# On rational transformations of linear functionals: direct problem

Manuel Alfaro [a,1], Francisco Marcellán [b,*,2], Ana Peña [a,3], M. Luisa Rezola [a,1]

[a] *Departamento de Matemáticas, Universidad de Zaragoza, Spain*
[b] *Departamento de Matemáticas, Universidad Carlos III de Madrid, 28911 Leganés, Spain*

**Abstract**

Let $u$ be a quasi-definite linear functional. We find necessary and sufficient conditions in order to the linear functional $v$ satisfying $(x-\tilde{a})u = \lambda(x-a)v$ be a quasi-definite one. Also we analyze some linear relations linking the polynomials orthogonal with respect to $u$ and $v$.

*Keywords* Orthogonal polynomials; Recurrence relations; Spectral transformations; Linear functionals

* Corresponding author. Fax: 34-9162-49151.
*E-mail address* pacomarc@ing.uc3m.es (F. Marcellán).

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## 1. Introduction

Let $u$ be a linear functional in the linear space $\mathbb{P}$ of polynomials with complex coefficients and denote by $\{u_n\}_{n\geqslant 0}$ the sequence of the moments associated with $u$, $u_n = \langle u, x^n\rangle$, $n \geqslant 0$, where $\langle\cdot\,,\cdot\rangle$ means the duality bracket.

The linear functional $u$ is said to be quasi-definite if the Hankel matrix $H = (u_{i+j})_{i,j=0}^{\infty}$ is quasi-definite, i.e., the principal submatrices $H_n = (u_{i+j})_{i,j=0}^{n}$, $n \in \mathbb{N}\cup\{0\}$, are nonsingular.

The linear functional $\delta_a$ given by $\langle\delta_a, P\rangle = P(a)$, for every $P \in \mathbb{P}$, is not a quasi-definite linear functional since rank $H_n = 1$ for every $n \geqslant 0$. This linear functional is said to be either the Dirac linear functional or the Dirac mass at the point $a$.

To the linear functional $u$ we can associate a formal power series $S_u(z) = \sum_{n=0}^{\infty}\frac{u_n}{z^{n+1}}$ which is related with the $z$-transform of the sequence $\{u_n\}$ of moments of $u$. $S_u$ is said to be the Stieltjes function of $u$. For the Dirac linear functional $u = \delta_a$ given as above, we have $S_u(z) = 1/(z-a)$ in a neighborhood of infinite.

Assuming $u$ quasi-definite, there exists a sequence of monic polynomials $\{P_n\}_{n\geqslant 0}$ such that (see [2])

(i) $\deg P_n = n$, $n \geqslant 0$,
(ii) $\langle u, P_n P_m\rangle = k_n\delta_{n,m}$ with $k_n \neq 0$.

The sequence $\{P_n\}_{n\geqslant 0}$ is said to be the sequence of monic orthogonal polynomials (SMOP) with respect to the linear functional $u$.

If $\{P_n\}_{n\geqslant 0}$ is an SMOP with respect to the quasi-definite linear functional $u$, then it is well known (see [2]) that it satisfies a three-term recurrence relation

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geqslant 0, \tag{1.1}$$

with $\gamma_n \neq 0$ and $P_{-1}(x) = 0$, $P_0(x) = 1$.

Conversely, given a sequence of monic polynomials generated by a recurrence relation as above, there exists a unique quasi-definite linear functional $u$ such that the family $\{P_n\}_{n\geqslant 0}$ is the corresponding SMOP. Such a result is known as the Favard theorem (see [2]).

For an SMOP $\{P_n\}_{n\geqslant 0}$ relative to $u$, let $\{P_n^{(1)}\}_{n\geqslant 0}$ be the sequence of monic polynomials such that

$$\begin{aligned} &P_{n+1}^{(1)}(x) = (x-\beta_{n+1})P_n^{(1)}(x) - \gamma_{n+1}P_{n-1}^{(1)}(x), \quad n \geqslant 0,\\ &P_{-1}^{(1)}(x) = 0, \qquad P_0^{(1)}(x) = 1. \end{aligned}$$

According to the Favard theorem there exists a quasi-definite linear functional $u^{(1)}$ such that $\{P_n^{(1)}\}_{n\geqslant 0}$ is the corresponding SMOP. The family $\{P_n^{(1)}\}_{n\geqslant 0}$ is said to be the sequence of polynomials of first kind associated with the linear functional $u$.

Another representation of $\{P_n^{(1)}\}_{n\geqslant 0}$ is given by

$$P_n^{(1)}(y) = \frac{1}{u_0}\left\langle u, \frac{P_{n+1}(y) - P_{n+1}(x)}{y-x}\right\rangle,$$

$n \geqslant 0$ (see [2, Chapter 3]).

Notice that $P_n^{(1)}(z)/P_{n+1}(z)$ is the $(n+1)$-convergent of the continued fraction

$$\cfrac{1}{z-\beta_0-\cfrac{\gamma_1}{z-\beta_1-\ddots}}.$$

Thus

$$S_u(z) = \cfrac{u_0}{z-\beta_0-\cfrac{\gamma_1}{z-\beta_1-\ddots}} \tag{1.2}$$

from a formal point of view (see [2]).

For simplicity we will assume $u_0 = 1$.

