Stochastic Orderings and Aging Notions based on Percentile Residual Life Functions

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To my mother

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Resumen en castellano

Los órdenes estocásticos constituyen un tema de investigación consolidado pero que a la vez ofrece multitud de líneas abiertas de investigación. En situaciones en las que modelos estocásticos realistas son demasiado complejos, los órdenes estocásticos proporcionan métodos de aproximación y cotas valiosas. También resultan de utilidad cuando sólo se conocen las distribuciones de los modelos parcialmente. Algunos campos de aplicación de los órdenes estocásticos son, entre otros, economía, teoría de colas, teoría de fiabilidad, física, estadística, epidemiología y el campo actuarial.

La función de vida media residual es un concepto muy utilizado en el contexto de fiabilidad. Esta función caracteriza la distribución que la origina, lo que ha sido explotado en la literatura. Sin embargo, la estimación con la versión muestral de la función de vida media residual presenta algunos inconvenientes si existen datos censurados o si la distribución subyacente es asimétrica o de cola pesada. Una alternativa robusta a la función de vida media residual es la función de vida mediana residual o, de forma más general, la función de vida cuantílica residual cuya estimación es factible y su interpretación más lógica en muchos casos. Un inconveniente es que, a diferencia de la función de vida media residual, la función de vida cuantílica residual no caracteriza la distribución, sino una familia de distribuciones.

Dadas las ventajas que presenta la función de vida cuantílica residual a nivel práctico, en el Capítulo 2 de la tesis presentamos los órdenes de vida cuantílica residual. Fijado $\gamma \in (0, 1)$, dos variables aleatorias están ordenadas en el sentido del orden de vida γ-cuantílica residual si sus correspondientes funciones de vida γ-cuantílica residual están ordenadas. Estudiamos sus propiedades de clausura, sus relaciones con otros órdenes existentes en la literatura y sus posibles aplicaciones en diversas disciplinas. En concreto, el orden de vida mediana residual supone una alternativa al orden de vida media residual que, al estar basado en la comparación de funciones de vida media residual, resulta más dependiente de la distribución subyacente. Por lo que hemos comentado anteriormente de que la función de vida cuantílica residual no caracteriza a la distribución de partida, los órdenes de vida cuantílica residual son relaciones binarias que no verifican la propiedad antisimétrica; es decir, no son órdenes sino preórdenes. Esto también pasa con los órdenes definidos en el Capítulo 3, que comparan las funciones de vida cuantílica residual, no en todo el soporte de la distribución, pero para todos los posibles valores de los cuantiles desde un cierto instante t_0 dado. Los órdenes definidos en el Capítulo 4 de la tesis también están basados en la comparación de funciones

de vida cuantílica residual y sí son propiamente órdenes. Estas ordenaciones comparan las funciones de vida cuantílica residual hasta t_0 para todos los posibles valores de los cuantiles. En estos dos capítulos se ilustran algunas aplicaciones de estos órdenes, como la comparación de productos una vez vencido su periodo de garantía o durante este periodo, y la comparación de artículos de segunda mano.

Otro concepto muy importante en fiabilidad es la noción de envejecimiento. 'No envejecimiento' significa que la edad de una componente no tiene ningún efecto en la distribución de su vida residual. 'Envejecimiento positivo' describe la situación en la cual la vida residual tiende a decrecer, en algún sentido probabilístico, cuando la edad de la componente aumenta. Esta situación es muy común en ingeniería, cuando las componentes tienden a empeorar debido al desgaste. Por otra parte, el 'envejecimiento negativo' tiene un efecto opuesto en la vida residual. Aunque esto es menos común, cuando un sistema supera ciertos tests y mejora, existen clases de distribuciones que se ajustan a este fenómeno en el que la fiabilidad tiende a incrementarse con el paso del tiempo. Los diferentes conceptos de envejecimiento describen como una componente o un sistema mejora o empeora con la edad. Muchas clases de distribuciones de vida se caracterizan o definen en la literatura según sus propiedades de envejecimiento. Un aspecto importante en esta clasificación es el hecho de que la distribución exponencial pertenece, en la mayoría de los casos, a todas las clases. Esto es debido a la propiedad de falta de memoria de la distribución exponencial. En el quinto capítulo de la tesis estudiamos en profundidad el concepto de envejecimiento conocido como función de vida cuantílica residual decreciente, que fue introducido por primera vez en Haines y Singpurwalla (1974). Además, presentamos resultados que permiten caracterizar esta noción a partir de ciertas propiedades del orden de vida cuantílica residual estudiado en el Capítulo 2. En este mismo capítulo definimos nuevas nociones de envejecimiento basadas también en el comportamiento monótono de las funciones de vida cuantílica residual y que pueden ser caracterizadas a partir de ciertas propiedades de los órdenes estocásticos definidos en los capítulos tercero y cuarto. Además, completamos algunos resultados demostrados en Launer (1993) que relacionan el comportamiento de la función tasa de fallo y la función de vida cuantílica residual. En particular, damos condiciones necesarias para las distribuciones bathtub y bathtub invertida.

Dada una cierta ordenación estocástica y las muestras aleatorias de dos poblaciones, es común que se presenten en la literatura ciertas técnicas estadísticas que permitan decidir cuándo las variables aleatorias subyacentes están ordenadas o no. En el sexto capítulo de la tesis presentamos un nuevo procedimiento para construir bandas de confianza para la diferencia de dos funciones de vida cuantílica residual. Estas bandas proporcionan un criterio mediante el cual poder decidir cuando dos variables aleatorias están cerca o no en el sentido del orden de vida cuantílica residual. La metodología que hemos aplicado requiere el uso de técnicas bootstrap y del concepto de profundidad estadística para funciones.

Por último, en el séptimo, resumimos nuestras principales aportaciones y explicamos cuáles serán las futuras líneas de investigación en las que trabajaremos a partir de ahora y que están íntimamente relacionados con el trabajo presentado en esta tesis.

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Chapter 1

Background and Motivation

This chapter gives an overview of the basic concepts, terminology, and related work regarding the topics of this dissertation. The focus of this dissertation is five-fold:

- to introduce and study new families of stochastic orders: the percentile residual life orders, the percentile residual life orders from time t_0 on, and the percentile residual life orders up to time t_0,
- to study in depth the decreasing percentile residual life aging notions and to characterize them in terms of the percentile residual life orders,
- to introduce new aging notions and characterize them in terms of the percentile residual life orders from time t_0 on and the percentile residual life orders up to time t_0,
- to introduce new results that relate the behavior of the hazard rate function and the behavior of the percentile residual life function providing necessary conditions for bathtub distributions and upside-down bathtub distributions, and
- to develop a new procedure for constructing bootstrap confidence bands for the difference of two percentile residual life functions.

In order to explain the interest of the topics of this dissertation, we present an example. The data were taken from Apendix I of Kalbfleisch and Prentice (1982) (Data Set II) and are part of a large clinical trial carried out by the Radiation Therapy Oncology Group in the United States. The full study included patients with squamous carcinoma of 15 sites in the mouth and throat, with 16 participating institutions, though in the book only data on three sites in the oropharynx reported by the six largest institutions are considered. Patients entering the study were randomly assigned to one of two treatment groups, radiation therapy alone or radiation therapy together with a chemotherapeutic agent. We are interested in the survival times (in days) of the patients after the treatments. Approximately 30% of the survival times are censored owing primarily to patients surviving to the time of analysis.

Some patients were lost to follow-up because the patient moved or were transferred to an institution not participating in the study, though these cases were relatively rare. From a statistical point of view, the main feature of these data that distinguishes this example from others is the considerable lack of homogeneity between individuals being studied. We have deleted the females in order to make the populations more homogeneous (this way we avoid possible differences due to gender).

In Figure 1.1 the histograms of the survival times of the patients belonging to the two treatment groups are shown. This figure reveals that both groups of data present asymmetry: the right tail (tail at small end of the distribution) is more pronounced than the left tail (tail at the large end of the distribution). Skewness is a measure of the degree of asymmetry of a distribution. We have computed the skewness of both groups of data and obtained 0.8386 and 1.3463, respectively.

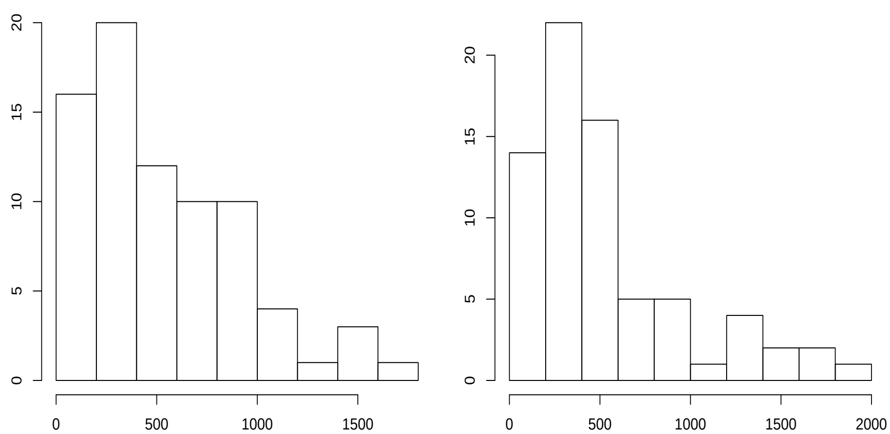

Figure 1.1: Histograms of the survival times of the two groups of patients

In Figure 1.2 the box-and-whisker plots of the survival times of the patients belonging to the two treatment groups are shown. It is clear from the figure that there exist several outliers within the second group of patients, those undergoing radiotherapy together with a chemotherapeutic agent.

Given these data, researchers may be interested in studying the effectiveness of the treatments. For doing that, different reliability measures can be considered. Besides, since there exist two treatment groups, one may be interested in comparing them. For comparing the two treatments, different stochastic orderings can be considered.

This chapter reviews some literature in reliability theory and stochastic orderings. It is organized as follows. In Section 1.1 we give a brief description of different reliability measures. Some very well known stochastic orders, based on the comparison of these reliability measures, are reviewed on Section 1.2. We will explain the deficiencies of some of these orders

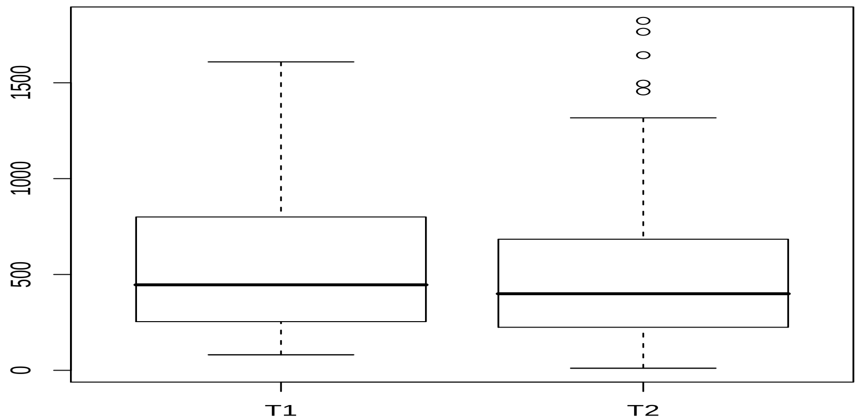

Figure 1.2: Box-and-whisker plots of the survival times of the two groups of patients

and motivate the introduction of new ones. Finally, in Section 1.3 we recall the definition of several aging notions.

The rest of the document is organized as follows. New families of stochastic orders: the percentile residual life orders, the percentile residual life orders from time t_0 on, and the percentile residual life orders up to time t_0 are explored in Chapter 2, Chapter 3, and Chapter 4, respectively. In Chapter 5 we study the decreasing percentile residual life aging notions and establish some characterizations of these notions based on the percentile residual life orders. Also in Chapter 5, new aging notions are defined and characterized in terms of the percentile residual life orders from time t_0 on and the percentile residual life orders up to time t_0. Besides, we complete a study carried out by Launer (1993) providing necessary conditions for bathtub and upside-down bathtub distributions. In Chapter 6 we describe a new procedure for constructing bootstrap confidence bands for the difference of two percentile residual life functions. Finally, Chapter 7 outlines our contributions and gives directions for future work.

Some conventions that we use in this dissertation are the following. By "increasing" and "decreasing" we mean "nondecreasing" and "nonincreasing", respectively. For any distribution function F we let function F^{-1} be the left continuous version of the inverse of F, that is

$$F^{-1}(p) = \inf\{x : F(x) \geq p\}, \quad p \in (0, 1).$$

1.1 Reliability measures

Let X be a random variable and let u_X be the right endpoint of its support. For any $t < u_X$, the *residual life* at time t is the random variable that has the conditional distribution of

$X - t$ given that $X > t$. We denote it by

$$X_t = [X - t \big| X > t], \quad t < u_X. \tag{1.1}$$

The residual life is of interest in many areas of applied probability and statistics such as actuarial studies, biometry, survivorship analysis, and reliability — see, for example, Lillo (2005) for a list of references.

We introduce the definitions of four very well known reliability measures.

Definition 1.1. *If F_X denotes the distribution function of X, then $\overline{F}_X = 1 - F_X$ denotes the corresponding* ***survival function****. That is,*

$$\bar{F}_X(t) = P(X > t).$$

Let $X_1, \ldots, X_n$ be the lifetimes of the patients in the example previously described undergoing Treatment 1 (radiotherapy alone). Only their right censored versions are observed, leading to the information $(\delta_1, Z_1), \ldots, (\delta_n, Z_n)$ where for $i = 1, \ldots, n$,

$$\delta_i = I_{\{X_i \leq Y_i\}} \quad \text{and} \quad Z_i = X_i \wedge Y_i = \max\{X_i, Y_i\},$$

with Y_i representing the i-th censoring random variable. ($I_{\{A\}}$ is one if A is true, and zero otherwise). It is assumed that $Y_1, \ldots, Y_n$ are i.i.d. with $G(y) = P(Y > y) > 0$ and that G is continuous. The survival function of X can be estimated by

$$\bar{F}_{X,n}(x) = \frac{N^+(x) + 1}{n+1} \prod_{j=1}^{n} \Big(\frac{2 + N^+(Z_j)}{1 + N^+(Z_j)} \Big)^{I_{\{\delta_j = 0, Z_j \leq x\}}}, \tag{1.2}$$

where $N^+(x)$ denotes the number of censored and uncensored observations greater than x. Analogously, the survival function of the patients undergoing Treatment 2 (radiotherapy together with a chemotherapeutic agent) is estimated.

In Figure 1.3 the estimated survival functions of the patients belonging to the two treatment groups are shown.

Definition 1.2. *If X is a random variable with an absolutely continuous distribution function F_X, then the* ***hazard rate*** *of X at t is defined as $r_X(t) = (d/dt)(-log(1 - F_X(t)))$. The hazard rate can alternatively be expressed as*

$$r_X(t) = \lim_{\Delta t \downarrow 0} \frac{P\{t < X \leq t + \Delta t | X > t\}}{\Delta t} = \frac{f_X(t)}{\bar{F}_X(t)}, t \in \mathbb{R}, \tag{1.3}$$

where f_X is the density function of X.

As it can be seen from (1.3), the hazard rate $r_X(t)$ can be thought of as the intensity of failure of a device, with a random lifetime X, at time t. Clearly, the higher the hazard rate is, the smaller X should be stochastically.

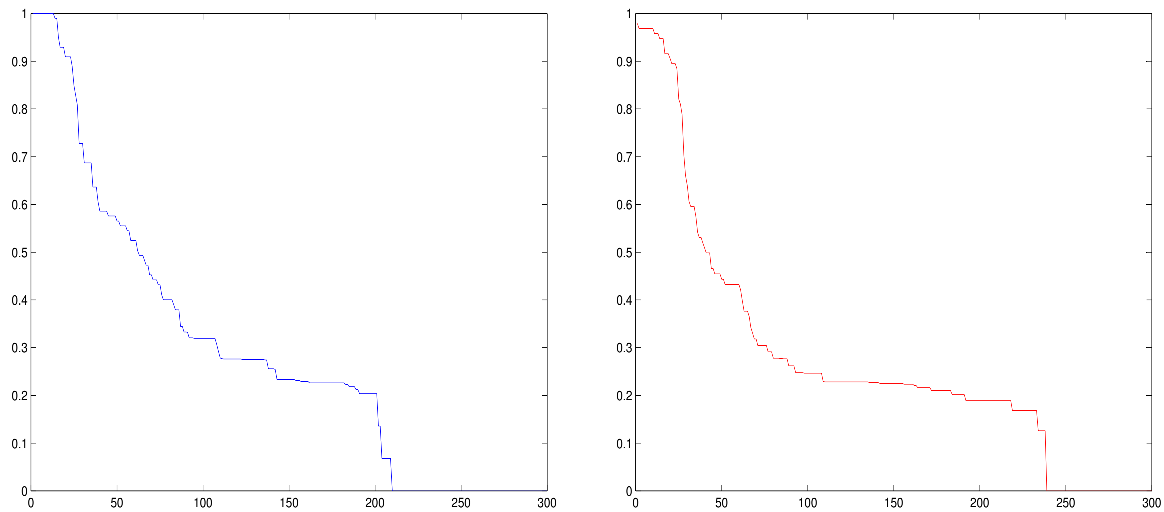

Figure 1.3: Estimated survival functions of the patients undergoing Treatment 1 (blue) and Treatment 2 (red)

Definition 1.3. *If X is a random variable with an absolutely continuous distribution function F_X, then the* ***reversed hazard rate*** *of X at the point t is defined as*

$$\tilde{r}_X(t) = (d/dt)(\log F_X(t)). \tag{1.4}$$

One interpretation of the reversed hazard rate at time t is the following. Suppose that X is nonnegative with distribution function F_X. Then X can be thought of as the lifetime of some device. Given that the device has already failed by time t, the probability that it survived up to time $t - \varepsilon$ (for small $\varepsilon > 0$) is approximately $\varepsilon \cdot \tilde{r}_X(t)$.

Definition 1.4. *The* ***mean residual life function*** *that is associated with X is given by*

$$m_X(t) = \begin{cases} E[X - t \big| X > t], & t < u_X; \\ 0, & t \geq u_X, \end{cases} \tag{1.5}$$

provided the expectation exists.

The mean residual life function is a useful tool for analyzing important properties of X when it exists because it characterizes the distribution:

$$\bar{F}_X(t) = \frac{m_X(0)}{m_X(t)} exp\Big\{ - \int_0^t \frac{1}{m_X(x)} dx \Big\}, \tag{1.6}$$

for all t such that $P(X > t) > 0$. However, the mean residual life function may not exist. Even when it exists it may have some practical shortcomings, especially in situations where the data are censored, or when the underlying distribution is skewed or heavy-tailed. In such cases, either the empirical mean residual life function cannot be calculated, or a single long-term survivor can have a marked effect upon it which will tend to be unstable due to its strong dependence on very long durations.

To estimate the mean residual life function of this group of patients, we have computed the estimator proposed by Ghorai, Susarla, Susarla, and van Ryzin (1980), given by

$$\hat{m}_{X,n}(x) = \frac{1}{(n+1)\bar{F}_{X,n}(x)} \sum_{l=1}^{n} \delta_l \cdot Z_l \cdot I_{\{Z_l > x\}} \cdot \prod_{j=1}^{n} \left(\frac{2 + N^+(Z_j)}{1 + N^+(Z_j)} \right)^{I_{\{\delta_j = 0, Z_j \leq Z_l\}}},$$

where $\bar{F}_{X,n}$ is computed as in equation (1.2). Analogously, the mean residual life function of the patients undergoing Treatment 2 (radiotherapy together with a chemotherapeutic agent) is estimated.

In Figure 1.4 the estimated mean residual life functions of the patients belonging to the two treatment groups are shown.

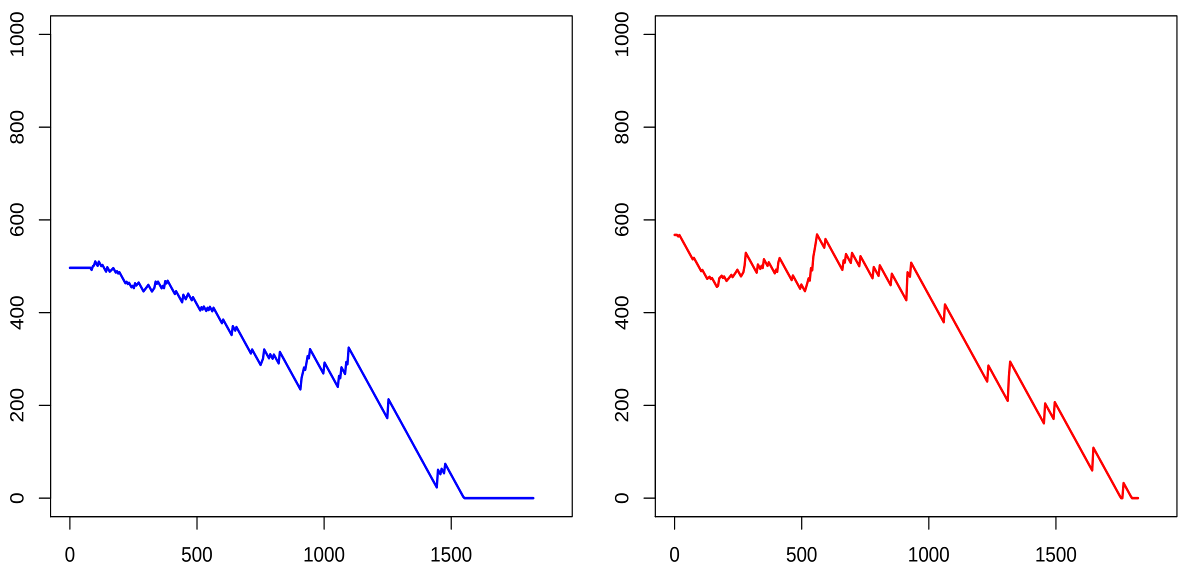

Figure 1.4: Estimated mean residual life functions of the patients undergoing Treatment 1 (blue) and Treatment 2 (red)

An alternative to the mean residual life function is the **γ-percentile residual life function**, denoted by $q_{X,\gamma}$, where γ is some number between 0 and 1. This function is defined for any $t < u_X$ by letting $q_{X,\gamma}(t)$ be the γ-percentile of X_t. The formal definition of the percentile residual life function will be given in Chapter 2.

The γ-percentile residual life functions were studied in some detail by Arnold and Brockett (1983), Gupta and Langford (1984), Joe and Proschan (1984a), and Joe (1985), as well as by Haines and Singpurwalla (1974). Families of distributions for which simple expressions for the γ-percentile residual life functions can be obtained, are identified in Raja Rao, Alhumoud, and Damaraju (2006).

A particular γ-percentile residual life function of interest is the median residual life function given by $q_{X,0.5}$. This function was studied in detail by Lillo (2005) and Gelfand and Kottas (2003) used it for Bayesian semiparametric modeling. See the above two references for further references to papers that studied the γ-percentile and the median residual life functions, and that used them in practical applications.

In Figure 1.5, the estimated median residual life functions of the two groups of patients are shown. Since there exist censored data, we have computed the estimator proposed in Csörgő (1987) which consists in calculating the empirical median residual life function associated with the survival function given in equation (1.2). Analogously, the median residual life function of the second group of patients is estimated.

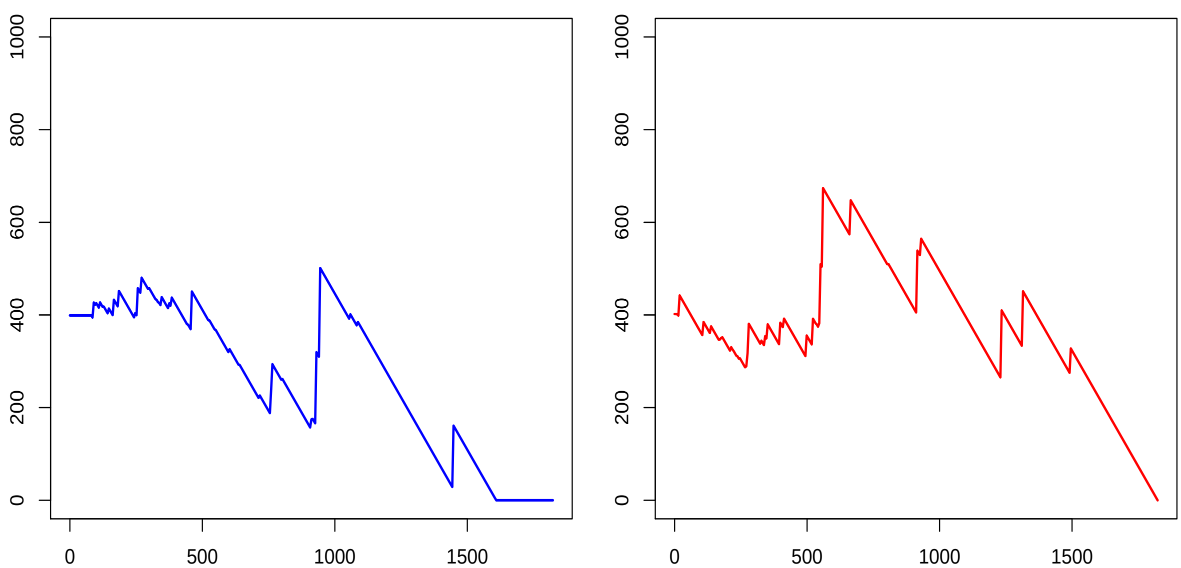

Figure 1.5: Estimated median residual life functions of the patients undergoing Treatment 1 (blue) and Treatment 2 (red)

As we have already said, the mean residual life function has a strong dependence to the underlying distribution which is inconvenient, specially in situations where the distribution is skewed or heavy-tailed. In this case, the survival data of both groups of patients present asymmetry. We have also pointed out the presence of some outliers that take large values, within the second group of patients. By a visual examination we can confirm that these outliers affect to the whole mean residual life function, but they only affect to the tail of the median residual life function.

A researcher should find all four reliability measures, whose definitions we have just recalled, interesting and complementary. To illustrate their different perspective, consider one of the cancer patients in the example undergoing radiotherapy alone and let us suppose that his radiotherapy began 60 days ago. It may be interesting to know how long he should expect to live. His expected remaining life is given by $m_X(60)$. In the same manner, the median of his remaining life is $q_{X,0.5}(60)$. The proportion of patients like him who survive more than 60 days is given by $\bar{F}_X(60)$ and his instantaneous probability of dying tomorrow is $r_X(60)$.

As we pointed out before, it is well known that if X has finite mean, then F_X is uniquely determined by its mean residual life function (see equation (1.6)). However, there can be an infinite number of life distributions with the same $q_{X,\gamma}(t)$ and, in particular, with the same median residual life function. Gupta and Langford (1984) proved this result by solving the functional equation that relates the survival function and the percentile residual life function,

$$\overline{F}_X(t + q_{X,\gamma}(t)) = \overline{\gamma}\overline{F}_X(t) \quad \text{for all } t,$$

where $\overline{\gamma} = 1-\gamma$. More generally, they studied the problem of solving the functional equation

$$g(\phi(t)) = sg(t), \tag{1.7}$$

called Schröder's equation. Here ϕ is a known function, s is a constant, and g is to be solved for. They proved that, under mild assumptions on $\phi(t)$, the solution is of the form

$$g(t) = g_0(t)K(\log(g_0(t))), \qquad \text{for all } t,$$

where g_0 is a well-behaved particular solution of (1.7) which can be constructed, and K is a periodic function of period $|\log(s)|$; thus the solution of (1.7) is not unique and, specifically, either the median residual life function or the γ-percentile residual life function does not characterize the distribution function as the mean residual life function does.

In spite of that, and as we have illustrated in the example, the median residual life function (and, in general, the percentile residual life function) has some practical advantages compared to the mean residual life function since it is not so sensitive to the underlying distribution (in particular to the presence of outliers). Besides, the γ-percentile residual life function always exists and to compute its empirical version we do not need all the data, just the $(1-\gamma)\cdot 100\%$ of the data. Given all these advantages in practical situations, we focus our attention on this reliability measure, and in this thesis we define several stochastic orders and aging notions based on this function.

1.2 Stochastic orderings

Stochastic orders and inequalities have been used during the last 40 years, at an accelerated rate, in different areas of probability and statistics. These areas include reliability theory, queueing theory, survival analysis, biology, economics, insurance, actuarial science, operations research, and management science.

The simplest way of comparing two distribution functions is by comparing the associated means. However, such a comparison is based on only two single numbers (the means), and therefore it is often not very informative. In addition to this, the means sometimes do not exist. In many instances in applications one has more detailed information, for the purpose of comparison of two distribution functions, than just the two means. The most important and common stochastic orders that compare the 'location' or the 'magnitude' of random variables are the usual stochastic order, the hazard rate order and the mean residual life order which are reviewed next.

Definition 1.5. *Let X and Y be two random variables such that*

$$\bar{F}_X(t) \leq \bar{F}_Y(t) \text{ for all } t \in (-\infty, \infty). \tag{1.8}$$

Then X is said to be ***smaller than Y in the usual stochastic order*** *(denoted by $X \leq_{st} Y$).*

Roughly speaking, (1.8) says that X is less likely than Y to take on large values, where 'large' means any value greater than t, and that is the case for all t's. Note that (1.8) is the same as

$$F_X(t) \geq F_Y(t) \text{ for all } t \in (-\infty, \infty). \tag{1.9}$$

An important characterization of the usual stochastic order is the following theorem ($=_{st}$ denotes equality in law), whose proof can be read in Shaked and Shanthikumar (2007).

Theorem 1.1. *Two random variables X and Y satisfy $X \leq_{st} Y$ if, and only if, there exist two random variables $\hat{X}$ and $\hat{Y}$, defined on the same probability space, such that*

$$\hat{X} =_{st} \hat{X},$$
$$\hat{Y} =_{st} \hat{Y},$$

and

$$P(\hat{X} \leq \hat{Y}) = 1.$$

In Figure 1.6, the estimated survival functions for the two groups of patients are compared.

From (1.9) and Theorem 1.1 it follows that the random variables X and Y, with the respective distribution functions F_X and F_Y, satisfy $X \leq_{st} Y$ if, and only if,

$$F_X^{-1}(p) \leq F_X^{-1}(p), \text{ for all } p \in (0, 1); \tag{1.10}$$

see, for example, (1.A.12) in Shaked and Shanthikumar (2007).

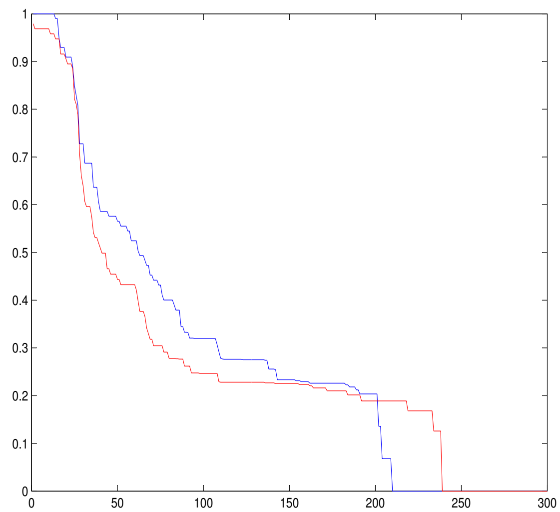

Figure 1.6: Comparing estimated survival functions of the patients undergoing Treatment 1 (blue) and Treatment 2 (red)

Definition 1.6. *Recall from* (1.3) *the definition of the hazard rate function* r_X *of a random variable* X*. Let* r_Y *be the hazard rate of another random variable* Y*. If*

$$r_X(t) \leq r_Y(t), \text{ for all } t \in \mathbb{R}, \tag{1.11}$$

then X *is said to be* ***smaller than*** Y ***in the hazard rate order*** *(denoted by* $X \leq_{hr} Y$*).*

Although the hazard rate order is usually applied to random lifetimes (that is, nonnegative random variables), definition (1.11) may also be used to compare more general random variables. In fact, even the absolute continuity, which is required in (1.11), is not really needed. It is easy to verify that (1.11) holds if, and only if,

$$\frac{\bar{F}_Y(t)}{\bar{F}_X(t)} \text{ increases in } t \in (-\infty, \max(u_X, u_Y)) \tag{1.12}$$

($a/0$ is taken to be equal to ∞ whenever $a > 0$). Here u_X and u_Y denote the corresponding right endpoints of the supports of X and Y. Equivalently, (1.12) can be written as

$$\bar{F}_X(x)\bar{F}_Y(Y) \geq \bar{F}_X(y)\bar{F}_Y(x) \text{ for all } x \leq y. \tag{1.13}$$

Recall from (1.4) the definition of the reversed hazard rate function $\tilde{r}_X$ of a random variable X. Let $\tilde{r}_Y$ be the reversed hazard rate of another random variable Y.

Definition 1.7. *If*

$$\tilde{r}_X(t) \leq \tilde{r}_Y(t), \text{ for all } t \in \mathbb{R}, \tag{1.14}$$

then X is said to be ***smaller than** Y **in the reversed hazard rate order*** *(denoted by $X \leq_{rh} Y$).*

The absolute continuity, which is required in (1.14), is not really needed. It is easy to verify that (1.14) holds if, and only if,

$$\frac{F_Y(t)}{F_X(t)} \text{ increases in } t \in (\min(l_X, l_Y), \infty) \tag{1.15}$$

($a/0$ is taken to be equal to ∞ whenever $a > 0$). Here l_X and l_Y denote the corresponding left endpoints of the supports of X and Y. Equivalently, (1.14) can be written as

$$F_X(x)F_Y(Y) \geq F_X(y)F_Y(x) \text{ for all } x \leq y. \tag{1.16}$$

Recall from (1.5) the definition of the mean residual life function m_X of a random variable X. Let m_Y be the mean residual life function of another random variable Y.

Definition 1.8. *If*

$$m_X(t) \leq m_Y(t) \quad \textit{for all } t \in \mathbb{R}, \tag{1.17}$$

then X is said to be ***smaller than** Y **in the mean residual life order*** *(denoted by $X \leq_{mrl} Y$).*

In Figure 1.7, the estimated mean residual life functions and the estimated median residual life functions for the two groups of patients are compared.

As we pointed out in the previous section, there exist several outliers within the second group of patients. Since these outliers affect the whole mean residual life function of the second group of patients, using the mean residual life order to compare the efficiency of the treatments may lead us to wrong conclusions. The comparison of the mean residual life functions leads us to conclude that the first treatment is better than the second one when the patients survive at least about 100 days after the treatment. Since the presence of these outliers only affects to the tail of the median residual life function of the second group of patients, we can conclude that, in this case, the comparison of median residual life functions is more reliable in some sense. If we do not take into account the information provided by the tail of the median residual life functions ($t > 600$) we would conclude that the second treatment is better than the second one when the patients survive at least about 100 days after the treatment. This is exactly the opposite we conclude if we compare the mean residual life functions.

Let us now consider a different example from the field of Finance. Every day investors and financial professionals look to one firm for financial market intelligence that is authoritative,

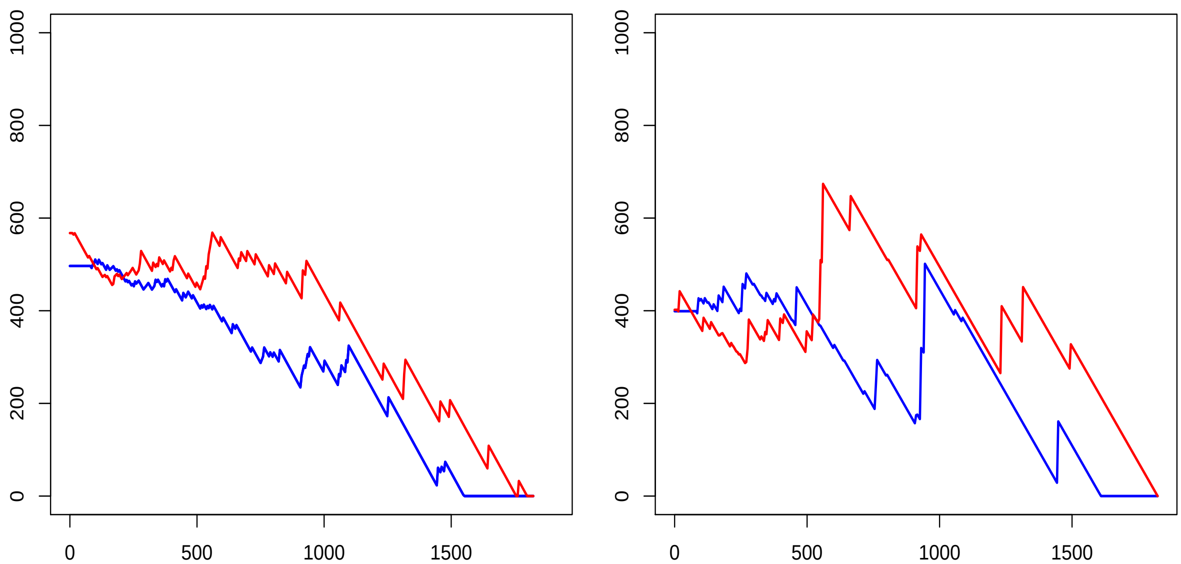

Figure 1.7: Estimated mean residual life functions (left) and estimated median residual life functions (right) of the patients undergoing Treatment 1 (blue) and Treatment 2 (red)

objective and credible. Standard & Poors credit ratings, indices, investment research and data provide financial decision-makers with the information and opinions they need to feel confident about their decisions. The following example shows how the comparison of median residual life functions is also more useful to interpret financial data than the median residual life order.

