Analysis of Stresses in Deep Beams using Displacement Potential Function

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This paper describes a numerical approach for the analysis of stresses of two dimensional boundary value elastic problems. Specifically, it is on the investigation of stresses and deformations in various critical regions of deep beams. An ideal mathematical model has been used to formulate the problem. The rationality and practicability of the present formulation are firmly established here through the finite difference solutions of deep beams. Results are also compared with the elementary solutions and as expected, the discrepancy is observed to be quite appreciable, specifically at the fixed ends of the beams. The present investigation shows that any fixed end of a deep beam is an extremely critical zone and the elementary theory of beams is quite helpless in predicting the stresses in this zone.

Keywords: Stress, Beam, Displacement potential function

NOTATION

a	: length of the beam in the y-direction
b	: depth of the beam in the x-direction
E	: elastic modulus of the material
q	: applied maximum stress at the boundary of deep beam
u	: displacement component in the x-direction
$\bar{u}$	: u/b
V	: displacement component in the y-direction
$\bar{v}$	: v/b
$\bar{x}$	: x/b
$\bar{y}$	: y/a
x, y	: rectangular coordinates
ν	: Poisson's ratio
σ_x	: normal stress component in the x-direction
σ_y	: normal stress component in the y-direction
σ_{xy}	: shearing stress component in the xy-plane
ϕ	: Airy's stress function
ψ	: potential function defined in terms of displacements
$\bar{\sigma}_x$	: σ_x/q
$\bar{\sigma}_y$	: σ_y/q
$\bar{\sigma}_{xy}$	: σ_{xy}/q

INTRODUCTION

Problems of solid mechanics, have hardly lost interest of researchers because they never got solved entirely to the satisfaction of all the physical conditions imposed on them by the practical world. Even the most classical problems of bending and twisting of a slender rod or beam by end forces are constantly looked into with increasing sophistication in their method of analysis[1-4]. The age-old Saint-Venant's principle is still applied and its merit is evaluated in solving problems of solid mechanics in which full boundary effects could not be taken into account in the solutiont process[3, 5].

The management of boundary conditions and boundary shapes have remained as the biggest hurdle in the solution processes of the problems of solid mechanics. Photo-elastic studies are being carried out for classical problems like uniformly loaded beams on two supports[6-7] only because boundary effects could not be fully taken into account in their analytical methods of solutions.

The difficulties involved in trying to solve practical elastic problems using Airy's[8] stress function are pointed out by Uddin[9] and also by Durelli[7]. The problem of boundary management persists even in the finite difference solutions of elastic problems using the displacements formulation. The u, v- formulation involves finding two functions simultaneously from two second order partial differential equations. But solving for two functions satisfying two simultaneous second order equations and variously mixed conditions on the boundary is almost impossible[9]. In circumventing this problem, Dow, Jones and Harwood[10] have introduced a new boundary modelling approach for finite- difference applications of displacement formulation of solid mechanics and solved the problem of uniformly loaded cantilever beam. In this connection, they reported that the accuracy of the finite difference method in reproducing the state of stresses along the boundary was much higher than that of finite element analysis. However, they have noted that the computational effort of the finite difference analysis under the new boundary modelling is even somewhat greater than that of the finite element analysis.

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Considering all these shortcomings, an ideal numerical model has been developed based on a new formulation[11] to solve the present problem. Here, the elastic problem has been formulated in terms of a potential function ψ, defined in terms of displacement components, which is considered as parallel to Airy's stress function ϕ, since both of them have to satisfy the same bi-harmonic equation, but free from its inability of accounting the mixed boundary conditions.

FORMULATION OF THE PROBLEM

Analysis of stress in a material body is usually a three-dimensional problem. In most cases, the stress analysis of three- dimensional bodies can easily be treated as two-dimensional problem, because most practical problems are often found to conform to the states of plane stress or plain strain. In case of the absence of any body forces, the equations governing the three stress components σ_x, σ_y and σ_{xy} under the states of plane stress or plane strain are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad (1)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \quad (2)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0 \quad (3)$$

If the stress functions be replaced in equations (1), (2) and (3) by displacement function u and v, which are related to stress functions through the expressions,

$$\sigma_x = \frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right] \quad (4)$$

$$\sigma_y = \frac{E}{1-\nu^2}\left[\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right] \quad (5)$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] \quad (6)$$

then equation (3) is redundant and equations (1) and (2) transform to

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 u}{\partial y^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 v}{\partial x \partial y} = 0 \quad (7)$$

$$\frac{\partial^2 v}{\partial y^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 v}{\partial x^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 u}{\partial x \partial y} = 0 \quad (8)$$

The problem thus reduces to finding u and v in a two-dimensional field satisfying the two elliptic partial differential equations, (7) and (8).

