Relative asymptotics and Fourier series of orthogonal polynomials with a discrete Sobolev inner product

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Abstract

Let μ be a finite positive Borel measure supported in $[-1,1]$ and introduce the discrete Sobolev type inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,d\mu(x) + \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k)\; g^{(i)}(a_k),$$

where the mass points a_k belong to $[-1,1]$, $M_{k,i} \geqslant 0$, $i = 0, \ldots, N_k - 1$, and $M_{k,N_k} > 0$. In this paper, we study the asymptotics of the Sobolev orthogonal polynomials by comparison with the orthogonal polynomials with respect to the measure μ and we prove that they have the same asymptotic behaviour. We also study the pointwise convergence of the Fourier series associated to this inner product provided that μ is the Jacobi measure. We generalize the work done by F. Marcellán and W. Van Assche where they studied the asymptotics for only one mass point in $[-1,1]$. The same problem with a finite number of mass points off $[-1,1]$ was solved by G. López, F. Marcellán and W. Van Assche in a more general setting: they consider the constants $M_{k,i}$ to be complex numbers. As regards the Fourier series, we continue the

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results achieved by F. Marcellán, B. Osilenker and I.A. Rocha for the Jacobi measure and mass points in $\mathbb{R}\backslash[\ \ 1,1]$.

Keywords: Orthogonal polynomials; Sobolev inner product; Fourier series

1. Introduction

Let μ be a finite positive Borel measure supported on the interval $[-1,1]$ with infinitely many points at the support and let $a_k,\ k=1,\dots,K$, be real numbers such that $a_k \in [-1,1]$. For f and g in $L^2(\mu)$ such that there exist the derivatives in a_k, we can introduce the Sobolev-type inner product

$$\langle f,g\rangle = \int_{1}^{1} f(x)g(x)\,d\mu(x) + \sum_{k=1}^{K}\sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k)\ g^{(i)}(a_k), \tag{1}$$

where $M_{k,i} \geqslant 0$ for $i=0,\dots,N_k-1$, and $M_{k,N_k}>0$ when $k=1,\dots,K$. We assume $\mu(\{a_k\})=0$, otherwise the corresponding $M_{k,0}$ should be modified. Let $(\widehat{B}_k)_{k=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to this inner product,

$$\langle \widehat{B}_n, \widehat{B}_k \rangle = \delta_{n,k}, \quad k,n=0,1,\dots$$

López et al. [1] deduced the relative asymptotics for the orthogonal polynomials with respect to the Sobolev inner product with mass points outside $[-1,1]$ and complex constants $M_{k,i}$. Marcellán and Van Assche [4] analyzed such a question when there is only one mass point inside $[-1,1]$. Here we deal with an extension of this last problem with a finite number of masses. We compare the polynomials $\widehat{B}_n$ with the polynomials $(p_n)_{n=0}^{\infty}$ orthonormal with respect to μ. The technique used in this paper is a generalization of the one used for obtaining estimates of the Sobolev orthogonal polynomials in [2,3]. There, F. Marcellán, B. Osilenker and I. A. Rocha studied the pointwise convergence of the Fourier series for sequences of orthogonal polynomials with respect to the inner product (1) for the Jacobi measure and with mass points outside $[-1,1]$.

The main results concerning asymptotic properties are given in Section 2. In Theorem 2.1 we prove that $\frac{\widehat{B}_n(x)}{p_n(x)}$ tends to 1, in Theorem 2.2 we obtain for $\widehat{B}_n(x)$ the usual weak asymptotics, and, in Theorem 2.3, the asymptotics for the coefficients in the recurrence relation of the Sobolev orthonormal polynomials are given.

In Section 3, we consider the pointwise convergence of the Fourier series with respect to (1) provided that μ is the Jacobi measure. We continue the work achieved in [2,3] and prove the pointwise convergence for the Fourier series of functions which satisfy some standard sufficient conditions as in the previous mentioned papers. Although the techniques used there are not valid when the mass points lie in $[-1,1]$, following the same idea, they can be generalized and, apart from the estimates of $\widehat{B}_n$

which allow more precise results for the behaviour of the kernels, the same conclusions follow.

2. Asymptotics

From now on $\kappa(\Pi_n)$ denotes the leading coefficient of any polynomial Π_n with real coefficients, and n is the degree of the polynomial.

Let N_k^* be the positive integer number defined by

$$N_k^* = \begin{cases} N_k + 1 & \text{if } N_k \text{ is odd,} \\ N_k + 2 & \text{if } N_k \text{ is even,} \end{cases}$$

and let $w_N(x) = \prod_{k=1}^{K} (x - a_k)^{N_k^*}$ where $N = \sum_{k=1}^{K} N_k^*$. Let $(p_n)_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to μ.

Lemma 2.1.

$$w_N(x)\widehat{B}_n(x) = \sum_{j=0}^{2N} A_{n,j} p_{n+N-j}(x), \quad A_{n,0} \neq 0.$$

Moreover, $A_{n,j}$ *are bounded and* $A_{n,2N} = \frac{\kappa(p_{n-N})}{\kappa(p_{n+N})} \frac{1}{A_{n,0}} \neq 0$.

Proof. Since $w_N(x)\widehat{B}_n(x) = \sum_{j=0}^{n+N} \alpha_{n,j} p_j(x)$ and

$$\alpha_{n,j} = \int_{-1}^{1} w_N(x)\widehat{B}_n(x) p_j(x)\, d\mu(x) = \langle \widehat{B}_n, w_N p_j \rangle = 0, \quad j < n - N,$$

we have the first assertion with $A_{n,j} = \alpha_{n,n+N-j}$. Furthermore,

$$\sum_{j=0}^{2N} A_{n,j}^2 = \int_{-1}^{1} \widehat{B}_n^2(x) w_N^2(x)\, d\mu(x) \leqslant \max_{x \in [-1,1]} w_N^2(x)$$

and thus $(A_{n,j})$ are bounded. Also $A_{n,0} = \int_{-1}^{1} w_N(x)\widehat{B}_n(x) p_{n+N}(x)\, d\mu(x) = \frac{\kappa(\widehat{B}_n)}{\kappa(p_{n+N})}$ as well as

$$\begin{aligned} A_{n,2N} &= \int_{-1}^{1} w_N(x)\widehat{B}_n(x) p_{n-N}(x)\, d\mu(x) = \langle \widehat{B}_n, w_N p_{n-N} \rangle = \frac{\kappa(p_{n-N})}{\kappa(\widehat{B}_n)} \\ &= \frac{\kappa(p_{n-N})}{\kappa(p_{n+N})} \frac{1}{A_{n,0}} \end{aligned}$$

and the lemma holds. □

Let Λ be a sequence of nonnegative integers such that $\lim_{n \in \Lambda} A_{n,j} = A_j$ for $j = 0, \ldots, 2N$. When $\mu'(x) > 0$ a.e., since $A_0 < \infty$ and $\lim_{n \to \infty} \frac{\kappa(p_{n-N})}{\kappa(p_{n+N})} = \frac{1}{2^{2N}}$ as it is well known (see [5,6]), A_{2N} has to be greater than zero. Let

$$\Pi_{2N}(x) = \sum_{j=0}^{2N} \frac{A_j}{A_{2N}} T_j(x),$$

where, for each j, $T_j(x)$ is the Chebyshev polynomial of the first kind and degree j.

Lemma 2.2. *If $\mu'(x) > 0$ a.e. then the polynomial Π_{2N} satisfies $\Pi_{2N}^{(i)}(a_k) = 0$ for $i = 0, 1, \ldots, N_k^* - 1$ and $k = 1, \ldots, K$.*

Proof. For a given $k \in \{1, 2, \ldots, K\}$, let $\varepsilon > 0$ and $i \in \{1, 2, \ldots, N_k^*\}$. Consider the function

$$\psi_{i,\varepsilon}(x) = \begin{cases} 0 & \text{if } x \in [-1, a_k + \varepsilon], \\ \dfrac{1}{(x - a_k)^i} & \text{if } x \in (a_k + \varepsilon, 1]. \end{cases}$$

This function is bounded in $[-1, 1]$ and satisfies the condition $\max_{x \in [\ 1,1]} |w_N(x)\psi_{i,\varepsilon}(x)| \leqslant C$ for some constant C independent of ε.

