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TAYLOR-NEWTON HOMOTOPY METHOD FOR COMPUTING THE DEPTH OF FLOW RATE FOR A CHANNEL

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Abstract

Homotopy approximation methods (HAM) can be considered as one of the new methods belong to the general classification of the computational methods which can be used to find the numerical solution of many types of the problems in science and engineering. The general problem relates to the flow and the depth of water in open channels such as rivers and canals is a nonlinear algebraic equation which is known as *continuity equation*. The solution of this equation is the depth of the water. This paper represents attempt to solve the equation of depth and flow using Newton homotopy based on Taylor series. Numerical example is given to show the effectiveness of the purposed method using MATLAB language.

Key Words: Newton homotopy, Taylor-Newton homotopy method, open channel, MATLAB.

1. Introduction

It was known that finding the depth of the water and its velocity in open channel flow is one of the most important problems in civil engineering. To determine it, we need to solve a nonlinear algebraic equation called *continuity equation*. There are many classical numerical methods used to solve this equation, such as Newton-Raphson method. This method has been proven globally convergent only under unrealistically restrictive conditions. They sometimes fail because it is difficult to provide a starting point sufficiently close to an unknown solution [1].

To overcome this convergence problem, globally convergent homotopy methods have been studied by many researchers from various view points. By these studies, the application of the homotopy methods in practical engineering simulation has been remarkably developed homotopy problem with the theoretical guarantee of global convergence. However, the programming of sophisticated homotopy methods is often difficult for non-experts or beginners.[1-23, 25-29]

There are several types of homotopy methods, as one of the efficient methods for solving *Van der Waals* problem [11]; the Newton homotopy (NH) method is well known [6]. For this method, many studies have been formed from various viewpoints. Since the idea of Newton homotopy is introduced, the path following often becomes smooth. However, in this method, the initial point is sometimes far from the solution [3].

In this paper, we discuss the use of Newton homotopy method (NHM) based on Taylor series of the multiple version to solve the nonlinear algebraic equation called *continuity equation* for finding the depth of the water in open channel flow. Newton homotopy connects a trivial solution of this nonlinear algebraic equation to the desired unknown solution. The proposed method is almost globally convergent algorithm [10]. By numerical examples, it is shown that the proposed method efficient than the conventional methods. It is also shown that the proposed method can be easily implemented in MATLAB software.

2. The Equation of Open-Channel Flow

The most fundamental relationship between flow q (m^3/s) and depth h (m) is the *continuity equation*

$$q = ua_c \tag{1}$$

where a_c = the cross-sectional area of the channel (m^2) and u = a specific velocity (m/s). Depending on the channel shape, the area can be related to the depth by some functional relationship. For the rectangular channel depicted in figure 1,

$$a_c = bh \tag{2}$$

where b = the width (m)

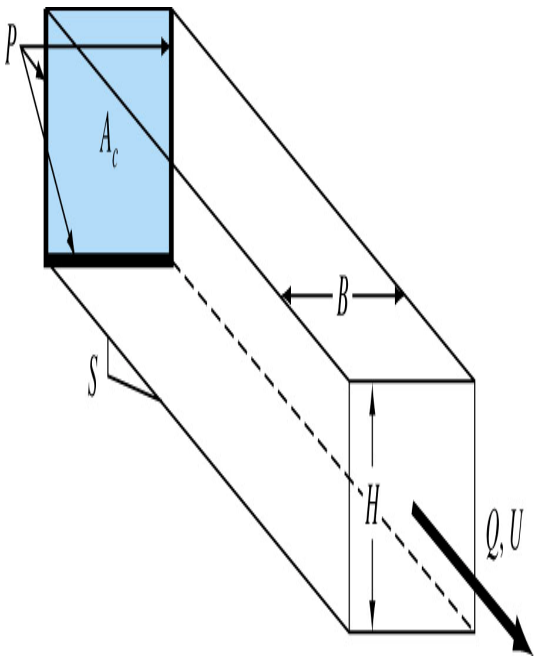

Figure 1

Substituting this relation into equation (1) gives

$$q = ubh \qquad (3)$$

Now, although equation (3) certainly relates the channel parameters, it is not sufficient to find the depth *h* and the velocity *u* because assuming that *b* is specified, we have one equation and two unknowns (*u* and *h*). We therefore require an additional equation. For uniform flow (meaning that the flow does not vary spatially and tempo7rally, the Irish engineer Robert Manning proposed the following semi-empirical formula (appropriately called the *Manning equation*)

