangle})$ is locally L-convex.

An interesting geometrical interpretation is given by the following proposition which shows the equivalence between Definition (3.1) and a previous definition given by Llinares [19] for an L- structure.

PROPOSITION (3.2). *E can be endowed with an L-structure if and only if for any nonempty finite subset $A = \{a_0, a_1, ..., a_n\}$ of X, there is a family of elements $\{b_0, b_1, ..., b_n\}$ in X and a family of functions $P_i^A : X \times [0,1] \to X$, $i = 0, 1, ..., n$ such that:*

1. $P_i^A(x, 0) = x$, , $P_i^A(x, 1) = b_i$, $i = 0, 1, ..., n$,

2. The function $G_A : [0,1]^n \to X$ defined by:

$$G_A(t_0, t_1, ..., t_{n-1}) = P_0^A(P_1^A(...(P_{n-1}^A(P_n^A(a_n, 1), t_{n-1}), ...), t_1), t_0)$$

is a continuous function.

Proof. If E has a L-structure, then we can define $\forall A \in \langle E \rangle$, $A = \{a_0, a_1, ..., a_n\}$, the functions P_i^A as follows:

$$P_n^A(a_n, 1) = f^A(e_n)$$

$$P_{n-1}^A(P_n^A(a_n, 1), t_{n-1}) = f^A(t_{n-1}e_{n-1} + (1 - t_{n-1})e_n)$$

$$P_{n-2}^A P_{n-1}^A(P_n^A(a_n, 1), t_{n-1}), t_{n-2}) = f^A(t_{n-2}e_{n-2} + (1 - t_{n-2})[t_{n-1}e_{n-1} + (1 - t_{n-1})e_n])$$

and so on. Moreover, the functions P_i^A will be defined in those values not considered until now so as to verify $P_i^A(x, 0) = x$, and $P_i^A(x, 1) = f^A(e_i)$. Therefore, the function

$$G_A(t_0, t_1, ..., t_{n-1}) = f^A(\sum_{i=0}^{n} \alpha_i e_i)$$

where coeficients α_i depend continuously on t_j, $j = 0, 1, ..., n$, will be continuous.

On the other hand, if for any finite set $A \in \langle E \rangle$, there exists a family of functions P_i^A satisfying the conditions 1 and 2, then we define the function $f^A : \Delta_n \to E$ by

$$f^A(\lambda) = G_A(T(\lambda))$$

where $T(\lambda) = (t_0(\lambda), t_1(\lambda), ..., t_{n-1}(\lambda))$ and for each $i = 0, 1, ..., n$,

$$t_i(\lambda) = \begin{cases} 0 & \text{if } \lambda_i = 0 \\ \frac{\lambda_i}{\sum_{j=i}^{n} \lambda_j} & \text{if } \lambda_i > 0 \end{cases}$$

It is not hard to prove that composition $G_A \circ T$ is continuous.

To define the multifunction $\Gamma : \langle E \rangle \to E$ in order to obtain the same convexity structure, we need to introduce a new concept, the restriction of function G_A to some subset B of E:

$$\forall B \in E,\ A \cap B \neq \emptyset,\ G_{A|B}(t_{i_0}, t_{i_1}, ..., t_{i_{m-1}}) = P_{i0}^A(...(P_{i_{m-1}}^A(P_{l_m}^A(a_{i_m}, 1), t_{im-1})..., t_{i_0})$$

where $P_{i_j}^A$ are the functions associated to the elements $a_{i_j} \in A \cap B$. We now construct the multifunction $\Gamma : \langle E \rangle \to E$ as follows

$$\Gamma(B) = \cup \{G_{A|B}([0,1]^m) \mid B \subset A,\ A \in \langle E \rangle\}$$

($m = \sharp B$). One easily verifies that if $A = \{x_0, \ldots, x_n\}$, $J \subset \{0, \ldots, n\}$ and $\lambda \in \triangle_J$, then

$$f^A(\lambda) = G_A(T(\lambda)) \in G_{A|\{x_j | j \in J\}}([0,1]^J) \subset \Gamma(\{x_j \mid j \in J\}).$$

□

Note that in the previous proposition, function $P_i^A(x, .)$ can be interpreted as a path joining x and b_i , while the composition of these paths (function G_A) can be seen as an abstract convex combination of the finite subset A. So, a subset X is L-convex if it contains every abstract combination of any finite subset of X.