As it is well known (see [5,6]), since $\mu'(x) > 0$ a.e.,

$$\lim_{n \to \infty} \int_{1}^{1} f(x) p_{n+v}(x) p_n(x)\, d\mu(x) = \frac{1}{\pi} \int_{1}^{1} f(x) T_v(x) \frac{dx}{\sqrt{1 - x^2}}$$

for all Borel measurable function f bounded on $[-1, 1]$. As a consequence, the expression of $w_N(x)\widehat{B}_n(x)$ in terms of $(p_j)_{j=0}^{n+N}$ of Lemma 2.1 gives

$$\lim_{n \in \Lambda} \int_{1}^{1} w_N(x) \widehat{B}_n(x) p_{n+N}(x) \psi_{i,\varepsilon}(x)\, d\mu(x) = \frac{1}{\pi} \int_{1}^{1} \sum_{v=0}^{2N} A_v T_v(x) \psi_{i,\varepsilon}(x) \frac{dx}{\sqrt{1 - x^2}}.$$

From the Cauchy–Schwarz inequality,

$$\left| \int_{1}^{1} \widehat{B}_n(x) p_{n+N}(x) w_N(x) \psi_{i,\varepsilon}(x)\, d\mu(x) \right| \leqslant C \left(\int_{1}^{1} \widehat{B}_n^2(x)\, d\mu(x) \right)^{1/2} \left(\int_{1}^{1} p_{n+N}^2(x)\, d\mu(x) \right)^{1/2} \leqslant C$$

and we get

$$\limsup_{\varepsilon \to 0} \left| \frac{1}{\pi} \int_{1}^{1} \sum_{v=0}^{2N} A_v T_v(x) \psi_{i,\varepsilon}(x) \frac{dx}{\sqrt{1 - x^2}} \right| \leqslant C \quad \text{for } i = 1, \ldots, N_k^*. \tag{2}$$

When $i = 1$ we have

$$\int_{1}^{1} \Pi_{2N}(x) \psi_{1,\varepsilon}(x) \frac{dx}{\sqrt{1 - x^2}} = \Pi_{2N}(a_k) \int_{1}^{1} \psi_{1,\varepsilon}(x) \frac{dx}{\sqrt{1 - x^2}} + \int_{1}^{1} (\Pi_{2N}(x) - \Pi_{2N}(a_k)) \psi_{1,\varepsilon}(x) \frac{dx}{\sqrt{1 - x^2}}.$$

Thus, condition (2) holds if and only if $\Pi_{2N}(a_k) = 0$. If we suppose now $\Pi_{2N}^{(j)}(a_k) = 0$ for $0 \leqslant j \leqslant i - 1 \leqslant N_k^* - 2$ and write $\Pi_{2N}(x) = (x - a_k)^i \Pi_{2N\ \ i}(x)$, with the same argument for $\int_{1}^{1} (x - a_k)^i \Pi_{2N\ \ i}(x) \psi_{i+1,\varepsilon}(x) \frac{dx}{\sqrt{1\ \ x^2}}$ we get $\Pi_{2N}^{(i)}(a_k) = 0$. Note that when $a_k = 1$ the definition of $\psi_{i,\varepsilon}$ must be modified in the obvious way. □

Lemma 2.3. *If* $\mu'(x) > 0$ *a.e. then*

$$\Pi_{2N}(x) = 2^N T_N(x) \prod_{k=1}^{K} (x - a_k)^{N_k^*}.$$

Proof. For a given $k \in \{1, \ldots, K\}$, let $\psi_{i,\varepsilon}(x)$, $\varepsilon > 0$ and $i = 1, \ldots, N_k^*$, be the functions given by

$$\psi_{i,\varepsilon}(x) = \begin{cases} \dfrac{1}{(x - a_k)^i} & \text{if } |x - a_k| > \varepsilon, \\ 0 & \text{if } |x - a_k| \leqslant \varepsilon. \end{cases}$$

Using Lemma 2.2, write $\sum_{v=0}^{2N} A_v T_v(x) = \prod_{k=1}^{K} (x - a_k)^{N_k^*} R_N(x)$ where $R_N(x)$ is a polynomial of degree N. From the boundedness of $\psi_{i,\varepsilon}(x)$ we get

$$\begin{aligned} &\lim_{\varepsilon \to 0} \lim_{n \in \Lambda} \int_{1}^{1} w_N(x) \widehat{B}_n(x) p_{n+N}(x) \psi_{i,\varepsilon}(x)\, d\mu(x) \\ &\quad = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{1}^{1} \sum_{v=0}^{2N} A_v T_v(x) \psi_{i,\varepsilon}(x) \frac{dx}{\sqrt{1 - x^2}} \\ &\quad = \frac{1}{\pi} \int_{1}^{1} \frac{\prod_{j=1}^{K} (x - a_j)^{N_j^*}}{(x - a_k)^i} R_N(x) \frac{dx}{\sqrt{1 - x^2}} \end{aligned} \tag{3}$$

because $\prod_{j=1}^{K} (x - a_j)^{N_j^*} \psi_{i,\varepsilon}(x)$ are bounded and as a consequence of the Lebesgue dominated convergence Theorem. Moreover

$$\begin{aligned} &\left| \int_{1}^{1} \widehat{B}_n(x) p_{n+N}(x) \frac{w_N(x)}{(x - a_k)^i} d\mu(x) - \frac{1}{\pi} \int_{1}^{1} \sum_{v=0}^{2N} A_v T_v(x) \frac{1}{(x - a_k)^i} \frac{dx}{\sqrt{1 - x^2}} \right| \\ &\quad \leqslant \left| \int_{1}^{1} \widehat{B}_n(x) p_{n+N}(x) \frac{w_N(x)}{(x - a_k)^i} d\mu(x) - \int_{1}^{1} \widehat{B}_n(x) p_{n+N}(x) w_N(x) \psi_{i,\varepsilon}(x)\, d\mu(x) \right| \\ &\qquad + \left| \int_{1}^{1} \widehat{B}_n(x) p_{n+N}(x) w_N(x) \psi_{i,\varepsilon}(x)\, d\mu(x) \right. \\ &\qquad \left. - \frac{1}{\pi} \int_{1}^{1} \sum_{v=0}^{2N} A_v T_v(x) \frac{1}{(x - a_k)^i} \frac{dx}{\sqrt{1 - x^2}} \right| \\ &\quad = I_{n,\varepsilon}^{(1)} + I_{n,\varepsilon}^{(2)}. \end{aligned}$$

Given $\delta > 0$, from (3), $\lim_{n\in\Lambda} I^{(2)}_{n,\varepsilon} < \delta$ for $\varepsilon > 0$ small enough. On the other hand,

$$I^{(1)}_{n,\varepsilon} = \left| \int_{a_k-\varepsilon}^{a_k+\varepsilon} \widehat{B}_n(x) p_{n+N}(x) \frac{w_N(x)}{(x-a_k)^i}\, d\mu(x) \right|$$

and, since there is a constant C, independent from ε and i, such that $\left|\frac{w_N(x)}{(x-a_k)^i}\right| \leqslant C$, from the Cauchy–Schwarz inequality,

$$I^{(1)}_{n,\varepsilon} \leqslant C \left(\int_{a_k-\varepsilon}^{a_k+\varepsilon} p^2_{n+N}(x)\, d\mu(x) \right)^{1/2}.$$

But $p^2_{n+N}(x)\, d\mu(x) \xrightarrow{*} \frac{1}{\pi} \frac{dx}{\sqrt{1-x^2}}$ and it means that, for ε small enough, $\limsup_{n\to\infty} I^{(1)}_{n,\varepsilon} < \delta$. As a consequence,

$$\limsup_{n\in\Lambda} \left| \int_{-1}^{1} \widehat{B}_n(x) p_{n+N}(x) \frac{w_N(x)}{(x-a_k)^i}\, d\mu(x) - \frac{1}{\pi} \int_{-1}^{1} \sum_{v=0}^{2N} A_v T_v(x) \frac{1}{(x-a_k)^i} \frac{dx}{\sqrt{1-x^2}} \right| < 2\delta$$

for $\delta > 0$. By orthogonality,

$$\int_{-1}^{1} \widehat{B}_n(x) p_{n+N}(x) \frac{w_N(x)}{(x-a_k)^i}\, d\mu(x) = 0, \quad i = 1, \ldots, N^*_k, \;\; k = 1, \ldots, K$$

and $R_N(x)$ satisfies

$$\int_{-1}^{1} \prod_{k=1}^{K} (x-a_k)^{N^*_k} \frac{1}{(x-a_k)^i} R_N(x) \frac{dx}{\sqrt{1-x^2}} = 0, \quad i = 1, \ldots, N^*_k, \;\; k = 1, \ldots, K.$$

Since $\{\frac{1}{(x-a_k)^i} \prod_{k=1}^{K} (x-a_k)^{N^*_k} \colon k = 1, \ldots, K; \; i = 1, \ldots, N^*_k\}$ is a basis of the space of polynomials of degree less than or equal to $N-1$, $R_N(x)$ is the Chebyshev polynomial of the first kind and degree N up to a constant factor. If we compare the leading coefficients, $\sum_{v=0}^{2N} \frac{A_v}{A_{2N}} T_v(x) = 2^N T_N(x) \prod_{k=1}^{K} (x-a_k)^{N^*_k}$ and the proof is complete. $\square$

Denote $\varphi^{\pm}(x) = x \pm \sqrt{x^2-1}$ where the square root is such that $|\varphi^{-}(x)| < 1$, $x \in \mathbb{C} \setminus [-1, 1]$.

