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Instability of Cup-Cylinder Compound Shell Under Uniform External Pressure

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Instability of compound cup-end cylindrical shells under uniform external pressure is studied. Nonlinear differential equations governing the large axisymmetric deformations of shells of revolution which ensure the unique states of lowest potential energy of the shells under a given pressure are solved. The method of solution is multisegment integration, developed by Kalnins and Lestingi, for predicting the mode of buckling and the critical pressure of these compound shells. Results show that, when simple cylindrical and spherical shells which develop the same membrane stress under pressure are used as a compound cup-end cylindrical shell, buckling takes place in the cylinder portion, near the cup-cylinder junction, at loads a few times higher than the buckling load of conventional dome-cylinder shells.

Introduction

HULLS OF SUBMARINES are generally constructed from combinations of cylinders, cones, and domes as shown in Fig. 1. The ends of conventional submarine hulls are convex domes and are often the zone of danger under external pressure loading. This is because the pressure sustaining capacity of these shells is limited by the buckling strength of the dome end instead of the yielding strength of the dome material. Ross (1987) in his paper on the design of dome ends introduced a new idea of using inverted spherical domes shown in Fig. 2 as ends of submarine hulls. He argued that as the inverted dome end, concave to external pressure, will be in tension, the possibility of dome buckling will thus be virtually eliminated. In the absence of the possibility of failure due to instability, the pressure sustaining capability of the hull will be enormously increased as now it will fail due to yielding. In his paper Ross (1987) analyzed the stresses in the compound cup-cylinder shell (Fig. 3) by the finite-element method and made conclusions in support of his new idea.

The present paper studies the instability of the cup-cylinder compound shell in order to throw more light on the suitability of this compound shell as a submarine hull. The cup-end submarine hull, as proposed by Ross, is shown in Fig. 2.

The literature on the elastic instability of structures is richer in theories than in solutions. Most of the earlier works on finding solutions of problems of instability of shells of revolution were confined to shallow shells. The main reason for keeping these studies of instability limited to shallow shells is that the large deflection equations of axisymmetric shells could be solved only when the simplification pertaining to the shallowness of the shells was made. The simplified nonlinear differential equations were then solved by different numerical and analytical methods (Akkas 1971a, Akkas 1971b, Akkas 1971c, Koiter 1967, Pope 1968, Reissner 1970, Tillman 1965, Walker 1969).

Recent efforts include development of a number of general-purpose computer programs as surveyed by Uddin (1987) for shells of revolution based either on finite-element or on finite-difference methods of solution. These programs determine the critical load from an eigenvalue formulation of the problem in which the results of the prebuckled nonlinear analyses of shells, based on the minimization of potential energy, are used. But as pointed out by Thompson (1973), eigenvalue analysis may lead to unreliable prediction if the prebuckled state is not accurately determined or not fully taken into account in the eigenvalue formulation. Uddin (1987) and Yamada et al (1993) discussed the shortcomings of classical eigenvalue analysis in detail.

From Patel et al (1982), and Sepetoski et al (1962) it is found that nonlinear equations of general shells of revolution are not usually amenable to solution by the presently used numerical methods (i.e., finite difference and finite element) as these methods ultimately lead to the solution of a large number of nonlinear algebraic equations which have to be solved by iterative techniques and often fail due to problems of nonconvergence. On the other hand, the method of direct integration of the shell equations requires the specification of the unknown boundary values for some of the dependent variables at the starting boundary which have to be determined from their known values at the other boundary through an iterative process. This iterative process for determining the unknown initial values would always fail if the length of the shell meridian over which the shell equations have to be integrated exceeds a certain critical value as defined by Sepetoski et al (1962). The method of collocation has also been tried for solving the nonlinear equations of a shell by a number of authors (e.g., Nath et al 1983, Dumir et al 1984), but these efforts have succeeded only in simple cases of shallow spherical shells and circular plates. For these reasons Kalnins & Lestingi (1967) developed the method of multisegment integration for solving the nonlinear equations of axisymmetric shells whose meridional length is many times the critical meridional length as defined by Sepetoski et al (1962).

In the present paper Reissner's nonlinear theory of axisymmetric deformations of shells of revolution has been used for investigating the meridional mode of buckling of cup-cylinder compound shells.

