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Upper-contour Strategy-proofness in the Random Assignment Problem

랜덤 배정 문제에서의 전략 무용성 연구

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Abstract

Upper-contour Strategy-proofness in the Random Assignment Problem

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Bogomolnaia and Moulin (2001) showed that the mechanism that satisfies sd-efficiency and equal treatment of equals cannot be sd-strategy-proof. Also, Mennle and Seuken (2017) showed a decomposition result of strategy-proofness and presented partial strategy-proofness, which is a weak notion of strategy-proofness used by Mennle and Seuken's paper. In this paper, we show other strategy-proofness notion under the random assignment problem. In this paper, we present a weakened notion of strategy-proofness which is related to the upper-contour set, upper-contour strategy-proofness. Our main result is even though sd-strategy-proofness is weakened to upper-contour strategy-proofness, we end up with impossibility results.

Keywords : Random assignment problem, indivisible goods, sd-efficiency, equal treatment of equals, strategy-proofness
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Table of Contents

1. Introduction

We consider the problem of allocating indivisible goods or objects among a group of agents without transfering money and each agent can receive at most one of them. For example, housing allocation of universities' dormitory, allocating assignment to workers and student placement in schools can be examples of allocating indivisible goods without monetary transfer. Each agent has a complete, transitive, and strict binary relation over objects.

Achieving fairness is one of the important aims of assignment. However, under the circumstance of allocating deterministic objects, it is very difficult to satisfy the fairness condition. For example, suppose there are two desirable objects to be allocated to two agents and they prefer the same object. It is clear that each of the two possible allocations will violate any reasonable notion of fairness. Therefore, instead of allocating deterministic objects, we assign a probability of each object for each agent. This way of assignment is called the random assignment.

A mechanism designer usually wants to achieve three goals. The three goals are efficiency, strategy-proofness and fairness. A mechanism is strategy-proof if truth-telling is a dominant strategy of the preference revelation game. However, achieving strategy-proofness is not easy because it makes the conflict with other properties. Zhou (1990) showed that there does not exist random assignment mechanism achieving strategy-proofness, efficiency with respect to cardinal utility, and equal treatment of equals.

Because we must give up some efficiency or fairness property in order to achieve strategy-proofness, mechanism designers are interested in studying non-strategy-proof mechanisms. Also, they have a weaker version strategy-proofness to make mechanism which is compatible with efficiency and fairness property. In this study, we suggest weaker strategy-proofness and show that even though strategy-proofness is weakened, the impossibility result still holds.

The assignment problem has attracted much attention after Hylland and Zeckhauser (1979). They proposed pseudomarkets and they associated them with the random assignment problem. Also, they showed that even though there exists competitive equilibrium when the agents have the same incomes, this mechanism is not strategy-proof.

The papers which have close relationships with our paper are as follows. Bogomolnaia and Moulin (2001) introduced the probabilistic serial mechanism and sd-efficiency. Also, they showed that when the number of agents is three, the mechanism that satisfies this efficiency and equal treatment of equals can be strategy-proof. However, when the number of agents is larger than three, these axioms are not compatible. These results also hold when strategy-proofness in their paper is weakened. Mennle and Seuken (2017) showed a decomposition result of strategy-proofness and presented partial strategy-proofness, which is a weak notion of strategy-proofness. In this paper, we show that another decomposition result about weakened strategy-proof axiom. Nesterov (2017) suggested impossibility results when rules in the model satisfy strategy-proofness. One of the results is that when the agents are at least three, ex-post efficiency, lower invariance, which is one of the axioms related to strategy-proofness, and upper envy-free, which is one of fairness axioms. This paper suggests that when the agents are more than three, impossibility result can be derived using weak axiom compared to upper envy-free.

There have been two representative ways to weaken the previous sd-strategy-proofness notion. One way is to use weak sd-strategy-proofness axiom. Because stochastic dominance relation is a partial ordering, we can make weakened axiom which requires the lottery where each agent tell the truth not to be stochastically dominated by any other lotteries. In contrast to weak sd-strategy-proofness, sd-strategy-proofness requires the lottery to be stochastically dominate any other lotteries. In other word, weak sd-strategy-proofness requires each agent should not benefit by misreporting his preferences. Bogomolnaia and Moulin (2001) shows that the probabilistic serial rule satisfies weak sd-strategy-proofness. Second way is to lessen the number of the misreport cases and pairs of probabilities to be considered. One example is limited invariance. It rules out profitable misrepresentation for certain types of preference, but not all of preference. With all other agents' preferences fixed, assume that the preference of an agent changes but the rankings from his most preferred object down to a certain object do not change. In this case, the probability of his receiving the object remain the same. This axiom is mainly used in order to characterize the probabilistic serial rule, which means the probabilistic serial rule satisfies limited invariance (Hashimoto et al., 2014; Heo, 2014; Heo and Yilmaz, 2015). In this paper, we focus on this type of weaking and we strengthen limited invariance axiom. Upper-contour strategy-proofness weaken the condition of limited invariance that the rankings must be the same. Instead, it requires the condition that the upper contour sets at an object must be the same. Therefore, the fact that upper-contour strategy-proofness implies limited invariance is obvious.

This paper is constructed as follows. In section 2, we introduce the model and three properties. Also, we introduce a new notion of strategy-proofness, upper-contour strategy-proofness. In section 3, after we propose the decomposition result of upper-contour strategy-proofness, we present the main impossibility results and characterization results. In section 4, we consider variations in the random assignment problem.

2. The model

First we define the random assignment problem proposed by Bogomolania and Moulin. Let I ={$1, 2, ..., n$} be the set of agents and A ={$1, 2, ..., n$} be the set of objects. We assume that $|I|$=$|A| = n$. Each agent i is equipped with a complete, transitive and antisymmetric binary relation P_i over A. Let $\mathcal{R}$ denote the set consisting of all strict preferences over A. Because we fix I and A, we write a *problem* as a list $R \in \mathcal{R}^N$.

Given $P_i \in \mathcal{R}$ and $a \in A$, let $r_k(P_i)$, $k = 1, ..., n$, denote the k-th ranked object according to P_i, and $U(P_i, a) = \{x \in A|xR_ia\}$ denote the upper contour set of a in P_i. Also, let $L(P_i, a) = \{x \in A|aR_ix\}$ denote the lower contour set of a in P_i. Also, let $rank(P_i, a)$ denote the a's ranking according to P_i.

Let $\Delta(A)$ denote the set of lotteries, or *probability distributions* over A. Given $\lambda \in \Delta(A)$, λ_a denotes the probability assigned to object a.

A (random) *assignment* is a bi-stochastic matrix $L = [L_{ia}]_{i\in I, a\in A}$, namely a non-negative square matrix of which elements in each row and each column sum to unity. Let $\mathcal{L}$ denotes the set of all bi-stochastic matrices.

Given $P_i \in \mathcal{R}$ and lotteries $\lambda, \lambda' \in \Delta(A)$, λ *stochastically dominates* λ' according to P_i, denoted $\lambda R_i^{sd} \lambda'$, if $\sum_{l=1}^{k} \lambda_{r_l(P_i)} \geq \sum_{l=1}^{k} \lambda'_{r_l(P_i)}$ for all $1 \leq k \leq n$. Analogously, given $R \in \mathcal{R}^n$, an assignment L stochastically dominates L' according to R, denoted $LR^{sd}L'$, if $L_i R_i^{sd} L'_i$ for all $i \in I$.

A *rule* is a mapping $\varphi : \mathcal{R}^n \to \mathcal{L}$. Given $R \in \mathcal{R}^n$, $\varphi_{ia}(R)$ denotes the probability of agent i receiving object a, and thus $\varphi_i(R)$ denotes the lottery assigned to agent i.

We introduce requirements imposed on rules. First condition is efficiency. An assignment L is sd-efficient if it is not stochastically dominated by any other assignment L'. Accordingly, a rule φ is *sd-efficient* if the assignment $\varphi(R)$ is sd-efficient for all $R \in \mathcal{R}^n$. A rule φ is *ex-post efficient* if each assignment induced by the rule can be represented as a probability distribution over efficient deterministic assignments. In this model, sd-efficient rule implies ex-post efficient rule but the converse is not true.