Lemma 2.4. *If* $\mu'(x) > 0$ *a.e., the coefficients* $A_{n,v}$ *satisfy*

(i) $\lim_{n\to\infty} A_{n,v} = A_v$, $v = 0, \ldots, 2N$ *where* $\sum_{v=0}^{2N} A_v T_v(x) = \prod_{k=1}^{K} (x-a_k)^{N^*_k} T_N(x)$.
(ii) $\sum_{v=0}^{2N} A_v (\varphi^{-}(x))^v = \frac{1}{2^N} \prod_{k=1}^{K} ((\varphi^{-}(x))^2 - 2a_k \varphi^{-}(x) + 1)^{N^*_k}$.

Proof. From Lemma 2.1 and the ratio asymptotics of p_n with $\mu'(x)>0$ a.e., we get

$$\lim_{n\in\Lambda}\frac{w_N(x)\widehat{B}_n(x)}{p_{n+N}(x)}=\lim_{n\in\Lambda}\sum_{v=0}^{2N}A_{n,v}\frac{p_{n+N\quad v}(x)}{p_{n+N}(x)}=\sum_{v=0}^{2N}A_v[\varphi\quad(x)]^v$$

uniformly in compact sets of $\mathbb{C}\backslash[-1,1]$. Denoting again $\Pi_{2N}(x)=\sum_{v=0}^{2N}\frac{A_v}{A_N}T_v(x)$, since

$$\sum_{v=0}^{2N}\frac{A_v}{A_N}(\varphi\quad(x))^v=\frac{\sqrt{x^2-1}}{\pi}\int_{\ \ 1}^{1}\frac{\Pi_{2N}(t)}{x-t}\frac{dt}{\sqrt{1-t^2}}$$

as it can be deduced from the residue theorem after the change $t=\cos\theta$, the expression of Π_{2N} in Lemma 2.3 gives

$$\begin{aligned}\sum_{v=0}^{2N}\frac{A_v}{A_{2N}}(\varphi\quad(x))^v&=\frac{-\sqrt{x^2-1}}{2\pi i}\int_{|\xi|=1}\frac{(1+\xi\quad^{2N})\prod_{k=1}^{K}(\xi^2-2a_k\xi+1)^{N_k^*}}{(\xi-\varphi\quad(x))(\xi-\varphi^+(x))}d\xi\\&=\frac{-\sqrt{x^2-1}}{\pi i}\int_{|\xi|=1}\frac{\prod_{k=1}^{K}(\xi^2-2a_k\xi+1)^{N_k^*}}{(\xi-\varphi\quad(x))(\xi-\varphi^+(x))}d\xi\\&=\prod_{k=1}^{K}((\varphi\quad(x))^2-2a_k\varphi\quad(x)+1)^{N_k^*}.\end{aligned}$$

In particular, this means that $\frac{A_0}{A_{2N}}=\lim_{x\to\infty}\sum_{v=0}^{2N}\frac{A_v}{A_{2N}}(\varphi\quad(x))^v=1$; but, from Lemma 2.1, $A_{2N}=\lim_{n\to\infty}\frac{\kappa(\,p_n\ \ _N)}{\kappa(\,p_{n+N})}\frac{1}{A_0}=\frac{1}{2^{2N}A_0}$ and $A_0=A_{2N}=\frac{1}{2^N}$ follows.

Now the coefficients A_j are completely determined for any subsequence Λ and we can assert that $\lim_{n\to\infty}A_{n,v}=A_v$, $v=0,1\ldots,2N$, with

$$\sum_{v=0}^{2N}A_vT_v(x)=\prod_{k=1}^{K}(x-a_k)^{N_k^*}T_N(x),$$

$$\sum_{v=0}^{2N}A_v[\varphi\quad(x)]^v=\frac{1}{2^N}\prod_{k=1}^{K}((\varphi\quad(x))^2-2a_k\varphi\quad(x)+1)^{N_k^*}.\qquad\square$$

Theorem 2.1. *If $\mu'(x)>0$ a.e. then*

(i)
$$\lim_{n\to\infty}\frac{\widehat{B}_n(x)}{p_n(x)}=1$$
uniformly on compact subsets of $\mathbb{C}\backslash[-1,1]$.

(ii) *$n-N$ zeros of $\widehat{B}_n(x)$ belong to $[-1,1]$ and the other N zeros accumulate in $[-1,1]$.*

(iii)
$$\lim_{n\to\infty}\frac{\widehat{B}_{n+1}(x)}{\widehat{B}_n(x)}=x+\sqrt{x^2-1}$$
uniformly on compact subsets of $\mathbb{C}\backslash[-1,1]$.

(iv) *If* $\int_{-1}^{1} \log \mu'(x) \frac{dx}{\sqrt{1-x^2}} > -\infty$ *then*

$$\lim_{n\to\infty} \frac{\widehat{B}_n(x)}{(x+\sqrt{x^2-1})^n} = S_{\mu'}(x)$$

uniformly on compact subsets of $\mathbb{C}\setminus[-1,1]$. *Here* $S_{\mu'}(x)$ *denotes the Szegő function of* $\mu'(x)$. (*see* [9, *Theorem* 12.1.2] *as well as the definition in p*. 276).

Proof. Item (ii) follows from $\int_{-1}^{1} x^k \widehat{B}_n(x) w_N(x)\, d\mu(x) = 0$ for $k+N<n$ and formula (i). Items (iii) and (iv) are consequences of (i) and the well-known ratio and strong asymptotics of p_n. So, we only need to prove (i).

From Lemma 2.4 we have

$$\lim_{n\to\infty} w_N(x) \frac{\widehat{B}_n(x)}{p_{n+N}(x)} = \sum_{j=0}^{2N} A_j(\varphi^{-}(x))^j = \frac{1}{2^N} \prod_{k=1}^{K} ((\varphi^{-}(x))^2 - 2a_k \varphi^{-}(x) + 1)^{N_k^*},$$

which yields

$$\lim_{n\to\infty} w_N(x) \frac{\widehat{B}_n(x)}{p_n(x)} = (\varphi^{+}(x))^N \frac{1}{2^N} \prod_{k=1}^{K} ((\varphi^{-}(x))^2 - 2a_k \varphi^{-}(x) + 1)^{N_k^*} = w_N(x). \qquad \square$$

Remark.

1. This relative asymptotics is the same as the relative asymptotics analyzed by Nevai [7, Lemma 16, p. 132], where he adds to a measure μ in the class $M(0,1)$ a Dirac mass located in $[-1,1]$.
2. Formula (i) of Theorem 2.1 remains true for a measure μ in the Nevai class $M(0,1)$ because the only facts we use are the ratio and weak asymptotics of $(p_n)_{n=0}^{\infty}$ and the asymptotics of $\kappa(p_n)$, and all these properties are still valid for a measure in $M(0,1)$. We only should replace $[-1,1]$ with *supp*(μ) and everything works in the same way.

Now we will prove weak asymptotic properties for the Sobolev polynomials $\widehat{B}_n$. To do it we need some auxiliary results.