$$u = \frac{1}{n} r^{2/3} s^{2/3} \qquad (4)$$

where n = the Manning roughness coefficient (a dimensionless number used to parameterize the channel friction), s = the channel slope (dimensionless, meters drop per meter length), and r = the hydraulic radius (m), which is related to more fundamental parameters by

$$r = \frac{a_c}{p} \qquad (5)$$

where p = the wetted perimeter (m). As the name implies, the wetted perimeter is the length of the channel sides and bottom that is under water. For example, for a rectangular channel, it is defined as

$$p = b + 2h \qquad (6)$$

Although the system of nonlinear equations (3) and (4) can be solved simultaneously using Newton-Raphson approach, an easier approach would be to combine the equations. Equation (a) can be substituted into equation (3) to give

$$q = \frac{bh}{n} r^{2/3} s^{1/3} \qquad (7)$$

Then the hydraulic radius, Equation (5), along with the various relationships for the rectangular channel can be substituted,

$$q = \frac{s^{1/2}}{n} \frac{(bh)^{5/3}}{(b+2h)^{2/3}} \qquad (8)$$

Thus, the equation now contains a single unknown *h* along with the given value for *q* and the channel parameters (*n*, *s*, and *b*).

Although we have one equation with an unknown, it is impossible to solve explicitly for *h*. However, the depth can be determined numerically by reformulating the equation as a root problem,

$$f(h) = \frac{s^{1/2} (bh)^{5/3}}{n(b+2h)^{2/3}} - q = 0 \qquad (9)$$

Equation (9) can be solved readily with any of the root-location methods either bracketing methods such as bisection and false position or open methods like the Newton-Raphson and secant method. The complexity of bracketing methods is that to determine whether we can estimate lower and upper guesses that always bracket a single root. For the cases of initial guesses, the problem of initial guesses is complicated by the issue of convergence. For this reason, we will apply the method of Taylor Newton Homotopy described in section 3 to solve equation (9) [44].

3. Taylor-Newton Homotopy Method

First, we describe roughly the theoretical basis of the method. Suppose we need to solve the following equation:

$$f(x) = 0 \qquad (10)$$

where f is a continuously differentiable function from R^n into itself. To find the solution, we can construct a homotopy. The term *homotopy* means a continuous mapping H is defined on the product R^{n+1}: $R^n \times I$ to R^n, H: $R^n \times I \rightarrow R^n$ where I is the unit interval [0, 1] such that

$$H(x, t) = tf(x) + (1 - t)g(x) \qquad (11)$$

from $g(x) = 0$ to $f(x) = 0$, where the solution of $g(x) = 0$ can be found trivially. For example, choose

$$g(x) = f(x) - f(x^0) \qquad (12)$$

Substitute equation (5) in equation (4) and suppose that x is a function of t, we obtained a new version of homotopy

$$H(x, t) = f(x) + (t - 1)f(x^0) \qquad (13)$$

This form of the homotopy is termed the Newton homotopy because some of the ideas behind it come from the work of *Sir Isaac Newton* (1642-1727) himself [8].

