Reliability Evaluation of Scalable Complex Networks through Delta-Star Conversion

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Abstract

Exact reliability evaluation of large size complex networks becomes intractable with conventional techniques due to the exponential scaling of the computation complexity as the size of network scales up. In this paper we develop a scalable model for the exact evaluation of system reliability of scalable complex networks of the n-tuple bridge type based on scaled delta-star conversion. The number of steps as well as the computation overhead is kept within practical limits as they scale up linearly with the size of the network. The proposed model enables simple numerical evaluation either manually or through spread-sheets.

I. INTRODUCTION

Computation complexity in connection with the exact reliability evaluation of n-tuple bridges scales up exponentially in numerous conventional techniques. The complexity In the historical Reliability Polynomial (RP) is $O(2^C)$ [1-2] . The complexity is equivalent to the enumeration of all states in which the bridge is connected between the two end points, where *C* is the total number of the links in the n-tuple bridge, Figure 1. With $C=3n+2$, where n is the number of cross links, the complexity becomes $O(2^{3n+2})$. In the Cut-set method (CS) [3-4], a popular technique that is easily programmable in digital computers, the number of reliability terms in the system reliability expression scales up exponentially, due to the probability of union of non-disjoint events to $2^{3n-1}-1$, where *n* is again the number of cross links in the bridge network. In a third technique, the Conditional Probability (CP) [3-4], the complexity is based on two factors. First, the equations needed to decompose the complex networks into simple sub-networks is $O(2^n-1)$. Second, the evaluation of the sub-networks reliabilities is also $O(2^n-1)$. The proposed technique in this paper transforms the complex structure of the *n-tuple* bridge into a simple network through *n* successive delta-star conversions with a linear complexity *O(n)*. In each conversion, the three components of the delta structure are transformed into their star equivalent through three computations, hence the reliability terms scale *O(3n)*.

This paper is organized as follows. Section II presents an overview of the *n-tuple* bridge. In section III, the basic concept of the delta-star conversion is summarized. Section IV implements the basic delta star conversion to evaluate the reliability of the single- and double-bridge. The mathematical relation between the components of the two subsequent conversions is identified and implemented on the subsequent conversions. Section V expands the recursive nature between the single- and double-bridge to develop a model for the evaluation of the reliability of an *n-tuple* bridge through *n* successive conversions. The final model develops the mathematical expressions for the system reliability.

II. THE *N-TUPLE* BRIDGE COMPLEX NETWORK

The *n-tuple* bridge network became the focus for the backbone of high reliability next-generation Internet [5-7] . It is considered under the framework of the Multiprotocol Label Switching (MPLS) networks. With its two points, *a* and *b* defined as the ingress and egress of a network domain, the reliability of the core network is identical with the evaluation of two-point reliability R_{ab} of the complex network.

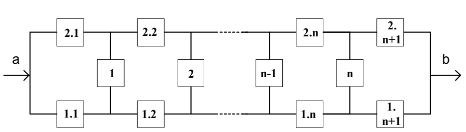

Figure 1. The *n-tuple* Bridge

The topology of the n-tuple bridge, Figure 1, can be considered as a scaled complex bridge network of *n* cross links, *1, 2, ...n*, and *2n+2* horizontal links. The first set of horizontal links, i.e. 1.1, 1.2, …, 1.n, 1.n+1 represents the main route, while the second set, 2.1, 2.2,…, 2.n, 2.n+1 is the backup route, which serves rerouting of the traffic in case of the main route failure. The routers between the horizontal links are assumed to be perfectly reliable.