The problem is reduced to the determination of a single function instead of two functions u and v simultaneously, satisfying the equilibrium equations (7) and (8). In this formulation, as in the case of Airy's stress function ϕ[12] a potential function $\psi(x, y)$ is defined in terms of displacement components as

$$u = \frac{\partial^2 \psi}{\partial x \partial y}$$

$$v = \frac{1}{1+\nu}\left[(1-\nu)\frac{\partial^2 \psi}{\partial y^2} + 2\frac{\partial^2 \psi}{\partial x^2}\right]$$

When the displacement components in the equations (7) and (8) are replaced by $\psi(x, y)$, equation (7) is automatically satisfied and the only condition that ψ has to satisfy, becomes

$$\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \quad (9)$$

Therefore, the problem is now formulated in such a way that a single function ψ has to be evaluated from the bi-harmonic equation (9), satisfying the boundary conditions that are specified at the edges of the beam.

In order to solve the problem using the present formulation, the boundary conditions are also needed to be expressed in terms of ψ and thus the corresponding relations between known functions on the boundary and he function $\psi$are,

$$u = \frac{\partial^2 \psi}{\partial x \partial y} \quad (10)$$

$$v = \frac{1}{1+\nu}\left[(1-\nu)\frac{\partial^2 \psi}{\partial y^2} + 2\frac{\partial^2 \psi}{\partial x^2}\right] \quad (11)$$

$$\sigma_x = \frac{E}{(1+\nu)^2}\left[\frac{\partial^3 \psi}{\partial x^2 \partial y} - \nu\frac{\partial^3 \psi}{\partial y^3}\right] \quad (12)$$

$$\sigma_y = -\frac{E}{(1+\nu)^2}\left[\frac{\partial^3 \psi}{\partial y^3} + (2+\nu)\frac{\partial^3 \psi}{\partial x^2 \partial y}\right] \quad (13)$$

$$\sigma_{xy} = \frac{E}{(1+\nu)^2}\left[\nu\frac{\partial^3 \psi}{\partial x \partial y^2} - \frac{\partial^3 \psi}{\partial x^3}\right] \quad (14)$$

As far as numerical method of solution of equation (9) is concerned, it is evident from the expressions of boundary conditions (10) to (14) that, no matter what combinations of two conditions are specified on the boundary, the whole range of conditions that ψ has to satisfy [equation (9) within the body and any two of the equations (10) to (14) at points on the boundary] can be expressed as finite difference equations in terms of $\psi(x, y)$. Here, it should be pointed out that, for Airy's stress function formulation, the boundary conditions known in terms of displacement components cannot be expressed in finite difference equations and hence the stress function formulation cannot be used for mixed boundary value problems[9].

SOLUTION PROCEDURE

The essential feature of this numerical approach is to get the solution of the bi-harmonic equation (9) subject to the appropriate boundary conditions prescribed over the edges of the beam. As far as the solution through the proposed displacement function, ψ-formulation is concerned, attention may be drawn to the following points.

Method of Solution

The limitation and complexity associated with analytical solution ultimately give rise to the fact that the numerical solution for this class of problems is the only plausible approach. Here, finite difference technique is used to discretize the bi-harmonic partial differential equation and also the differential equations associated with the boundary conditions. The discrete values of the displacement function ψ(x, y), at mesh points of the domain concerned (Fig 1) is solved from the system of linear algebraic equations resulting from the discretizaion of the bi-harmonic equation and the associated boundary conditions.

