



Solutions of time-dependent Emden–Fowler type equations by homotopy-perturbation method

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Abstract

In this Letter, we apply the homotopy-perturbation method (HPM) to obtain approximate analytical solutions of the time-dependent Emden–Fowler type equations. We also present a reliable new algorithm based on HPM to overcome the difficulty of the singular point at $x = 0$. The analysis is accompanied by some linear and nonlinear time-dependent singular initial value problems. The results prove that HPM is very effective and simple.

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1. Introduction

Recently, a lot of attention has been focused on the studies of singular initial value problems (IVPs) in the second-order ordinary differential equations (ODEs). Many problems in mathematical physics and astrophysics related to the diffusion of heat perpendicular to the surface of parallel planes can be modelled by the heat equation [1],

$$y_{xx} + \frac{r}{x}y_x + af(x,t)g(y) + h(x,t) = y_t, \quad 0 < x \leq L, \quad 0 < t < T, \quad r > 0 \quad (1)$$

subject to the boundary conditions

$$y(0,t) = \alpha, \quad y'(0,t) = 0, \quad (2)$$

where α is a constant and $f(x,t)g(y) + h(x,t)$ is the nonlinear heat source, $y(x,t)$ is the temperature, and t is the dimensionless time variable. For steady-state case, and $r = 2$, $h(x,t) = 0$, Eq. (1) becomes

$$y'' + \frac{2}{x}y' + af(x)g(y) = 0, \quad y(0,t) = \alpha, \quad y'(0,t) = 0, \quad (3)$$

which is known as Emden–Fowler equation where $f(x)$ and $g(y)$ are some given function of x and y respectively. When $f(x) = 1$ and $a = 1$, Eq. (3) reduces to the Lane–Emden equation with specified $f(y)$ was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere and theory of thermionic currents [2–4]. In this analysis, we also study the wave type equations with singular behavior of the form

$$y_{xx} + \frac{r}{x}y_x + af(x,t)g(y) + h(x,t) = y_{tt}, \quad 0 < x \leq L, \quad 0 < t < T, \quad r > 0 \quad (4)$$

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where $y(x, t)$ is the displacement of the wave at the position x and time t . The approximate solutions to the above problems were presented by Shawagfeh [5] and Wazwaz [6–8] using the Adomian decomposition method (ADM) [9]. Recently Wazwaz [10] employed ADM to solve Eq. (3). Sometimes it is a very intricate problem to calculate the so-called Adomian polynomials involved in ADM. Another powerful and more convenient analytical technique, called the homotopy-perturbation method (HPM), was first proposed by He in [11] and was further developed and improved by He [12–17]. In recent years, the application of HPM in nonlinear problems has been devoted by scientists and engineers were appeared [18–36]. Recently, Ramos [37] applied HPM to the Volterra integral form of the Lane–Emden equation without time dependence for y . Very recently, in our work [38] we applied the standard HPM successfully to solve various types of singular initial value problems but there is no time dependence for y .

The solution of the time-dependent Emden–Fowler equation as well as variety of linear, nonlinear singular IVPs in quantum mechanics and astrophysics is numerically challenging because of singularity behavior at the origin. The singularity behavior that occurs at the point $x = 0$, is the main difficulty in the analysis of Eqs. (4) and (1). In this Letter, we present a reliable algorithm based on the HPM to obtain the exact and/or approximate analytical solutions of the time-dependent Emden–Fowler type equations where y depends on the position x and on the time t as well. Comparisons with the solutions obtained by the ADM [1] shall be made.

2. Basic ideas of HPM

Homotopy-perturbation method (HPM) is a novel and effective method, and can solve various nonlinear equations. To illustrate the basic ideas of the HPM, we consider the following general nonlinear differential equation:

$$A(y) - f(r) = 0, \quad r \in \Omega, \quad (5)$$

with boundary conditions

$$B(y, \partial y / \partial n) = 0, \quad r \in \Gamma, \quad (6)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω .

