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Modulational Instability In Salerno Model

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Abstract: We investigate the properties of modulational instability in the Salerno equation in quasi-one dimension in Bose-Einstein condensate (BEC). We analyze the regions of modulational instability of nonlinear plane waves and determine the conditions of its existence in BEC.

Key words: Modulational instability; Salerno model; Bose-Einstein condensate.

INTRODUCTION

The modulational instability is known phenomenon to be fundamental subject of the theory of nonlinear waves. This phenomenon consists of the instability of nonlinear plane waves against weak long-scale modulations with wave numbers (frequency) lower than some critical value. It has been predicted by Banjamen and Feir (Benjamin., 1967) for waves in deep water and by Bespalov and Talanov (Bespalov., 1967) for electro-magnetic waves in nonlinear media with cubic nonlinearity. Later, it was observed in nonlinear optics (Karpman, 1967; Ostrovskii., 1967; Tanuiti, 1968), plasma physics (Gomez-Gardenes., 2006; Hasegawa, 1970) and condensate matter (Bose-Einstein Condensate, long Josephson junction,...) (Nicolin., 2007; Strecker., 2002).

In this work, we determine the regions and conditions of existence of modulational instability in Bose-Einstein condensate in periodic potential trap (optical lattices).

II. The Model:

It is very important to investigate the modulational instability in different models. We restrict ourselves to the Salerno model in Bose-Einstein condensate in optical lattice in quasi-1D. This model is a combination of the discrete non-linear Schrodinger (DNLS) equation (Chris Eilbeck.) with cubic nonlinearity and Ablowitz-Ladik (AL) equation[1]. It is given by (Gomez-Gardenes., 2006)

$$i\,\dot{\phi} + (1 + \mu|\phi_n|^2)(\phi_{n+1} + \phi_{n-1}) + 2v|\phi_n|^2\phi_n = 0, \qquad (1)$$

where φ_n is the complex field amplitude at site of the lattice, μ is the nonlinearity of (AL) equation, is the nonlinearity of (DNLS) equation.

III. Modulational Instability of Nonlinear Plane Wave:

The nonlinear discrete equation Eq.(1) has the plane wave solution

$$\phi_n = \phi_0 \exp[i(qn + \omega t)], \qquad (2)$$

where φ_o, q, ω are amplitude, wave number and frequency respectively. Substituting Eq.(2) into Eq.(1), we obtain.

$$[-\omega + 2\cos(q)(1 + \mu\phi_0^2) + 2v\phi_0^2]\phi_n = 0. \qquad (3)$$

The Eq.(3) has two solution, one is trivial solution for $\varphi_{n} = 0$ and other is

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$$-\omega + 2\cos(q)(1 + \mu\phi_0^2) + 2\upsilon\phi_0^2 = 0$$

then, we find the relation between frequency and wave number, which called " dispersion relation".

$$\omega = 2\cos(q)(1 + \mu\phi_0^2) + 2\upsilon\phi_0^2 \,. \tag{4}$$

For unstaggered solution , the dispersion becomes , for the staggered solution the dispersion becomes $\omega = 2 + 2(\mu + \upsilon)\phi_0^2$ (Hasegawa, 1970),

We investigate when the plane wave solution Eq.(2) is stable against small perturbation , we perturb the plane wave solution slightly such that

$$\phi_n = (\phi_0 + \delta\phi_n(t))\exp[i(qn + \omega t)] \tag{5}$$

and examine evolution of the perturbation using linear stability analysis. then substituting Eq.(5) into Eq.(1) and linearizing at $\delta\phi$

After simplifying, it become

$$i\delta\dot{\phi}_n - 2\cos(q)\delta\phi_n + 2\cos(q)\mu\phi_0^2\,\delta\overline{\phi}_n + (1 + \mu\phi_0^2)(\exp[iq]\delta\phi_{n+1} + \exp[-iq]\delta\phi_{n-1})$$
$$+2\upsilon\phi_0^2(\delta\phi_n + \delta\overline{\phi}_n) = 0 \tag{6}$$

where q = 0 or q = π then sin (q) = 0 so the equation (6) becomes

$$i\,\delta\dot{\phi}_n - 2\cos(q)\delta\phi_n + 2\cos(q)\mu\phi_0^2\,\delta\overline{\phi}_n + \cos(q)\,(1 + \mu\phi_0^2)(\delta\phi_{n+1} + \delta\phi_{n-1}) + 2\upsilon\phi_0^2(\delta\phi_n + \delta\overline{\phi}_n) = 0$$

simplifying the above equation, we obtain

$$i\delta\dot{\phi}_n + \cos(q)\,(1 + \mu\phi_0^2)(\delta\phi_{n+1} + \delta\phi_{n-1}) + 2\cos(q)[\mu\phi_0^2\,\delta\overline{\phi}_n - \delta\phi_n] + 2\upsilon\phi_0^2(\delta\phi_n + \delta\overline{\phi}_n) = 0 \tag{7}$$

Considering the modulational in the form below $\delta\phi_n = u_n + iv_n$ then its complex conjugate is $\overline{\phi}_n = u_n - iv_n$, substituting the terms above into Eq.(6), we find

$$i(\dot{u}_n + i\dot{v}_n) + \cos(q)\,(1 + \mu\phi_0^2)(u_{n+1} + u_{n-1} + i(v_{n+1} + v_{n-1})) +$$
$$2\cos(q)[\mu\phi_0^2(u_n - iv_n) - u_n - iv_n] + 2\upsilon\phi_0^2(2u_n) = 0 \tag{8}$$

