# Comparison of High-order Accurate Schemes for Solving the Nonlinear Viscous Burgers Equation

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**Abstract:** In this paper, a comparison between higher order schemes has been performed in terms of numerical accuracy. Four finite difference schemes, the explicit fourth-order compact Pade scheme, the implicit fourth-order Pade scheme, flowfield dependent variation (FDV) method and high order compact flowfield dependent variation (HOC-FDV) scheme are tested. The FDV scheme is used for time discretization and the fourth-order compact Pade scheme is used for spatial derivatives. The solution procedures consist of a number of tri-diagonal matrix operations and produce an efficient solver. The comparisons are performed using one dimensional nonlinear viscous Burgers equation to demonstrate the accuracy and the convergence characteristics of the high-resolution schemes. The numerical results show that HOC-FDV is highly accurate in comparison with analytical and with other higher order schemes.

**Key words:** Flowfield-dependent variation (FDV), Higher-order compact (HOC), Burgers' equation, Finite Difference Method, Pade scheme.

## INTRODUCTION

Higher-order-accurate methods (greater than second-order) are used for direct numerical simulation (DNS) in order to minimize errors. The advantages of using higher-order compact (HOC) scheme over traditional finite difference methods include the high order of accuracy, better stability, better resolution and fewer boundary points to be applied at boundaries. The fundamental idea behind higher-order compact schemes, also known as Padé schemes, is that the derivatives are treated as unknowns at each point of the computational grid. To evaluate the derivatives, high order relations are provided and solved simultaneously with the governing equations of the problem considered. The high-order relations are derived by reconstruction of the weighted average of the mesh function including neighboring point derivatives in order to obtain a high order difference relation with a narrow stencil (Lele 1992). DNS recent algorithms have used high-order accuracy and the resolution power of HOC finite difference schemes such as Yee et al (1997), Adams (1998), Freund et al (2000), Nagarajan et al (2003) among others. High-order compact schemes have also found their way into convection diffusion problems (Spotz 1995).

Hirsh (1975) has shown that the fourth-order compact scheme has better accuracy than the non-compact one due to the smaller coefficients of the truncation error terms, and has also discussed the stability properties of the scheme for a linearized model problem. Two different techniques proposed by Adam (1977) to eliminate the second-order derivatives in parabolic equations, while keeping the fourth-order accuracy and the tri-diagonal nature of the scheme. Lele (1992) has presented and analyzed more generalized forms of the Hermitian schemes and introduced the notion of resolution efficiency as a measure of accuracy. Asrar et al (2002) and Jiun et al (2003) have shown that a GEB fourth-order compact discretization of the one-dimensional viscous Burgers equation gives more accurate results than the Hermitian discretization for the same order of accuracy. In the FDV method recently proposed by Chung (1999), the characteristic parameters of the flow field are calculated to guide the numerical scheme to a solution. The basic idea is to extend the conservation of flow variables into a Taylor series in terms of FDV parameters, which are related to changes in physical parameters such as the Reynolds number and Mach number. The high order compact flowfield dependent variation (HOC-FDV) scheme proposed by Elfaghi et al (2007) has been used to solve Burgers' equation. The scheme has shown more accurate results over FDV and traditional second order schemes.

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### Governing Equation:

The Navier-Stokes equations (without the source term) can be written in conservation form as:

$$\frac{\partial U}{\partial t} + \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} = 0 \tag{1}$$

Where:

$$U = \begin{bmatrix} \rho \\ \rho V_i \\ \rho E \end{bmatrix}, \quad F_i = \begin{bmatrix} \rho V_i \\ \rho V_i V_j + p\delta_{ij} \\ \rho E V_j + p V_j \end{bmatrix}, \quad G_i = \begin{bmatrix} 0 \\ -\tau_{ij} \\ -\tau_{ij} V_j + q_i \end{bmatrix}$$

Derivations of the FDV equations, as introduced by Chung (2002), begin with the expansion of Eq. (1) in a special form of the Taylor series about $U^n$ and introducing the parameters $s_a$ and $s_b$ for the first and second order derivatives of U with respect to time, respectively. The compact form of FDV equation is (Chung 2002)

$$\Delta U^{n+1} + \frac{\partial_i}{\partial x_i}\left( D_i\, \Delta U^{n+1} \right) + \frac{\partial^2}{\partial x_i\,\partial x_j}\left( D_{ij}\, \Delta U^{n+1} \right) = -Q^n \tag{2}$$