Basic Delta-Star Conversion

This technique is useful for simplifying reliability calculation of complex networks as long as the network topology shows the potential for such transformation. It converts three components in a delta configuration to another set of three imaginary components in star configuration. In order for the two structures to be equivalent from the reliability point of view, the components A, B, and C from the delta configuration, Figure 2, may be interchanged with the components D, E, and F from the star configuration such that the two networks are equivalent and interchangeable. The reliabilities of the star components D, E, and F can be expressed in term of the reliabilities of the delta components A, B, and C [8-9] as follows:

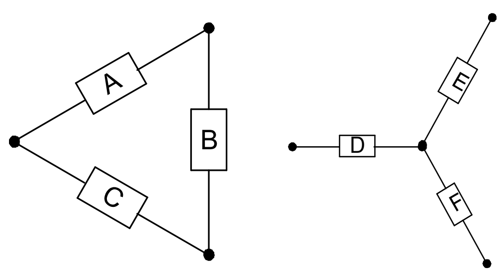

Figure 2. Equivalent delta and star components

$$R_D = \frac{K_1 K_2}{K_1 + K_2 + K_3}$$

$$R_E = \frac{K_1 + K_2 - K_3}{K_1} \qquad (1)$$

$$R_F = \frac{K_1 + K_2 - K_3}{K_2}$$

Where the parameters K_1, K_2, and K_3 are functions of the delta components as follows

$$K_1 = R_A R_B + R_C - R_A R_B R_C$$

$$K_2 = R_A + R_B R_C - R_A R_B R_C \qquad (2)$$

$$K_3 = R_A + R_C - R_A R_C$$

III. CONVERSION IN SINGLE- AND DOUBLE-BRIDGE

A. Conversion in single-bridge

For the single bridge, one conversion is needed to transform the complex structure of the bridge into a simple structure of series/parallel type.

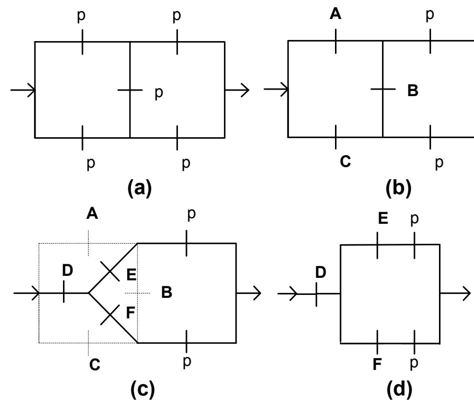

Figure 3. Delta-star conversion of single-bridge

Figure 3 shows the conversion of the single bridge into a simple network of series parallel structure. Figure 3 (a) shows the bridge with identical links of reliability *p.*. Figure 3 (b) replaces the component reliabilities *p* of the left delta structure by their assumed symbols R_A, R_B, *and* R_C *and* subsequently by the abbreviated symbols *A, B*, and *C*. Figure 3 (c) shows the conversion of the delta components *A, B*, and *C* into their star equivalent *D, E*, and *F*. Finally, Figure 3 (d) shows the simple structure of the bridge after the conversion process. The reliability of the bridge is:

$$R_1 = D\,(1-(1-pE)\,(1-pF)) \qquad (3)$$

With: $A = B = C = p$, the reliability of the single-bridge is found according to equations *(1)* and *(2).*

B. Conversion in double-bridge

Two successive conversions are needed to transform the complex structure of the double bridge into a simple structure. The first conversion transforms the double-bridge into the star component *D* that resulted from the first conversion in series with a second delta structure that incorporate the two remaining star components *E* and *F* from the first conversion. To differentiate between the components of the delta and star structures of successive conversions, we replace the delta components *A, B, C* of the single bridge by the components A_1, B_1, C_1, and the star components *D, E, F*, by D_1, E_1, *and* F_1. Figure

4 shows how the complex structure of the double bridge transforms into a simple structure through two successive delta-star conversions. Figure 4a shows the conversion of the first delta structure A_1, B_1, and C_1 into the equivalent star components D_1, E_1, and F_1. Figure 4b shows the emergence of a new delta structure A_2, B_2, C_2, where $A_2 = E_1 x\ p$, $B_2 = p$, and $C_2 = F_1 x\ p$. Hence, the first conversion transforms the double-bridge structure into a single bridge in series with the star component D_1. The remaining two star components E_1, and F_1 are integrated in the components A_2, and C_2 of the new delta structure A_2, B_2, C_2.

