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# OPTIMIZATION PROBLEM ON PERMUTATIONS WITH LINEAR-FRACTIONAL OBJECTIVE FUNCTION: PROPERTIES OF THE SET OF ADMISSIBLE SOLUTIONS

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We consider an optimization problem on permutations with a linear-fractional objective function. We investigate the properties of the domain of admissible solutions of the problem.

In recent years, a great number of works devoted to the investigation of problems of combinatorial optimization (in particular, problems on Euclidean combinatorial sets and with linear-fractional objective functions) have been published.

The aim of these investigations is to study the properties of objective functionals on combinatorial sets, determine and justify the properties of admissible sets in problems of this type, and develop methods for their solution. In [1–15], various aspects of the solution of the problems indicated are considered. For this purpose, one often uses the methods developed in [16]. In the present paper, we consider an optimization problem with linear-fractional objective function on permutations.

Let us introduce necessary terminology and present certain facts from [5] that are necessary for what follows. A collection of elements that may contain equal elements is called a multiset. A multiset $A$ is defined by its support $S(A)$, i.e., by the set of all different elements of this multiset, and multiplicity, i.e., the number of repetitions of each element of the support of this multiset. We denote the set of the first $k$ natural numbers by $J_k$ and $J_k^0 = J_k \cup \{0\}$.

Consider a multiset of real numbers $G = \{g_1, \ldots, g_k\}$ with support $s(G) = \{e_1, \ldots, e_n\}$, where $e_j \in R^1$ for any $i \in J_n$, and the multiplicities of elements $k(e_j) = \eta_j$, where $i \in J_n$, $n \le k$, and, furthermore,

$$g_1 \ge g_2 \ge \ldots \ge g_k \ge 0, \qquad e_1 > e_2 > \ldots > e_n \ge 0, \tag{1}$$

$$k_0 = 0, \quad k_1 = \eta_1, \quad k_2 = \eta_1 + \eta_2, \ldots, k_n = \eta_1 + \eta_2 + \ldots + \eta_n, \tag{2}$$

and $g_1 = \ldots = g_{k_1} = e_1, \ldots, g_{k_{n-1}+1} = \ldots = g_k = e_n$.

The set of all ordered $k$-samples from the multiset $G$ forms the general set of permutations $E_{kn}(G) \subset R^k$. The convex hull of the set $E_{kn} = E_{kn}(G)$ is called the general permutable polyhedron $\Pi_{kn}(G)$, which is described [5, 9] by the system

$$\sum_{i \in \omega} x_i \le \sum_{i=1}^{|\omega|} g_i \qquad \forall \omega \subset J_k, \tag{3}$$

$$\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} g_i, \tag{4}$$

where $|\omega|$ is the number of elements in $\omega$.

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Let us give a criterion for a vertex of $\Pi_{kn}(G)$ [5], which is necessary for what follows. If $x = (x_1, \dots, x_k)$ is a vertex of $\Pi_{kn}(G)$, then

$$\{\alpha_1^1\} \subset \{\alpha_1^2, \alpha_2^2\} \subset \dots \subset \{\alpha_1^{k-1}, \dots, \alpha_{k-1}^{k-1}\} \subset \{\alpha_1^k, \dots, \alpha_k^k\} = J_k,$$

$$\sum_{t=1}^{i} x_{\alpha_t^i} = \sum_{t=1}^{i} g_i \qquad \forall i \subset J_k,$$

where $\alpha_k^k$ is the last element in $J_k$ (the superscript denotes the number of the subset and the subscript denotes the number of the element in the subset) and, conversely, if the conditions indicated above are satisfied, then $x$ is a vertex of the general permutable polyhedron $\Pi_{kn}(G)$.

***Statement of the Problem.*** Let us find a pair $\langle F(x^*), x^* \rangle$ such that

$$F(x^*) = \max_{x \in R^k} \frac{\sum_{i=1}^k c_i x_i}{\sum_{i=1}^k d_i x_i}, \qquad x^* = \arg\max_{x \in R^k} \frac{\sum_{i=1}^k c_i x_i}{\sum_{i=1}^k d_i x_i} \tag{5}$$

under the condition

$$X = (x_1, \dots, x_k) \in E_{kn}(G) \subset R^k, \tag{6}$$

$$c_i, d_i \in R^1.$$

From problem (5), (6), we pass to the problem with a linear objective function. For this purpose, we denote

$$y_0 = \left\{ \sum_{i=1}^{k} d_i x_i \right\}^{-1}, \qquad y_i = x_i y_0, \qquad i \in J_k. \tag{7}$$

Then $\alpha\big(E_{kn}(G)\big) = E \subset R^{k+1}$, where $\alpha$ is mapping (7). Assume that $y_0 > 0$ (otherwise, we can change the sign of the numerator) and $y_i \ge 0$, $i \in J_k$. In the problems on permutations with linear objective functions, the properties of $\Pi_{kn}(G)$ play an important role because, as is known [5, 9], $\operatorname{vert}\Pi_{kn}(G) = E_{kn}(G)$, where $\operatorname{vert} M$ is the set of vertices of the polyhedron $M$.

Consider the image of $\Pi_{kn}(G)$ under the mapping $\alpha$. Substituting $y_0$ and $y_i$ into (3) and (4), we obtain

$$\sum_{i \in \omega} y_i \le \sum_{i=1}^{|\omega|} g_i y_0 \qquad \forall \omega \subset J_k, \tag{8}$$

$$\sum_{i=1}^{k} y_i = \sum_{i=1}^{k} g_i y_0, \tag{9}$$

$$\sum_{i=1}^{k} d_i y_i = 1, \tag{10}$$

$$y_0 > 0, \quad y_i \ge 0, \quad \forall i \subset J_k.$$

The set defined by system (8)–(10) is denoted by $Q_{kn}(G) \subset R^{k+1}$. Then problem (5), (6) reduces to the determination of the ordered pair

$$F(x^*) = F'(y^*) = \operatorname*{extr}_{y \in R^{k+1}} \sum_{i=1}^{k} c_i y_i, \qquad y^* = \operatorname*{arg\,extr}_{y \in R^{k+1}} \sum_{i=1}^{k} c_i y_i \tag{11}$$

under the condition

$$y = (y_0, ,y_1, \ldots, y_k) \in E \subset R^{k+1}. \tag{12}$$

Thus, let us investigate the structure of the convex hull of $E$. In the present paper, we consider the problem of investigation of the properties of the admissible set for problem (11), (12), to which problem (5), (6) is reduced. First, we consider the properties of $Q_{kn}(G)$.