Lemma 2.5. *With the previous notation, if* $\mu'(x) > 0$ *a.e., we get*

$$w_N^2(x) = \sum_{j=0}^{2N} A_j^2 T_0(x) + 2 \sum_{v=1}^{2N} \sum_{j=0}^{2N\ v} A_j A_{j+v} T_v(x).$$

Proof. From Lemma 2.4, $\sum_{j=0}^{2N} A_j T_j(x) = T_N(x) \prod_{k=1}^{K} (x - a_k)^{N_k^*}$. Besides, as it was proved, $A_{2N} = A_0$, and for $j = 1, \ldots, N-1$, we get

$$\begin{aligned}
\frac{1}{2} A_{N+j} =& \frac{1}{\pi} \int_{1}^{1} \prod_{k=1}^{K} (x - a_k)^{N_k^*} T_N(x) T_{N+j}(x) \frac{dx}{\sqrt{1 - x^2}} \\
=& \frac{1}{2\pi} \int_{1}^{1} \prod_{k=1}^{K} (x - a_k)^{N_k^*} (T_{2N+j}(x) + T_j(x)) \frac{dx}{\sqrt{1 - x^2}} \\
=& \frac{1}{2\pi} \int_{1}^{1} \prod_{k=1}^{K} (x - a_k)^{N_k^*} T_j(x) \frac{dx}{\sqrt{1 - x^2}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2} A_{N\ j} =& \frac{1}{\pi} \int_{1}^{1} \prod_{k=1}^{K} (x - a_k)^{N_k^*} T_N(x) T_{N\ j}(x) \frac{dx}{\sqrt{1 - x^2}} \\
=& \frac{1}{2\pi} \int_{1}^{1} \prod_{k=1}^{K} (x - a_k)^{N_k^*} (T_{2N\ j}(x) + T_j(x)) \frac{dx}{\sqrt{1 - x^2}} \\
=& \frac{1}{2\pi} \int_{1}^{1} \prod_{k=1}^{K} (x - a_k)^{N_k^*} T_j(x) \frac{dx}{\sqrt{1 - x^2}},
\end{aligned}$$

which yields $A_{N+j} = A_{N\ j}$ for $j = 1, \ldots, N-1$. As a consequence

$$T_N(x) \prod_{k=1}^{K} (x - a_k)^{N_k^*} = A_N T_N(x) + \sum_{j=1}^{N} A_{N+j} (T_{N+j}(x) + T_{N\ j}(x))$$

and thus $\prod_{k=1}^{K} (x - a_k)^{N_k^*} = A_N T_0(x) + 2 \sum_{j=1}^{N} A_{N+j} T_j(x)$. Now, if we work out the coefficients of $w_N^2(x) = (A_N T_0(x) + 2 \sum_{j=1}^{N} A_{N+j} T_j(x))^2$ in terms of the polynomials $(T_v)_{v=0}^{2N}$, the statement of the Lemma follows. □

Lemma 2.6. *If* $\mu'(x) > 0$ *a.e. and* f *is a Borel measurable function bounded on* $[-1, 1]$ *then*

$$\begin{aligned}
&\lim_{n \to \infty} \int_{1}^{1} f(x) w_N^2(x) \widehat{B}_n(x) \widehat{B}_{n+k}(x)\, d\mu(x) \\
&= \frac{1}{\pi} \int_{1}^{1} f(x) w_N^2(x) T_k(x) \frac{dx}{\sqrt{1 - x^2}}, \quad k = 0, 1, \ldots
\end{aligned}$$

Proof. Let f be a Borel measurable function bounded on $[-1,1]$. Writing the polynomials $w_N(x)\widehat{B}_n$ in terms of $(p_n)_{n=0}^{\infty}$ as in Lemma 2.1, from the asymptotics of the polynomials p_n we get

$$\begin{aligned}
&\lim_{n\to\infty}\int_{-1}^{1} f(x)w_N(x)\widehat{B}_n(x)w_N(x)\widehat{B}_{n+k}(x)\,d\mu(x)\\
&=\lim_{n\to\infty}\int_{-1}^{1} f(x)\sum_{j=0}^{2N}A_{n,j}p_{n+N-j}(x)\sum_{v=0}^{2N}A_{n+k,v}p_{n+k+N-v}(x)\,d\mu(x)\\
&=\lim_{n\to\infty}\int_{-1}^{1} f(x)\left(\sum_{j=v}+\sum_{j>v}+\sum_{j<v}\right)(A_{n,j}A_{n+k,v}p_{n+N-j}(x)p_{n+k+N-v}(x))\,d\mu(x)\\
&=\frac{1}{\pi}\int_{-1}^{1} f(x)\left\{\sum_{j=0}^{2N}A_j^2T_k(x)+\sum_{j>v}A_jA_v(T_{k+j-v}(x)+T_{k-(j-v)}(x))\right\}\frac{dx}{\sqrt{1-x^2}}\\
&=\frac{1}{\pi}\int_{-1}^{1} f(x)\left\{\sum_{j=0}^{2N}A_j^2+2\sum_{j>v}A_jA_vT_{j-v}(x)\right\}T_k(x)\frac{dx}{\sqrt{1-x^2}}\\
&=\frac{1}{\pi}\int_{-1}^{1} f(x)\left\{\sum_{j=0}^{2N}A_j^2+2\sum_{v=1}^{2N}\sum_{j=0}^{2N-v}A_jA_{j+v}T_v(x)\right\}T_k(x)\frac{dx}{\sqrt{1-x^2}}\\
&=\frac{1}{\pi}\int_{-1}^{1} f(x)w_N^2(x)T_k(x)\frac{dx}{\sqrt{1-x^2}}
\end{aligned}$$

according to Lemma 2.5. □

Lemma 2.7. *If* $\mu'(x)>0$ *a.e. then*

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{a_k-\varepsilon}^{a_k+\varepsilon}\widehat{B}_n^2(x)\,d\mu(x)=0,\quad k=1,\ldots,K.$$

Proof. Denoting by $\|f\|=\langle f,f\rangle^{1/2}$ the Sobolev norm, for $k=1,\ldots,K$ and $i=0,\ldots,N_k$ we have

$$M_{k,i}\leqslant\inf\{\|\pi_n\|^2:\ \deg\pi_n\leqslant n,\ \pi^{(i)}(a_k)=1\}=\frac{1}{\sum_{v=0}^{n}(\widehat{B}_v^{(i)}(a_k))^2}$$

because $1=\sum_{v=0}^{n}c_v\widehat{B}_v^{(i)}(a_k)\leqslant\sum_{v=0}^{n}c_v^2\sum_{v=0}^{n}(\widehat{B}_v^{(i)}(a_k))^2$ and $\left\|\frac{\sum_{v=0}^{n}\widehat{B}_v^{(i)}(a_k)\widehat{B}_v(x)}{\sum_{v=0}^{n}(\widehat{B}_v^{(i)}(a_k))^2}\right\|^2=\frac{1}{\sum_{v=0}^{n}(\widehat{B}_v^{(i)}(a_k))^2}$. Then, for any (k,i) such that $M_{k,i}>0,\sum_{v=0}^{n}(\widehat{B}_v^{(i)}(a_k))^2\leqslant\frac{1}{M_{k,i}}$ and, in particular, $\widehat{B}_n^{(i)}(a_k)\to 0$ for $0\leqslant i\leqslant N_k$ and $1\leqslant k\leqslant K$ provided that the corresponding coefficient satisfies the condition $M_{k,i}>0$. As a consequence,

$$\lim_{n\to\infty}\int_{-1}^{1}\widehat{B}_n^2(x)\,d\mu(x)=1. \tag{4}$$

For $\varepsilon > 0$, let ψ_ε be the function defined by

$$\psi_\varepsilon(x) = \begin{cases} \frac{1}{w_N(x)} & \text{if } x \in [-1,1] \setminus \bigcup_{k=1}^{K} [a_k - \varepsilon, a_k + \varepsilon], \\ 0 & \text{if } x \in \bigcup_{k=1}^{K} [a_k - \varepsilon, a_k + \varepsilon]. \end{cases}$$

Then, using Eq. (4), Lemma 2.6 and dominated convergence theorem, we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \lim_{n \to \infty} \sum_{k=1}^{K} \int_{a_k\ \varepsilon}^{a_k + \varepsilon} \widehat{B}_n^2(x)\, d\mu(x) \\ &= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{1}^{1} (1 - \psi_\varepsilon^2(x) w_N^2(x)) \widehat{B}_n^2(x)\, d\mu(x) \\ &= \lim_{\varepsilon \to 0} \left(1 - \frac{1}{\pi} \int_{1}^{1} \psi_\varepsilon^2(x) w_N^2(x) \frac{dx}{\sqrt{1 - x^2}} \right) = 0, \end{aligned}$$

which gives the lemma. □

Now we can prove the weak convergence for the Sobolev orthonormal polynomials.