It is seen from equation (13) that, at $t = 0$, the solution of the equation (13) is already known. For different values of t, the equation will result in different solutions. At t =1, the solution of equation (13) is identical to the desired unknown solution. To follow the above predetermined trajectory, define

$$H^{-1} = \{(x, t) = 0 / H(x, t) = 0\} \qquad (14)$$

as the set of all solutions $(x, t) \in R^{n+1}$ to the system $H(x, t) = 0$. If H is continuously differentiable, we can make a linear approximation of it near (x^k, t^k) using *multivariable version of the Taylor series*:

$$H(x^{k+1}, t^{k+1}) = H(x^k, t^k) + (x^{k+1} - x^k)H_x(x^k, t^k) + (t^{k+1} - t^k)H_t(x^k, t^k) + \ldots \qquad (15)$$

or

$$H(x^{k+1}, t^{k+1}) \approx H(x^k, t^k) + (x^{k+1} - x^k)H_x(x^k, t^k) + (t^{k+1} - t^k)H_t(x^k, t^k) \qquad (16)$$

The symbol ≈ means approximately equal and

$$H_x = \frac{\partial H}{\partial x}, \text{ and } H_t = \frac{\partial H}{\partial t}.$$

Study points (x^{k+1}, t^{k+1}) in H^{-1} that are near the point (x^k, t^k), as we want to show that they are on a path through. By definition, all such points must satisfy

$$H(x^{k+1}, t^{k+1}) = 0. \qquad (17)$$

Moreover, if we assume that H is actually linear near (x^k, t^k), then by (16) and (17), we get

$$0 = H(x^k, t^k) + (x^{k+1} - x^k)H_x(x^k, t^k) + (t^{k+1} - t^k)H_t(x^k, t^k) \qquad (18)$$

In other words, if H where actually linear near (x^k, t^k), then any $(x^{k+1}, t^{k+1}) \in H^{-1}$ point near (x^k, t^k) would have to satisfy this equation [8]. Assuming that H_x is invertible at (x^k, t^k),

$$x^{k+1} = x^k - \frac{H(x^k, t^k) + \Delta t H_t(x^k, t^k)}{H_x(x^k, t^k)};$$
$$\Delta t = t^{k+1} - t^k \text{ and } k = 0, 1, 2, \ldots \qquad (19)$$

or

$$x^{k+1} = x^k - [H_x(x^k, t^k)]^{-1}[H(x^k, t^k) + \Delta t H_t(x^k, t^k)];$$
$$k = 0, 1, 2, \ldots \qquad (20)$$

Here $t^k \in [0, 1]$. We named (19) & (20) as Taylor-Newton homotopy (T-NH) formulas.

4. The Proposed Algorithm (T-NH Algorithm)

Based on the above theory, we propose a new application of Newton homotopy based on Taylor series (T-NH) for solving *the continuity equation* as follows:

$$f(h) = \frac{s^{1/2}(bh)^{5/3}}{n(b+2h)^{2/3}} - q$$
$$H(h, t) = f(h) + (1-t)f(h^0) \qquad (21)$$

where h is the depth of the water. It is observed that the first term of the above equation is equal to *te continuity equation* and h^0 is the initial value for the depth of the water h. The value of h can be obtained by solving (21) using (22);

$$h^{k+1} = h^k - \frac{H(h^k, t^k) + \Delta t H_t(h^k, t^k)}{H_h(x^k, t^k)};$$
$$k = 0, 1, 2, \ldots \qquad (22)$$

Thus, the proposed method is summarized as follows:

1. Choose the initial value h^0.
2. Choose Δt.
3. Follow the trajectory from $t = 0$ to $t = 1$ to find the desired solution using (22).
4. Stop when the norm less than or equal the allowance tolerance ε (epsilon: Small positive real number) ;

$$\|f(h)\| \leq \varepsilon \text{ .or } \|f(h)\| \to 0$$

5. Numerical Example

Compute the depth for the water in open vhannel using the following provided data:

$$q = 5 \text{ m}^3/\text{s},\ n = 0.03,\ b = 20\text{m},\ s = 0.0002,$$

Solution

$$f(h) = \frac{0.471405(20h)^{5/3}}{(20+2h)^{2/3}} - 5$$

$$f'(h) = \frac{15.7135(20h)^{2/3}}{(20+2h)^{2/3}} - \frac{12.5708(20h)^{5/3}}{(20+2h)^{5/3}}.$$