In order to prove a selection theorem, let us now introduce a stronger convexity requirement which however weakens a definition given by Bielawski [6].

DEFINITION (3.3). A *B'-simplicial convexity* on E is given by a family of functions $(\Phi^A)_{A \in \langle E \rangle}$ such that:

(i) if $A = \{x_0, x_1, ..., x_n\}$, $\Phi^A : \Delta_n \to E$ is continuous and if $\lambda \in \partial \Delta_n$, then $\Phi^A(\lambda) = \Phi^B(\lambda)$, where B is obtained from A by removing x_i whenever $\lambda_i = 0$.

Defining $\Gamma(A) = \Phi^A(\Delta_n)$ for $A = \{x_0, x_1, ..., x_n\}$, it is quite obvious that a B'-simplicial convexity on E is a L-structure on E. Then, as previously, $X \subset E$ is said to be *B'-convex* if $\forall A \in \langle X \rangle$, $\Gamma(A) \subset X$.

The class of B'-spaces contains topological vector spaces and their convex subsets as well as a number of spaces with different abstract topological convexities that can be found in the literature.Two typical examples are the following:

DEFINITION (3.4). A *B-simplicial convexity* on E is given by a family of functions $(\Phi^A)_{A\in\langle E\rangle}$ satisfying condition (i) of the previous definition and:
(ii) $\Phi^{\{x\}}(1) = x$.

This is the definition given by Bielawski [6] with the same convex sets.

DEFINITION (3.5). An *H-structure* is given by a set-valued map $\Gamma : \langle E\rangle \to E$ such that $\Gamma(A) \subset \Gamma(B)$ if $A \subset B$ and assumed to have nonempty C^∞ values (any continuous function defined on the (relative) boundary of a finite simplex with values in $\Gamma(A)$ can be extended to a continuous function defined on the whole simplex with values in $\Gamma(A)$).
$X \subset E$ is said to be *H-convex* if $\forall A \in \langle X\rangle,\ \Gamma(A) \subset X$.

Definition (3.5) is due to Horvath [13-15] who uses the terminology c-structure and c-sets for H-structure and H-convex sets defined above . We adopt here the terminology of Bardaro and Cepitelli [3], Ding and Tan [8], Tarafdar [26-27], Park and Kim [21]. As noted by Park and Kim [21], it follows from Theorem 1, Section 1 of Horvath [14] that if Γ defines an H-structure, then (X,Γ) is a L-space. Moreover, we have the following:

PROPOSITION (3.6). *Let (E,Γ) be a H-space. Then it is possible to define a B'-simplicial convexity such that H-convex sets are B'-convex sets.*

Proof. Since (E,Γ) is a H-space, with every $x \in E$ we can associate an element $z_x \in \Gamma(\{x\})$. Then we define functions Φ^A for the singletons as follows

$$\forall x \in E, \qquad \Phi^{\{x\}}(1) = z_x$$

Next we construct functions Φ^A associated to finite subsets A of E by induction on the number of elements of A. Suppose that for any finite subset $B = \{b_o, \dots, b_k\}$ $(k < n)$, there exists $\Phi^B : \Delta_k \to \Gamma(B)$ satisfying condition (i) of Definition (3.3). Let $A = \{x_0, \dots, x_n\}$. For $\lambda \in \partial\Delta_n$, we define $\Psi^A(\lambda) = \Phi^{A_{J(\lambda)}}(\lambda)$ where $J(\lambda) = \{i \mid \lambda_i > 0\}$ and $A_{J(\lambda)} = \{x_i \mid \lambda_i > 0\}$. By the induction hypothesis, $\Psi^A(\lambda) = \Phi^{A_{J(\lambda)}}(\lambda) \in \Gamma(A_{J(\lambda)}) \subset \Gamma(A)$ and $\Psi^A(\lambda)$ coincides with $\Phi^{A_J}(\lambda)$ for all $J \subset \{O, \dots, n\}$, $J \neq \{O, \dots, n\}$, such that $J(\lambda) \subset J$, so that the continuity of Ψ^A on $\partial\Delta_n$ follows from the continuity of all Φ^B. Since $\Gamma(A)$ is C^∞, we can find an extension Φ^A of Ψ^A, continuous on Δ_n with values in $\Gamma(A)$, which satisfies by construction condition (i) of Definition (3.3).