Lagging Di and Dij one time step behind,

$$\left( I + D_i^n \frac{\partial_i}{\partial x_i} + D_{ij}^n \frac{\partial^2}{\partial x_i\,\partial x_j} \right) \Delta U^{n+1} = -Q^n \tag{3}$$

Where:

$$D_i^n = \Delta t \left( s_1 a_i + s_3 b_i \right)^n \tag{4}$$

$$D_{ij}^n = \left\{ \Delta t\, s_3 c_{ij} - \frac{\Delta t^2}{2}\left[ s_2\left( a_i a_j + b_i a_j \right) + s_4\left( a_i b_j + b_i b_j \right) \right] \right\}^n \tag{5}$$

$$Q^n = \Delta t \frac{\partial}{\partial x_i}\left( F_i^n + G_i^n \right) - \frac{\Delta t^2}{2} \frac{\partial^2}{\partial x_i\,\partial x_j}\left[ \left( a_i + b_i \right)\left( F_j^n + G_j^n \right) \right] \tag{6}$$

The Jacobeans, $a_i$, $b_i$, and $c_{ij}$ are based on the convection, diffusion, and diffusion gradient terms, respectively, and are defined by:

$$a_i = \frac{\partial F_i}{\partial U}, \quad b_i = \frac{\partial G_i}{\partial U}, \quad c_{ij} = \frac{\partial G_i}{\partial U,_j} \tag{7}$$

and

$$\Delta U^{n+1} = U^{n+1} - U^n \tag{8}$$

The concept of flow field-dependent variation theory provides a modified form of incremental partial differential equations. The physical interpretation of the FDV parameters $s_a$ and $s_b$ is the foundation of the FDV method. Large values of these parameters reflect large changes in the conservation variables. These changes may occur between adjacent nodal points within the special nodes as well as between adjacent time steps. The first-order FDV parameter, $s_a$, is separated into a convection parameter, ($s_1$), and diffusion and diffusion gradient parameter, ($s_3$). Similar arguments apply to the second-order FDV parameter, $s_b$ leading to $s_2$ for convection, and $s_4$ for diffusion and diffusion gradients. These second-order FDV parameters are chosen to be exponentially proportional to the first-order FDV parameters. This choice is based on the fact that the first-order FDV parameters tend to assure accuracy of the solution, whereas the second-order FDV parameters provide numerical stability (diffusion), exponentially proportional to the first-order FDV parameters. These properties lead to the following definitions for the first-order and second-order variation parameters in terms of the Mach number (M), and Reynolds number (Re):

First and second-order convection variation parameter s1 and s2:

$$s_1 = \begin{cases} \min(r,1), & r > \alpha, \quad \alpha = 0.01 \\ 0 & r < \alpha, \quad for \ M_{min} \neq 0 \\ 1 & M_{min} = 0 \end{cases} \tag{9}$$

$$s_2 = \frac{1}{2}\left(1 + s_3^{\eta}\right), \qquad 0.05 < \eta < 0.2 \tag{10}$$

with

$$r = \sqrt{M_{max}^2 - M_{min}^2} / M_{min} \tag{11}$$

First and second -order diffusion parameter s3 and s4:

$$s_3 = \begin{cases} \min(r,1), & r > \alpha \quad \alpha = 0.01 \\ 0 & r < \alpha, \quad Re_{min} \neq 0 \\ 1 & Re_{min} = 0 \end{cases} \tag{12}$$

$$s_4 = \frac{1}{2}\left(1 + s_1^{\eta}\right), \qquad 0.05 < \eta < 0.2 \tag{13}$$

with

$$r = \sqrt{Re_{max}^2 - Re_{min}^2} / Re_{min} \tag{14}$$

The variation parameters introduced in the above equations are used for a variety of purposes. All the variation parameters fall between 0 and 1 and are calculated locally at each element making them flow field dependent. The values of $s_1$ and $s_3$ are high in regions of high gradients and small in regions of small gradients.