Second condition is strategy-proofness. Agents cannot have an incentive to misreport their preferences in order to improve their utilities. A rule φ is *sd-strategy-proof* if for all $i \in I$, all $P_i, P'_i \in \mathcal{R}$, and all $P_{-i} \in \mathcal{R}^{n-1}$, $\varphi_i(P_i, P_{-i}) R_i^{sd} \varphi_i(P'_i, P_{-i})$.

Final condition is fairness. An assignment L is sd-envyfree if $L_i R_i^{sd} L_j$ for all $i, j \in I$. Accordingly, a rule $\varphi : \mathcal{R}^n \to \mathcal{L}$ is *sd-envyfree* if $\varphi(R)$ is sd-envyfree for all $R \in \mathcal{R}^n$. A rule $\varphi : \mathcal{R}^n \to \mathcal{L}$ is *upper envyfree* if for all $i, j \in I, R \in \mathcal{R}^n$ and all $a, b \in A$, if $U(P_i, a) = U(P_j, a)$, then $\varphi_{ia}(R) = \varphi_{ja}(R)$. A rule φ satisfies *strong equal treatment of equals* if for all $i, j \in I$, all $P_i, P_j \in \mathcal{R}$ and all object $a \in A$, if $U(P_i, a) = U(P_j, a)$ and $rank(P_i, k) = rank(P_j, k)$ for all

$k \in U(P_i, a)$, then $\varphi_{ik}(R) = \varphi_{jk}(R)$. An assignment $L \in \mathcal{L}$ satisfies equal treatment of equals if for all $i, j \in I$, $[P_i = P_j] \Rightarrow [L_i = L_j]$. Similarly, a rule φ satisfies *equal treatment of equals* if $\varphi(R)$ satisfies equal treatment of equals for all $R \in \mathcal{R}^n$. In this model, sd-envyfree rule implies upper envyfree rule but the converse is not true. upper envyfree implies strong equal treatment of equals but the converse is not true. Strong equal treatment of equals imples equal treatment of equals but the converse is not true. The proofs of these relationships are written in Nesterov (2017).

In this study, we present a relaxed notion of strategy-proofness which is related to the upper contour set.

Definition 1. *A rule φ is* ***upper-contour strategy-proof*** *if for all $i \in I$, all $P_i, P_i' \in \mathcal{R}$, all $P_{-i} \in \mathcal{R}^{n-1}$, and all $a, b \in A$, if $U(P_i, a) = U(P_i', b)$, then $\sum_{l=1}^{rank(P_i,a)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) = \sum_{l=1}^{rank(P_i',b)} \varphi_{i(r_l(P_i'))}(P_i', P_{-i})$.*

When a rule satisfies upper-contour strategy-proofness, if upper contour set of P_i and upper contour set of P_i' are the same regardless of the objects preference ordering, then the sum of probability to get these objects in the upper contour set must be the same. Suppose, for example, that an agent misreports his preferences but his upper contour set of some objects remains the same. There is a possibility that this agent finds objects at least as good as the objects much more desirable than the remaining objects . If he is assigned a greater probability of getting these objects, then he is obviously better off misrepresenting his preferences; if he is assigned a smaller probability for the same object, then a similar argument applies by switching his true preferences and his misrepresented preference. Therefore, upper-contour strategy-proofness rules out such a profitable misrpresentaion by making the sum of probability to get these objects the same.

When we consider sd-strategy-proofness, we have to consider all possible misreports and all probabilities of getting at least kth ranked objects for all k. However when we consider upper-contour strategy-proofness, we only consider some of these misreports and probabilities. When the number of agents is three and we consider sd-strategy-proofness, each agent has to consider five misreports and compare ten pairs of probabilities. On the other hand, when we consider upper-contour strategy-proofness, each agent only considers two misreports and compares two pairs of probabilities. When the number of agents is four and we consider sd-strategy-proofness, each agent has to consider twenty three misreports and compare sixty nine pairs of probabilities. On the other hand, when we consider upper-contour strategy-proofness, each agent only considers ten misreports and compares fourteen pairs of probabilities. However, we show that previous results in other papers hold even though we replace sd-strategy-proofness with upper-contour strategy-proofness in section 3. By definition, sd-strategy-proofness implies upper-contour strategy-proofness. Also, it is not difficult to find a rule which satisfies upper-contour strategy-proofness but does not satisfy sd-strategy-proofness. However, when the number of agents is two, we cannot consider the case where two different preferences have the same upper-contour set. Therefore, every rule in the model satisfies upper-contour strategyproofness trivially. Therefore, from now on, we consider the number of agents is at least three.

Example 1. *Let $A = \{a, b, c\}$ and $I = \{1, 2, 3\}$. Assume that agent 1's allocation according to her preference is as follows and the others' allocations are fixed regardless of their preferences.*

•P_1^1 : aP_1bP_1c
•P_1^2 : aP_1cP_1b
•P_1^3 : bP_1aP_1c
•P_1^4 : bP_1cP_1a
•P_1^5 : cP_1aP_1b
•P_1^6 : cP_1bP_1a

Then, for all $P_2, P_3 \in \mathcal{R}$, the lottery assigned to agent 1 is as follows.

$$\varphi_1(P_1^1, P_2, P_3) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \qquad \varphi_1(P_1^2, P_2, P_3) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$\varphi_1(P_1^3, P_2, P_3) = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \qquad \varphi_1(P_1^4, P_2, P_3) = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$\varphi_1(P_1^5, P_2, P_3) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \qquad \varphi_1(P_1^6, P_2, P_3) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Because agent 2 and agent 3 cannot affect the result of the rule, we consider only agent 1's allocation. This rule satisfies upper-contour strategy-proofness but does not satisfy sd-strategy-proofness. Because when the agent 1s preference ordering is P_1^6, she may have incentive to misreport her preference as P_1^5. It is because the probability to receive c or b is $\frac{3}{4}$ under P_1^5 compared to $\frac{1}{2}$ under P_1^6. The probability to receive c or b is higher under P_1^5 than under P_1^6.

3. Main results

Mennle and Seuken (2017) showed that strategy-proofness can be decomposed into three axioms. First, for each pair $P_i, P_i' \in \mathcal{R}$, P_i' *is adjacent to* P_i if P_i' is obtained from P_i only by switching two consecutively ranked objects. In other words, when we compare P_i with P_i', just two consecutive objects ranks are swapped and other objects ranks are the same. First axiom is swap monotonicity. For all agents and all other agents preferences, if one agent changed her report into another report which is adjacent to original report, then either the agent's lotteries of the two cases are the same or she has to receive higher probabilities to get her more preferred object in her report. Second axiom is upper invariance. Upper invariance guarantees that agents cannot influence their probabilities to obtain more-preferred objects by changing the ranks of less-preferred objects. Similarly, lower invariance, which is third axiom, guarantees that agents cannot influence their probabilities to obtain less preferred objects by changing the ranks of more-preferred objects.

Definition 2. *A rule φ is **swap monotonic** if for all $i \in I$, all $P_i, P_i' \in \mathcal{R}$, all $P_{-i} \in \mathcal{R}^{n-1}$, and all $a, b \in A$, if P_i' is adjacent to P_i, $P_i : aP_ib$ and $P_i' : bP_i'a$ then either $\varphi_i(P_i, P_{-i}) = \varphi_i(P_i', P_{-i})$ or $\varphi_{ia}(P_i, P_{-i}) > \varphi_{ia}(P_i', P_{-i})$.*

Definition 3. *A rule φ is **upper invariant** if for all $i \in I$, all $P_i, P_i' \in \mathcal{R}$, all $P_{-i} \in \mathcal{R}^{n-1}$, and all $a, b \in A$, if P_i' is adjacent to P_i, $P_i : aP_ib$ and $P_i' : bP_i'a$ then $\varphi_{ik}(P_i, P_{-i}) = \varphi_{ik}(P_i', P_{-i})$ for all $k \in U(P_i, a) \setminus \{a\}$.*

Definition 4. *A rule φ is **lower invariant** if for all $i \in I$, all $P_i, P_i' \in \mathcal{R}$, all $P_{-i} \in \mathcal{R}^{n-1}$, and all $a, b \in A$, if P_i' is adjacent to P_i, $P_i : aP_ib$ and $P_i' : bP_i'a$ then $\varphi_{ik}(P_i, P_{-i}) = \varphi_{ik}(P_i', P_{-i})$ for all $k \in L(P_i, b) \setminus \{b\}$.*

In Mennle and Seuken (2017), they showed that a rule φ is sd-strategy-proof if and only if it is swap monotonic, upper invariant, and lower invariant. Here, we show that upper-contour strategy-proofness is equivalent to upper invariance and lower invariance. Therefore, upper-contour strategy-proofness ensures that agents cannot influence their probabilities to obtain objects of which ranks are not changed.