The operator A can be generally divided into two parts L and N , where L is linear, while N is nonlinear. Therefore Eq. (5) can be written as follows:

$$L(y) + N(y) - f(r) = 0. \quad (7)$$

We construct a homotopy of Eq. (5) $y(r, p) : \Omega \times [0, 1] \rightarrow \Re$ which satisfies

$$H(y, p) = (1 - p)[L(y) + L(y_0)] + p[A(y) - f(r)] = 0, \quad r \in \Omega, \quad (8)$$

which is equivalent to

$$H(y, p) = L(y) - L(y_0) + pL(y_0) + p[N(y) - f(r)] = 0, \quad (9)$$

where $p \in [0, 1]$ is an embedding parameter and y_0 is an initial approximation which satisfies the boundary conditions. It follows from Eqs. (8) and (9) that

$$H(y, 0) = L(y) - L(y_0) = 0 \quad \text{and} \quad H(y, 1) = A(y) - f(r) = 0. \quad (10)$$

Thus, the changing process of p from 0 to 1 is just that of $y(r, p)$ from $y_0(r)$ to $y(r)$. In topology this is called deformation and $L(y) - L(y_0)$ and $A(y) - f(r)$ are called homotopic. Here the embedding parameter is introduced much more naturally, unaffected by artificial factors; further it can be considered as a small parameter for $0 \leq p \leq 1$. So it is very natural to assume that the solution of (8) and (9) can be expressed as

$$y(x) = u_0(x) + pu_1(x) + p^2u_2(x) + \dots \quad (11)$$

According to HPM, the approximate solution of Eq. (5) can be expressed as a series of the power of p , i.e.,

$$y = \lim_{p \rightarrow 1} y = u_0 + u_1 + u_2 + \dots \quad (12)$$

The convergence of series (12) has been proved by He in his paper [13].

3. Alternative approach of HPM

In this section, we shall introduce a reliable new algorithm to handle time-dependent singular initial value problems (IVPs) in a realistic and efficient way considering time-dependent Emden–Fowler equation as a model problem. The HPM will be applied in a straightforward manner, but with a new choice for the differential operator L . It is well known that HPM generally begins by separating linear and nonlinear parts in the problem, but this procedure does not always give satisfactory result in the singular IVPs [39]. However, a slight change is necessary to overcome the singularity behavior at $x = 0$. The alternative approach here is to define the operator L in terms of the second order derivatives, $y_{xx} + ry_x/x$, contained in the problem.

Now we construct a homotopy into Eq. (1) which satisfies the following relation

$$y_{xx} + \frac{r}{x}y_x - y_{0xx} - \frac{r}{x}y_{0x} + p\left(y_{0xx} + \frac{r}{x}y_{0x} + af(x, t)g(y) + h(x, t) - y_t\right) = 0, \tag{13}$$

where $p \in [0, 1]$ is an embedding parameter and y_0 is an initial approximation which satisfies the boundary conditions. Let us consider the solution form of Eq. (1)

$$y(x) = u_0(x, t) + pu_1(x, t) + p^2u_2(x, t) + \dots, \tag{14}$$

and the initial approximation

$$y_0 = \alpha + \int_0^x x^{-r} \int_0^x x^r h(x, t) dx dx. \tag{15}$$

Now substituting (14) into (13) and (2) and equating the coefficients of like powers of p , we get

$$u_{0xx} + \frac{r}{x}u_{0x} - y_{0xx} - \frac{r}{x}y_{0x} = 0, \quad u_0(0, t) = \alpha, \quad u_{0x}(0, t) = 0, \tag{16}$$

$$u_{1xx} + \frac{r}{x}u_{1x} + y_{0xx} + \frac{r}{x}y_{0x} + af(x, t)g(u_0) + h(x, t) - u_{0t} = 0, \quad u_1(0, t) = 0, \quad u_{1x}(0, t) = 0, \tag{17}$$

$$u_{2xx} + \frac{r}{x}u_{2x} + af(x, t)g(u_1) - u_{1t} = 0, \quad u_2(0, t) = 0, \quad u_{2x}(0, t) = 0, \tag{18}$$

$$u_3'' + \frac{r}{x}u_{3x} + af(x, t)g(u_2) - u_{2t} = 0, \quad u_3(0, t) = 0, \quad u_{3x}(0, t) = 0. \tag{19}$$