Let $\{P_n(x,\alpha)\}_{n\geqslant 0}$ be the sequence of monic polynomials satisfying (1.1) with initial conditions $P_0(x,\alpha) = 1$, $P_1(x,\alpha) = P_1(x) - \alpha$. Taking into account the Favard theorem, there exists a quasi-definite linear functional $u_\alpha$ such that $\{P_n(x,\alpha)\}_{n\geqslant 0}$ is the corresponding SMOP. This sequence is said to be the co-recursive SMOP of parameter $\alpha$ associated with the linear functional $u$. It is known see [2,7] that $P_n(x,\alpha) = P_n(x) - \alpha P_{n-1}^{(1)}(x)$.

From (1.2) we get

$$S_{u^{(1)}}(z) = \frac{1}{\gamma_1}\left[z-\beta_0-\frac{1}{S_u(z)}\right],$$
$$S_{u_\alpha}(z) = \left[\frac{1}{S_u(z)}-\alpha\right]^{-1} = \frac{S_u(z)}{1-\alpha S_u(z)}.$$

These two bilinear rational transforms are related to self-similar reductions and spectral transformations in the theory of nonlinear integrable systems (see [12]).

For a linear functional $u$, a polynomial $\pi$, and a complex number $a$, let $\pi u$, $(x-a)^{-1}u$, and $Du$ be the linear functionals defined on $\mathbb{P}$ by

$$\langle \pi u, P\rangle = \langle u, \pi P\rangle,$$
$$\left\langle (x-a)^{-1}u, P\right\rangle = \left\langle u, \frac{P(x)-P(a)}{x-a}\right\rangle,$$
$$\langle Du, P\rangle = -\langle u, P'\rangle,$$

where $P \in \mathbb{P}$.

A Cauchy product of two linear functionals $u, v$ can be defined as the linear functional $uv$ such that $\langle uv, x^n\rangle = \sum_{h=0}^{n} u_h v_{n-h}$, $n \geqslant 0$. Obviously, $uv = vu$ and $\delta_0 u = u\delta_0 = u$. Since $u_0 = 1$, there exists a unique linear functional $v$ such that $uv = vu = \delta_0$. This linear functional $v$ is said to be the inverse linear functional of $u$ and it will be denoted by $u^{-1}$. Notice that $(u^{-1})_0 = 1$ and $(u^{-1})_n = -\sum_{h=0}^{n-1} u_{n-h}(u^{-1})_h$, $n \geqslant 1$ (see [10]).

Since $z^2 S_{u^{-1}}(z) S_u(z) = 1$, we have $S_{u^{(1)}}(z) = \frac{1}{\gamma_1}[z-\beta_0-z^2 S_{u^{-1}}(z)]$. Taking into account $(u^{-1})_0 = 1$ and $(u^{-1})_1 = -\beta_0$, we get $u^{(1)} = -\frac{1}{\gamma_1}x^2 u^{-1}$. Concerning the linear functional $u_\alpha$, it is easy to check that $u_\alpha = (u^{-1} + \alpha\delta_0')^{-1}$. This is an alternative proof of the result of [10] but notice that there the Stieltjes function has an opposite sign.

In the constructive theory of orthogonal polynomials the so-called direct problem is considered. A direct problem for linear functionals can be stated as follows: given two linear functionals $u, v$ such that $v = F(u)$, where $F$ is a function defined in $\mathbb{P}'$, the dual space of $\mathbb{P}$, to find necessary and sufficient conditions in order to $F$ preserves quasi-definiteness. As a subsequent question, to find the explicit relations between the corresponding SMOP $\{P_n\}$ and $\{Q_n\}$ associated with $u$ and $v$, respectively.

If $u$ is a linear functional defined by a nonnegative measure $\mu$ on some interval $I$ of the real line, with an infinite set of increasing points such that the moments exist, i.e., $\langle u, x^n \rangle = \int_I x^n \, d\mu < \infty$ then we can introduce the linear functional $v$ such that

$$\langle v, x^n \rangle = \int_I x^n \frac{p(x)}{q(x)} \, d\mu, \tag{1.3}$$

where $p, q$ are two polynomials with pairwise distinct zeros that has constant sign on $I$. If we assume (1.3) is finite for every $n$, the generalized Christoffel theorem gives the SMOP with respect to $v$ in terms of polynomials of the SMOP with respect to $u$ (see [4,11]). In terms of linear functionals, the above transform reads $qv = pu$. Notice that $pu = qv$ is a more general transform because of Dirac measures and derivatives of Dirac measures at the zeros of $q(x)$ can be considered for $v$ in addition in such a general problem.

When $q(x) = 1$ and $p(x) = x - \tilde{a}$, the transform for linear functionals is said to be a Christoffel transform (see [12]). Using the Jacobi matrix $J$ associated with the linear functional $u$, the shifted Darboux transform of $J$ without free parameter yields the Jacobi matrix of $v$ (see [6]).

It is known that $v$ is quasi-definite if and only if $P_n(\tilde{a}) \neq 0$, $n \geqslant 1$, and

$$(x - \tilde{a}) Q_n(x) = P_{n+1}(x) - \frac{P_{n+1}(\tilde{a})}{P_n(\tilde{a})} P_n(x)$$

as well as

$$\frac{Q_n(x) P_n(\tilde{a})}{\langle u, P_n^2 \rangle} = \sum_{k=0}^{n} \frac{P_k(x) P_k(\tilde{a})}{\langle u, P_k^2 \rangle}.$$

The polynomials $\{Q_n\}_{n \geqslant 0}$ are said to be the monic kernel polynomials of parameter $\tilde{a}$ associated with the linear functional $u$ (see [2]).