Proposition 1. *A rule φ is upper-contour strategy-proof if and only if it is upper invariant and lower invariant.*

Proof.

In this proof, we denote objects by o_a, o_b, ... , o_n.

($\Rightarrow$) Assume to the contrary that φ is upper-contour strategy-proof but not upper invariant. Then, there exist some agent $i \in I$, some preference profiles P_i and P_i' and object o_a and o_b such that P_i' is adjacent to P_i, $P_i : o_a \succ o_b$ and $P_i' : o_b \succ o_a$ and $\varphi_{io_k}(P_i, P_{-i}) \neq \varphi_{io_k}(P_i', P_{-i})$ for some $o_k \in U(P_i, o_a) \setminus \{o_a\} = U(P_i', o_b) \setminus \{o_b\}$. Also, because φ is upper-contour strategy-proof, for all $o_c, o_d \in A$, if $U(P_i, o_c) = U(P_i', o_d)$, then $\sum_{l=1}^{rank(P_i, o_c)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) = \sum_{l=1}^{rank(P_i', o_d)} \varphi_{i(r_l(P_i'))}(P_i', P_{-i})$. $U(P_i, o_a) \setminus \{o_a\} = U(P_i', o_b) \setminus \{o_b\}$ and their ranks in this upper contour set are also the same because there is just one swap between o_a and o_b. Also, the above statement and the fact that $o_k \in U(P_i, o_a) \setminus \{o_a\} = U(P'_i, o_b) \setminus \{o_b\}$ mean $U(P_i, o_k) = U(P'_i, o_k)$. Therefore, by using $U(P_i, o_k) = U(P'_i, o_k)$ and upper-contour strategy-proofness,

$$\sum_{l=1}^{rank(P_i,o_k)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) = \sum_{l=1}^{rank(P'_i,o_k)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i}) \cdots (1)$$

Let $rank(P_i, o_k)$=ω.
If ω=1, the above equation is contradiction because of above assumption, $\varphi_{io_k}(P_i, P_{-i}) \neq \varphi_{io_k}(P'_i, P_{-i})$.
Also, if $\omega \geq 2$, let $r_{\omega-1}(P_i) = o_j$. Then the following two equations hold.

$$\sum_{l=1}^{rank(P_i,o_k)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) - \varphi_{io_k}(P_i, P_{-i}) = \sum_{l=1}^{rank(P_i,o_j)} \varphi_{i(r_l(P_i))}(P_i, P_{-i})$$
$$\sum_{l=1}^{rank(P'_i,o_k)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i}) - \varphi_{io_k}(P'_i, P_{-i}) = \sum_{l=1}^{rank(P'_i,o_j)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i}).$$

Therefore by subtracting $\varphi_{io_k}(P_i, P_{-i})$ from left side of (1) and $\varphi_{ik}(P'_i, P_{-i})$ from right side of (1) and using $\varphi_{io_k}(P_i, P_{-i}) \neq \varphi_{io_k}(P'_i, P_{-i})$, we can get the equation

$$\sum_{l=1}^{rank(P_i,o_j)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) \neq \sum_{l=1}^{rank(P'_i,o_j)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i}).$$

However, because $U(P_i, o_j) = U(P'_i, o_j)$ and by the upper-contour strategy-proofness,

$$\sum_{l=1}^{rank(P_i,o_j)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) = \sum_{l=1}^{rank(P'_i,o_j)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i})$$

Contradiction.

Similarly, we can prove the case of lower invariance. Assume to the contrary that φ is upper-contour strategy-proof but not lower invariant. Then, there exist some agent $i \in I$, some preference profiles P_i and P'_i and object o_a and o_b such that P'_i is adjacent to P_i, $P_i : o_a \succ o_b$ and $P'_i : o_b \succ o_a$ and $\varphi_{io_k}(P_i, P_{-i}) \neq \varphi_{io_k}(P'_i, P_{-i})$ for some $o_k \in L(P_i, o_b) \setminus \{o_b\} = L(P'_i, o_a) \setminus \{o_a\}$. Also, because φ is upper-contour strategy-proof, for all $o_c, o_d \in A$, if $U(P_i, o_c) = U(P'_i, o_d)$, then $\sum_{l=1}^{rank(P_i,o_c)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) = \sum_{l=1}^{rank(P'_i,o_d)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i})$. Because $U(P_i, o_b) = U(P'_i, o_a)$ and their ranks in $L(P_i, o_b) \setminus \{o_b\} = L(P'_i, o_a) \setminus \{o_a\}$ are also the same, $U(P_i, o_k) = U(P'_i, o_k)$. Therefore, by using $U(P_i, o_k) = U(P'_i, o_k)$ and upper-contour strategy-proofness,

$$\sum_{l=1}^{rank(P_i,o_k)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) = \sum_{l=1}^{rank(P'_i,o_k)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i}) \cdots (2)$$

Let $rank(P_i, o_k)$=ω.
Let $r_{\omega-1}(P_i) = o_j$. Then the following two equations hold.

$$\sum_{l=1}^{rank(P_i,o_k)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) - \varphi_{io_k}(P_i, P_{-i}) = \sum_{l=1}^{rank(P_i,o_j)} \varphi_{i(r_l(P_i))}(P_i, P_{-i})$$
$$\sum_{l=1}^{rank(P'_i,o_k)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i}) - \varphi_{io_k}(P'_i, P_{-i}) = \sum_{l=1}^{rank(P'_i,o_j)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i}).$$

Therefore by subtracting $\varphi_{io_k}(P_i, P_{-i})$ from left side of (2) and $\varphi_{io_k}(P'_i, P_{-i})$ from right side of (1) and using $\varphi_{io_k}(P_i, P_{-i}) \neq \varphi_{io_k}(P'_i, P_{-i})$, we can get the equation

$$\sum_{l=1}^{rank(P_i,o_j)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) \neq \sum_{l=1}^{rank(P'_i,o_j)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i}).$$

However, because $U(P_i, o_j) = U(P'_i, o_j)$ and by the upper-contour strategy-proofness,

$$\sum_{l=1}^{rank(P_i,o_j)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) = \sum_{l=1}^{rank(P'_i,o_j)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i})$$

Contradiction.