Now we can easily solve the above equations for u_0, u_1, u_2 and u_3 , etc. using the Maple package. Finally, if four-term approximation is enough, then we can write,

$$y \simeq u_0 + u_1 + u_2 + u_3. \tag{20}$$

The above procedure we also can apply to solve the wave type equations (4).

4. Applications of alternative approach of HPM

In order to assess both the applicability and the accuracy of HPM, we apply the proposed alternative approach of HPM to several singular time-dependent Emden–Fowler type equations as indicated in the following examples.

4.1. Case: time-dependent Lane–Emden type

4.1.1. Example 1

First we consider the following linear, homogeneous time-dependent Lane–Emden equation,

$$y_{xx} + \frac{2}{x}y_x - (6 + 4x^2 - \cos t)y = y_t, \tag{21}$$

subject to the boundary conditions

$$y(0, t) = e^{\sin t}, \quad y_x(0, t) = 0. \tag{22}$$

We now construct a homotopy which satisfies the following relation:

$$y_{xx} + \frac{2}{x}y_x - y_{0xx} - \frac{2}{x}y_{0x} + p\left(y_{0xx} + \frac{2}{x}y_{0x} - (6 + 4x^2 - \cos t)y - y_t\right) = 0, \tag{23}$$

where $p \in [0, 1]$ is an embedding parameter and y_0 is an initial approximation which satisfies the boundary conditions. We assume the initial approximation $y_0 = e^{\sin t}$.

Now substituting (14) into (23) and (22) and equating the coefficients of like powers of p , we get

$$u_{0xx} + \frac{2}{x}u_{0x} - y_{0xx} - \frac{2}{x}y_{0x} = 0, \quad u_0(0, t) = e^{\sin t}, \quad u_{0x}(0, t) = 0, \quad (24)$$

$$u_{1xx} + \frac{2}{x}u_{1x} + y_{0xx} + \frac{2}{x}y_{0x} - (6 + 4x^2 - \cos t)u_0 - u_{0t} = 0, \quad u_1(0, t) = 0, \quad u_{1x}(0, t) = 0, \quad (25)$$

$$u_{2xx} + \frac{2}{x}u_{2x} - (6 + 4x^2 - \cos t)u_1 - u_{1t} = 0, \quad u_2(0, t) = 0, \quad u_{2x}(0, t) = 0, \quad (26)$$

$$u_{3xx} + \frac{2}{x}u_{3x} - (6 + 4x^2 - \cos t)u_2 - u_{2t} = 0, \quad u_3(0, t) = 0, \quad u_{3x}(0, t) = 0. \quad (27)$$

Solving these equations, we obtain the following solutions for u_0 , u_1 , u_2 and u_3 , etc.,

$$u_0(x, t) = e^{\sin t}, \quad (28)$$

$$u_1(x, t) = e^{\sin t} \left(x^2 + \frac{1}{5}x^4 \right), \quad (29)$$

$$u_2(x, t) = e^{\sin t} \left(\frac{3}{10}x^4 + \frac{13}{105}x^6 + \frac{1}{90}x^8 \right), \quad (30)$$

$$u_3(x, t) = e^{\sin t} \left(\frac{3}{70}x^6 + \frac{17}{630}x^8 + \frac{59}{11550}x^{10} + \frac{1}{3510}x^{12} \right). \quad (31)$$

Finally, the approximate solution in a series form is

$$y(x, t) \simeq e^{\sin t} \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots \right), \quad (32)$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$y(x, t) = e^{x^2 + \sin t}, \quad (33)$$

which is the same as the solution obtained by Wazwaz [1] using ADM.