We are interested in comparing the most solvent firms with the less ones in terms of the financial burden of the company with respect to its business volume. This information is measured by the ratio *Financial Expenditures/Sales*. For doing that, we have considered the Standard & Poors credit ratings of 394 firms in 2000; 358 of those firms were classified as triple-A, AA or A (we denote them as A* firms) and 36 were classified as triple-C, CC or C (we denote them as C* firms).

The histograms of the ratios for the two groups of firms are shown in Figure 1.8. This figure reveals that both groups of data present asymmetry. As in the oncological example, the right tail is more pronounced than the left tail. We have computed the skewness of both groups of data and obtained 2.5831 and 3.2584, respectively.

In Figure 1.9 the box-and-whisker plots of the ratios of the two groups of firms are shown. It is clear from the figure that there exist several outliers within both groups of firms. The number of outliers is larger in the A* firms; however, the outliers within the C* firms take more extreme values.

In Figure 1.10, the estimated mean residual life functions and the estimated median

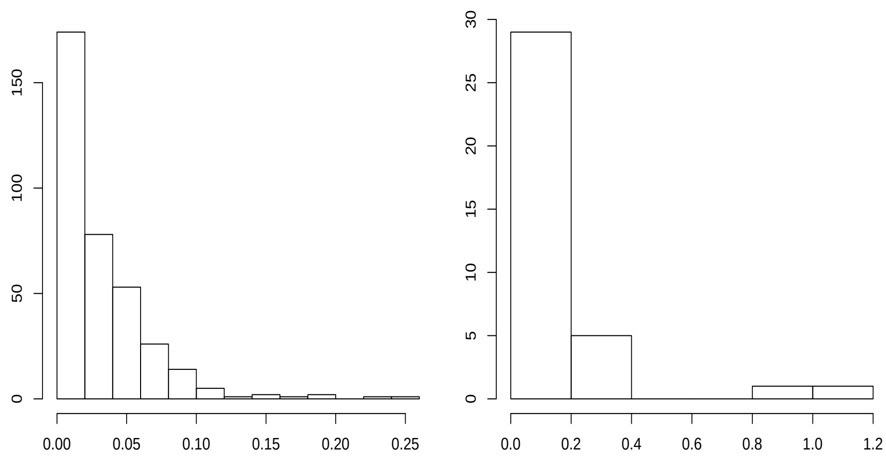

Figure 1.8: Histograms of the ratios for the two groups of firms

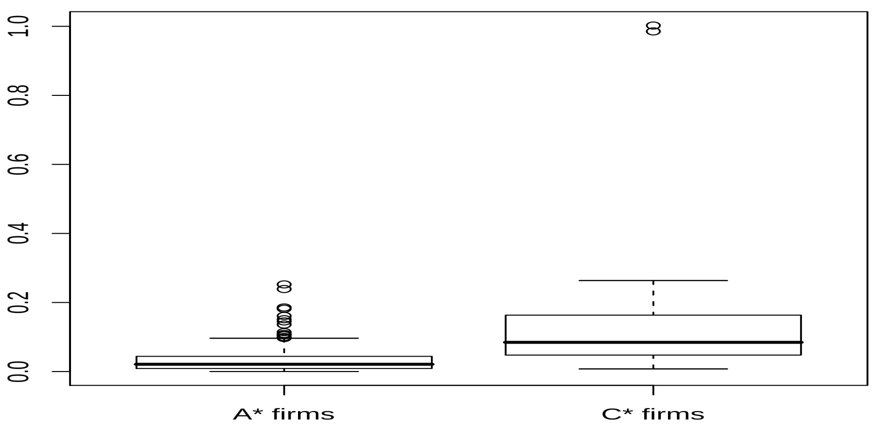

Figure 1.9: Box-and-whisker plots of the ratios for the two groups of firms

residual life functions for the two groups of firms are compared.

From Figure 1.7 we can affirm that, in this case, the comparison of the mean or median residual life functions leads us to the same conclusion. That is, that those firms which are less solvent, tend to invest more with respect to their sales benefits than those which are more solvent. However, this difference seems to be larger if we compare the mean residual life functions instead of the median residual life function.

In this example it is even more clear how the presence of outliers affects the mean and the median residual life functions. It is seen from the figure that the few outliers that belong to the second group of firms affect to the whole mean residual life function. However, they do only affect to the tail of the median residual life function of the second group of firms.

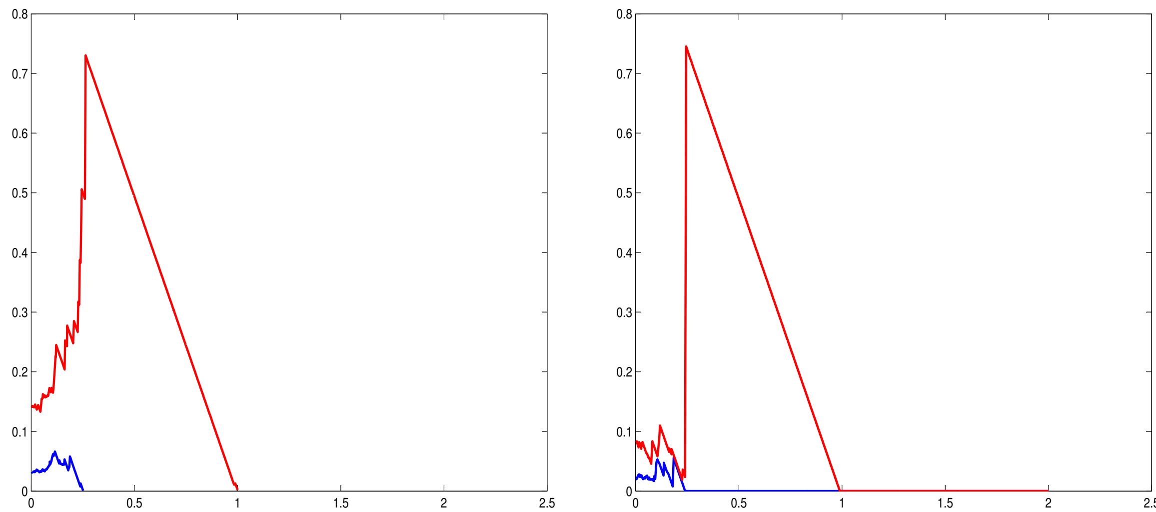

Figure 1.10: Estimated mean residual life functions (left) and estimated median residual life functions (right) of the A* firms (blue) and the C* firms (red)

Therefore, we can conclude that, also in this example, the comparison of median residual life functions is more reliable that the comparison of the mean residual life functions.

Given the advantages of comparing the median residual life functions instead of the mean residual life functions of random variables, in Chapter 2 we have introduced and studied a new family of stochastic orderings which is based on the comparison of percentile residual life functions: the percentile residual life orders. One of these advantages is that the percentile residual life orders are less sensitive to outliers than the mean residual life order, as we have illustrated through these real data examples. However, since the γ-percentile residual life function does not characterize the distribution, the γ-percentile residual life orders are not orders but preorders.

Motivated by the applicability of the percentile residual life orders for comparing items after initial warranty or to compare used items as we show in Chapter 2, in Chapter 3 we have proposed and studied new stochastic orderings which can be used with the same purpose but these orders are based on the comparison of all the percentile residual life functions of two random variables, not in the whole support but from a certain moment $t_0 > 0$ on. They are called the percentile residual life orders from time t_0 on. Analogously to the percentile residual life order from time t_0 on, we have defined and studied new stochastic orders that allow us to compare random variables until t_0. These orders are useful to compare items during the warranty period or in medical trials. They were introduced and studied in Chapter 4.

Let X and Y be two nonnegative random variables with mean residual life functions m_X and m_Y, respectively, and suppose that the harmonic averages of m_X and m_Y are comparable as follows:

$$[\frac{1}{x}\int_0^x \frac{1}{m_X(u)}du]^{-1} \leq [\frac{1}{x}\int_0^x \frac{1}{m_Y(u)}du]^{-1} \text{ for all } x > 0.$$

Then X is said to be **smaller than Y in the harmonic mean residual life order** (denoted by $X \leq_{hmrl} Y$).

Since the harmonic averages of m_X and m_Y are increasing functionals of m_X and m_Y, respectively, it follows that

$$X \leq_{mrl} Y \Rightarrow X \leq_{hmrl} Y. \tag{1.18}$$

The previous stochastic orders compare the size of the variables. Very often, however, also the variability of a random variable is of interest, since it describes the risk of an uncertain outcome. If two random variables X and Y with the same mean describe the returns of two risky investments, then every risk-averse decision maker will choose that one with the lower variability. Therefore variability orderings are of special interest in the context of decision making under risk.

It turns out that there is a natural connection between variability of random variables and stochastic orders based on convex functions. Recall that a real function g is called convex if

$$g(ax + (1-a)y) \leq ag(x) + (1-a)g(y),$$

for all x and y and all $0 < a < 1$. Let X and Y be random variables with finite means. If

$$Eg(X) \leq Eg(Y), \tag{1.19}$$

for all increasing convex functions f such that the expectations exist, then X is said to be **smaller than** Y in the **increasing convex order**.

1.3 Aging notions

The concept of aging is very important in reliability analysis. 'No aging' means that the age of a component has no effect on the distribution of the residual lifetime of the component. 'Positive aging' describes the situation where residual lifetime tends to decrease, in some probabilistic sense, with increasing age of a component. This situation is common in reliability engineering as components tend to become worse with time due to increased wear and tear. On the other hand, 'negative aging' has an opposite effect on the residual lifetime. Although this is less common, when a system undergoes regular testing and improvement, there are cases for which we have reliability growth phenomenon.

Concepts of aging describe how a component or a system improves or deteriorate with age. Many classes of life distributions are categorized or defined in the literature according to their aging properties. An important aspect of such classifications is that the exponential distribution is nearly always a member of each class. This is due to the *memorylessness property* of the exponential distribution.

From the definitions of the life distribution classes, results may be derived concerning such things as properties of systems (based upon properties of components), bounds for survival functions, moment inequalities, and algorithms for use in maintenance policies (Hollander and Proschan, 1984).

Next we introduce the definition of very well known aging notions for nonnegative random variables. Most of the definitions we provide here are based on the verification of a stochastic order relation between the random variable and its residual life. For alternative definitions and more characterizations of these concepts see, for example, Müller and Stoyan (2002).

A random variable X, or a lifetime distribution, is said to have an **increasing hazard rate** distribution (IHR distribution), if its hazard rate $r_X(t)$ is increasing. Some authors speak of IFR distributions, standing for **increasing failure rate**. This notion has several characterizations. Some of them are collected in the following proposition.

Proposition 1.1. *The following statements are equivalent:*

(i) *X has an IHR distribution;*

(ii) *$X_t \leq_{hr} X_s$ for all $s < t$;*

(iii) *$X_t \leq_{st} X_s$ for all $s < t$.*

The **bathtub shaped hazard rate life distributions**, often known simply as **bathtub distributions**, have a hazard rate curve that resembles to a bathtub shape. There are several variants of the definition of a bathtub shaped hazard rate but they are essentially the same. The main difference is whether the assumption of having two change points is imposed. See, for example, Lai and Xie (2006) for different definitions of this concept. Here we consider the following. Let X be a random variable with hazard rate function r_X continuous. Then X has a bathtub distribution (BT distribution) if there exist $t_1 \leq t_2$ such that

(i) $r_X(t)$ is strictly decreasing for $t < t_1$,

(ii) $r_X(t)$ is a constant for $t_1 \leq t \leq t_2$, and

(iii) $r_X(t)$ is strictly increasing for $t > t_2$.

Another important family of distributions is known as the **upside-down bathtub hazard rate** class consisting of distributions whose hazard rates are reverse bathtub shaped.

A weaker notion of aging than IHR, is obtained if we replace the requirement $X_t \leq_{st} X_s$ for all $s < t$ in Proposition 1.1 by the weaker requirement $X_t \leq_{icx} X_s$ for all $s < t$. Distributions with this property are said to have the DMRL property, standing for **decreasing**

mean residual life. Equivalently, X has a DMR distribution if, and only if, the mean residual life function of X, that is m_X, is decreasing.

Another important concept of aging is the NBU property, an abbreviation for **new better than used**. This holds if $X_t \leq_{st} X$ for all $t > 0$. It is easy to see that the NBU property is weaker than IHR.

Finally, for any $\gamma \in (0, 1)$, a distribution is said to be **decreasing γ-percentile residual life** (DPRL(γ)) if the γ-percentile residual life function of X is decreasing.

In Chapter 5 we give some characterization results of the decreasing percentile residual life aging notions in terms of the percentile residual life orders we study in Chapter 2 and we introduce new aging notions which are also based on the monotonous behavior of percentile residual life functions. We show the usefulness of these concepts providing necessary conditions for bathtub and upside-down bathtub distributions.

Chapter 2

The γ-percentile residual life orders

In this chapter we study a family of stochastic orders indexed by $\gamma \in (0, 1)$. For a fixed $\gamma \in (0, 1)$ the γth order compares pointwise the percentile residual life functions of two random variables. These stochastic orders were introduced in Joe and Proschan (1984b), but they were not extensively studied there.

This chapter is organized as follows. The γ-percentile residual life stochastic orders are formally defined in Section 2.1. We also give some equivalent ways of describing these orders that turn up to be useful in the sequel. Section 2.2 consists of a thorough study of the relationships among the γ-percentile residual life orders and other stochastic orders in the literature. Some useful properties of the γ-percentile residual life orders are given in Section 2.3. Finally, some applications in reliability theory and finance are described in Section 2.4. Most of the results of this chapter will be found in Franco-Pereira, Lillo, Romo and Shaked (2009).

2.1 Definition

Let X be a random variable and let u_X be the right endpoint of its support. Recall from (1.1) that the residual life of X is given by

$$X_t = [X - t \big| X > t], \quad t < u_X.$$

If F_X denotes the distribution function of X and $\overline{F}_X = 1 - F_X$ denotes the corresponding survival function, then the survival function of X_t is given by

$$\overline{F}_{X_t}(x) = \frac{\overline{F}_X(t + x)}{\overline{F}_X(t)}, \quad x \geq 0.$$

If γ is some number between 0 and 1, the γ-percentile residual life function of X, denoted by $q_{X,\gamma}$, is defined for any $t < u_X$ by letting $q_{X,\gamma}(t)$ be the γ-percentile of X_t. That is,

$$q_{X,\gamma}(t) = \begin{cases} F_{X_t}^{-1}(\gamma), & t < u_X; \\ 0, & t \geq u_X, \end{cases} \tag{2.1}$$

A straightforward computation shows that

$$q_{X,\gamma}(t) = \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) - t, \quad t < u_X, \tag{2.2}$$

where $\overline{\gamma} = 1 - \gamma$. Alternatively,

$$q_{X,\gamma}(t) = F_X^{-1}(\gamma + \overline{\gamma}F_X(t)) - t, \quad t < u_X. \tag{2.3}$$

Similar expressions can be found in Joe and Proschan (1984b). Note that, unlike Joe and Proschan (1984a,b), we do not assume here that X is a nonnegative random variable.

Now let Y be another random variable, and let $q_{Y,\gamma}$ be its γ-percentile residual life function. If

$$q_{X,\gamma}(t) \leq q_{Y,\gamma}(t) \quad \text{for all } t, \tag{2.4}$$

then we say that X is smaller than Y in the *γ-percentile residual life order*, and we denote it as $X \leq_{\gamma\text{-rl}} Y$. The γ-percentile residual life orders were introduced in Joe and Proschan (1984b), but these orders were not extensively studied there. The focus of Joe and Proschan (1984b) was to test the hypothesis $H_0 : F_X = F_Y$ versus $H_1 : q_{X,\gamma} \leq q_{Y,\gamma}$.

Note that (2.4) defines a family of stochastic orders, indexed by $\gamma \in (0, 1)$. It follows from (2.1) and (2.4) that if $X \leq_{\gamma\text{-rl}} Y$ then

$$u_X \leq u_Y, \tag{2.5}$$

where u_X and u_Y are the right endpoints of corresponding supports.

The following proposition states equivalent conditions for the γ-percentile residual life order to hold.

Proposition 2.1. *Let γ be in $(0, 1)$ and let X and Y be two random variables.*

(i) *The random variables X and Y satisfy $X \leq_{\gamma\text{-rl}} Y$ if, and only if,*

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) \leq \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(t)) \quad \textit{for all } t.$$

(ii) *The random variables X and Y satisfy $X \leq_{\gamma\text{-rl}} Y$ if, and only if,*

$$F_X^{-1}(\gamma + \overline{\gamma}F_X(t)) \leq F_Y^{-1}(\gamma + \overline{\gamma}F_Y(t)) \quad \textit{for all } t.$$

(iii) *Suppose that F_X and F_Y are continuous. Then $X \leq_{\gamma\text{-rl}} Y$ if, and only if,*

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(u))}{u} \leq \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}u))}{\overline{\gamma}u} \quad \textit{for all } u \in (0, 1).$$

Proof. Parts (i) and (ii) follow at once from (2.2), (2.3), and (2.4). In order to prove part (iii) we note that under the stated assumptions we have that $\overline{F}_X\big(\overline{F}_X^{-1}(p)\big) = p$ and $\overline{F}_Y\big(\overline{F}_Y^{-1}(p)\big) = p$ for all $p \in (0,1)$. Now, by part (i), we have that $X \leq_{\gamma\text{-rl}} Y$ is equivalent to

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) \leq \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(t)) \quad \text{for all } t.$$

Applying $\overline{F}_Y$ to both sides of the above inequality we get that it is equivalent to

$$\overline{F}_Y\big(\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t))\big) \geq \overline{\gamma}\overline{F}_Y(t) \quad \text{for all } t.$$

Letting $t = \overline{F}_X^{-1}(u)$ in the latter inequality we see that it is equivalent to

$$\frac{\overline{F}_Y\big(\overline{F}_X^{-1}(\overline{\gamma}u)\big)}{\overline{\gamma}u} \geq \frac{\overline{F}_Y\big(\overline{F}_X^{-1}(u)\big)}{u} \quad \text{for all } u \in (0,1),$$

completing the proof. □

The γ-percentile residual life orders indicate comparisons of size or magnitude. For example, letting $t \to -\infty$ in (2.4) we see that if $X \leq_{\gamma\text{-rl}} Y$ then the γ-percentile of X is smaller than (or at least not larger than) the γ-percentile of Y. Inequality (2.5) is another indication of comparisons of size or magnitude. Now let l_X and l_Y are the left endpoints of corresponding supports. One may wonder whether $X \leq_{\gamma\text{-rl}} Y$ implies $l_X \leq l_Y$. The following counterexample shows that this is not necessarily the case.

Counterexample 2.1. *For some $\gamma \in (0,1)$, let $X(\gamma)$ have the distribution function given by*

$$F_{X(\gamma)}(t) = \begin{cases} 0, & t < \gamma; \\ t, & \gamma \leq t < 1; \\ 1, & t \geq 1; \end{cases}$$

that is, $F_{X(\gamma)}$ is a mixture of a uniform distribution on $(\gamma, 1)$ with probability $1-\gamma$, and a degenerate variable at γ with probability γ. Let Y have the uniform distribution on $(0,1)$. We compute

$$q_{X(\gamma),\gamma}(t) = \begin{cases} \gamma - t, & t < \gamma; \\ \gamma(1-t), & \gamma \leq t < 1; \\ 0, & t \geq 1; \end{cases}$$

and

$$q_{Y,\gamma}(t) = \begin{cases} \gamma - t, & t < 0; \\ \gamma(1-t), & 0 \leq t < 1; \\ 0, & t \geq 1. \end{cases} \tag{2.6}$$

It is easy to check that $X(\gamma) \leq_{\gamma\text{-rl}} Y$ but $l_{X(\gamma)} = \gamma \nleq 0 = l_Y$. ◀

We end this section with an example that describes a family of random variables that are ordered with respect to $\leq_{\gamma\text{-rl}}$. It will be used in the sequel.

Example 2.1. *Let X have the Pareto distribution:*

$$F_X(t) = 1 - \Big(\frac{\lambda}{\lambda + t}\Big)^{\nu}, \quad t \geq 0,$$

where $\lambda > 0$ and $\nu > 0$. Then, for any $\gamma \in (0,1)$,

$$q_{X,\gamma}(t) = \begin{cases} ((1-\gamma)^{-1/\nu} - 1)\lambda - t, & t < 0; \\ ((1-\gamma)^{-1/\nu} - 1)(\lambda + t), & t \geq 0. \end{cases}$$

Let Y have the Pareto distribution:

$$F_Y(t) = 1 - \Big(\frac{\delta}{\delta + t}\Big)^{\mu}, \quad t \geq 0,$$

where $\delta > 0$ and $\mu > 0$. Then, for any $\gamma \in (0,1)$,

$$q_{Y,\gamma}(t) = \begin{cases} ((1-\gamma)^{-1/\mu} - 1)\delta - t, & t < 0; \\ ((1-\gamma)^{-1/\mu} - 1)(\delta + t), & t \geq 0. \end{cases}$$

It follows that

$$X \leq_{\gamma\text{-rl}} Y \iff \begin{cases} \mu \leq \nu \quad \textit{and} \\ \frac{(1-\gamma)^{-1/\nu}-1}{(1-\gamma)^{-1/\mu}-1} \leq \frac{\delta}{\lambda}. \end{cases}$$ ◀

2.2 Relationship to other stochastic orders

Recall from (1.10) that a random variable X is said to be smaller than the random variable Y in the usual stochastic order (denoted as $X \leq_{\text{st}} Y$) if, and only if,

$$F_X^{-1}(p) \leq F_Y^{-1}(p) \quad \text{for all } p \in (0,1). \tag{2.7}$$

Next recall from (1.13) that a random variable X is said to be smaller than the random variable Y in the hazard rate order (denoted as $X \leq_{\text{hr}} Y$) if $\overline{F}_X(y)\overline{F}_Y(x) \leq \overline{F}_X(x)\overline{F}_Y(y)$ for all $x \leq y$. If X_t and Y_t denote the residual lives that are associated with X and Y respectively, it is known that $X \leq_{\text{hr}} Y$ if, and only if,

$$X_t \leq_{\text{st}} Y_t \quad \text{for all } t < u_X; \tag{2.8}$$

see, for example, (1.B.6) in Shaked and Shanthikumar (2007). Equivalently, recalling the notation $q_{X,\gamma}$ and $q_{Y,\gamma}$ from (2.1), we can apply (2.7) to (2.8) and see that $X \leq_{\text{hr}} Y$ if, and only if,

$$q_{X,\gamma}(t) \leq q_{Y,\gamma}(t) \quad \text{for all } t < u_X \text{ and } \gamma \in (0,1).$$

From (2.4) we thus obtain the following result (which has already been observed in Joe and Proschan (1984b)).

Theorem 2.1. *Let X and Y be two random variables. Then $X \leq_{\mathrm{hr}} Y$ if, and only if,*

$$X \leq_{\gamma\text{-rl}} Y \quad \textit{for all } \gamma \in (0,1). \tag{2.9}$$

In particular, for any $\gamma \in (0,1)$,

$$\leq_{\mathrm{hr}} \Longrightarrow \leq_{\gamma\text{-rl}} .$$

Joe and Proschan (1984b) stated that there is no relationship between the orders $\leq_{\mathrm{st}}$ and $\leq_{\gamma\text{-rl}}$. However, they gave no proof of this statement. In the following discussion, especially in Remarks 2.1 and 2.2 below, it is formally shown, among other things, that indeed there is no relationship between these orders.

Recall from (1.17) that two random variables X and Y with mean residual life functions m_X and m_Y respectively, are ordered with respect to the mean residual life order if

$$m_X(t) \leq m_Y(t) \quad \text{for all } t \in \mathbb{R}.$$

Remark 2.1. *The random variables in Counterexample 2.1 have expectations $E[X(\gamma)] = \frac{\gamma^2+1}{2}$ and $E[Y] = \frac{1}{2}$. Thus, although $X(\gamma) \leq_{\gamma\text{-rl}} Y$ we have $E[X(\gamma)] > E[Y]$. That is, the γ-percentile residual life orders do not preserve expectations. It follows that any stochastic order that preserves expectations cannot be implied by any γ-percentile residual life order. In particular, for any $\gamma \in (0,1)$,*

$$\leq_{\gamma\text{-rl}} \not\Longrightarrow \leq_{\mathrm{st}},$$
$$\leq_{\gamma\text{-rl}} \not\Longrightarrow \leq_{\mathrm{mrl}},$$

and

$$\leq_{\gamma\text{-rl}} \not\Longrightarrow \leq_{\mathrm{hmrl}};$$

see Section 1.2 for the definitions of the above orders and see Shaked and Shanthikumar (2007) for the fact that they preserve expectations. $\triangleleft$

Recall from (1.16) that a random variable X is said to be smaller than the random variable Y in the reversed hazard rate order (denoted as $X \leq_{\mathrm{rh}} Y$) if $F_X(y)F_Y(x) \leq F_X(x)F_Y(y)$ for all $x \leq y$. Since the order $\leq_{\mathrm{rh}}$ implies the order $\leq_{\mathrm{st}}$ (see, for example, Shaked and Shanthikumar (2007)), it follows from Remark 2.1 that, for any $\gamma \in (0,1)$,

$$\leq_{\gamma\text{-rl}} \not\Longrightarrow \leq_{\mathrm{rh}} .$$

It is known (see, for example, Shaked and Shanthikumar (2007)) that $\leq_{\mathrm{rh}} \not\Longrightarrow \leq_{\mathrm{hr}}$. It thus follows from Theorem 2.1 that $\leq_{\mathrm{rh}} \not\Longrightarrow \leq_{\gamma\text{-rl}}$ for *some* $\gamma \in (0,1)$. In the next remark we will show a stronger result.

Remark 2.2. *Note that the distribution $F_{X(\gamma)}$ of the random variable $X(\gamma)$ in Counterexample 2.1 can be obtained from the distribution of the random variable Y there by shifting some of the mass of F_Y to the right. Thus Counterexample 2.1 shows in a simple way that shifting some mass of a distribution of a random variable to the right can actually decrease it in the γ-percentile residual life order. This shows that, for any $\gamma \in (0,1)$,*

$$\leq_{\rm st} \not\Longrightarrow \leq_{\gamma\text{-rl}} . \tag{2.10}$$

In fact, it is easy to verify that $F_{X(\gamma)}$ and F_Y in Counterexample 2.1 satisfy $F_Y(y)F_{X(\gamma)}(x) \leq F_Y(x)F_{X(\gamma)}(y)$ for all $x \leq y$; that is, $Y \leq_{\rm rh} X(\gamma)$. It follows that, for any $\gamma \in (0,1)$,

$$\leq_{\rm rh} \not\Longrightarrow \leq_{\gamma\text{-rl}} . \tag{2.11}$$

Note that (2.11) is a stronger statement than (2.10) because the order $\leq_{\rm rh}$ implies the order $\leq_{\rm st}$. ◁

Let us now return to the consideration of the relationship between the orders $\leq_{\gamma\text{-rl}}$ and $\leq_{\rm hr}$. In Theorem 2.1 it is shown that condition (2.9) implies $X \leq_{\rm hr} Y$ (actually these two conditions are equivalent). The question that now arises is whether a weaker condition, such as

$$X \leq_{\gamma\text{-rl}} Y \quad \text{for all } \gamma \in (0,\beta)$$

for some $\beta \in (0,1)$, implies the same conclusion. It turns out that this is indeed the case, no matter how small β is (provided it is positive). In order to show it we need the following lemma.

Lemma 2.1. *Let $\gamma \in (0,1)$ and let X and Y be two random variables with continuous distributions. If $X \leq_{\gamma\text{-rl}} Y$ then*

$$X \leq_{(1-\overline{\gamma}^{2^m})\text{-rl}} Y \quad \textit{for all } m = 1,2,\ldots.$$

Proof. By Proposition 2.1(iii), if $X \leq_{\gamma\text{-rl}} Y$ then

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(u))}{u} \leq \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}u))}{\overline{\gamma}u} \quad \text{for all } u \in (0,1).$$

Replacing above u by $\overline{\gamma}u$ we get

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}u))}{\overline{\gamma}u} \leq \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}^2u))}{\overline{\gamma}^2u} \quad \text{for all } u \in (0,1),$$

and by induction

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}^{2^{m-1}}u))}{\overline{\gamma}^{2^{m-1}}u} \leq \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}^{2^m}u))}{\overline{\gamma}^{2^m}u} \quad \text{for all } u \in (0,1) \text{ and } m = 1,2,\ldots.$$

Multiplying the above inequalities we get

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(u))}{u} \le \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}^{2^m}u))}{\overline{\gamma}^{2^m}u} \quad \text{for all } u \in (0,1) \text{ and } m = 1,2,\dots,$$

and, by Proposition 2.1(iii), this yields the stated result. □

Theorem 2.2. *Let $\beta \in (0,1)$ and let X and Y be two random variables with continuous distributions. If*

$$X \le_{\gamma\text{-rl}} Y \quad \textit{for all } \gamma \in (0,\beta)$$

then $X \le_{\text{hr}} Y$.

Proof. For any $\gamma \in (0,\beta)$, since $X \le_{\gamma\text{-rl}} Y$, it follows from Lemma 2.1 that

$$X \le_{(1-\overline{\gamma}^{2^m})\text{-rl}} Y \quad \text{for all } m = 1,2,\dots.$$

Now, let $\delta \in [\beta,1)$, and consider

$$\gamma \stackrel{\text{def}}{=} 1-(1-\delta)^{\frac{1}{2^m}} \quad \text{where} \quad m = \left[\frac{\log\left(\frac{\log(1-\delta)}{\log(1-\beta)}\right)}{\log 2}\right]+1;$$

here $[s]$ denotes the integer part of s. It is straightforward to verify that $\gamma < \beta$. Plugging this γ in the inequality $X \le_{(1-\overline{\gamma}^{2^m})\text{-rl}} Y$ we obtain $X \le_{\delta\text{-rl}} Y$. Since this is true for every $\delta \in [\beta,1)$ we get $X \le_{\text{hr}} Y$ from (2.9). □

Remark 2.3. *Looking at condition* (2.9) *and at Theorem 2.2 it is natural to wonder whether a condition such as*

$$X \le_{\gamma\text{-rl}} Y \quad \textit{for all } \gamma \in (\delta,\beta),$$

for some $0 < \delta < \beta < 1$ (note that here we do not allow $\delta = 0$), implies $X \le_{\text{hr}} Y$. It turns out that this is not the case. In order to see it, fix a $\delta \in (0,1)$, and consider the random variables $X(\delta)$ and Y from Counterexample 2.1. For any $\gamma \in (\delta,1)$ we have

$$q_{X(\delta),\gamma}(t) = \begin{cases} \gamma - t, & t < \delta; \\ \gamma(1-t), & \delta \le t < 1; \\ 0, & t \ge 1; \end{cases}$$

whereas $q_{Y,\gamma}$ is given in (2.6)*. It is now easy to verify that $X(\delta) \le_{\gamma\text{-rl}} Y$ (and this is true for all $\gamma \in (\delta,1)$), but $X(\delta) \not\le_{\text{hr}} Y$.* ◁

In the next counterexample it is shown that for any $\gamma \in (0,1)$ we have

$$\le_{\text{mrl}} \not\Longrightarrow \le_{\gamma\text{-rl}}.$$

Counterexample 2.2. *For $(\omega, \delta, \lambda) \in (0,1)^3$, let X have the survival function given by*

$$\overline{F}_X(t) = \begin{cases} 1, & t < 0; \\ 1 - \omega t, & 0 \le t < \delta; \\ \omega(1-t), & \delta \le t < 1; \\ 0, & t \ge 1; \end{cases} \tag{2.12}$$

that is, F_X is a mixture of a uniform distribution on $(0,1)$ with probability ω, and a degenerate variable at δ with probability $1-\omega$. Let Y have the survival function given by

$$\overline{F}_Y(t) = \begin{cases} 1, & t < 0; \\ 1 - \lambda t, & 0 \le t < 1; \\ 0, & t \ge 1; \end{cases} \tag{2.13}$$

that is, F_Y is a mixture of a uniform distribution on $(0,1)$ with probability λ, and a degenerate variable at 1 with probability $1-\lambda$. Lengthy computations show that the mean residual life functions of X and Y, respectively, are given by

$$m_X(t) = \begin{cases} \frac{\omega}{2} + \delta(1-\omega) - t, & t < 0; \\ \frac{\omega(1-t^2)+2\delta(1-\omega)}{2(1-\omega t)} - t, & 0 \le t < \delta; \\ \frac{1-t}{2}, & \delta \le t < 1; \\ 0, & t \ge 1, \end{cases}$$

and

$$m_Y(t) = \begin{cases} 1 - \frac{\lambda}{2} - t, & t < 0; \\ \frac{2-\lambda-\lambda t^2}{2(1-\lambda t)} - t, & 0 \le t < 1; \\ 0, & t \ge 1. \end{cases}$$

Now let ω and λ be such that

$$0 < \omega < \lambda < 1, \tag{2.14}$$

and set

$$\delta = \frac{1-\lambda}{1-\omega}; \tag{2.15}$$

from (2.14) it follows that $0 < \delta < 1$.

For $t < 0$ we see that

$$m_X(t) = \frac{\omega}{2} + \delta(1-\omega) - t = \frac{\omega}{2} + 1 - \lambda - t \le 1 - \frac{\lambda}{2} - t = m_Y(t),$$

where the second equality follows from (2.15), and the inequality follows from (2.14).

For $0 \le t < \delta$ note, by (2.14) and (2.15), that $2(1-\omega t) \ge 2(1-\lambda t)$ and that

$$\omega(1-t^2) + 2\delta(1-\omega) = \omega(1-t^2) + 2(1-\lambda) \le \lambda(1-t^2) + 2(1-\lambda) = 2 - \lambda - \lambda t^2.$$

Thus,

$$m_X(t) = \frac{\omega(1-t^2)+2\delta(1-\omega)}{2(1-\omega t)} - t \leq \frac{2-\lambda-\lambda t^2}{2(1-\lambda t)} - t = m_Y(t).$$

Finally, for $\delta \leq t < 1$, *we have*

$$m_X(t) = \frac{1-t}{2} \leq \frac{2-\lambda-\lambda t^2}{2(1-\lambda t)} - t = m_Y(t),$$

where the inequality follows (after some straightforward manipulations) from $0 < \lambda < 1$ *and* $0 \leq t \leq 1$. *Thus*

$$X \leq_{\mathrm{mrl}} Y.$$

Now consider $\gamma \in (0,1)$. *If*

$$\lambda > \gamma \tag{2.16}$$

then the γ*-percentile of the random variable* Y *(with survival function given in* (2.13)*) is easily seen to be*

$$q_{Y,\gamma}(0) = \frac{\gamma}{\lambda}.$$

If

$$\omega\delta > \gamma \tag{2.17}$$

then the γ*-percentile of the random variable* X *(with survival function given in* (2.12)*) is easily seen to be*

$$q_{X,\gamma}(0) = \frac{\gamma}{\omega}.$$

Note that if (2.14) *holds then*

$$q_{X,\gamma}(0) > q_{Y,\gamma}(0),$$

and therefore $X \not\leq_{\gamma\text{-rl}} Y$. *For* δ *in* (2.15) *we can rewrite the inequality* (2.17) *as*

$$\frac{\omega(1-\lambda)}{1-\omega} > \gamma. \tag{2.18}$$

In summary, consider the following task:

$$\boxed{\text{For } \gamma \in (0,1), \text{ find } (\omega,\lambda) \in (0,1)^2 \text{ that satisfies the inequalities (2.14), (2.16), and (2.18).}} \tag{2.19}$$

If we can find a solution to the task (2.19), *then the corresponding* X *and* Y, *with survival functions given in* (2.12) *and* (2.13), *will satisfy* $X \leq_{\mathrm{mrl}} Y$ *and* $X \not\leq_{\gamma\text{-rl}} Y$.