***Properties of the Set $Q_{kn}(G)$.*** As is known [1, p. 17], a convex cone is the set of solutions of a system of homogeneous linear inequalities, and a pyramid is the convex hull of a polyhedron $Q$ (the base of the pyramid) and a point that does not belong to $Q$ (the vertex of the pyramid). Therefore, system (8), (9) defines a convex polyhedral cone $Q$ with the vertex $O(0, \ldots, 0)$, and relation (10) defines a hyperplane that intersects the cone and does not contain its vertex. The polyhedron $Q_{kn}(G)$ is the base of a pyramid. In order to consider the faces of the polyhedron $Q_{kn}(G)$, we first prove the following lemma on the faces of the cone $K_{kn}^0(G)$:

**Lemma 1.**

*I. If $F$ is an $m$-face ($m \in J_{k-1}$) of $K_{kn}^0(G)$, then there exist sets $\omega_1 \subset \ldots \subset \omega_{k+1-m} = J_k$ for which the inequalities in (8) turn into the equalities for any $y \in F$ ($F$ is the set of solutions of the system obtained from (8), (9) by the replacement of the inequalities in (8) by the inequalities for $\omega = \omega_\sigma$ with $\sigma \in J_{k-m}$).*

*II. If, for sets $\omega_1 \subset \ldots \subset \omega_\lambda = J_k$, the inequalities in (8) are replaced by the equalities, then the set $F$ of solutions of (8), (9) is an $m$-face of $K_{kn}^0(G)$, where*

$$m = \dim F = (k+1) - \left\{ \lambda + \sum \left( |\omega_\sigma| - |\omega_{\sigma-1}| - 1 \right) \right\} \tag{13}$$

*and the summation is carried out over all $\sigma \in J_\lambda$ for each of which there exists $j \in J_n$ such that $k_{j-1} \le |\omega_{\sigma-1}| \le |\omega_\sigma| \le k_j$ ($|\omega_0| = 0$).*

***Proof.*** I. Assume that $\Omega$ is the collection of all subsets $\omega \subseteq J_k$ for which the corresponding restrictions in (8), (9) are severe for $F, \omega', \omega'' \in \Omega$. We show that $\omega' \cup \omega'' \in \Omega$ and $\omega' \cap \omega'' \in \Omega$. If $y = (y_0, y_1, \ldots, y_k) \in F$, then

$$\sum_{i \in \omega' \cup \omega''} y_i + \sum_{i \in \omega' \cap \omega''} y_i = \sum_{i \in \omega'} y_i + \sum_{i \in \omega''} y_i = \sum_{i=1}^{|\omega'|} g_i y_0 + \sum_{i=1}^{|\omega''|} g_i y_0 \ge \sum_{i=1}^{|\omega' \cap \omega''|} g_i y_0 + \sum_{i=1}^{|\omega' \cup \omega''|} g_i y_0. \tag{14}$$

Relation (14) follows from the equalities $\omega' = A_1 \cup A_{12}$ and $\omega'' = A_2 \cup A_{12}$, where $A_1 = \omega' \setminus \omega''$ and $A_2 = \omega'' \setminus \omega'$. Then $\omega' \cap \omega'' = A_{12}$ and $\omega' \cup \omega'' = A_1 \cup A_2 \cup A_{12}$. We investigate the inequality in (14). Let

$$|A_1| = \alpha_1, \quad |A_2| = \alpha_2, \quad |A_{12}| = \alpha_{12}\colon \ |\omega'| = \alpha_1 + \alpha_{12},$$

$$|\omega''| = \alpha_2 + \alpha_{12}, \quad |\omega' \cup \omega''| = \alpha_1 + \alpha_{12} + \alpha_2, \quad |\omega' \cap \omega''| = \alpha_{12}. \tag{15}$$

Substituting (15) in (14), we get

$$\sum_{i=1}^{\alpha_1+\alpha_{12}} g_i y_0 + \sum_{i=1}^{\alpha_2+\alpha_{12}} g_i y_0 \ge \sum_{i=1}^{\alpha_{12}} g_i y_0 + \sum_{i=1}^{\alpha_1+\alpha_{12}+\alpha_2} g_i y_0 .$$

We divide this inequality by $y_0$ ($y_0 > 0$). Note that the second term on the right-hand side of this inequality contains the most part of the numbers $i$ from 1 to $\alpha_1 + \alpha_{12} + \alpha_2$. Since relation (1) is true, we establish (if all $g_i$ from the first one to the $(\alpha_1 + \alpha_{12} + \alpha_2)$th one are not equal) that the right-hand side of the last inequality can be smaller than the left-hand side. On the other hand, system (8), (9) contains restrictions for $\omega = \omega' \cup \omega''$ and $\omega = \omega' \cap \omega''$, which turn into the equalities for $y \in F$. Therefore, $\omega' \cap \omega'' \in \Omega$ and $\omega' \cup \omega'' \in \Omega$, which was to be proved.