($\Leftarrow$) Assume that φ is upper invariant and lower invariant but not upper-contour strategy-proof. Then, there exist some agent $i \in I$, some preference profiles P_i and P'_i and object o_a and o_b such that when $U(P_i, o_a) = U(P'_i, o_b)$,

$$\sum_{l=1}^{rank(P_i,o_a)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) \neq \sum_{l=1}^{rank(P'_i,o_b)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i}). \cdots (3)$$

Then by swapping two adjacent ranks in $U(P'_i, o_b)$ finite times, we can always make new preference profile P''_i such that $rank(P_i, o_k) = rank(P''_i, o_k)$ for all $o_k \in U(P_i, o_a)$. In this case, by lower invariance, the agent i cannot affect the sum of probabilities to get one of $A \setminus U(P_i, o_a)$. That means

$$1 - \sum_{l=1}^{rank(P'_i,o_a)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i}) = 1 - \sum_{l=1}^{rank(P''_i,o_a)} \varphi_{i(r_l(P''_i))}(P''_i, P_{-i}).$$
$$\sum_{l=1}^{rank(P'_i,o_a)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i}) = \sum_{l=1}^{rank(P''_i,o_a)} \varphi_{i(r_l(P''_i))}(P''_i, P_{-i}).$$

Therefore, by replacing $\sum_{l=1}^{rank(P'_i,o_b)} \varphi_{i(r_l(P'_i))}(P'_i, P_{-i})$ in (3) with $\sum_{l=1}^{rank(P''_i,o_a)} \varphi_{i(r_l(P''_i))}(P''_i, P_{-i})$,

$$\sum_{l=1}^{rank(P_i,o_a)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) \neq \sum_{l=1}^{rank(P''_i,o_a)} \varphi_{i(r_l(P''_i))}(P''_i, P_{-i}). \cdots (4)$$

Similarly, by changing two adjacent ranks in $A \setminus U(P_i, o_a)$ finite times, we can change P''_i to P'''_i such that $rank(P_i, o_m) = rank(P'''_i, o_m)$ for all $o_m \in A \setminus U(P_i, o_a)$. Also, by upper invariance, the agent i cannot affect the probability to get $U(P_i, o_a)$. Therefore,

$$\sum_{l=1}^{rank(P''_i,o_a)} \varphi_{i(r_l(P''_i))}(P''_i, P_{-i}) = \sum_{l=1}^{rank(P'''_i,o_a)} \varphi_{i(r_l(P'''_i))}(P'''_i, P_{-i}).$$

Therefore, by replacing $\sum_{l=1}^{rank(P''_i,o_b)} \varphi_{i(r_l(P''_i))}(P''_i, P_{-i})$ in (4) with $\sum_{l=1}^{rank(P'''_i,o_a)} \varphi_{i(r_l(P'''_i))}(P'''_i, P_{-i})$,

$$\sum_{l=1}^{rank(P_i,o_a)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) \neq \sum_{l=1}^{rank(P'''_i,o_a)} \varphi_{i(r_l(P'''_i))}(P'''_i, P_{-i}).$$

When we compare P_i and P'''_i, all ranks in two preference profiles are the same. Therefore, $P_i = P'''_i$. Then,

$$\sum_{l=1}^{rank(P_i,o_a)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}) \neq \sum_{l=1}^{rank(P_i,o_a)} \varphi_{i(r_l(P_i))}(P_i, P_{-i}).$$

Contradiction. □

An important solution to the random assignment problem is the random serial dictatorship. The random serial dictatorship orders the agents with equal probability and the first agent receives her most preferred good, the next agent obtains her most preferred good among the remaining ones, and so on. The random serial dictatorship is known to satisfy sd-strategy-proofness and ex-post efficiency. In Bogomolnaia & Moulin (2001), they showed when $n = 3$, the random serial dictatorship is characterized by the combination of three axioms: sd-efficiency, sd-strategy-proofness, and equal treatment of equals. Here, we show that even though sd-strategy-proofness is weakened to upper-contour strategy-proofness, their characterization result still holds. It means that swap monotonicity is redundant in this characterization result.

Proposition 2. *Assume $n = 3$. Then the random serial dictatorship is characterized by the combination of three axioms: sd-efficiency, equal treatment of equals, and upper-contour strategy-proofness.*

Proof.
For $n = 3$, there are six types of preference profiles (Bogomolnaia and Moulin, 2001). Actually, any other preference profiles in the same type can be represented as one of these types after renaming agents and objects. These preference profiles are as follows.

$$\text{Type 1 (48 profiles)} \begin{cases} aP_1(b,c) \\ bP_2(a,c) \\ cP_3(a,b) \end{cases} \qquad \text{Type 2 (6 profiles)} \begin{cases} aP_1bP_1c \\ aP_2bP_2c \\ aP_3bP_3c \end{cases}$$

$$\text{Type 3 (18 profiles)} \begin{cases} aP_1bP_1c \\ aP_2bP_2c \\ aP_3cP_3b \end{cases} \qquad \text{Type 4 (36 profiles)} \begin{cases} aP_1cP_1b \\ aP_2cP_2b \\ bP_3(a,c) \end{cases}$$

$$\text{Type 5 (36 profiles)} \begin{cases} aP_1bP_1c \\ aP_2bP_2c \\ bP_3(a,c) \end{cases} \qquad \text{Type 6 (72 profiles)} \begin{cases} aP_1bP_1c \\ aP_2cP_2b \\ bP_3(a,c) \end{cases}$$

Type 1 : By sd-efficiency, $\varphi_{1a}(R^1) = \varphi_{2b}(R^1) = \varphi_{3c}(R^1) = 1$.

$$\varphi(R^1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Type 2 : By equal treatment of equals, $\varphi_{ia}(R^2) = \varphi_{ib}(R^2) = \varphi_{ic}(R^2) = \frac{1}{3}$ for all i.

$$\varphi(R^2) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Type 3 : By upper-contour strategy-proofness from type 2, $\varphi_{3a}(R^3) = \frac{1}{3}$. By sd-efficiency, $\varphi_{3b}(R^3) = 0$. Therefore, $\varphi_{3c}(R^3) = \frac{2}{3}$. We can easily derive other elements by equal treatment of equals.

$$\varphi(R^3) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

Also, we can consider $R^{3'}$ in Type 3. $R^{3'}$: For $i = 1, 3$, aP_ibP_ic. aP_2cP_2b.
As we wrote above, any other preference profiles in the same type can be represented as one of these types after renaming agents using the same logic above.

$$\varphi(R^{3'}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

Type 4 : By sd-efficiency, $\varphi_{3b}(R^4) = 1$. We can easily derive other elements by equal treatment of equals.

$$\varphi(R^4) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

Type 5-1 : Assume that bP_3aP_3c. By upper-contour strategy-proofness from type 2, $\varphi_{3c}(R^{5-1}) = \frac{1}{3}$. By sd-efficiency, $\varphi_{3a}(R^{5-1}) = 0$. Therefore, $\varphi_{3b}(R^{5-1}) = \frac{2}{3}$. We can easily derive other elements by equal treatment of equals.

$$\varphi(R^{5-1}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Type 5-2 : Assume that bP_3cP_3a. By sd-efficiency, $\varphi_{3a}(R^{5-2}) = 0$. By upper-contour-strategyprrofness from type 5-1, $\varphi_{3b}(R^{5-2}) = \frac{2}{3}$. Therefore, $\varphi_{3c}(R^{5-2}) = \frac{1}{3}$. We can easily derive other elements by equal treatment of equals.

$$\varphi(R^{5-2}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Type 6-1 : Assume that bP_3aP_3c. By upper-contour strategy-proofness from $R^{3'}$, $\varphi_{3c}(R^{6-1}) = \frac{1}{6}$. By sd-efficiency, $\varphi_{3a}(R^{6-1}) = 0$. Therefore, $\varphi_{3b}(R^{6-1}) = \frac{5}{6}$. By upper-contour strategy-proofness from type 5-1, $\varphi_{2a}(R^{6-1}) = \frac{1}{2}$. By sd-efficiency, $\varphi_{2b}(R^{6-1}) = 0$. Therefore, $\varphi_{2c}(R^{6-1}) = \frac{1}{2}$.

$$\varphi(R^{6-1}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{5}{6} & \frac{1}{6} \end{pmatrix}$$

Type 6-2 : Assume that bP_3cP_3a. By upper-contour strategy-proofness from type 6-1, $\varphi_{3b}(R^{6-2}) = \frac{5}{6}$. By sd-efficiency, $\varphi_{3a}(R^{6-2}) = 0$. Therefore, $\varphi_{3c}(R^{6-2}) = \frac{1}{6}$. By upper-contour strategy-proofness from type 5-2, $\varphi_{2a}(R^{6-2}) = \frac{1}{2}$. By sd-efficiency, $\varphi_{2b}(R^{6-2}) = 0$. Therefore, $\varphi_{2c}(R^{6-2}) = \frac{1}{2}$.

$$\varphi(R^{6-2}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{5}{6} & \frac{1}{6} \end{pmatrix}$$

□

However, when the agent's number is larger than three, their characterization result does not hold anymore. They showed that when the number of agent is larger than three, there is no rule meeting the three following axioms: sd-efficiency, sd-strategy-proofness, and equal treatment of equals. Also, the same impossibility result is true with upper-contour strategy-proofness, not sd-strategy-proofness. It means that swap monotonicity is redundant not only in characterization result but also in impossibility result.