We choose the minimum value of h as the initial value of ,

$$h^0 = 0.001$$
$$f(0.001) = -4.999$$
$$f'(0.001) = 0.1571$$
$$H_t = f(h^0) \ \&\ H_v = f'(h)$$

By choosing $\Delta t = 0.25$, the calculations to determine h are shown by the table in below,

Table(1)

No.	t^k	h^k	h^{k+1}	N
1	0	0.001	3.1833	49.0168
2	0.25	3.1833	2.3094	28.1215
3	0.50	2.3094	1.0200	04.1334
4	0.75	1.0200	0.7023	4.4e-05

From the table above, it appears that when the parameter $t^k = 0.75$, $k = 4$ then $t^{k+1} = 1$ and the depth $h^{k+1} = 0.7023$ with norm $N = 4.4288\text{e-}005$. Since this norm is very close to zero then the obtained value of h; $h = 0.7023$ can be the desired value for the depth of the water.

6. Conclusion

The Taylor-Newton homotopy method is conceptually very simple in finding the depth of the water in open channel. We conclude that the norm becomes smaller when the parameter t increases from $t = 0$ to $t = 1$. The values of norm showed that this method is globally convergent. To gain more effective results, we can choose Δt smaller and repeat the same calculations. Another advantage is with certain number of iterations, the desired solution will be obtained. Based this advantage, in this method, we didn't need to control the value of the error as in the classical methods as Bisection, fixed-point and Newton-Raphson methods. In particular, the proposed method is especially attractive compared with the existing methods. Naturally, the proposed method take more computing time, this is the common disadvantage of homotopy methods. At the expense of computing speed we can achieve a much wider convergent region.

7. References

[1] A. Kevorkian, "A Decomposition Methodology and a Class of Algorithms for the Solution of Nonlinear Equations" Mathematics of operations research, Vol. 6, No. 2, 1981, pp. 277-292.

[2] A. Ushida and L. Chua, "Tracing Solution Curves of Nonlinear Equations with Sharp Turning Points" Circuit Theory and Applications, Vol. 12, 1984, pp. 1-21.

[3] Andrew J. Sommese and Charles W. Wampler, The numerical solution of systems of polynomials. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[4] Andrew Sommese, "Solving Polynomial Systems by Homotopy Continuation, Algebraic Geometry and Applications" Seminar, IMA, 2006.

[5] Anton Leykin and Jan Verschelde. "Interfacing with the numerical homotopy algorithms in PHCpack", In Nobuki Takayama and Andres Iglesias, editors, Proceedings of ICMS 2006, pp. 354–360, 2006.

[6] C. B. Garcia and W. B Zangwill, "Pathways to Solutions, Fixed Points, and Equilibrium" Prentice Hall, 1981.

[7] C. Desa, K. M. Irani, C. J. Ribbens, L. T. Watson, and H. F. Walker, , "Preconditioned Iterative Methods for Homotopy Curve Tracking" SIAM J. SCI. STAT. COMPUT., Vol. 13, No. 1, 1992, pp. 33-46.

[8] C. Garcia, and W. Zangwill, Pathways to Solutions, Fixed Points, and Equilibrium Handbook, Prentice Hall. U.S.A, 1981.