Assume that, for $\omega' \in \Omega$ and $\omega'' \in \Omega$, the following condition is satisfied for certain $j \in J_n$:

$$|\omega'| \le k_j \le |\omega''| . \tag{16}$$

We prove that $\omega' \subseteq \omega''$ by contradiction. Let $\omega' \not\in \omega''$. Consider

$$\sum_{i=1}^{|\omega'|} g_i y_0 + \sum_{i=1}^{|\omega''|} g_i y_0 = \sum_{i=1}^{|\omega' \cap \omega''|} g_i y_0 + \sum_{i=|\omega' \cap \omega''|+1}^{|\omega'|} g_i y_0 + \sum_{i=1}^{|\omega' \cup \omega''|} g_i y_0 - \sum_{i=|\omega''|+1}^{|\omega' \cup \omega''|} g_i y_0 . \tag{17}$$

The second and the fourth sum on the right-hand side of (17) contain the same number of terms. This can be verified by the substitution of (15) into (17), namely,

$$\sum_{i=1}^{\alpha_1+\alpha_{12}} g_i y_0 + \sum_{i=1}^{\alpha_2+\alpha_{12}} g_i y_0 = \sum_{i=1}^{\alpha_{12}} g_i y_0 + \sum_{i=1+\alpha_{12}}^{\alpha_1+\alpha_{12}} g_i y_0 + \sum_{i=1}^{\alpha_1+\alpha_{12}+\alpha_2} g_i y_0 - \sum_{i=1+\alpha_{12}+\alpha_2}^{\alpha_1+\alpha_{12}+\alpha_2} g_i y_0 . \tag{18}$$

Taking (1) into account, we get

$$\sum_{i=1+\alpha_{12}}^{\alpha_1+\alpha_{12}} g_i y_0 - \sum_{i=1+\alpha_{12}+\alpha_2}^{\alpha_1+\alpha_{12}+\alpha_2} g_i y_0 > 0,$$

whence, omitting the difference and taking (16) into account, we obtain

$$\sum_{i=1}^{|\omega'|} g_i y_0 + \sum_{i=1}^{|\omega''|} g_i y_0 > \sum_{i=1}^{|\omega' \cap \omega''|} g_i y_0 + \sum_{i=1}^{|\omega' \cup \omega''|} g_i y_0 ,$$

which contradicts the conclusion about the equality in (14). In view of (1), it is necessary to consider the following cases:

(i) the second and the fourth sum with respect to $i$ in (18) begin with the same number;

(ii) the second and the fourth sum with respect to $i$ in (18) begin with different numbers $i$, provided that $g_i$ are equal.

*Case (i).* Consider the difference from (17), namely,

$$\sum_{i=|\omega'\cap\omega''|+1}^{|\omega'|} g_i y_0 - \sum_{i=|\omega''|+1}^{|\omega'\cup\omega''|} g_i y_0.$$

Since the summation begins with the same number $i$, we equate the lower limits. As a result, we get $|\omega'\cap\omega''|+1 = |\omega''|+1$, or $\alpha_{12} = \alpha_{12} + \alpha_2$, whence $\alpha_2 = 0$. Therefore, $\omega'' \subseteq \omega'$, which contradicts the assumption made above.

*Case (ii).* Consider $G = \{g_1,\ldots, g_z,\ldots, g_\alpha \ldots, g_\beta,\ldots, g_n,\ldots, g_k\}$. Let $g_z = \ldots = g_\alpha = \ldots = g_\beta = \ldots = g_n = g$, $\omega' = \{\alpha, \alpha+1, \ldots, \alpha+\alpha_{12}+\alpha_1\}$, and $\omega'' = \{\beta, \beta+1, \ldots, \beta+\alpha_{12}+\alpha_2\}$. Then $|\omega'| = \alpha_1 + \alpha_{12}$ and $|\omega''| = \alpha_2 + \alpha_{12}$. It follows from (1) that

$$k_0 = 0, \ldots, k_{t-2} = z-1, \ k_{t-1} = n, \quad k_t = p, \ldots, k_n = k, \tag{19}$$

whence $0 \le z-1 < n < p \le k$. It follows from the second and the fourth sum in equality (18) that

$$z \le \alpha_{12} + 1 \le n, \quad z \le \alpha_1 + \alpha_{12} + \alpha_2 \le n, \tag{20}$$

where $z$ is the index of a certain element $g_j$, $j \in J_k$, in the multiset $G$.

Taking into account condition (16) and relation (15), we get

$$\alpha_1 + \alpha_{12} \le k_j \le \alpha_2 + \alpha_{12}. \tag{21}$$

For equal $g_i$, where $i \in J_n \backslash J_{z-1}$, we have $\alpha > z-1$, $\beta \ge \alpha$, $\alpha_1 + \alpha_{12} + \alpha \le n$, and $\beta + \alpha_{12} + \alpha_2 \le n$.

By virtue of (20) and (21), we obtain

$$z \le \alpha_{12} + 1 \le \alpha_1 + \alpha_{12} \le \alpha_2 + \alpha_{12} < n. \tag{22}$$

Consider the following two cases:

(a) $\alpha_{12} + 1 = \alpha_1 + \alpha_{12}$;

(b) $\alpha_1 + \alpha_{12} = \alpha_2 + + \alpha_{12}$.

We have $\alpha_1 = 1$ in case (a) and $\alpha_1 = \alpha_2$ in case (b). Combining these cases and taking (21) into account, we get $k_j = 1 + \alpha_{12}$ and, by virtue of (20), we have $z \le k_j \le n$. Relations (19) yield $k_{t-2} \le k_j \le k_{t-1}$, whence $k_j = k_{t-2} = z - 1$ and $k_{j+1} = k_{t-1} = n$. Then

$$k_j \le |\omega'| \le |\omega''| \le k_{j+1}. \tag{23}$$

Hence, $\omega' \subseteq \omega''$. Consider $\omega', \omega'' \in \Omega$ for which relation (23) holds for every $j \in J_{p-1}$. The condition

$$|\omega'| + |\omega''| = |\omega' \cup \omega''| + |\omega' \cap \omega''| \tag{24}$$

and relation (23) yield

$$k_j \le |\omega' \cap \omega''| \le |\omega' \cup \omega''| \le k_{j+1}. \tag{25}$$