If $p(x) = 1$ and $q(x) = \lambda(x - a)$ then the transform is said to be the Geronimus transform of the linear functional $u$ (see [10,12]). The Jacobi matrix of $v$ is the shifted Darboux transform with free parameter of the Jacobi matrix of $u$ (see [6]).

Notice that in such a case, $v = \lambda^{-1}(x - a)^{-1} u + \delta_a$ is a quasi-definite linear functional if and only if $P_n(a, -\lambda^{-1}) \neq 0$, $n \geqslant 1$, and then

$$Q_n(x) = P_n(x) - \frac{P_n(a, -\lambda^{-1})}{P_{n-1}(a, -\lambda^{-1})} P_{n-1}(x)$$

(see [9]).

In our contribution, we analyze the direct problem stated as above for the case $p(x) = (x - \tilde{a})$ and $q(x) = \lambda(x - a)$. For $a \neq \tilde{a}$ this situation has not been studied in the literature as far as we know up to in the so-called positive definite case (see [4]).

In Section 2, given a quasi-definite linear functional $u$ and complex numbers $a, \tilde{a},$ and $\lambda$ with $a \neq \tilde{a}$ and $\lambda \neq 0$, we characterize the quasi-definiteness of the linear functional $v = \frac{1}{\lambda}(x-a)^{-1}(x-\tilde{a})u + (1-\frac{1}{\lambda})\delta_a$. Instead of the analysis of the quasi-definiteness of the linear functional $v$ in two steps (first, the rational perturbation and, second, the addition of the Dirac linear functional) we consider the whole transformation taking into account the first one cannot preserve the quasi-definiteness of the linear functional $u$. Indeed in [4] this constraint must be emphasized when polynomial perturbations are introduced. Further, we show that $(x-\tilde{a})Q_n$ is a linear combination of three consecutive polynomials of the SMOP $\{P_n\}_{n\geqslant 0}$.

Notice that the confluent case $a = \tilde{a}$ yields a perturbation of $u$ via the addition of a Dirac mass at the point $x = a$. This corresponds to the Uvarov transform of the linear functional $u$ (see [12]). The direct problem has been solved in [8]. We point out that the results for $a \neq \tilde{a}$ extend in a natural way those already known for $a = \tilde{a}$.

In Section 3, under the thesis of Section 2 we characterize when the relation between $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$, obtained there, can be reduced to a relation $P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x)$ with $s_n t_n \neq 0$ for every $n \geqslant 1$, and $s_1 \neq t_1$. This last type of relation, as an inverse problem, has been analyzed in [1]. The motivation for such a kind of problems is reflected in [3] when an extension of the concept of coherent pairs of measures associated with Sobolev inner products is considered.

We also observe that there is an important difference for the cases $a = \tilde{a}$ and $a \neq \tilde{a}$. Namely, if $a = \tilde{a}$ then $s_n \neq t_n$ for every $n \geqslant 1$ while if $a \neq \tilde{a}$ both situations, i.e., either $s_n \neq t_n$ for every $n \geqslant 1$ or $s_n = t_n$ for some values of $n$, can appear as we show in some examples.

## 2. Direct problem

In this section, we study the direct problem for $v = \frac{1}{\lambda}(x-a)^{-1}(x-\tilde{a})u + (1-\frac{1}{\lambda})\delta_a$ where $u$ is a given quasi-definite linear functional, and $a, \tilde{a}, \lambda \in \mathbb{C}$ with $a \neq \tilde{a}$, $\lambda \neq 0$.

**Theorem 2.1.** *Let $u, v$ be two linear functionals related by*

$$(x-\tilde{a})u = \lambda(x-a)v, \quad a, \tilde{a}, \lambda \in \mathbb{C}. \tag{2.1}$$

*Assume $u_0 = 1 = v_0$ and $a \neq \tilde{a}$. If $u$ is a quasi-definite linear functional with corresponding SMOP $\{P_n\}_{n\geqslant 0}$ then, the linear functional $v$ is quasi-definite if and only if*

$$\Delta_n = \begin{vmatrix} P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_n(a) & R_{n-1}(a) \end{vmatrix} \neq 0, \quad n \geqslant 1,$$

*where $R_n(x) = (\lambda-1)P_n(x) + (a-\tilde{a})P^{(1)}_{n-1}(x)$. Furthermore, if $\{Q_n\}_{n\geqslant 0}$ is the SMOP associated with $v$ then*

$$(x-\tilde{a})Q_n(x) = \Delta_n^{-1} \begin{vmatrix} P_{n+1}(x) & P_n(x) & P_{n-1}(x) \\ P_{n+1}(\tilde{a}) & P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_{n+1}(a) & R_n(a) & R_{n-1}(a) \end{vmatrix}, \quad n \geqslant 1. \tag{2.2}$$

**Proof.** Assume $v$ is a quasi-definite linear functional and $\{Q_n\}_{n\geqslant 0}$ is its corresponding SMOP.

Consider the Fourier expansion of $(x-\tilde{a})Q_n$ in terms of the polynomials $P_n$, that is

$$(x-\tilde{a})Q_n(x)=P_{n+1}(x)+\sum_{j=0}^{n}\alpha_{n,j}P_j(x),\quad n\geqslant 1,$$

where $\alpha_{nj}=\langle u,P_j^2\rangle^{-1}\langle u,(x-\tilde{a})Q_nP_j\rangle$. From formula (2.1) we get

$$(x-\tilde{a})Q_n(x)=P_{n+1}(x)+\alpha_{n,n}P_n(x)+\alpha_{n,n-1}P_{n-1}(x) \tag{2.3}$$

with $\alpha_{n,n-1}=\lambda\frac{\langle v,Q_n^2\rangle}{\langle u,P_{n-1}^2\rangle}\neq 0$.