4.1.2. Example 2

Now we consider the following linear nonhomogeneous equation

$$y_{xx} + \frac{2}{x}y_x - (5 + 4x^2)y = y_t + (6 - 5x^2 - 4x^4), \quad (34)$$

subject to the boundary conditions

$$y(0, t) = e^t, \quad y_x(0, t) = 0. \quad (35)$$

We construct a homotopy in the following form:

$$y_{xx} + \frac{2}{x}y_x - y_{0xx} - \frac{2}{x}y_{0x} + p \left(y_{0xx} + \frac{2}{x}y_{0x} - (5 + 4x^2)y - y_t - 6 + 5x^2 + 4x^4 \right) = 0. \quad (36)$$

By the same manipulations as in the previous examples and assuming the initial approximation $y_0(x) = e^t + x^2 - \frac{1}{4}x^4 - \frac{2}{21}x^6$, we have

$$u_{0xx} + \frac{2}{x}u_{0x} - y_{0xx} - \frac{2}{x}y_{0x} = 0, \quad u_0(0, t) = e^t, \quad u_{0x}(0, t) = 0, \quad (37)$$

$$u_{1xx} + \frac{2}{x}u_{1x} + y_{0xx} + \frac{2}{x}y_{0x} - (5 + 4x^2)u_0 - u_{0t} - 6 + 5x^2 + 4x^4 = 0, \quad u_1(0, t) = 0, \quad u_{1x}(0, t) = 0, \quad (38)$$

$$u_{2xx} + \frac{2}{x}u_{2x} - (5 + 4x^2)u_1 - u_{1t} = 0, \quad u_2(0, t) = 0, \quad u_{2x}(0, t) = 0, \quad (39)$$

$$u_{3xx} + \frac{2}{x}u_{3x} - (5 + 4x^2)u_2 - u_{2t} = 0, \quad u_3(0, t) = 0, \quad u_{3x}(0, t) = 0. \quad (40)$$

Solving these equations, we obtain the following solutions for u_0 , u_1 , u_2 and u_3 , etc.,

$$u_0(x, t) = e^t + x^2 - \frac{1}{4}x^4 - \frac{2}{21}x^6, \quad (41)$$

$$u_1(x, t) = x^2 e^t + \frac{1}{5}x^4 e^t + \frac{1}{4}x^4 + \frac{11}{168}x^6 - \frac{31}{1512}x^8 - \frac{4}{1155}x^{10}, \quad (42)$$

$$u_2(x, t) = \frac{3}{10}x^4e^t + \frac{13}{105}x^6e^t + \frac{5}{168}x^6 + \frac{223}{12096}x^8 + \frac{1}{90}x^8e^t + \frac{241}{166320}x^{10} - \frac{59}{92664}x^{12} - \frac{8}{121275}x^{14}, \tag{43}$$

$$u_3(x, t) = \frac{3}{70}x^6e^t + \frac{25}{12096}x^8 + \frac{17}{630}x^8e^t + \frac{73}{38016}x^{10} + \frac{59}{11550}x^{10}e^t + \frac{1}{3510}x^{12}e^t + \frac{449}{864864}x^{12} + \frac{8473}{681080400}x^{14} - \frac{16327}{1543782240}x^{16} - \frac{16}{20738025}x^{18}. \tag{44}$$

Finally, the approximate solution in a series form is

$$y(x, t) \simeq x^2 + e^t \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots \right), \tag{45}$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$y(x, t) = x^2 + e^{x^2+t}, \tag{46}$$

which is the same as the solution obtained by Wazwaz [1] using ADM.