Theorem 2.2. *If $\mu'(x) > 0$ a.e. and f is a Borel measurable function bounded on $[-1,1]$ then*

$$\lim_{n \to \infty} \int_{1}^{1} f(x) \widehat{B}_n(x) \widehat{B}_{n+k}(x)\, d\mu(x) = \frac{1}{\pi} \int_{1}^{1} f(x) T_k(x) \frac{dx}{\sqrt{1 - x^2}}, \quad k = 0, 1, \ldots .$$

Proof. For $\varepsilon > 0$, let ψ_ε be the function defined in the previous lemma. Let f be a Borel measurable function bounded on $[-1,1]$. Since $f(x)\psi_\varepsilon^2(x)$ is also bounded, according to Lemma 2.6,

$$\begin{aligned} &\lim_{n \to \infty} \int_{1}^{1} f(x) \psi_\varepsilon^2(x) w_N^2(x) \widehat{B}_n(x) \widehat{B}_{n+v}(x)\, d\mu(x) \\ &= \frac{1}{\pi} \int_{1}^{1} f(x) \psi_\varepsilon^2(x) w_N^2(x) T_v(x) \frac{dx}{\sqrt{1 - x^2}} \end{aligned}$$

and, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} &\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{1}^{1} f(x) \psi_\varepsilon^2(x) w_N^2(x) \widehat{B}_n(x) \widehat{B}_{n+v}(x)\, d\mu(x) \\ &= \frac{1}{\pi} \int_{1}^{1} f(x) T_v(x) \frac{dx}{\sqrt{1 - x^2}}. \end{aligned} \tag{5}$$

Moreover

$$\left|\int_{-1}^{1} f(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)\,d\mu(x) - \frac{1}{\pi}\int_{-1}^{1} f(x)T_v(x)\frac{dx}{\sqrt{1-x^2}}\right|$$
$$\leqslant \left|\int_{-1}^{1} f(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)\,d\mu(x) - \int_{-1}^{1} f(x)\psi_\varepsilon^2(x)w_N^2(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)\,d\mu(x)\right|$$
$$+\left|\int_{-1}^{1} f(x)\psi_\varepsilon^2(x)w_N^2(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)\,d\mu(x) - \frac{1}{\pi}\int_{-1}^{1} f(x)T_v(x)\frac{dx}{\sqrt{1-x^2}}\right|$$
$$= I_{n,\varepsilon}^{(1)} + I_{n,\varepsilon}^{(2)}.$$

Given $\delta>0$, from (5), $\lim_{n\to\infty} I_{n,\varepsilon}^{(2)}<\delta$ for $\varepsilon>0$ small enough. On the other hand,

$$I_{n,\varepsilon}^{(1)} \leqslant \sum_{k=1}^{K}\left|\int_{a_k-\varepsilon}^{a_k+\varepsilon} f(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)\,d\mu(x)\right|.$$

Since f is bounded on $[-1,1]$, there exists a constant C such that $|f(x)|\leqslant C,\ x\in[-1,1]$, and we get

$$I_{n,\varepsilon}^{(1)} \leqslant C\sum_{k=1}^{K}\int_{a_k-\varepsilon}^{a_k+\varepsilon}|\widehat{B}_n(x)\widehat{B}_{n+v}(x)|\,d\mu(x) \leqslant C\sum_{k=1}^{K}\left(\int_{a_k-\varepsilon}^{a_k+\varepsilon}\widehat{B}_n^2(x)\,d\mu(x)\right)^{1/2}.$$

By Lemma 2.7, $\limsup_{n\to\infty} I_{n,\varepsilon}^{(1)}<\delta$ for ε small enough. Then

$$\limsup_{n\to\infty}\left|\int_{-1}^{1} f(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)\,d\mu(x) - \int_{-1}^{1} f(x)T_v(x)\frac{dx}{\pi\sqrt{1-x^2}}\right|<2\delta$$

and the proof is complete. □

Theorem 2.3. *The polynomials* $\widehat{B}_n$ *satisfy the recurrence relation*

$$w_N(x)\widehat{B}_n(x) = \sum_{j=-N}^{N}\alpha_{n,j}\widehat{B}_{n+j}(x),\quad \alpha_{n,-j}=\alpha_{n+j,j},\ j=1,\dots,N,\ \alpha_{n,N}\neq 0.$$

Furthermore if $\mu'(x)>0$ *a.e. then* $\lim_{n\to\infty}\alpha_{n,j}=\alpha_j,\ j=0,\dots,N,$ *where*

$$w_N(x) = \alpha_0 + 2\sum_{j=1}^{N}\alpha_j T_j(x)$$

and are given by $\alpha_j = \frac{1}{2^N}\frac{W_{2N}^{(N-j)}(0)}{(N-j)!},\ j=0,\dots,N,$ *with* $W_{2N}(\xi)=\prod_{k=1}^{K}(\xi^2-2a_k\xi+1)^{N_k^*}$.

Proof. We can write $w_N(x)\widehat{B}_n(x)=\sum_{j=0}^{n+N}\lambda_{n,j}\widehat{B}_j(x)$ where

$$\lambda_{n,j}=\langle w_N\widehat{B}_n,\widehat{B}_j\rangle = \int_{-1}^{1} w_N(x)\widehat{B}_n(x)\widehat{B}_j(x)\,d\mu(x) = \langle \widehat{B}_n, w_N\widehat{B}_j\rangle = 0$$
$$\text{for } j<n-N.$$

Thus we get the recurrence relation with $\alpha_{n,j} = \lambda_{n,n+j},\ j = -N, \ldots, N$. Moreover, for $j = 1, \ldots, N,\ \alpha_{n,\ j} = \langle w_N \widehat{B}_n, \widehat{B}_{n\ j} \rangle = \langle w_N \widehat{B}_{n\ j}, \widehat{B}_n \rangle = \alpha_{n\ j, j}$.

On the other hand, if $\mu'(x) > 0$ a.e., for $j = 0, \ldots, N$, from Theorem 2.2

$$\lim_{n\to\infty} \alpha_{n,j} = \lim_{n\to\infty} \int_{1}^{1} w_N(x)\widehat{B}_n(x)\widehat{B}_{n+j}(x)\, d\mu(x) = \frac{1}{\pi}\int_{1}^{1} w_N(x) T_j(x) \frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{1}{\pi i}\int_{|\xi|=1} \prod_{k=1}^{K} (\xi^2 - 2a_k\xi + 1)^{N_k^*} \frac{\xi^{j\ 1\ N}}{2^{N+1}}\, d\xi$$
$$= \frac{1}{2^N (N-j)!} W_{2N}^{(N\ j)}(0). \qquad \square$$

In terms of linear operator theory, the recurrence relation may be more useful in the form given in the following theorem.

Theorem 2.4. *If $\mu'(x) > 0$ a.e., the Sobolev polynomials satisfy the recurrence relation*

$$x\widehat{B}_n(x) = h_n \widehat{B}_{n+1}(x) + v_n \widehat{B}_n(x) + h_{n\ 1}\widehat{B}_{n\ 1}(x) + F_n(x),$$

where h_n and v_n are the coefficients of the recurrence relation $xp_n(x) = h_n p_{n+1}(x) + v_n p_n(x) + h_{n\ 1} p_{n\ 1}(x)$, and $F_n(x)$ are functions such that

$$\lim_{n\to\infty} \frac{F_n(x)}{\widehat{B}_n(x)} = 0 \text{ uniformly on compact subsets of } \mathbb{C}\backslash[-1,1].$$

Proof. From Lemma 2.1,

$$x\ w_N(x)\widehat{B}_n(x) = \sum_{j=0}^{2N} A_{n,j} x p_{n+N\ j}(x)$$
$$= \sum_{j=0}^{2N} A_{n,j}(h_{n+N\ j} p_{n+1+N\ j}(x) + v_{n+N\ j} p_{n+N\ j}(x)$$
$$+\ h_{n\ 1+N\ j} p_{n\ 1+N\ j}(x))$$
$$= w_N(x)(h_n \widehat{B}_{n+1}(x) + v_n \widehat{B}_n(x) + h_{n\ 1}\widehat{B}_{n\ 1}(x))$$
$$+ \sum_{j=0}^{2N} (A_{n,j} h_{n+N\ j} - h_n A_{n+1,j}) p_{n+1+N\ j}(x)$$
$$+ \sum_{j=0}^{2N} A_{n,j}(v_{n+N\ j} - v_n) p_{n+N\ j}(x)$$
$$+ \sum_{j=0}^{2N} (A_{n,j} h_{n\ 1+N\ j} - h_{n\ 1} A_{n\ 1,j}) p_{n\ 1+N\ j}(x)$$

and the lemma follows from Lemma 2.4, Theorem 2.1 (i) and the asymptotics of the sequence $(p_n(x))_{n=0}^{\infty}$. $\square$

Remark. Once again, we may replace $[-1,1]$ with $supp(\mu)$ and consider, in Lemmas 2.6 and 2.7 and Theorems 2.2–2.4, μ in the Nevai class $M(0,1)$ instead of $\mu'(x)>0$ a.e.