For $x=\tilde{a}$

$$0=P_{n+1}(\tilde{a})+\alpha_{n,n}P_n(\tilde{a})+\alpha_{n,n-1}P_{n-1}(\tilde{a}). \tag{2.4}$$

On the other hand,

$$(a-\tilde{a})Q_n(a)=P_{n+1}(a)+\alpha_{n,n}P_n(a)+\alpha_{n,n-1}P_{n-1}(a). \tag{2.5}$$

Subtracting (2.5) to (2.3) and dividing by $x-a$, we can apply $u$ in order to get

$$\begin{aligned}&\left\langle u,\frac{(x-\tilde{a})Q_n(x)-(a-\tilde{a})Q_n(a)}{x-a}\right\rangle\\&\quad=P_n^{(1)}(a)+\alpha_{n,n}P_{n-1}^{(1)}(a)+\alpha_{n,n-1}P_{n-2}^{(1)}(a).\end{aligned} \tag{2.6}$$

The left-hand side becomes

$$\begin{aligned}\left\langle u,(x-\tilde{a})\frac{Q_n(x)-Q_n(a)}{x-a}\right\rangle+Q_n(a)&=\lambda\langle v,Q_n(x)-Q_n(a)\rangle+Q_n(a)\\&=(1-\lambda)Q_n(a)\end{aligned}$$

and therefore

$$(1-\lambda)Q_n(a)=P_n^{(1)}(a)+\alpha_{n,n}P_{n-1}^{(1)}(a)+\alpha_{n,n-1}P_{n-2}^{(1)}(a). \tag{2.7}$$

Thus, (2.5) and (2.7) yield

$$0=R_{n+1}(a)+\alpha_{n,n}R_n(a)+\alpha_{n,n-1}R_{n-1}(a). \tag{2.8}$$

Since the system of Eqs. (2.4) and (2.8) in $\alpha_{n,n}$ and $\alpha_{n,n-1}$ has a non-zero solution, then we get $\Delta_n\neq 0$ for every $n\geqslant 1$.

Besides, from (2.3), (2.4), and (2.8) we obtain (2.2).

Conversely, if $\Delta_n\neq 0$ for every $n\geqslant 1$ we will prove that the polynomials $Q_n$ defined by

$$(x-\tilde{a})Q_n(x)=\Delta_n^{-1}\begin{vmatrix}P_{n+1}(x)&P_n(x)&P_{n-1}(x)\\P_{n+1}(\tilde{a})&P_n(\tilde{a})&P_{n-1}(\tilde{a})\\R_{n+1}(a)&R_n(a)&R_{n-1}(a)\end{vmatrix},\quad n\geqslant 1,$$

are orthogonal with respect to $v$. Indeed, for $0\leqslant j\leqslant n-2$,

$$\lambda\langle v,Q_n(x)(x-a)P_j(x)\rangle=\langle u,(x-\tilde{a})Q_n(x)P_j(x)\rangle=0$$

and for $j = n-1$,

$$\lambda\big\langle v, Q_n(x)(x-a)P_{n-1}(x)\big\rangle = \big\langle u, (x-\tilde{a})Q_n(x)P_{n-1}(x)\big\rangle = \Delta_{n+1}\Delta_n^{-1}\big\langle u, P_{n-1}^2\big\rangle \neq 0.$$

Thus, we only need to prove that $\langle v, Q_n\rangle = 0$ for every $n \geqslant 1$. In order to do this, observe that

$$\begin{aligned}
\lambda\langle v, Q_n\rangle &= \lambda\left[\left\langle v, (x-a)\frac{Q_n(x)-Q_n(a)}{x-a}\right\rangle + Q_n(a)\right] \\
&= \left\langle (x-\tilde{a})u, \frac{Q_n(x)-Q_n(a)}{x-a}\right\rangle + \lambda Q_n(a) \\
&= \left\langle u, \frac{(x-\tilde{a})Q_n(x)-(a-\tilde{a})Q_n(a)}{x-a}\right\rangle + (\lambda-1)Q_n(a).
\end{aligned}$$

Applying the expression of $(x-\tilde{a})Q_n(x)$ in terms of the polynomials $P_n(x)$ and (2.7) we get

$$\begin{aligned}
&\left\langle u, \frac{(x-\tilde{a})Q_n(x)-(a-\tilde{a})Q_n(a)}{x-a}\right\rangle \\
&\quad = \Delta_n^{-1}\begin{vmatrix} P_n^{(1)}(a) & P_{n-1}^{(1)}(a) & P_{n-2}^{(1)}(a) \\ P_{n+1}(\tilde{a}) & P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_{n+1}(a) & R_n(a) & R_{n-1}(a) \end{vmatrix} = (1-\lambda)Q_n(a).
\end{aligned}$$

So $\langle v, Q_n\rangle = 0$ for every $n \geqslant 1$.

As a conclusion, $\langle v, Q_n^2\rangle = \langle v, Q_n(x-a)P_{n-1}\rangle \neq 0$, and $\langle v, Q_n p\rangle = 0$ for every polynomial $p$ of degree less than $n$. □

**Corollary 2.2.** *Under the conditions of Theorem* 2.1 *the linear functional* $v$ *is quasi-definite if and only if* $1 + \sum_{j=0}^{n-1} \frac{P_j(\tilde{a})R_j(a)}{\langle u, P_j^2\rangle} \neq 0$, *for every* $n \geqslant 1$.