FIGURE 1
DISCRETIZATION OF THE DOMAIN IN RELATION TO COORDINATE SYSTEM

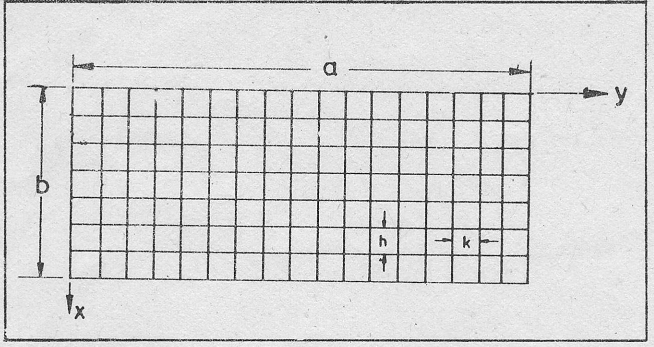


FIGURE 2
BOTH ENDS FIXED DEEP BEAM SUBJECTED TO UNIFORM LOADING AT THE TOP

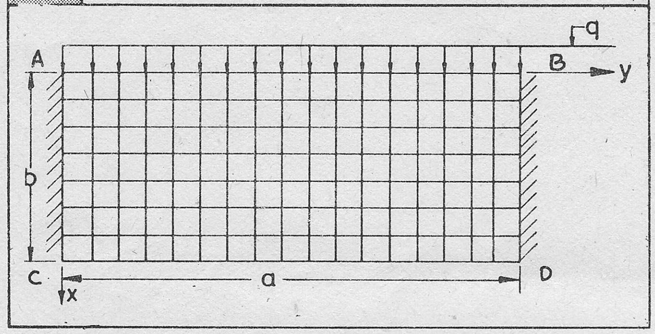
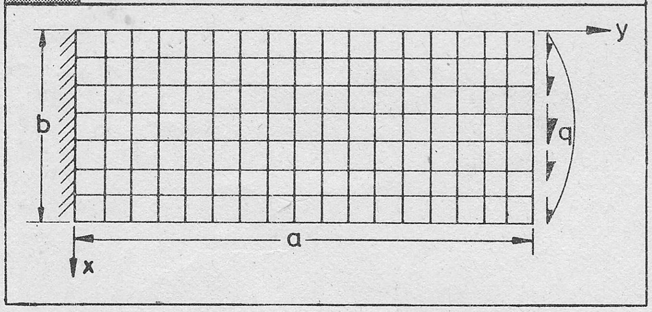


FIGURE 3
DEEP CANTILEVER SUBJECTED TO PARABOLIC SHEAR AT THE FREE END



Discretization of the Domain

The division of the domain into mesh points can be done in any regular or irregular manner, but considering the rectangular shape of the boundary and also the nature of the differential equations involved, rectangular grid points are used all over the region concerned for numerical computation, Moreover, to keep the order of error of the difference equations resulting from the boundary conditions to a minimum, an additional false boundary, exterior to the physical boundary, is introduced.

Specification of the Boundary Conditions

As each mesh point on the physical boundary of he elastic body always entertains two conditions at a time, one is used to evaluate ψat the points on the physical boundary itself and the other for the corresponding points on the exterior false boundary. Finally, the governing bi-harmonic equation is used only to evaluate ψat the interior mesh points of the two-dimensional body.

Solution of Resulting Algebraic Equations

There are numerous existing methods of solving a system of algebraic equations. The iteration method is advocated for large sparse system of linear algebraic equations, but considering the most unfortunate part associated with the iteration method not always converging to a solution and sometimes converging but very slowly, the present problem is solved by the use of the direct method of solution which ensures better reliability as well as possibility of getting more accurate solution in a shorter period of time.

Evaluation of Stress and Displacement Components

Finally, the same difference equations are again organized for the evaluation of all the parameters of interest in the solution of deep beams from the ψ values at different points of the body as all the components of stress and displacement are expressed as summation of different derivatives of the function ψ.

EXEMPLARY PROBLEMS AND DISCUSSIONS

Numerical solution of elastic problems with mixed and changeable boundary conditions had rarely been attempted in the past because of the inability of handling these boundary conditions in Airy's stress function formulation of the problem. This problem is now satisfactorily tackled by present formulation. The proposed displacement formulation is successfully applied to analyze the state of stresses as well as deformations in deep beams.

In obtaining numerical values for the present problem, the beam as the elastic body, having a narrow rectangular cross section of unit width, is assumed to be made of ordinary steel ($v = 0.3$, $E = 200$ GPa). Solutions are presented mainly in the form of graphs. In order to make the results non-dimensional, the displacements are expressed as the ratio of actual displacement to the depth of the beam and the stresses are expressed as the ratio of the actual stress to the maximum stress applied at the boundary. Here, two specific problems of deep beams are solved—one is a both ends fixed beam and the other a cantilever.

Solution of Both Ends Fixed Deep Beams

For the problem of both ends fixed beam, shown in Fig 2, the opposing lateral edges ($\bar{y} = 0$ and 1) are fixed, bottom edge ($\bar{x} = 1$) is free of loading and the top edge ($\bar{x} = 1$) is subjected to uniformly distributed compressive loading.

When the boundary conditions are expressed mathematically, then, for the opposing lateral edges, AC and BD, the normal and tangential displacements,

$$u(\bar{x},\bar{y}) = v(\bar{x},\bar{y}) = 0 \qquad \text{for } 0 \le \bar{x} \le 1,\ \bar{y} = 0 \text{ and } 1;$$

for the bottom edge, CD, the normal and tangential stresses,

$$\sigma_x(\bar{x},\bar{y}) = \sigma_{xy}(\bar{x},\bar{y}) = 0 \quad \text{for } 0 \le \bar{y} \le 1,\ \bar{x} = 1;$$

and for the top edge, AB, the normal and tangential stresses;

$$\sigma_x(\bar{x},\bar{y}) = -5 \text{ MPa}$$

$$\sigma_{xy}(\bar{x},\bar{y}) = 0 \qquad \text{for } 0 \le \bar{y} \le 1,\ \bar{x} = 0$$

The results obtained for this problem are shown in Figs 4 to 8. In the following discussions, three different solutions of the same problem are referred to.