4.2. Case: Singular wave-type equations

4.2.1. Example 3

In this section, we consider the following nonhomogeneous singular wave-type equation

$$y_{xx} + \frac{2}{x}y_x - (5 + 4x^2)y = y_{tt} + (12x - 5x^3 - 4x^5), \tag{47}$$

subject to the boundary conditions

$$y(0, t) = e^{-t}, \quad y_x(0, t) = 0. \tag{48}$$

We construct a homotopy in the following form:

$$y_{xx} + \frac{2}{x}y_x - y_{0xx} - \frac{2}{x}y_{0x} + p \left(y_{0xx} + \frac{2}{x}y_{0x} - (5 + 4x^2)y - y_{tt} - 12x + 5x^3 + 4x^5 \right) = 0. \tag{49}$$

Carrying out the steps involved in the alternative approach of HPM and assuming the initial approximation $y_0(x) = e^{-t} + x^3 - \frac{1}{6}x^5 - \frac{1}{14}x^7$, we have

$$u_{0xx} + \frac{2}{x}u_{0x} - y_{0xx} - \frac{2}{x}y_{0x} = 0, \quad u_0(0, t) = e^{-t}, \quad u_{0x}(0, t) = 0, \tag{50}$$

$$u_{1xx} + \frac{2}{x}u_{1x} + y_{0xx} + \frac{2}{x}y_{0x} - (5 + 4x^2)u_0 - u_{0tt} - 12x + 5x^3 + 4x^5 = 0, \quad u_1(0, t) = 0, \quad u_{1x}(0, t) = 0, \tag{51}$$

$$u_{2xx} + \frac{2}{x}u_{2x} - (5 + 4x^2)u_1 - u_{1tt} = 0, \quad u_2(0, t) = 0, \quad u_{2x}(0, t) = 0, \tag{52}$$

$$u_{3xx} + \frac{2}{x}u_{3x} - (5 + 4x^2)u_2 - u_{2tt} = 0, \quad u_3(0, t) = 0, \quad u_{3x}(0, t) = 0. \tag{53}$$

Solving these equations, we obtain the following solutions for u_0, u_1, u_2 and u_3 , etc.,

$$u_0(x, t) = e^{-t} + x^3 - \frac{1}{6}x^5 - \frac{1}{14}x^7, \tag{54}$$

$$u_1(x, t) = x^2e^{-t} + \frac{1}{5}x^4e^{-t} + \frac{1}{6}x^5 + \frac{19}{336}x^7 - \frac{43}{3780}x^9 - \frac{1}{462}x^{11}, \tag{55}$$

$$u_2(x, t) = \frac{3}{10}x^4e^{-t} + \frac{13}{105}x^6e^{-t} + \frac{5}{336}x^7 + \frac{1}{90}x^8e^{-t} + \frac{319}{30240}x^9 + \frac{8}{6237}x^{11} - \frac{1171}{3783780}x^{13} - \frac{1}{27720}x^{15}, \tag{56}$$

$$u_3(x, t) = \frac{3}{70}e^{-t}x^6 + \frac{17}{630}e^{-t}x^8 + \frac{5}{6048}x^9 + \frac{59}{11550}e^{-t}x^{10} + \frac{97}{114048}x^{11} + \frac{1}{3510}e^{-t}x^{12} + \frac{12127}{45405360}x^{13} + \frac{1627}{108972864}x^{15} - \frac{10733}{2315673360}x^{17} - \frac{1}{2633400}x^{19}. \tag{57}$$

Thus, finally, the approximate solution in a series form is

$$y(x, t) \simeq x^3 + e^{-t} \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots \right), \tag{58}$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$y(x, t) = x^3 + e^{x^2-t}, \tag{59}$$

which is the same as the solution obtained by Wazwaz [1] using ADM.

4.2.2. Example 4

Now we consider the following nonhomogeneous wave-type equation

$$y_{xx} + \frac{4}{x}y_x - (18x + 9x^4)y = y_{tt} - 2 - (18x + 9x^4)t^2, \tag{60}$$

subject to the boundary conditions

$$y(0, t) = 1 + t^2, \quad y_x(0, t) = 0. \tag{61}$$

We construct a homotopy in the following form:

$$y_{xx} + \frac{4}{x}y_x - y_{0xx} - \frac{4}{x}y_{0x} + p\left(y_{0xx} + \frac{4}{x}y_{0x} - (18x + 9x^4)y - y_{tt} + 2 + (18x + 9x^4)t^2\right) = 0. \tag{62}$$