*Furthermore, we have*

$$(x-\tilde{a})Q_n(x) = P_{n+1}(x) + a_n(a,\tilde{a})P_n(x) + b_n(a,\tilde{a})P_{n-1}(x), \quad n \geqslant 1 \tag{2.9}$$

*with*

$$a_n(a,\tilde{a}) = \beta_n - \tilde{a} + (a-\tilde{a})\Delta_n^{-1}P_{n-1}(\tilde{a})R_n(a) \tag{2.10}$$

*and*

$$b_n(a,\tilde{a}) = \gamma_n + (\tilde{a}-a)\Delta_n^{-1}P_n(\tilde{a})R_n(a). \tag{2.11}$$

**Proof.** From the expression of $\Delta_n$, using the Christoffel–Darboux formula (see [2]), we have for $n \geqslant 1$

$$\Delta_n = (a-\tilde{a})\big[(1-\lambda)K_{n-1}(a,\tilde{a};u)\big\langle u, P_{n-1}^2\big\rangle + B_n(a,\tilde{a})\big],$$

where $K_n(x,y;u)$ denotes the reproducing kernel of degree $n$ associated with $u$ and

$$B_n(a,\tilde{a}) = \begin{vmatrix} P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ P_{n-1}^{(1)}(a) & P_{n-2}^{(1)}(a) \end{vmatrix}.$$

Inserting the three-term recurrence relation for both polynomials $P_n$ and $P_{n-1}^{(1)}$, we get

$$\frac{B_n(a,\tilde a)}{\langle u, P_{n-1}^2\rangle} = (\tilde a - a)\frac{P_{n-1}(\tilde a)P_{n-2}^{(1)}(a)}{\langle u, P_{n-1}^2\rangle} + \frac{B_{n-1}(a,\tilde a)}{\langle u, P_{n-2}^2\rangle}, \quad n \geqslant 2.$$

Iteration yields

$$\frac{B_n(a,\tilde a)}{\langle u, P_{n-1}^2\rangle} = (\tilde a - a)\sum_{j=0}^{n-1}\frac{P_j(\tilde a)P_{j-1}^{(1)}(a)}{\langle u, P_j^2\rangle} - 1, \quad n \geqslant 1. \tag{2.12}$$

Therefore

$$\begin{aligned}\Delta_n &= (\tilde a - a)\langle u, P_{n-1}^2\rangle\left[1 + (\lambda - 1)K_{n-1}(a,\tilde a; u) + (a - \tilde a)\sum_{j=0}^{n-1}\frac{P_j(\tilde a)P_{j-1}^{(1)}(a)}{\langle u, P_j^2\rangle}\right]\\ &= (\tilde a - a)\langle u, P_{n-1}^2\rangle\left[1 + \sum_{j=0}^{n-1}\frac{P_j(\tilde a)R_j(a)}{\langle u, P_j^2\rangle}\right],\end{aligned} \tag{2.13}$$

and the first part of the corollary follows from Theorem 2.1.

On the other hand, we can write formula (2.2) as follows

$$(x - \tilde a)Q_n(x) = P_{n+1}(x) + a_n(a,\tilde a)P_n(x) + b_n(a,\tilde a)P_{n-1}(x), \quad n \geqslant 1.$$

Using the three-term recurrence relation for $P_{n+1}(\tilde a)$ and $R_{n+1}(a)$ we get

$$\begin{aligned}a_n(a,\tilde a) &= \beta_n - \Delta_n^{-1}\big[\tilde a P_n(\tilde a)R_{n-1}(a) - aP_{n-1}(\tilde a)R_n(a)\big]\\ &= \beta_n - \tilde a + (a - \tilde a)\Delta_n^{-1}P_{n-1}(\tilde a)R_n(a).\end{aligned}$$

Besides, from (2.13) we obtain

$$\frac{\Delta_{n+1}}{\langle u, P_n^2\rangle} = \frac{\Delta_n}{\langle u, P_{n-1}^2\rangle} + (\tilde a - a)\frac{P_n(\tilde a)R_n(a)}{\langle u, P_n^2\rangle}$$

and, since $b_n(a,\tilde a) = \Delta_{n+1}/\Delta_n$ and $\gamma_n = \langle u, P_n^2\rangle/\langle u, P_{n-1}^2\rangle$, then

$$b_n(a,\tilde a) = \gamma_n + (\tilde a - a)\Delta_n^{-1}P_n(\tilde a)R_n(a). \qquad \square$$

In Theorem 2.1 and Corollary 2.2 we have assumed $a \neq \tilde a$. Notice that if $a = \tilde a$ the relation (2.1) between the linear functionals $u$ and $v$ becomes $u = \lambda v + (1 - \lambda)\delta_a$. In this situation it is well known (see [8]) that $v$ is quasi-definite if and only if for every $n \geqslant 1$

$$1 + (\lambda - 1)K_n(a,a;u) \neq 0$$

and then

$$(x - a)Q_n(x) = P_{n+1}(x) + a_n(a)P_n(x) + b_n(a)P_{n-1}(x), \quad n \geqslant 1, \tag{2.14}$$

holds, where

$$a_n(a) = \beta_n - a - \frac{(\lambda - 1)P_{n-1}(a)P_n(a)}{\langle u, P_{n-1}^2\rangle[1 + (\lambda - 1)K_{n-1}(a,a;u)]}$$

and

$$b_n(a) = \gamma_n \frac{1 + (\lambda - 1)K_n(a, a; u)}{1 + (\lambda - 1)K_{n-1}(a, a; u)}.$$

Notice that, these results can be recovered from Corollary 2.2, when $\tilde{a}$ tends to $a$.