Let us verify this fact. Taking conditions (15) and (23) into account, we get

$$k_j \le \alpha_1 + \alpha_{12} \le \alpha_2 + \alpha_{12} \le k_{j+1}. \tag{26}$$

The first inequality in (26) is true for every $\alpha_1 \ge 0$. Consider $\min(\alpha_1 + \alpha_{12})$. For $\alpha_1 = 0$, the minimum $\alpha_{12} = |\omega' \cap \omega''|$. Therefore, the first inequality in (25) is true because

$$k_j \le \alpha_{12} \le k_{j+1}. \tag{27}$$

Consider the second inequality, representing it, in view of (15), in the following form: $\alpha_1 + \alpha_2 + \alpha_{12} \le k_{j+1}$. For $\omega', \omega'' \in \Omega$, according to (23) the following conditions are satisfied:

$$k_j \le |\omega'| \le k_{j+1}, \tag{28}$$

$$k_j \le |\omega''| \le k_{j+1}. \tag{29}$$

First, we add relations (28) and (29) together. As a result, we obtain

$$2k_j \le |\omega'| + |\omega''| \le 2k_{j+1}, \tag{30}$$

$$2k_j \le \alpha_1 + \alpha_{12} + \alpha_2 + \alpha_{12} \le 2k_{j+1}. \tag{31}$$

Then, subtracting (27) from (31), we get

$$k_j + (k_j - k_{j+1}) \le \alpha_1 + \alpha_2 + \alpha_{12} \le k_{j+1} + (k_{j+1} - k_j). \tag{32}$$

The first inequality in (32) has been proved above. Consider the second inequality, taking into account that $k_{j+1} - k_j > 0$. The following two cases are possible:

(i) $\alpha_1 + \alpha_2 + \alpha_{12} \le k_{j+1} \le k_{j+1} + (k_{j+1} - k_j)$;

(ii) $k_{j+1} \le \alpha_1 + \alpha_2 + \alpha_{12} \le k_{j+1} + (k_{j+1} - k_j)$.

We begin with case (ii), taking (27) into account. Since the number of elements in $\omega'$ and $\omega''$ is determined by formulas (15), we establish, by virtue of the fact that $\alpha_1$, $\alpha_2$, and $\alpha_{12}$ may be equal to zero, that the following cases are possible:

(a) $\alpha_1 = 0$ and $k_j \le \alpha_{12} \le k_{j+1} \le \alpha_2 + \alpha_{12}$, which is impossible according to (23);

(b) $\alpha_2 = 0$ and $k_j \le \alpha_{12} + \alpha_1 \le k_{j+1} \le \alpha_1 + \alpha_{12}$, which is impossible according to what has been proved above;

(c) $\alpha_{12} = 0$ and $k_j \le \alpha_1 \le k_{j+1} \le \alpha_2 + \alpha_{12}$, which is impossible according to (23).

According to (a)–(c), we can conclude that case (ii) is impossible, i.e., case (i) takes place. Therefore, for $\omega', \omega'' \in \Omega$, relation (25) is true. For every $j \in J_{p-1}$, we denote by $\Omega_j$ the collection of all subsets $\omega \in \Omega$ for which

$$k_j \le |\omega| \le k_{j+1}, \tag{33}$$

and by $\omega^*$ and $\omega^{**}$ the subsets from $\Omega_j$ with the minimum and maximum number of elements. It follows from the results obtained above that if $\omega \in \Omega_j$, then

$$\omega^* \subseteq \omega \subseteq \omega^{**}. \tag{34}$$

The converse statement is also true, namely, if $\omega$ satisfies (34), then $\omega \in \Omega_j$, i.e., if $y \in F$ and $s \in \omega^{**} \setminus \omega^*$, then

$$\sum_{s \in \omega^{**} \setminus \omega^*} y_s = \left|\omega^{**} \setminus \omega^*\right| e_{j+1} y_0. \tag{35}$$

Let us verify this statement. Let $\omega^* = \left\{\alpha_1, \ldots, \alpha_{k_j}\right\}$ and $\omega^{**} = \left\{\alpha_1, \ldots, \alpha_{k_{j+1}}\right\}$, $\omega^*, \omega^{**} \in \Omega_j$, i.e., the restrictions in (8) are severe for these subsets. Therefore,

$$y_{\alpha_1} + y_{\alpha_2} + y_{\alpha_{k_j}} = (g_1 + g_2 + \ldots + g_{k_j}) y_0,$$

$$y_{\alpha_1} + y_{\alpha_2} + \ldots + y_{\alpha_{k_j}} + y_{\alpha_{k_j+1}} + \ldots + y_{\alpha_{k_{j+1}}} = (g_1 + g_2 + \ldots + g_{k_j} + g_{k_j+1} + \ldots + g_{k_{j+1}}) y_0.$$

Subtracting the first equality from the second one and taking into account (25) and (1), we obtain equality (35), which was to be proved.

If $\overline{\omega} \in \Omega$ and $|\overline{\omega}| < k_1$, then we have $\omega \in \Omega$ for $\omega \subset \overline{\omega}$. In $\Omega$, we consider the chain of sets $\omega_1 \subset \ldots \subset \omega_\lambda = J_k$. It follows from the reasoning presented above that if $y \in K_{kn}^0(G)$ turns the inequalities in (8) into the equalities for $\omega_1 \subset \ldots \subset \omega_\lambda = J_k$, then, for any $\omega \in \Omega$, the corresponding inequalities in (8) turn into the equalities at the point $y$. In other words, the system of severe restrictions for $F$ defined by the subsets $\omega_1, \ldots, \omega_\lambda$ is complete. On the other hand, the matrix of these restrictions has a triangular form and, therefore, they are linearly independent. Hence, taking into account that $\dim F = m$, we get $\lambda = (k+1) - m$. Consequently, assertion I of Lemma 1 is proved.