Contrary to the Beam-Warming scheme (1978), the FDV approach is to obtain the implicitness parameters from the current flowfield variables at each and every nodal point rather than by fixing the implicitness parameters to certain predetermined numbers and using them for the entire flow domain irrespective of local flowfield variation from one point to another.

*Applications:*

The one dimensional non-linear viscous Burgers equation is solved numerically used explicit and implicit HOC scheme, FDV scheme and HOC-FDV scheme. The equation is in the form (Hoffmann, 2000) :

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \tag{15}$$

Initial distribution is given by the following equation.

$$u = -\frac{2 \sin(x)}{\cos(x) - e^{-t}} \tag{16}$$

at t = 0.1 and x varies from -9.0 – 9.0, the boundary conditions are: at x = - 9.0, u= 2.0 and at x = 9.0, u = -2.0, the spatial step size: dx = 0.2 and the time step dt = 0.01

*Explicit HOC Burgers Equation:*

To solve equation (15) using explicit HOC, the following three equations are solved simultaneously (Hirsh, 1975):

$$\left[ \left( u_i^{n+1} - u_i^n \right) \right] + u_i^n f_i^n = S_i^n \tag{17}$$

$$f_{i-1}^n + 4 f_i^n + f_{i+1}^n = \frac{3}{h} \left( u_{i-1}^n + u_{i+1}^n \right) \tag{18}$$

$$S_{i-1}^n + 10 S_i^n + S_{i+1}^n = \frac{12}{h^2} \left( \Delta u_{i-1}^n - 2 \Delta u_i^n + \Delta u_{i+1}^n \right) \tag{19}$$

*Implicit FDV Burgers Equation:*

Fully implicit HOC solution is achieved by solving the following system of equations for (*u, f*) and S simultaneously at the time level n+1.

$$\left[ \left( u_i^{n+1} - u_i^n \right) \right] + u_i^n f_i^{n+1} = S_i^{n+1} \tag{20}$$

$$f_{i-1}^{n+1} + 4 f_i^{n+1} + f_{i+1}^{n+1} = \frac{3}{h} \left( \Delta u_{i-1}^{n+1} + \Delta u_{i+1}^{n+1} \right) \tag{21}$$

$$S_{i-1}^{n+1} + 10 S_i^{n+1} + S_{i+1}^{n+1} = \frac{12}{h^2} \left( \Delta u_{i-1}^{n+1} - 2 \Delta u_i^{n+1} + \Delta u_{i+1}^{n+1} \right) \tag{22}$$

A block tri-diagonal matrix will be generated at each time step which can be solved using available algorithms.

*FDV Burgers Equation:*

For solving the Burger's equation, Eq. (3) is rewritten for the one-dimensional momentum equation without the pressure gradients.

$$\Delta u^{n+1} = -\Delta t\,(s_1\,a)^n\,\frac{\partial \Delta u^{n+1}}{\partial x} + \left[\Delta t\,s_3\,c - \frac{\Delta t^2}{2}(s_2\,a^2)^n\right]\frac{\partial^2 \Delta u^{n+1}}{\partial x^2}$$

$$+ \Delta t\,\frac{\partial}{\partial x}\left(F^n + G^n\right) - \frac{\Delta t^2}{2}\,a\,\frac{\partial^2}{\partial x^2}\left(F^n + G^n\right) \tag{23}$$

Where $F^n = \left(\dfrac{u^2}{2}\right)^n$, $\quad G^n = \left(\dfrac{\partial u}{\partial x}\right)^n$, $\quad a = u^n \quad and \quad c = -1,$

Equation (23) is solved for the values of velocity at time $n+1$. The first derivative, $\dfrac{\partial \Delta u^{n+1}}{\partial x}$ and the second derivative, $\dfrac{\partial^2 \Delta u^{n+1}}{\partial x^2}$ re approximated at each node by second order central differences. The resulting finite difference equations are then solved using tri-diagonal matrix solver to calculate $u^{n+1}$ at all grid points.