3. Fourier series

In this section we are focused on the study of the pointwise convergence of the Fourier series expansions in terms of the sequence of polynomials $(\widehat{B}_n)_{n=0}^{\infty}$ orthonormal with respect to the inner product (1) provided that μ is the Jacobi measure. In order to do this we need some previous results and, in what follows, we will denote by $||f|| = \langle f,f \rangle^{1/2}$ the Sobolev norm of a function f.

Lemma 3.1. *Given a positive Borel measure μ supported on $[-1,1]$ with infinitely many points at the support, the polynomials $\widehat{B}_n(x)$ satisfy*

(i) *If $M_{k,i}>0$ then $\sum_{n=0}^{\infty} (\widehat{B}_n^{(i)}(a_k))^2 = \frac{1}{M_{k,i}}$.*
(ii) *If $M_{k,i}>0$ then $\sum_{n=0}^{\infty} \widehat{B}_n^{(i)}(a_k)\widehat{B}_n^{(j)}(a_t) = 0$ for $(t,j)\neq(k,i)$ such that $M_{t,j}>0$.*
(iii) *If $M_{k,i}>0$ then $\lim_{n\to\infty} \int_{-1}^{1}(\sum_{v=0}^{n} \widehat{B}_v^{(i)}(a_k)\widehat{B}_v(x))^2\, d\mu(x) = 0$.*

Proof. For $i=0,1,\dots,N_k$ and $k=1,\dots,K$, let $\ell_{n,k}^{(i)} = \inf\{||\pi_n||^2\colon \deg \pi_n \leqslant n,\ \pi^{(i)}(a_k)=1\}$. It is clear that for all n, $M_{k,i}\leqslant \ell_{n,k}^{(i)}$ and, as it was proved at the beginning of the proof of Lemma 2.7,

$$\ell_{n,k}^{(i)} = \frac{1}{\sum_{v=0}^{n} (\widehat{B}_v^{(i)}(a_k))^2}.$$

Let $N^* = \sum_{k=0}^{K} (N_k+1)$ and introduce the function

$$\varphi(x) = \begin{cases} \exp\{-\frac{x^{2N^*}}{1-x^2}\}, & |x|<1,\\ 0, & |x|\geqslant 1.\end{cases}$$

Then $\varphi\in\mathscr{C}^{\infty}(\mathbb{R})$, $\varphi(0)=1$, $|\varphi(x)|\leqslant 1$ for every $x\in\mathbb{R}$, and $\varphi^{(i)}(0)=0$ for $i=1,\dots,N_k$, $k=1,\dots,K$. For a fixed $k\in\{1,\dots,K\}$ and $\varepsilon>0$ such that $a_t\notin[a_k-\varepsilon,a_k+\varepsilon]$ for $t\neq k$, let us consider the function $\varphi_{k,\varepsilon}(x) = \varphi(\frac{x-a_k}{\varepsilon})$. For $i\in\{0,1,\dots,N_k\}$, let $w_{k,i}(x) = \frac{(x-a_k)^i}{i!}$ and consider a polynomial $\Pi(x)$ such that $\Pi^{(i)}(a_k)=1$ and satisfies

$$\max_{x\in[-1,1]} |\Pi^{(j)}(x) - (w_{k,i}\varphi_{k,\varepsilon})^{(j)}(x)| < \varepsilon, \quad j = 0, 1 \ldots, N^*.$$

Since $(w_{k,i}\varphi_{k,\varepsilon})^{(j)}(a_t) = 0$ for $t \neq k$ and $j = 0, 1, \ldots, N_t$, and $(w_{k,i}\varphi_{k,\varepsilon})^{(j)}(a_k) = \delta_{i,j}$ when $0 \leqslant j \leqslant N_k$, we have

$$\begin{aligned} ||\Pi|| &\leqslant ||w_{k,i}(x)\varphi_{k,\varepsilon}(x)|| + ||\Pi(x) - w_{k,i}(x)\varphi_{k,\varepsilon}(x)|| \\ &\leqslant \Bigg\{ \mu([a_k - \varepsilon, a_k + \varepsilon]) \max_{x\in[-1,1]} w_{k,i}^2(x) + M_{k,i} \\ &\quad + \varepsilon^2 \left(\mu([-1,1]) + \sum_{t=1}^{K} \sum_{j=0}^{N_t} M_{t,j} \right) \Bigg\}^{1/2} \\ &= (M_{k,i} + h(\varepsilon))^{1/2}, \end{aligned}$$

where $\lim_{\varepsilon\to 0} h(\varepsilon) = 0$ because $\mu(\{a_k\}) = 0$. As a consequence,

$$M_{k,i} \leqslant \lim_{n\to\infty} \ell_{n,k}^{(i)} = \frac{1}{\sum_{v=0}^{\infty} (\widehat{B}_v^{(i)}(a_k))^2} \leqslant ||\Pi||^2 \leqslant M_{k,i} + h(\varepsilon)$$

and thus $M_{k,i} = \frac{1}{\sum_{n=0}^{\infty} (\widehat{B}_n^{(i)}(a_k))^2}$.

Moreover, for (k, i) such that $M_{k,i} > 0$,

$$\begin{aligned} \sum_{v=0}^{n} (\widehat{B}_v^{(i)}(a_k))^2 &= \left|\left| \sum_{v=0}^{n} \widehat{B}_v^{(i)}(a_k)\widehat{B}_v(x) \right|\right|^2 \\ &= \int_{-1}^{1} \left(\sum_{v=0}^{n} \widehat{B}_v^{(i)}(a_k)\widehat{B}_v(x) \right)^2 d\mu(x) + M_{k,i} \left\{ \sum_{v=0}^{n} (\widehat{B}_v^{(i)}(a_k))^2 \right\}^2 \\ &\quad + \sum_{j\neq i} M_{k,j} \left(\sum_{v=0}^{n} \widehat{B}_v^{(i)}(a_k)\widehat{B}_v^{(j)}(a_k) \right)^2 \\ &\quad + \sum_{t\neq k} \sum_{j=0}^{N_t} M_{t,j} \left(\sum_{v=0}^{n} \widehat{B}_v^{(i)}(a_k)\widehat{B}_v^{(j)}(a_t) \right)^2. \end{aligned}$$

Multiplying this equality by $\ell_{n,k}^{(i)}$ and taking limit when $n \to \infty$, (ii) and (iii) follows from (i) and the proof is complete. □

Corollary 3.1. *Let μ be a positive Borel measure supported on $[-1, 1]$ with infinitely many points at the support and let f be a function of $L^2(\mu)$ such that there exist the derivatives $f^{(i)}(a_k)$ for $i = 0, 1, \ldots, N_k$ and $k = 1, \ldots, K$. If $M_{k,i} > 0$ then*

$$\sum_{n=0}^{\infty} \langle f, \widehat{B}_n \rangle \widehat{B}_n^{(i)}(a_k) = f^{(i)}(a_k), \quad i = 0, 1, \ldots, N_k, \ k = 1, \ldots, K.$$

Proof.

$$\sum_{v=0}^{n} \langle f, \widehat{B}_v \rangle \widehat{B}_v^{(i)}(a_k) = \int_{1}^{1} f(x) \sum_{v=0}^{n} \widehat{B}_v^{(i)}(a_k)\widehat{B}_v(x)\, d\mu(x)$$
$$+ M_{k,i} f^{(i)}(a_k) \sum_{v=0}^{n} (\widehat{B}_v^{(i)}(a_k))^2$$
$$+ \sum_{j \neq i} M_{k,j} f^{(j)}(a_k) \sum_{v=0}^{n} \widehat{B}_v^{(i)}(a_k)\widehat{B}_v^{(j)}(a_k)$$
$$+ \sum_{t \neq k} \sum_{j=0}^{N_t} M_{t,j} f^{(j)}(a_t) \sum_{v=0}^{n} \widehat{B}_v^{(i)}(a_k)\widehat{B}_v^{(j)}(a_t).$$

Since

$$\left| \int_{1}^{1} f(x) \sum_{v=0}^{n} \widehat{B}_v^{(i)}(a_k)\widehat{B}_v(x)\, d\mu(x) \right|$$
$$\leqslant \left(\int_{1}^{1} f^2(x)\, d\mu(x) \right)^{1/2} \left\{ \int_{1}^{1} \left(\sum_{v=0}^{n} \widehat{B}_v^{(i)}(a_k)\widehat{B}_v(x) \right)^2 d\mu(x) \right\}^{1/2},$$

taking limit in n, the statement follows from Lemma 3.1. □

So, we have convergence at the mass points for any function belonging to $L^2(\mu)$ and with derivatives at such points. But for the convergence at other points, more conditions are needed and, in order to study this problem, we start with some straightforward estimates for the polynomials $\widehat{B}_n$. The polynomial $w_N(x)$ defined in Section 2 will be used again.