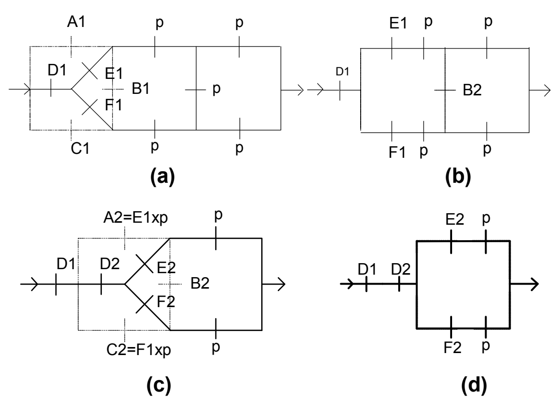

Figure 4. Delta star conversion of the double bridge

Figure 4c shows the second conversion in which the remaining single bridge transforms into the star component D_2 in series with the simple series/parallel connection of the components E_2, p, and F_2, p. The reliability of the double-bridge after the two consecutive conversions is:

$$R_2 = D_1 D_2 \left(1-(1-pE_2)(1-pF_2)\right) \quad (4)$$

In order to deal with delta-star conversions beyond the single- and double-bridge, we need to model the system with parameters of general format. The general form parameters are the components of the delta structure: A_i, B_i, C_i, $(i = 1, 2, 3, \ldots.n)$. The index "i" refers to the i^{th} cross link where $i=1$ is referring to the leftmost cross link and $i=n$ to the n^{th} or the rightmost link. In the same time "i" identifies the i^{th} delta structure, hence $i=1$ refers to the left most delta structure, $i=2$ refers to the left most delta structure following the first conversion. and $i=n$ refers to the n^{th} or last delta structure. In a similar way we identify the components D_i, E_i, and F_i as the star equivalent components of the i^{th} conversion. The parameters K_{i1}, K_{i2}, K_{i3}, $(i=1, 2, \ldots n)$ are used in the evaluation of the star components in each conversion. Hence K_{11}, K_{12}, K_{13} are used in connection with the first conversion, K_{21}, K_{22}, K_{23}, in connection with the second and K_{n1}, K_{n2}, K_{n3} in connection with the last conversion. Hence, the delta-star components and the respective K parameters in connection with the conversion of the double-bridge can be formulated as follows:

$$D_1 = \frac{K_{11}K_{12}}{K_{11}+K_{12}+K_{13}} \quad D_2 = \frac{K_{21}K_{22}}{K_{21}+K_{22}-K_{32}}$$

$$E_1 = \frac{K_{11}+K_{12}-K_{13}}{K_{11}} \quad E_2 = \frac{K_{21}+K_{22}-K_{23}}{K_{21}} \quad (5)$$

$$F_1 = \frac{K_{11}+K_{12}-K_{13}}{K_{12}} \quad F_2 = \frac{K_{21}+K_{22}-K_{23}}{K_{22}}$$

and

$$\begin{aligned} K_{11} &= A_1 B_1 + C_1 - A_1 B_1 C_1 \\ K_{12} &= A_1 + B_1 C_1 - A_1 B_1 C_1 \\ K_{13} &= A_1 + C_1 - A_1 C_1 \\ K_{21} &= A_2 B_2 + C_2 - A_2 B_2 C_2 \\ K_{22} &= A_2 + B_2 C_2 - A_2 B_2 C_2. \\ K_{23} &= A_2 + C_2 - A_2 C_2 \end{aligned} \quad (6)$$

Now, equation sets *(5) and (6)* are used to evaluate the system reliability of the double-bridge in *(4)*.

IV. DELTA-STAR CONVERSION IN N-TUPLE BRIDGE

Figure 4 (c) can be considered as the starting point for further scaled conversions. A third conversion converts a triple-bridge into a simple network of three star components D_1, D_2, and D_3 in series, and a parallel connection of pE_3 and pF_3. Similarly, a total of n conversions will transform an *n-tuple* bridge into a simple network of $n x D_i$ star components in series :D_1, D_2, D_3, ...D_{n-1}, D_n, and a parallel connection of pE_n and pF_n, Figure 5.