So, we separate the last equation into real part and imaginary part

$$\begin{cases} \dot{u}_n + \cos(q)(1 + \mu\phi_0^2)(v_{n+1} + v_{n-1} - 2v_n) = 0 \\ -\dot{v}_n + \cos(q)(1 + \mu\phi_0^2)(u_{n+1} + u_{n-1} - 2u_n) + 4v\phi_0^2 u_n = 0 \end{cases} \tag{9}$$

Then, we consider the modulation in the form below

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \exp\,[i(Qn + \Omega t)] \tag{10}$$

Where Q and Ω are the wave number and frequency of the perturbation respectively.
Substituting Eq.(10) into Eq.(9), then we get

$$\begin{cases} i\Omega\alpha - 4\cos(q)\sin^2\left(\frac{Q}{2}\right)(1+\mu\phi_0^2)\beta = 0 \\ \left[4\nu\phi_0^2 - 4\cos(q)\left(\sin^2\left(\frac{Q}{2}\right)(1+\mu\phi_0^2) - \mu\phi_0^2\right)\right]\alpha - i\Omega\beta = 0 \end{cases}$$

We rewrite this system of equations in matrix form

$$\begin{bmatrix} i\Omega & -4\cos(q)\sin^2\left(\frac{Q}{2}\right)(1+\mu\phi_0^2) \\ \nu\phi_0^2 - 4\cos(q)\left(\sin^2\left(\frac{Q}{2}\right)(1+\mu\phi_0^2) - \mu\phi_0^2\right) & -i\Omega \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$$

We have trivial solution if $\det(A) \neq 0$ where

$$A = \begin{bmatrix} i\Omega & -4\cos(q)\sin^2\left(\frac{Q}{2}\right)(1+\mu\phi_0^2) \\ \nu\phi_0^2 - 4\cos(q)\left(\sin^2\left(\frac{Q}{2}\right)(1+\mu\phi_0^2) - \mu\phi_0^2\right) & -i\Omega \end{bmatrix}$$

We compute the determinant of matrix, where ,det (A) = 0 we get

$$\Omega^2 + 16\cos(q)\sin^2\left(\frac{Q}{2}\right)(1+\mu\phi_0^2)\left[\nu\phi_0^2 - \cos(q)\sin^2\left(\frac{Q}{2}\right)(1+\mu\phi_0^2) + \mu\phi_0^2\right] = 0$$

then

$$-\Omega^2 = 16\cos(q)\sin^2\left(\frac{Q}{2}\right)(1+\mu\phi_0^2)\left[\nu\phi_0^2 - \cos(q)\sin^2\left(\frac{Q}{2}\right)(1+\mu\phi_0^2) + \mu\phi_0^2\right] = 0$$

Where det(A) = 0 Then, we derive the gain which has form

$$g = \mathrm{Im}|\Omega| = 16\cos(q)\sin^2\left(\frac{Q}{2}\right)(1+\mu\phi_0^2)\sqrt{\left[\nu\phi_0^2 - \cos(q)\sin^2\left(\frac{Q}{2}\right)(1+\mu\phi_0^2) + \mu\phi_0^2\right]} = 0 \quad (11)$$

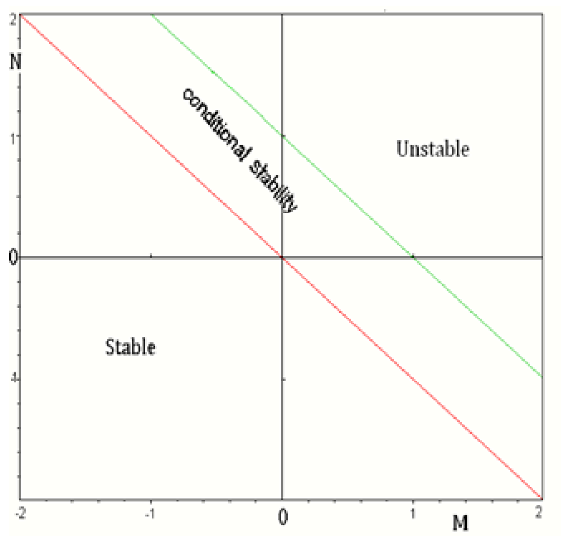

Fig. 1: Modulational Instability regions $N = \frac{\nu\phi_0^2}{\cos(q)(1+\mu\phi_0^2)}$, $M = \frac{\mu\phi_0^2}{(1+\mu\phi_0^2)}$, with $\cos(q) > 0$

From the gain of equation (11), we determine the regions of modulational instability. If the expression under root square is negative then the frequency of perturbation is real, and the modulational is stable. Hence the perturbation leads to small oscillations. If the expression under root square is positive so is positive, then modulational is unstable, it means that the modulational is growth exponentially when evolution ---ti-ne.

From the gain of equation, we can see that the unstaggered solution q = 0 is unstable whenever $\upsilon > \frac{1}{\phi_0^2}$

and the staggered solution q = π is unstable whenever $\upsilon < -\frac{1}{\phi_0^2}$ in general case where q can take any value,

We can distinguish two regions of modulational instability as shown in figure.

The one region is fully unstable and other is conditionally stable, it mean that it depends on the wave number of carrier waves and wave number of perturbation as shown in the figure.

IV. Conclusion:

To conclude, we have investigated the modulational instability in a one-dimension described by Salerno equation. The model can applied to BECs in deep optical lattice. Analytical expression for the MI gain spectra is obtained, the regions and conditions of instability of plane wave solutions in the parameter space of the governing Salerno model is determined.

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