***HOC-FDV Burgers Equation:***

To solve the Burgers equation using the higher order compact-flowfield dependent variation (HOC-FDV) scheme, the first and second derivatives of $\Delta u^{n+1}$ in equation (23) are approximated by using the implicit fourth order compact differencing scheme proposed by Hirsh (1975) in the following form:

$$f_{i-1}^{n+1} + 4 f_i^{n+1} + f_{i+1}^{n+1} = \frac{3}{h}\left(\Delta u_{i-1}^{n+1} + \Delta u_{i+1}^{n+1}\right) \tag{24}$$

$$S_{i-1}^{n+1} + 10\,S_i^{n+1} + S_{i+1}^{n+1} = \frac{12}{h^2}\left(\Delta u_{i-1}^{n+1} - 2\,\Delta u_i^{n+1} + \Delta u_{i+1}^{n+1}\right) \tag{25}$$

Where: $f^{n+1} = \left(\dfrac{\partial \Delta u^{n+1}}{\partial x}\right)$ $S^{n+1} = \left(\dfrac{\partial^2 \Delta u^{n+1}}{\partial x^2}\right)$

Equations (23), (24) and (25) are solved for the three unknowns $\Delta u^{n+1}$ $f^{n+1}$ and $S^{n+1}$ it each time step to form a fully implicit system of equations which are coupled and solved simultaneously using the block tri-diagonal matrix inversion.

## RESULTS AND DISCUSSION

The schemes are tested by calculating the absolute error between the numerical solution of each scheme and the analytical solutions at different time steps. Figure 1 shows the results for FDV method for times t = 0.1s, 0.4s, 0.7s and 1.0s. The numerical and analytical solutions are visually identical. Therefore the absolute errors between the numerical and the exact solutions are required to compare the accuracy of the FDV scheme.

Figure 2 shows the comparison between absolute errors of implicit higher order compact and implicit higher order compact schemes. The implicit higher order compact scheme gives accurate results over explicit scheme. The maximum error reduced by 69% when using implicit higher order compact scheme compared to explicit higher order compact scheme.
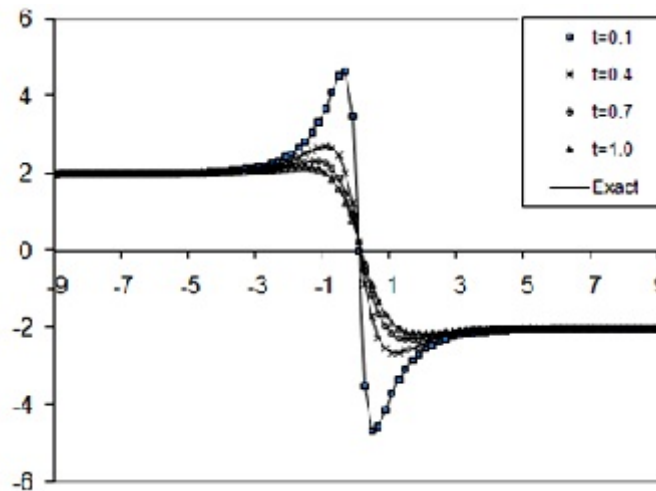
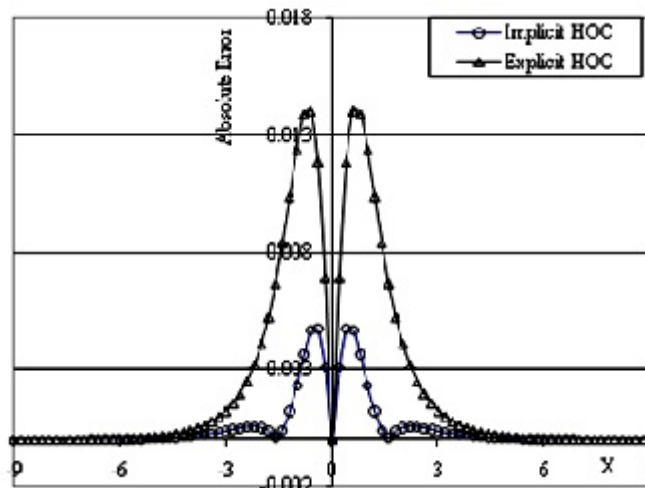**Fig. 1:** FDV solution at times t = 0.1s, 0.4s, 0.7s, and 1.0



**Fig. 2:** Comparison between Implicit and Explicit HOC Schemes for Non-linear Viscous Burger's Equation at t=1.0s

The plot for the error calculated by subtracting the numerical values from the exact solution for time t = 1.0 sec is shown in Fig. 3. This figure shows a reduction by 80% in the maximum error at the discontinuity region when using FDV method comparing with the BTCS method, Hoffmann (2000).