Theorem 1. *Assume $n \geq 4$. Then there is no rule meeting the three following axioms: sd-efficiency, equal treatment of equals, and upper-contour strategy-proofness.*

Proof.

Suppose that there exists a rule that satisfies sd-efficiency, equal treatment of equals, and upper-contour strategy-proofness. We will reach a contradiction after considering preference profiles and assginments induced by the rule satisfies above three desirable axioms. First, assume that $n = 4$ and the case $n > 4$ will be proved using the impossibility result in case $n = 4$.

We use Fact 1 (Bogomolnaia and Moulin, 2001) throughout the proof of Theorem 1 and Theorem 2.

Fact 1 (Bogomolnaia and Moulin, 2001). Suppose that bP_ia, while aP_jb for all $j \neq i$. Then sd-efficiency implies $\varphi_{ia} = 0$. Also, let bP_ia for $i \in I$, while aP_jb for $j \notin I$. Then sd-efficiency implies $\varphi_{ia} = 0$ for all $i \in I$ or $p_{jb} = 0$ for all $j \notin I$.

Profile 1. R^1 : For all i, $aP_ibP_icP_id$.

By equal treatment of equals, this result is trivial.

$$\varphi(R^1) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Profile 2. R^2 : For $i = 1, 2, 3$, $aP_ibP_icP_id$. For $i = 4$, $bP_iaP_icP_id$

By upper-contour strategy-proofness from Profile 1, $\varphi_{4c}(R^2) = \varphi_{4d}(R^2) = \frac{1}{4}$. By Fact 1, $\varphi_{4a}(R^2) = 0$. Therefore, $\varphi_{4b}(R^2) = \frac{1}{2}$. We can easily derive other elements by equal treatment of equals.

$$\varphi(R^2) = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Profile 3. R^3 : For $i = 1, 2$, $aP_ibP_icP_id$. For $i = 3, 4$, $bP_iaP_icP_id$

By upper-contour strategy-proofness from Profile 2 and equal treatment of equals, $\varphi_{3c}(R^3) = \varphi_{3d}(R^3) = \varphi_{4c}(R^3) = \varphi_{4d}(R^3) = \frac{1}{4}$. By equal treatment of equals, $\varphi_{1c}(R^3) = \varphi_{1d}(R^3) = \varphi_{2c}(R^3) = \varphi_{2d}(R^3) = \frac{1}{4}$. By Fact 1, $\varphi_{1b}(R^3) = \varphi_{2b}(R^3) = \varphi_{3a}(R^3) = \varphi_{4a}(R^3) = 0$. Therefore, $\varphi_{1a}(R^3) = \varphi_{2a}(R^3) = \varphi_{3b}(R^3) = \varphi_{4b}(R^3) = \frac{1}{2}$.

$$\varphi(R^3) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Profile 4. R^4 : For $i = 1, 2, 3$, $aP_ibP_icP_id$. For $i = 4$, $bP_icP_iaP_id$

By upper-contour strategy-proofness from Profile 2, $\varphi_{4d}(R^4) = \frac{1}{4}$ and $\varphi_{4b}(R^4) = \frac{1}{2}$. By Fact 1, $\varphi_{4a}(R^4) = 0$. Therefore, $\varphi_{4c}(R^4) = \frac{1}{4}$. We can easily derive other elements by equal treatment of equals.

$$\varphi(R^4) = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Profile 5. R^5 : For $i = 1, 2$, $aP_ibP_icP_id$. For $i = 3$, $bP_iaP_icP_id$. For $i = 4$, $bP_icP_iaP_id$

By upper-contour strategy-proofness from Profile 4, $\varphi_{3c}(R^5) = \varphi_{3d}(R^5) = \frac{1}{4}$. By upper-contour strategy-proofness from Profile 3, $\varphi_{4b}(R^5) = \frac{1}{2}$ and $\varphi_{4d}(R^5) = \frac{1}{4}$. By Fact 1(a and c), $\varphi_{4a}(R^5) = 0$. Then, $\varphi_{4c}(R^5) = \frac{1}{4}$. By equal treatment of equals, $\varphi_{1c}(R^5) = \varphi_{1d}(R^5) = \varphi_{2c}(R^5) = \varphi_{2d}(R^5) = \frac{1}{4}$. By Fact 1(a and b), $\varphi_{1b}(R^5) = \varphi_{2b}(R^5) = \varphi_{3a}(R^5) = 0$.

$$\varphi(R^5) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Profile 6. R^6 : For $i = 1, 2$, $aP_ibP_icP_id$. For $i = 3, 4$, $bP_icP_iaP_id$

By upper-contour strategy-proofness from Profile 5 and equal treatment of equals, $\varphi_{3b}(R^6) = \varphi_{4b}(R^6) = \frac{1}{2}$ and $\varphi_{3d}(R^6) = \varphi_{4d}(R^6) = \frac{1}{4}$. Then, $\varphi_{1b}(R^6) = \varphi_{2b}(R^6) = 0$ and $\varphi_{1d}(R^6) = \varphi_{2d}(R^6) = \frac{1}{4}$. By Fact 1(a and c), $\varphi_{3a}(R^6) = \varphi_{4a}(R^6) = 0$. We can easily derive other elements.

$$\varphi(R^6) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Profile 7. R^7 : For $i = 1, 2, 3$, $aP_ibP_icP_id$. For $i = 4$, $bP_icP_idP_ia$

By upper-contour strategy-proofness from Profile 4, $\varphi_{4b}(R^7) = \frac{1}{2}$ and $\varphi_{4c}(R^7) = \frac{1}{4}$. By Fact 1(a and b), $\varphi_{4a}(R^7) = 0$. Therefore, $\varphi_{4d}(R^7) = \frac{1}{4}$. We can easily derive other elements by equal treatment of equals.

$$\varphi(R^7) = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Profile 8. R^8 : For $i = 1, 2$, $aP_ibP_icP_id$. For $i = 3$, $bP_iaP_icP_id$. For $i = 4$, $bP_icP_idP_ia$

By upper-contour strategy-proofness from Profile 7, $\varphi_{3c}(R^8) = \varphi_{3d}(R^8) = \frac{1}{4}$. By upper-contour strategy-proofness from Profile 5, $\varphi_{4b}(R^8) = \frac{1}{2}$ and $\varphi_{4c}(R^8) = \frac{1}{4}$.. By Fact 1(a and d), $\varphi_{4a}(R^8) = 0$. Therefore, $\varphi_{1c}(R^8) = \varphi_{1d}(R^8) = \varphi_{2c}(R^8) = \varphi_{2d}(R^8) = \frac{1}{4}$. By Fact 1(a and b), $\varphi_{1b}(R^8) = \varphi_{2b}(R^8) = \varphi_{3a}(R^8) = 0$.

$$\varphi(R^8) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Profile 8′. $R^{8'}$: For $i = 1$, $bP_iaP_icP_id$. For $i = 2, 3$, $aP_ibP_icP_id$. For $i = 4$, $bP_icP_idP_ia$
By the same logic from Profile 1 to Profile 8, we can derive $\varphi(R^{8'})$.

$$\varphi(R^{8'}) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Profile 8″. $R^{8''}$: For $i = 1, 3$, $aP_ibP_icP_id$. For $i = 2$, $bP_iaP_icP_id$. For $i = 4$, $bP_icP_idP_ia$
By the same logic from Profile 1 to Profile 8, we can derive $\varphi(R^{8''})$.

$$\varphi(R^{8''}) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Profile 9. R^9 : For $i = 1, 2, 3, 4$, $bP_iaP_icP_id$.
By equal treatment of equals, this result is trivial.