The first of these three solutions is the elementary solution which is based on the assumption that the plane cross-sections of the beam remain plane under loading and the elastic curve, that is the deformed shape of the neutral axis, is such that it has a zero slope at the fixed end. The elementary solutions for this problem are as follows:

$$\bar{\sigma}_x = 0$$

$$\bar{\sigma}_y = (a/b)^2 \left[3\bar{y} - 3\bar{y}^2 - 1/2\right] (2\bar{x} - 1)$$

$$\bar{\sigma}_{xy} = -3\,(a/b)\,(2\bar{y}-1)\left[\bar{x} - \bar{x}^2\right]$$

The second one is the Airy's stress function ϕ-solution which is, of course, free from the assumption of plane sections remaining plane but the end fixity is accounted in the same way as in the elementary solution and the results are claimed to be accurate at a distance equal to or greater than the depth of the beam from the fixed end, on accounts of Saint-Venant's Principle. The relevant ϕ-solutions for this problem are,

$$\bar{\sigma}_x = -\left[2\bar{x}^3 - 3\bar{x}^2 + 1\right]$$

$$\bar{\sigma}_y = (2\bar{x}-1)\left[(a/b)^2(3\bar{y} - 3\bar{y}^2 - 1/2) + (2\bar{x}^2 - 2\bar{x} + 1/5)\right]$$

$$\bar{\sigma}_{xy} = -3\,(a/b)\,(2\bar{y}-1)\left[\bar{x} - \bar{x}^2\right]$$

The third is the displacement potential function ψ- solution which is based on the complete satisfactions of the conditions imposed on the beam in the real world. This solution is thus expected to be identical with that of ϕ- solution at distances of the order of depth from the restrained edges and would differ only in the regions near and at the restrained edges.

The present numerical solution predicts almost the same results as that of both elementary and ϕ- solutions for sections at a considerable distance from the ends and the relevant comparison of the three solutions are shown in Fig 4. This establishes the fact that the ψ-formulation of the problem is free from any conceptual error and mathematical procedure, and no error is committed in the computational procedure for ψ. Fig 5 shows the distribution of displacement component $\bar{u}$ along the neutral axis of the beam of Fig 2, for different length to depth ratios. Here, the effect of a/b ratio on the distribution conforms to the fact that, at lower a/b ratio, the end effects become very prominent and provides restriction to the deflection of the beams.

Fig 6 shows the distribution of normal stress component $\bar{\sigma}_x$ with respect to x at various transverse sections of a particular beam (a/b = 5). From the graph, it is evident that the distribution of $\bar{\sigma}_x$ at the fixed ends is quite different from other sections of the beams and also the fixed end is observed as the most critical section of the beam as far as $\bar{\sigma}_x$ is concerned. As appears from the graph (Fig 6), magnitude of $\bar{\sigma}_x$ at the top face ($\bar{x} = 0$) is always unity and zero at the bottom face ($\bar{x} = 1$) which, in turn conforms to the fact that the numerical formulation is capable of reproducing the state of stresses exactly either at or away from the boundaries. Here, also the ϕ -solution of the problem, independent of $\bar{y}$, is compared with ψ-solution and is observed to be identical at section $\bar{y} = 0.5$.

Fig 7 shows the variation of the normal stress component $\bar{\sigma}_y$ at various transverse sections of the beam. Here also the fixed ends are observed as the most critical section with respect to other sections of the beam. This variation of normal stress components in the direction of y is analyzed mainly to compare how the elementary solutions differ from that of the numerical solutions. In elementary solution it is assumed that the distribution varies linearly with distance from neutral axis and the magnitude is equal at the top and bottom fibers. As evident from Fig 7, the solutions obtained through the present formulation differ from that of elementary solution in the sense that the distribution is far off from linear, rather it approaches to linear distribution only at the mid section of the beam which of course, conforms to the famous Saint Venant's principle that the effect of end fixity does not disturb the stress distribution far away from the edges. Moreover, the magnitude of $\bar{\sigma}_y$ at the top corner of the fixed end is higher than that at the bottom corner. But in case of elementary solution this magnitude is essentially the same for both top and bottom corners of the fixed end. It is also observed that the fixed

FIGURE 4
COMPARISON OF THREE DIFFERENT SOLUTIONS AT VARIOUS TRANSVERSE SECTIONS OF BOTH ENDS FIXED DEEP BEAM (a/b = 5)