Carrying out the steps involved in the alternative approach of HPM and assuming the initial approximation $y_0(x) = 1 + t^2 - t^2x^3 - \frac{1}{5}x^2 - \frac{1}{6}t^2x^6$, we have

$$u_{0xx} + \frac{4}{x}u_{0x} - y_{0xx} - \frac{4}{x}y_{0x} = 0, \quad u_0(0, t) = 1 + t^2, \quad u_{0x}(0, t) = 0, \tag{63}$$

$$u_{1xx} + \frac{4}{x}u_{1x} + y_{0xx} + \frac{4}{x}y_{0x} - (18x + 9x^4)u_0 - u_{0tt} + 2 + (18x + 9x^4)t^2 = 0, \quad u_1(0, t) = 0, \quad u_{1x}(0, t) = 0, \tag{64}$$

$$u_{2xx} + \frac{4}{x}u_{2x} - (18x + 9x^4)u_1 - u_{1tt} = 0, \quad u_2(0, t) = 0, \quad u_{2x}(0, t) = 0, \tag{65}$$

$$u_{3xx} + \frac{4}{x}u_{3x} - (18x + 9x^4)u_2 - u_{2tt} = 0, \quad u_3(0, t) = 0, \quad u_{3x}(0, t) = 0. \tag{66}$$

Solving these equations, we obtain the following solutions for u_0, u_1, u_2 and u_3 , etc.,

$$u_0(x, t) = 1 + t^2 - t^2x^3 - \frac{1}{5}x^2 - \frac{1}{6}t^2x^6, \tag{67}$$

$$u_1(x, t) = \frac{1}{5}x^2 + x^3 + t^2x^3 - \frac{7}{50}x^5 + \frac{1}{6}x^6 - \frac{1}{6}t^2x^6 - \frac{4}{165}x^8 - \frac{1}{9}t^2x^9 - \frac{1}{120}x^{12}t^2, \tag{68}$$

$$u_2(x, t) = \frac{7}{50}x^5 + \frac{1}{3}x^6 + \frac{1}{3}t^2x^6 - \frac{79}{6600}x^8 + \frac{1}{9}x^9 + \frac{1}{18}x^9t^2 - \frac{9497}{762300}x^{11} - \frac{7}{360}x^{12}t^2 + \frac{1}{120}x^{12} - \frac{31}{31416}x^{14} - \frac{23}{5400}x^{15}t^2 - \frac{1}{5040}x^{18}t^2, \tag{69}$$

$$u_3(x, t) = \frac{239}{6600}x^8 + \frac{1}{18}x^9t^2 + \frac{1}{18}x^9 + \frac{11441}{1524600}x^{11} + \frac{1}{45}x^{12}t^2 + \frac{1}{36}x^{12} - \frac{565423}{362854800}x^{14} + \frac{1}{1800}x^{15}t^2 + \frac{23}{5400}x^{15} - \frac{215233}{528731280}x^{17} - \frac{151}{226800}x^{18}t^2 + \frac{1}{5040}x^{18} - \frac{1093}{54192600}x^{20} - \frac{11}{132300}x^{21}t^2 - \frac{1}{362880}x^{24}t^2. \tag{70}$$

Thus, the approximate solution in a series form is

$$y(x, t) \simeq t^2 + \left(1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \dots\right), \tag{71}$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$y(x, t) = t^2 + e^{x^3}, \tag{72}$$

which is the same as the solution obtained by Wazwaz [1] using ADM.