The plot for the absolute errors calculated by subtracting the numerical values from the exact solution of Equation (16) available in Hoffmann and Chiang (2000) and Edward and George (1972), for time t = 1.0 sec is shown in Fig. 4. This figure shows a reduction by 88% in the maximum error at the discontinuity region when using FDV method when compared with the BTCS method, Hoffmann and Chiang (2000). High order compact scheme with flowfield-dependent variation method, (HOC-FDV), gives better results than the results obtained from using flowfield-dependent variation method with second order central approximations and the maximum error is further reduced by 56% as compared to the FDV technique.

***Conclusion:***
A numerical simulation of the non-linear viscous Burger's equation using explicit higher order compact scheme, fully implicit high order compact scheme, flow field-dependent variation (FDV) and high order compact-flow field-dependent variation (HOC-FDV) method have been obtained. The results have been compared with the standard BTCS scheme and with the analytical results. It can be concluded based on the results that the HOC-FDV method is more accurate compared with the other approaches.
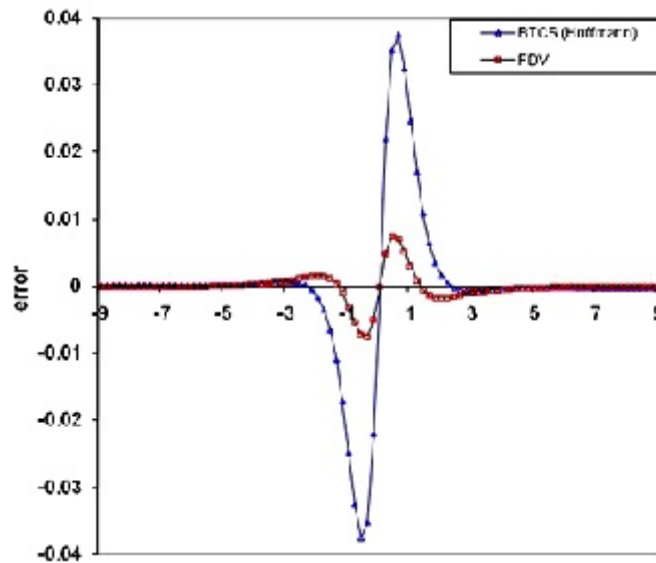
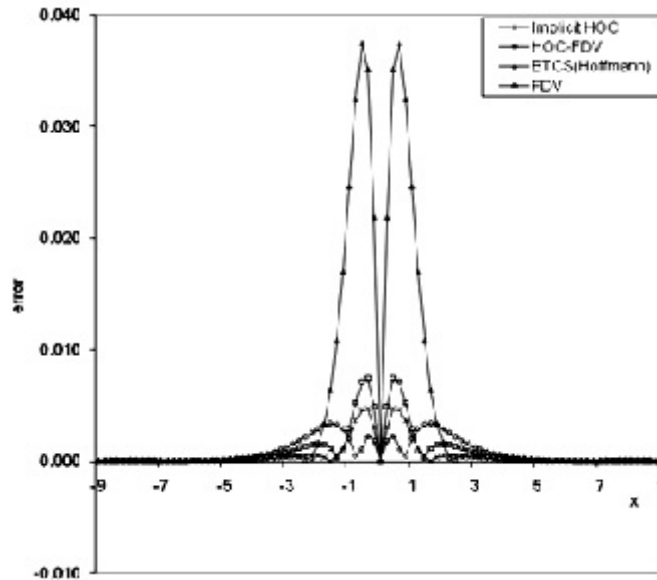**Fig. 3:** Comparison of error distributions for FDV and BTCS (Hoffmann) at t=1.0sec



**Fig. 4:** Comparison of absolute error distribution at t=1.0 sec, BTCS (Hoffman), Implicit HOC, FDV, and HOC-FDV schemes

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