It only remains to show that H-convex sets are B'-convex sets. If X is a H-convex set, then it is verified that $\forall A \in \langle X \rangle$, then $\Gamma(A) \subset X$. Therefore, in order to prove that X is a B'-convex set, we have to verify that $\forall A = \{x_0, \ldots, x_n\} \in \langle X \rangle$, $\Phi^A(\Delta_n) \subset X$ but, by construction, $\Phi^A(\Delta_n) \subset \Gamma(A) \subset X$. □

COROLLARY (3.7) *If (E, Γ) is a H-space such that $\forall x \in E$, $x \in \Gamma(\{x\})$, then E has a simplicial convexity.*

The two previous examples encompass (sometimes obviously) topological convex structures introduced in [1], [2], [12], [16], [17], [18], [20], [21], [22], [23], [24], [25]. [16] gives an order theoretical version of the H-convexity. The proof that hyperconvex spaces ([1], [2], [23]) are locally H-convex spaces is not trivial and can be found in Horvath [15].

The interest of Definition (3.3) is in the following selection theorem which extends Proposition (3.9) in Bielawski [6] and for which we give the same simple proof.

PROPOSITION (3.8) *Let C be a B'-convexity on a topological space Y. Let X be a paracompact space and let $\phi : X \to Y$ be a correspondence with nonempty, C-convex values and open lower sections $\phi^{-1}(y) = \{x \in X \mid y \in \phi(x)\}$. Then ϕ has a continuous selection.*

Proof. Let $(U_i)_{i \in I}$ be an open neighborhood finite refinement of the open covering $\left(\phi^{-1}(y)\right)_{y \in Y}$ of X and let $(\lambda_i)_{i \in I}$ be a continuous partition of unity subordinated to $(U_i)_{i \in I}$. To each $i \in I$, we associate $y_i \in Y$ such that $U_i \subset \phi^{-1}(y_i)$. We can assume that I is a well-ordered (hence completely ordered) indexing set and define for $x \in X$,

$$f(x) = \Phi^{\{y_{i_1}, \ldots, y_{i_m}\}}\left(\lambda_{i_1}(x), \ldots, \lambda_{i_m}(x)\right)$$

where $\{i_1, \ldots, i_m\} = \{i \in I \mid \lambda_i(x) > 0\}$ and $i_1 < i_2 < \ldots < i_m$.

Let Γ be the map associated to the B'-convexity. Then $f(x) \in \Gamma(\{y_{i_1}, \ldots, y_{i_m}\})$ with $y_{i_k} \in \phi(x)$, $k = 1, \ldots, m$. Since $\phi(x)$ is B'-convex, $f(x) \in \phi(x)$. On the other hand, let V_x be a neighborhood of x which intersects only a finite number of U_i, let say $U_{j_0}, \ldots, U_{j_n}$, where $j_0 < j_1 < \ldots < j_n$. For all $x' \in V_x$, if $J(x) = \{i \mid \lambda_i(x') > 0\}$, then $J(x') \subset \{j_0, \ldots, j_n\}$ and

$$f(x') = \Phi^{\{y_{j_0}, \ldots, y_{j_n}\}}\left(\lambda_{j_0}(x'), \ldots, \lambda_{j_n}(x')\right).$$

It then follows from *(i)* in Definition (3.3) that f is continuous at x. □

We are now ready to state the main result of this section. Let X and Y be two topological spaces. For a given convexity structure C on Y, let us define the following classes of correspondences:

$$\mathcal{C}(X,Y) := \{\Phi : X \to Y \mid \Phi \text{ is u.s.c. with nonempty } \mathcal{C}-\text{convex values}\}.$$

If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are uniform spaces, we use the continuity property (given in Section 2) :

$$\forall V \in \mathcal{V},\ \forall x \in X,\ \exists U \in \mathcal{U},\ \Phi(U[x]) \subset V[\Phi(x)] \tag{1}$$

and set:

$$\mathcal{C}(X,Y) := \{\Phi : X \to Y \mid \Phi \text{ satisfies (1) and has nonempty } \mathcal{C}-\text{convex values}\}.$$

We will write $\mathcal{C}(X)$ for $\mathcal{C}(X,X)$.

PROPOSITION (3.9). *In all the three following cases:*
(i) $\mathcal{C}$ is a H-structure,
(ii) $\mathcal{C}$ is a B-simplicial convexity,
(iii) $\mathcal{C}$ is a B'-convexity,
we have $\mathcal{C}(X,Y) \subset \mathcal{A}(X,Y)$ provided X is paracompact and Y is a locally $\mathcal{C}$-convex space.
The same is true if $\mathcal{C}$ is an L-structure and if X is compact.