Analysis

The nonlinear governing equations of equilibrium for axisymmetric deformation of a shell of revolution applied to its

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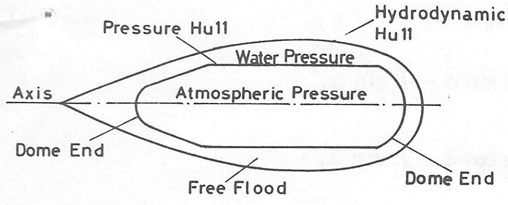

Fig. 1 Submarine pressure hull (conventional)

deformed shape, as developed by Reissner (1949) and modified by Uddin (1969), are solved by the method of multisegment integration, developed by Kalnins and Lestingi (1967), using the computer code developed by Uddin (1986).

The buckling phenomenon of the shells is interpreted here by the so-called "classical criterion" of buckling. According to this criterion, a given state of equilibrium becomes unstable when there are equilibrium positions infinitesimally near to that state of equilibrium for the same external load. These unstable states of equilibrium are the critical points on the path of equilibrium configurations. Accordingly, the critical value of the external load corresponds to the first bifurcation point on the path of equilibrium configurations. The critical points appear automatically if the equilibrium configurations based on the principle of stationary potential energy are determined corresponding to all the values of the load parameter. Thus, the nonlinear differential equations of shells, which embody the principle of minimum potential energy, are solved for the increasing values of load parameter until the first unstable state of equilibrium is reached. The unstable state of equilibrium is signalled by a sudden change in mode and a high rate of change of deformation with a slight increase of loading parameter.

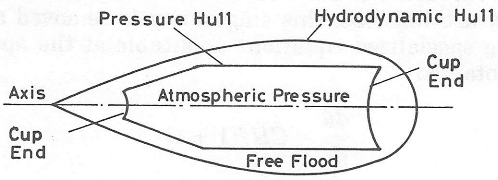

Fig. 2 Submarine pressure hull with inverted dome end

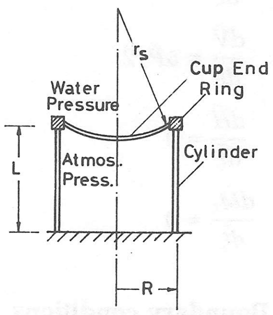

Fig. 3 Cup/cylinder combination under uniform external pressure

According to Thompson & Hunt (1973) the buckling characteristics of any structure, irrespective of the type of buckling, may be best comprehended if the equilibrium path of the deformed structure under load is determined for both the prebuckling and post-buckling zones. Thompson's two theorems on buckling (1973) point out that the first instability of the equilibrium equations on the primary equilibrium configuration path would correspond to the critical load of the structure, irrespective of its type of buckling.

The nonlinear governing equations used in this analysis are as follow (the symbols in the equations are defined in Figs. 4–6):

$$\bar{\epsilon}_\theta = \frac{\bar{u}}{\bar{r}_0} \tag{1a}$$

$$\phi = \phi_0 - \beta \tag{1b}$$

$$\bar{k}_\theta = (\sin\phi_0 - \sin\phi)/\bar{r}_0 \tag{1c}$$

$$\bar{N}_\xi = \bar{H}\cos\phi + \bar{V}\sin\phi \tag{1d}$$

$$\bar{\epsilon}_\xi = \bar{C}\bar{N}_\xi - \nu\bar{\epsilon}_\theta \tag{1e}$$

$$\bar{k}_\xi = \bar{M}_\xi/\bar{D} - \bar{k}_\theta\nu \tag{1f}$$

$$\bar{N}_\theta = (\bar{\epsilon}_\theta + \nu\bar{\epsilon}_\xi)/\bar{C} \tag{1g}$$