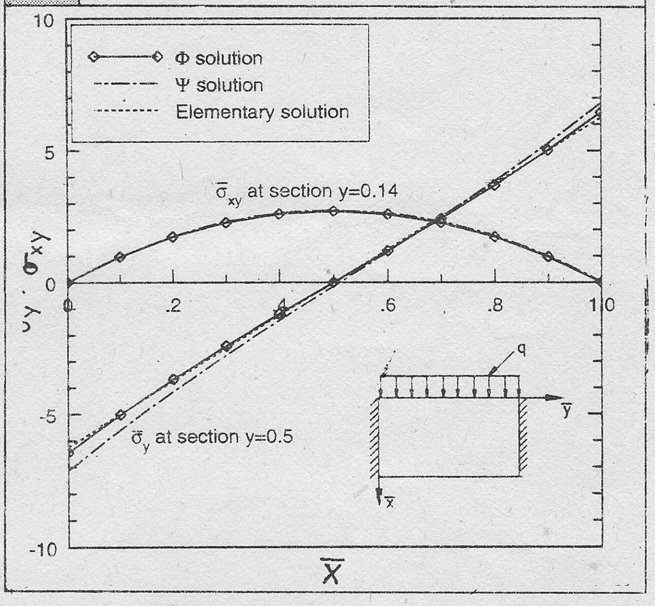
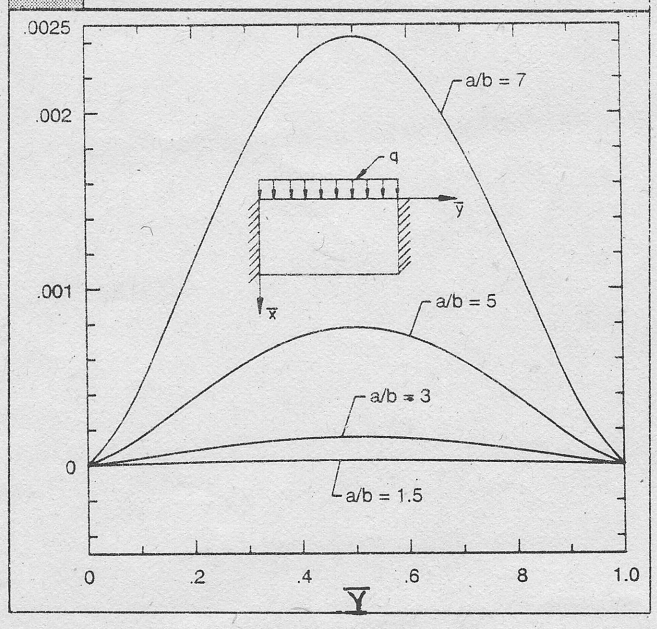


FIGURE 5
DISTRIBUTION OF DISPLACEMENT COMPONENT *u* ALONG THE NEUTRAL AXIS OF BOTH ENDS FIXED DEEP BEAMS



boundaries become more critical when the length of the beam is increased, keeping the loading constant.

The shearing stress distribution is shown in Fig 8 which shows that there is no shearing stress at the mid section of the beam (obvious from the problem) and it increases towards the ends as expected. From the distribution, it is observed that variation of this stress component over the depth of the beam is almost similar to that of elementary solutions over transverse sections everywhere except near the fixed edges. Away from the boundary, the distribution is parabolic in nature and, they are identical in nature and magnitude. From the elementary solution it is observed that the magnitude of shearing stress is always maximum at the mid section of the beam. This is not

FIGURE 6
DISTRIBUTION OF NORMAL STRESS COMPONENT $\bar{\sigma}_x$ AT VARIOUS TRANSVERSE SECTIONS OF BOTH ENDS FIXED DEEP BEAM (a/b = 5)

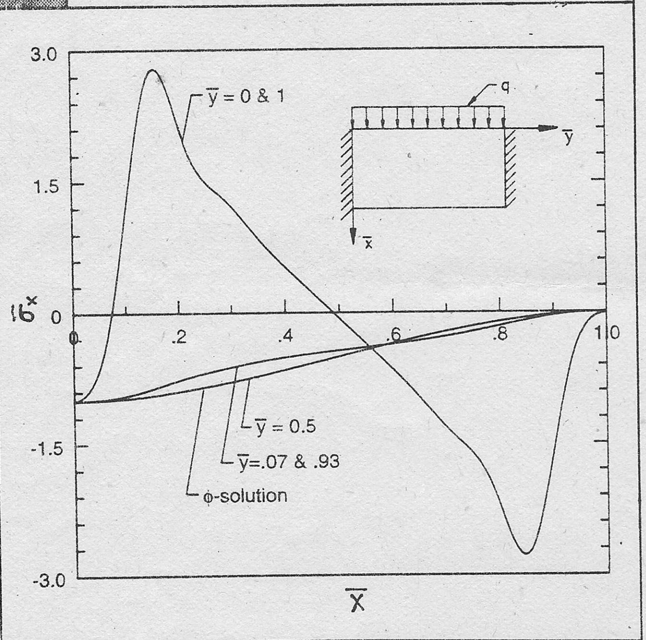


FIGURE 7
DISTRIBUTION OF LONGITUDINAL STRESS $\bar{\sigma}_y$ AT VARIOUS TRANSVERSE SECTIONS OF BOTH ENDS FIXED DEEP BEAM (a/b = 5)