Proof. Recalling that a paracompact topological space is uniformizable, let $\mathcal{U}$ be a basis of symmetric entourages for the uniformity on X. Let us also assume that $(Y,\mathcal{V})$ is endowed with a $\mathcal{C}$-convexity such that for every $V \in \mathcal{V}$, $V[A]$ is $\mathcal{C}$-convex whenever A is $\mathcal{C}$-convex. Given $\Phi \in \mathcal{C}(X,Y)$ and $(U,V) \in \mathcal{U} \times \mathcal{V}$, let $\Psi_{U,V} : X \to Y$ be the open-graph majorant of Φ given by Lemma (2.3). By construction, the values of $\Psi_{U,V}$ being finite intersections of $\mathcal{C}$-convex sets (namely, V-neighborhoods of values of Φ) are also $\mathcal{C}$-convex. By Proposition (3.8), in cases (i), (ii) and (iii), the map $\Psi_{U,V}$ admits a continuous selection $s_{U,V}$ satisfying $s_{U,V}(x) \in \Psi_{U,V}(x) \subseteq V[\Phi(U[x])], \forall x \in X$. The entourages U and V being arbitrary, it follows that $\Phi \in \mathcal{A}(X,Y)$.

If X is compact and $\mathcal{C}$ is an L-structure, the existence of a selection follows from Llinares [19]. □

COROLLARY (3.10). *Under the assumptions of the previous proposition, if X is compact, for every $(U,V) \in \mathcal{U} \times \mathcal{V}$, there is a finite subset A of Y such that $s_{U,V}(X) \subset \Gamma(A)$. Moreover, in all cases where $s_{U,V}$ is proved to exist, it can be chosen so that $s_{U,V}(X)$ is contained in any $\mathcal{C}$-convex subset of Y containing $\Phi(X)$.*

Proof. The first assertion is obvious given the proof of the existence of $s_{U,V}$. To prove our last assertion, it suffices to apply Proposition (3.9), replacing Y by $\mathrm{co}_{\mathcal{C}}(\Phi(X))$. □

Corollary (3.10) extends Theorem 6 in Horvath [15], stated for the case where X and Y are metric spaces. Both have as an obvious corollary Theorem 1 in Cellina [7], stated for the case where X is a metric space, Y a Frechet linear space (where balls are convex) and the correspondences Φ are compact.

For other examples of non-convex approachable maps, the reader is referred to Ben-El-Mechaiekh and Deguire [5] and Ben-El-Mechaiekh [4].

4. THE MAIN THEOREM

The Himmelberg fixed point theorem for self-correspondences defined on convex subsets of locally convex topological vector spaces was extended to the class $\mathcal{A}$ by the first author as follows.

LEMMA (4.1), [4]. *If X is a nonempty convex subset of a locally convex topological vector space and if $\Phi \in \mathcal{A}(X)$ is upper semicontinuous with nonempty closed values, then Φ has a fixed point provided $\Phi(X)$ is contained in a compact subset K of X.*

We provide now a generalization of this result to a class of uniform L-spaces satisfying an additional property borrowed from Horvath [14].

THEOREM (4.2). *Assume that $(X, \mathcal{U}, \Gamma)$ is a uniform L-space such that for every $U \in \mathcal{U}$, there exist two correspondences $S : X \to X$ and $T : X \to X$ (depending on U) satisfying :*

(i) $\forall x \in X,\ S(x) \subset T(x)$

(ii) $\forall x \in X,\ \forall A \in \langle S(x) \rangle,\ \Gamma(A) \subset T(x)$

(iii) $X = \bigcup \{ intS^{-1}(y),\ y \in X \}$

(iv) $\forall x \in X,\ T(x) \subset U[x]$.

Assume also that $\Phi \in \mathcal{A}(X)$ is u.s.c. with nonempty closed values. Then Φ has a fixed point, provided $\Phi(X)$ is contained in a compact subset K of X.

Proof. Let $U \in \mathcal{U}$ be arbitrary but fixed and consider a cover of K by a finite collection $\{\mathrm{int}S^{-1}(y_i)\}_{i=0}^n$ and a continuous partition of unity $\alpha = (\alpha_i)_{i=0}^n$ subordinated to this cover.