II. Let us prove that the set $F$ is a face of $K_{kn}^0(G)$. As in the proof of assertion I, we denote by $\Omega$ the collection of $\omega \subseteq J_k$ that define severe restrictions for $F$ in (8). We prove that $\omega \in \Omega$ if and only if either $\omega$ coincides with one of the sets $\omega_1 \subset \ldots \subset \omega_\lambda$ or there exist $j \in J_k$ and $\sigma \in J_\lambda$ such that $\omega_{\sigma-1} \subseteq \omega \subseteq \omega_\sigma$ and relation (23) is true, namely, $k_{j-1} \le \left|\omega_{\sigma-1}\right| \le |\omega_\sigma| \le k_j$. Without loss of generality, we can assume that $\omega_\sigma = \{1, \ldots, |\omega_\sigma|\}$ for $\sigma \in J_\lambda$. Consider the point $y = (y_0, y_1, \ldots, y_k)$, where

$$y_0 = \begin{cases} \left(\sum_{i=1}^{k} d_i g_{\alpha_i}\right)^{-1} & i \in \omega_\sigma \backslash \omega_{\sigma-1},\ k_{j-1} \le |\omega_{\sigma-1}| \le |\omega_\sigma| \le k_j,\ j \in J_k,\ \alpha_j \in J_k, \\ \left(\sum_{i=1}^{k} d_i (g_{\alpha_i} + \varepsilon_i)\right)^{-1} & \text{if } \ i = |\omega_\sigma| > k_j > |\omega_{\sigma-1}|,\ j \in J_k,\ \alpha_i \in J_k, \\ \left(\sum_{i=1}^{k} d_i (g_{\alpha_i} - \varepsilon_i)\right)^{-1} & \alpha_i \in J_k, \ \text{otherwise}, \end{cases}$$

$$y_i = \begin{cases} g_i y_0 & \text{if } \ i \in \omega_\sigma \backslash \omega_{\sigma-1}, \ \text{where } \ k_{j-1} \le |\omega_{\sigma-1}| \le |\omega_\sigma| \le k_j \ \text{ for } j \in J_k, \\ (g_i + \varepsilon_i) y_0 & \text{if } \ i = |\omega_\sigma| > k_j > |\omega_{\sigma-1}| \ \text{ for } j \in J_k, \\ (g_i - \varepsilon_i) y_0, & \text{otherwise}. \end{cases} \tag{36}$$

It is easy to choose small $\varepsilon_i > 0$ so that $y \in F$. It is obvious that, for $\omega \subset J_k$ for which (36) is not true, the corresponding inequalities hold for $y$ as strict ones, i.e., $\omega \notin \Omega$. The sufficiency of condition (36) for the inclusion $\omega \in \Omega$ is obvious. The maximum number of linearly independent severe restrictions for $F$ can be calculated by using formula (13). Lemma 1 is proved.

***Corollary 1.*** *The vertex of the cone $K^0_{kn}(G)$ is determined by the system of $k + 1$ equations $k$ of which are obtained from (8), (9) by the replacement of the inequalities in (8) by the equalities for sets $\omega_1 \subset \ldots \subset \omega_k = J_k$, and the $(k+1)$th equation is obtained by the replacement of one inequality in (8) by the equality for an arbitrary set $\omega_i \subset J_k$, where $i \in J_k$.*

**Theorem 1.**

*I. If $F$ is an $m$-face of $Q_{kn}(G)$ defined by system (8)–(10), then there exist sets $\omega_1 \subset \omega_2 \subset \ldots \subset \omega_{k-m} = J_k$, $m \in J^0_{k-1}$, for which the inequalities in (8) turn into the equalities for $y \in F$ ($F$ is the set of solutions of the system obtained from (8)–(10) by the replacement of the inequalities in (8) by the equalities for $\omega = \omega_\sigma$ with $\sigma \in J_{k-m-1}$).*

*II. If, for sets $\omega_1 \subset \omega_2 \subset \ldots \subset \omega_\lambda = J_k$, the inequalities in (8) are replaced by the equalities, then the set $F$ of solutions of the system obtained from (8)–(10) is an $m$-face of $Q_{kn}(G)$, whose dimension is determined by the formula*

$$m = \dim F = k - \left\{\lambda + \sum \left(|\omega_\sigma| - |\omega_{\sigma-1}| - 1\right)\right\}, \tag{37}$$

*where the summation is carried out over $\sigma \in J_\lambda$ for each of which there exists $j \in J_n$ such that $k_{j-1} \le |\omega_{\sigma-1}|$ and $|\omega_\sigma| \le k_j$ (we assume that $|\omega_0| = 0$).*

***Proof.*** The proof of Theorem 1 follows from Lemma 1. For the description of a face of the base of the pyramid, we choose the same sets $\omega_i \subseteq J_k$, $i \in J_k$, as for the description of the face of the cone and add (10); dimension (37) of a face obtained under the indicated choice of severe restrictions is smaller than the dimension of a face of the cone by 1 because we add an equality to the system of restrictions and, as is known [1], the dimension of a polyhedron in $R^k$ is equal to $k - r$, where $r$ is the rank of the matrix of severe restrictions for a polyhedron. Theorem 1 is proved.

Assume that $G$ is the set of numbers $g_1 > \ldots > g_k \geq 0$, i.e., $k = n$. Then we denote the polyhedron $Q_{kn}(G)$ by $Q_k^+(G)$.

***Corollary 2.*** *The set of solutions of system (8)–(10) is an $i$-face of $Q_k^+(G)$ if and only if each of these solutions turns the inequalities in (8) into the equalities for sets $\omega_1, \ldots, \omega_{k-i-1}$, $i \in J_{k-1}^0$, such that*

$$\omega_1 \subset \omega_2 \subset \ldots \subset \omega_{k-i-1} \subset J_k. \tag{38}$$

Let $V_k$ denote the set of vertices of the base of the pyramid $Q_{kn}(G)$; it is clear that $V_k \subset R^{k+1}$.