$$\bar{M}_\theta = \bar{D}(\bar{k}_\theta + \nu\bar{k}_\xi) \tag{1h}$$

$$\bar{\alpha} = \bar{L} + \bar{\epsilon}_\xi \tag{1i}$$

Nomenclature

- E,ν = Young's modulus, Poisson's ratio
- h,R = shell thickness, radius of cylindrical shell
- r_S = radius of spherical cap or cup
- C,D = extensional rigidity Eh, bending rigidity $Eh^3/[12(1-\nu^2)]$
- $\bar{C},\bar{D}$ = $(1-\nu^2)\xi_e/R$, $1/\{12\bar{P}\bar{T}^2\bar{R}(1-\nu^2)\}$
- ξ_e = meridional length between apex and base circle
- k_Θ,k_ξ = curvature changes
- $\bar{k}_\theta,\bar{k}_\xi$ = $k_\Theta\xi_e$, $k_\xi\xi_e$
- N_ξ,N_Θ = meridional and circumferential stress resultants
- H,V = horizontal and vertical stress resultants
- $\bar{N}_\xi,\bar{N}_\theta$ = N_ξ/PR, N_Θ/PR
- $\bar{H},\bar{V}$ = $H/PR, V/PR$
- $\epsilon_\xi,\epsilon_\Theta$ = middle surface strains
- $\bar{\epsilon}_\xi,\bar{\epsilon}_\theta$ = $\epsilon_\xi Eh\xi_e/PR^2$, $\epsilon_\Theta Eh\xi_e/PR^2$
- Φ_0,Φ = angle between axis of symmetry and normal to undeformed and deformed middle surface
- $\xi,\bar{\xi}$ = distance measured along meridian from the apex of cup-end, ξ/ξ_e
- M_Θ,M_ξ = circumferential and meridional couple resultants
- $\bar{M}_\theta,\bar{M}_\xi$ = M_Θ/PRh, M_ξ/PRh
- $P,\bar{P},\bar{R}$ = external pressure, P/E, ξ_e/R
- $r_0,\bar{r}_0$ = radial distance of points on undeformed middle surface from axis of symmetry, r_0/ξ_e
- u,w = radial and axial displacements
- $\bar{u},\bar{w}$ = uEh/PR^2, wEh/PR^2
- α,β,r = shell parameter, $\Phi_0 - \Phi$, $r_0 + u$
- σ_{ci},σ_{co} = circumferential stress index at inner and outer surfaces
- $\bar{\sigma}_{ci},\bar{\sigma}_{co}$ = σ_{ci}/E, σ_{co}/E
- $\bar{L},\bar{T}$ = $\xi_e Eh/PR^2$, R/h

$$\bar{r} = \bar{L}\bar{r}_0 + \bar{u} \quad (1j)$$

$$\frac{d\bar{w}}{d\bar{\xi}} = \bar{\alpha} \sin \phi - \bar{L} \sin \phi_0 \quad (1k)$$

$$\frac{d\bar{u}}{d\bar{\xi}} = \bar{\alpha} \cos \phi - \bar{L} \cos \phi_0 \quad (1l)$$

$$\frac{d\bar{\beta}}{d\bar{\xi}} = \bar{k}_\xi \quad (1m)$$

$$\frac{d\bar{V}}{d\bar{\xi}} = -\bar{\alpha} \cos \phi(\bar{V}/\bar{r} - \bar{P}\bar{T}) \quad (1n)$$

$$\frac{d\bar{H}}{d\bar{\xi}} = -\bar{\alpha}\{(\bar{H} \cos \phi - \bar{N}_\theta)/\bar{r} + \bar{P}\bar{T} \sin \phi\} \quad (1o)$$

$$\frac{d\bar{M}}{d\bar{\xi}} = \bar{\alpha} \cos \phi(\bar{M}_\theta - \bar{M}_\xi)/\bar{r} - \bar{\alpha}\bar{P}\bar{T}^2(\bar{H} \sin \phi - \bar{V} \cos \phi) \quad (1p)$$

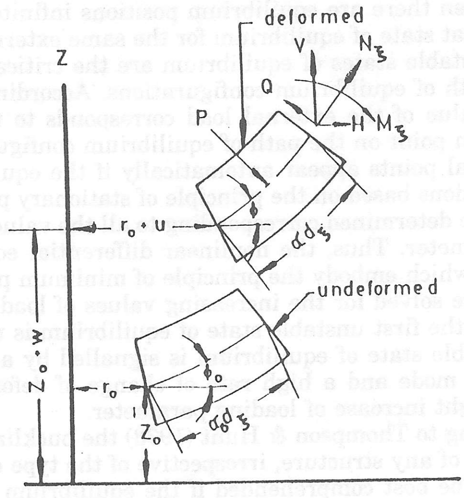

Fig. 4 Side view of elements of shell in deformed and undeformed states

These governing equations are singular at the apex of those shells which are continuous at the apex. When the conditions (*a*) that all the dependent variables are regular and (*b*) that the curvature of the undeformed shell is continuous at the apex are imposed, this singularity is removed and the following specialized equations applicable at the apex ($\bar{\xi}$ = 0) are obtained:

$$\frac{d\bar{u}}{d\bar{\xi}} = \bar{C}\bar{H}/(1 + \nu) \quad (2a)$$

$$\frac{d\bar{w}}{d\bar{\xi}} = 0 \quad (2b)$$

$$\frac{d\bar{\beta}}{d\bar{\xi}} = \bar{M}_\xi/\{\bar{D}(1 + \nu)\} \quad (2c)$$

$$\frac{d\bar{V}}{d\bar{\xi}} = \bar{\alpha}\bar{P}/2 \quad (2d)$$

$$\frac{d\bar{H}}{d\bar{\xi}} = 0 \quad (2e)$$

$$\frac{d\bar{M}_\xi}{d\bar{\xi}} = 0 \quad (2f)$$

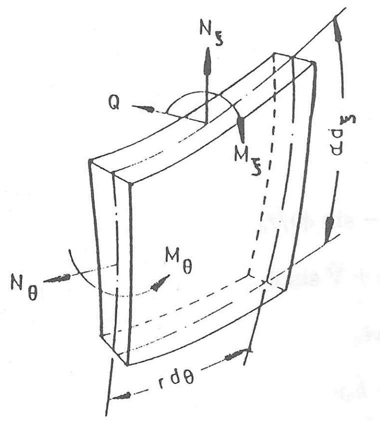

Fig. 5 Elements of shell showing stress resultants and couples

Boundary conditions

For the general case of axisymmetric deformations of shells of revolution, it was shown by Uddin (1969) that the boundary conditions on the edges require specification of

$$\bar{H} \text{ or } \bar{u}, \bar{M}_\xi \text{ or } \beta, \text{ and } \bar{V} \text{ or } \bar{w} \quad (3)$$

In the present analysis, the boundary conditions at the center of the spherical cup become

$$\bar{u} = 0, \beta = 0, \text{ and } \bar{V} = 0 \quad (4)$$

and those at a point in the cylinder far from cup-cylinder junctions are

$$\bar{H} = 0, \beta = 0, \text{ and } \bar{w} = 0 \quad (5)$$

but, in order to keep the analysis parallel to that of Ross, the boundary conditions in the cylindrical portion are taken as

$$\bar{u} = \beta = 0 \text{ and } \bar{w} = 0 \quad (6)$$

as shown in Fig. 3.

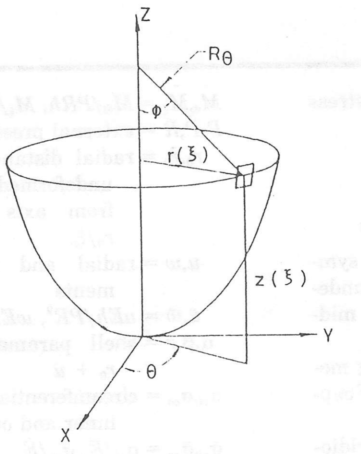

Fig. 6 Middle surface of shell

Table 1 Particulars of shells. C and D in shell numbers stands for cup-cylinder and dome-cylinder compound shells, respectively

Shell No.	R/h	L/R	r_S/R	Type of Junction
1C, 1D	100	0.40	2.0	no ring
2C, 2D	100	0.40	2.0	with ring
3C, 3D	250	0.40	2.0	no ring
4C, 4D	250	0.40	2.0	with ring

Solution

The same method of multisegment integration as used by Uddin (1986) for nonlinear analysis of pressure vessels has been employed with boundary conditions given in equations (4) and (6). The program developed by Uddin (1986) can handle different combination of axisymmetric shells with positive curvature only. For the present analysis the program has been modified to handle a combination of shells with positive as well as negative curvature and is used for the determination of the buckling pressure. To determine buckling pressure the program starts with an assumed load $\bar{P}$ and a load step $\Delta\bar{P}$; then the nonlinear governing equation at each load step with a preassigned convergence criterion is solved. If the solution fails to converge at any load step, the load step $\Delta\bar{P}$ is automatically halved and the solution is again attempted. When $\Delta\bar{P}$ becomes very small compared with the value of $\bar{P}$, then $\bar{P}$ is taken as the critical pressure for the buckling of the given shell.