Applying condition $(*)$ in the definition of L-convexity, there exists a continuous function $f : \Delta_N \to X$ such that $\forall J \in \langle N \rangle,\ f(\Delta_J) \subset \Gamma(\{y_i \mid i \in J\}$. Note that for every $x \in K$,

$$f \circ \alpha(x) \in \Gamma(\{y_i \mid x \in \mathrm{int}S^{-1}[y_i]\}) \subset \Gamma(\{y_i \mid y_i \in S(x)\} \subset T(x)$$

so that $f \circ \alpha(x) \in U[x]$.

Let us now consider the correspondence

$$\Psi = \alpha \circ \Phi \circ f \ : \Delta_n \xrightarrow{f} X \xrightarrow{\Phi} K \xrightarrow{\alpha} \Delta_n.$$

Since Δ_n is compact, it follows from Lemma 2.3 that Ψ is approachable. By Lemma 4.1, Ψ has a fixed point $\bar{s} \in \Delta_n$, that is $\bar{s} \in \alpha \circ \Phi\big(f(\bar{s})\big)$. Recalling that the previous results depend on the choice of U, let $\bar{x}_U = f(\bar{s})$. Then,

$$\bar{x}_U \in f \circ \alpha \circ \Phi(\bar{x}_U) \in U[\Phi(\bar{x}_U].$$

Clearly, x_U is a U- approximative fixed point for Φ. Since U was arbitrarily chosen and Φ is u.s.c. with compact values, the net $\{x_U\}$ has an accumulation point which is a fixed point for Φ. □

Note that the condition (iv) in the previous theorem means that T and Id_X are U-near.

Clearly, if $(X, \mathcal{U}, \Gamma)$ is such that for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$, $V \subset U$ such that $\forall x \in X$, $\forall A \in \langle V[x] \rangle$, $\Gamma(A) \subset U[x]$, then the correspondences defined by $T(x) = U[x]$ and $S(x) = (\mathrm{int}V)[x]$ satisfy the conditions (i) - (iv) in the previous theorem, so that we have the following corollaries.

COROLLARY (4.3). *Assume that $(X, \mathcal{U}, \Gamma)$ is a uniform L-space such that for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$, $V \subset U$ such that $\forall x \in X$, $\forall A \in \langle V[x] \rangle$, $\Gamma(A) \subset U[x]$. Assume also that $\Phi \in \mathcal{A}(X)$ is u.s.c. with non empty closed values. Then Φ has a fixed point, provided $\Phi(X)$ is contained in a compact subset K of X.*

COROLLARY (4.4). *Assume that $(X, \mathcal{U}, \Gamma)$ is a uniform L-space such that for every $U \in \mathcal{U}$ and for every $x \in X$, the ball $U[x]$ is convex. Assume also that $\Phi \in \mathcal{A}(X)$ is u.s.c. with non empty closed values. Then Φ has a fixed point, provided $\Phi(X)$ is contained in a compact subset K of X.*

REMARK (4.5). In Corollary (4.3), the condition on $(X, \mathcal{U}, \Gamma)$ corresponds to what is called by Bielawski [6] "local simplicial convexity" when the convexity on X is a simplicial convexity.

REMARK (4.6). Obviously, under the condition of the previous theorem (or of Corollaries (4.3) and (4.4)) on $(X, \mathcal{U}, \Gamma)$, a compact continuous function from X to X has a fixed point. This remark generalizes Theorem 4, Section 4 in Horvath [14] and Remark (2.4) in Bielawski [6].

Therefore, combining Corollary (4.4) with Proposition (3.9), we obtain the two following consequences.

COROLLARY (4.7). *Assume that C is either an H-structure or alternatively a simplicial convexity or a B'- simplicial convexity, that $(X,\mathcal{U})$ is a nonempty paracompact locally C-convex space where the balls $U[x], x \in X$, are C-convex (or equivalently, since X is Hausdorff, such that $\Gamma(\{x\}) = \{x\}, x \in X$) and that $\Phi \in \mathcal{C}(X)$ has nonempty closed values in X. Then Φ has a fixed point, provided $\Phi(X)$ is contained in a compact subset K of X.*

The same is true if C is an L-structure and if X is compact.

COROLLARY (4.8). *In the previous statement, the condition that X is paracompact can be replaced by X metric.*

Recalling that X as in Corollary (4.8) is called an lc-metric space by Horvath, Corollary (4.8) generalizes in several respects the Corollary to Theorem 6 in Horvath [15] and also, in the particular case where the correspondence is a single-valued function, the Corollary 4 to Proposition 1, Section 4 in Horvath [14].

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