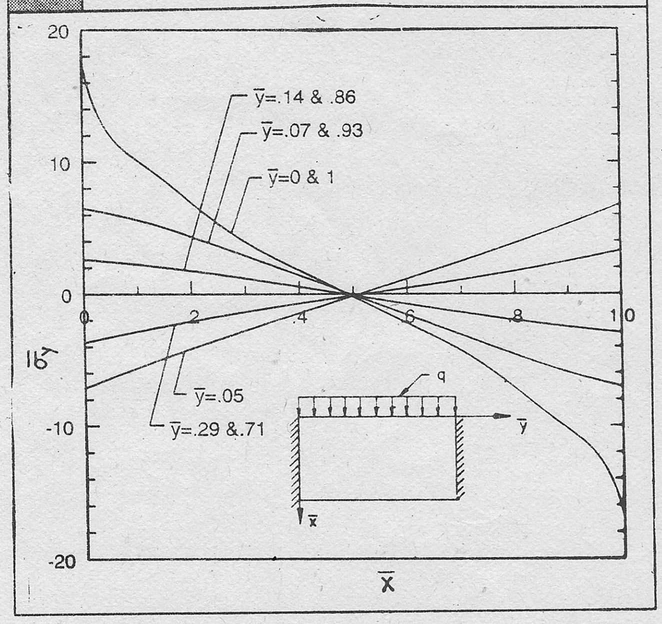
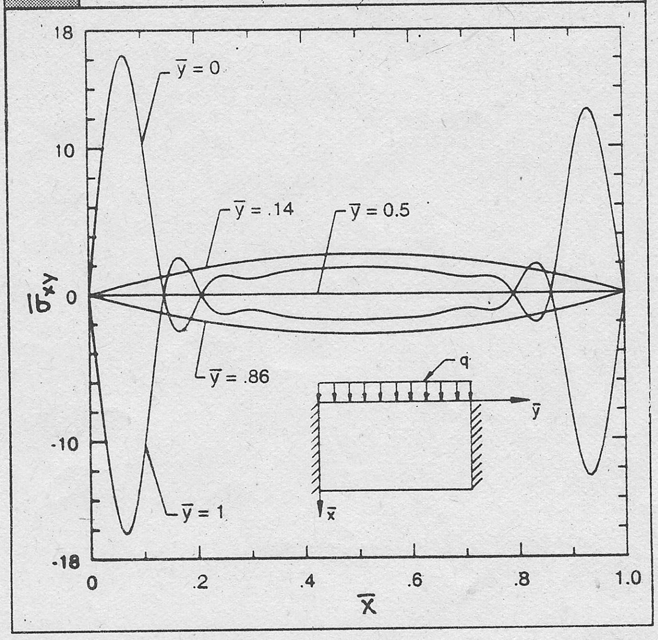


FIGURE 8
DISTRIBUTION OF SHEARING STRESS $\bar{\sigma}_{xy}$ AT VARIOUS TRANSVERSE SECTIONS OF BOTH ENDS FIXED DEEP BEAM (a/b = 5)



agreed by numerical solution and it differs mainly at the fixed ends. The distribution is not parabolic at the fixed ends, rather it is similar to that found by Filon[13] for the case of two opposing forces on a long strip displaced by a short distance[12] This similarity in solution is observed here because the fixed edges are subjected to the same kind of shear loading as that of Filon and Timpe. Here, also the upper corner zone is more critical than the lower zone and the most critical section of the beam in terms of shearing stress is at about $\bar{x} = 0.07$

Solution of Deep Cantilevers

In case of deep cantilevers subjected to end shear, shown in Fig 3, the boundary conditions, stated mathematically, are as follows:

At both the top and bottom edges, AB and CD,

$$\sigma_x(\bar{x}, \bar{y}) = \sigma_{xy}(\bar{x}, \bar{y}) = 0 \quad \text{for } 0 \le \bar{y} \le 1,\ \bar{x} = 0 \text{ and } 1$$

At the left lateral edge, AC,

$$u(\bar{x}, \bar{y}) = v(\bar{x}, \bar{y}) = 0 \qquad \text{for } 0 \le \bar{x} \le 1,\ \bar{y} = 0,$$

and at the right lateral edge, BD,

$$\sigma_y(\bar{x}, \bar{y}) = 0$$

$$\sigma_{xy}(\bar{x}, \bar{y}) = -4\, q\, (\bar{x}^2 - \bar{x}) \quad \text{for } 0 \le \bar{x} \le 1,\ \bar{y} = 1$$

where, q = 7.5 MPa

For this problem, the exact analytical solution is not known. Both the elementary and the ϕ-solutions for this problem are,

$$\bar{\sigma}_x = 0$$

$$\bar{\sigma}_y = 4\,(a/b)\,(\bar{y} - 1)\,(2\bar{x} - 1)$$

$$\bar{\sigma}_{xy} = 4\,(\bar{x} - \bar{x}^2)$$

Here also the present numerical solutions are found to be identical with that of both the elementary and ϕ-solutions at distances sufficiently away from the end. Distribution of displacement component $\bar{u}$ along the neutral axis of the beams is shown in Fig 9, which is observed to be identical with the elementary solution having third order polynomial like behav-