***Corollary 3.*** *If $y = (y_0, y_1, \ldots, y_k) \in V_k$, then the following conditions are satisfied:*

$$\{\alpha_1^1\} \subset \{\alpha_1^2, \alpha_2^2\} \subset \ldots \subset \{\alpha_1^{k-1}, \ldots, \alpha_{k-1}^{k-1}\} \subset \{\alpha_1^k, \ldots, \alpha_k^k\} = J_k, \tag{39}$$

$$\sum_{t=1}^{i} y_{\alpha_t^i} = \sum_{t=1}^{i} g_t y_0 \qquad \forall\, i \in J_k, \tag{40}$$

$$\sum_{t=1}^{k} d_t y_t = 1. \tag{41}$$

*Conversely, if conditions (39)–(41) are satisfied, then $y \in V_k$.*

**Proposition 1.** *The mapping $\alpha$ defined by formulas (7) determines a one-to-one correspondence between the points $x = (x_1, x_2, \ldots, x_k) \in E_{kn}(G) \subset R^k$ and $y = (y_0, y_1, \ldots, y_k) \in V_k$.*

***Proof.*** Assume that relations (1) hold for $G$. The points

$$x^* = \{g_1, g_2, \ldots, g_\alpha, g_{\alpha+1}, \ldots, g_\beta, g_{\beta+1}, \ldots, g_t, \ldots, g_k\},$$

$$x^{**} = \{g_1, g_2, \ldots, g_\beta, g_{\alpha+1}, \ldots, g_\alpha, g_{\beta+1}, \ldots, g_t, \ldots, g_k\}$$

are the vertices of $\Pi_{kn}(G)$ obtained one from another by the permutation of the components equal to $e_i$ and $e_j$ or the coordinates $g_\alpha$ and $g_\beta$. Then, according to the criterion of a vertex of $\Pi_{kn}(G)$ given in [5], which is formulated at the beginning of the present paper, we have

$$\{\alpha_1^1\} \subset \{\alpha_1^2, \alpha_2^2\} \subset \ldots \subset \{\alpha_1^\alpha, \ldots, \alpha_\alpha^\alpha\} \subset \ldots \subset \{\alpha_1^\beta, \ldots, \alpha_\alpha^\beta, \ldots, \alpha_\beta^\beta\}$$

$$\ldots \subset \{\alpha_1^k, \ldots, \alpha_\alpha^k, \ldots, \alpha_\beta^k, \ldots, \alpha_k^k\} = J_k,$$

$$\sum_{t=1}^{i} x_{\alpha_t^i} = \sum_{t=1}^{i} g_t \qquad \forall\, i \in J_k.$$

Hence, for the point $x^*$, the inclusions take the form

$$\{1\} \subset \ldots \subset \{1, 2, \ldots, \alpha\} \subset \{1, 2, \ldots, \alpha, \alpha+1\}$$

$$\ldots \subset \{1, 2, \ldots, \alpha, \alpha+1, \ldots, \beta\} \subset \{1, \ldots, \alpha, \alpha+1, \ldots, \beta+1\}$$

$$\ldots \subset \{1, 2, \ldots, \alpha, \alpha+1, \ldots, \beta-1, \beta, \beta+1, \ldots, k\} = J_k$$

and the system is as follows:

$$x_1 = g_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_1 + x_2 + \ldots + x_{\alpha-1} + x_\alpha = g_1 + g_2 + \ldots + g_{\alpha-1} + g_\alpha,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_1 + \ldots + x_{\alpha-1} + x_\alpha + x_{\alpha+1} + \ldots + x_\beta + x_{\beta+1} = g_1 + \ldots + g_{\alpha-1} + g_\alpha + \ldots + g_\beta + g_{\beta+1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_1 + \ldots + x_\alpha + x_{\alpha+1} + \ldots + x_\beta + x_{\beta+1} + \ldots + x_k = g_1 + \ldots + g_\alpha + \ldots + g_\beta + g_{\beta+1} + \ldots + g_k.$$

For the point $x^{**}$, the conditions

$$\{1\} \subset \ldots \subset \{1, 2, \ldots, \beta\} \subset \{1, 2, \ldots, \beta, \alpha+1\}$$

$$\ldots \subset \{1, \ldots, \beta, \alpha+1, \ldots, \beta-1, \alpha\} \subset \{1, \ldots, \beta, \alpha+1, \ldots, \alpha, \beta+1\}$$

$$\ldots \subset \{1, 2, \ldots, \beta, \alpha+1, \ldots, \beta-1, \alpha, \beta+1, \ldots, k\} = J_k$$

are satisfied and the system has the form

$$x_1 = g_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_1 + x_2 + \ldots + x_{\alpha-1} + x_\beta = g_1 + g_2 + \ldots + g_{\alpha-1} + g_\alpha,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_1 + \ldots + x_{\alpha-1} + x_\beta + x_{\alpha+1} + \ldots + x_\alpha + x_{\beta+1} = g_1 + \ldots + g_{\alpha-1} + g_\alpha + \ldots + g_\beta + g_{\beta+1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_1 + \ldots + x_\beta + x_{\alpha+1} + \ldots + x_\alpha + x_{\beta+1} + \ldots + x_k = g_1 + \ldots + g_\alpha + \ldots + g_\beta + g_{\beta+1} + \ldots + g_k.$$