$$\varphi(R^9) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Profile 10. R^{10} : For $i = 1, 2, 3$, $bP_iaP_icP_id$. For $i = 4$, $bP_icP_iaP_id$.
By upper-contour strategy-proofness from Profile 9, $\varphi_{4b}(R^{10}) = \frac{1}{4}$ and $\varphi_{4d}(R^{10}) = \frac{1}{4}$. By Fact 1, $\varphi_{4a}(R^{10}) = 0$. Therefore, $\varphi_{4c}(R^{10}) = \frac{1}{2}$. We can easily derive other elements by equal treatment of equals.

$$\varphi(R^{10}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{6} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{6} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{6} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

Profile 11. R^{11} : For $i = 1, 2, 3$, $bP_iaP_icP_id$. For $i = 4$, $bP_icP_idP_ia$.
By upper-contour strategy-proofness from Profile 10, $\varphi_{4b}(R^{11}) = \frac{1}{4}$ and $\varphi_{4c}(R^{11}) = \frac{1}{2}$. By Fact 1, $\varphi_{4a}(R^{11}) = 0$. Therefore, $\varphi_{4d}(R^{11}) = \frac{1}{4}$. We can easily derive other elements by equal treatment of equals.

$$\varphi(R^{11}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{6} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{6} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{6} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

Profile 12. R^{12} : For $i = 1, 2, 4$, $bP_iaP_icP_id$. For $i = 3$, $aP_ibP_icP_id$.
By upper-contour strategy-proofness from Profile 9, $\varphi_{3c}(R^{12}) = \frac{1}{4}$ and $\varphi_{3d}(R^{12}) = \frac{1}{4}$. By Fact 1, $\varphi_{3b}(R^{12}) = 0$. Therefore, $\varphi_{3a}(R^{12}) = \frac{1}{2}$. We can easily derive other elements by equal treatment of equals.

$$\varphi(R^{12}) = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Profile 13. R^{13} : For $i = 1, 2$, $bP_iaP_icP_id$. For $i = 3$, $aP_ibP_icP_id$. For $i = 4$, $bP_icP_idP_ia$.
By upper-contour strategy-proofness from Profile $8'$, $\varphi_{2c}(R^{13}) = \varphi_{2d}(R^{13}) = \frac{1}{4}$. By upper-contour strategy-proofness from Profile $8''$, $\varphi_{1c}(R^{13}) = \varphi_{1d}(R^{13}) = \frac{1}{4}$ By upper-contour strategy-proofness from Profile 11, $\varphi_{3c}(R^{13}) = \frac{1}{6}$ and $\varphi_{3d}(R^{13}) = \frac{1}{4}$. By Fact 1(a and b), $\varphi_{3b}(R^{13})$. By Fact 1(a and d) $\varphi_{4a}(R^{13}) = 0$. Then, $\varphi_{3a}(R^{13}) = \frac{7}{12}$. Also, $\varphi_{1a}(P^{13}) = \frac{5}{24}$. By upper-contour strategy-proofness from Profile 12, $\varphi_{4b}(R^{13}) = \frac{1}{3}$. Then, by equal treatment of equals, $\varphi_{1b}(R^{13}) = \varphi_{2b}(R^{13}) = \frac{1}{3}$. However, $\sum_{k\in A} \varphi_{1k}(R^{13}) = \sum_{k\in A} \varphi_{2k}(R^{13}) = \frac{5}{24} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} > 1$. Contradiction.

$$\varphi(R^{13}) = \begin{pmatrix} \frac{5}{24} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{24} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} \\ \frac{7}{12} & 0 & \frac{1}{6} & \frac{1}{4} \\ 0 & \frac{1}{3} & \cdot & \cdot \end{pmatrix}$$

If $n > 4$, we can construct a preference profile by making the agent k who is newly added prefer the object l which is newly added to any other objects. Also, we can make the other agents's worst object be l. Then, by sd-efficiency, $\varphi_{kl}(R) = 1$. Therefore, the assignment problem is reduced to the first four agents. Hence, it is enough to consider the case $n = 4$. □

Serial dictatorship satisfies sd-efficiency, upper invariance and lower invariance. Random serial dictatorship satisfies equal treatment of equals, upper invariance, and lower invariance. Probabilistic serial rule satisfies sd-efficiency, equal treatment of equals and upper invariance. The existence of a rule which satisfies sd-efficiency, equal treatment of equals, and lower invariance is an open question. Also, because the domain which is used for this proof is single-peaked preferences domain, the following corollary is also true.

Corollary 1. *Assume $n \geq 4$. In single-peaked preferences domain, there is no rule meeting the three following axioms: sd-efficiency, ucs-strategy-proofness, and equal treatment of equals.*

In Nesterov (2017), he shows that when the number of agents is at least three, there is no rule meeting the three following axioms: ex-post efficiency, lower invariance, and upper envyfree. However, when the number of agents is three, there exists rule meeting three following axioms: ex-post efficiency, upper invariance, and upper envyfree. It means that even though upper invariance and lower invariance look similar, lower invariance is more restrictive than upper invariance when we use these axioms with other desirable axioms. Also, by changing upper envyfree into sd-envyfree, we can characterize the probabilistic serial rule when the number of agents is three. You can find a formal definition of the probabilistic serial rule in Bogomolnaia & Moulin (2001).

We introduce the probabilistic serial rule briefly. Before introducing the probabilistic serial rule, we introduce eating algorithm. Each object is suppoesed as being infinitely divisible. A quantity of object a, given to agent i, represents the probability with which agent i is assigned object a. For each agient i, let $\omega_i : [0, 1] \to \mathbb{R}_+$ be a function such that $\int_0^1 \omega_i(t)dt = 1$. The eating algorithm lets agent i eat his favorite available object at the speed $\omega_i(t)$: the objects $a, b, c, ...$ have been entirely eaten and objects $x, y, z, ...$ have not, he eats his favorite object among $x, y, z, ...$ at the speed $\omega_i(t)$. The probabilistic serial rule is obtained by choosing uniform eating speeds: for each agent i, and for $0 \leq t \leq 1$, $\omega_i(t) = 1$. The probabilistic serial rule satisfies sd-efficiency and sd-envyfree, but it does not satisfy sd-strategy-proofness. Bogomolnaia and Heo (2012) show that the probabilistic serial rule is characterized by sd-efficiency, sd-envyfree, and bounded invariance. Bounded invariance requires a rule that changing the ranks of less preferred objects cannot influence that each agents' probabilities to get more preferred objects. Therefore, bounded invariance implies upper invariance. This result implies that when we weaken bounded invariance to upper invariance, the characterization result only holds when the number of agents is three. Note that when the number of agents is three, sd-efficiency is equivalent to ex-post efficiency.

Proposition 3. *Assume $n = 3$. Then the probabilistic serial rule is characterized by the combination of three axioms: ex-post efficiency, sd-envyfreeness and upper invariance.*

Proof.
For $n = 3$, there are six types of preference profiles (Bogomolnaia and Moulin, 2001). These preference profiles are as follows.

$$\text{type 1 (48 profiles)} \begin{cases} aP_1(b,c) \\ bP_2(a,c) \\ cP_3(a,b) \end{cases} \qquad \text{type 2 (6 profiles)} \begin{cases} aP_1bP_1c \\ aP_2bP_2c \\ aP_3bP_3c \end{cases}$$

$$\text{type 3 (18 profiles)} \begin{cases} aP_1bP_1c \\ aP_2bP_2c \\ aP_3cP_3b \end{cases} \qquad \text{type 4 (36 profiles)} \begin{cases} aP_1cP_1b \\ aP_2cP_2b \\ bP_3(a,c) \end{cases}$$

$$\text{type 5 (36 profiles)} \begin{cases} aP_1bP_1c \\ aP_2bP_2c \\ bP_3(a,c) \end{cases} \qquad \text{type 6 (72 profiles)} \begin{cases} aP_1bP_1c \\ aP_2cP_2b \\ bP_3(a,c) \end{cases}$$

Type 1 : By ex-post efficiency, $\varphi_{1a}(R^1) = \varphi_{2b}(R^1) = \varphi_{3c}(R^1) = 1$.

$$\varphi(R^1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Type 2 : By sd-envyfree, $\varphi_{ia}(R^2) = \varphi_{ib}(R^2) = \varphi_{ic}(R^2) = \frac{1}{3}$ for all i.