## 3. Linear relations between the polynomials $\{P_n\}$ and $\{Q_n\}$

Let $u$ and $v$ be quasi-definite linear functionals with corresponding SMOP $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$, respectively. In Section 2, we have obtained that if $u$ and $v$ satisfy the relation $(x - \tilde{a})u = \lambda(x - a)v$ with $a, \tilde{a}, \lambda \in \mathbb{C}$ then an expression of the form

$$(x - \tilde{a})Q_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x), \quad n \geqslant 1, \tag{3.1}$$

holds (see formulas (2.9) and (2.14)). That is, a linear combination of three consecutive polynomials $P_n$ coincides with a linear combination of three consecutive polynomials $Q_n$.

On the other hand, in [1], it was proved that if the linear functionals $u$ and $v$ are quasi-definite and they are related as above, then there exists a relation $P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x)$ with $s_n t_n \neq 0$, $n \geqslant 1$, and $s_1 \neq t_1$ if and only if for every $n \geqslant 1$, $P_n \neq Q_n$.

Thus, at the present, we have two expressions linking the polynomials $P_n$ and $Q_n$, the last quoted and the one given in formula (3.1).

We see below that if $P_n \neq Q_n$, $n \geqslant 1$, then both formulas are not independent. In fact, one of them can be reduced to the other.

**Theorem 3.1.** *Let $u, v$ be two different quasi-definite linear functionals normalized by $u_0 = 1 = v_0$ and related by*

$$(x - \tilde{a})u = \lambda(x - a)v, \quad a, \tilde{a}, \lambda \in \mathbb{C}.$$

*Let $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ be their corresponding SMOP. The following conditions are equivalent*:

(i) *Formula* (3.1) *can be reduced to an expression*

$$P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x) \tag{3.2}$$

*with $s_n t_n \neq 0$ for every $n \geqslant 1$ and $s_1 \neq t_1$.*

(ii) *For all $n \geqslant 1$, $R_n(a) = (\lambda - 1)P_n(a) + (a - \tilde{a})P^{(1)}_{n-1}(a) \neq 0$.*

**Proof.** Suppose that (i) holds. In [1, Theorem 2.4] it has been proved that whenever such a relation (3.2) is satisfied then $P_n \neq Q_n$, for every $n$, and besides $P_n(x) = Q_n(x) + \lambda^{-1}R_n(a)K_{n-1}(x, a; v)$, $n \geqslant 1$ (see formula (2.24) in [1]). So, (ii) follows.

In order to derive the converse result we will first consider the case $a \neq \tilde{a}$. Inserting the three-term recurrence relation in (3.1) successively for $P_{n+1}$ and $P_n$ we get, for $n \geqslant 2$,

$$(x - \tilde{a})Q_n(x) = (x - \tilde{a})P_n(x) + (\tilde{a} - \beta_n + a_n)P_n(x) + (b_n - \gamma_n)P_{n-1}(x)$$

$$
\begin{aligned}
&= (x-\tilde a)\big[P_n(x) + (\tilde a - \beta_n + a_n)P_{n-1}(x)\big] \\
&\quad + \big[(\tilde a - \beta_n + a_n)(\tilde a - \beta_{n-1}) + b_n - \gamma_n\big]P_{n-1}(x) \\
&\quad - \gamma_{n-1}(\tilde a - \beta_n + a_n)P_{n-2}(x).
\end{aligned} \tag{3.3}
$$

The first part of the formula (3.3) for $n-1$ reads:

$$
\begin{aligned}
(x-\tilde a)Q_{n-1}(x) &= (x-\tilde a)P_{n-1}(x) + (\tilde a - \beta_{n-1} + a_{n-1})P_{n-1}(x) \\
&\quad + (b_{n-1} - \gamma_{n-1})P_{n-2}(x).
\end{aligned} \tag{3.4}
$$

Taking into account (2.10) and (2.11), the above two formulas can be written

$$
\begin{aligned}
(x-\tilde a)Q_n(x) &= (x-\tilde a)\left[P_n(x) + \frac{(a-\tilde a)}{\Delta_n}R_n(a)P_{n-1}(\tilde a)P_{n-1}(x)\right] \\
&\quad + \frac{(a-\tilde a)}{\Delta_n}R_n(a)\gamma_{n-1}\big[P_{n-2}(\tilde a)P_{n-1}(x) - P_{n-1}(\tilde a)P_{n-2}(x)\big],
\end{aligned}
$$

$$
\begin{aligned}
&(x-\tilde a)Q_{n-1}(x) \\
&\quad = (x-\tilde a)P_{n-1}(x) + \frac{(a-\tilde a)}{\Delta_{n-1}}R_{n-1}(a)\big[P_{n-2}(\tilde a)P_{n-1}(x) - P_{n-1}(\tilde a)P_{n-2}(x)\big].
\end{aligned}
$$