We now prove that the mapping $\alpha$ defined by condition (7) transforms the points $x^*$ and $x^{**}$, which are vertices of $\Pi_{kn}(G)$, into points from $V_k$. Under the mapping $\alpha$, the point $x^*$ turns into the point $y^*$ with the coordinates

$$y_0 = \left\{d_1g_1 + \ldots + d_\alpha g_\alpha + d_{\alpha+1}g_{\alpha+1} + \ldots + d_{\beta-1}g_{\beta-1} + d_\beta g_\beta + d_{\beta+1}g_{\beta+1} + \ldots + d_k g_k\right\}^{-1},$$

$$y_1 = g_1 y_0, \quad y_2 = g_2 y_0, \ldots, y_{\alpha-1} = g_{\alpha-1} y_0, \quad y_\alpha = g_\alpha y_0,$$

$$y_{\alpha+1} = g_{\alpha+1} y_0, \ldots, y_{\beta-1} = g_{\beta-1} y_0, \quad y_\beta = g_\beta y_0,$$

$$y_{\beta+1} = g_{\beta+1} y_0, \ldots, y_{k-1} = g_{k-1} y_0, \quad y_k = g_k y_0,$$

and the point $x^{**}$ turns into the point $y^{**}$ with the coordinates

$$\bar{y}_0 = \left\{d_1g_1 + \ldots + d_\alpha g_\beta + d_{\alpha+1}g_{\alpha+1} + \ldots + d_{\beta-1}g_{\beta-1} + d_\beta g_\alpha + d_{\beta+1}g_{\beta+1} + \ldots + d_k g_k\right\}^{-1},$$

$$\bar{y}_1 = g_1 y_0, \quad \bar{y}_2 = g_2 y_0, \ldots, \bar{y}_{\alpha-1} = g_{\alpha-1} y_0, \quad \bar{y}_\alpha = g_\beta y_0,$$

$$\bar{y}_{\alpha+1} = g_{\alpha+1} y_0, \ldots, \bar{y}_{\beta-1} = g_{\beta-1} y_0, \quad \bar{y}_\beta = g_\alpha y_0,$$

$$\bar{y}_{\beta+1} = g_{\beta+1} y_0, \ldots, \bar{y}_{k-1} = g_{k-1} y_0, \quad \bar{y}_k = g_k y_0.$$

Comparing the coordinates of these points, we get $y^* \neq y^{**}$ because $\bar{y}_\alpha \neq y_\alpha$ and $\bar{y}_\beta \neq y_\beta$. We now prove that $y^*, y^{**} \in V_k$. According to Corollary 3, if conditions (39)–(41) are satisfied for $y \in R^{k+1}$, then $y \in V_k$. Let us verify this statement for $y^*$ and $y^{**}$. Condition (41) is satisfied for these points by virtue of (7). For $y^*$, on the basis of the indices of coordinates, we construct the following chain:

$$\{1\} \subset \{1, 2\} \subset \ldots \subset \{1, 2, \ldots, \alpha - 1, \alpha\}$$

$$\subset \{1, 2, \ldots, \alpha - 1, \alpha, \alpha + 1\} \subset \ldots \subset \{1, 2, \ldots, \alpha, \alpha + 1, \ldots, \beta - 1, \beta\}$$

$$\subset \{1, 2, \ldots, \alpha, \alpha + 1, \ldots, \beta - 1, \beta, \beta + 1\} \ldots$$

$$\subset \{1, 2, \ldots, \alpha, \alpha + 1, \ldots, \beta - 1, \beta, \beta + 1, \ldots, k\} = J_k.$$

Then we obtain the following system of severe restrictions for this point:

$$y_1 = g_1 y_0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1 + y_2 + \ldots + y_{\alpha-1} + y_\alpha = (g_1 + g_2 + \ldots + g_{\alpha-1} + g_\alpha) y_0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1 + \ldots + y_\alpha + y_{\alpha-1} + \ldots + y_\beta + y_{\beta+1} = \left(g_1 + \ldots + g_{\alpha-1} + g_\alpha + g_{\alpha+1} + \ldots + g_\beta + g_{\beta+1}\right) y_0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1 + \ldots + y_\alpha + \ldots + y_\beta + y_{\beta+1} + \ldots + y_k = \left(g_1 + \ldots + g_\alpha + \ldots + g_\beta + g_{\beta+1} + \ldots + g_k\right) y_0.$$

Hence, conditions (39) and (40) are satisfied. Analogous results hold for the point $y^{**}$. Therefore, $y^*, y^{**} \in V_k$ are the images of the points $x^*, x^{**} \in \operatorname{vert} \Pi_{kn}(G)$.

The first part of Proposition 1 is proved. To prove its second part, it suffices to choose an arbitrary point $y \in V_k$ and show by analogy that a point $x \in \operatorname{vert} \Pi_{kn}(G)$ is its preimage. Proposition 1 is proved.

It follows from Proposition 1 and the equality $\alpha\left(E_{kn}(G)\right) = E \subset R^{k+1}$ that $E \subset V_k$ and $V_k \subset E$, i.e., $V_k = E$ and, therefore, $\operatorname{vert} Q_{kn}(G) = E$.

As is known [1, p. 53], adjacent vertices of a polyhedron are two vertices that lie on the same edge. Taking into account different forms of definition of polyhedron, one can obtain different criteria for the adjacency of its vertices. If a polyhedron $M$ is defined in the space $R^k$ in its canonical form, then an edge of this polyhedron is defined by $(k-1)$ linearly independent severe restrictions. Hence, according to [1. p. 53], the definition of adjacency of vertices of a polyhedron can be formulated as follows: two vertices of a polyhedron given in its canonical form are adjacent if the systems of their linearly independent severe restrictions differ only by one equation.

Consider the following criterion of adjacency of vertices in $Q_{kn}(G)$:

**Theorem 2.** *The vertices of* $Q_{kn}(G)$ *adjacent to the vertex*

$$\bar{g} = \left( \frac{1}{\sum_{t=1}^{k} g_{\alpha_t} d_t}; \frac{g_{\alpha_1}}{\sum_{t=1}^{k} g_{\alpha_t} d_t}; \frac{g_{\alpha_2}}{\sum_{t=1}^{k} g_{\alpha_t} d_t}; \ldots; \frac{g_{\alpha_k}}{\sum_{t=1}^{k} g_{\alpha_t} d_t} \right),$$

*where* $\alpha_j \in J_k$ *and* $j \in J_k$, *are the vertices obtained from* $\bar{g}$ *by the permutation of the components equal to* $e_i$ *and* $e_{i+1}$, $i \in J_{n-1}$; *moreover, only these vertices are adjacent to* $\bar{g}$.