Lemma 3.2. *Let $(p_n)_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to the measure μ. Then there exists a positive constant C such that*

$$|w_N(x)\widehat{B}_n(x)| \leqslant C \sum_{j=\ N}^{N} |p_{n+j}(x)| \quad \textit{for every } x \in \mathbb{R}.$$

This lemma is an obvious consequence of Lemma 2.1.

When $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}dx,\ \alpha > -1,\ \beta > -1$, i.e. the Jacobi measure, as it is well known (see [8, Theorem 3.14, p. 101]), the orthonormal polynomials p_n satisfy

$$(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}(1+x)^{\frac{\beta}{2}+\frac{1}{4}}|p_n(x)| \leqslant C, \quad \alpha > -\tfrac{1}{2},\ \beta > -\tfrac{1}{2}, \tag{6}$$

$$|p_n(x)| \leqslant C, \quad -1 < \alpha \leqslant -\tfrac{1}{2},\ -1 < \beta \leqslant -\tfrac{1}{2},$$

for $x \in [-1, 1]$ and, as a consequence of the previous lemma, the corresponding

Jacobi–Sobolev polynomials $\widehat{B}_n$ satisfy the condition

$$|\widehat{B}_n(x)| \leqslant Ch(x) \tag{7}$$

for $x \in (-1,1) \backslash \bigcup_{k=1}^{K} \{a_k\}$ and for all n, where $h(x)$ is the function which depends on α and β deduced from (6) and Lemma 3.2.

Lemma 3.1 gives some properties of the Dirichlet kernels $D_n(x,t) = \sum_{v=0}^{n} \widehat{B}_v(x)\widehat{B}_v(t)$ and, as it was proved in [3] for the case $|a_k| > 1$, they satisfy a Christoffel–Darboux formula deduced from the recurrence relation. If $x_0 \in [-1,1]$ the polynomial $w_N(x) - w_N(x_0)$ may have more than one zero at $[-1,1]$ and this is not convenient for the representation of the Dirichlet kernel. Instead of $w_N(x)$ we will consider the polynomial $w_{N+1}(x) = \int_{-1}^{x} w_N(t)\,dt$ and, from the positivity of $w_N(x)$, when $x_0 \neq a_k,\ k = 1, \ldots, K,\ x_0$ is the only zero of $w_{N+1}(x) - w_{N+1}(x_0)$ in $[-1,1]$. Because the derivatives of $w_{N+1}(x)$ vanish at the $a_k's$, we have $\langle w_{N+1}\widehat{B}_n, \widehat{B}_m \rangle = \langle \widehat{B}_n, w_{N+1}\widehat{B}_m \rangle$ and this means that the Sobolev polynomials $\widehat{B}_n$ satisfy the recurrence relation

$$w_{N+1}(x)\widehat{B}_n(x) = \sum_{v=0}^{N+1} \alpha_{n,v}\widehat{B}_{n+v}(x) + \sum_{v=1}^{N+1} \alpha_{n-v,v}\widehat{B}_{n-v}(x). \tag{8}$$

Moreover, the coefficients $\alpha_{n,v}$ are bounded because

$$\begin{aligned} |\alpha_{n,v}| &= |\langle w_{N+1}\widehat{B}_n, \widehat{B}_{n+v} \rangle| \\ &\leqslant \left| \int_{-1}^{1} \widehat{B}_n(x)\widehat{B}_{n+v}(x) w_{N+1}(x)\,d\mu(x) \right| \\ &\quad + \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} |w_{N+1}(a_k)| |\widehat{B}_n^{(i)}(a_k)\widehat{B}_{n+v}^{(i)}(a_k)| \\ &\leqslant \max_{x \in [-1,1]} |w_{N+1}(x)| \left(1 + \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} |\widehat{B}_n^{(i)}(a_k)\widehat{B}_{n+v}^{(i)}(a_k)| \right), \end{aligned}$$

and, from Lemma 3.1, $\widehat{B}_n^{(i)}(a_k)$ are bounded when $M_{k,i} > 0$.

Christoffel–Darboux formula now takes the following form,

Lemma 3.3. *The orthonormal polynomials with respect to the inner product* (1) *satisfy the following Christoffel–Darboux type formula*:

$$
\begin{aligned}
&\{w_{N+1}(x) - w_{N+1}(y)\} \sum_{n=0}^{v} \widehat{B}_n(x)\widehat{B}_n(y) \\
&\quad = \alpha_{v,1}(\widehat{B}_{v+1}(x)\widehat{B}_v(y) - \widehat{B}_{v+1}(y)\widehat{B}_v(x)) \\
&\qquad + \alpha_{v,2}(\widehat{B}_{v+2}(x)\widehat{B}_v(y) - \widehat{B}_{v+2}(y)\widehat{B}_v(x)) \\
&\qquad + \alpha_{v\ 1,2}(\widehat{B}_{v+1}(x)\widehat{B}_{v\ 1}(y) - \widehat{B}_{v+1}(y)\widehat{B}_{v\ 1}(x)) \\
&\qquad + \cdots + \alpha_{v,N+1}(\widehat{B}_{v+N+1}(x)\widehat{B}_v(y) - \widehat{B}_{v+N+1}(y)\widehat{B}_v(x)) \\
&\qquad + \cdots + \alpha_{v\ N,N+1}(\widehat{B}_{v+1}(x)\widehat{B}_{v\ N}(y) - \widehat{B}_{v+1}(y)\widehat{B}_{v\ N}(x))
\end{aligned}
$$

with bounded coefficients.

Corollary 3.2. *Let* $x \in (-1,1)\setminus\bigcup_{k=1}^{K}\{a_k\}$ *and* μ *the Jacobi measure. If* $M_{k,i} > 0$ *then*

$$\sum_{n=0}^{\infty} \widehat{B}_n^{(i)}(a_k)\widehat{B}_n(x) = 0$$

and the convergence is uniform in compact subsets of $(-1,1)\setminus\bigcup_{k=1}^{K}\{a_k\}$.

Proof. Let (k,i) be such that $M_{k,i} > 0$. From the Christoffel–Darboux formula of Lemma 3.3 it is clear that $\sum_{v=0}^{n} \widehat{B}_v^{(i)}(a_k)\widehat{B}_v(x)$ is a sum of a finite—depending on N—number of terms of the following type:

$$\alpha_{n\ \ v,j}\frac{\widehat{B}_{n\ \ v+j}(x)\widehat{B}^{(i)}_{n\ \ v}(a_k)}{w_{N+1}(x) - w_{N+1}(a_k)}.$$

Since the coefficients $\alpha_{n\ \ v,j}$ are bounded, $|B_n(x)| \leqslant h(x)$ with $h(x)$ a continuous function in compact subsets of $(-1,1)\setminus\bigcup_{k=1}^{K}\{a_k\}$ and $\lim_{n\to\infty} \widehat{B}_n^{(i)}(a_k) = 0$, the lemma is proved. □

Theorem 3.1. *Let* $x_0 \in (-1,1)\setminus\bigcup_{i=1}^{K}\ \{a_i\}$ *and let* f *be a function with derivatives at the points* a_k *such that* $\frac{f(x_0)\ \ f(t)}{x_0\ \ t}$ *belongs to* $L^2(\mu)$ *when* μ *is the Jacobi measure. Then*

(i) $\sum_{n=0}^{\infty} \langle f, \widehat{B}_n \rangle \widehat{B}_n(x_0) = f(x_0)$.
(ii) *If* $M_{k,i} > 0$ *then* $\sum_{n=0}^{\infty} \langle f, \widehat{B}_n \rangle \widehat{B}_n^{(i)}(a_k) = f^{(i)}(a_k)$.