FIGURE 9
DISTRIBUTION OF DISPLACEMENT COMPONENT u ALONG THE NEUTRAL AXIS OF DEEP CANTILEVERS

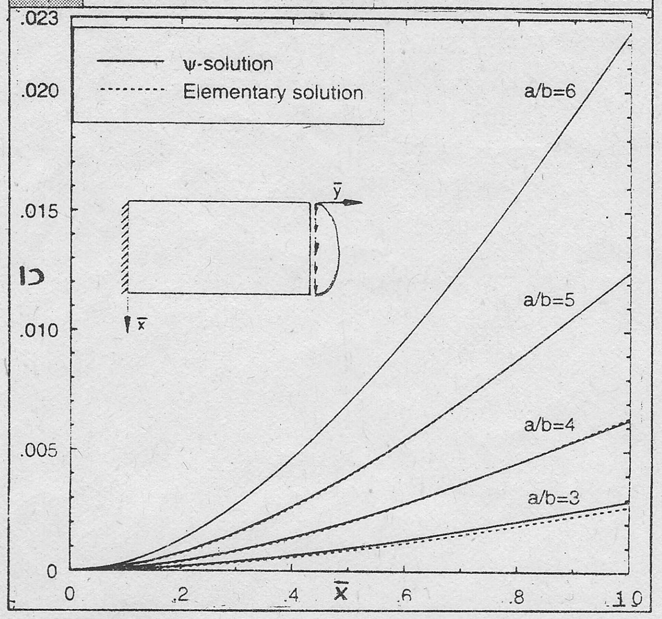
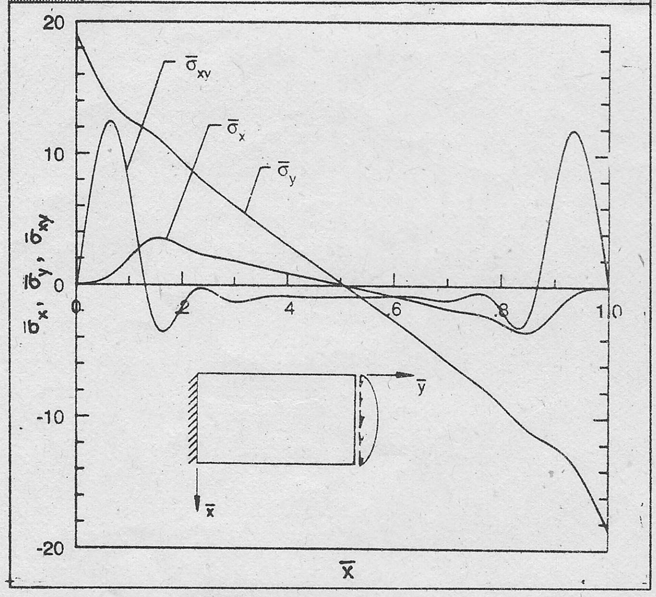


FIGURE 10
DISTRIBUTION OF NORMAL AND SHEARING STRESSES AT THE FIXED END (y = 0) OF A DEEP CANTILEVER (a/b = 4)



iour. This figure also shows the elastic curve of the cantilevers by elementary theory which is supposed to be valid for long beams; the longer is the beam, the higher is its accuracy. Of course the beams under consideration here is rather short and thus the predictions of elementary solution is supposed to suffer from inaccuracy. Here, the percentage discrepancy in deflection at the free end is observed to be almost zero for the beam a/b = 5. Moreover, the general trend of the curve and the effect of a/b ratio on the distribution are also in good agreement with the physical characteristics of the cantilever.

Distributions of relevant stress components at various sections of the beam (a/b = 4) are shown in Figs 10 and 11. From the graphs shown in Fig 10 it is evident that the fixed end of deep cantilever is the most critical section of the beam as far as stresses are concerned. Here it is also seen that shear stress does not resemble the parabolic distribution given by the elementary theory, and there are very large stresses at the top and bottom of the fixed end while the middle portion is practically free from shearing stresses. One interesting thing in noted that at higher a/b ratio the middle portion of the fixed edge is under negative shear, while stresses are positive at the upper and lower portions. This phenomenon is absent in case of lower a/b ratio as well as in both ends fixed beams.