[9] D. Wolf and S. Sanders, "Multiparameter

Homotopy Methods for Solving DC Operating Points of Nonlinear Circuits" IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, Vol. 43, No. 10, 1996, pp. 824-838.
[10] E Allgower and K. Georg, “Numerical Path Following” Internet Document, Colorado, U.S.A., 1994.
[11] E. Allgower and K. Georg, “Simplicial and Continuation Methods for Approximating Fixed Poits and Solutions to Systems of Equations” SIAM REVIEW, Vol. 22, No. 1, 1980, pp. 28-85.
[12] E. Allgower, and K. Georg, 1994, "Numerical Path Following" handbook of numerical analysis, volume 5, pp. 3-207, North-Holland.
[13] E. Allgower, and K. Georg, "Piecewise Linear Methods for Nonlinear Equations and Optimization",http://www.math.colostate.edu/~georg/, 1999, pp. 1-18.
[14] J. M. Todd, “Exploiting Structur in Piecewise-Linear Homotopy Algoritms for Solving Equations” School of Operation Research and Industrial Engineering, College of Engineering, Cornell University, Ithaca, NY, U.S.A., 1978.
[15] J. Verschelde and A. Haegemans, "Homotopies for solving polynomial systems within a bounded domain" Theoretical Computer Science 133, 1994, pp. 165-185.
[16] Kapil Ahuja, Probability-One Homotopy Maps for Mixed Complementarity Problems, Thesis submitted to the faculty of the Virginia Polytechnic, Institute and State Universityin partial fulfillment of the requirements for the degree of Master of Science in Computer Science & Applications, 2007.
[17] K. M. Irani, M. P. Kamat, C. J. Ribbens, H. F. Walker and L. T. Watson, “Experiments with Conjugate Gradient Algorithms for Homotopy Curve Tracking” SIAM J. Optimization, Vol. 1, N0. 2, 1991, pp. 222-251.
[18] K. Yamamura and K. Horiuchi, “Quadratic Convergence of the Homotopy Method Using A Rectangular subdivision” Trans. IEICE, Vol. E 72, No. 3, 1989, pp. 188-193.
[19] K. Yamamura and K. Horiuchi, “Solving Nonlinear Resistive Networks by a Homotopy Method Using a Rectangular Subdivision” The Transactions of the IEICE, Vol. E 72, No. 5, 1989, pp. 584-549.
[20] K. Yamamura and K. Horiuchi, “A Globally and Quadratically Convergent Algorithm for Solving Nonlinear Resistive Networks” IEEE transactions on computer-aided design, Vol. 9, No. 5, 1990, pp. 487-499.
[21] K. Yamamura and M. K. Kiyo, “A Piecewise-Linear Homotopy Method with the Use of the Newton Homotopy and a Polyhedral Subdivision” the transactions of the IEICE, Vol. E 73, No. 1, 1990, pp 140-148.
[22] K. Yamamura K., K. Horiuchi. and S. Oishi, , "A Decomposition Method Based on Simplicial Approximation for the Numerical Analysis of Nonlinear Systems" proceedings of ISCAS, 1985, pp. 635-638.
[23] K. Yamamura, H. Kubo and K. Horiuchi, “A Globally Convergent Algorithm Based on Fixed-Point Homotopy for the Solution of Nonlinear Resistive Circuits”, Trans. IEICE, J70-A, 10, 1987, pp. 1430-1438.
[24] S. C. Chapra and R. P. Canale, Numerical Methods for Engineers, Fifth Edition, 2008.
[25] S. Lui, B. Keller and T. Kwok, "Homotopy Methods for the Large, Sparse, Real Nonsymmetric Eigenvalue Problems" SIAM J. Matrix Anal. Appl. Vol.18, No. 2, 1997. pp 312-333.
[26] Talib. Hashim and F. Soesianto, "A Newton Homotopy Method for Solving Systems of Nonlinear Equations" to appear. 2000.
[27] W. Kuroki, K. Yamamura, and S. Furuki, “An Efficient Variable Gain Homotopy Method Using the SPICE-Oriented Approach”, IEEE Transactions on Circuits and Systems-II: Express Briefs, VOL. 54, No. 7., 2007.
[28] W. Lin, and G. Lutzer, "An application of the homotopy method to the ceneralized symmetric eigenvalue problem" J. Austral. Math. Soc. Ser. B 30, 1988. pp. 230-249.
[29] Xudong Zha and Qiuming Xiao, “Homotopy Method for Back calculation of Pavement Layer Moduli”, International Symposium on Non-Destructive Testing in Civil Engineering, 2003.

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