Thus, for any $t_n \in \mathbb{R}$, $n \geqslant 2$

$$
\begin{aligned}
&(x-\tilde a)\big[Q_n(x) + t_n Q_{n-1}(x)\big] \\
&\quad = (x-\tilde a)\left[P_n(x) + \left(\frac{(a-\tilde a)}{\Delta_n}R_n(a)P_{n-1}(\tilde a) + t_n\right)P_{n-1}(x)\right] \\
&\qquad + (a-\tilde a)\left[\frac{R_n(a)}{\Delta_n}\gamma_{n-1} + \frac{R_{n-1}(a)}{\Delta_{n-1}}t_n\right]\big[P_{n-2}(\tilde a)P_{n-1}(x) - P_{n-1}(\tilde a)P_{n-2}(x)\big].
\end{aligned}
$$

Now, since by hypothesis $R_n(a) \neq 0$ for all $n$, if we take

$$
t_n = -\frac{R_n(a)}{R_{n-1}(a)}\frac{\Delta_{n-1}}{\Delta_n}\gamma_{n-1}, \quad n \geqslant 2,
$$

we get $t_n \neq 0$ as well as

$$
Q_n(x) + t_n Q_{n-1}(x) = P_n(x) + s_n P_{n-1}(x),
$$

where $s_n = (a-\tilde a)\Delta_n^{-1}R_n(a)P_{n-1}(\tilde a) + t_n$.

Observe that, using (2.11), we can obtain

$$
s_n = -\frac{R_n(a)}{R_{n-1}(a)} \neq 0, \quad n \geqslant 2.
$$

For $n = 1$, from the values of $a_1$ and $b_1$, the first part of formula (3.3) becomes $Q_1(x) = P_1(x) + \frac{(a-\tilde a)}{\Delta_1}R_1(a)$. Then $P_1(x) + s_1 = Q_1(x) + t_1$ holds with $s_1 t_1 \neq 0$ and $s_1 - t_1 \neq 0$.

Finally, notice that the case $a = \tilde a$ can be derived in a similar way. □

**Remarks.** (1) In Section 2, we have seen that the linear functional $v$ is quasi-definite if and only if $1 + \sum_{j=0}^{n} \frac{P_j(\tilde a)R_j(a)}{\langle u, P_j^2\rangle} \neq 0$, $n \geqslant 1$. It is worth noticing that the parameters $\{R_n(a)\}_{n\geqslant 0}$, which appear in the above result, also characterize the existence of formula (3.2).

(2) In terms of the linear functionals, we have that $R_n(a) \neq 0$ $(n \geqslant 1)$ if and only if the linear functional $(x-a)w$ is quasi-definite, where $w$ is either the linear functional $u$ (case $a = \tilde{a}$, $\lambda \neq 1$), or the linear functional $u^{(1)}$ (case $a \neq \tilde{a}$, $\lambda = 1$) or the linear functional associated with the co-recursive polynomials (case $a \neq \tilde{a}$, $\lambda \neq 1$).

(3) If $a \neq \tilde{a}$ and $\lambda \neq 1$ it was proved in [9] that $R_n(a) \neq 0$ for every $n \geqslant 1$ if and only if the linear functional $\frac{a-\tilde{a}}{\lambda-1}(x-a)^{-1}u + \delta_a$ is quasi-definite. When $u$ and $v$ are related as in Theorem 3.1, this last condition is equivalent to the quasi-definiteness of the linear functional $\lambda v - u$. Moreover, in this case the SMOP associated with $\lambda v - u$ is $\{P_n - \frac{R_n(a)}{R_{n-1}(a)} P_{n-1}\}_{n\geqslant 0}$.

Next, we want to point out that a difference appears between the cases $a = \tilde{a}$ and $a \neq \tilde{a}$ with respect to the parameters $s_n$ and $t_n$ in formula (3.2).

In Theorem 3.1, it has been shown that there exists a relation of the form

$$P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x) \tag{3.5}$$

with $s_n t_n \neq 0$, $n \geqslant 1$, and $s_1 \neq t_1$ if and only if $R_n(a) \neq 0$, $n \geqslant 1$. Moreover, we get for every $n \geqslant 1$

$$t_n - s_n = \frac{P_{n-1}(\tilde{a}) R_n(a)}{\langle u, P_{n-1}^2\rangle [1 + \sum_{j=0}^{n-1} \frac{P_j(\tilde{a}) R_j(a)}{\langle u, P_j^2\rangle}]}.$$

Then, whenever $a = \tilde{a}$ and $\lambda \neq 1$, (3.5) holds if and only if the linear functional $(x-\tilde{a})u$ is quasi-definite. Besides $s_n \neq t_n$, for $n \geqslant 1$.

However, if $a \neq \tilde{a}$, even if the condition $R_n(a) \neq 0$ is satisfied for all $n \geqslant 1$ then both situations either $(x-\tilde{a})u$ is quasi-definite or $(x-\tilde{a})u$ is not quasi-definite can appear. In fact, an example of the first situation was given in [1] being $u$ and $v$ the Jacobi linear functionals with parameters $\alpha - 1, \beta$ and $\alpha, \beta - 1$ $(\alpha, \beta > 0)$, respectively, and $a = -1$, $\tilde{a} = 1$, $\lambda = -\alpha\beta^{-1}$. In this case, also $s_n \neq t_n$ for every $n \geqslant 1$.

Next, we are going to show an example of the second situation, that is, when the linear functional $(x-\tilde{a})u$ is not quasi-definite and, as a consequence, the condition $s_n \neq t_n$ is not satisfied for every $n \geqslant 1$.