In order to find a solution to the task (2.19) *(for any fixed* γ*), let* $b > 1$ *be a number such that*

$$b^{-1} > \gamma.$$

For a small positive ε (that will be shown below to exist), define

$$\begin{aligned} \omega &= \gamma + \varepsilon \quad \textit{and} \\ \lambda &= \gamma + b\,\varepsilon; \end{aligned}$$

of course, ε should be small enough so that $\lambda < 1$. Then (2.14) *and* (2.16) *hold. To see that* (2.18) *also holds, we rewrite it as*

$$\frac{(\gamma+\varepsilon)(1-\gamma-b\,\varepsilon)}{1-\gamma-\varepsilon} > \gamma.$$

This simplifies to

$$1 - b\,\gamma - b\,\varepsilon > 0.$$

Since $\gamma < 1/b$ we can find such an $\varepsilon > 0$, and the resulting ω and λ will satisfy (2.14), (2.16), *and* (2.18). ◀

Since $X \leq_{\text{mrl}} Y \Longrightarrow X \leq_{\text{hmrl}} Y$ (see Shaked and Shanthikumar (2007, page 95)), it follows from Counterexample 2.2 that for any $\gamma \in (0,1)$ we have

$$X \leq_{\text{hmrl}} Y \not\Longrightarrow X \leq_{\gamma\text{-rl}} Y.$$

One may wonder whether the orders $\leq_{\gamma\text{-rl}}$ and $\leq_{\beta\text{-rl}}$ imply each other when $\gamma \neq \beta$. The following counterexample shows that if $\beta < \gamma$ then $X \leq_{\gamma\text{-rl}} Y$ does not necessarily imply that $X \leq_{\beta\text{-rl}} Y$. The counterexample after that (Counterexample 2.4) will show that also if $\beta < \gamma$ then $X \leq_{\gamma\text{-rl}} Y$ does not necessarily imply that $X \leq_{\beta\text{-rl}} Y$.

Counterexample 2.3. *Let $0 < \beta < \gamma < 1$. Let X and Y have the Pareto distributions, given in Example 2.1, with $\mu = 1$ and $\nu = 2$. Choose λ and δ such that $\frac{(1-\gamma)^{-1/2}-1}{(1-\gamma)^{-1}-1} = \frac{\delta}{\lambda}$. Then, by Example 2.1, $X \leq_{\gamma\text{-rl}} Y$. It is not hard to verify that $\frac{(1-\gamma)^{-1/2}-1}{(1-\gamma)^{-1}-1}$ is strictly decreasing in $\gamma \in (0,1)$. Therefore $\frac{(1-\beta)^{-1/2}-1}{(1-\beta)^{-1}-1} > \frac{\delta}{\lambda}$. It follows from Example 2.1 that $X \not\leq_{\beta\text{-rl}} Y$.* ◀

The basic idea in the following counterexample has been inspired by the study of Gupta and Langford (1984) that we explained in Section1.1. They presented an example to show that the percentile residual life function does not characterize a distribution. For simplicity we consider a special case of their example (that is, their a and b are taken here to be both equal to 1) that still provides us with our objective.

Counterexample 2.4. *For $\gamma \in (0,1)$, let X has the Pareto distribution with survival function*

$$\overline{F}_X(t) = \left(\frac{1}{1+t}\right)^{\frac{-\log(1-\gamma)}{\log 2}}, \quad t \geq 0. \tag{2.20}$$

Now define

$$k_\varepsilon(x) = 1 + \varepsilon \sin\left(\frac{2\pi x}{\log 2}\right), \quad x \in \mathbb{R},$$

where $\varepsilon > 0$, *and consider the function* H_ε *given by*

$$H_\varepsilon(t) = \left(\frac{1}{1+t}\right)^{\frac{-\log(1-\gamma)}{\log 2}} \cdot k_\varepsilon(\log(1+t)), \quad t \geq 0.$$

Obviously, $H_\varepsilon(0) = 1$ *and* $\lim_{t\to\infty} H_\varepsilon(t) = 0$. *If we can find an* $\varepsilon > 0$ *such that* $H_\varepsilon(t)$ *is decreasing in* $t \geq 0$, *then it would follow that* H_ε *is a survival function. In order to identify such an* ε, *we note that the derivative of* k_ε *is given by*

$$k'_\varepsilon(x) = \varepsilon \cos\left(\frac{2\pi x}{\log 2}\right) \cdot \frac{2\pi}{\log 2}, \quad x \in \mathbb{R},$$

and thus the derivative of H_ε *is given by*

$$\begin{aligned} H'_\varepsilon(t) = \frac{\log(1-\gamma)}{\log 2} \cdot \left(\frac{1}{1+t}\right)^{\frac{-\log(1-\gamma)}{\log 2}} \cdot \frac{1}{1+t}\left[1 + \varepsilon \sin\left(\frac{2\pi \log(1+t)}{\log 2}\right)\right] \\ + \frac{2\pi}{\log 2} \cdot \frac{1}{1+t} \cdot \left(\frac{1}{1+t}\right)^{\frac{-\log(1-\gamma)}{\log 2}} \cdot \varepsilon \cos\left(\frac{2\pi \log(1+t)}{\log 2}\right), \quad t \geq 0. \end{aligned}$$

Therefore H_ε *is decreasing if, and only if,*

$$\varepsilon\left[\log(1-\gamma) \sin\left(\frac{2\pi \log(1+t)}{\log 2}\right) + 2\pi \cos\left(\frac{2\pi \log(1+t)}{\log 2}\right)\right] \leq -\log(1-\gamma), \quad t \geq 0. \quad (2.21)$$

Since

$$\varepsilon\left[\log(1-\gamma) \sin\left(\frac{2\pi \log(1+t)}{\log 2}\right) + 2\pi \cos\left(\frac{2\pi \log(1+t)}{\log 2}\right)\right] \leq \varepsilon(-\log(1-\gamma) + 2\pi), \quad t \geq 0,$$

we see that if

$$\varepsilon \leq \frac{-\log(1-\gamma)}{-\log(1-\gamma) + 2\pi} \quad (2.22)$$

then (2.21) *holds. Thus, for such an* ε *the function* H_ε *is a survival function.*

Let Y *be a random variable with survival function* H_ε, *namely,*

$$\overline{F}_Y(t) = \left(\frac{1}{1+t}\right)^{\frac{-\log(1-\gamma)}{\log 2}} \cdot k_\varepsilon(\log(1+t)), \quad t \geq 0.$$

Recall the random variable X *with survival function given in* (2.20). *From Gupta and Langford (1984) we know that* $q_{X,\gamma}(t) = q_{Y,\gamma}(t)$ *for all* t. *So,*

$$X \leq_{\gamma\text{-rl}} Y.$$

Let $\beta \in (\gamma, 1)$. *We are going to identify a* $t_0 > 0$ *such that*

$$q_{X,\beta}(t_0) > q_{Y,\beta}(t_0). \quad (2.23)$$

(It would then follow that $X \not\leq_{\beta\text{-rl}} Y$*.) Rewriting* (2.23) *it is seen to be equivalent to*

$$\overline{F}_Y(\overline{F}_X^{-1}(\overline{\beta}\,\overline{F}_X(t_0))) < \overline{\beta}\,\overline{F}_Y(t_0).$$

Setting $u_0 = \overline{F}_X(t_0)$*, it is seen that rather than identifying* t_0 *that satisfies* (2.23)*, we may as well identify* $u_0 \in (0,1)$ *such that*

$$\overline{F}_Y(\overline{F}_X^{-1}(\overline{\beta}u_0)) < \overline{\beta}\,\overline{F}_Y(\overline{F}_X^{-1}(u_0)). \tag{2.24}$$

We now compute

$$\overline{F}_X^{-1}(u) = u^{\frac{\log 2}{\log(1-\gamma)}} - 1, \quad u \in (0,1),$$

and

$$\overline{F}_Y(\overline{F}_X^{-1}(u)) = uk_\varepsilon(\log(1+\overline{F}_X^{-1}(u))) = uk_\varepsilon\Big(\frac{\log 2}{\log(1-\gamma)} \cdot \log u\Big).$$

So (2.24) *is the same as*

$$k_\varepsilon\Big(\frac{\log 2}{\log(1-\gamma)} \cdot \log(\overline{\beta}u_0)\Big) < k_\varepsilon\Big(\frac{\log 2}{\log(1-\gamma)} \cdot \log u_0\Big),$$

which is the same as

$$\sin\left(2\pi \cdot \frac{\log\overline{\beta} + \log u_0}{\log(1-\gamma)}\right) < \sin\left(2\pi \cdot \frac{\log u_0}{\log(1-\gamma)}\right). \tag{2.25}$$

Now take $u_0 = \exp\left\{\frac{\log(1-\gamma)}{4}\right\}$*. Then* $u_0 \in (0,1)$*, as well as* $\sin\left(2\pi \cdot \frac{\log\overline{\beta}+\log u_0}{\log(1-\gamma)}\right) < 1$ *and* $\sin\left(2\pi \cdot \frac{\log u_0}{\log(1-\gamma)}\right) = 1$*. So* (2.25)*, and therefore also* (2.24)*, hold for this* u_0*. It follows that* $X \not\leq_{\beta\text{-rl}} Y$*.*

Besides, we are going to identify a $t_1 > 0$ *such that*

$$q_{X,\beta}(t_1) < q_{Y,\beta}(t_1). \tag{2.26}$$

(It would then follow that $X \not\geq_{\beta\text{-rl}} Y$ *either.) Rewriting* (2.26) *it is seen to be equivalent to identify a* $u_1 \in (0,1)$ *such that*

$$\sin\left(2\pi \cdot \frac{\log\overline{\beta} + \log u_1}{\log(1-\gamma)}\right) > \sin\left(2\pi \cdot \frac{\log u_1}{\log(1-\gamma)}\right). \tag{2.27}$$

We take $u_1 = \exp\left\{\frac{3\log(1-\gamma)}{4}\right\}$*. Then* $u_1 \in (0,1)$*, as well as* $\sin\left(2\pi \cdot \frac{\log\overline{\beta}+\log u_1}{\log(1-\gamma)}\right) > -1$ *and* $\sin\left(2\pi \cdot \frac{\log u_1}{\log(1-\gamma)}\right) = -1$*. So* (2.27)*, and therefore also* (2.26)*, hold for this* u_1*. It follows that* $X \not\geq_{\beta\text{-rl}} Y$*.* ◀

Figure 2.1 summarizes some of the results shown in this section.

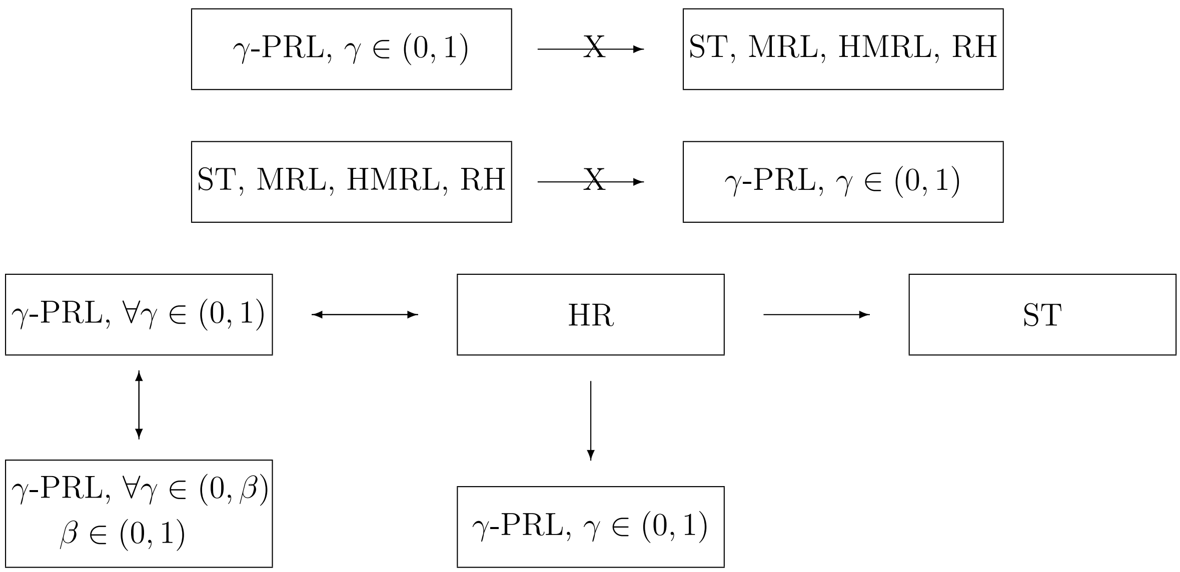

Figure 2.1: Relationship among some common stochastic orders

2.3 Closure properties

The γ-percentile residual life orders satisfy some desirable closure properties. These are described and discussed in this section.

First we show that the γ-percentile residual life orders are preserved under strictly increasing transformations.

Theorem 2.3. *Let X and Y be random variables, let $\gamma \in (0,1)$, and let ϕ be a strictly increasing function. Then $X \leq_{\gamma\text{-rl}} Y$ if, and only if, $\phi(X) \leq_{\gamma\text{-rl}} \phi(Y)$.*

Proof. Let $\overline{F}_{\phi(X)}$ and $\overline{F}_{\phi(Y)}$ denote the survival functions of the indicated random variables. Since ϕ is strictly increasing we have

$$\overline{F}_{\phi(X)}(t) = \overline{F}_X(\phi^{-1}(t)) \quad \text{and} \quad \overline{F}_{\phi(Y)}(t) = \overline{F}_Y(\phi^{-1}(t)) \quad \text{for all } t,$$

and

$$\overline{F}^{-1}_{\phi(X)}(u) = \phi(\overline{F}^{-1}_X(u)) \quad \text{and} \quad \overline{F}^{-1}_{\phi(Y)}(u) = \phi(\overline{F}^{-1}_Y(u)) \quad \text{for all } u \in (0,1).$$

Therefore, by Proposition 2.1(i), $\phi(X) \leq_{\gamma\text{-rl}} \phi(Y)$ if, and only if,

$$\phi(\overline{F}^{-1}_X(\overline{\gamma}\overline{F}_X(\phi^{-1}(t)))) \leq \phi(\overline{F}^{-1}_Y(\overline{\gamma}\overline{F}_Y(\phi^{-1}(t)))) \quad \text{for all } t.$$

By the strict monotonicity of ϕ, the latter condition is equivalent to

$$\overline{F}^{-1}_X(\overline{\gamma}\overline{F}_X(\phi^{-1}(t))) \leq \overline{F}^{-1}_Y(\overline{\gamma}\overline{F}_Y(\phi^{-1}(t))) \quad \text{for all } t.$$

Letting $t' = \phi^{-1}(t)$, this condition is the same as

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t')) \le \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(t')) \quad \text{for all } t',$$

and the stated result follows from Proposition 2.1(i). □

For the next result we need the following lemma from van der Vaart (1998, page 305). Note that the assumption requiring that a distribution function has interval support, means that the distribution function has no “flats” on that interval.

Lemma 2.2. *Let $\{F_n\}$ be a sequence of distribution functions that converges in distribution to F. Suppose that F is continuous and has interval support. Then F_n^{-1} converges to F^{-1} on $(0,1)$.*

The following result gives conditions under which the γ-percentile residual life orders are closed under limits in distribution.

Theorem 2.4. *Let $\{X_n,\ n = 1,2,\dots\}$ and $\{Y_n,\ n = 1,2,\dots\}$ be two sequences of random variables such that $X_n \to_{\mathrm{st}} X$ and $Y_n \to_{\mathrm{st}} Y$ as $n \to \infty$, where “$\to_{\mathrm{st}}$” denotes convergence in distribution. Suppose that both X and Y have continuous distribution functions with interval supports. For any $\gamma \in (0,1)$, if $X_n \le_{\gamma\text{-rl}} Y_n$, $n = 1,2,\dots$, then $X \le_{\gamma\text{-rl}} Y$.*

Proof. From Lemma 2.2 we know that

$$F_X^{-1}(\gamma + \overline{\gamma}F_X(t)) = \lim_{n\to\infty} F_{X_n}^{-1}(\gamma + \overline{\gamma}F_{X_n}(t))$$

and that

$$F_Y^{-1}(\gamma + \overline{\gamma}F_Y(t)) = \lim_{n\to\infty} F_{Y_n}^{-1}(\gamma + \overline{\gamma}F_{Y_n}(t))$$

for all t. If $X_n \le_{\gamma\text{-rl}} Y_n$, $n = 1,2,\dots$, then, using Proposition 2.1(ii), we have

$$F_X^{-1}(\gamma + \overline{\gamma}F_X(t)) = \lim_{n\to\infty} F_{X_n}^{-1}(\gamma + \overline{\gamma}F_{X_n}(t)) \le \lim_{n\to\infty} F_{Y_n}^{-1}(\gamma + \overline{\gamma}F_{Y_n}(t)) = F_Y^{-1}(\gamma + \overline{\gamma}F_Y(t)),$$

and the stated result follows from Proposition 2.1(ii). □

Without the assumption of interval supports in Theorem 2.4 the conclusion of the theorem may not hold. This is shown next.

Counterexample 2.5. *Let $\gamma \in (0,1)$. For every $n > \frac{1}{\gamma}$ let X_n be a random variable whose distribution is the following mixture:*

$$\begin{cases} \textit{uniform on } [1.5, 2.5] & \textit{with probability } \frac{\gamma n}{n+1}, \\ \textit{uniform on } [2.5, 3.5] & \textit{with probability } \frac{1}{n+1}, \\ \textit{standard exponential with shift } 4.5 & \textit{with probability } \frac{(1-\gamma)n}{n+1}; \end{cases}$$

that is

$$F_{X_n}(t) = \begin{cases} 0, & t < 1.5; \\ \frac{\gamma n(t-1.5)}{n+1}, & 1.5 \le t < 2.5; \\ \frac{\gamma n + t - 2.5}{n+1}, & 2.5 \le t < 3.5; \\ \frac{\gamma n+1}{n+1}, & 3.5 \le t < 4.5; \\ 1 - \frac{(1-\gamma)ne^{-(t-4.5)}}{n+1}, & t \ge 4.5. \end{cases}$$

It is easy to see that X_n converges in distribution to X whose distribution is

$$\begin{cases} \textit{uniform on } [1.5, 2.5] & \textit{with probability } \gamma, \\ \textit{standard exponential with shift 4.5} & \textit{with probability } (1-\gamma); \end{cases}$$

that is

$$F_X(t) = \begin{cases} 0, & t < 1.5; \\ \gamma(t-1.5), & 1.5 \le t < 2.5; \\ \gamma, & 2.5 \le t < 4.5; \\ 1-(1-\gamma)e^{-(t-4.5)}, & t \ge 4.5. \end{cases}$$

Next, for every $n > \frac{1}{\gamma}$ let Y_n be a random variable whose distribution is the following mixture:

$$\begin{cases} \textit{uniform on } [0.5, 1.5] & \textit{with probability } \frac{\gamma n}{n+1}, \\ \textit{uniform on } [2.5, 3.5] & \textit{with probability } \frac{1}{n+1}, \\ \textit{standard exponential with shift 4.5} & \textit{with probability } \frac{(1-\gamma)n}{n+1}; \end{cases}$$

that is

$$F_{Y_n}(t) = \begin{cases} 0, & t < 0.5; \\ \frac{\gamma n(t-0.5)}{n+1}, & 0.5 \le t < 1.5; \\ \frac{\gamma n}{n+1}, & 1.5 \le t < 2.5; \\ \frac{\gamma n + t - 2.5}{n+1}, & 2.5 \le t < 3.5; \\ \frac{\gamma n+1}{n+1}, & 3.5 \le t < 4.5; \\ 1 - \frac{(1-\gamma)ne^{-(t-4.5)}}{n+1}, & t \ge 4.5. \end{cases}$$

It is easy to see that Y_n converges in distribution to Y whose distribution is

$$\begin{cases} \textit{uniform on } [0.5, 1.5] & \textit{with probability } \gamma, \\ \textit{standard exponential with shift 4.5} & \textit{with probability } (1-\gamma); \end{cases}$$

that is

$$F_Y(t) = \begin{cases} 0, & t < 0.5; \\ \gamma(t-0.5), & 0.5 \le t < 1.5; \\ \gamma, & 1.5 \le t < 4.5; \\ 1-(1-\gamma)e^{-(t-4.5)}, & t \ge 4.5. \end{cases}$$

Computing the γ-percentile residual life functions that are associated with X_n and with Y_n, we get

$$q_{X_n,\gamma}(t) = \begin{cases} 2.5+\gamma-t, & t<1.5; \\ 2.5+\gamma+n\gamma(1-\gamma)(t-1.5)-t, & 1.5\le t<1.5+\frac{1}{\gamma n}; \\ 4.5-\log\left(\frac{n+1-\gamma n(t-1.5)}{n}\right)-t, & 1.5+\frac{1}{\gamma n}\le t<2.5; \\ 4.5-\log\left(\frac{n-\gamma n-t+3.5}{n}\right)-t, & 2.5\le t<3.5; \\ 4.5-\log(1-\gamma)-t, & 3.5\le t<4.5; \\ -\log(1-\gamma), & t\ge 4.5; \end{cases}$$

and

$$q_{Y_n,\gamma}(t) = \begin{cases} 2.5+\gamma-t, & t<0.5; \\ 2.5+\gamma+n\gamma(1-\gamma)(t-0.5)-t, & 0.5\le t<0.5+\frac{1}{\gamma n}; \\ 4.5-\log\left(\frac{n+1-\gamma n(t-0.5)}{n}\right)-t, & 0.5+\frac{1}{\gamma n}\le t<1.5; \\ 4.5-\log\left(\frac{n-\gamma n+1}{n}\right)-t, & 1.5\le t<2.5; \\ 4.5-\log\left(\frac{n-\gamma n-t+3.5}{n}\right)-t, & 2.5\le t<3.5; \\ 4.5-\log(1-\gamma)-t, & 3.5\le t<4.5; \\ -\log(1-\gamma), & t\ge 4.5. \end{cases}$$

It is straightforward to verify that $q_{X_n,\gamma}(t)\le q_{Y_n,\gamma}(t)$ for all t. Thus $X_n \le_{\gamma\text{-rl}} Y_n$, $n>\frac{1}{\gamma}$. On the other hand, by our convention that the inverse distribution function is the left continuous version of it, we see that the γ-percentile of X is 2.5 while the γ-percentile of Y is 1.5. So $X \not\le_{\gamma\text{-rl}} Y$. ◀

The following two lemmas, that deal with simple mixtures, will yield a general closure under mixtures property of the γ-percentile residual life orders.

Lemma 2.3. *Let X, Y, U, and V be random variables with continuous distribution functions, and let W be a random variable with distribution function*

$$F_W = pF_X + (1-p)F_Y,$$

for some $p\in[0,1]$.

(i) *If $U\le_{\gamma\text{-rl}} X$ and $U\le_{\gamma\text{-rl}} Y$ then $U\le_{\gamma\text{-rl}} W$.*

(ii) *If $X\le_{\gamma\text{-rl}} V$ and $Y\le_{\gamma\text{-rl}} V$ then $W\le_{\gamma\text{-rl}} V$.*

Proof. First we prove (i). From $U\le_{\gamma\text{-rl}} X$ and $U\le_{\gamma\text{-rl}} Y$, using Proposition 2.1(i), we obtain

$$\overline{F}_U^{-1}(\overline{\gamma}\,\overline{F}_U(t)) \le \overline{F}_X^{-1}(\overline{\gamma}\,\overline{F}_X(t)) \quad \text{and} \quad \overline{F}_U^{-1}(\overline{\gamma}\,\overline{F}_U(t)) \le \overline{F}_Y^{-1}(\overline{\gamma}\,\overline{F}_Y(t)) \quad \text{for all } t.$$

It follows, by the continuity of F_X and of F_Y, that

$$\overline{F}_X(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) \geq \overline{\gamma}\overline{F}_X(t) \quad \text{and} \quad \overline{F}_Y(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) \geq \overline{\gamma}\overline{F}_Y(t) \quad \text{for all } t.$$

Therefore,

$$p\overline{F}_X(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) + (1-p)\overline{F}_Y(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) \geq \overline{\gamma}p\overline{F}_X(t) + \overline{\gamma}(1-p)\overline{F}_X(t) \quad \text{for all } t;$$

that is,

$$\overline{F}_W(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) \geq \overline{\gamma}\overline{F}_W(t) \quad \text{for all } t.$$

By the continuity of F_W we get

$$\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t)) \leq \overline{F}_W^{-1}(\overline{\gamma}\overline{F}_W(t)) \quad \text{for all } t;$$

that is, by Proposition 2.1(i), $U \leq_{\gamma\text{-rl}} W$.

Now we prove (ii). From $X \leq_{\gamma\text{-rl}} V$ and $Y \leq_{\gamma\text{-rl}} V$, using Proposition 2.1(i), we obtain

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) \leq \overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t)) \quad \text{and} \quad \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(t)) \leq \overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t)) \quad \text{for all } t.$$

It follows, by the continuity of F_X and of F_Y, that

$$\overline{\gamma}\overline{F}_X(t) \geq \overline{F}_X(\overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t))) \quad \text{and} \quad \overline{\gamma}\overline{F}_Y(t) \geq \overline{F}_Y(\overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t))) \quad \text{for all } t.$$

Therefore,

$$\overline{\gamma}p\overline{F}_X(t) + \overline{\gamma}(1-p)\overline{F}_Y(t) \geq p\overline{F}_X(\overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t))) + (1-p)\overline{F}_Y(\overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t))) \quad \text{for all } t;$$

that is,

$$\overline{\gamma}\overline{F}_W(t) \geq \overline{F}_W(\overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t))) \quad \text{for all } t.$$

By the continuity of F_W we get

$$\overline{F}_W^{-1}(\overline{\gamma}\overline{F}_W(t)) \leq \overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t)) \quad \text{for all } t;$$

that is, by Proposition 2.1(i), $W \leq_{\gamma\text{-rl}} V$. □

Lemma 2.4. *Let X_1, X_2, Y_1, and Y_2 be random variables with continuous distribution functions, and let W and Z be random variables with distribution functions*

$$F_W = pF_{X_1} + (1-p)F_{X_2} \quad \textit{and} \quad F_Z = pF_{Y_1} + (1-p)F_{Y_2},$$

for some $p \in [0,1]$. If there exists a random variable S such that

$$X_1 \leq_{\gamma\text{-rl}} S, \quad X_2 \leq_{\gamma\text{-rl}} S, \quad S \leq_{\gamma\text{-rl}} Y_1, \quad S \leq_{\gamma\text{-rl}} Y_2,$$

then $W \leq_{\gamma\text{-rl}} Z$.

Proof. Since $X_1 \leq_{\gamma\text{-rl}} S$ and $X_2 \leq_{\gamma\text{-rl}} S$, it follows from Lemma 2.3 (ii) that $W \leq_{\gamma\text{-rl}} S$. Furthermore, since $S \leq_{\gamma\text{-rl}} Y_1$ and $S \leq_{\gamma\text{-rl}} Y_2$, it follows from Lemma 2.3 (i) that $S \leq_{\gamma\text{-rl}} Z$. By the transitivity property of the order $\leq_{\gamma\text{-rl}}$ we get $W \leq_{\gamma\text{-rl}} Z$. □

By repeated application of Lemma 2.4, and convergence arguments, we obtain the following result.

Theorem 2.5. *Let $\{X_\theta,\ \theta \in \Theta\}$ and $\{Y_\theta,\ \theta \in \Theta\}$ be two families of random variables with continuous distribution functions. Let W and Z be random variables with distribution functions given by*

$$F_W(t) = \int_\Theta F_{X_\theta}(t)dH(\theta) \quad \textit{and} \quad F_Z(t) = \int_\Theta F_{Y_\theta}(t)dH(\theta), \quad t \in \mathbb{R},$$

where H is some distribution function on Θ. Suppose that there exists a random variable S such that

$$X_\theta \leq_{\gamma\text{-rl}} S \leq_{\gamma\text{-rl}} Y_\theta \quad \textit{for all } \theta \in \Theta. \tag{2.28}$$

Then $W \leq_{\gamma\text{-rl}} Z$.

Note that condition (2.28) can be rewritten as

$$X_\theta \leq_{\gamma\text{-rl}} Y_{\theta'} \quad \text{for all } \theta, \theta' \in \Theta.$$

It is worth noting that results that are similar to Theorem 2.5 hold for the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the mean residual life order (see, respectively, Theorems 1.B.8, 1.B.46, 1.C.15, and 2.A.13 in Shaked and Shanthikumar, 2007).

A special case of Theorem 2.5 is the following result which shows that a random variable, whose distribution is a mixture of two distributions of γ-percentile residual life ordered random variables, is bounded from below and from above, in the γ-percentile residual life order sense, by these two random variables.

Corollary 2.1. *Let X and Y be two random variables with continuous distribution functions, and let W be a random variable with distribution function*

$$F_W = pF_X + (1-p)F_Y,$$

for some $p \in [0,1]$. If $X \leq_{\gamma\text{-rl}} Y$ then $X \leq_{\gamma\text{-rl}} W \leq_{\gamma\text{-rl}} Y$.

Again, note that similar results hold for the hazard rate order, the likelihood ratio order, and the mean residual life order (see, respectively, Theorems 1.B.22, 1.C.30, and 2.A.18 in Shaked and Shanthikumar, 2007).

The possible preservation of a stochastic order under the formation of coherent systems is a useful property that has important applications in reliability theory (see, for example, Barlow and Proschan, 1975, for the definition and the use of coherent systems). Thus it is of interest to ask whether the γ-percentile residual life orders are closed under this formation. Boland, El-Neweihi, and Proschan (1994) showed that the hazard rate order is not preserved under the formation of coherent systems. It follows from Theorem 2.1 that, *for some* γ, the γ-percentile residual life order is not closed under this formation. In the next counterexample it is shown that in fact, *for all* γ, the γ-percentile residual life order is not closed under this formation. This is shown by considering a parallel system of size 2 whose lifetime is the maximum of the lifetimes of its two components.

Counterexample 2.6. *For any* $\gamma \in (0,1)$*, let* X *be an exponential random variable with rate* $-\log(1-\gamma)$*. That is,*

$$F_X(t) = \begin{cases} 0, & t < 0; \\ 1 - e^{(\log(1-\gamma))t}, & t \geq 0. \end{cases}$$

Let Y *be a random variable that is degenerate at 0, and let* Z *be a random variable that is degenerate at 1. Note that* $\max\{X,Y\} =_{\text{st}} X$*. Note also that* $Y \leq_{\gamma\text{-rl}} Z$*, and, of course,* $X \leq_{\gamma\text{-rl}} X$*. Now we compute*

$$q_{\max\{X,Y\},\gamma}(t) = q_{X,\gamma}(t) = \begin{cases} 1-t, & t < 0; \\ 1, & t \geq 0, \end{cases}$$

and

$$q_{\max\{X,Z\},\gamma}(t) = \begin{cases} 1-t, & t < 1; \\ 1, & t \geq 1. \end{cases}$$

It is seen that $\max\{X,Y\} \nleq_{\gamma\text{-rl}} \max\{X,Z\}$ *(in fact,* $\max\{X,Y\} \geq_{\gamma\text{-rl}} \max\{X,Z\}$ *strictly). Thus the* γ*-percentile residual life order is not closed under the maximum operation.* ◀

In fact, unlike the hazard rate order, for every $\gamma \in (0,1)$, the γ-percentile residual life order is not even closed under the formation of series systems (that is, under the minimum operation). This is shown in the next counterexample.

Counterexample 2.7. *Let* X_1 *and* X_2 *be two random variables that are degenerate at 1. For any* $\gamma \in (0,1)$*, let* Y_1 *and* Y_2 *be two independent exponential random variables, each with rate* $-\log(1-\gamma)$*. The corresponding* γ*-percentile residual life functions are*

$$q_{X_1,\gamma}(t) = q_{X_2,\gamma}(t) = \begin{cases} 1-t, & t < 1; \\ 0, & t \geq 1; \end{cases}$$

and

$$q_{Y_1,\gamma}(t) = q_{Y_2,\gamma}(t) = \begin{cases} 1-t, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

It is easy to see that $X_1 \le_{\gamma\text{-rl}} Y_1$ *and* $X_2 \le_{\gamma\text{-rl}} Y_2$. *Now we compute*

$$q_{\min\{X_1,X_2\},\gamma}(t) = q_{X_1,\gamma}(t) = \begin{cases} 1-t, & t<1; \\ 0, & t\ge 1; \end{cases}$$

and (note that $\min\{Y_1,Y_2\}$ *is an exponential random variable with rate 2)*

$$q_{\min\{Y_1,Y_2\},\gamma}(t) = \begin{cases} 1/2-t, & t<0; \\ 1/2, & t\ge 0. \end{cases}$$

It is seen that $\min\{X_1,X_2\} \not\le_{\gamma\text{-rl}} \min\{Y_1,Y_2\}$. *Thus the* γ*-percentile residual life order is not closed under the minimum operation.* ◀

We point out that some comparisons of minima in percentile residual life orders are given in Corollary 2.2 below.

In relation to Counterexample 2.7 it is worthwhile to note that if X and Y are continuous random variables, then, for any $\gamma \in (0,1)$ we have

$$\min\{X,Y\} \le_{\gamma\text{-rl}} X. \tag{2.29}$$

In order to see it we note that $\overline{F}_X(t) \ge \overline{\gamma}\overline{F}_X(t)$ for all t. Therefore $t \le \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t))$ and hence $\overline{F}_Y(t) \ge \overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)))$ for all t. It follows that

$$\overline{\gamma}\overline{F}_X(t)\overline{F}_Y(t) \ge \overline{F}_X(\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)))\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t))) \quad \text{for all } t.$$

Since $\overline{F}_{\min\{X,Y\}} = \overline{F}_X\overline{F}_Y$, the last inequality can be written as

$$\overline{\gamma}\overline{F}_{\min\{X,Y\}}(t) \ge \overline{F}_{\min\{X,Y\}}(\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t))) \quad \text{for all } t,$$

or, equivalently,

$$\overline{F}_{\min\{X,Y\}}^{-1}(\overline{\gamma}\overline{F}_{\min\{X,Y\}}(t)) \le \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) \quad \text{for all } t.$$

Thus (2.29) follows from Proposition 2.1(i).

2.4 Some applications

2.4.1 Risk management

Consider a firm confronted with a risky business over some time period, and let the random variable X represent the loss that the firm incurs at the end of the period. A common measurement of the risk is the *value at risk*, denoted by $\mathrm{VaR}_\gamma[X]$, which is defined as the γ-percentile of the loss distribution for some prescribed confidence level $\gamma \in (0,1)$; see, for example, Hürlimann (2002, 2003). That is, if F_X denotes the distribution function of X, then

$$\mathrm{VaR}_\gamma[X] = F_X^{-1}(\gamma).$$

In practice γ is "large" in the sense that $\gamma = 95\%$, 99%, 99.75%, etc.

The risk measure $\mathrm{VaR}_\gamma[X]$ is widely used. However, it is not very informative in the following sense: Suppose that we want to compare the losses X and Y of two risky items that satisfy $\mathrm{VaR}_\gamma[X] = \mathrm{VaR}_\gamma[Y]$. We then know that losses, larger than the common value of $\mathrm{VaR}_\gamma[X]$ and $\mathrm{VaR}_\gamma[Y]$, will occur with the same probability $1-\gamma$, but we do not know the magnitudes of these losses. A risk measure that captures this information is the *average value at risk* defined, for a random loss X, by

$$\mathrm{AVaR}_\gamma[X] = \frac{1}{1-\gamma}\int_\gamma^1 \mathrm{VaR}_u[X]\,du. \tag{2.30}$$

The average value at risk is also called the *conditional value at risk*, or the *expected shortfall*, or the *tail conditional expectation*. The expression (2.30) for $\mathrm{AVaR}_\gamma[X]$ is only one of a long list of expressions that are given in Hürlimann (2003).

Suppose now that the firm with the risky asset X insures itself against heavy losses, that is, against losses above some deductible t. Then the loss that the reinsurer experiences (if it does) is $X_t = [X - t \big| X > t]$. Its corresponding value at risk is

$$\mathrm{VaR}_\gamma[X_t] = q_{X,\gamma}(t),$$

where $q_{X,\gamma}(t)$ is defined, for instance, in (2.1). Consider another risky asset Y with the same deductible t, and the reinsurer's loss $Y_t = [Y - t \big| Y > t]$, and its corresponding value at risk $\mathrm{VaR}_\gamma[Y_t] = q_{Y,\gamma}(t)$. Obviously, for any fixed deductible t, we have that

$$X \le_{\gamma\text{-rl}} Y \Longrightarrow \mathrm{VaR}_\gamma[X_t] \le \mathrm{VaR}_\gamma[Y_t]. \tag{2.31}$$

However, it may be more interesting to compare the corresponding average value at risk measures $\mathrm{AVaR}_\gamma[X_t]$ and $\mathrm{AVaR}_\gamma[Y_t]$. This is done in the next theorem.

Theorem 2.6 (Comparison of AVaR's). *Let $\gamma \in (0,1)$ and let X and Y be two risky assets with continuous distributions. Let t be a fixed deductible. If*

$$X \le_{\beta\text{-rl}} Y \quad \textit{for all } \beta \in [\gamma, 1) \tag{2.32}$$

then

$$\mathrm{AVaR}_\gamma[X_t] \le \mathrm{AVaR}_\gamma[Y_t]. \tag{2.33}$$

In particular, (2.33) *holds if $X \le_{\mathrm{hr}} Y$.*

Proof. From (2.31) and (2.32) it is seen that

$$\int_\gamma^1 \mathrm{VaR}_\beta[X_t]\,d\beta \le \int_\gamma^1 \mathrm{VaR}_\beta[Y_t]\,d\beta.$$

The inequality (2.33) now follows from (2.30). The last statement of the theorem follows from the fact, shown in Theorem 2.1, that $X \le_{\mathrm{hr}} Y$ implies (2.32). □

It is of interest to note, following Remark 2.2, that, for a fixed $\gamma \in (0,1)$, it is not necessary to assume that $X \le_{\mathrm{hr}} Y$ in order for (2.33) to hold. Since γ is usually "large", the condition (2.32) seems to be significantly weaker than $X \le_{\mathrm{hr}} Y$.