$$\varphi(R^2) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Type 3 : By sd-envyfree, $\varphi_{1a}(R^3) = \varphi_{2a}(R^3) = \varphi_{3a}(R^3) = \frac{1}{3}$. By sd-efficiency, $\varphi_{3b}(R^3) = 0$. Therefore, $\varphi_{3c}(R^3) = \frac{2}{3}$. We can easily derive other elements by sd-envyfree.

$$\varphi(R^3) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

Type 4 : By ex-post efficiency, $\varphi_{3b}(R^4) = 1$. We can easily derive other elements by sd-envyfree.

$$\varphi(R^4) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

Type 5-1 : Assume that bP_3aP_3c By ex-post efficiency, $\varphi_{3a}(R^{5-1}) = 0$. By sd-envyfree, $\varphi_{1a}(R^{5-1}) = \varphi_{2a}(R^{5-1}) = \frac{1}{2}$. Also, by sd-envyfree, $\varphi_{1c}(R^{5-1}) = \varphi_{2c}(R^{5-1}) = \varphi_{3c}(R^{5-1}) = \frac{1}{3}$. We can easily derive other elements.

$$\varphi(R^{5-1}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Type 5-2 : Assume that bP_3cP_3a. By ex-post efficiency, $\varphi_{3a}(R^{5-2}) = 0$. By sd-envyfree, $\varphi_{1a}(R^{5-2}) = \varphi_{2a}(R^{5-2}) = \frac{1}{2}$. By upper invariance from type 5-1,$\varphi_{3b}(R^{5-2}) = \frac{2}{3}$. Therefore, $\varphi_{3c}(R^{5-2}) = \frac{1}{3}$. We can easily derive other elements by sd-envyfree.

$$\varphi(R^{5-2}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Type 6-1 : Assume that bP_3aP_3c. By ex-post efficiency, $\varphi_{3a}(R^{6-1}) = 0$. By sd-envyfree, $\varphi_{1a}(R^{6-1}) = \varphi_{2a}(R^{6-1}) = \frac{1}{2}$.
By ex-post efficiency, $\varphi_{2b}(R^{6-1}) = 0$. Therefore, $\varphi_{2c}(R^{6-1}) = \frac{1}{2}$. By sd-envyfree, $\varphi_{1c}(R^{6-1}) = \varphi_{3c}(R^{6-1}) = \frac{1}{4}$. We can easily derive other elements.

$$\varphi(R^{6-1}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Type 6-2 : Assume that bP_3cP_3a. By upper invariance from type 6-1, $\varphi_{3b}(R^{6-2}) = \frac{3}{4}$. By ex-post efficiency, $\varphi_{3a}(R^{6-2}) = 0$. Therefore, $\varphi_{3c}(R^{6-2}) = \frac{1}{4}$. By sd-envyfree, $\varphi_{1a}(R^{6-2}) =$

$\varphi_{2a}(R^{6-2}) = \frac{1}{2}$. By ex-post efficiency, $\varphi_{2b}(R^{6-2}) = 0$. Therefore, $\varphi_{2c}(R^{6-2}) = \frac{1}{2}$.

$$\varphi(R^{6-2}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

□

However, we use lower invariance instead of upper invariance, we can derive an impossibility result. As we wrote above, Nesterov (2017) shows that when the number of agents is at least three, there is no rule meeting the three following axioms: ex-post efficiency, lower invariance, and upper envyfree. We show that when $n \geq 4$, impossibility result can be still derived from even weaker fairness axiom than upper envyfree. We can show impossibility result by using strong equal treatment of equals and sd-efficiency. The random priority rule does not satisfies upper invariance but satisfies strong equal treatment of equals. However, when n is at least 4, the random priority rule does not satisfies sd-efficiency.

Theorem 2. *Assume $n \geq 4$. Then there is no rule meeting the three following axioms: sd-efficiency, strong equal treatment of equals, and lower invariance.*

Proof.

Similar to Theorem 1, we suppose that there exists a rule that satisfies sd-efficiency, strong equal treatment of equals, and lower invariance. We will reach a contradiction after considering preference profiles and assginments induced by the rule satisfies above three desirable axioms. First, assume that $n = 4$ and the case $n > 4$ will be proved using the impossibility result in case $n = 4$.

Profile 1. R^1 : For $i = 1, 2, 3$, $aP_icP_ibP_id$. For $i = 4$, $aP_idP_icP_ib$.

By Fact 1, $\varphi_{4b}(R^1) = \varphi_{4c}(R^1) = 0$. By strong equal treatment of equals, $\varphi_{1a}(R^1) = \varphi_{2a}(R^1) = \varphi_{3a}(R^1) = \varphi_{4a}(R^1) = \frac{1}{4}$. Then, $\varphi_{4a}(R^1) = \frac{3}{4}$. We can easily derive other elements by strong equal treatment of equals.

$$\varphi(R^1) = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{12} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}$$

Profile 2. R^2 : For $i = 1, 2$, $aP_icP_ibP_id$. For $i = 3$, $aP_ibP_icP_id$. For $i = 4$, $aP_idP_icP_ib$.

By Fact 1, $\varphi_{4b}(R^2) = \varphi_{4c}(R^2) = 0$. By lower invariance from Profile 1, $\varphi_{3d}(R^2) = \frac{1}{12}$. By strong equal treatment of equals, $\varphi_{1a}(R^2) = \varphi_{2a}(R^2) = \varphi_{3a}(R^2) = \varphi_{4a}(R^2) = \frac{1}{4}$. Then, $\varphi_{4d}(R^2) = \frac{3}{4}$. By Fact 1(b and c), $\varphi_{3c}(R^2) = 0$. Then, $\varphi_{3b}(R^2) = \frac{2}{3}$. We can easily derive other elements by strong equal treatment of equals.

$$\varphi(R^2) = \begin{pmatrix} \frac{1}{4} & \frac{1}{6} & \frac{1}{2} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{6} & \frac{1}{2} & \frac{1}{12} \\ \frac{1}{4} & \frac{2}{3} & 0 & \frac{1}{12} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}$$

Profile 2′. $R^{2'}$: For $i = 1$, $aP_ibP_icP_id$. For $i = 2, 3$, $aP_icP_ibP_id$. For $i = 4$, $aP_idP_icP_ib$. By the same logic from Profile 1 to Profile 2, we can derive $\varphi(R^{2'})$.

$$\varphi(R^{2'}) = \begin{pmatrix} \frac{1}{4} & \frac{2}{3} & 0 & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{6} & \frac{1}{2} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{6} & \frac{1}{2} & \frac{1}{12} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}$$

Profile 3. R^3 : For $i = 1, 2$, $aP_icP_ibP_id$. For $i = 3$, $bP_iaP_icP_id$. For $i = 4$, $aP_idP_icP_ib$. By Fact 1, $\varphi_{4b}(R^3) = \varphi_{4c}(R^3) = 0$. By lower invariance from Profile 2, $\varphi_{3c}(R^3) = 0$ and $\varphi_{3d}(R^3) = \frac{1}{12}$. By Fact 1, $\varphi_{3a}(R^3) = 0$. Then, $\varphi_{3b}(P^3) = \frac{11}{12}$. By strong equal treatment of equals, $\varphi_{1a}(R^3) = \varphi_{2a}(R^3) = \varphi_{4a}(R^3) = \frac{1}{3}$. Then, $\varphi_{4d}(R^3) = \frac{2}{3}$. We can easily derive other elements by strong equal treatment of equals.

$$\varphi(R^3) = \begin{pmatrix} \frac{1}{3} & \frac{1}{24} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{24} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{11}{12} & 0 & \frac{1}{12} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

Profile 4. R^4 : For $i = 1, 2, 3$, $bP_iaP_icP_id$. For $i = 4$, $aP_idP_icP_ib$. By Fact 1, $\varphi_{4b}(R^4) = \varphi_{4c}(R^4) = 0$. By strong equal treatment of equals, $\varphi_{1b}(R^4) = \varphi_{2b}(R^4) = \varphi_{3b}(R^4) = \frac{1}{3}$ and $\varphi_{1c}(R^4) = \varphi_{2c}(R^4) = \varphi_{3c}(R^4) = \frac{1}{3}$.