Proof. Because of $f \in L^2(\mu)$ when $\frac{f(x_0)\ \ f(t)}{x_0\ \ t} \in L^2(\mu)$, Corollary 3.1 yields (ii). Now, we denote by $S_n(x_0;f)$ the nth partial sum of the Fourier Sobolev expansion and by $D_n(x,t)$ the Dirichlet kernel $\sum_{v=0}^{n} \widehat{B}_v(x)\widehat{B}_v(t)$. Then

$$
\begin{aligned}
f(x_0) - S_n(x_0;f) &= \langle f(x_0) - f(t), D_n(x_0,t) \rangle \\
&= \int_{1}^{1} (f(x_0) - f(t)) D_n(x_0,t)\, d\mu(t) \\
&\quad + \sum_{k=1}^{K} M_{k,0}(f(x_0) - f(a_k)) D_n(x_o, a_k) \\
&\quad - \sum_{k=1}^{K} \sum_{i=1}^{N_k} M_{k,i} f^{(i)}(a_k) \frac{\partial^i D_n}{\partial t^i}(x_0, a_k).
\end{aligned}
$$

From Corollary 3.2 we get

$$
\lim_{n\to\infty} (f(x_0) - S_n(x_0;f)) = \lim_{n\to\infty} \int_{1}^{1} (f(x_0) - f(t)) D_n(x_0,t)\, d\mu(t).
$$

Using the Christoffel–Darboux type formula, the above expression is the limit of a sum of a finite—depending on N—number of terms

$$
\int_{1}^{1} (f(x_0) - f(t)) \alpha_{n\ \ i,j} \frac{\widehat{B}_{n\ \ i+j}(x_0) \widehat{B}_{n\ \ i}(t)}{w_{N+1}(x_0) - w_{N+1}(t)}\, d\mu(t).
$$

But

$$
\begin{aligned}
&\left| \int_{1}^{1} (f(x_0) - f(t)) \alpha_{n\ \ i,j} \frac{\widehat{B}_{n\ \ i+j}(x_0) \widehat{B}_{n\ \ i}(t)}{w_{N+1}(x_0) - w_{N+1}(t)}\, d\mu(t) \right| \\
&\quad = |\alpha_{n\ \ i,j}|\, |\widehat{B}_{n\ \ i+j}(x_0)| \left| \int_{1}^{1} \frac{f(x_0) - f(t)}{x_0 - t} \frac{x_0 - t}{w_{N+1}(x_0) - w_{N+1}(t)} \widehat{B}_{n\ \ i}(t)\, d\mu(t) \right|,
\end{aligned}
$$

where the coefficients $|\alpha_{n\ \ i,j}|$ are bounded and $|\widehat{B}_{n\ \ i+j}(x_0)| \leqslant h(x_0)$ from Lemma 3.2 and the comments after the lemma.

Since the function

$$
g_{x_0}(t) = \frac{f(x_0) - f(t)}{x_0 - t} \frac{x_0 - t}{w_{N+1}(x_0) - w_{N+1}(t)}
$$

belongs to $L^2(\mu)$ and there exist the derivatives $g_{x_0}^{(i)}(a_k)$, then

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \langle g_{x_0}, \widehat{B}_n \rangle^2 \\
&\quad = \sum_{n=0}^{\infty} \left(\int_{1}^{1} g_{x_0}(t) \widehat{B}_n(t)\, d\mu(t) + \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} g_{x_0}^{(i)}(a_k) \widehat{B}_n^{(i)}(a_k) \right)^2 \leqslant ||g_{x_0}||^2
\end{aligned}
$$

and, as a consequence, $\lim_{n\to\infty} \langle g_{x_0}, \widehat{B}_n \rangle = 0$. Taking into account that, when $M_{k,i} > 0$, $\lim_{n\to\infty} \widehat{B}_n^{(i)}(a_k) = 0$, we have $\lim_{n\to\infty} \int_{1}^{1} g_{x_0}(t) \widehat{B}_n(t)\, d\mu(t) = 0$. This means that $\lim_{n\to\infty} (f(x_0) - S_n(x_0;f)) = 0$. $\square$

Theorem 3.2. *Let $f(x)$ be a function with derivatives at the points a_k satisfying a Lipschitz condition of order $0<\eta<1$ uniformly in $[-1,1]$, i.e. $|f(x+h)-f(x)|\leqslant M|h|^{\eta}$ for $|h|<\delta$ and for some $\delta>0$. For the Jacobi measure, if $c_n=\langle f,\widehat{B}_n\rangle$ then*

$$\sum_{n=0}^{\infty} c_n\widehat{B}_n(x)=f(x), \quad x\in(-1,1),$$

and the convergence is uniform in compact subsets of $(-1,1)\setminus\bigcup_{k=1}^{K}\{a_k\}$. Moreover, at the mass points, $\sum_{n=0}^{\infty} c_n\widehat{B}_n^{(i)}(a_k)=f^{(i)}(a_k)$ provided that $M_{k,i}>0$.

Proof. As in the previous theorem, we only need to prove that $\int_{-1}^{1} f(t)D_n(x,t)\,d\mu(t)$ converges to $f(x)$ for $x\neq a_k,\ k=1,\dots,K$. Besides,

$$\begin{aligned}
&\left|\int_{-1}^{1}(f(x)-f(t))D_n(x,t)\,d\mu(t)\right|\\
&\quad\leqslant\left|\int_{|x-t|\geqslant\delta}(f(x)-f(t))D_n(x,t)\,d\mu(t)\right|\\
&\qquad+\left|\int_{|x-t|<\delta}(f(x)-f(t))D_n(x,t)\,d\mu(t)\right|\\
&\quad=I_n^{(1)}(x)+I_n^{(2)}(x).
\end{aligned}$$

Since $\frac{f(x)-f(t)}{w_{N+1}(x)-w_{N+1}(t)}(1-\chi_{(x-\delta,x+\delta)}(t))$, where $\chi_{(x-\delta,x+\delta)}(t)$ is the characteristic function of the interval $(x-\delta,x+\delta)$, belongs to $L^2(\mu)$, using Christoffel–Darboux type formula and the same procedure as in the previous theorem, the term $I_n^{(1)}(x)$ tends to zero.

On the other hand, $I_n^{(2)}(x)$ is a sum of a finite number of terms

$$\alpha_{n-i,j}\widehat{B}_{n-i+j}(x)\int_{|x-t|<\delta}\frac{f(x)-f(t)}{x-t}\,\frac{x-t}{w_{N+1}(x)-w_{N+1}(t)}\,\widehat{B}_{n-i}(t)\,d\mu(t),$$

where the coefficients $\alpha_{n-i,j}\widehat{B}_{n-i+j}(x)$ are uniformly bounded in closed sets of $(-1,1)\setminus\bigcup_{k=1}^{K}\{a_k\}$. Furthermore, when x belongs to a compact subset F of $(-1,1)\setminus\bigcup_{k=1}^{K}\{a_k\}$, Lipschitz condition gives

$$\begin{aligned}
&\left|\int_{|x-t|<\delta}\frac{f(x)-f(t)}{x-t}\,\frac{x-t}{w_{N+1}(x)-w_{N+1}(t)}\,\widehat{B}_{n-i}(t)\,d\mu(t)\right|\\
&\quad\leqslant C\int_{|x-t|<\delta}\frac{d\mu(t)}{|x-t|^{1-\eta}},
\end{aligned}$$

where the constant C depends on $\max\{\frac{|x-t|}{|w_{N+1}(x)-w_N(t)|}:\ t\in[-1,1],\ x\in F\}$, the constant of the Lipschitz condition, and $\max\{h(x):\ x\in F\}$, $h(x)$ being the function such that $|\widehat{B}_n(x)|\leqslant h(x)$ on the interval $(-1,1)\setminus\bigcup_{k=1}^{K}\{a_k\}$. Hence, since μ is the Jacobi measure, for $\varepsilon > 0$ there exists $\delta > 0$ such that $|I_n^{(2)}(x)| < \varepsilon$ and the pointwise convergence is proved. The uniform convergence is an easy consequence of the uniform continuity of $\frac{f(y)\quad f(t)}{w_{N+1}(y)\quad w_{N+1}(t)}$ when (y, t) belongs to $\{(y,t)\colon |y - x| \leqslant \frac{\delta}{2},\ |t - x| \geqslant \delta,\ x, y \in F\}$ for a fixed $x \in F$ and for a fixed δ such that $\int_{|x\quad t| < \delta} \frac{d\mu(t)}{|x\quad t|^{1\quad \eta}} < \varepsilon$, and the compactness of F. $\square$

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