As it is indicated in the proof of Theorem 2.7, the condition (2.32) in Theorem 2.7 can be verified through Theorem 2.1. Alternatively, using empirical data, condition (2.32) can be verified through Theorem 2.4.

2.4.2 Reliability theory

Let X be a random variable with survival function $\overline{F}_X$. For $\theta > 0$, let $X(\theta)$ denote a random variable with survival function $\overline{F}_X^\theta$. In the theory of statistics, $\overline{F}_X^\theta$ is often referred to as the *Lehmann's alternative*. In reliability theory terminology, different $X(\theta)$'s are said to have *proportional hazards*. If $\theta < 1$ then $X(\theta)$ is the lifetime of a component with lifetime X which is subjected to *imperfect repair* procedure where θ is the probability of minimal (rather than perfect) repair (see Brown and Proschan (1983)). If $\theta = n$, where n is a positive integer, then $\overline{F}_X^n$ is the survival function of $\min\{X_1, X_2, \dots, X_n\}$ where $X_1, X_2, \dots, X_n$ are independent copies of X; that is, $\overline{F}_X^n$ is the survival function of a series system of size n where the component lifetimes are independent copies of X. Similarly, if Y is a random variable with survival function $\overline{F}_Y$, then denote by $Y(\theta)$ a random variable with survival function $\overline{F}_Y^\theta$. The following result compares $X(\theta)$ and $Y(\theta)$.

Theorem 2.7. *Let X and Y be two random variables with continuous distributions on interval supports. Let $\gamma \in (0,1)$ and $\theta > 0$. If $X \le_{\gamma\text{-rl}} Y$ then*

$$X(\theta) \le_{\beta\text{-rl}} Y(\theta), \tag{2.34}$$

where $\beta = 1 - (1-\gamma)^\theta$.

Proof. It is not hard to verify that, under the continuity assumptions above, we have

$$(\overline{F}_X^\theta)^{-1}(u) = \overline{F}_X^{-1}(u^{1/\theta}) \quad \text{and} \quad (\overline{F}_Y^\theta)^{-1}(u) = \overline{F}_Y^{-1}(u^{1/\theta}), \quad u \in (0,1),$$

or, equivalently,

$$\overline{F}_X^{-1}(u) = (\overline{F}_X^\theta)^{-1}(u^\theta) \quad \text{and} \quad \overline{F}_Y^{-1}(u) = (\overline{F}_Y^\theta)^{-1}(u^\theta), \quad u \in (0,1).$$

Now, by Proposition 2.1(i), $X \le_{\gamma\text{-rl}} Y$ means

$$\overline{F}_X^{-1}(\overline{\gamma}\,\overline{F}_X(t)) \le \overline{F}_Y^{-1}(\overline{\gamma}\,\overline{F}_Y(t)) \quad \text{for all } t,$$

that is,

$$(\overline{F}_X^{\theta})^{-1}(\overline{\gamma}^{\theta}\,\overline{F}_X^{\theta}(t)) \le (\overline{F}_Y^{\theta})^{-1}(\overline{\gamma}^{\theta}\,\overline{F}_Y^{\theta}(t)) \quad \text{for all } t,$$

and the result follows from Proposition 2.1(i). □

As a corollary of Theorem 2.7 we have the following "preservation property" of the γ-percentile residual life order under formation of series systems.

Corollary 2.2. *Let $X_1, X_2 \dots, X_n$ be independent and identically distributed random variables with a continuous distribution function on an interval support. Also let $Y_1, Y_2 \dots, Y_n$ be independent and identically distributed random variables with a continuous distribution function on an interval support. If $X_1 \le_{\gamma\text{-rl}} Y_1$ then*

$$\min\{X_1, X_2 \dots, X_n\} \le_{\beta\text{-rl}} \min\{Y_1, Y_2 \dots, Y_n\}, \tag{2.35}$$

where $\beta = 1 - (1-\gamma)^n$.

It is of interest to contrast Corollary 2.2 with the result in Counterexample 2.7.

It is worthwhile to remark that each of the conclusions of Theorem 2.7 and Corollary 2.2 (that is, (2.34) with $\theta > 0$, or (2.35) with $n \ge 1$) is sufficient for $X \le_{\gamma\text{-rl}} Y$ or $X_1 \le_{\gamma\text{-rl}} Y_1$, respectively.

Corollary 2.2 can be useful in reliability theory when it is of importance to compare a particular percentile (say, the median, that is, $\gamma = .5$) of the residual life of a series system that survived up to time t_0, with the same percentile (again, say, the median) of the residual life of another series system, with different components, that survived up to time t_0. This can be useful, for instance, when t_0 is the time at which the initial warranty of the system expires.

For example, if the series systems consist of $n = 4$ components, then the second one will be preferred to the first one, in the median residual life order, if the lifetimes of the components of the first system are smaller than the lifetimes of the components of the second system with respect to the order $\le_{.169\text{-rl}}$ (since $(1 - .169)^4 \approx .5$). An engineer who is familiar with the possible components of these systems can usually tell whether the two types of components have lifetimes that are ordered with respect to $\le_{.169\text{-rl}}$. ◁

Similar applications can be described in biometry and in statistics.

2.4.3 Market of used items

The order $\le_{\gamma\text{-rl}}$ can also be useful in a market of used items. Assume that an engineer (or any individual) is considering a purchase of a used machine (or a car, say). Suppose that she has a choice among a few equally aged machines (or cars). If the original machine lifetimes are ordered with respect to the hazard rate order, and if the engineer wishes to maximize a certain γ-percentile of the remaining life of the purchased machine, then, obviously (for example, by Theorem 2.1), she should select the machine whose lifetime is the highest with respect to the order $\leq_{\text{hr}}$.

Note, however, that the requirement that the machine lifetimes are ordered with respect to $\leq_{\text{hr}}$ is a very strong requirement that may be hard to verify (or that actually may not even hold) in practice. On the other hand, verification of the order $\leq_{\gamma\text{-rl}}$ may be a simpler matter — and it yields the same decision!

Moreover, if the above engineer (or individual) has a choice between two markets that have different mixtures of aged machines, and if the original machine lifetimes in these markets satisfy (2.28) [here X_θ and Y_θ, $\theta \in \Theta$, are the original machine lifetimes that are mixed in the two markets], then Theorem 2.5 determines which market is preferable.

Chapter 3

The PRL orders from time t_0 on

Motivated by the applicability of the percentile residual life orders for comparing items after initial warranty or to compare used items, here we propose new stochastic orderings which can be used with this purpose. These orders are based on the comparison of all the percentile residual life functions of two random variables, not in the whole support but from a certain moment $t_0 > 0$ on.

The formal definition of the percentile residual life orders from time t_0 on is given in Section 3.1. Some interpretations of these stochastic orders are given, and various properties of them are derived in Section 3.2. In Section 3.3 the relationships to other stochastic orders are studied. Finally, some applications in reliability theory are described in Section 3.4.

3.1 Definition

Let X be a random variable. The γ-percentile residual life function $q_{X,\gamma}$ is defined by

$$q_{X,\gamma}(t) = \begin{cases} F_{X_t}^{-1}(\gamma), & t < u_X; \\ 0, & t \geq u_X. \end{cases} \tag{3.1}$$

As we have already pointed out, the γ-percentile residual life function can also be written as

$$q_{X,\gamma}(t) = \overline{F}_X^{-1}(\overline{\gamma}\,\overline{F}_X(t)) - t, \quad t < u_X, \tag{3.2}$$

where $\overline{\gamma} = 1 - \gamma$ or, equivalently,

$$q_{X,\gamma}(t) = F_X^{-1}(\gamma + \overline{\gamma} F_X(t)) - t, \quad t < u_X. \tag{3.3}$$

Now let Y be another random variable, and let $q_{Y,\gamma}$ be its γ-percentile residual life function. Let $t_0 < u_X$. If

$$q_{X,\gamma}(t) \leq q_{Y,\gamma}(t) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0,1), \tag{3.4}$$

then we say that X is smaller than Y in the *percentile residual life order from time t_0 on*, and we denote it as $X \le_{t_0}^{\mathrm{prl}} Y$.

The following proposition states equivalent conditions for the percentile residual life from time t_0 order to hold.

Proposition 3.1. *Let $t_0 < u_X$ and let X and Y be two random variables.*

(i) *The random variables X and Y satisfy $X \le_{t_0}^{\mathrm{prl}} Y$ if, and only if,*

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) \le \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(t)) \quad \text{for all } t \ge t_0 \text{ and all } \gamma \in (0,1).$$

(ii) *The random variables X and Y satisfy $X \le_{t_0}^{\mathrm{prl}} Y$ if, and only if,*

$$F_X^{-1}(\gamma + \overline{\gamma}F_X(t)) \le F_Y^{-1}(\gamma + \overline{\gamma}F_Y(t)) \quad \text{for all } t \ge t_0 \text{ and all } \gamma \in (0,1).$$

(iii) *Suppose that F_X and F_Y are continuous. Then $X \le_{t_0}^{\mathrm{prl}} Y$ if, and only if,*

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(u))}{u} \le \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}u))}{\overline{\gamma}u} \quad \text{for all } u \le \bar{F}_X(t_0) \text{ and all } \gamma \in (0,1).$$

Proof. Parts (i) and (ii) follow at once from (3.2), (3.3), and (3.4). In order to prove part (iii) we note that, under the stated assumptions, we have that $\overline{F}_X\big(\overline{F}_X^{-1}(p)\big) = p$ and $\overline{F}_Y\big(\overline{F}_Y^{-1}(p)\big) = p$ for all $p \in (0,1)$. Now, by part (i), we have that $X \le_{t_0}^{\mathrm{prl}} Y$ is equivalent to

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) \le \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(t)) \quad \text{for all } t \ge t_0 \text{ and for all } \gamma \in (0,1).$$

Applying $\overline{F}_Y$ to both sides of the above inequality, we get that it is equivalent to

$$\overline{F}_Y\big(\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t))\big) \ge \overline{\gamma}\overline{F}_Y(t) \quad \text{for all } t \ge t_0 \text{ and for all } \gamma \in (0,1).$$

Letting $t = \overline{F}_X^{-1}(u)$ in the latter inequality we see that it is equivalent to

$$\frac{\overline{F}_Y\big(\overline{F}_X^{-1}(\overline{\gamma}u)\big)}{\overline{\gamma}u} \ge \frac{\overline{F}_Y\big(\overline{F}_X^{-1}(u)\big)}{u} \quad \text{for all } u \le \bar{F}_X(t_0) \text{ and for all } \gamma \in (0,1),$$

completing the proof. □

The question now is whether a weaker condition, such as

$$q_{X,\gamma}(t) \le q_{Y,\gamma}(t) \quad \text{for all } \gamma \in (0,\beta) \text{ and all } t \ge t_0,$$

for some $\beta \in (0,1)$, implies $X \le_{t_0}^{\mathrm{prl}} Y$. It turns out that this is indeed the case, no matter how small β is (provided it is positive). In order to show it we need the following lemma.

Lemma 3.1. *Let $\gamma \in (0,1)$ and let X and Y be two random variables with continuous distributions. If $q_{X,\gamma}(t) \le q_{Y,\gamma}(t)$ for all $t \ge t_0$, then*

$$q_{X,1-\overline{\gamma}^{2m}}(t) \le q_{Y,1-\overline{\gamma}^{2m}}(t) \text{ for all } t \ge t_0 \text{ and all } m = 1,2,\ldots.$$

Proof. By Proposition 3.1(iii), if $X \le_{t_0}^{\mathrm{prl}} Y$ then

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(u))}{u} \le \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}u))}{\overline{\gamma}u} \quad \text{for all } u \le \bar{F}_X(t_0) \text{ and for all } \gamma \in (0,1).$$

Replacing above u by $\overline{\gamma}u$ we get

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}u))}{\overline{\gamma}u} \le \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}^2u))}{\overline{\gamma}^2u} \quad \text{for all } u \le \bar{F}_X(t_0) \text{ and for all } \gamma \in (0,1),$$

and by induction, for all $\gamma \in (0,1)$,

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}^{2^{m-1}}u))}{\overline{\gamma}^{2^{m-1}}u} \le \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}^{2^m}u))}{\overline{\gamma}^{2^m}u} \quad \text{for all } u \le \bar{F}_X(t_0) \text{ and } m = 1,2,\ldots.$$

Multiplying the above inequalities we get

$$\frac{\overline{F}_Y(\overline{F}_X^{-1}(u))}{u} \le \frac{\overline{F}_Y(\overline{F}_X^{-1}(\overline{\gamma}^{2^m}u))}{\overline{\gamma}^{2^m}u} \quad \text{for all } u \le \bar{F}_X(t_0) \text{ and } m = 1,2,\ldots,$$

and, by Proposition 3.1(iii), this yields the stated result. □

Theorem 3.1. *Let $\beta \in (0,1)$ and let X and Y be two random variables with continuous distributions. If*

$$q_{X,\gamma}(t) \le q_{Y,\gamma}(t) \quad \text{for all } \gamma \in (0,\beta) \text{ and all } t \ge t_0,$$

then $X \le_{t_0}^{\mathrm{prl}} Y$.

Proof. For any $\gamma \in (0,\beta)$, since $q_{X,\gamma}(t) \le q_{Y,\gamma}(t)$ for every $t \ge t_0$, it follows from Lemma 3.1 that

$$q_{X,1-\overline{\gamma}^{2m}}(t) \le q_{Y,1-\overline{\gamma}^{2m}}(t) \quad \text{for all } t \ge t_0 \text{ and for all } m = 1,2,\ldots.$$

Now, let $\delta \in [\beta,1)$, and consider

$$\gamma \stackrel{\text{def}}{=} 1-(1-\delta)^{\frac{1}{2^m}} \quad \text{where} \quad m = \left[\frac{\log\left(\frac{\log(1-\delta)}{\log(1-\beta)}\right)}{\log 2}\right]+1;$$

here $[s]$ denotes the integer part of s. It is straightforward to verify that $\gamma < \beta$. Plugging this γ in the inequality $q_{X,1-\overline{\gamma}^{2m}}(t) \le q_{Y,1-\overline{\gamma}^{2m}}(t)$ we obtain $q_{X,\delta}(t) \le q_{Y,\delta}(t)$ for all $t \ge t_0$. Since this is true for every $\delta \in [\beta,1)$ we get $X \le_{t_0}^{\mathrm{prl}} Y$. □

Looking at Theorem 3.1 it is natural to wonder whether a condition such as

$$q_{X,\delta}(t) \le q_{Y,\delta}(t) \quad \text{for all } \gamma \in (\delta,\beta) \text{ and all } t \ge t_0,$$

for some $0 < \delta < \beta < 1$ (note that here we do not allow $\delta = 0$), implies $X \le_{t_0}^{\mathrm{prl}} Y$. It turns out that this is not the case. See Example given in Remark 2.3.

3.2 Relationship to other stochastic orders

Recall from (1.10) that a random variable X is said to be smaller than the random variable Y in the ordinary stochastic order (denoted as $X \leq_{\mathrm{st}} Y$) if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ for all $x \in \mathbb{R}$. It is known that $X \leq_{\mathrm{st}} Y$ if, and only if,

$$F_X^{-1}(p) \leq F_Y^{-1}(p) \quad \text{for all } p \in (0,1); \tag{3.5}$$

see, for example, (1.A.12) in Shaked and Shanthikumar (2007).

Recall also from (1.12) that a random variable X is said to be smaller than the random variable Y in the hazard rate order (denoted as $X \leq_{\mathrm{hr}} Y$) if $\frac{\overline{F}_Y(t)}{\overline{F}_X(t)}$ is increasing in t. Recalling the notation X_t and Y_t for the residual lives that are associated with X and Y, it is known that $X \leq_{\mathrm{hr}} Y$ if, and only if,

$$X_t \leq_{\mathrm{st}} Y_t \quad \text{for all } t < u_X; \tag{3.6}$$

see, for example, (1.B.6) in Shaked and Shanthikumar (2007).

The following proposition states an equivalent condition for the percentile residual life order from time t_0 on to hold.

Proposition 3.2. *Let $t_0 < u_X$, and let X and Y be two random variables. The random variables X and Y satisfy $X \leq_{t_0}^{\mathrm{prl}} Y$ if, and only if,*

$$X_t \leq_{st} Y_t \quad \textit{for all } t \geq t_0. \tag{3.7}$$

Proof. For every $t \geq t_0$,

$$X_t \leq_{st} Y_t \Leftrightarrow F_{X_t}^{-1}(\delta) \leq F_{Y_t}^{-1}(\delta) \quad \text{for all } \delta \in (0,1).$$

Then, by (3.1), $q_{X,\delta}(t) \leq q_{Y,\delta}(t)$ for all $t \geq t_0$ and all $\delta \in (0,1)$. That is $X \leq_{t_0}^{\mathrm{prl}} Y$. □

Corollary 3.1. *If X and Y are two nonnegative random variables and $t_0 \leq 0$, then $\leq_{t_0}^{\mathrm{prl}} \Rightarrow \leq_{st}$.*

Proof. Let $t_0 < u_X$, by the second condition in Proposition 3.1,

$$X \leq_{t_0}^{\mathrm{prl}} Y \Leftrightarrow F_X^{-1}(\gamma + (1-\gamma)F_X(t)) \leq F_Y^{-1}(\gamma + (1-\gamma)F_Y(t)), \text{ for all } t \geq t_0 \text{ and all } \gamma \in (0,1).$$

In particular, the latter inequality is true when $t = 0$. Therefore,

$$F_X^{-1}(\gamma) \leq F_Y^{-1}(\gamma), \text{ for all } \gamma \in (0,1) \Leftrightarrow X \leq_{st} Y.$$

□

From (3.6) we obtain the following result.

Theorem 3.2. *Let X and Y be two random variables and $t_0 < u_X$. If $X \leq_{hr} Y$ then, $X \leq_{t_0}^{\mathrm{prl}} Y$.*

Remark 3.1. *When X and Y are two nonnegative random variables and $t_0 \leq 0$, the percentile residual life order from time t_0 on is an order between the usual stochastic order and the hazard rate order.*

Corollary 3.2. *If $t_0 = -\infty$,*

$$\leq_{hr} \Leftrightarrow \leq_{t_0}^{\mathrm{prl}} .$$

From Theorem 3.2 and Corollary 3.2 it is seen that when we consider nonnegative random variables, the percentile residual life orders from time t_0 are indeed orders and not only preorders.

In the next counterexample it is shown that for any $t_0 < u_X$ we have

$$\leq_{st} \not\Rightarrow \leq_{t_0}^{\mathrm{prl}} .$$

Counterexample 3.1. *Let $t_0 < u_X$ and $k \in (t_0, u_X)$. Assume that X is uniformly distributed on (0,k+2) and that Y is a random variable whose distribution is the following mixture:*

$$F_Y(x) = \begin{cases} \text{uniform on } [0,k] & \text{with probability } a, \\ \text{uniform on } [k,k+1] & \text{with probability } \frac{k+1}{k+2} - a, \\ \text{uniform on } [k+1,k+2] & \text{with probability } \frac{1}{k+2}; \end{cases}$$

with $a < \frac{k}{k+2}$. Then $X \leq_{st} Y$ holds. But X_k is uniformly distributed on $(k, k+2)$ and the distribution function of Y_k is given by the following mixture:

$$F_{Y_k}(x) = \begin{cases} \text{uniform on } [k,k+1] & \text{with probability } \frac{1}{1-a}\left(\frac{k+1}{k+2} - a\right), \\ \text{uniform on } [k+1,k+2] & \text{with probability } \frac{1}{(k+2)(1-a)}; \end{cases}$$

so that $X_k \geq_{st} Y_k$. And therefore $X \not\leq_{t_0}^{\mathrm{prl}} Y$.

From this result, it follows that $\leq_{t_0}^{\mathrm{prl}}$ does not imply $\leq_{t_1}^{\mathrm{prl}}$ when $t_1 < t_0$. Obviously, $\leq_{t_0}^{\mathrm{prl}}$ imply $\leq_{t_1}^{\mathrm{prl}}$ when $t_1 > t_0$.

Let us now return to the consideration of the relationship between the orders $\leq_{t_0}^{\mathrm{prl}}$ and $\leq_{\mathrm{hr}}$.

Recall from (1.17) the definition of the mean residual life function m_X of a random variable X. Similarly the mean residual life function m_Y, of another random variable Y, is defined. If

$$m_X(t) \leq m_Y(t) \quad \text{for all } t \in \mathbb{R},$$

then X is said to be smaller than Y in the mean residual life order (denoted as $X \leq_{\mathrm{mrl}} Y$); see Shaked and Shanthikumar (2007).

Since $\leq_{\text{mrl}} \not\Longrightarrow \leq_{hr}$, there exist two random variables X and Y such that $X \leq_{\text{mrl}} Y$ but $X \nleq_{hr} Y$. By Theorem 3.2 $\leq_{\text{hr}} \Rightarrow \leq_{t_0}^{\text{prl}}$ for all $t_0 < u_X$, the same example shows that there exists $t_0 < u_X$ such that

$$\leq_{\text{mrl}} \not\Longrightarrow \leq_{t_0}^{\text{prl}} .$$

Since $X \leq_{\text{mrl}} Y \Longrightarrow X \leq_{\text{hmrl}} Y$, it follows from the above that for any $t_0 < u_X$ we have

$$X \leq_{\text{hmrl}} Y \not\Longrightarrow X \leq_{\text{prl}}^{t_0} Y;$$

where $\leq_{\text{hmrl}}$ denotes the harmonic mean residual life stochastic order (see Shaked and Shanthikumar (2007) for the definition, and for the fact that the mean residual life order implies the harmonic mean residual life order).

The following example shows that the percentile residual life order from time t_0 on does not imply the mean residual life order. Therefore, since the *hr* order implies the *mrl* order) the percentile residual life order from time t_0 on does not imply the *hr* order.

Counterexample 3.2. *Let $k > 0$ and $\frac{k+1}{k+2} < w < 1$. Let X have the uniform distribution on $(0, k+2)$ and let Y be distributed as a mixture of a degenerate random variable at $k+1$ with probability w, and a degenerate random variable at $k+2$ with probability $1-w$.*

For every $k+1 < t < k+2$, X_t is uniformly distributed on $(t, k+2)$ and Y_t is distributed as a degenerate random variable in $k+2$. It is easy to verify that $X_t \leq_{st} Y_t$ for every $k+1 < t < k+2 = u_X$. Therefore, $X \leq_{t_0}^{prl} Y$, where $t_0 \geq k+1$.

Now, take $\varepsilon > 0$, such that $\varepsilon < 2w - 1$. Then, $X_{k+1-\varepsilon}$ is uniformly distributed on $(k+1-\varepsilon, k+2)$ and $Y_{k+1-\varepsilon} =_{st} Y$. Besides, since $w > \frac{k+1}{k+2} > \frac{1}{2}$, then $k + \frac{3}{2} = E(X_{k+1}) = m_X(k+1) > m_Y(k+1) = E(Y_{k+1}) = k+2-w$. Therefore, $X \nleq_{mrl} Y$.

However, the following result shows that there exists a relationship between the percentile residual life order from time t_0 on and the mean residual life order.

Theorem 3.3. *Let X and Y be two random variables and $t_0 < u_X$. If $X \leq_{prl}^{t_0} Y$ then $m_X(t) \leq m_Y(t)$, for all $t \geq t_0$.*

Proof. By Proposition 3.2,

$$X \leq_{\text{prl}}^{t_0} Y \Leftrightarrow X_t \leq_{st} Y_t \text{ for all } t \geq t_0.$$

And, $X_t \leq_{st} Y_t$ implies $m_X(t) = E(X_t) \leq E(Y_t) = m_Y(t)$. □

Recall that a random variable X is said to be smaller than the random variable Y in the reversed hazard rate order (denoted as $X \leq_{\text{rh}} Y$) if $F_X(y)F_Y(x) \leq F_X(x)F_Y(y)$ for all $x \leq y$. The following counterexample shows that $\leq_{\text{rh}} \not\Rightarrow \leq_{t_0}^{\text{prl}}$.

Counterexample 3.3. *Let $t_0 < u_X$ and $k \in (t_0, u_X)$. Consider any $\gamma \in (0,1)$ and let X have the distribution function given by*

$$F_X(x) = \begin{cases} 0, & x < k+\gamma; \\ x-k, & k+\gamma \le x < k+1; \\ 1, & t \ge k+1; \end{cases}$$

that is, F_X is a mixture of a uniform distribution on $(k+\gamma, k+1)$ with probability $1-\gamma$, and a degenerate variable at $k+\gamma$ with probability γ; and let Y have the uniform distribution on $(k, k+1)$. We compute

$$q_{X,\gamma}(x) = \begin{cases} k+\gamma-x, & x < k+\gamma; \\ \gamma(k+1-x), & k+\gamma \le x < k+1; \\ 0, & t \ge k+1; \end{cases}$$

and

$$q_{Y,\gamma}(x) = \begin{cases} k+\gamma-x, & x < k; \\ \gamma(k+1-x), & k \le x < k+1; \\ 0, & t \ge k+1; \end{cases}$$

It is easy to verify that F_X and F_Y satisfy $F_Y(y)F_X(x) \le F_Y(x)F_X(y)$ for all $x \le y$; that is, $X \le_{\mathrm{rh}} Y$. However $q_{X,\gamma}(t) > q_{Y,\gamma}(t)$ for all $t \in (k, k+\gamma)$, and, since $k > t_0$, $X \not\le_{t_0}^{\mathrm{prl}} Y$.

Nanda and Shaked (2002) showed the following result. Let g be a continuous strictly decreasing function. Then,

$$X \le_{hr} Y \Leftrightarrow g(X) \le_{rh} g(Y).$$

So, in general, $X \le_{hr} Y \not\Rightarrow X \le_{rh} Y$. Therefore, there exist X and Y two random variables such that $X \le_{hr} Y$ (or, equivalently, $X \le_{t_0}^{\mathrm{prl}} Y$ for all $t_0 < u_X$) and $X \not\le_{rh} Y$. This example shows that

$$X \le_{t_0}^{\mathrm{prl}} Y \not\Longrightarrow X \le_{\mathrm{rh}} Y.$$

Figure 3.1 summarizes some of the results shown in this section. Here PRL-t_0 denotes the percentile residual life order from time t_0 on.

3.3 Closure properties

The percentile residual life orders from time t_0 on satisfy some desirable closure properties. These are described and discussed in this section.

First we show that the percentile residual life orders from time t_0 on are preserved under strictly increasing transformations.

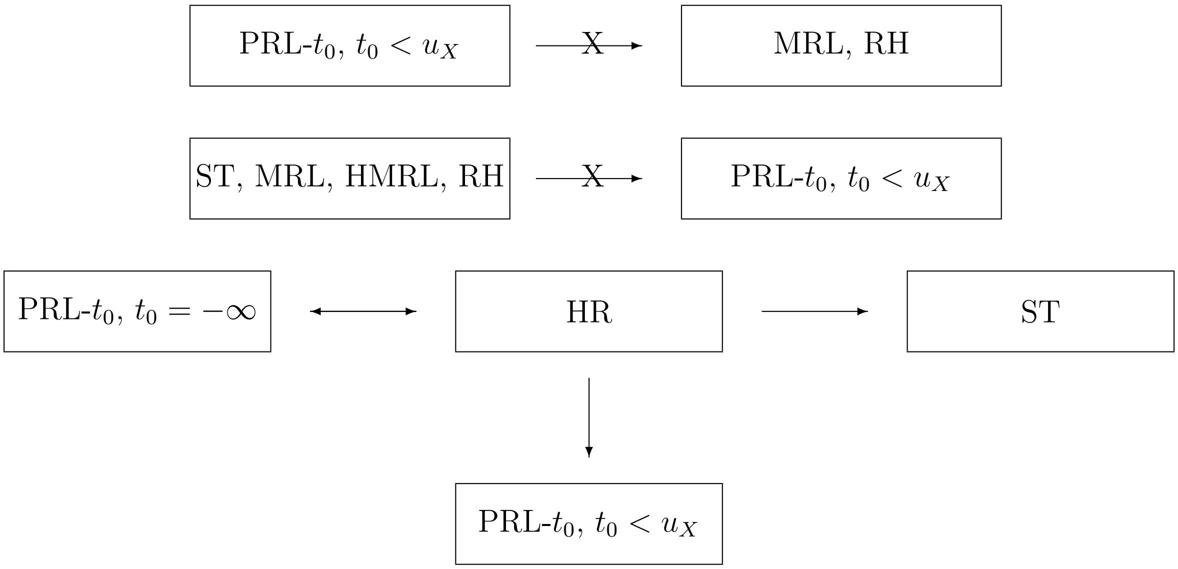

Figure 3.1: Relationship among some common stochastic orders

Theorem 3.4. *Let X and Y be random variables, let $t_0 < u_X$, and let ϕ be a strictly increasing function. Then $X \leq_{t_0}^{prl} Y$ if, and only if, $\phi(X) \leq_{\phi^{-1}(t_0)}^{prl} \phi(Y)$.*

Proof. Let $\overline{F}_{\phi(X)}$ and $\overline{F}_{\phi(Y)}$ denote the survival functions of the indicated random variables. Since ϕ is strictly increasing we have

$$\overline{F}_{\phi(X)}(t) = \overline{F}_X(\phi^{-1}(t)) \quad \text{and} \quad \overline{F}_{\phi(Y)}(t) = \overline{F}_Y(\phi^{-1}(t)) \quad \text{for all } t,$$

and

$$\overline{F}^{-1}_{\phi(X)}(u) = \phi(\overline{F}^{-1}_X(u)) \quad \text{and} \quad \overline{F}^{-1}_{\phi(Y)}(u) = \phi(\overline{F}^{-1}_Y(u)) \quad \text{for all } u \in (0,1).$$

Therefore, by Proposition 3.1(i), $\phi(X) \leq_{t_0}^{\text{prl}} \phi(Y)$ if, and only if,

$$\phi(\overline{F}^{-1}_X(\overline{\gamma}\overline{F}_X(\phi^{-1}(t)))) \leq \phi(\overline{F}^{-1}_Y(\overline{\gamma}\overline{F}_Y(\phi^{-1}(t)))) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0,1).$$

By the strict monotonicity of ϕ, the latter condition is equivalent to

$$\overline{F}^{-1}_X(\overline{\gamma}\overline{F}_X(\phi^{-1}(t))) \leq \overline{F}^{-1}_Y(\overline{\gamma}\overline{F}_Y(\phi^{-1}(t))) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0,1).$$

Letting $t' = \phi^{-1}(t)$, this condition is the same as

$$\overline{F}^{-1}_X(\overline{\gamma}\overline{F}_X(t')) \leq \overline{F}^{-1}_Y(\overline{\gamma}\overline{F}_Y(t')) \quad \text{for all } t' \geq \phi^{-1}(t_0) \text{ and all } \gamma \in (0,1),$$

and the stated result follows from Proposition 3.1(i). □

The percentile residual life orders from time t_0 on are closed under limits in distribution.

Theorem 3.5. *Let $\{X_n,\ n = 1, 2, \dots\}$ and $\{Y_n,\ n = 1, 2, \dots\}$ be two sequences of random variables such that $X_n \to_{\text{st}} X$ and $Y_n \to_{\text{st}} Y$ as $n \to \infty$, where "$\to_{\text{st}}$" denotes convergence in distribution. For any $t_0 < u_X$, if $X_n \leq_{t_0}^{prl} Y_n$, $n = 1, 2, \dots$, then $X \leq_{t_0}^{prl} Y$.*

Proof. For every $n = 1, 2, \dots$, by Proposition 3.2,

$$X_n \leq_{t_0}^{\text{prl}} Y_n \Leftrightarrow (X_n)_t \leq_{st} (Y_n)_t \text{ for all } t \geq t_0,$$

where $(A_i)_t = [A_i - t | A_i > t]$, for every random variable A.

Since the usual stochastic order is closed with respect to weak convergence, then

$$\lim (X_n)_t \leq_{st} \lim (Y_n)_t.$$

On the other hand, for every $n = 1, 2, \dots$, it holds that $(X_n)_t \to_{\text{st}} X_t$ and $(Y_n)_t \to_{\text{st}} Y_t$. Then

$$X_t \leq_{st} Y_t$$

for all $t \geq t_0$ and, by Proposition 3.2 the claim is true. □

The following two lemmas, that deal with simple mixtures, will yield a general closure under mixtures property of the percentile residual life orders from time t_0 on.

Lemma 3.2. *Let X, Y, U, and V be random variables with continuous distribution functions, and let W be a random variable with distribution function*

$$F_W = pF_X + (1 - p)F_Y,$$

for some $p \in [0, 1]$.

(i) *If $U \leq_{t_0}^{prl} X$ and $U \leq_{t_0}^{prl} Y$ then $U \leq_{t_0}^{prl} W$.*

(ii) *If $X \leq_{t_0}^{prl} V$ and $Y \leq_{t_0}^{prl} V$ then $W \leq_{t_0}^{prl} V$.*

Proof. First we prove (i). From $U \leq_{t_0}^{\text{prl}} X$ and $U \leq_{t_0}^{\text{prl}} Y$, using Proposition 3.1(i), we obtain

$$\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t)) \leq \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) \quad \text{and} \quad \overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t)) \leq \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(t)) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0, 1).$$

It follows, by the continuity of F_X and of F_Y, that

$$\overline{F}_X(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) \geq \overline{\gamma}\overline{F}_X(t) \quad \text{and} \quad \overline{F}_Y(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) \geq \overline{\gamma}\overline{F}_Y(t) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0, 1).$$

Therefore, for every $\gamma \in (0, 1)$,

$$p\overline{F}_X(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) + (1 - p)\overline{F}_Y(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) \geq \overline{\gamma}p\overline{F}_X(t) + \overline{\gamma}(1 - p)\overline{F}_X(t) \quad \text{for all } t \geq t_0;$$

that is,

$$\overline{F}_W(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) \geq \overline{\gamma}\overline{F}_W(t) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0,1).$$

By the continuity of F_W we get

$$\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t)) \leq \overline{F}_W^{-1}(\overline{\gamma}\overline{F}_W(t)) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0,1);$$

that is, by Proposition 3.1(i), $U \leq_{t_0}^{\text{prl}} W$.

Now we prove (ii). From $X \leq_{t_0}^{\text{prl}} V$ and $Y \leq_{t_0}^{\text{prl}} V$, using Proposition 3.1(i), we obtain

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) \leq \overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t)) \quad \text{and} \quad \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(t)) \leq \overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t)) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0,1).$$

It follows, by the continuity of F_X and of F_Y, that

$$\overline{\gamma}\overline{F}_X(t) \geq \overline{F}_X(\overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t))) \quad \text{and} \quad \overline{\gamma}\overline{F}_Y(t) \geq \overline{F}_Y(\overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t))) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0,1).$$

Therefore, for every $\gamma \in (0,1)$,

$$\overline{\gamma}p\overline{F}_X(t) + \overline{\gamma}(1-p)\overline{F}_Y(t) \geq p\overline{F}_X(\overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t))) + (1-p)\overline{F}_Y(\overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t))) \quad \text{for all } t \geq t_0;$$

that is,

$$\overline{\gamma}\overline{F}_W(t) \geq \overline{F}_W(\overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t))) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0,1).$$

By the continuity of F_W we get

$$\overline{F}_W^{-1}(\overline{\gamma}\overline{F}_W(t)) \leq \overline{F}_V^{-1}(\overline{\gamma}\overline{F}_V(t)) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0,1);$$

that is, by Proposition 3.1(i), $W \leq_{t_0}^{\text{prl}} V$. □

Lemma 3.3. *Let X_1, X_2, Y_1, and Y_2 be random variables with continuous distribution functions, and let W and Z be random variables with distribution functions*

$$F_W = pF_{X_1} + (1-p)F_{X_2} \quad \textit{and} \quad F_Z = pF_{Y_1} + (1-p)F_{Y_2},$$

for some $p \in [0,1]$. If there exists a random variable S such that

$$X_1 \leq_{t_0}^{prl} S, \quad X_2 \leq_{t_0}^{prl} S, \quad S \leq_{t_0}^{prl} Y_1, \quad S \leq_{t_0}^{prl} Y_2,$$

then $W \leq_{t_0}^{prl} Z$.

Proof. Since $X_1 \leq_{t_0}^{\text{prl}} S$ and $X_2 \leq_{t_0}^{\text{prl}} S$, it follows from Lemma 4.1(ii) that $W \leq_{t_0}^{\text{prl}} S$. Furthermore, since $S \leq_{t_0}^{\text{prl}} Y_1$ and $S \leq_{t_0}^{\text{prl}} Y_2$, it follows from Lemma 4.1(i) that $S \leq_{t_0}^{\text{prl}} Z$. By the transitivity property of the order $\leq_{t_0}^{\text{prl}}$ we get $W \leq_{t_0}^{\text{prl}} Z$. □

By repeated application of Lemma 4.2, and convergence arguments, we obtain the following result.