The system reliability of the n-tiuple bridge is found according to Figure 5 as:

$$R_n = D_1 x D_2 x \ldots. x D_{n-1} x\, D_n \left[1-(1-p\,E_n)(1-p\,F_n)\right] \quad (7)$$

where

$$D_i = \frac{K_{i1}K_{i2}}{K_{i1}+K_{i2}-K_{i3}} \qquad i=1 \text{ to } n$$

$$E_i = \frac{K_{i1}+K_{i2}-K_{i3}}{K_{i1}} \quad (7)$$

$$F_i = \frac{K_{i1}+K_{i2}-K_{i3}}{K_{i2}}$$

and

$$K_{i1} = A_i B_i + C_i - A_i B_i C_i$$
$$K_{i2} = A_i + B_i C_i - A_i B_i C_i \qquad (8)$$
$$K_{i3} = A_i + C_i - A_i C_i$$

The current delta components and the previous star components are related as follows:

$A_i = p E_{i-1}$ *and* $C_i = p F_{i-1}$ (9)

Finally, the general system reliability of the n-tuple bridge can be expressed as:

$$R_n = [1 - (1 - p E_n)(1 - p F_n)] \prod_{i=1}^{n} D_i \qquad (10)$$

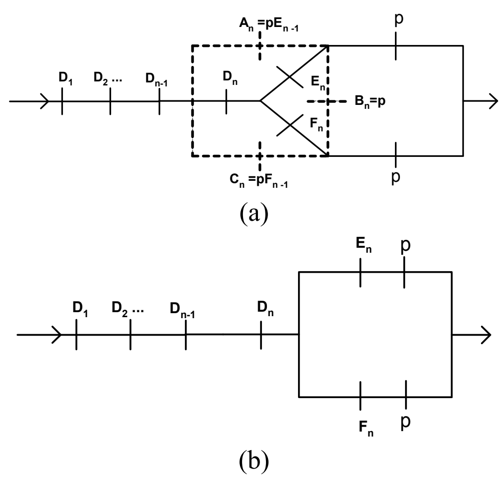

Figure 5. Delta-star conversion of *n-tuple* Bridge

V. NUMERICAL EVALUATION AND SCALING POWER

A. Numerical Evaluation

The recursive nature of the delta-star conversion over the *n-tuple* bridge and the interrelation between the delta-star components of the subsequent steps enable simple spreadsheet evaluations of the n-tuple bridge of any size. Table 1 shows the relation between the delta components, A_i, B_i, C_i, the star components D_i, E_i, F_i, and the parameters K_{i1}, K_{i2}, K_{i3} used in connection with the evaluation. Column 1 lists the components and parameters of the single-bridge. The star components D_1, E_1, F_1 are computed through the parameters K_{11}, K_{12}, K_{13} which in turn are found from the delta components $A_1 = B_1 = C_1 = p$. The star components D_2, E_2, F_2 in column 2, which refers to the double-Bridge, are computed similarly through the parameters K_{21}, K_{22}, K_{23} which are found from components A_2, B_2, C_2, of the second delta structure. Now, the delta components A_2 *and* C_2 of column 2 are related to the previous column 1 through the relations $A_2 = p\ E_1$ and $C_2 = p\ F_1$. The recursive nature of the process in columns 1 and 2 can be extended to column n where the star components D_n, E_n, and F_n are computed through K_{n1}, K_{n2}, K_{n3} which are found from the components of the last delta structure A_n, B_n, C_n. These depend on the star components of the previous column, i.e. $n-1$, through the relations $A_n = p\ E_{n-1}$ and $C_n = p\ F_{n-1}$.

TABLE 1: DELTA-STAR COMPONENTS AND THE PARAMETERS *K*

i	*1*	*2*	*3*	……	*n-1*	*n*
A_i	A_1	A_2	A_3	……	$A_{(n-1)}$	A_n
B_i	B_1	B_2	B_3	……	$B_{(n-1)}$	B_n
C_i	C_1	C_2	C_3	……	$C_{(n-1)}$	C_n
K_{i1}	K_{11}	K_{21}	K_{31}	……	$K_{(n-1)1}$	K_{n1}
K_{i2}	K_{12}	K_{22}	K_{32}	……	$K_{(n-1)2}$	K_{n2}
K_{i3}	K_{13}	K_{23}	K_{33}	……	$K_{(n-1)3}$	K_{n3}
D_i	D_1	D_2	D_3	……	$D_{(n-1)}$	D_n
E_i	E_1	E_2	E_3	……	$E_{(n-1)}$	E_n
F_i	F_1	F_2	F_3	……	$F_{(n-1)}$	F_n
R_{si}	R_{s1}	R_{s2}	R_{s3}	……	$R_{(n-1)}$	R_n