From Fig 11, it is clear that the variation of stresses at section sufficiently away from the boundary ($\bar{y}$ = 0.5) is in good agreement with the elementary solution where the shear distribution is parabolic and the longitudinal stress varies linearly with distance from neutral axis. It is noted from the variation of stresses at different section of the beam that the closer the section is to the ends, the more is the deviations from the

FIGURE 11
DISTRIBUTION OF NORMAL AND SHEARING STRESSES AT SECTION y = 0.5 OF A DEEP CANTILEVER (a/b = 4)

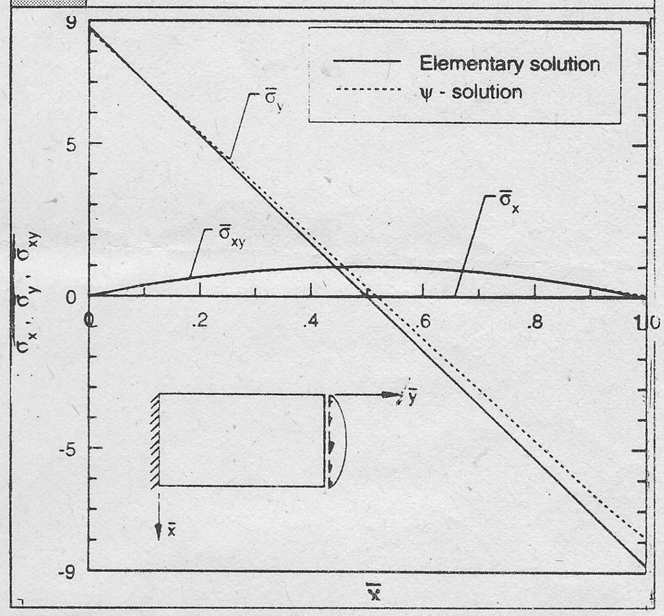
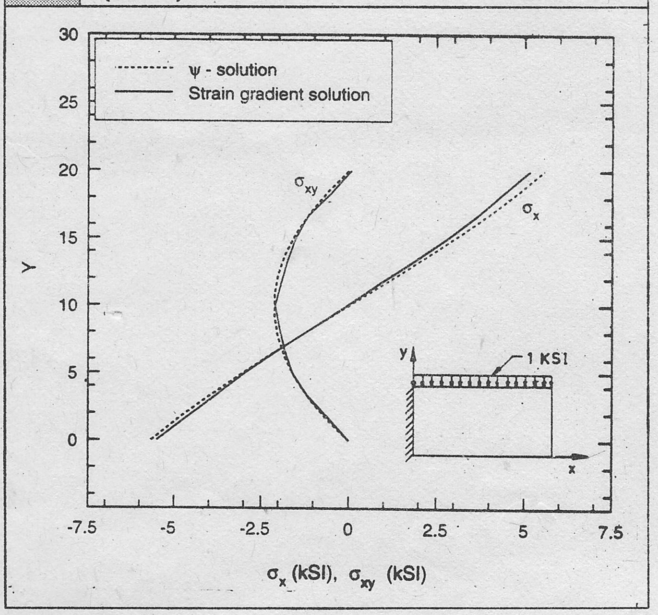


FIGURE 12
COMPARISON OF ψ-SOLUTION AND STRAIN GRADIENT SOLUTION (1990) AT SECTION x = 12.5 OF UNIFORMLY LOADED DEEP CANTILEVER (20× 40)



elementary solution. It is also observed from numerical investigation that the beam becomes more critical in terms of various stress components when the length of the beam is increased, while the loading remains constant.

For comparison, the same problem is analyzed with equivalent uniform shear at the free end. When the results are compared with that of parabolic end shear, they are found to be identical. The major discrepancy appears at around the free end. Moreover, the same example problem of Dow, Jones and Harwood[10] is also solved and the results are compared with our ψ-solutions, shown in Fig 12. The results of the two finite difference analyses are found to be virtually identical.

CONCLUSIONS

This result has introduced a modification to the usual approach to the solution of boundary value elastic problems. Here, attempt is made to obtain the numerical solutions of deep beams through a new displacement formulation which has a bright prospect in handling two-dimensional mixed boundary-value stress problems of elasticity.

Earlier mathematical models of elasticity were very deficient in handling practical problems. No appropriate approach was available in literature which could provide the explicit information about he distribution of stresses at he critical regions of boundaries. The reason for the superiority of the present displacement potential formulation over the existing approaches is its ability in satisfying the boundary conditions exactly, whether they are specified in terms of loading or restraints or any combination of them and thus he solutions obtained are promising and satisfactory for entire region of interest. Moreover, the comparative study with elementary solutions verifies that the elementary solutions are highly approximate as they fail to provide the solutions in the neighbourhood of restrained boundaries.

The developments and example problems of deep beams have displayed the successful implementation of the potential displacement approach. This work is also a precursor to the exact solution of more practical problems like stresses in gear teeth and screw threads. It is thus expected that, with time, this finite difference version of the potential displacement formulation would provide a powerful tool for solving problems in solid mechanics and other engineering applications.

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