Theorem 3.6. *Let $\{X_\theta,\ \theta \in \Theta\}$ and $\{Y_\theta,\ \theta \in \Theta\}$ be two families of random variables with continuous distribution functions. Let W and Z be random variables with distribution functions given by*

$$F_W(t) = \int_\Theta F_{X_\theta}(t)dH(\theta) \quad \textit{and} \quad F_Z(t) = \int_\Theta F_{Y_\theta}(t)dH(\theta), \quad t \in \mathbb{R},$$

where H is some distribution function on Θ. Suppose that there exists a random variable S such that

$$X_\theta \leq_{t_0}^{prl} S \leq_{t_0}^{prl} Y_\theta \quad \textit{for all } \theta \in \Theta. \tag{3.8}$$

Then $W \leq_{t_0}^{prl} Z$.

Note that condition (4.8) can be rewritten as

$$X_\theta \leq_{t_0}^{\mathrm{prl}} Y_{\theta'} \quad \text{for all } \theta, \theta' \in \Theta.$$

It is worth noting that results that are similar to Theorem 4.6 hold for the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the mean residual life order (see, respectively, Theorems 1.B.8, 1.B.46, 1.C.15, and 2.A.13 in Shaked and Shanthikumar, 2007).

A special case of Theorem 4.6 is the following result which shows that a random variable, whose distribution is a mixture of two distributions of random variables which are ordered with respect to the percentile residual life order from time t_0 on, is bounded from below and from above, in the percentile residual life order from time t_0 on sense, by these two random variables.

Corollary 3.3. *Let X and Y be two random variables with continuous distribution functions, and let W be a random variable with distribution function*

$$F_W = pF_X + (1-p)F_Y,$$

for some $p \in [0,1]$. If $X \leq_{t_0}^{prl} Y$ then $X \leq_{t_0}^{prl} W \leq_{t_0}^{prl} Y$.

Again, note that similar results hold for the hazard rate order, the likelihood ratio order, and the mean residual life order (see, respectively, Theorems 1.B.22, 1.C.30, and 2.A.18 in Shaked and Shanthikumar, 2007).

The possible preservation of a stochastic order under the formation of coherent systems is a useful property that has important applications in reliability theory (see, for example, Barlow and Proschan, 1975, for the definition and the use of coherent systems). Thus it is of interest to ask whether the percentile residual life orders from time t_0 on are closed under this formation. Boland, El-Neweihi, and Proschan (1994) showed that the hazard rate order is not preserved under the formation of coherent systems. In the next counterexample it is shown that, *for all $t_0 < u_X$*, the percentile residual life order from time t_0 on is not closed under this formation. This is shown by considering a parallel system of size 2 whose lifetime is the maximum of the lifetimes of its two components.

Counterexample 3.4. *Let X be an exponential random variable with rate $\lambda > 0$. That is,*

$$F_X(t) = \begin{cases} 0, & t < 0; \\ 1 - e^{-\lambda t}, & t \geq 0. \end{cases}$$

Let Y be a random variable that is degenerate at 0, and let Z be a random variable that is degenerate at 1. Note that $\max\{X, Y\} =_{\text{st}} X$. *Note also that for every $t_0 < u_X$, $Y \leq_{t_0}^{prl} Z$, and, of course, $X \leq_{t_0}^{prl} X$. Now we compute*

$$q_{\max\{X,Y\},\gamma}(t) = q_{X,\gamma}(t) = \begin{cases} \frac{-\log(1-\gamma)}{\lambda} - t, & t < 0; \\ \frac{-\log(1-\gamma)}{\lambda}, & t \geq 0, \end{cases}$$

and

$$q_{\max\{X,Z\},\gamma}(t) = \begin{cases} \frac{-\log(1-\gamma)}{\lambda} - t, & t < 1; \\ \frac{-\log(1-\gamma)}{\lambda}, & t \geq 1. \end{cases}$$

It is seen that $\max\{X, Y\} \not\leq_{prl}^{t_0} \max\{X, Z\}$ *(in fact,* $\max\{X, Y\} \geq_{prl}^{t_0} \max\{X, Z\}$ *for every $t_0 < u_X$). Thus the percentile residual life order from time t_0 on is not closed under the maximum operation.* ◀

For every $t_0 < u_X$, the percentile residual life order from time t_0 on is closed under the formation of series systems (that is, under the minimum operation). This is shown in the next theorem.

Theorem 3.7. *Let $X_1, X_2 \ldots, X_n$ and $Y_1, Y_2 \ldots, Y_n$ be independent random variables with $X_i \leq_{t_0}^{prl} Y_i$, for $i = 1, \ldots, n$. Then*

$$\min\{X_1, X_2 \ldots, X_n\} \leq_{t_0}^{prl} \min\{Y_1, Y_2 \ldots, Y_n\}. \tag{3.9}$$

Proof. For every $i = 1, \ldots, n$, by Proposition 3.2,

$$X_i \leq_{t_0}^{\text{prl}} Y_i \Leftrightarrow (X_i)_t \leq_{st} (Y_i)_t \text{ for all } t \geq t_0,$$

where $(A_i)_t = [A_i - t | A_i > t]$, for every random variable A.

Since the usual stochastic order is closed under the minimum operation, we have that

$$\min\{(X_1)_t, (X_2)_t \ldots, (X_n)_t\} \leq_{st} \min\{(Y_1)_t, (Y_2)_t \ldots, (Y_n)_t\}.$$

On the other hand, for every $n = 1, 2, \ldots$ it holds that $\min\{(X_1)_t, (X_2)_t \ldots, (X_n)_t\} =_{st} (\min\{X_1, X_2 \ldots, X_n\})_t$ and $\min\{(Y_1)_t, (Y_2)_t \ldots, (Y_n)_t\} =_{st} (\min\{Y_1, Y_2 \ldots, Y_n\})_t$. Then,

$$(\min\{X_1, X_2 \ldots, X_n\})_t \leq_{st} (\min\{Y_1, Y_2 \ldots, Y_n\})_t$$

for all $t \geq t_0$ and, by Proposition 3.2, the claim is true. □

3.4 Some applications

Besides the practical applications we have enumerated on the introduction of the chapter, here we show some other practical features of the new order which may be considered as applications.

As we already explained in Section 2.4, the following result is useful in reliability theory.

Theorem 3.8. *Let X and Y be two random variables with continuous distributions on interval supports. Let $t_0 < u_X$ and $\theta > 0$. If $X \leq_{t_0}^{prl} Y$ then*

$$X(\theta) \leq_{t_0}^{prl} Y(\theta). \tag{3.10}$$

Proof. It is not hard to verify that under the continuity assumptions above we have

$$(\overline{F}_X^{\theta})^{-1}(u) = \overline{F}_X^{-1}(u^{1/\theta}) \quad \text{and} \quad (\overline{F}_Y^{\theta})^{-1}(u) = \overline{F}_Y^{-1}(u^{1/\theta}), \quad u \in (0,1),$$

or, equivalently,

$$\overline{F}_X^{-1}(u) = (\overline{F}_X^{\theta})^{-1}(u^{\theta}) \quad \text{and} \quad \overline{F}_Y^{-1}(u) = (\overline{F}_Y^{\theta})^{-1}(u^{\theta}), \quad u \in (0,1).$$

Now, by Proposition 3.1(i), $X \leq_{t_0}^{\mathrm{prl}} Y$ means

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) \leq \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(t)) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0,1),$$

that is,

$$(\overline{F}_X^{\theta})^{-1}(\overline{\gamma}^{\theta}\overline{F}_X^{\theta}(t)) \leq (\overline{F}_Y^{\theta})^{-1}(\overline{\gamma}^{\theta}\overline{F}_Y^{\theta}(t)) \quad \text{for all } t \geq t_0 \text{ and all } \gamma \in (0,1),$$

and the result follows from Proposition 3.1(i). □

Chapter 4

The PRL orders up to time t_0

In the two previous chapters, we have introduced different tools that are useful to compare items from a certain moment on. We think that it is interesting too, to develop a technique for comparing items until a certain instant. This would be useful in medical trials, when there exists a time for testing different treatments, for example, and to compare items during the warranty period.

The percentile residual life order up to time t_0 is a stochastic ordering for nonnegative random variables which compares all their percentile residual life functions until a certain moment t_0. This order is stronger than the usual stochastic order and weaker than the hazard rate order.

The percentile residual life orders that we have defined and studied in Chapter 2 and in Chapter 3 are not orders but preorders. The reason is that these binary relations do not verify the *antisymmetry property*. That is, $X \leq_{\gamma\text{-rl}} Y$ and $Y \leq_{\gamma\text{-rl}} X$ does not necessarily imply $X =_{st} Y$. And, analogously, $X \leq_{t_0}^{\text{prl}} Y$ and $Y \leq_{t_0}^{\text{prl}} X$ does not necessarily imply $X =_{st} Y$. However, since the percentile residual life order up to time t_0 implies the usual stochastic order, then this new order is an order not only a preorder.

In this chapter the percentile residual life orders up to time t_0 are formally defined in Section 4.1. We also give some equivalent ways of describing these orders that turn up to be useful in the sequel. Section 4.2 consists of a thorough study of the relationships among the percentile residual life orders up to time t_0 and other stochastic orders in the literature. Some useful properties of the percentile residual life orders up to time t_0 are given in Section 4.3. Finally, some applications in reliability theory are described in Section 4.4. We will assume that all random variables considered along this chapter are nonnegative, unless stated otherwise.

4.1 Definition

Let X be a random variable. Recall that the γ-percentile residual life function $q_{X,\gamma}$ is defined by

$$q_{X,\gamma}(t) = \begin{cases} F_{X_t}^{-1}(\gamma), & t < u_X; \\ 0, & t \geq u_X. \end{cases} \tag{4.1}$$

Recall from the two previous chapters that the γ-percentile residual life function can be written

$$q_{X,\gamma}(t) = \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) - t, \quad t < u_X, \tag{4.2}$$

where $\overline{\gamma} = 1 - \gamma$. Alternatively,

$$q_{X,\gamma}(t) = F_X^{-1}(\gamma + \overline{\gamma}F_X(t)) - t, \quad t < u_X. \tag{4.3}$$

Now let Y be another random variable, and let $q_{Y,\gamma}$ be its γ-percentile residual life function. Let $t_0 > 0$. If

$$q_{X,\gamma}(t) \leq q_{Y,\gamma}(t) \quad \text{for all } t \leq t_0 \text{ and for all } \gamma \in (0,1), \tag{4.4}$$

then we say that X is smaller than Y in the *percentile residual life order up to time* t_0, and we denote it as $X \leq_{\text{prl}}^{t_0} Y$.

The following proposition states equivalent conditions for the percentile residual life order up to time t_0 to hold. The proof follows straightforward from (4.2), (4.3) and (4.4).

Proposition 4.1. *Let $t_0 > 0$ and let X and Y be two random variables.*

(i) *The random variables X and Y satisfy $X \leq_{\text{prl}}^{t_0} Y$ if, and only if,*

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) \leq \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(t)) \quad \textit{for all } t \leq t_0 \textit{ and all } \gamma \in (0,1).$$

(ii) *The random variables X and Y satisfy $X \leq_{\text{prl}}^{t_0} Y$ if, and only if,*

$$F_X^{-1}(\gamma + \overline{\gamma}F_X(t)) \leq F_Y^{-1}(\gamma + \overline{\gamma}F_Y(t)) \quad \textit{for all } t \leq t_0 \textit{ and all } \gamma \in (0,1).$$

4.2 Relationship to other stochastic orders

Recall from (1.10) that a random variable X is said to be smaller than the random variable Y in the ordinary stochastic order (denoted as $X \leq_{\text{st}} Y$) if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ for all $x \in \mathbb{R}$. It is known that $X \leq_{\text{st}} Y$ if, and only if,

$$F_X^{-1}(p) \leq F_Y^{-1}(p) \quad \text{for all } p \in (0,1); \tag{4.5}$$

see, for example, (1.A.12) in Shaked and Shanthikumar (2007).

Recall also from (1.12) that a random variable X is said to be smaller than the random variable Y in the hazard rate order (denoted as $X \leq_{\text{hr}} Y$) if $\frac{\overline{F}_Y(t)}{\overline{F}_X(t)}$ is increasing in t. Recalling the notation X_t and Y_t for the residual lives that are associated with X and Y, it is known that $X \leq_{\text{hr}} Y$ if, and only if,

$$X_t \leq_{\text{st}} Y_t \quad \text{for all } t < u_X; \tag{4.6}$$

see, for example, (1.B.6) in Shaked and Shanthikumar (2007).

The following proposition states an equivalent condition for the percentile residual life order up to time t_0 to hold.

Proposition 4.2. *Let $t_0 > 0$, and let X and Y be two random variables. The random variables X and Y satisfy $X \leq_{prl}^{t_0} Y$ if, and only if,*

$$X_t \leq_{st} Y_t \quad \textit{for all } t \leq t_0. \tag{4.7}$$

Proof. For every $t \leq t_0$, by (4.5),

$$X_t \leq_{st} Y_t \Leftrightarrow F_{X_t}^{-1}(\gamma) \leq F_{Y_t}^{-1}(\gamma) \quad \text{ for all } \gamma \in (0,1).$$

Then, by (4.1), $q_{X,\gamma}(t) \leq q_{Y,\gamma}(t)$ for all $t \leq t_0$ and all $\gamma \in (0,1)$. That is $X \leq_{\text{prl}}^{t_0} Y$. □

In particular, we get the following corollary which indicates that the percentile residual life orders up to time t_0 indicate comparisons of size or magnitude and that the percentile residual life orders up to time t_0 are indeed orders, not only preorders.

Corollary 4.1. *Let $t_0 > 0$. Then,*

$$\leq_{prl}^{t_0} \Rightarrow \leq_{st} .$$

From (4.6) we obtain the following result.

Theorem 4.1. *Let X and Y be two random variables and $t_0 > 0$. If $X \leq_{hr} Y$ then, $X \leq_{prl}^{t_0} Y$.*

Theorem 4.2. *If $t_0 \geq u_X$,*

$$\leq_{hr} \Leftrightarrow \leq_{prl}^{t_0} .$$

Remark 4.1. *When $t_0 < u_X$, the percentile residual life order up to time t_0 is an order between the usual stochastic order and the hazard rate order.*

In the next counterexample it is shown that for any $t_0 > 0$ we have

$$\leq_{st} \not\Rightarrow \leq_{\text{prl}}^{t_0} .$$

Counterexample 4.1. *Let $t_0 > 0$ and $0 < k < t_0$. Assume that X is uniformly distributed on (0,k+2) and that Y is a random variable whose distribution is the following mixture:*

$$F_Y(x) = \begin{cases} \text{uniform on } [0,k] & \text{with probability } a, \\ \text{uniform on } [k,k+1] & \text{with probability } \frac{k+1}{k+2} - a, \\ \text{uniform on } [k+1,k+2] & \text{with probability } \frac{1}{k+2}; \end{cases}$$

with $a < \frac{k}{k+2}$. Then $X \le_{st} Y$ holds. Now consider $t = k$, X_k is uniformly distributed on $(k, k+2)$ and the distribution function of Y_k is given by the following mixture:

$$F_{Y_k}(x) = \begin{cases} \text{uniform on } [k,k+1] & \text{with probability } \frac{1}{1-a}\left(\frac{k+1}{k+2} - a\right), \\ \text{uniform on } [k+1,k+2] & \text{with probability } \frac{1}{(k+2)(1-a)}; \end{cases}$$

so that $X_k \ge_{st} Y_k$, and therefore $X \not\le_{prl}^{t_0} Y$.

From this result, it follows that $\le_{\text{prl}}^{t_0}$ does not imply $\le_{\text{prl}}^{t_1}$ when $t_1 > t_0$. It is obvious that $\le_{\text{prl}}^{t_0}$ implies $\le_{\text{prl}}^{t_1}$ when $t_1 < t_0$.

Recall from (1.17) the definition of the mean residual life function m_X of a random variable X. Similarly the mean residual life function m_Y, of another random variable Y, is defined. If

$$m_X(t) \le m_Y(t) \quad \text{for all } t \in \mathbb{R},$$

then X is said to be smaller than Y in the mean residual life order (denoted as $X \le_{\text{mrl}} Y$); see Shaked and Shanthikumar (2007).

In Counterexample 2.2 it is shown that for any $\gamma \in (0,1)$ we have

$$\le_{\text{mrl}} \not\Longrightarrow \le_{\gamma\text{-rl}} .$$

In that counterexample, X and Y are two nonnegative random variables such that $X \le_{\text{mrl}} Y$ but $q_{X,\gamma}(0) > q_{Y,\gamma}(0)$. Therefore, the same counterexample shows that $\le_{\text{mrl}} \not\Rightarrow \le_{\text{prl}}^{t_0}$ for any $t_0 > 0$.

Since $X \le_{\text{mrl}} Y \Longrightarrow X \le_{\text{hmrl}} Y$, it follows from the above that for any $t_0 > 0$ we have

$$X \le_{\text{hmrl}} Y \not\Longrightarrow X \le_{\text{prl}}^{t_0} Y;$$

where $\le_{\text{hmrl}}$ denotes the harmonic mean residual life stochastic order (see Shaked and Shanthikumar (2007) for the definition, and for the fact that the mean residual life order implies the harmonic mean residual life order).

The following example shows that the percentile residual life order up to time t_0 does not imply the mean residual life order. Therefore, since the *hr* order implies the *mrl* order, the same example shows that percentile residual life order up to time t_0 does not imply the *hr* order.

Counterexample 4.2. *Let $k > 0$ and $\frac{1}{2} < w < \frac{k+1}{k+2}$. Let X have the uniform distribution on $(0, k+2)$ and let Y be distributed as a mixture of a degenerate random variable at $k+1$ with probability w, and a degenerate random variable at $k+2$ with probability $1-w$.*

For every $0 < t \leq k+2-\frac{1}{1-w}$, X_t is uniformly distributed on $(t, k+2)$ and $Y_t =_{st} Y$. That is

$$F_{X_t}(x) = \begin{cases} 0, & x < t; \\ \frac{x-t}{k+2-t}, & t \leq x < k+2; \\ 1 & x \geq k+2; \end{cases}$$

and

$$F_{Y_t}(x) = \begin{cases} 0, & x < k+1; \\ w, & k+1 \leq x < k+2; \\ 1 & x \geq k+2. \end{cases}$$

It is easy to verify that $X_t \leq_{st} Y_t$ for every $t \leq k+2-\frac{1}{1-w}$ (notice that, since $w < \frac{k+1}{k+2}$, then $k+2-\frac{1}{1-w} > 0$). Therefore, $X \leq_{prl}^{t_0} Y$, for $t_0 \leq k+2-\frac{1}{1-w}$.

Now, take $\varepsilon > 0$ such that $\varepsilon < 2w-1$ (note that such an ε exists because $w > \frac{1}{2}$). Then, $X_{k+1-\varepsilon}$ is uniformly distributed on $(k+1-\varepsilon, k+2)$ and $Y_{k+1-\varepsilon} =_{st} Y$. We compute

$$k + \frac{3-\varepsilon}{2} = E(X_{k+1-\varepsilon}) = m_X(k+1-\varepsilon) > m_Y(k+1-\varepsilon) = E(Y_{k+1-\varepsilon}) = k+2-w.$$

Therefore, $X \nleq_{mrl} Y$.

However, the following result shows that there exists a relationship between the percentile residual life order up to time t_0 and the mean residual life order.

Theorem 4.3. *Let X and Y be two random variables and $t_0 > 0$. If $X \leq_{prl}^{t_0} Y$ then $m_X(t) \leq m_Y(t)$, for all $t \leq t_0$.*

Proof. By Proposition 4.2,

$$X \leq_{\text{prl}}^{t_0} Y \Leftrightarrow X_t \leq_{st} Y_t \text{ for all } t \leq t_0.$$

And, $X_t \leq_{st} Y_t$ for all $t \leq t_0$ implies $m_X(t) = E(X_t) \leq E(Y_t) = m_Y(t)$, for all $t \leq t_0$. □

Recall that a random variable X is said to be smaller than the random variable Y in the reversed hazard rate order (denoted as $X \leq_{\text{rh}} Y$) if $F_X(y)F_Y(x) \leq F_X(x)F_Y(y)$ for all $x \leq y$. The following counterexample shows that $\leq_{\text{rh}} \not\Rightarrow \leq_{\text{prl}}^{t_0}$.

Counterexample 4.3. *Let $t_0 > 0$ and take any $\gamma \in (0,1)$ such that $t_0 - \gamma > 0$. Let $k = t_0 - \gamma$ and let X have the distribution function given by*

$$F_X(x) = \begin{cases} 0, & x < k+\gamma; \\ x-k, & k+\gamma \leq x < k+1; \\ 1, & x \geq k+1; \end{cases}$$

that is, F_X is a mixture of a uniform distribution on $(k+\gamma, k+1)$ with probability $1-\gamma$, and a degenerate variable at $k+\gamma$ with probability γ; and let Y have the uniform distribution on $(k, k+1)$. We compute

$$q_{X,\gamma}(x) = \begin{cases} k+\gamma-x, & x < k+\gamma; \\ \gamma(k+1-x), & k+\gamma \leq x < k+1; \\ 0, & x \geq k+1; \end{cases}$$

and

$$q_{Y,\gamma}(x) = \begin{cases} k+\gamma-x, & x < k; \\ \gamma(k+1-x), & k \leq x < k+1; \\ 0, & x \geq k+1. \end{cases}$$

It is easy to verify that F_X and F_Y satisfy $F_Y(y)F_X(x) \leq F_Y(x)F_X(y)$ for all $x \leq y$; that is, $X \leq_{\mathrm{rh}} Y$. However $q_{X,\gamma}(t) > q_{Y,\gamma}(t)$ for all $t \in (k, k+\gamma)$, and, since $k+\gamma = t_0$, $X \nleq_{prl}^{t_0} Y$.

Nanda and Shaked (2001) showed the following result. Let g be a continuous strictly decreasing function. Then,

$$X \leq_{hr} Y \Leftrightarrow g(X) \leq_{rh} g(Y).$$

So, in general, $X \leq_{hr} Y \nRightarrow X \leq_{rh} Y$. Therefore, since $X \leq_{hr} Y$ is equivalent to $X \leq_{\mathrm{prl}}^{t_0} Y$ for all $t_0 > 0$, then

$$X \leq_{\mathrm{prl}}^{t_0} Y \nRightarrow X \leq_{\mathrm{rh}} Y.$$

Figure 4.1 summarizes some of the results shown in this section. Here t_0-PRL denotes the percentile residual life order up to time t_0.

4.3 Closure properties

The percentile residual life orders up to time t_0 satisfy some desirable closure properties. These are described and discussed in this section. First we show that the percentile residual life orders up to time t_0 are preserved under strictly increasing transformations.

Theorem 4.4. *Let X and Y be random variables, let $t_0 > 0$, and let ϕ be a strictly increasing function. Then $X \leq_{prl}^{t_0} Y$ if, and only if, $\phi(X) \leq_{prl}^{\phi^{-1}(t_0)} \phi(Y)$.*

Proof. Let $\overline{F}_{\phi(X)}$ and $\overline{F}_{\phi(Y)}$ denote the survival functions of the indicated random variables. Since ϕ is strictly increasing, we have

$$\overline{F}_{\phi(X)}(t) = \overline{F}_X(\phi^{-1}(t)) \quad \text{and} \quad \overline{F}_{\phi(Y)}(t) = \overline{F}_Y(\phi^{-1}(t)) \quad \text{for all } t,$$

and

$$\overline{F}^{-1}_{\phi(X)}(u) = \phi(\overline{F}^{-1}_X(u)) \quad \text{and} \quad \overline{F}^{-1}_{\phi(Y)}(u) = \phi(\overline{F}^{-1}_Y(u)) \quad \text{for all } u \in (0,1).$$

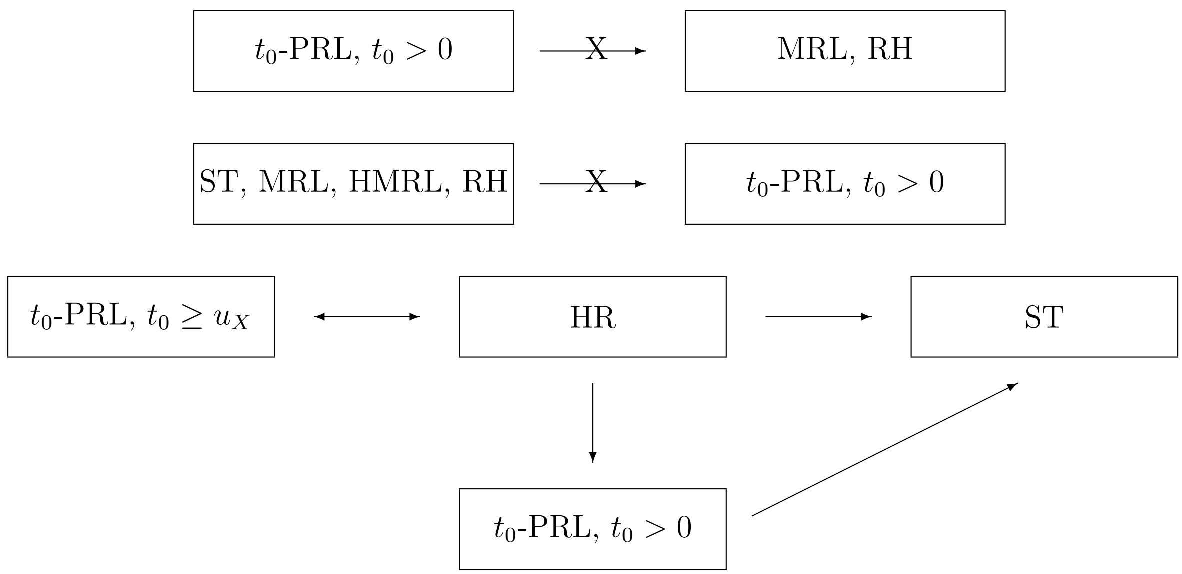

Figure 4.1: Relationship among some common stochastic orders

Therefore, by Proposition 4.1(i), $\phi(X) \leq_{\text{prl}}^{t_0} \phi(Y)$ if, and only if,

$$\phi(\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(\phi^{-1}(t)))) \leq \phi(\overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(\phi^{-1}(t)))) \quad \text{for all } t \leq t_0 \text{ and all } \gamma \in (0,1).$$

By the strict monotonicity of ϕ, the latter condition is equivalent to

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(\phi^{-1}(t))) \leq \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(\phi^{-1}(t))) \quad \text{for all } t \leq t_0 \text{ and all } \gamma \in (0,1).$$

Letting $t' = \phi^{-1}(t)$, this condition is the same as

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t')) \leq \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(t')) \quad \text{for all } t' \leq \phi^{-1}(t_0) \text{ and all } \gamma \in (0,1),$$

and the stated result follows from Proposition 4.1(i). □

The percentile residual life orders up to time t_0 are closed under limits in distribution.

Theorem 4.5. *Let $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ be two sequences of random variables such that $X_n \to_{\text{st}} X$ and $Y_n \to_{\text{st}} Y$ as $n \to \infty$, where "$\to_{\text{st}}$" denotes convergence in distribution. For any $t_0 > 0$, if $X_n \leq_{prl}^{t_0} Y_n$, $n = 1, 2, \ldots$, then $X \leq_{prl}^{t_0} Y$.*

Proof. For every $n = 1, 2, \ldots$, by Proposition 4.2,

$$X_n \leq_{\text{prl}}^{t_0} Y_n \Leftrightarrow (X_n)_t \leq_{st} (Y_n)_t \text{ for all } t \leq t_0,$$

where $(A_i)_t = [A_i - t | A_i > t]$, for every random variable A.

Since the usual stochastic order is closed with respect to weak convergence, then

$$\lim (X_n)_t \leq_{st} \lim (Y_n)_t.$$

On the other hand, for every $n = 1, 2, \ldots,$ it holds that $(X_n)_t \to_{\mathrm{st}} X_t$ and $(Y_n)_t \to_{\mathrm{st}} Y_t$. Then

$$X_t \leq_{st} Y_t$$

for all $t \leq t_0$ and, by Proposition 3.2 the claim is true. □

The following two lemmas, that deal with simple mixtures, will yield a general closure under mixtures property of the percentile residual life orders up to time t_0.

Lemma 4.1. *Let X, Y, U, and V be random variables with continuous distribution functions, and let W be a random variable with distribution function*

$$F_W = pF_X + (1-p)F_Y,$$

for some $p \in [0,1]$.

(i) *If $U \leq_{prl}^{t_0} X$ and $U \leq_{prl}^{t_0} Y$ then $U \leq_{prl}^{t_0} W$.*

(ii) *If $X \leq_{prl}^{t_0} V$ and $Y \leq_{prl}^{t_0} V$ then $W \leq_{prl}^{t_0} V$.*

Proof. First we prove (i). From $U \leq_{\mathrm{prl}}^{t_0} X$ and $U \leq_{\mathrm{prl}}^{t_0} Y$, using Proposition 4.1(i), we obtain

$$\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t)) \leq \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) \quad \text{and} \quad \overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t)) \leq \overline{F}_Y^{-1}(\overline{\gamma}\overline{F}_Y(t)) \quad \text{for all } t \leq t_0 \text{ and all } \gamma \in (0,1).$$

It follows, by the continuity of F_X and F_Y, that

$$\overline{F}_X(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) \geq \overline{\gamma}\overline{F}_X(t) \quad \text{and} \quad \overline{F}_Y(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) \geq \overline{\gamma}\overline{F}_Y(t) \quad \text{for all } t \leq t_0 \text{ and all } \gamma \in (0,1).$$

Therefore, for every $\gamma \in (0,1)$,

$$p\overline{F}_X(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) + (1-p)\overline{F}_Y(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) \geq \overline{\gamma}p\overline{F}_X(t) + \overline{\gamma}(1-p)\overline{F}_X(t) \quad \text{for all } t \leq t_0;$$

that is,

$$\overline{F}_W(\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t))) \geq \overline{\gamma}\overline{F}_W(t) \quad \text{for all } t \leq t_0 \text{ and all } \gamma \in (0,1).$$

By the continuity of F_W, we get

$$\overline{F}_U^{-1}(\overline{\gamma}\overline{F}_U(t)) \leq \overline{F}_W^{-1}(\overline{\gamma}\overline{F}_W(t)) \quad \text{for all } t \leq t_0 \text{ and all } \gamma \in (0,1);$$

that is, by Proposition 4.1(i), $U \leq_{\mathrm{prl}}^{t_0} W$.

Now we prove (ii). From $X \leq_{\rm prl}^{t_0} V$ and $Y \leq_{\rm prl}^{t_0} V$, using Proposition 4.1(i), we obtain

$$\overline{F}_X^{-1}(\gamma\overline{F}_X(t)) \leq \overline{F}_V^{-1}(\gamma\overline{F}_V(t)) \quad \text{and} \quad \overline{F}_Y^{-1}(\gamma\overline{F}_Y(t)) \leq \overline{F}_V^{-1}(\gamma\overline{F}_V(t)) \quad \text{for all } t \leq t_0 \text{ and all } \gamma \in (0,1).$$

It follows, by the continuity of F_X and F_Y, that

$$\gamma\overline{F}_X(t) \geq \overline{F}_X(\overline{F}_V^{-1}(\gamma\overline{F}_V(t))) \quad \text{and} \quad \gamma\overline{F}_Y(t) \geq \overline{F}_Y(\overline{F}_V^{-1}(\gamma\overline{F}_V(t))) \quad \text{for all } t \leq t_0 \text{ and all } \gamma \in (0,1).$$

Therefore, for every $\gamma \in (0,1)$,

$$\gamma p\overline{F}_X(t) + \gamma(1-p)\overline{F}_Y(t) \geq p\overline{F}_X(\overline{F}_V^{-1}(\gamma\overline{F}_V(t))) + (1-p)\overline{F}_Y(\overline{F}_V^{-1}(\gamma\overline{F}_V(t))) \quad \text{for all } t \leq t_0;$$

that is,

$$\gamma\overline{F}_W(t) \geq \overline{F}_W(\overline{F}_V^{-1}(\gamma\overline{F}_V(t))) \quad \text{for all } t \leq t_0 \text{ and all } \gamma \in (0,1).$$

By the continuity of F_W we get

$$\overline{F}_W^{-1}(\gamma\overline{F}_W(t)) \leq \overline{F}_V^{-1}(\gamma\overline{F}_V(t)) \quad \text{for all } t \leq t_0 \text{ and all } \gamma \in (0,1);$$

that is, by Proposition 4.1(i), $W \leq_{\rm prl}^{t_0} V$. □

Lemma 4.2. *Let X_1, X_2, Y_1, and Y_2 be random variables with continuous distribution functions, and let W and Z be random variables with distribution functions*

$$F_W = pF_{X_1} + (1-p)F_{X_2} \quad \text{and} \quad F_Z = pF_{Y_1} + (1-p)F_{Y_2},$$

for some $p \in [0,1]$. If there exists a random variable S such that

$$X_1 \leq_{prl}^{t_0} S, \quad X_2 \leq_{prl}^{t_0} S, \quad S \leq_{prl}^{t_0} Y_1, \quad S \leq_{prl}^{t_0} Y_2,$$

then $W \leq_{prl}^{t_0} Z$.

Proof. Since $X_1 \leq_{\rm prl}^{t_0} S$ and $X_2 \leq_{\rm prl}^{t_0} S$, it follows from Lemma 4.1(ii) that $W \leq_{\rm prl}^{t_0} S$. Furthermore, since $S \leq_{\rm prl}^{t_0} Y_1$ and $S \leq_{\rm prl}^{t_0} Y_2$, it follows from Lemma 4.1(i) that $S \leq_{\rm prl}^{t_0} Z$. By the transitivity property of the order $\leq_{\rm prl}^{t_0}$ we get $W \leq_{\rm prl}^{t_0} Z$. □

By repeated application of Lemma 4.2, and convergence arguments, we obtain the following result.

Theorem 4.6. *Let $\{X_\theta,\ \theta \in \Theta\}$ and $\{Y_\theta,\ \theta \in \Theta\}$ be two families of random variables with continuous distribution functions. Let W and Z be random variables with distribution functions given by*

$$F_W(t) = \int_\Theta F_{X_\theta}(t)dH(\theta) \quad \text{and} \quad F_Z(t) = \int_\Theta F_{Y_\theta}(t)dH(\theta), \quad t \in \mathbb{R},$$

where H is some distribution function on Θ. Suppose that there exists a random variable S such that

$$X_\theta \leq_{prl}^{t_0} S \leq_{prl}^{t_0} Y_\theta \quad \text{for all } \theta \in \Theta. \tag{4.8}$$

Then $W \leq_{prl}^{t_0} Z$.

Note that condition (4.8) can be rewritten as

$$X_\theta \leq_{\mathrm{prl}}^{t_0} Y_{\theta'} \quad \text{for all } \theta, \theta' \in \Theta.$$

It is worth noting that results that are similar to Theorem 4.6 hold for the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the mean residual life order (see, respectively, Theorems 1.B.8, 1.B.46, 1.C.15, and 2.A.13 in Shaked and Shanthikumar, 2007).

A special case of Theorem 4.6 is the following result which shows that a random variable, whose distribution is a mixture of two distributions of random variables which are ordered in the sense of the percentile residual life order up to time t_0, is bounded from below and from above, in the percentile residual life order up to time t_0 sense, by these two random variables.

Corollary 4.2. *Let X and Y be two random variables with continuous distribution functions, and let W be a random variable with distribution function*

$$F_W = pF_X + (1-p)F_Y,$$

for some $p \in [0,1]$. If $X \leq_{prl}^{t_0} Y$ then $X \leq_{prl}^{t_0} W \leq_{prl}^{t_0} Y$.

Again, note that similar results hold for the hazard rate order, the likelihood ratio order, and the mean residual life order (see, respectively, Theorems 1.B.22, 1.C.30, and 2.A.18 in Shaked and Shanthikumar, 2007).

The possible preservation of a stochastic order under the formation of coherent systems is a useful property that has important applications in reliability theory (see, for example, Barlow and Proschan, 1975, for the definition and the use of coherent systems). Thus it is of interest to ask whether the percentile residual life orders up to time t_0 are closed under this formation. Boland, El-Neweihi, and Proschan (1994) showed that the hazard rate order is not preserved under the formation of coherent systems. In the next counterexample it is shown that, *for all* $t_0 > 0$, the percentile residual life order up to time t_0 is not closed under this formation. This is shown by considering a parallel system of size 2 whose lifetime is the maximum of the lifetimes of its two components.