Figure 6 shows the reliability variation of *n-tuple* bridge with n ranging between 1 and 7 for five different link reliability p ranging between 0.9 and 0.995.

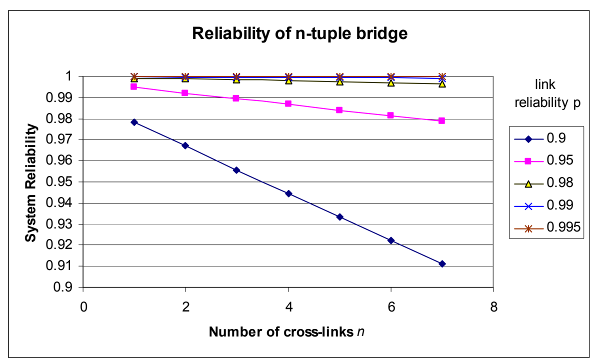

Figure 6. Reliability of n-tuple bridge

B. Scaling Power

The major advantage of the proposed approach in this paper is its scaling power. It scales linearly in two aspects. First, it scales at *O(n)* with regard to the number of steps required to transform the complex structure of the n-tuple bridge into a simple network structure. This is equivalent to the number of complex

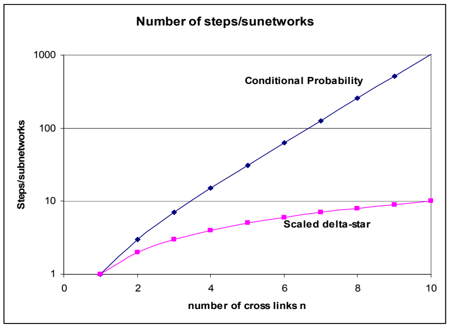

Figure 7. Comparing CP and Scaled Delta-star

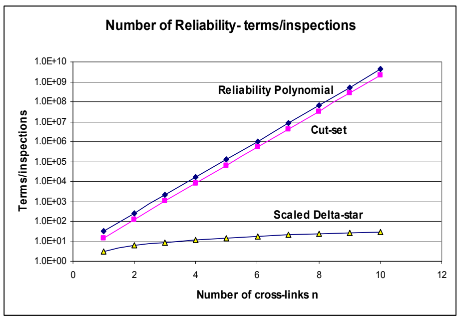

Figure 8. Comparing CS, RP & Scaled Delta-star

subnetworks involved in the entire process of conversions. Second, it scales at O(3n) with regard to the total number of reliability terms involved in evaluating the system reliability of the n-tuple bridge, i.e. three terms with each conversions. Figures 7 and 8 compare these scaling with those of other techniques mentioned earlier under Section I. Figure 7 compares the total number of steps/subnetworks in each of the proposed technique and the conditional probability approach. For the 10-tuple bridge, 1023 steps with 1023 complex subnetworks are involved in the evaluation process in the CP approached compared to 10 conversion steps and 10 complex subnetworks in the scaled delta star conversion. Figure 8 compares, between the total number of inspections in the (RP), the total number of the reliability terms in the system reliability in the (CS) method, and the total number of the component terms developed in the proposed scaled delta-star conversion.

VI. CONCLUSION

In this paper, a scalable technique is developed for the exact reliability evaluation of the *n-tuple* bridge. It transforms the complex structure of the bridge into a simple structure by repeated delta-star conversion, whose reliability can easily determined through the reliability/unreliability product rules. The recursive nature of the conversion and the linear scaling of the model make the numerical evaluation of the system reliability through the spread sheet a simple task. The scaling power of the modeling was quantified by comparing it with the exponential scaling in three other techniques.

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