4.3. Case: Nonlinear models

4.3.1. Example 5

We now consider the following nonlinear time dependent equation

$$y_{xx} + \frac{5}{x}y_x + (24t + 16t^2x^2)e^y - 2x^2e^{y/2} = y_t, \tag{73}$$

subject to the boundary conditions

$$y(0, t) = 0, \quad y_x(0, t) = 0. \tag{74}$$

Now, based on our alternative approach of HPM, we now construct a homotopy of Eq. (73) which satisfies the following relation:

$$y_{xx} + \frac{5}{x}y_x - y_{0xx} - \frac{5}{x}y_{0x} + p\left(y_{0xx} + \frac{5}{x}y_{0x} - (24t + 16t^2x^2)e^y - 2x^2e^{y/2} - y_t\right) = 0. \tag{75}$$

Now we assume the initial approximation $y_0 = 0$ and substituting (14) into (75) and (74) and equating the coefficients of like powers of p , we get

$$u_{0xx} + \frac{5}{x}u_{0x} - y_{0xx} - \frac{5}{x}y_{0x} = 0, \quad u_0(0, t) = 0, \quad u_{0x}(0, t) = 0, \tag{76}$$

$$u_{1xx} + \frac{5}{x}u_{1x} + y_{0xx} + \frac{5}{x}y_{0x} + (24t + 16t^2x^2)\left(1 + u_0 + \frac{1}{2}u_0^2 + \frac{1}{6}u_0^3\right) - 2x^2\left(1 + \frac{1}{8}u_0^2 + \frac{1}{2}u_0 + \frac{1}{48}u_0^3\right) - u_{0t} = 0, \tag{77}$$

$$u_1(0, t) = 0, \quad u_{1x}(0, t) = 0,$$

$$u_{2xx} + \frac{5}{x}u_{2x} + (24t + 16t^2x^2)\left(u_1 + u_0u_1 + \frac{1}{2}u_0^2u_1 + \frac{1}{6}u_0^3u_1\right) - 2x^2\left(\frac{1}{2}u_1 + \frac{1}{16}u_0^2u_1 + \frac{1}{4}u_0u_1\right) - u_{1t} = 0, \tag{78}$$

$$u_2(0, t) = 0, \quad u_{2x}(0, t) = 0,$$

$$u_{3xx} + \frac{5}{x}u_{3x} - (24t + 16t^2x^2)\left(\frac{1}{2}u_0^2u_2 + \frac{1}{2}u_0u_1^2 + u_0u_2 + u_2 + \frac{1}{2}u_1^2 + \frac{1}{4}u_0^2u_1^2 + \frac{1}{6}u_0^3u_2\right) - 2x^2\left(\frac{1}{4}u_0u_2 + \frac{1}{2}u_2 + \frac{1}{16}u_0^2u_2 + \frac{1}{8}u_1^2 + \frac{1}{16}u_0u_1^2\right) - u_{2t} = 0, \quad u_3(0, t) = 0, \quad u_{3x}(0, t) = 0. \tag{79}$$

Solving these equations, we obtain the following solutions for u_0, u_1, u_2 and u_3 , etc.,

$$u_0(x, t) = 0, \tag{80}$$

$$u_1(x, t) = -2tx^2 + \left(\frac{1}{16} - \frac{1}{2}t^2\right)x^4, \tag{81}$$

$$u_2(x, t) = \left(\frac{3}{2}t^2 - \frac{1}{16}\right)x^4 + \left(\frac{11}{15}t^3 - \frac{3}{40}t\right)x^6 + \left(\frac{1}{12}t^4 - \frac{1}{64}t^2 + \frac{1}{1536}\right)x^8, \tag{82}$$

$$u_3(x, t) = \left(\frac{3}{40}t - \frac{7}{5}t^3\right)x^6 + \left(\frac{7}{64}t^2 - \frac{61}{60}t^4 - \frac{11}{7680}\right)x^8 + \left(\frac{101}{2400}t^3 - \frac{491}{2100}t^5 - \frac{37}{22400}t\right)x^{10} + \left(\frac{43}{9216}t^4 - \frac{5}{288}t^6 - \frac{7}{18432}t^2 + \frac{5}{589824}\right)x^{12}. \tag{83}$$

Hence, the approximate series solution is

$$y(x, t) \simeq -2\left(tx^2 - \frac{t^2x^4}{2} + \frac{t^3x^6}{3} - \frac{t^4x^8}{4} + \dots\right), \tag{84}$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$y(x, t) = -2\ln(1 + tx^2), \tag{85}$$

which is the same as the closed-form solution obtained by Wazwaz [1] using ADM.