Counterexample 4.4. *Let X be an exponential random variable with rate $\lambda > 0$. That is,*

$$F_X(t) = \begin{cases} 0, & t < 0; \\ 1 - e^{-\lambda t}, & t \geq 0. \end{cases}$$

Let Y be a random variable that is degenerate at 0, and let Z be a random variable that is degenerate at 1. Note that $\max\{X, Y\} =_{\mathrm{st}} X$. *Note also that for every $t_0 > 0$, $Y \leq_{prl}^{t_0} Z$ ($Y \leq_{hr} Z$), and, of course, $X \leq_{prl}^{t_0} X$ ($X \leq_{hr} X$). Now we compute*

$$q_{\max\{X,Y\},\gamma}(t) = q_{X,\gamma}(t) = \begin{cases} \frac{-\log(1-\gamma)}{\lambda} - t, & t < 0; \\ \frac{-\log(1-\gamma)}{\lambda}, & t \geq 0, \end{cases}$$

and

$$q_{\max\{X,Z\},\gamma}(t) = \begin{cases} \frac{-\log(1-\gamma)}{\lambda} - t, & t < 1; \\ \frac{-\log(1-\gamma)}{\lambda}, & t \geq 1. \end{cases}$$

It is seen that $\max\{X,Y\} \not\leq^{t_0}_{prl} \max\{X,Z\}$ *(in fact,* $\max\{X,Y\} \geq^{t_0}_{prl} \max\{X,Z\}$ *for every* $t_0 > 0$ *because* $\max\{X,Y\} \geq_{hr} \max\{X,Z\}$*). Thus the percentile residual life order up to time* t_0 *is not closed under the maximum operation.* ◀

For every $t_0 > 0$, the percentile residual life order up to time t_0 is closed under the formation of series systems (that is, under the minimum operation). This is shown in the next theorem.

Theorem 4.7. *Let* $X_1, X_2 \ldots, X_n$ *and* $Y_1, Y_2 \ldots, Y_n$ *be independent random variables with* $X_i \leq^{t_0}_{prl} Y_i$*, for* $i = 1, \ldots, n$*. Then*

$$\min\{X_1, X_2 \ldots, X_n\} \leq^{t_0}_{prl} \min\{Y_1, Y_2 \ldots, Y_n\}. \tag{4.9}$$

Proof. For every $i = 1, \ldots, n$, by Proposition 4.2,

$$X_i \leq^{t_0}_{\text{prl}} Y_i \Leftrightarrow (X_i)_t \leq_{st} (Y_i)_t \text{ for all } t \leq t_0,$$

where $(A_i)_t = [A_i - t | A_i > t]$, for every random variable A.

Since the usual stochastic order is closed under the minimum operation, we have that

$$\min\{(X_1)_t, (X_2)_t \ldots, (X_n)_t\} \leq_{st} \min\{(Y_1)_t, (Y_2)_t \ldots, (Y_n)_t\}.$$

On the other hand, for every $n = 1, 2, \ldots$ it holds that $\min\{(X_1)_t, (X_2)_t \ldots, (X_n)_t\} =_{st} (\min\{X_1, X_2 \ldots, X_n\})_t$ and $\min\{(Y_1)_t, (Y_2)_t \ldots, (Y_n)_t\} =_{st} (\min\{Y_1, Y_2 \ldots, Y_n\})_t$. Then,

$$(\min\{X_1, X_2 \ldots, X_n\})_t \leq_{st} (\min\{Y_1, Y_2 \ldots, Y_n\})_t$$

for all $t \leq t_0$ and, by Proposition 3.2, the claim is true. □

4.4 Some applications

Besides the practical applications we have enumerated on the introduction to the chapter, here we show some other practical features of the new order which may be considered as applications.

As we already explained in Section 2.4, the following result is useful in reliability theory.

Theorem 4.8. *Let* X *and* Y *be two random variables with continuous distributions on interval supports. Let* $t_0 > 0$ *and* $\theta > 0$*. If* $X \leq^{t_0}_{prl} Y$ *then*

$$X(\theta) \leq^{t_0}_{prl} Y(\theta). \tag{4.10}$$

Proof. It is not hard to verify that under the continuity assumptions above we have

$$(\overline{F}^{\theta}_X)^{-1}(u) = \overline{F}^{-1}_X(u^{1/\theta}) \quad \text{and} \quad (\overline{F}^{\theta}_Y)^{-1}(u) = \overline{F}^{-1}_Y(u^{1/\theta}), \quad u \in (0,1),$$

or, equivalently,

$$\overline{F}^{-1}_X(u) = (\overline{F}^{\theta}_X)^{-1}(u^{\theta}) \quad \text{and} \quad \overline{F}^{-1}_Y(u) = (\overline{F}^{\theta}_Y)^{-1}(u^{\theta}), \quad u \in (0,1).$$

Now, by Proposition 4.1(i), $X \leq^{t_0}_{\text{prl}} Y$ means

$$\overline{F}^{-1}_X(\overline{\gamma}\overline{F}_X(t)) \leq \overline{F}^{-1}_Y(\overline{\gamma}\overline{F}_Y(t)) \quad \text{for all } t \leq t_0 \text{ and all } \gamma \in (0,1),$$

that is,

$$(\overline{F}^{\theta}_X)^{-1}(\overline{\gamma}^{\theta}\overline{F}^{\theta}_X(t)) \leq (\overline{F}^{\theta}_Y)^{-1}(\overline{\gamma}^{\theta}\overline{F}^{\theta}_Y(t)) \quad \text{for all } t \leq t_0 \text{ and all } \gamma \in (0,1),$$

and the result follows from Proposition 4.1(i). $\square$

Chapter 5

Aging notions

As we pointed out in Section 1.3, statisticians find it useful to categorize life distributions according to different aging properties. These categories of distributions are useful for modeling situations where items deteriorate with age. Haines and Singpurwalla (1974) and Joe and Proschan (1984a) studied some aspects of the classes of distribution functions with decreasing γ-percentile residual life (DPRL(γ)), $0 < \gamma < 1$. The formal definition of these classes were given in Section 1.3 and will be recalled in Section 5.1.1 in which we will also present some equivalent conditions. In Section 5.1.2 we derive the new properties of these classes by employing results involving the γ-percentile residual life orders that were obtained in Chapter 2.

In Section 5.2.1 we introduce four new aging notions for nonnegative random variables and derive some equivalent conditions. Some characterization results of one of these notions in terms of a stochastic ordering are presented in Section 5.2.2. In Section 5.2.3 we complete some of the results relating the behavior of the hazard rate function and the percentile residual life function given in Launer (1993).

5.1 The γ-DPRL aging notion

5.1.1 Definition

Let X be a random variable, and let u_X be the right endpoint of its support. Recall from Section 2.1 that the γ-percentile residual life function $q_{X,\gamma}$, where γ is some number between 0 and 1, is defined by

$$q_{X,\gamma}(t) = \begin{cases} F_{X_t}^{-1}(\gamma), & t < u_X; \\ 0, & t \geq u_X. \end{cases}$$

It is useful to also recall that $q_{X,\gamma}$ satisfies

$$\overline{F}_X(t + q_{X,\gamma}(t)) = \overline{\gamma}\overline{F}_X(t) \quad \text{for all } t, \tag{5.1}$$

where $\overline{\gamma} = 1 - \gamma$. Also,

$$q_{X,\gamma}(t) = \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) - t, \quad t < u_X. \tag{5.2}$$

Let $0 < \gamma < 1$. A random variable X is said to have (or to be) DPRL(γ) if $q_{X,\gamma}(t)$ is decreasing in t. It is also possible to similarly define the notion of increasing γ-percentile residual life (IPRL(γ)). However, note that with our definition of $q_{X,\gamma}$, in order for a random variable to be IPRL(γ) it is necessary that $u_X = \infty$.

Some useful equivalent conditions for the DPRL(γ) notion are given in the following proposition for absolutely continuous random variables with interval support (which may be finite or infinite). For such random variable X we denote by f_X its density function and by $r_X \equiv f_X/\overline{F}_X$ its hazard rate function.

Proposition 5.1. *Let X be an absolutely continuous random variable with interval support (l_X, u_X). The following conditions are equivalent:*

(i) X *is* DPRL(γ)*;*

(ii) $\overline{\gamma} f_X(t) \le f_X(\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)))$ *for all* $t \in (l_X, u_X)$*;*

(iii) $\overline{\gamma} f_X(\overline{F}_X^{-1}(p)) \le f_X(\overline{F}_X^{-1}(\overline{\gamma}p))$ *for all* $p \in (0,1)$*;*

(iv) $r_X(t) \le r_X(t + q_{X,\gamma}(t))$ *for all* $t \in (l_X, u_X)$.

Proof. Assume (i). Then $q_{X,\gamma}(t)$ is decreasing in $t \in (l_X, u_X)$. Therefore, by differentiating (5.2) we see that

$$0 \ge \frac{d}{dt} q_{X,\gamma}(t) = \frac{\overline{\gamma} f_X(t)}{f_X(\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)))} - 1,$$

and (ii) follows. In fact, the proof shows that (i)$\Longleftrightarrow$(ii).

Next assume (ii). Putting there $t = \overline{F}^{-1}(p)$ we obtain (iii). In fact, the proof shows that (ii)$\Longleftrightarrow$(iii).

Finally, assume (ii) again. For $t \in (l_X, u_X)$, divide the left hand side by $\overline{\gamma}\overline{F}_X(t)$ and the right hand side by $\overline{F}_X(t + q_{X,\gamma}(t))$, which are equal by (5.1). We obtain

$$r_X(t) \le \frac{f_X(\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)))}{\overline{F}_X(t + q_{X,\gamma}(t))} = \frac{f_X(t + q_{X,\gamma}(t))}{\overline{F}_X(t + q_{X,\gamma}(t))},$$

where the last equality follows from (5.2). This gives (iv). In fact, the proof shows that (ii)$\Longleftrightarrow$(iv). $\square$

The equivalence (i)$\Longleftrightarrow$(iv) can be found already in Haines and Singpurwalla (1974) and in Joe and Proschan (1984).

From (iv) it is seen that if r_X is increasing (that is, if X has an increasing hazard rate (IHR)) then X is DPRL(γ) for any $\gamma \in (0,1)$. On the other hand, if X is DPRL(γ) for some $\gamma \in (0,1)$ then it is not necessary that X be IHR. In fact, the following example shows that, given any $\varepsilon > 0$, then, even if X is DPRL(γ) for every $\gamma \geq \varepsilon$, it is not necessary that X be IHR. A related result, with a more positive flavor, will be given later in Proposition 5.3.

Example 5.1. *Let us fix an $\varepsilon \in (0,1)$. Denote*

$$a = (-\log \varepsilon)^{1/2}.$$

Consider a random variable X with the hazard rate function (see Figure 5.1)

$$r_X(t) = \begin{cases} a - t, & 0 \leq t \leq a; \\ t - a, & t > a. \end{cases}$$

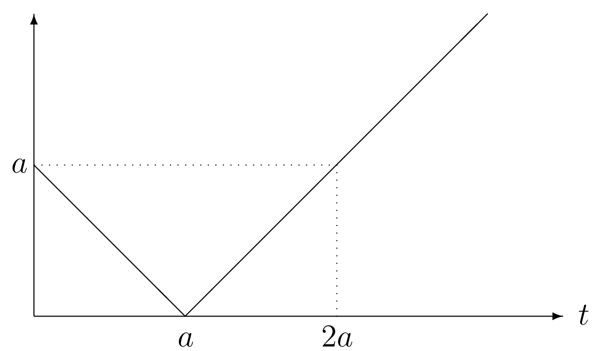

Figure 5.1: Hazard rate function r_X in Example 5.1

A straightforward computation yields the survival function of X:

$$\overline{F}_X(t) = \begin{cases} \exp\{-at + \frac{t^2}{2}\}, & t \leq a; \\ \exp\{-\frac{a^2}{2} - \frac{(t-a)^2}{2}\}, & t > a. \end{cases}$$

Note that

$$\overline{F}(2a) = \exp\{-a^2\} = \varepsilon.$$

From a Remark in page 672 of Joe and Proschan (1984) it follows that X is DPRL(γ) for every $\gamma \geq \varepsilon$. However, obviously, X is not IHR. ◀

From Example 5.1 it is seen that, given any $\gamma \in (0,1)$, it is possible to find a random variable that is DPRL(γ), but that is not DPRL(β) for $\beta < \gamma$.

A natural question to ask now is whether X being DPRL(γ) implies that X is also DPRL(β) for $\beta > \gamma$. In the next example we show that the answer to this question is negative. That is, the following example shows that, given $\gamma \in (0,1)$, it is possible to find a random variable X, and a $\beta \in (\gamma, 1)$, such that X is DPRL(γ) but it is not DPRL(β).

Example 5.2. *Fix an $\gamma \in (0,1)$ and let θ be such that*

$$\theta > \frac{3\log(1-\gamma)}{2\log\left(\frac{-\log(1-\gamma)}{4\pi-\log(1-\gamma)}\right)}; \tag{5.3}$$

it is not hard to verify that the right hand side of (5.3) is positive. Furthermore, let ε be such that

$$\frac{-\log(1-\gamma)(1-(1-\gamma)^{\frac{3}{2\theta}})}{2\pi(1+(1-\gamma)^{\frac{3}{2\theta}})} < \varepsilon \le \frac{-\log(1-\gamma)}{-\log(1-\gamma)+2\pi}; \tag{5.4}$$

it is not hard to verify that, when (5.3) holds, then the left hand side of (5.4) is smaller than the right hand side of (5.4).

Now define

$$k(x) = 1 + \varepsilon \sin\left(\frac{2\pi x}{\log(1-\gamma)}\right), \quad x \in \mathbb{R};$$

and

$$H(t) = (1-t)^\theta \cdot k\left(\log\left[(1-t)^\theta\right]\right), \quad 0 \le t \le 1.$$

Below, first we show that H is a survival function. Second, we show that a random variable X that has the survival function H is DPRL(γ). Finally, we show that there exists a $\beta > \gamma$ such that X is not DPRL(β).

Obviously, $H(0) = 1$ and $H(1) = 0$. If we can find an $\varepsilon > 0$ such that $H(t)$ is decreasing in $0 \le t \le 1$, then it would follow that H is a survival function. In order to identify such an ε, we note that the derivative of k is given by

$$k'(x) = \varepsilon \cos\left(\frac{2\pi x}{\log(1-\gamma)}\right) \cdot \frac{2\pi}{\log(1-\gamma)}, \quad x \in \mathbb{R},$$

and thus the derivative of H is given by

$$H'(t) = -\theta(1-t)^{\theta-1}\left[1 + \varepsilon \sin\left(\frac{2\pi\theta\log(1-t)}{\log(1-\gamma)}\right) - \frac{2\pi\varepsilon}{-\log(1-\gamma)}\cos\left(\frac{2\pi\theta\log(1-t)}{\log(1-\gamma)}\right)\right], \quad 0 \le t \le 1.$$

Therefore H is decreasing if, and only if,

$$\varepsilon\left[\log(1-\gamma)\sin\left(\frac{2\pi\theta\log(1-t)}{\log(1-\gamma)}\right) + 2\pi\cos\left(\frac{2\pi\theta\log(1-t)}{\log(1-\gamma)}\right)\right] \le -\log(1-\gamma), \quad 0 \le t \le 1. \tag{5.5}$$

Since

$$\varepsilon\left[\log(1-\gamma)\sin\left(\frac{2\pi\theta\log(1-t)}{\log(1-\gamma)}\right) + 2\pi\cos\left(\frac{2\pi\theta\log(1-t)}{\log(1-\gamma)}\right)\right] \le \varepsilon(-\log(1-\gamma)+2\pi), \ 0 \le t \le 1,$$

we see that if

$$\varepsilon \leq \frac{-\log(1-\gamma)}{-\log(1-\gamma)+2\pi} \tag{5.6}$$

then (5.5) *holds. But* (5.6) *is the right hand side of* (5.4)*, and therefore H is a survival function.*

Now, let X have the survival function H, and let Y be a random variable with survival function $\overline{F}_Y$ given by

$$\overline{F}_Y(t) = (1-t)^{\theta}, \quad 0 \leq t \leq 1.$$

From Gupta and Langford (1984) we know that $q_{X,\gamma}(t) = q_{Y,\gamma}(t)$ for all t. Computing $q_{Y,\gamma}$, and using the equality $q_{X,\gamma} = q_{Y,\gamma}$, we obtain

$$q_{X,\gamma}(t) = \begin{cases} 1-\overline{\gamma}^{1/\theta}-t, & t<0, \\ (1-\overline{\gamma}^{1/\theta})(1-t), & 0 \leq t < 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Thus, X is DPRL(γ).

For the reminder of this example, let H and $-H'$ be denoted by $\overline{F}_X$ and f_X, respectively. We will now show that there exists a $\beta \in (\gamma, 1)$ such that X is not DPRL(β). Specifically, let $\beta = 1-(1-\gamma)^{\frac{3}{2}}$ ($>\gamma$). We will show that for $t_0 = 1-(1-\gamma)^{\frac{3}{2\theta}}$ we have

$$\overline{\beta} f_X(t_0) > f_X(\overline{F}_X^{-1}(\overline{\beta}\,\overline{F}_X(t_0))), \tag{5.7}$$

and then use Proposition 5.1(ii).

We compute

$$\begin{aligned} \overline{F}_X(t_0) &= \overline{F}_X(1-(1-\gamma)^{\frac{3}{2\theta}}) = (1-\gamma)^{\frac{3}{2}} k(\theta \log\left[(1-\gamma)^{\frac{3}{2\theta}}\right]) \\ &= (1-\gamma)^{\frac{3}{2}} k\big(\tfrac{3}{2}\log(1-\gamma)\big) = (1-\gamma)^{\frac{3}{2}}(1+\varepsilon\sin(3\pi)) = (1-\gamma)^{\frac{3}{2}}. \end{aligned}$$

So

$$\overline{\beta}\,\overline{F}_X(t_0) = (1-\gamma)^3 = (1-\gamma)^3[1+\varepsilon\sin(6\pi)] = \overline{F}_X(1-(1-\gamma)^{\frac{3}{\theta}}).$$

Hence

$$\overline{F}_X^{-1}(\overline{\beta}\,\overline{F}_X(t_0)) = 1-(1-\gamma)^{\frac{3}{\theta}}.$$

So, (5.7) *is equivalent to*

$$\overline{\beta} f_X(1-(1-\gamma)^{\frac{3}{2\theta}}) > f_X(1-(1-\gamma)^{\frac{3}{\gamma}}),$$

and this is equivalent to

$$\begin{aligned} (1-\gamma)^{\frac{3}{2}}\theta(1-\gamma)^{\frac{3(\theta-1)}{2\theta}} &\left[1+\varepsilon\sin(3\pi)+\tfrac{2\pi\varepsilon}{\log(1-\gamma)}\cos(3\pi)\right] \\ &> \theta(1-\gamma)^{\frac{3(\theta-1)}{\theta}}\left[1+\varepsilon\sin(6\pi)+\tfrac{2\pi\varepsilon}{\log(1-\gamma)}\cos(6\pi)\right], \end{aligned}$$

which is equivalent to

$$(1-\gamma)^{\frac{3}{2\theta}}\left[1+\tfrac{2\pi\varepsilon}{-\log(1-\gamma)}\right] > \left[1-\tfrac{2\pi\varepsilon}{-\log(1-\gamma)}\right],$$

which is equivalent to

$$\varepsilon > \frac{-\log(1-\gamma)(1-(1-\gamma)^{\frac{3}{2\theta}})}{2\pi(1+(1-\gamma)^{\frac{3}{2\theta}})}.$$

The last inequality is the left hand side of (5.4)*. So* (5.7) *holds, and therefore X is not DPRL(β).* ◀

The previous example shows that if X is DPRL(γ), it does not necessarily follow that X is DPRL(β) for $\beta > \gamma$. In the next proposition we notice that if the density function of X is decreasing on a specific region of its support, then, if X is DPRL(γ), it does follow that X is DPRL(β) for $\beta > \gamma$.

Proposition 5.2. *Let X be an absolutely continuous random variable with interval support (l_X, u_X), such that $u_X < \infty$, and with density and survival functions f_X and $\overline{F}_X$, respectively. Let $\gamma \in (0,1)$. If X is* DPRL(γ) *and if f_X is increasing on $[\overline{F}_X^{-1}(\overline{\gamma}), u_X]$, then X is* DPRL(β) *for all $\beta > \gamma$.*

Proof. Let $\beta > \gamma$. Then, for all $p \in (0,1)$ we have

$$\begin{aligned}\overline{\beta} f_X(\overline{F}_X^{-1}(p)) &\le \overline{\gamma} f_X(\overline{F}_X^{-1}(p))\\ &\le f_X(\overline{F}_X^{-1}(\overline{\gamma}p))\\ &\le f_X(\overline{F}_X^{-1}(\overline{\beta}p)),\end{aligned}$$

where the second inequality follows from Proposition 5.1(iii), and the last inequality follows from the increasingness of f_X. The stated result now follows from Proposition 5.1(iii). □

Note that if f_X is increasing on its support, then the monotonicity condition on f_X in Proposition 5.2 obviously holds, but this is not a useful observation because if f_X is increasing on its support then X is IHR, and, as we noted after Proposition 5.1, this implies that X is DPRL(γ) for all $\gamma \in (0,1)$.

5.1.2 More properties

We recall the following family of stochastic orders that was recently studied in Chapter 2. Let $0 < \gamma < 1$. Let X and Y be two random variables with percentile residual life functions $q_{X,\gamma}$ and $q_{Y,\gamma}$, respectively. If

$$q_{X,\gamma}(t) \le q_{Y,\gamma}(t) \quad \text{for all } t$$

then X is said to be smaller than Y in the γ-percentile residual life order (denoted as $X \leq_{\gamma\text{-rl}} Y$).

In the following result we provide some characterizations of the DPRL(γ) aging notion in term of the γ-percentile residual life order. Recall the definition of X_t from (1.1):

$$X_t = [X - t | X > t], \quad t < u_X.$$

Theorem 5.1. *Let X be an absolutely continuous random variable with interval support. Then X is* DPRL(γ) *if, and only if, any of the following equivalent conditions holds:*

(i) $X_t \geq_{\gamma\text{-rl}} X_{t'}$ *whenever* $t \leq t' < u_X$*;*

(ii) $X \geq_{\gamma\text{-rl}} X_t$ *whenever* $0 \leq t < u_X$ *(when X is a nonnegative random variable);*

(iii) $X + t \leq_{\gamma\text{-rl}} X + t'$ *whenever* $t \leq t'$*.*

Proof. For all $t < u_X$,

$$\bar{F}_{X_t}(x) = \frac{\bar{F}_X(t+x)}{\bar{F}_X(t)}, \quad x \geq 0.$$

It is easy to verify that

$$q_{X_t,\gamma}(x) = \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t+x)) - (t+x) \quad \text{for all } 0 < x < u_X - t.$$

Now, let $t \leq t' < u_X$. Then $X_t \geq_{\gamma\text{-rl}} X_{t'}$ if, and only if,

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t+x)) - (t+x) \geq \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t'+x)) - (t'+x) \quad \text{for all } x < u_X - t';$$

that is (by (5.2)), $q_{X,\gamma}(t+x) \geq q_{X,\gamma}(t'+x)$ whenever $t + x \leq t' + x < u_X$; that is, $q_{X,\gamma}$ is decreasing. This proves the equivalence of DPRL(γ) and (i).

Next, let $0 \leq t < u_X$. Then $X \geq_{\gamma\text{-rl}} X_t$ if, and only if,

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(x)) - x \geq \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t+x)) - (t+x) \quad \text{for all } x < u_X - t;$$

that is (by (5.2)), $q_{X,\gamma}(x) \geq q_{X,\gamma}(t+x)$ whenever $t + x \leq u_X$; that is, $q_{X,\gamma}$ is decreasing. This proves the equivalence of DPRL(γ) and (ii).

In order to prove the equivalence of DPRL(γ) and (iii), let $t \leq t'$, and denote $a = t' - t$. Then condition (iii) is equivalent to

$$X \leq_{\gamma\text{-rl}} X + a \quad \text{for all } a > 0. \tag{5.8}$$

Now, from (5.2) we have

$$q_{X,\gamma}(t) = \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) - t \quad \text{for all } t < u_X,$$

and, for $a > 0$ we have

$$q_{X+a,\gamma}(t) = \overline{F}_{X+a}^{-1}(\overline{\gamma}\overline{F}_{X+a}(t)) - t = \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t-a)) - t + a = q_{X,\gamma}(t-a) \quad \text{for all } t < u_X + a.$$

That is, condition (5.8) is equivalent to the decreasingness of $q_{X,\gamma}$. □

There are results in the literature that are similar to Theorem 5.1, but which involve aging notions other than DPRL(γ). For example, Theorems 1.A.30, 1.B.38, 3.B.24, 3.B.25, and 4.A.53 in Shaked and Shanthikumar (2007) give similar characterizations for the IHR aging notion. Theorems 2.A.23, 2.B.17, 3.A.56, 3.C.13, and 4.A.51 in Shaked and Shanthikumar (2007), as well as a result in Belzunce, Gao, Hu, and Pellerey (2004), give similar characterizations for the decreasing mean residual life (DMRL) aging notion.

A classical result (Joe and Proschan, 1984b) shows that

$$X \leq_{\gamma\text{-rl}} Y \quad \text{for all } \gamma \in (0,1)$$

if, and only if,

$$X \leq_{\text{hr}} Y, \tag{5.9}$$

where $\leq_{\text{hr}}$ denotes the hazard rate stochastic order (see 1.11). This result was strengthened in the result given in who showed that if X is a continuous random variable, and if for some fixed $\varepsilon \in (0,1)$ we have that

$$X \leq_{\gamma\text{-rl}} Y \quad \text{for all } \gamma \in (0,\varepsilon) \tag{5.10}$$

then (5.9) still holds. Now, suppose that some continuous random variable X is DPRL(γ) for all $\gamma \in (0,\varepsilon)$. From Theorem 5.1(iii) we see that (5.10) holds if we replace X and Y there by $X+t$ and $X+t'$, for any t and t' such that $t \leq t'$. Thus, from (5.9) we get

$$X + t \leq_{\text{hr}} X + t' \quad \text{whenever } t \leq t',$$

which means, by Shaked and Shanthikumar (2007, Theorem 1.B.38(iii)), that X is IHR. We have thus proven the following positive result that may be contrasted with the negative result shown in Example 5.1.

Proposition 5.3. *Let X be a random variable with a continuous distribution function, and let $\varepsilon \in (0,1)$. If X is* DPRL(γ) *for all $\gamma \in (0,\varepsilon)$ then X is* IHR.

Intuitively speaking, the order $\leq_{\gamma\text{-rl}}$ is an order of magnitude in the sense that a "larger" random variable may be expected to be larger with respect to this order. However, Theorem 5.1(iii) shows that that is not always the case. A natural condition under which indeed $X + t'$ is larger than $X + t$ with respect to this order, when $t \leq t'$, is that X is DPRL(γ). The next result highlights the usefulness of the DPRL(γ) notion in a similar situation. The following result is an analog of Theorem 1.B.21 in Shaked and Shanthikumar (2007) which involves the IHR aging notion, and of Theorem 2.A.17 in Shaked and Shanthikumar (2007) which involves the DMRL aging notion.

Theorem 5.2. *Let X be a positive, absolutely continuous,* DPRL(γ) *random variable with interval support. Then*

$$X \le_{\gamma\text{-rl}} aX \quad \text{for all } a > 1. \tag{5.11}$$

Proof. By (5.2),

$$q_{X,\gamma}(t) = \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) - t \quad \text{for all } t < u_X,$$

and

$$q_{aX,\gamma}(t) = \overline{F}_{aX}^{-1}(\overline{\gamma}\overline{F}_{aX}(t)) - t = a\overline{F}_X^{-1}\left(\overline{\gamma}\overline{F}_X\left(\frac{t}{a}\right)\right) - t = aq_{X,\gamma}\left(\frac{t}{a}\right)$$
$$\text{for all } t < u_X \text{ and for all } a > 1.$$

If X is DPRL(γ) then

$$q_{X,\gamma}(t) \le q_{X,\gamma}\left(\frac{t}{a}\right) \le aq_{X,\gamma}\left(\frac{t}{a}\right) = q_{aX,\gamma}(t) \quad \text{for all } 0 < t < u_X \text{ and for all } a > 1,$$

which yields (5.11). $\square$

If X is not DPRL(γ) then (5.11) may not hold. More explicitly, for any $\gamma \in (0,1)$, if X is not DPRL(γ) then (5.11) need not all for any $a > 1$; this is shown in the following example.

Example 5.3. *For a fixed $\gamma \in (0,1)$ and a fixed $a > 1$, let X be a random variable with the following distribution function:*

$$F_X(t) = \begin{cases} 0, & t < 0, \\ (1+\gamma)at, & 0 \le t < \frac{1}{2a}, \\ 1 - \frac{a(1-\gamma)(1-t)}{(2a-1)}, & \frac{1}{2a} \le t < 1. \end{cases}$$

A lengthy straightforward computation yields

$$q_{X,\gamma}(t) = \begin{cases} \frac{\gamma}{a(1+\gamma)} - t, & t < 0, \\ \gamma(\frac{1}{a(1+\gamma)} - t), & 0 \le t < \frac{1}{2a(1+\gamma)}, \\ \big[(1+\gamma)(2a-1) - 1\big]t - \frac{a-1}{a}, & \frac{1}{2a(1+\gamma)} \le t < \frac{1}{2a}, \\ \gamma(1-t), & \frac{1}{2a} \le t < 1, \\ 0, & t \ge 1. \end{cases}$$

The graph of $q_{X,\gamma}(t)$ is shown in Figure 5.2. Obviously X is not DPRL(γ) $\big[q_{X,\gamma}(t)$ is increasing in the interval $\big(\frac{1}{2a(1+\gamma)}, \frac{1}{2a}\big)\big]$. We want to show (see the proof of Theorem 5.2) that

$$aq_{X,\gamma}\left(\frac{\tilde{t}}{a}\right) < q_{X,\gamma}(\tilde{t}) \tag{5.12}$$

for some $\tilde{t} \in (0,1)$*. In order to do that take* $\tilde{t} = \frac{1}{2(1+\gamma)}$*; again, see Figure 5.2. Then*

$$aq_{X,\gamma}\left(\frac{\tilde{t}}{a}\right) = aq_{X,\gamma}\left(\frac{1}{2a(1+\gamma)}\right) = \frac{\gamma}{2(1+\gamma)},$$

and

$$q_{X,\gamma}(\tilde{t}) = \begin{cases} \gamma\left(1 - \frac{1}{2(1+\gamma)}\right) = \frac{\gamma(1+2\gamma)}{2(1+\gamma)}; & \text{if } a \geq 1+\gamma, \\ \frac{2(a^2+1)(1+\gamma)-3a\gamma-4a}{2a(1+\gamma)}; & \text{if } a < 1+\gamma. \end{cases}$$

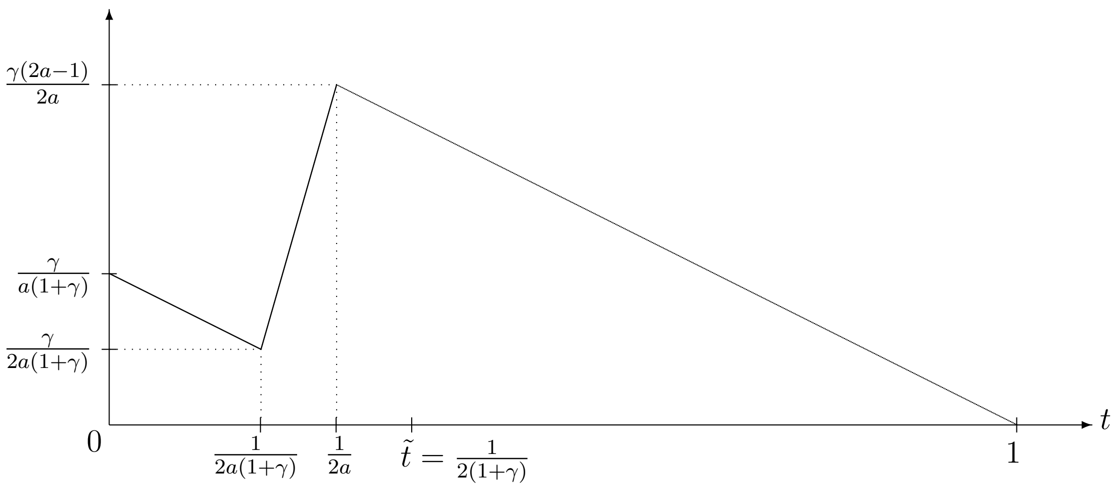

Figure 5.2: Graph of $q_{X,\gamma}(t)$ in Example 5.3

If $a \geq 1+\gamma$ *then*

$$aq_{X,\gamma}\left(\frac{\tilde{t}}{a}\right) = \frac{\gamma}{2(1+\gamma)} < \frac{\gamma(1+2\gamma)}{2(1+\gamma)} = q_{X,\gamma}(\tilde{t}),$$

where the inequality follows from $1 + 2\gamma > 1$*. So inequality* (5.12) *holds in this case.*

On the other hand, if $a < 1+\gamma$ *then a straightforward computation shows that inequality* (5.12) *is equivalent to* $(a-1)^2 > 0$*, which is always true. Therefore* $X \nleq_{\gamma\text{-rl}} aX$*.* ◀

Another situation in which the DPRL(γ) aging notion arises as a natural condition will be described next. The result below (Theorem 5.3), again, indicates a useful property of the order $\leq_{\gamma\text{-rl}}$ when one of the compared random variables is "larger in magnitude" than the other one. The following result from Chapter 2 will be used in the proofs of Theorems 5.3 and 5.4.

Proposition 5.4. *Let* $\{X_\theta,\ \theta \in \Theta\}$ *and* $\{Y_\theta,\ \theta \in \Theta\}$ *be two families of random variables with continuous distribution functions. Let* V *and* W *be random variables with distribution functions given by*

$$F_V(t) = \int_\Theta F_{X_\theta}(t)dH(\theta) \quad \textit{and} \quad F_W(t) = \int_\Theta F_{Y_\theta}(t)dH(\theta), \quad t \in \mathbb{R},$$

where H is some distribution function on Θ. If

$$X_\theta \leq_{\gamma\text{-rl}} Y_{\theta'} \quad \textit{for all } \theta, \theta' \in \Theta, \tag{5.13}$$

then $V \leq_{\gamma\text{-rl}} W$.

The following result is a generalization of the sufficiency part of Theorem 5.1(iii).

Theorem 5.3. *Let X be a continuous* DPRL(γ) *random variable. Let Z be a nonnegative continuous random variable that is independent of X. Then*

$$X \leq_{\gamma\text{-rl}} X + Z. \tag{5.14}$$

Proof. We write

$$F_X(x) = \int_{z=0}^{\infty} F_X(x)\, dF_Z(\theta)$$

and

$$F_{X+Z}(x) = \int_{z=0}^{\infty} F_{X+\theta}(x)\, dF_Z(\theta).$$

Denote $X_\theta = X$ and $Y_\theta = X + \theta$. Now, in Proposition 5.4, take $\Theta = [0, \infty)$ and $H = F_Z$. Then $V = X$ and $W = X + Z$. By Theorem 5.1(iii) we see that (5.13) holds. Therefore the stated result follows from Proposition 5.4. □

It is worthwhile to point out that if X in Theorem 5.3 is not DPRL(γ) then the conclusion of that theorem need not hold. In order to see this, note that Theorem 5.1(iii) actually says that X is DPRL(γ) if, and only if, $X \leq_{\gamma\text{-rl}} X + a$ for every $a \geq 0$. Thus, if X in Theorem 5.3 is not DPRL(γ) then there exists a degenerate Z such that (5.14) does not hold.

The DPRL(γ) aging notion is also useful as a condition under which the order $\leq_{\gamma\text{-rl}}$ is preserved under certain random additions. This is shown next.

Theorem 5.4. *Let X and Y be two* DPRL(γ) *random variables. Let Z be a random variable, independent of X and Y, with support in $[l, u]$, where $-\infty < l < u < \infty$. If $X + u \leq_{\gamma\text{-rl}} Y + l$, then*

$$X + Z \leq_{\gamma\text{-rl}} Y + Z.$$

Proof. Write

$$F_{X+Z}(x) = \int_{\theta=0}^{\infty} F_{X+\theta}(x)\, dF_Z(\theta)$$

and

$$F_{Y+Z}(x) = \int_{\theta=0}^{\infty} F_{Y+\theta}(x)\, dF_Z(\theta).$$

Denote $X_\theta = X + \theta$ and $Y_\theta = Y + \theta$. Take any $\theta, \theta' \in [l, u]$. Then

$$\begin{aligned} X_\theta = X + \theta &\leq_{\gamma\text{-rl}} X + u && \text{(by Theorem 5.1(iii) and } \theta \leq u) \\ &\leq_{\gamma\text{-rl}} Y + l && \text{(by assumption)} \\ &\leq_{\gamma\text{-rl}} Y + \theta' = Y_{\theta'} && \text{(by Theorem 5.1(iii) and } l \leq \theta'); \end{aligned}$$

that is, (5.13) holds for $\Theta = [l, u]$. So, taking $H = F_Z$ in Proposition 5.4, we obtain the stated result. □

5.2 Other aging notions

5.2.1 Definitions

In this section, we introduce four new definitions of aging notions. We will assume that all the variables considered along the section are nonnegative, unless stated otherwise.

Definition 5.1. *Let $t_0 > 0$. A random variable X is said to be decreasing percentile residual life up to time t_0, denoted t_0-DPRL, if its γ-percentile residual life function is decreasing for every $\gamma \in (0, 1)$ and for every $t \leq t_0$. That is,*

$$q_{X,\gamma}(t) \geq q_{X,\gamma}(t^{'}), \qquad \text{for all } t < t^{'} \leq t_0.$$

Analogously, we can define the t_0-IPRL aging notion.