Let $u$ be the Chebyshev linear functional of second kind, that is, the Jacobi linear functional with parameters $\alpha = \beta = 1/2$, and take $a = 1$, $\tilde{a} = 0$, and $\lambda = 3$. We denote by $\{P_n\}$ the monic polynomials associated with $u$ whose recurrence coefficients are $\beta_n = 0$ and $\gamma_n = 1/4$ (see [2]). Observe that the linear functional $xu$ is not quasi-definite.

With these conditions the co-recursive polynomials $R_n$ are given by

$$R_n(x) = 2\left[P_n(x) + \frac{1}{2}P_{n-1}(x)\right]. \tag{3.6}$$

Notice that $\frac{1}{2}R_n(x)$ are the monic Chebyshev polynomials of fourth kind, that is the monic Jacobi polynomials with parameters $\alpha = 1/2$ and $\beta = -1/2$, see [5].

First, we check that the linear functional $v$ defined by $xu = 3(x-1)v$ is quasi-definite. As we have introduced in Theorem 2.1

$$\Delta_n = \begin{vmatrix} P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_n(a) & R_{n-1}(a) \end{vmatrix}, \quad n \geqslant 1,$$

and since $P_{2n}(0) = (-1)^n/4^n$, $P_{2n+1}(0) = 0$, and $R_n(1) = (2n+1)/2^{n-1}$ we get

$$\Delta_{2n} = (-1)^n \frac{4n-1}{4^{2n-1}} \quad \text{and} \quad \Delta_{2n+1} = (-1)^{n+1} \frac{4n+3}{4^{2n}}.$$

Therefore, $\Delta_n \neq 0$ for every $n \geqslant 1$, and thus $v$ is quasi-definite. Observe that $v = -\frac{x}{3} w + \delta_1$ where $w$ denotes the Chebyshev linear functional of third kind.

As $R_n(1) \neq 0$, for $n \geqslant 1$, from Theorem 3.1 a relation of the form (3.5) holds with

$$s_n = -\frac{R_n(1)}{R_{n-1}(1)} = -\frac{2n+1}{2(2n-1)}, \quad n \geqslant 2,$$

and

$$t_n = \frac{\Delta_{n-1}}{4\Delta_n} s_n, \quad n \geqslant 2.$$

Therefore, taking into account $P_1(x) = Q_1(x) + 1$, we deduce

$$P_{2n}(x) - \frac{4n+1}{2(4n-1)} P_{2n-1}(x) = Q_{2n}(x) - \frac{4n+1}{2(4n-1)} Q_{2n-1}(x), \quad n \geqslant 1,$$
$$P_{2n+1}(x) - \frac{4n+3}{2(4n+1)} P_{2n}(x) = Q_{2n+1}(x) + \frac{4n-1}{2(4n+1)} Q_{2n}(x), \quad n \geqslant 0.$$

Notice that in this case $s_{2n} = t_{2n}$, $n \geqslant 1$.

Eventually, from the values of the recurrence coefficients of $\{P_n\}$ and Theorem 2.2 in [1], we can deduce that the recurrence parameters for $\{Q_n\}$ are $\tilde{\beta}_n = (-1)^n$, $n \geqslant 0$, and

$$\tilde{\gamma}_{2n+1} = -\frac{4n-1}{4(4n+3)}, \quad n \geqslant 0, \quad \text{and} \quad \tilde{\gamma}_{2n} = -\frac{4n+3}{4(4n-1)}, \quad n \geqslant 1.$$

## References


[1] M. Alfaro, F. Marcellán, A. Peña, M.L. Rezola, On linearly related orthogonal polynomials and their functionals, J. Math. Anal. Appl. 287 (2003) 307–319.

[2] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[3] A.M. Delgado, F. Marcellán, Companion linear functionals and Sobolev inner products: a case study, submitted for publication.

[4] W. Gautschi, An algorithmic implementation of the generalized Christoffel theorem, in: G. Hammerlin (Ed.), Numerical Integration, in: Internat. Ser. Numer. Math., vol. 57, Birkhäuser, Bassel, 1982, pp. 147–154.

[5] W. Gautschi, S. Li, A set of orthogonal polynomials induced by a given orthogonal polynomial, Aequationes Math. 46 (1993) 174–198.

[6] F.A. Grünbaum, L. Haine, Bispectral Darboux transformations: an extension of the Krall polynomials, Internat. Math. Res. Notices 8 (1997) 359–392.

[7] F. Marcellán, J.S. Dehesa, A. Ronveaux, On orthogonal polynomials with perturbed recurrence relations, J. Comput. Appl. Math. 30 (1990) 203–212.

[8] F. Marcellán, P. Maroni, Sur l'adjonction d'une masse de Dirac à une forme regulière et semi-classique, Ann. Mat. Pura Appl. 162 (1992) 1–22.

[9] P. Maroni, Sur la suite de polynômes orthogonaux associée à la forme $u = \delta_c + \lambda(x-c)^{-1}L$, Period. Math. Hungar. 21 (1990) 223–248.

[10] P. Maroni, Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques, in: C. Brezinski, et al. (Eds.), Orthogonal Polynomials and Their Applications, in: IMACS Ann. Comput. Appl. Math., vol. 9, 1991, pp. 95–130.

[11] V.B. Uvarov, The connection between systems of polynomials orthogonal with respect to different distribution functions, Comput. Math. Math. Phys. 9 (1969) 25–36.
[12] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, J. Comput. Appl. Math. 85 (1997) 67–86.