Results and discussion

To investigate the suitability and superiority of cup-cylinder compound shells as proposed by Ross (1987) over conventional dome-cylinder compound shells under uniform external pressure, four shells of each group are studied. Particulars of the compound shells are given in Table 1. As the maximum membrane stress for a thin-walled sphere is half the maximum membrane stress for a thin-walled cylinder of the same radius, it was decided to make the radius of the cup end or dome end twice the radius of the cylinder. With such a combination the inverted spherical cap of the cup-end compound shell extends 0.27 times the radius of the cylinder into the cylindrical space, reducing the total internal space of the submarine. The lost space may be compensated by lengthening the pressure hull. As the space available inside the spherical dome is only a fraction of the volume provided by cylinder segment of the pressure hull, the loss of buoyancy is not significant. Submarine hulls are strengthened by bulkheads at reasonably close intervals. As the end closures are the vulnerable zones for instability, it would be wise to use bulkheads near the cup-cylinder junction. In the present case, use of bulkhead very near the junction may help increase the stability pressure. In the case of the cup-cylinder shell, setting of an internal bulkhead at cylinder length to radius ratio (l/R) 0.3, that is, near the tip of the inverted cup, may make the whole space between the cup and the bulkhead useless; thus $l/R = 0.4$ may be assumed to be feasible.

Results of the investigation for the meridional mode of buckling of the cup-cylinder and dome-cylinder compound shells are presented in Table 2.

Instability pressures for the cylindrical portion of the shell, considering simple supports at the ends and subjected to combined action of uniform lateral and axial pressure, calculated from the Windenburg (1934) formula

Table 2 Critical loads of different shells. (1C), (1D) etc. indicate shell numbers

$P_{cr}/E \times 10^6$			
Present Analysis		Windenburg Equation for Cylinder	von Mises Equation for Cylinder
Cup-Cylinder	Dome-Cylinder		
86.6 (1C)	17.35 (1D)	27.3	27.3 (12)
103.3 (2C)	17.5 (2D)		
8.56 (3C)	2.63 (3D)	2.59	2.59 (16)
9.46 (4C)	2.98 (4D)		

$$\frac{P_{cr}}{E} = \frac{2.6\left(\dfrac{t}{d}\right)^{5/2}}{\dfrac{l}{d} - 0.45\left(\dfrac{t}{d}\right)^{1/2}}$$

are given in column 3 of Table 2.

The Windenburg equation is the modified version of the von Mises (1929) equation

$$\frac{P_{cr}}{E} = \frac{t/a}{[n^2 + 0.5(\pi a/l)^2]}\, x \left\{ \frac{1}{[n^2(l/\pi a)^2 + 1]^2} + \frac{t^2}{12a^2(1-\nu^2)}[n^2 + (\pi a/l)^2]^2 \right\}$$

The Windenburg equation is an approximation that minimizes the circumferential wave number (n). Therefore (n) is considered even though it does not appear in the equation. The critical load calculated from the von Mises equation is given in column 4 of Table 2 with circumferential wave numbers in brackets.

Critical loads of compound shells in Table 2 show that cup-cylinder compound shells are superior to the dome-cylinder

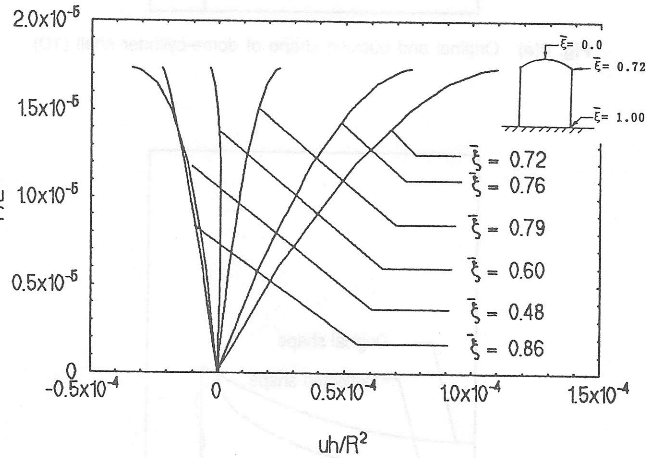

Fig. 7(a) Load versus radial deflection at critical load ($P_{cr}/E = 1.735 \times 10^{-5}$) for dome-cylinder compound shell (1D)