Definition 5.2. *Let $t_0 > 0$. A random variable X is said to be increasing percentile residual life from time t_0 on, denoted IPRL-t_0, if its γ-percentile residual life function is decreasing for every $\gamma \in (0, 1)$ and for every $t \geq t_0$. That is,*

$$q_{X,\gamma}(t) \leq q_{X,\gamma}(t^{'}), \qquad \text{for all } t_0 \leq t < t^{'}.$$

Analogously, we can define the DPRL-t_0 aging notion. Note that, if X is IPRL-t_0, it is necessary $u_X = \infty$.

Some useful equivalent conditions for the t_0-DPRL, the t_0-IPRL, the IPRL-t_0 and the DPRL-t_0 notions are given in the following propositions for absolutely continuous random variables with interval support (which may be finite or infinite). For such random variable X we denote by f_X its density function and by $r_X \equiv f_X/\overline{F}_X$ its hazard rate function.

Proposition 5.5. *Let X be an absolutely continuous random variable with interval support (l_X, u_X). Let $t_0 > 0$. The following conditions are equivalent:*

(i) *X is t_0-DPRL;*

(ii) $\overline{\gamma} f_X(t) \le f_X(\overline{F}_X^{-1}(\overline{\gamma}\,\overline{F}_X(t)))$ *for all* $t \in (l_X, t_0]$ *and all* $\gamma \in (0,1)$*;*

(iii) $\overline{\gamma} f_X(\overline{F}_X^{-1}(p)) \le f_X(\overline{F}_X^{-1}(\overline{\gamma} p))$ *for all* $p \in [\bar{F}_X(t_0), 1)$ *and all* $\gamma \in (0,1)$*;*

(iv) $r_X(t) \le r_X(t + q_{X,\gamma}(t))$ *for all* $t \in (l_X, t_0]$ *and all* $\gamma \in (0,1)$.

Proof. Assume (i). Then $q_{X,\gamma}(t)$ is decreasing in $t \in (l_X, t_0]$ for every $\gamma \in (0,1)$. Therefore, by differentiating $q_{X,\gamma}$ we see that

$$0 \ge \frac{d}{dt} q_{X,\gamma}(t) = \frac{\overline{\gamma} f_X(t)}{f_X(\overline{F}_X^{-1}(\overline{\gamma}\,\overline{F}_X(t)))} - 1,$$

for all $\gamma \in (0,1)$ and $t \le t_0$ and (ii) follows. In fact, the proof shows that (i)$\Longleftrightarrow$(ii).

Next assume (ii). Putting there $t = \overline{F}^{-1}(p)$ we obtain (iii). In fact, the proof shows that (ii)$\Longleftrightarrow$(iii).

Finally, assume (ii) again. For $t \in (l_X, t_0]$ divide the left hand side by $\overline{\gamma}\,\overline{F}_X(t)$ and the right hand side by $\overline{F}_X(t + q_{X,\gamma}(t))$, which are equal by the definition of percentile residual life function. We obtain

$$r_X(t) \le \frac{f_X(\overline{F}_X^{-1}(\overline{\gamma}\,\overline{F}_X(t)))}{\overline{F}_X(t + q_{X,\gamma}(t))} = \frac{f_X(t + q_{X,\gamma}(t))}{\overline{F}_X(t + q_{X,\gamma}(t))},$$

where the last equality follows from the definition of hazard rate function . This gives (iv). In fact, the proof shows that (ii)$\Longleftrightarrow$(iv). □

The equivalence (i)$\Longleftrightarrow$(iv) can be found already in Haines and Singpurwalla (1974) and in Joe and Proschan (1984). The difference here is that we consider all $\gamma \in (0,1)$ and $t \le t_0$.

From (iv) it is seen that if r_X is increasing (that is, if X has an increasing hazard rate (IHR)) then X is t_0-DPRL for any $\gamma \in (0,1)$ and every $t_0 > 0$.

Proposition 5.6. *Let X be an absolutely continuous random variable with interval support (l_X, u_X). Let $t_0 > 0$. The following conditions are equivalent:*

(i) X *is* t_0-IPRL*;*

(ii) $\overline{\gamma} f_X(t) \ge f_X(\overline{F}_X^{-1}(\overline{\gamma}\,\overline{F}_X(t)))$ *for all* $t \in (l_X, t_0]$ *and all* $\gamma \in (0,1)$*;*

(iii) $\overline{\gamma} f_X(\overline{F}_X^{-1}(p)) \ge f_X(\overline{F}_X^{-1}(\overline{\gamma} p))$ *for all* $p \in [\bar{F}_X(t_0), 1)$ *and all* $\gamma \in (0,1)$*;*

(iv) $r_X(t) \ge r_X(t + q_{X,\gamma}(t))$ *for all* $t \in (l_X, t_0]$ *and all* $\gamma \in (0,1)$.

The proof of this proposition is analog to the proof of Proposition 5.5. Analogously, the two following propositions hold.

Proposition 5.7. *Let X be an absolutely continuous random variable with interval support (l_X, u_X). Let $t_0 > 0$. The following conditions are equivalent:*

(i) *X is* IPRL-t_0*;*

(ii) $\overline{\gamma} f_X(t) \geq f_X(\overline{F}_X^{-1}(\overline{\gamma}\,\overline{F}_X(t)))$ *for all $t \geq t_0$ and all $\gamma \in (0,1)$;*

(iii) $\overline{\gamma} f_X(\overline{F}_X^{-1}(p)) \geq f_X(\overline{F}_X^{-1}(\overline{\gamma} p))$ *for all $p \in (0, \bar{F}_X(t_0)]$ and all $\gamma \in (0,1)$;*

(iv) $r_X(t) \geq r_X(t + q_{X,\gamma}(t))$ *for all $t \geq t_0$ and all $\gamma \in (0,1)$.*

Proposition 5.8. *Let $t_0 > 0$ and let X be an absolutely continuous random variable with no flats in (t_0, u_X). The following conditions are equivalent:*

(i) *X is DPRL-t_0;*

(ii) $\bar{\gamma} f(t) \leq f(\bar{F}^{-1}(\bar{\gamma}\bar{F}(t)))$ *for all $t \geq t_0$ and all $\gamma \in (0,1)$;*

(iii) $\bar{\gamma} f(\bar{F}^{-1}(p)) \leq f(\bar{\gamma}\bar{F}(p))$ *for all $p \in (0, \bar{F}_X(t_0)]$ and all $\gamma \in (0,1)$;*

(iv) $r(t) \leq r(t + q_\gamma(t))$ *for all $t \geq t_0$ and all $\gamma \in (0,1)$;*

5.2.2 Characterization of the t_0-DPRL aging notion

In the following result we provide some characterizations of the t_0-DPRL aging notion in terms of the percentile residual life order up to time t_0. Let X be a random variable and let u_X be the right endpoint of its support. Recall the definition of X_t,

$$X_t = [X - t | X > t], \quad t < u_X;$$

whose survival function is given by

$$\bar{F}_{X_t}(x) = \frac{\bar{F}_X(t+x)}{\bar{F}_X(t)}, \quad x \geq 0.$$

Theorem 5.5. *Let X be an absolutely continuous random variable with interval support and $t_0 > 0$. Then X is t_0-DPRL if, and only if, any of the following equivalent conditions holds:*

(i) $X_t \geq_{prl}^{t_0^*} X_{t'}$ *whenever $t \leq t'$ and $t_0^* = t_0 - t'$;*

(ii) $X \geq_{prl}^{t_0^*} X_t$ *whenever $0 \leq t \leq t_0$ and $t_0^* = t_0 - t$ (when X is a nonnegative random variable);*

(iii) $X + t \leq_{prl}^{t_0} X + t'$ *whenever $t \leq t'$.*

Proof. From the definition of the percentile residual life function it is easy to verify that

$$q_{X_t,\gamma}(x) = \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t+x)) - (t+x) \quad \text{for all } 0 < x < u_X - t.$$

Now, let $t \le t'$. Then $X_t \ge_{prl}^{t_0^*} X_{t'}$ if, and only if,

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t+x)) - (t+x) \ge \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t'+x)) - (t'+x) \quad \text{for all } x \le t_0^*;$$

that is, $q_{X,\gamma}(t+x) \ge q_{X,\gamma}(t'+x)$ whenever $t+x \le t'+x \le t'+t_0^*$; that is, $q_{X,\gamma}$ is decreasing for all $\gamma \in (0,1)$ and all $t \le t' + t_0^* = t_0$. This proves the equivalence of t_0-DPRL and (i).

Next, let $0 \le t \le t_0$. Then $X \ge_{prl}^{t_0^*} X_t$ if, and only if,

$$\overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(x)) - x \ge \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t+x)) - (t+x) \quad \text{for all } x \le t_0^*;$$

that is, $q_{X,\gamma}(x) \ge q_{X,\gamma}(t+x)$ whenever $x \le t+x \le t_0$; that is, $q_{X,\gamma}$ is decreasing for all $\gamma \in (0,1)$ and all $t \le t + t_0^* = t_0$. This proves the equivalence of t_0-DPRL and (ii).

In order to prove the equivalence of DPRL(γ) and (iii), let $t \le t'$, and denote $a = t' - t$. Then condition (iii) is equivalent to

$$X \le_{prl}^{t_0} X + a \quad \text{for all } a > 0. \tag{5.15}$$

Now, from the definition of percentile residual life function we have

$$q_{X,\gamma}(t) = \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) - t \quad \text{for all } t < u_X \text{ and all } \gamma \in (0,1),$$

and, for $a > 0$ we have

$$q_{X+a,\gamma}(t) = \overline{F}_{X+a}^{-1}(\overline{\gamma}\overline{F}_{X+a}(t)) - t = \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t-a)) - t + a = q_{X,\gamma}(t-a) \quad \text{for all } t < u_X + a.$$

That is, condition (5.15) is equivalent to the decreasingness of $q_{X,\gamma}$ for all $\gamma \in (0,1)$ and $t \le t_0$. □

In the literature there are results that are similar to Theorem 5.5, but which involve aging notions other than t_0-DPRL. See Theorem 5.1 in this document and the paragraph which follows its proof for a list of references.

Intuitively speaking, the order $\le_{prl}^{t_0}$ is an order of magnitude in the sense that a "larger" random variable may be expected to be larger with respect to this order. However, Theorem 5.5 (iii) shows that that is not always the case. A natural condition under which indeed $X+t'$ is larger than $X+t$ with respect to this order, when $t \le t' \le t_0$, is that X is t_0-DPRL. The next result highlights the usefulness of the t_0-DPRL notion in a similar situation. The following result is an analog of Theorem 1.B.21 in Shaked and Shanthikumar (2007) which involves the IHR aging notion, of Theorem 2.A.17 in Shaked and Shanthikumar (2007) which involves the DMRL aging notion, and of Theorem 5.2 which envolves the DPRL(γ) aging notion.

Theorem 5.6. *Let $t_0 > 0$. Let X be an absolutely continuous random variable with interval support, If X is t_0-DPRL, then*

$$X \leq_{prl}^{t_0} aX \quad \textit{for all } a > 1. \tag{5.16}$$

Proof. We know that

$$q_{X,\gamma}(t) = \overline{F}_X^{-1}(\overline{\gamma}\overline{F}_X(t)) - t \quad \text{for all } t < u_X,$$

and

$$q_{aX,\gamma}(t) = \overline{F}_{aX}^{-1}(\overline{\gamma}\overline{F}_{aX}(t)) - t = a\overline{F}_X^{-1}\left(\overline{\gamma}\overline{F}_X\left(\frac{t}{a}\right)\right) - t = aq_{X,\gamma}\left(\frac{t}{a}\right)$$
$$\text{for all } t < u_X \text{ and for all } a > 1.$$

If X is t_0-DPRL then

$$q_{X,\gamma}(t) \leq q_{X,\gamma}\left(\frac{t}{a}\right) \leq aq_{X,\gamma}\left(\frac{t}{a}\right) = q_{aX,\gamma}(t) \quad \text{for all } 0 < t \leq t_0 \text{ and for all } a > 1,$$

which yields (5.16). □

5.2.3 Necessary conditions for BT and UBT distributions

We recall the definition of bathtub distributions and upside-down bathtub distributions, already given in Section 1.3 .

Definition 5.3. *Let X be an absolutely continuous random variable with hazard rate function r_X continuous. Then X has a bathtub distribution (BT distribution) if there exists a $t_1 \leq t_2$ such that*

(i) *$r_X(t)$ is strictly decreasing for $t < t_1$,*

(ii) *$r_X(t)$ is a constant for $t_1 \leq t \leq t_2$, and*

(iii) *$r_X(t)$ is strictly increasing for $t > t_2$.*

Definition 5.4. *Let X be an absolutely continuous random variable with hazard rate function r_X continuous. Then X has an upside-down bathtub distribution (UBT distribution) if there exists a $t_1 \leq t_2$ such that*

(i) *$r_X(t)$ is strictly increasing for $t < t_1$,*

(ii) *$r_X(t)$ is a constant for $t_1 \leq t \leq t_2$, and*

(iii) *$r_X(t)$ is strictly decreasing for $t > t_2$.*

In Launer (1993) some results relating the behavior of the hazard rate function and the percentile residual life function are given. He states and illustrates how those relationships can be useful for studying the behavior of the empirical hazard rate function. In this paper he shows that the maximum of the γ-percentile residual life function precedes in time the minimum of the hazard rate (providing a minimum exists) and that the minimum of the γ-percentile residual life function precedes the maximum of the hazard rate. This set of relationships forms the basis for computational procedures in that paper.

The determination of the time at which the γ-percentile residual life function is a maximum can be important in fixing product warranty. For example, product burn-in could be used to eliminate the units which fail early, and thus, maximize the reliability of the remaining product.

In this section we complete the study carried out by Launer (1993), providing some new results in terms of the aging notions defined in Section 5.2.1.

First of all, we introduce the following propositions which show how the conditions t_0-DPRL and IPRL-t_0 have implications on the behavior of the hazard rate function.

Proposition 5.9. *Let $t_0 > 0$ and X be an absolutely continuous random variable with hazard rate r_X. Then, X is t_0-DPRL if, and only if, $r_X(t) \leq r_X(t^{'})$, for all $t < t^{'}$, $t \leq t_0$.*

Proof. In order to prove the conclusion of the theorem, consider $0 < t \leq t_0 < u_X$ and $t^{'} > t$. Since X is a nonnegative random variable, we can write $t^{'} = t + q_{X,\gamma}(t)$ where $\gamma = 1 - \frac{\bar{F}_X(t^{'})}{\bar{F}_X(t)} \in (0,1)$. That is,

$$t' = t + q_{X,\gamma}(t) = t + \bar{F}_X^{-1}(\bar{\gamma}\bar{F}_X(t)) - t = \bar{F}_X^{-1}(\bar{\gamma}\bar{F}_X(t)) = t'.$$

Therefore,

$$r_X(t) \leq r_X(t^{'}) \Leftrightarrow r_X(t) \leq r_X(t + q_{X,\gamma}(t)) \tag{5.17}$$

and, by Proposition 5.5(iv), the right side of equation (5.17) is equivalent to X being t_0-DPRL. □

Analogously, the three following results hold.

Proposition 5.10. *Let $t_0 > 0$ and X be an absolutely continuous random variable with hazard rate r_X. Then, X is t_0-IPRL if, and only if, $r_X(t) \geq r_X(t^{'})$, for all $t < t^{'}$, $t \leq t_0$.*

Proposition 5.11. *Let $t_0 > 0$ and X be an absolutely continuous random variable with hazard rate r_X. Then, X is IPRL-t_0 if, and only if, $r_X(t) \geq r_X(t^{'})$, for all $t < t^{'}$, $t \geq t_0$.*

Proposition 5.12. *Let $t_0 > 0$ and X be an absolutely continuous random variable with hazard rate r_X. Then, X is DPRL-t_0 if, and only if, $r_X(t) \leq r_X(t^{'})$, for all $t < t^{'}$, $t \geq t_0$.*

Launer (1993) stated the following result that gives necessary conditions for a special kind of bathtub distributions.

Theorem 5.7. *Let X be an absolutely continuous random variable with hazard rate function r_X. Let $t_* > 0$ be such that $r_X(0) = r_X(t_*)$. If r_X has a bathtub shape, there is a minimum $\gamma = \gamma_*$ for which $q_{X,\gamma}(t)$ is a decreasing function of t, for $\gamma > \gamma_*$. For $\gamma \leq \gamma_*$, however, $q_{X,\gamma}(t)$ attains a maximum for some $t > 0$.*

We give some results in terms of the aging notions defined on Section 5.2.1 and that complete the study carried out by Launer (1993). First, let us to introduce the following notation:

$$\begin{aligned} t_1^* &= \max\{t : X \text{ is } t\text{-DPRL}\}, \\ t_2^* &= \min\{t : X \text{ is IPRL-}t\}, \\ t_3^* &= \max\{t : X \text{ is } t\text{-IPRL}\}, \\ t_4^* &= \min\{t : X \text{ is DPRL-}t\}. \end{aligned}$$

Then, Theorem 5.7 is a particular case of the following result.

Theorem 5.8. *Let X be an absolutely continuous random variable with hazard rate function r_X. If r_X has a bathtub shape, then X is DPRL-t_4^* with $t_4^* \in (l_X, u_X)$. Besides, t_4^* is the point where r_X attains the minimum value.*

Proof. Since X is BT, there exits t_2 such that r_X is strictly increasing for $t > t_2$. Then, by Proposition 5.12, X is DPRL-t_4^* and $t_4^* = \min\{t : r_X(t) \text{ is increasing}\}$. □

Theorem 5.9. *Let X be an absolutely continuous random variable with hazard rate function r_X. Let $t_* > 0$ be such that $r_X(u_X) = r_X(t_*)$. If r_X has a bathtub shape, then X is DPRL-t_4^* and t_3^*-IPRL. Besides, t_4^* is the point where r_X attains the minimum value and $t_3^* = t_*$.*

Proof. Since X is BT, there exits t_2 such that r_X is strictly increasing for $t > t_2$. Then, by Proposition 5.12, X is DPRL-t_4^* and $t_4^* = \min\{t : r_X(t) \text{ is increasing}\}$. Besides, there exits t_1 such that r_X is strictly decreasing for $t < t_1$. Then, by Proposition 5.10, X is t_3^*-IPRL.

Obviously, $t_3^* \leq t_4^*$. Besides, since X is t_3^*-IPRL, by Proposition 5.6,

$$r_X(t) \geq r_X(t + q_{X,\gamma}(t)) \text{ for all } t \in (l_X, t_3^*] \text{ and all } \gamma \in (0, 1).$$

In particular, this inequality holds for $t = t_3^*$ and $\gamma = 1$, that is,

$$r_X(t_3^*) \geq r_X(t_3^* + q_{X,1}(t_3^*)) = r_X(u_X) = r_X(t_*). \tag{5.18}$$

And, since t_3^*, by definition, is the maximum value that verifies (5.18), this proofs that $t_* = t_3^*$. □

Remark 5.1. *Notice that X is IHR if $t_1^* = u_X$ and $t_4^* = l_X$. Analogously, X is DFR if $t_2^* = u_X$ and $t_3^* = l_X$.*

The following results give a necessary conditions for upside-down bathtub distributions.

Theorem 5.10. *Let X be an absolutely continuous random variable with hazard rate function r_X. If r_X has a upside-down bathtub shape, then X is IPRL-t_2^* with $t_2^* \in (l_X, u_X)$.*

Theorem 5.11. *Let X be an absolutely continuous random variable with hazard rate function r_X. Let $t_* > 0$ be such that $r_X(u_X) = r_X(t_*)$. If r_X has a upside-down bathtub shape, then X is t_1^*-DPRL and IPRL-t_2^*.*

Additionally, the two following results hold.

Theorem 5.12. *Let X be an absolutely continuous random variable with hazard rate function r_X. If X is t_1^*-DPRL and DPRL-t_4^*, then r_X has, at least, one maximum value.*

Theorem 5.13. *Let X be an absolutely continuous random variable with hazard rate function r_X. If X is t_3^*-IPRL and DPRL-t_4^*, then r_X has, at least, one minimum value.*

Chapter 6

Confidence bands for ordering percentile residual life functions

In this chapter we present a nonparametric method for constructing confidence bands for the difference of two percentile residual life functions. Given two random samples, we estimate the difference of their percentile residual life functions and its bootstrapped counterparts and apply statistical depth as a criteria for constructing the bands. These bands evidence whether two random variables are close with respect to the percentile residual life order or not. A simulation study support the results. We also present a application to real data.

This chapter is organized as follows. The problem is introduced in Section 6.1. In Section 6.2 we give an overview of some definitions and basic concepts on statistical depth. The methodology for constructing the confidence bands is described in Section 6.3. The practical performances of the bands are evaluated through simulation in Section 6.4. Finally, we show some applications to real data.

6.1 Description of the problem

Given the advantages of the percentile residual life orders, specially in practical situations, it is convenient to develop an statistical tool to test whether two independent random samples have underlying random variables which are close with respect to a γ-percentile residual life order or not, where $0 < \gamma < 1$.

In this work, we present a nonparametric method for constructing confidence bands for the difference of two percentile residual life functions. This confidence bands provide us with evidence of whether two random variables are close with respect to some percentile residual life order or not. These bands do not only allow us to compare the whole functions, but also in a given interval.

6.2 Definition and basic concepts on statistical depth

The analysis of functional data is one of the topics that, within the field of statistics, is receiving a steady increasing attention in recent years (see, for example Ramsay and Siverman (2005)). In particular, a robust methodology is important to study curves, which are the output of experiments in applied statistics. A natural tool to analyze functional data aspects is the idea of statistical depth. It has been introduced to measure the 'centrality' or the 'outlyingness' of an observation with respect to a given dataset or a population distribution.

The notion of depth was first considered for multivariate data to generalize order statistics, ranks, and medians to higher dimensions. Several depth definitions for multivariate data have been proposed and analyzed by Mahalanobis (1936), Tukey (1975), Oja (1983), Liu (1990), Singh (1991), Fraiman and Meloche (1999), Vardi and Zhang (2000), Koshevoy and Mosler (1997) and Zuo (2003).

Liu (1990) presented four four desirable properties that an ideal depth function should possess:

(i) AFFINE INVARIANCE. The depth of a point $x \in \mathbb{R}^d$ should not depend on the underlying coordinate system or, in particular, on the scales of the underlying measurements.

(ii) MAXIMALITY AT CENTER. For a distribution having a uniquely defined 'center' (e.g., the point of symmetry with respect to some notion of symmetry), the depth function should attain a maximum value at this center.

(iii) MONOTONICITY RELATIVE TO DEEPEST POINT. As a point $x \in \mathbb{R}^d$ moves away from the 'deepest point' (the point at which the depth function attains maximum value) along any fixed ray through the center, the depth at x should decrease monotonically.

(iv) VANISHING AT INFINITY. The depth of a point x should approach to zero as $||x||$ approaches to infinity.

Direct generalization of current multivariate depths to functional data often leads to either depths that are computationally intractable or depths that do not take into account some natural properties of the functions, such as shape. For that reason several specific definitions of depth for functional data were introduces. See for example, Vardi and Zhang (2000), Fraiman and Muniz (2001), López-Pintado and Romo (2005), Cuevas, Febrero and Fraiman (2007) and López-Pintado and Romo (2009). The definition of depth for curves provides us with a criteria to order the sample curves from the center-outward (from the deepest to the most extreme).

Here we recall the definition of band depth introduced in López-Pintado and Romo (2009). The band depth follows a graph-based approach.

Let $x_1(t), \cdots, x_n(t)$ be a collection of real functions. Although the following ideas can be introduced for more general functional observations, we will restrict the exposition to functions in the space $C(I)$ of real continuous functions on the compact interval I. Let the graph of a function x be the subset of the plane $G(x) = \{(t, x(t)) : t \in I\}$ and let the band in $\mathbb{R}^2$ delimited by the curves $x_{i_1}, ...x_{i_k}$ be

$$B(x_{i_1}, ...x_{i_k}) = \{(t, y) : t \in I, \min_{r=1,...,k} x_{i_r}(t) \leq y \leq \max_{r=1,...,k} x_{i_r}(t)\} =$$

$$= \{(t, y) : t \in I, y = \beta \min_{r=1,...,k} x_{i_r}(t) + (1 - \beta) \max_{r=1,...,k} x_{i_r}(t), \beta \in [0, 1]\}.$$

Now, for any function x in $x_1, \cdots, x_n$, and a fixed j value with $2 \leq j \leq n$, the quantity

$$BD_n^{(j)}(x) = \binom{n}{j}^{-1} \sum_{1 \leq i_1 < i_2 < ... < i_j \leq n} I_{\{G(x) \subset B(x_{i_1}, ...x_{i_j})\}},$$

expresses the proportion of bands $B(x_{i_1}, ...x_{i_j})$ determined by j different curves $x_{i_1}, ...x_{i_j}$ containing the whole graph of x. ($I_{\{A\}}$ is one if A is true, and zero otherwise).

Let J be a fixed value with $2 \leq J \leq n$. For functions $x_1, ...x_n$, the *band depth* of any of these curves x is

$$BD_{n,J}(x) = \sum_{j=2}^{J} BD_n^{(j)}(x).$$

If $X_1, \cdots, X_J$ are independent copies of the stochastic process X generating the observations $x_1, \cdots, x_n$, the corresponding population versions are $BD^{(j)}(x, P) = P\{G(x) \subset B(X_1, \cdots, X_j)\}$ and

$$BD_J(x, P) = \sum_{j=2}^{J} BD^{(j)}(x, P) = \sum_{j=2}^{J} P(G(x) \subset B(X_1, \cdots, X_j)).$$

López-Pintado and Romo (2009) recommend considering the definition of band depth with $J = 3$ for several reasons: (1) when J is larger than 3 the index $BD_{n,J}$ can be computationally intensive, (2) bands corresponding to large values of J do not resemble the shape of any of the curves from the sample, (3) the band depth induced order is very stable in J, and (4) the band depth with $J = 2$ is the easiest to compute but, if two curves cross over, the band delimited by them is degenerated in a point and, with probability one, no other curve will be inside this band.

Instead of considering the indicator function, a more flexible definition was introduced by measuring the set where the function is inside the band. For any of the functions x in $x_1, \cdots, x_n$ and for $2 \leq j \leq n$, let

$$A_j \equiv A(x; x_{i_1}, \cdots, x_{i_j}) \equiv \{t \in I : \min_{r=i_1,\cdots,i_j} x_r(t) \leq x(t) \leq \max_{r=i_1,\cdots,i_j} x_r(t)\}$$

be the set where the function x is in the band determined by the observations $x_{i_1}, \cdots, x_{i_j}$. If λ is the Lebesgue measure on I, $\lambda_r(A_j(x)) = \lambda(A_j(x))/\lambda(I)$ gives the 'proportion of time' that x is in the band. Now, for $2 \leq j \leq n$,

$$MBD_n^{(j)}(x) = \binom{n}{j}^{-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq n} \lambda_r(A(x; x_{i_1}, \cdots, x_{i_j}))$$

is a more flexible version of $BD_n^j(x)$: if x is always inside the band, the value $\lambda_r(A_j(x))$ is one as in the previous notion of depth.

Let J be a fixed value with $2 \leq J \leq n$. For functions $x_1, \cdots, x_n$, the *modified band depth* of any of these curves x is

$$MDB_{n,J}(x) = \sum_{j=2}^{J} MDB_n^J(x).$$

The population version of the modified band depth is $MDB_J(x) = \sum_{j=2}^{J} MDB^{(j)}(x)$, where $MDB^{(j)}(x) = E[\lambda_r(A(x; X_1, \cdots, X_j))]$.

It is straightforward to check that in the univariate case, the band depth and the modified band depth coincide. Moreover, the ordering induced in that case does not depend on J.

López-Pintado and Romo (2009) recommend considering the definition of modified band depth with $J = 2$ because it is computationally fast and the order induced is very stable in J, even if many curves from the sample cross over.

These notions satisfy the usual depth properties except affine invariance, which is not natural for functional data. They have also established the uniform consistency of the sample band depth in the finite and functional case. Robustness of these depths was illustrated with a simulation study and several real examples.

6.3 Our methodology

Let X be a random variable and let X_1, X_2,...,X_n be an i.i.d. random sample from X. Let $F_{X,n}$ be the empirical distribution function of X. The sample or empirical γ-percentile residual life of X at $t < u_X$ is

$$q_{X,n,\gamma}(t) = Q_n(\gamma + \bar{\gamma} F_n(t)) - t, \qquad 0 < \gamma < 1,$$

where Q_n is the sample quantile function given by:

$$\begin{array}{lll} Q_n(x) = X_k & \text{if} & \frac{(k-1)}{n} < x \leq \frac{k}{n} \qquad (k = 1, ..., n), \\ Q_n(x) = X_1 & \text{if} & x = 0. \end{array}$$

Section 2.3.1 of Serfling (1980) and the Glivenko-Cantelli theorem for F_n implies that

$$q_{X,n,\gamma}(t) \to q_{X,\gamma}(t), \tag{6.1}$$

almost surely as $n \to \infty$. That is, $q_{X,n,\gamma}(t)$ is a pointwise strongly consistent estimator of $q_{X,\gamma}(t)$. See, for example, Csörgő and Csörgő (1987).

Now, let Y be another random variable and let Y_1, Y_2,...,Y_m be an i.i.d. random sample from Y. Given $\gamma \in (0,1)$, plots of the empirical γ-percentile residual life functions of X and Y can give an indication of the plausibility of whether the γ-percentile residual life functions of X and Y are ordered; that is, if X and Y are ordered according to the γ-percentile residual life order. By equation (6.1),

$$q_{Y,m,\gamma}(t) - q_{X,n,\gamma}(t) \to q_{Y,\gamma}(t) - q_{X,\gamma}(t),$$

almost surely as $n, m \to \infty$.

For constructing the band we follow a three-step algorithm. Let B be the bootstrap size, $\alpha \in (0,1)$ the confidence level and $\gamma \in (0,1)$ the percentile.

(i) Consider the bootstrap replications from $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$. We denote them by $X_1^{*b}, \cdots, X_n^{*b}$ and $Y_1^{*b}, \cdots, Y_m^{*b}$, respectively; for $b = 1, \cdots, B$.

(ii) For every $b = 1, \cdots, B$, compute the empirical γ-percentile residual life functions which is associated to $X_1^{*b}, \cdots, X_n^{*b}$ and $Y_1^{*b}, \cdots, Y_m^{*b}$. We denote them by $q^{*b}_{X,n,\gamma}$ and $q^{*b}_{Y,m,\gamma}$, respectively. Then consider
$$q_b^* = q^{*b}_{Y,m,\gamma} - q^{*b}_{X,n,\gamma}.$$

(iii) Order the sample curves q_b^*, $b = 1, \ldots, B$, from inner to outer using any concept of depth for curves and take the band given by the $(1-\alpha) \cdot 100\%$ deepest curves.

The convex hull for these $(1-\alpha) \cdot 100\%$ deepest curves constitutes the confidence band.

Remark 6.1. *If there exist censored data in the sample, we can consider the percentile residual life estimator under censoring proposed in Csörgő (1987) and described in Section 1.1, and the bootstrap mechanism follows the approach proposed by Efron (1981).*

6.4 A simulation study

A simulation study has been carried out in order to evaluate the performance and to illustrate the consistence of our methodology. As we will see, the bootstrap procedures play a central role in the methods studied here. The computer codes written in R are available from the author. The details of the simulation study are explained in the next paragraphs.

The considered sampling models

We have examined several examples through simulation. The random variables we have considered follow Pareto distributions. The reason is, that for any $\gamma \in (0,1)$, the γ-percentile residual life function of a Pareto distribution is a line. Therefore, if we compare the γ-percentile residual life functions of two Pareto distributions there are only two possible situations: either their γ-percentile residual life functions are parallel or they cross. In the first case we can conclude that X and Y are ordered in the γ-percentile residual life order and in the second case they are not.

Recall from Example 2.1 that, if X have the Pareto distribution,

$$F_X(t) = 1 - \Big(\frac{\rho}{\rho+t}\Big)^{\nu}, \quad t \geq 0,$$

where $\rho > 0$ and $\nu > 0$. Besides, for any $\gamma \in (0,1)$,

$$q_{X,\gamma}(t) = \begin{cases} ((1-\gamma)^{-1/\nu}-1)\rho - t, & t < 0; \\ ((1-\gamma)^{-1/\nu}-1)(\rho+t), & t \geq 0. \end{cases}$$

That is, if X follows a Pareto distribution, its percentile residual life functions are lines.

Let Y follow a Pareto distribution whose distribution function is given by

$$F_Y(t) = 1 - \Big(\frac{\delta}{\delta+t}\Big)^{\mu}, \quad t \geq 0,$$

where $\delta > 0$ and $\mu > 0$. Then,

$$X \leq_{\alpha\text{-rl}} Y \Longleftrightarrow \begin{cases} \mu \leq \nu \quad \text{and} \\ \frac{(1-\alpha)^{-1/\nu}-1}{(1-\alpha)^{-1/\mu}-1} \leq \frac{\delta}{\gamma}. \end{cases}$$ ◀

It is straightforward to check that if X and Y have Pareto distribution, then

$$X \leq_{\gamma-rl} Y \Leftrightarrow X \leq_{\beta-rl} Y, \text{ for all } \beta \in (0,1) \Leftrightarrow X \leq_{hr} Y,$$

where $\leq_{hr}$ denotes the hazard rate order. The last equivalence follows from Theorem 7.

We have considered the following examples. The pairs of variables from Example 6.1 to Example 6.6 are ordered with respect to any percentile residual life order. In particular, they are ordered with respect to the median residual life order. The pairs of variables from Example 6.7 to Example 6.10 are not ordered with respect to any percentile residual life order. In particular, they are not ordered with respect to the median residual life order.

Example 6.1.

$$\begin{cases} X & \sim Pareto(\rho = 10, \nu = 10) \\ Y_1 & \sim Pareto(20, 10) \end{cases}$$

Example 6.2.

$$\begin{cases} X & \sim Pareto(10, 10) \\ Y_2 & \sim Pareto(40, 10) \end{cases}$$

Example 6.3.

$$\begin{cases} X & \sim Pareto(10, 10) \\ Y_3 & \sim Pareto(60, 10) \end{cases}$$

Example 6.4.

$$\begin{cases} X & \sim Pareto(10, 10) \\ Y_4 & \sim Pareto(80, 10) \end{cases}$$

Example 6.5.

$$\begin{cases} X & \sim Pareto(10, 10) \\ Y_5 & \sim Pareto(100, 10) \end{cases}$$

Example 6.6.

$$\begin{cases} X & \sim Pareto(10, 10) \\ Y_6 & \sim Pareto(110, 10) \end{cases}$$

The median residual life functions of the variables from Example 6.1 to Example 6.6 are represented in Figure 6.1. It can be seen that the distance between the median residual life function of X and the median residual life function of Y_i, $i = 1, \cdots, 6$ increases as i increases.

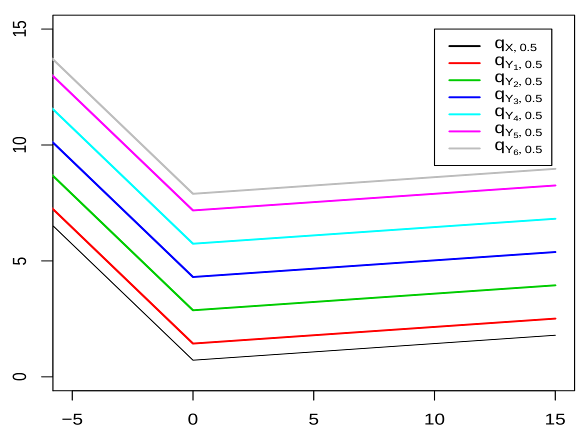

Figure 6.1: Median residual life functions of the variables from Example 6.1 to Example 6.6

Example 6.7.

$$\begin{cases} X_7 & \sim Pareto(10,10) \\ Y_7 & \sim Pareto(1,5) \end{cases}$$

Example 6.8.

$$\begin{cases} X_8 & \sim Pareto(20,5) \\ Y_8 & \sim Pareto(70,10) \end{cases}$$

Example 6.9.

$$\begin{cases} X_9 & \sim Pareto(160,20) \\ Y_9 & \sim Pareto(70,10) \end{cases}$$

Example 6.10.

$$\begin{cases} X_{10} & \sim Pareto(10,10) \\ Y_{10} & \sim Pareto(20,15) \end{cases}$$

The simulation mechanism

We have chosen the modified band depth with $J = 2$ proposed by López-Pintado and Romo (2009) and recalled in Section 6.2 because, contrary to most of other definitions of depth, this depth is not computationally intensive. However, any other definition of depth for curves can be considered.

The simulation results

The 90%-confidence bands for the difference of the median residual life functions of the variables in the first six examples are shown in Figure 6.2 to Figure 6.7. The corresponding 95%-confidence bands are shown in Figure 6.8 to Figure 6.13.

In the left graph of every figure, the estimations of the difference of the median residual life functions together with the confidence band is shown. In the right graph, only the band is shown.