4.3.2. Example 6

Finally, we consider the following nonlinear time dependent homogeneous equation

$$y_{xx} + \frac{6}{x}y_x + (14t + x^4)y + 4ty \ln y = y_{tt}, \quad (86)$$

subject to the boundary conditions

$$y(0, t) = 1, \quad y_x(0, t) = 0. \quad (87)$$

Now we construct a homotopy which satisfies the following relation

$$y_{xx} + \frac{6}{x}y_x - y_{0xx} - \frac{6}{x}y_{0x} + p \left(y_{0xx} + \frac{6}{x}y_{0x} - (14t + x^4)y + 4ty \ln y \right) - y_{tt} = 0. \quad (88)$$

Considering the initial approximation $y_0 = 1$ and substituting (14) into (88) and (87) and equating the coefficients of like powers of p , we get

$$u_{0xx} + \frac{6}{x}u_{0x} - y_{0xx} - \frac{6}{x}y_{0x} = 0, \quad u_0(0, t) = 1, \quad u_{0x}(0, t) = 0, \quad (89)$$

$$u_{1xx} + \frac{6}{x}u_{1x} + y_{0xx} + \frac{6}{x}y_{0x} + (14t + x^4)u_0 + 4tu_0 \ln u_0 - u_{0tt} = 0, \quad u_1(0, t) = 0, \quad u_{1x}(0, t) = 0, \quad (90)$$

$$u_{2xx} + \frac{6}{x}u_{2x} + (14t + x^4)u_1 + 4t(u_1 + u_1 \ln u_0) - u_{1tt} = 0, \quad u_2(0, t) = 0, \quad u_{2x}(0, t) = 0, \quad (91)$$

$$u_{3xx} + \frac{6}{x}u_{3x} + (14t + x^4)u_2 + 4tu_2 + \frac{2tu_1^2}{u_0} + 4tu_2 \ln u_0 - u_{2tt} = 0, \quad u_3(0, t) = 0, \quad u_{3x}(0, t) = 0. \quad (92)$$

Solving these equations, we obtain the following solutions for u_0, u_1, u_2 and u_3 , etc.,

$$u_0(x, t) = 1, \quad (93)$$

$$u_1(x, t) = -\frac{1}{66}x^6 - tx^2, \quad (94)$$

$$u_2(x, t) = \frac{1}{2}t^2x^4 + \frac{7}{572}tx^8 + \frac{1}{13464}x^{12}, \quad (95)$$

$$u_3(x, t) = \left(\frac{1}{66} - \frac{1}{6}t^3 \right) x^6 - \frac{67}{12870}t^2x^{10} - \frac{965}{18290844}tx^{14} - \frac{1}{5574096}x^{18}. \quad (96)$$

Hence, the approximate series solution is

$$y(x, t) \simeq 1 - tx^2 + \frac{t^2x^4}{2!} - \frac{t^3x^6}{3!} + \frac{t^4x^8}{4!} + \dots, \quad (97)$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$y(x, t) = e^{-tx^2}, \quad (98)$$

which is the same as the closed-form solution obtained by Wazwaz [1] using ADM.

5. Conclusions

In this Letter, we present a reliable algorithm based on the HPM to solve time-dependent singular IVPs. The obtained solutions are compared with the Adomian's decomposition method. All the examples show that the results of the present method are the same as with those obtained by the Adomian's decomposition method which illustrate the validity and accuracy of this procedure. The HPM has many merits and more advantages than the Adomian's decomposition method. The main advantage of this method is to overcome the difficulties arising in finding Adomian polynomials and also the calculation in HPM are very simple and straightforward. Very recently, the HPM has been used by many scientist and Engineers because of it's reliability and the reduction in the size of computations. It is shown that the homotopy-perturbation method is a promising tool for both linear and nonlinear time-dependent singular IVPs.

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