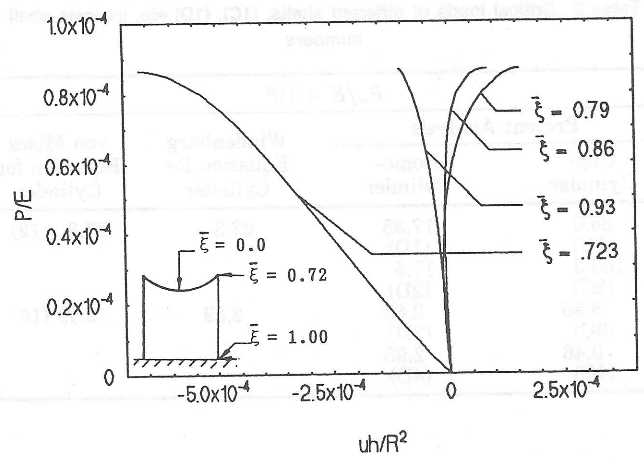

Fig. 7(*b*) Load versus radial deflection at critical load ($P_{cr}/E = 8.66 \times 10^{-5}$) for cup-cylinder compound shell (1C)

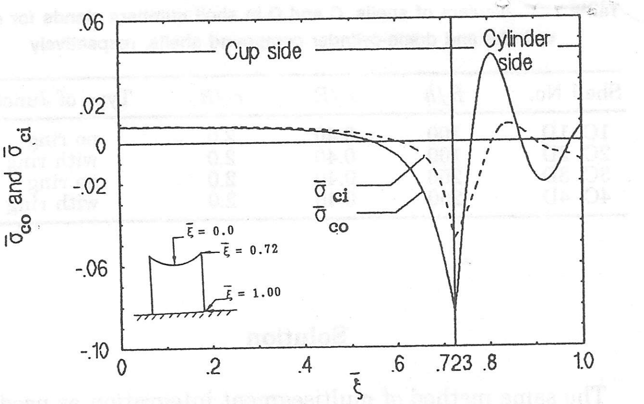

Fig. 9(*a*) Circumferential stresses at critical load ($P_{cr}/E = 8.66 \times 10^{-5}$) for cup-cylinder compound shell (1C)

shells of the same physical parameters and that the critical load for the former is a few times greater than that for the latter. Results for the dome cylinder and pure cylinder given in columns 2, 3, and 4 of Table 2 show that the instability load of the dome cylinder is dependent on the radius-to-thickness ratio. For thicker shells, dome-cylinder shells buckle axisymmetrically at loads much lower than that of the asymmetric instability load of a pure cylinder. The load-displacement curve for such a dome cylinder (shell No. 1D, Fig. 7(*a*)), shows that buckling takes place both at the cylinder and dome portion of the combination. From this discussion it is seen that the dome-cylinder combination may even reduce the instability load of the attached cylinder. In the case of cup-cylinder shells, only the cylindrical portion of the shell buckles. Figure 7(*b*) shows load versus radial

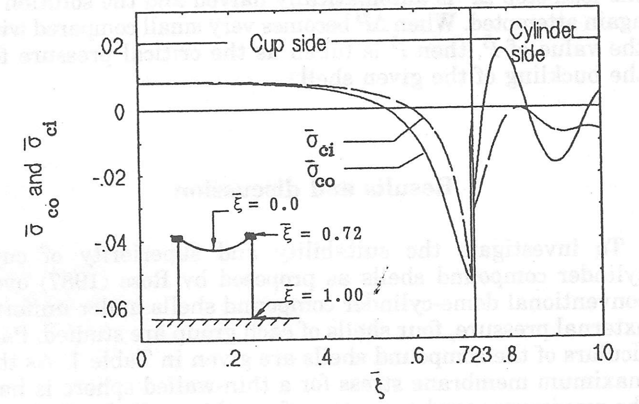

Fig. 9(*b*) Circumferential stresses at $P/E = 8.66 \times 10^{-5}$ for cup-cylinder compound shell (2C)