The 90%-confidence bands for the difference of the median residual life functions of the variables in the last four examples are shown in Figure 6.14 to Figure 6.17.

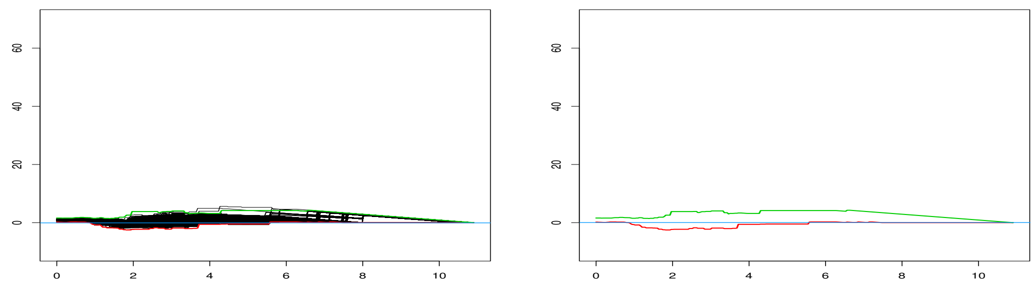

Figure 6.2: 90%-confidence band for $q_{Y_1,0.5} - q_{X,0.5}$

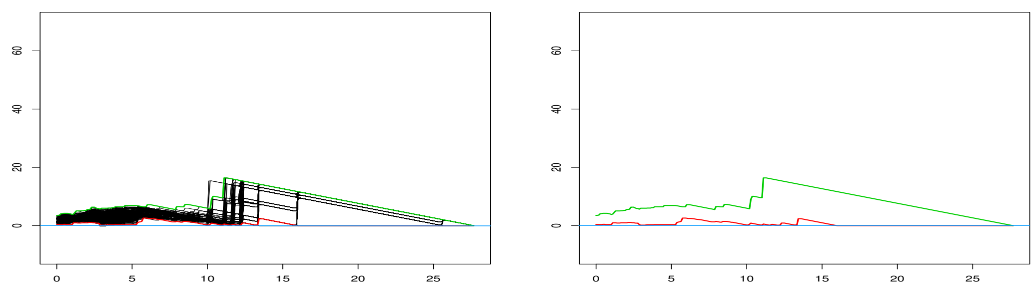

Figure 6.3: 90%-confidence band for $q_{Y_2,0.5} - q_{X,0.5}$

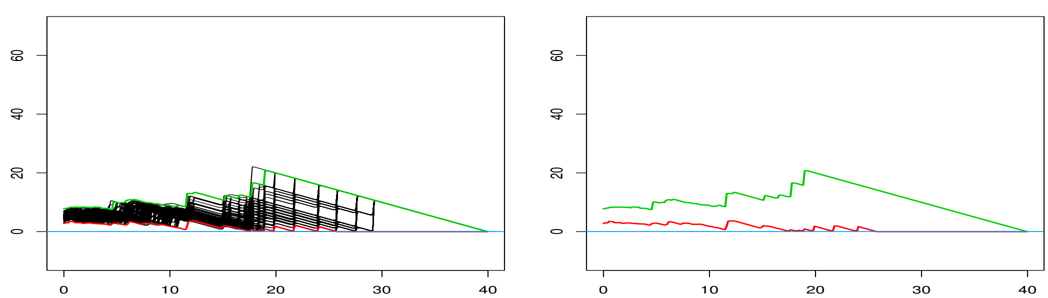

Figure 6.4: 90%-confidence band for $q_{Y_3,0.5} - q_{X,0.5}$

Conclusions of the simulation study

Figures 6.2 to 6.7 and Figures 6.8 to 6.13, which correspond to the examples where the variables are ordered in the sense of the median residual life order (Examples 6.1 to 6.6), it is seen that most of the region delimited by the band lies above the x-axis. Indeed, except when the difference of the variables is very small (Example 6.1), all this region lies above the x-axis.

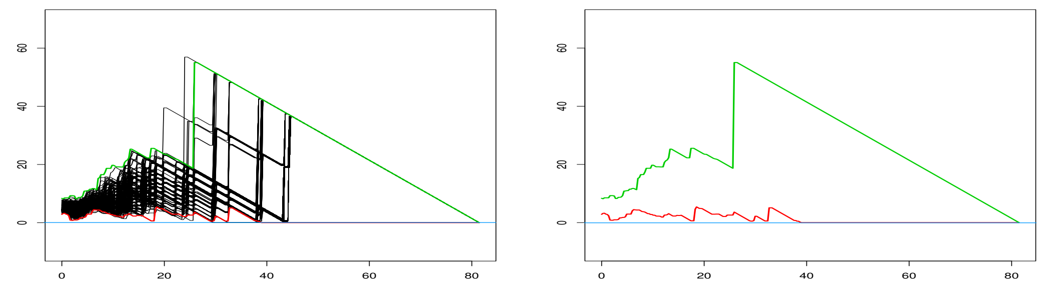

Figure 6.5: 90%-confidence band for $q_{Y_4,0.5} - q_{X,0.5}$

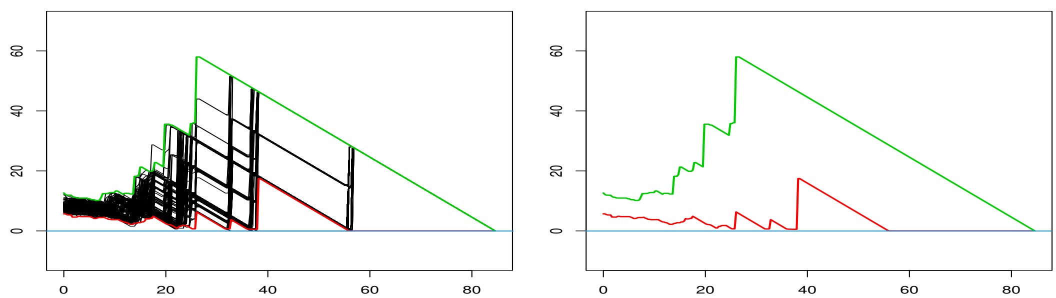

Figure 6.6: 90%-confidence band for $q_{Y_5,0.5} - q_{X,0.5}$

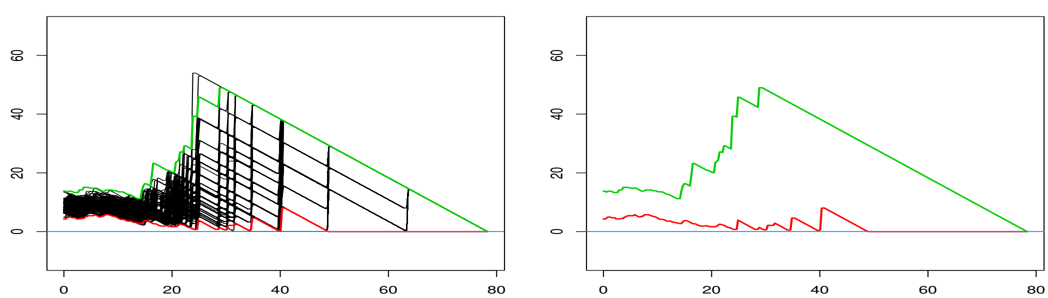

Figure 6.7: 90%-confidence band for $q_{Y_6,0.5} - q_{X,0.5}$

In Figure 6.2 and Figure 6.13, in which the variables involved are ordered but close with respect to the median residual life order (Example 6.1), it is seen that the lower limit of the band crosses the x-axis or the x-axis is contained in the band. However as the difference between the two median residual life functions increases, the distance of the lower limit of the band to the x-axis also increases. This fact shows the coherence of these bands.

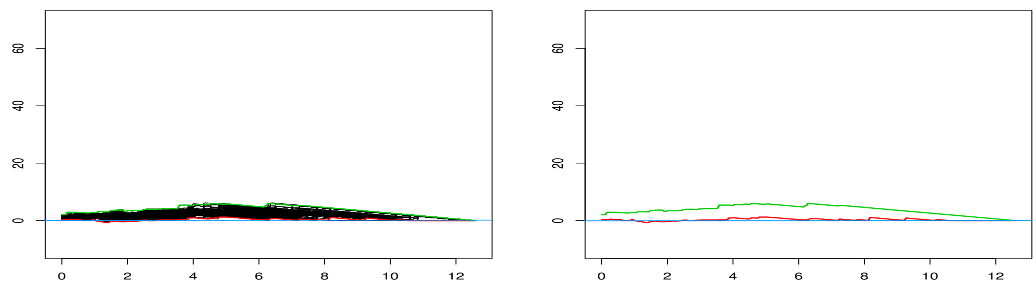

Figure 6.8: 95%-confidence band for $q_{Y_1,0.5} - q_{X,0.5}$

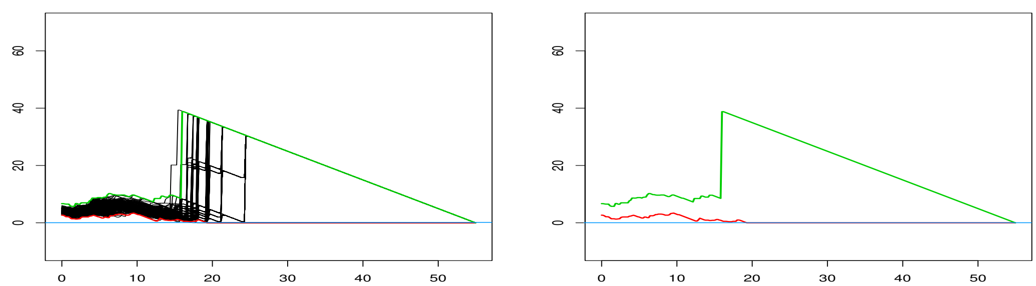

Figure 6.9: 95%-confidence band for $q_{Y_2,0.5} - q_{X,0.5}$

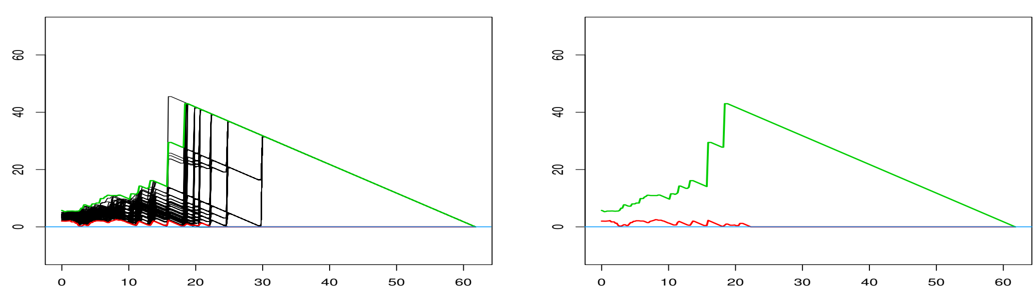

Figure 6.10: 95%-confidence band for $q_{Y_3,0.5} - q_{X,0.5}$

In Figures 6.14 to 6.17, which correspond to the examples where the variables are not ordered in the sense of the median residual life order (Examples 6.7 to 6.10), the x-axis either cross any of the limits of the band or is contained in the band.

We have computed the confidence band for the difference of two median residual life functions. However, as we have explained, the procedure is valid for every percentile $\gamma \in (0, 1)$.

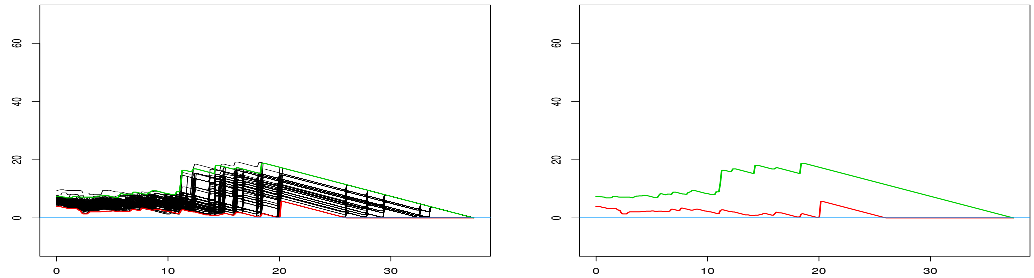

Figure 6.11: 95%-confidence band for $q_{Y_4,0.5} - q_{X,0.5}$

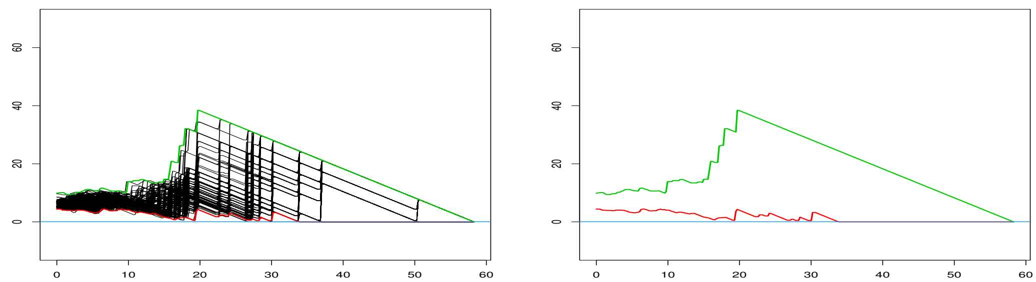

Figure 6.12: 95%-confidence band for $q_{Y_5,0.5} - q_{X,0.5}$

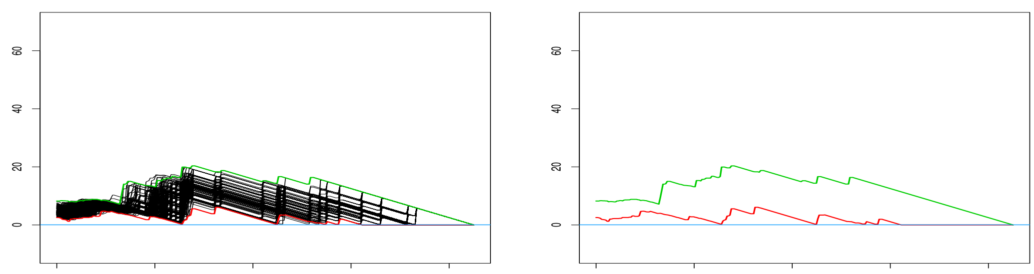

Figure 6.13: 95%-confidence band for $q_{Y_6,0.5} - q_{X,0.5}$

As we have illustrated with this example, the confidence bands for the different of two percentile residual lifetimes provide us with a criteria of whether two random variables are close or not with respect to a percentile residual life order and allow us to compare percentile residual life functions in a given interval in the following sense. When the lower limit of the band lies above the x-axis, we can conclude that the random variables are ordered with respect to the median residual life order. Analogously, if the upper limit of the band lies below the x-axis, the two random variables are also ordered but the first one dominates the

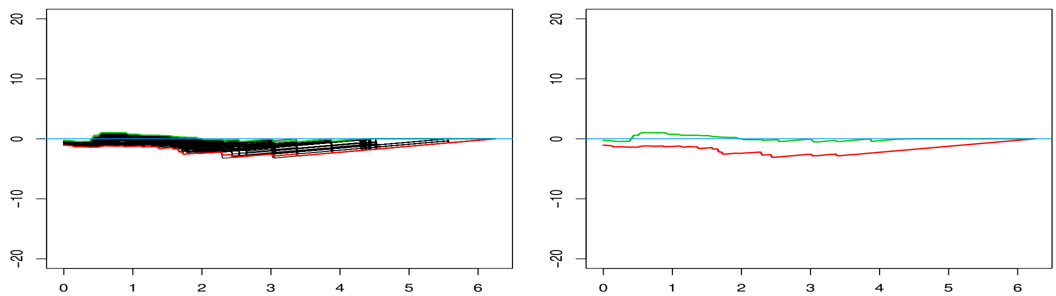

Figure 6.14: 90%-confidence band for $q_{Y_7,0.5} - q_{X,0.5}$

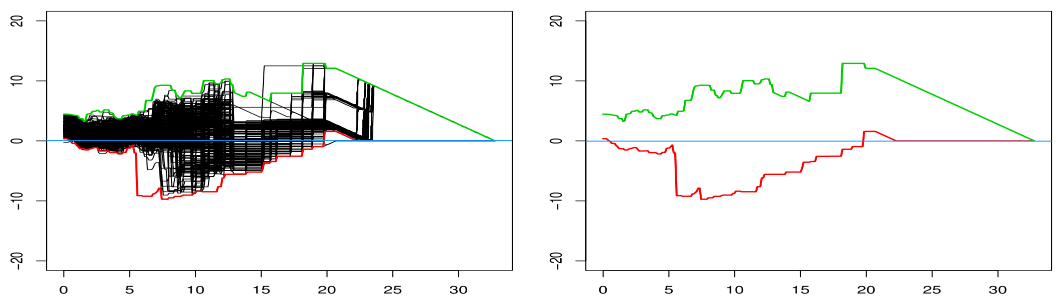

Figure 6.15: 90%-confidence band for $q_{Y_8,0.5} - q_{X,0.5}$

second one. When the lower limit of the band lies below the x-axis and the upper limit of the band lies above the x-axis, we can not say that one variable dominates the other in the sense of the median residual life order.

6.5 Application to real data

Failure time analysis (FTA) addresses data of the form 'time until an event occurs'. The survival times of medical patients or industrial products have been the usual subjects of FTA, but data from a wide variety of ecological studies may be cast in these terms, including survival times of organisms or part of organisms and times until certain behaviours are exhibited.

In the biomedical context, FTA has been also called 'survival analysis' since the event is commonly the death of a patient, so the time until the death is the survival time.

Failure time analysis accommodates 'censored data'. Censored data points are those in which the event was not observed, perhaps because the study ended before the event happened to some of the individuals under observation or because some of the individuals

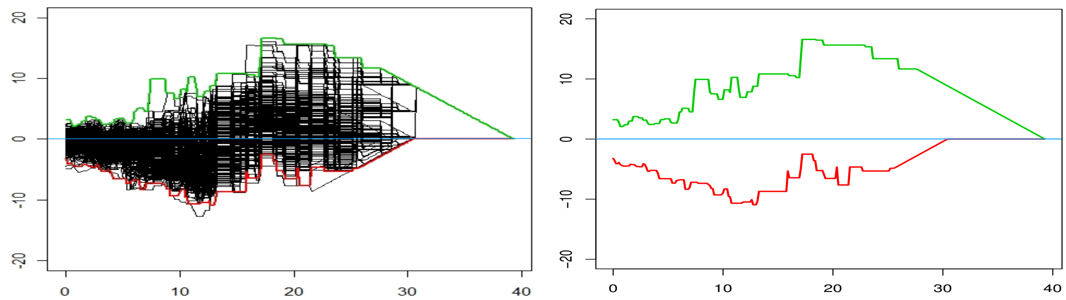

Figure 6.16: 90%-confidence band for $q_{Y_9,0.5} - q_{X,0.5}$

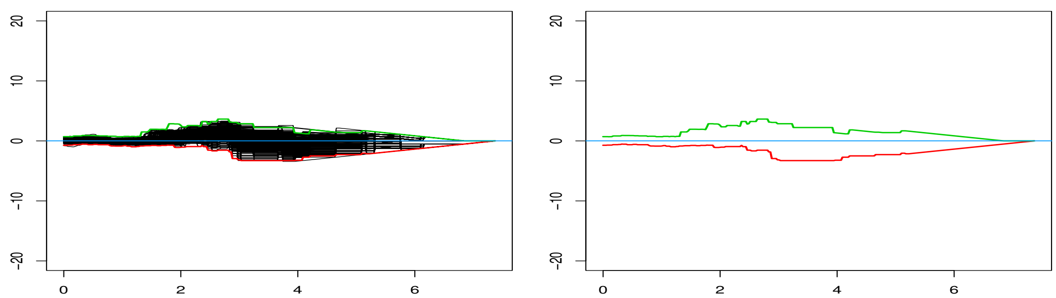

Figure 6.17: 90%-confidence band for $q_{Y_{10},0.5} - q_{X,0.5}$

were lost track of before the event occurred during the study. For these censored data points, the actual time of occurrence is not known. Instead we know a minimum length of time during which the event did not occur. Failure time analysis allows use of such censored data for their partial information. This feature is apt to be useful in the field biology, where identification markers may be lost, external conditions may cause the premature end of observations, or the observation period may be too brief for all positive events to occur.

Here we consider two examples in which the construction of confidence bands allows us to compare two treatments of cancer and to conclude whether male plants are more attractive to insects than female plants or not.

Application in medicine Consider the example introduced on Chapter 1. The data correspond to the survival times in days of patients of cancer who were randomly assigned to one of two treatment groups, radiation therapy alone or radiation therapy together with a chemotherapeutic agent. The data were taken from Apendix I of Kalbfleisch and Prentice (1982) (Data Set II) and are part of a large clinical trial carried out by the Radiation Therapy Oncology Group in the United States.

We recall that approximately 30% of the survival times after the treatment are censored owing primarily to patients surviving to the time of analysis. From a statistical point of view, the main feature of these data that distinguishes this example from others is the considerable lack of homogeneity between individuals being studied. We have deleted the females in order to make the populations more homogeneous (this way we avoid possible differences due to gender).

We have constructed the 90% and the 95%-confidence bands for the difference of the median residual life functions for the patients belonging to both groups, see Figure 6.18 and Figure 6.19, respectively. Since there exist censored data, we have considered the median residual life function estimator proposed in Csörgo (1987). From the figures we can not conclude that one treatment is better than the other.

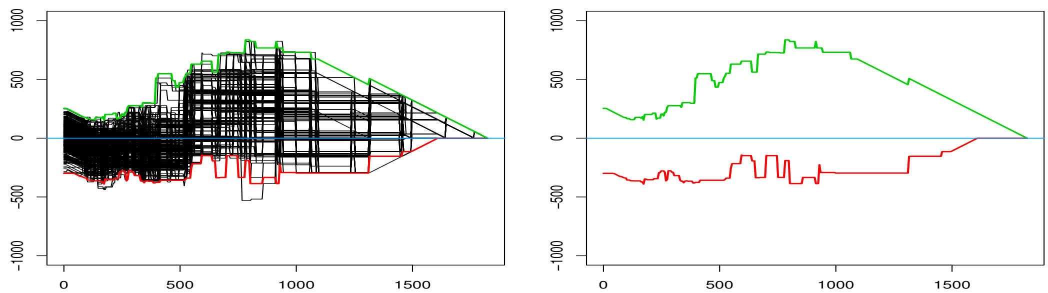

Figure 6.18: 90%-confidence bands for comparing the two treatments of cancer.

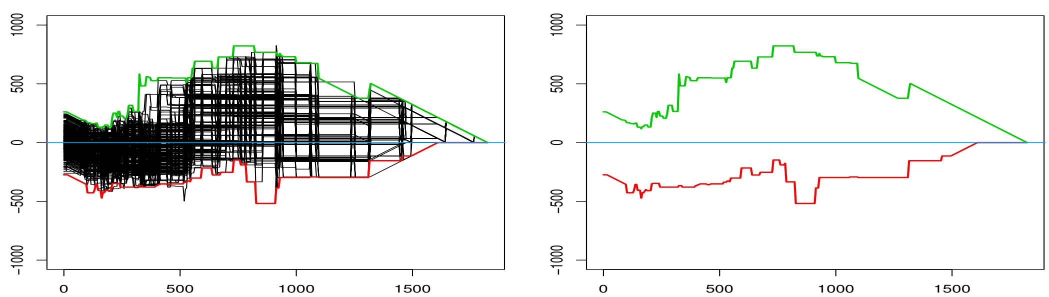

Figure 6.19: 95%-confidence bands for comparing the two treatments of cancer.

Application in ecology A number of ecological questions can be phrased in terms of 'time until an event occurs'. Events of interest might include the arrival of a migrant or parasite, the display of a particular behavior, the dispersal of a fruit or offspring, the germination of a seed, the abscission of a flower, or the death of an organisms or a part of an organism.

Male plants of dioecious species are often more floriferous than female plants, see Loyd and Webb (1977). This is true of *Clematis ligusticifolia* Nutt., the species we have considered in our example. These data were collected in Matthews-Winters Park, Jefferson County, Colorado, and are available in Muenchow (1986). In this paper it is tested whether males and females are equally attractive to insects against the alternative hypothesis that males are more attractive. The event was defined as the arrival of any flying insect at one of the flowers. He concluded that male flowers were visited at a significantly faster rate than were female flowers after carrying out the Cox-Mantel test.

We have constructed the 90% and the 95%-confidence bands for the difference of the median residual life functions for both groups of plants, see Figure 6.20 and Figure 6.21, respectively. Since there exist censored data, we have considered the median residual life function estimator proposed in Csörgo (1987). From the figures we can not conclude that one group of plants is more attractive than the other.

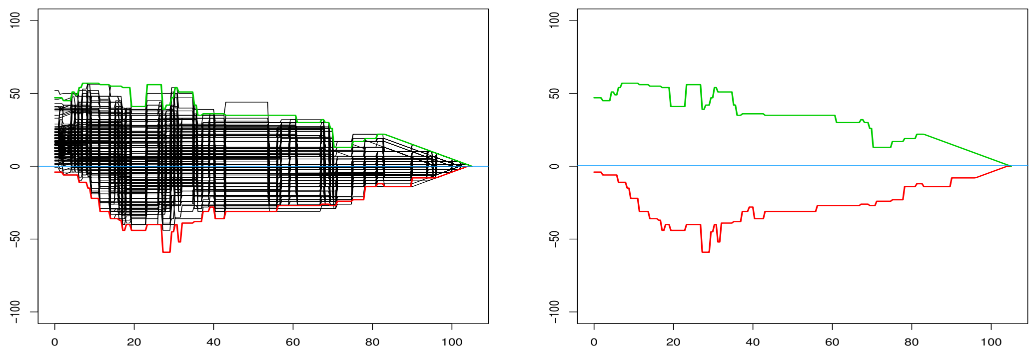

Figure 6.20: 90%-confidence bands for comparing the two groups of plants.

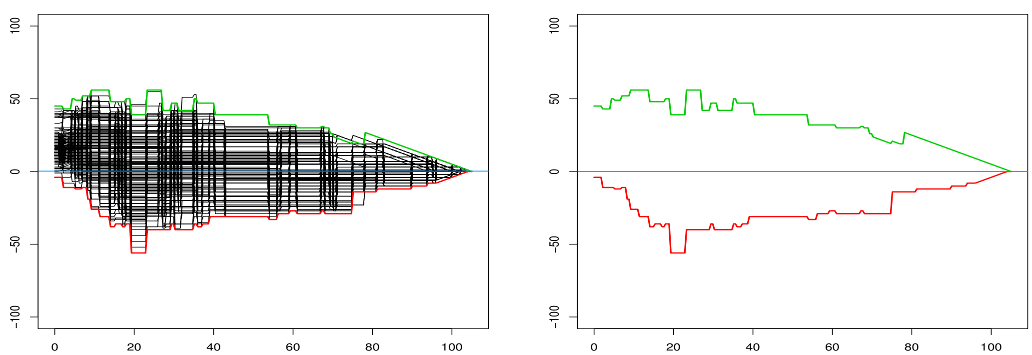

Figure 6.21: 95%confidence bands for comparing the two groups of plants.

Chapter 7

Main results and further research

Stochastic orders have been used during the last four decades in different areas of probability and statistics. These areas include reliability theory, queueing theory, survival analysis, biology, economics, insurance, actuarial science, operations research, and management science.

The simplest way of comparing two distribution functions is by comparing the associated means. However, such a comparison is based on only two single numbers (the means), and therefore it is often not very informative. Besides, the means sometimes do not exist. In many instances in applications, one has more detailed information, for the purpose of comparison of two distribution functions, than just the two means. The most important and common stochastic orders that take into account various forms of possible knowledge about the two underlying distributions are the usual stochastic order, the hazard rate order, and the mean residual life order.

In Chapter 2 we have introduced and studied a new family of stochastic orderings which are useful, specially in practical situations. These stochastic orderings are based on the comparison of percentile residual life functions. Given $\gamma \in (0, 1)$, two random variables are ordered with respect to the γ-percentile residual life order if the γ-percentile residual life functions of the variables are ordered for every t. Since the γ-percentile residual life function does not characterize the distribution, the γ-percentile residual life orders are not orders but preorders. One of the advantages of these orderings is that the percentile residual life orders are less sensitive to outliers than the mean residual life order, as we have illustrated through real data examples. We have also studied whether this family of stochastic orders verify or not several closure properties, we have established its relationship to other stochastic orders and described possible applications in diverse disciplines.

Motivated by the applicability of the percentile residual life orders for comparing items after initial warranty or to compare used items, in Chapter 3 we have proposed and studied new stochastic orderings which can be used with the same purpose but which are based on the comparison of all the percentile residual life functions of two random variables, not in the whole support but from a certain moment $t_0 > 0$ on. They are called percentile residual life orders from time t_0 on and they are also preorders. Analogously, we have defined and studied new stochastic orders that allow us to compare random variables until t_0. These orders are indeed orders, not only preorders. They were introduced and studied in Chapter 4 and are useful for comparing items during the warranty period and in medical trials.

The concept of aging is very important in reliability analysis. 'No aging' means that the age of a component has no effect on the distribution of the residual lifetime of the component. 'Positive aging' describes the situation where residual lifetime tends to decrease, in some probabilistic sense, with increasing age of a component. This situation is common in reliability engineering as components tend to become worse with time due to increased wear and tear. On the other hand, 'negative aging' has an opposite effect on the residual lifetime. Although this is less common, when a system undergoes regular testing and improvement, there are cases for which we have reliability growth phenomenon.

Concepts of aging describe how a component or a system improves or deteriorate with age. Many classes of life distributions are categorized or defined in the literature according to their aging properties. An important aspect of such classifications is that the exponential distribution is nearly always a member of each class. This is due to the *memorylessness property* of the exponential distribution.

From the definitions of the life distribution classes, results may be derived concerning such things as properties of systems (based upon properties of components), bounds for survival functions, moment inequalities, and algorithms for use in maintenance policies (Hollander and Proschan, 1984). In Chapter 5 we gave some characterization results of the classes of distribution functions with decreasing γ-percentile residual life (DPRL(γ)), $0 < \gamma < 1$, in terms of the percentile residual life orders introduced and studied in Chapter 2. In Launer (1993) some results relating the behavior of the hazard rate function and the percentile residual life function are given. He states and illustrates how those relationships can be useful for studying the behavior of the empirical hazard rate function. In this paper, he shows that the maximum of the γ-percentile residual life function precedes in time the minimum of the hazard rate (providing a minimum exists) and that the minimum of the γ-percentile residual life function precedes the maximum of the hazard rate. This set of relationships forms the basis for computational procedures in that paper. The determination of the time at which the γ-percentile residual life function is a maximum can be important in fixing product warranty. For example, product burn-in could be used to eliminate the units which fail early, and thus, maximize the reliability of the remaining product.

In Chapter 5 we have also defined two new aging notions for nonnegative random variables, also based on the percentile residual life function, and we have given some characterization results of one of these notions in terms of the stochastic order studied in Chapter 4. We have also completed the study of Launer (1993) providing some necessary conditions for bathtub and upside-down bathtub distributions. One of our future lines of research is to continue with this study trying to characterize the bathtub and upside-down bathtub distributions in terms of the aging notions defined in Chapter 5. Given a set of γ's $\in (0, 1)$

we wonder if it would be possible to define a criteria in order to conclude when a random variable has a bathtub distribution.

We are also interested in studying the possibility of extending these new families of stochastic orderings to the multivariate case. In the multivariate case the problem is how to define the multivariate percentiles. We will also study the possibility of defining new univariate stochastic orderings based on the comparison of γ-percentile residual life functions for all t and $\gamma \in (\alpha_1, \gamma_2)$, with $0 < \gamma_1 < \gamma_2 < 1$.

In Chapter 6, we presented a nonparametric method for constructing confidence bands for the difference of two percentile residual life functions. The methodology we have used involves bootstrap techniques and the concept of statistical depth for functional data. These confidence bands provide us with an evidence of whether two random variables are close with respect to some percentile residual life order or not. Besides, they do not only allow us to compare the whole functions, but also in a given interval. A simulation study support the results and we have presented applications to real data. The next natural step is to design a hypothesis test to compare percentile residual life functions.

As we have already pointed out in Section 1.2, stochastic orders have been an important topic in statistics and in many other disciplines. It is common in the literature to develop statistical tools to check, given two random samples, whether the underlying random variables are ordered or not with respect to a stochastic order. Some examples of papers which deal with this topic are Joe and Proschan (1984b) and Cheng (1985) in which tests for comparing hazard rate functions and percentile residual life functions are proposed, respectively. These tests, as well as many other tests proposed to compare functions, check the null hypothesis that two functions are equal for all t, versus the alternative that one dominates the other for all t. These models do not account for the realistic possibility that the functions cross. A test designed only to test the null hypothesis of equality may have a large probability of rejecting this null hypothesis for two populations whose functions cross. Rejection of the null hypothesis by such a test may be interpreted as evidence that one function dominates the other only if the possibility of crossing functions can be eliminated a priori. In order to avoid this fact, Berger, Boos and Guess (1988) proposed tests to compare mean residual life functions and median residual life functions that included the possibility that the functions cross. The null hypothesis in these tests is that the two functions are ordered in an interval (which may be the whole support) versus the alternative that they are not.

We are interested in testing whether two percentile residual life functions are ordered using a new methodology. This methodology is based on a new idea: *extreme measures for functional data.*

Finally, another interesting problem is to check whether the percentile residual life function or any kind of average of percentile residual life functions can be considered as a risk measure.

We recall here the definition of a *monetary measure of risk* (see, for example, in Föllmer and Schied (2002)). Let Ω be a fixed set of scenarios. A *financial position* is described by a mapping $X : \Omega \to \mathbb{R}$ where $X(w)$ is the discounted net worth of the position at the end of the trading period if the scenario $w \in \Omega$ is realized. A general aim is to quantify the risk of X by some number $\rho(X)$, where X belong to a given class χ of financial positions.

Definition 7.1. *A mapping $\rho : \chi \to \mathbb{R}$ is called a monetary measure of risk if it satisfies the following conditions for all X, $Y \in \chi$.*

(i) *Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.*

(ii) *Cash invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.*

The financial meaning of monotonicity is clear: The downside risk of a position is reduced if the payoff profile is increased. Cash invariance is also called translation invariance. It is motivated by the interpretation of $\rho(X)$ as a capital requirement, i.e., $\rho(X)$ is the amount which should be added to the position X in order to make it acceptable from the point of view of a supervising agency. Thus, if the amount m is added to the position and invested in a risk-free manner, the capital requirement is reduced by the same amount.

The following example shows that the percentile residual life function does not verify the monotonicity condition for being a measure of risk.

For every $\alpha \in (0, 1)$ and every $t \in \mathbb{R}$, there exist two random variables X and Y such that

$$X \leq_{st} Y \quad \text{and} \quad q_{X,\alpha}(t) > q_{Y,\alpha}(t).$$

Let us consider any $k \in \mathbb{R}$ such that $t \in (k, k + \alpha)$. Since $\alpha > 0$, this is always possible. Let X have the uniform distribution on $(k, k+1)$ and let Y have the distribution function given by

$$F_Y(x) = \begin{cases} 0, & x < k + \alpha; \\ x - k, & k + \alpha \leq x < k + 1; \\ 1, & t \geq k + 1; \end{cases}$$

that is, F_Y is a mixture of a uniform distribution on $(k + \alpha, k + 1)$ with probability $1 - \alpha$, and a degenerate variable at $k + \alpha$ with probability α. We compute

$$q_{X,\alpha}(x) = \begin{cases} k + \alpha - x, & x < k; \\ \alpha(k + 1 - x), & k \leq x < k + 1; \\ 0, & t \geq k + 1; \end{cases}$$

and

$$q_{Y,\alpha}(x) = \begin{cases} k + \alpha - x, & x < k + \alpha; \\ \alpha(k + 1 - x), & k + \alpha \leq x < k + 1; \\ 0, & t \geq k + 1; \end{cases}$$

It is easy to check that $F_X(x) \geq F_Y(x)$, for all x. Therefore $X \leq_{st} Y$. However $q_{X,\alpha}(t) > q_{Y,\alpha}(t)$.

Indeed, in this example we have $X \leq_{st} Y$ (strictly) and $X \geq_{\alpha-rl} Y$ (strictly). Therefore, it is not a good idea to consider an average in t. This example shows that any average in t would not be monotonous either.

We compute for $\beta \geq \alpha$,

$$q_{X,\beta}(x) = \begin{cases} k+\beta-x, & x < k; \\ \beta(k+1-x), & k \leq x < k+1; \\ 0, & t \geq k+1; \end{cases}$$

and

$$q_{Y,\beta}(x) = \begin{cases} k+\beta-x, & x < k+\alpha; \\ \beta(k+1-x), & k+\alpha \leq x < k+1; \\ 0, & t \geq k+1; \end{cases}$$

It is easy to check that $q_{X,\beta}(t) > q_{Y,\beta}(t)$ for every $\beta \geq \alpha$. Therefore, it is not a good idea to consider an average in $\beta \geq \alpha$. This example shows that any average in $\beta \geq \alpha$ would be not monotonous either. However, one of our future lines of research is to define a risk measure based on the percentile residual life function.

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