***Proof.*** Since the pyramid is located in the space $R^{k+1}$, according to Theorem 1 and Corollary 3 every vertex of the base of this pyramid is described by a system of linearly independent severe restrictions, which consists of $k+1$ equations of the system given by conditions (39)–(41). Since adjacent vertices lie on the same edge of the polyhedron, the system of linearly independent severe restrictions that describes this edge consists of $k$ common equations contained in the system of restrictions that describe adjacent vertices. Every $(k-1)$th equation from the system of severe restrictions that determine the adjacent vertices differs by its left-hand side [1]. This fact is explained by the permutation of the components $e_i$ and $e_{i+1}$, $i \in J_{n-1}$, that determine the coordinates of the point $\bar{g}$.

Hence, to determine the vertices of $Q_{kn}(G)$ that are adjacent to $\bar{g}$, it is necessary to perform the permutation of the components equal to $e_i$ and $e_{i+1}$, $i \in J_{n-1}$, which was to be proved.

***Corollary 4.*** *Each vertex of the polyhedron* $Q_{kn}(G)$ *is adjacent to the vertex of the pyramid located at the point* $O(0, \ldots, 0)$.

**Theorem 3.** *The number* $r$ *of vertices adjacent to an arbitrary vertex of* $Q_{kn}(G)$ *is equal to* $r = \eta_1\eta_2 + \eta_2\eta_3 + \eta_3\eta_4 + \ldots + \eta_{n-1}\eta_n + 1$.

***Proof.*** According to Theorem 2, to obtain a vertex adjacent to the vertex

$$\bar{g} = \left( \frac{1}{\sum_{t=1}^{k} g_{\alpha_t} d_t}; \frac{g_{\alpha_1}}{\sum_{t=1}^{k} g_{\alpha_t} d_t}; \frac{g_{\alpha_2}}{\sum_{t=1}^{k} g_{\alpha_t} d_t}; \ldots ; \frac{g_{\alpha_k}}{\sum_{t=1}^{k} g_{\alpha_t} d_t} \right),$$

where $\alpha_j \in J_k$ and $j \in J_k$, it is necessary to perform the permutation of $e_i$ and $e_{i+1}$, $i \in J_{n-1}$. The number of elements $e_i$ is equal to $\eta_i$, $i \in J_n$. The number of vertices adjacent to $\bar{g}$ that are obtained by the permutation of $e_i$ and $e_{i+1}$ is equal to $\eta_i \eta_{i+1}$, $i \in J_{n-1}$, and the number of all adjacent vertices is equal to $r = \eta_1\eta_2 + \eta_2\eta_3 + \eta_3\eta_4 + \ldots + \eta_{n-1}\eta_n$. Each vertex of $Q_{kn}(G)$ is adjacent to a vertex of the pyramid and, therefore, $r = \eta_1\eta_2 + \eta_2\eta_3 + \eta_3\eta_4 + \ldots + \eta_{n-1}\eta_n + 1$, which was to be proved.

As is known [5, p. 29], two $i$-faces $S_1^i$ and $S_2^i$ of a $(k-1)$-polyhedron $M$ are called adjacent if they intersect along a $(k-1)$-face $S^{i-1}$ of this polyhedron, i.e.,

$$S_1^i \cap S_2^i = S^{i-1}, \qquad i \in J_{k-2}. \tag{42}$$

According to Corollary 2, for an arbitrary $i$-face of $Q_k^+(G)$ there exist sets $\omega_1, \ldots, \omega_{k-i-1}$, $i \in J_{k-1}^0$. Denote the collection of these sets for $S_1^i$ and $S_2^i$ by $\Omega_1^i$ and $\Omega_2^i$, respectively. Let us formulate a criterion for the adjacency of faces of $Q_k^+(G)$.

**Theorem 4.** *In order that two i-faces* $S_1^i$ *and* $S_2^i$ *of the polyhedron* $Q_k^+(G)$ *be adjacent, it is necessary and sufficient that the set*

$$\Omega^{i-1} = \Omega_1^i \cup \Omega_2^i \tag{43}$$

*define an* $(i-1)$*-face* $S^{i-1}$, $i \in J_{k-2}$.

***Proof.*** *Necessity.* Assume that $i$-faces $S_1^i$ and $S_2^i$ of the polyhedron $Q_k^+(G)$ are adjacent. According to Corollary 2, there exist sets $\Omega_1^i = \{\omega_j^1\}_{j=1}^{k-i-1}$ for $S_1^i$ and $\Omega_2^i = \{\omega_j^2\}_{j=1}^{k-i-1}$ for $S_2^i$ that satisfy (38). Assume that condition (42) is satisfied. Then, according to Corollary 2, there exists a set $\Omega^{i-1} = \{\omega_j\}_{j=1}^{k-i}$ that corresponds to the face $S^{i-1}$ and satisfies (38). Since the restrictions describing the faces $S_1^i$ and $S_2^i$ must simultaneously hold at the points of the face $S^{i-1}$, we conclude that condition (43) is satisfied.

*Sufficiency.* Assume that condition (43) is satisfied. Then it follows from Corollary 2 that there exist faces $S_1^i$ and $S_2^i$ that are defined by the sets $\Omega_1^i$ and $\Omega_2^i$ and, by virtue of (43), we get (42). Thus, the criterion for the adjacency of the faces of the polyhedron $Q_k^+(G)$ is proved.

Note that the properties of the admissible domain for problem (5), (6) described in the present paper [in particular, the sets $E$ of admissible solutions of problem (11), (12)] enable one to apply the method of combinatorial truncation presented in [11, 12] to the solution of the problems indicated.

In our opinion, the other properties of problem (5), (6) established in this paper can be used for the construction of methods and algorithms for the solution of problems with linear-fractional objective function and certain additional (in particular, nonlinear) conditions.

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