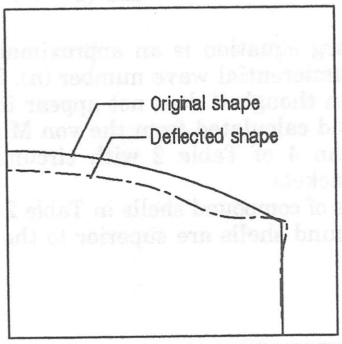

Fig. 8(*a*) Original and buckled shape of dome-cylinder shell (1D)

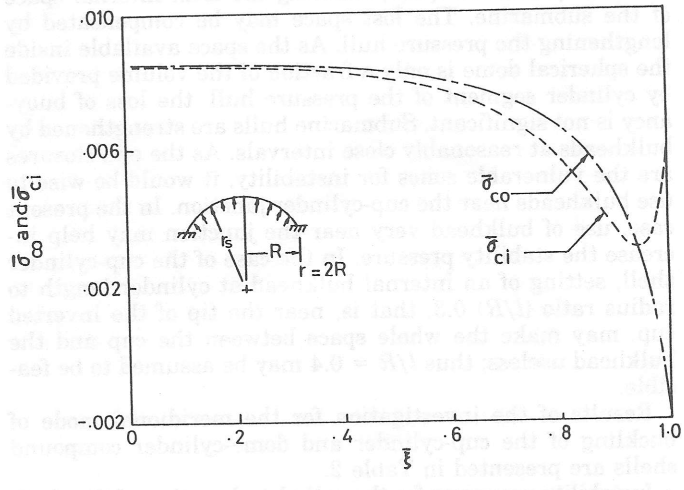

Fig. 10 Circumferential stresses in spherical cap under internal pressure of $P/E = 0.86 \times 10^{-4}$

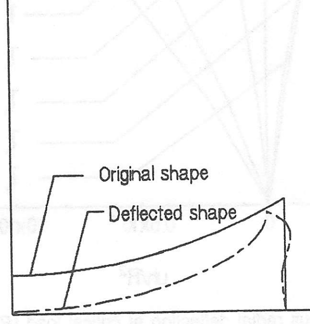

Fig. 8(*b*) Original and buckled shape of cup-cylinder shell (1C) deflection of the cylinder of the cup-cylinder shell (1C). In Figs. 8(*a*) and 8(*b*), the buckled modes of shells (1D) and (1C) are shown.

In Figs. 9(*a*) and 9(*b*) circumferential stresses developed in the compound shells (1C) and (2C) of Table 1 corresponding to the critical load of shell (1C) are plotted against meridional distance, and in Fig. 10 the circumferential stresses in the cup end of the compound shell with clamped boundary and (shown in the inset of Fig. 10) under a pressure equal to the critical load of shell (1C) are plotted. Figures 9(*a*) and 9(*b*) show that at the junction of cup and cylinder, stresses are maximum and compressive at both the inner and outer surfaces. The maximum stresses in Figs. 9(*a*) and 9(*b*) are around 40 and 25 times greater than that developed in the simple cup of Fig. 10. These high compressive stresses around the junction may lead to local circumferential instability and either of the shell components may buckle circumferentially before axisymmetric buckling.

Conclusion

Nonlinear axisymmetric analysis of the cup-cylinder shells as proposed by Ross (1987) and also of the conventional dome-cylinder shells under uniform external pressure show that cup-cylinder shells are superior to dome-cylinder compound shells. In the case of cup-cylinder shells, buckling takes place only in the cylinder portion, and the buckling load is a few times higher than that of the dome-cylinder shell of same physical parameters. Both the dome and the cylinder of a dome-cylinder shell buckle simultaneously. Sometimes the cylinder portion of a dome-cylinder shell is found to buckle even at loads lower than the buckling load of a pure cylinder under uniform external loading. But circumferential stresses developed at the junction of the cup-cylinder compound shells are very high and compressive both at the inner and outer surfaces, which may lead to asymmetric buckling before the occurrence of meridional buckling. Thus, asymmetric analysis is required to make an ultimate conclusion.

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