$$\varphi(R^4) = \begin{pmatrix} \cdot & \frac{1}{3} & \frac{1}{3} & \cdot \\ \cdot & \frac{1}{3} & \frac{1}{3} & \cdot \\ \cdot & \frac{1}{3} & \frac{1}{3} & \cdot \\ \cdot & 0 & 0 & \cdot \end{pmatrix}$$

Profile 5. R^5 : For $i = 1, 2$, $bP_iaP_icP_id$. For $i = 3$, $aP_ibP_icP_id$. For $i = 4$, $aP_idP_icP_ib$. By Fact 1, $\varphi_{4b}(R^5) = \varphi_{4c}(R^5) = 0$. By lower invariance from Profile 4, $\varphi_{3c}(R^5) = \frac{1}{3}$. By strong equal treatment of equals, $\varphi_{1c}(R^5) = \varphi_{2c}(R^5) = \frac{1}{3}$

$$\varphi(R^5) = \begin{pmatrix} \cdot & \cdot & \frac{1}{3} & \cdot \\ \cdot & \cdot & \frac{1}{3} & \cdot \\ \cdot & \cdot & \frac{1}{3} & \cdot \\ \cdot & 0 & 0 & \cdot \end{pmatrix}$$

Profile 6. R^6 : For $i = 1, 2, 3$, $aP_ibP_icP_id$. For $i = 4$, $aP_idP_icP_ib$.
By Fact 1, $\varphi_{4b}(R^6) = \varphi_{4c}(R^6) = 0$. By strong equal treatment of equals, $\varphi_{1a}(R^6) = \varphi_{2a}(R^6) = \varphi_{3a}(R^6) = \varphi_{4a}(R^6) = \frac{1}{4}$. Then, $\varphi_{4d}(R^6) = \frac{3}{4}$. We can easily derive other elements by strong equal treatment of equals.

$$\varphi(R^6) = \begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{12} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}$$

Profile 7. R^7 : For $i = 1, 2$, $aP_ibP_icP_id$. For $i = 3$, $bP_iaP_icP_id$. For $i = 4$, $aP_idP_icP_ib$.
By Fact 1, $\varphi_{4b}(R^7) = \varphi_{4c}(R^7) = 0$. By Fact 1, $\varphi_{3a}(R^7) = 0$. By strong equal treatment of equals, $\varphi_{1a}(R^7) = \varphi_{2a}(R^7) = \varphi_{4a}(R^7) = \frac{1}{3}$. Then, $\varphi_{4a}(R^7) = \frac{2}{3}$. By lower invariance from Profile 6 ,$\varphi_{3c}(R^7) = \frac{1}{3}$ and $\varphi_{3d}(R^7) = \frac{1}{12}$.Then, $\varphi_{3b}(R^7) = \frac{7}{12}$. We can easily derive other elements by strong equal treatment of equals.

$$\varphi(R^7) = \begin{pmatrix} \frac{1}{3} & \frac{5}{24} & \frac{1}{3} & \frac{1}{8} \\ \frac{1}{3} & \frac{5}{24} & \frac{1}{3} & \frac{1}{8} \\ 0 & \frac{7}{12} & \frac{1}{3} & \frac{1}{12} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

Profile 7′. $R^{7'}$: For $i = 1, 3$, $aP_ibP_icP_id$. For $i = 2$, $bP_iaP_icP_id$. For $i = 4$, $aP_idP_icP_ib$.
By the same logic from Profile 1 to Profile 7, we can derive $\varphi(R^{7'})$.

$$\varphi(R^{7'}) = \begin{pmatrix} \frac{1}{3} & \frac{5}{24} & \frac{1}{3} & \frac{1}{8} \\ 0 & \frac{7}{12} & \frac{1}{3} & \frac{1}{12} \\ \frac{1}{3} & \frac{5}{24} & \frac{1}{3} & \frac{1}{8} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

Profile 7″. $R^{7''}$: For $i = 1$, $bP_iaP_icP_id$. For $i = 2, 3$, $aP_ibP_icP_id$. For $i = 4$, $aP_idP_icP_ib$.
By the same logic from Profile 1 to Profile 7, we can derive $\varphi(R^{7''})$.

$$\varphi(R^{7''}) = \begin{pmatrix} 0 & \frac{7}{12} & \frac{1}{3} & \frac{1}{12} \\ \frac{1}{3} & \frac{5}{24} & \frac{1}{3} & \frac{1}{8} \\ \frac{1}{3} & \frac{5}{24} & \frac{1}{3} & \frac{1}{8} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

Profile 5. R^5 : For $i = 1, 2$, $bP_iaP_icP_id$. For $i = 3$, $aP_ibP_icP_id$. For $i = 4$, $aP_idP_icP_ib$. Using above results, we can fix the other elements of $\varphi(R^5)$. By lower invariance from Profile 7′, $\varphi_{1d}(R^5) = \frac{1}{8}$. By lower invariance from Profile 7″, $\varphi_{2d}(R^5) = \frac{1}{8}$. By Fact 1(a and b), $\varphi_{3b}(R^5) = 0$. By strong equal treatment of equals, $\varphi_{1b}(R^5) = \varphi_{2b}(R^5) = \frac{1}{2}$. Then, $\varphi_{1a}(R^5) = \varphi_{2a}(R^5) = \frac{1}{24}$. By strong equal treatment of equals, $\varphi_{3a}(R^5) = \varphi_{4a}(R^5) = \frac{11}{24}$. We can easily derive other elements.

$$\varphi(R^5) = \begin{pmatrix} \cdot & \cdot & \frac{1}{3} & \cdot \\ \cdot & \cdot & \frac{1}{3} & \cdot \\ \cdot & \cdot & \frac{1}{3} & \cdot \\ \cdot & 0 & 0 & \cdot \end{pmatrix} \Rightarrow \varphi(R^5) = \begin{pmatrix} \frac{1}{24} & \frac{1}{2} & \frac{1}{3} & \frac{1}{8} \\ \frac{1}{24} & \frac{1}{2} & \frac{1}{3} & \frac{1}{8} \\ \frac{11}{24} & 0 & \frac{1}{3} & \frac{5}{24} \\ \frac{11}{24} & 0 & 0 & \frac{13}{24} \end{pmatrix}$$

Profile 5′. $R^{5'}$: For $i = 1, 3$, $bP_iaP_icP_id$. For $i = 2$, $aP_ibP_icP_id$. For $i = 4$, $aP_idP_icP_ib$. By the same logic from Profile 1 to Profile 7, we can derive $\varphi(R^{5'})$.

$$\varphi(R^{5'}) = \begin{pmatrix} \frac{1}{24} & \frac{1}{2} & \frac{1}{3} & \frac{1}{8} \\ \frac{11}{24} & 0 & \frac{1}{3} & \frac{5}{24} \\ \frac{1}{24} & \frac{1}{2} & \frac{1}{3} & \frac{1}{8} \\ \frac{11}{24} & 0 & 0 & \frac{13}{24} \end{pmatrix}$$

Profile 8. R^8 : $aP_1bP_1cP_1d$. $aP_2cP_2bP_2d$. $bP_3aP_3cP_3d$. $aP_4dP_4cP_4b$. By Fact 1, $\varphi_{4b}(R^8) = \varphi_{4c}(R^8) = 0$. By Fact 1, $\varphi_{3a}(R^8) = 0$. By strong equal treatment of equals, $\varphi_{1a}(R^8) = \varphi_{2a}(R^8) = \varphi_{4a}(R^8) = \frac{1}{3}$. Then, $\varphi_{4d}(R^8) = \frac{2}{3}$. By lower invariance from Profile 3, $\varphi_{1d}(R^8) = \frac{1}{8}$. By lower invariance from Profile 7, $\varphi_{2d}(R^8) = \frac{1}{8}$. By lower invariance from Profile 2′, $\varphi_{3d}(R^8) = \frac{1}{12}$.

$$\varphi(R^8) = \begin{pmatrix} \frac{1}{3} & \cdot & \cdot & \frac{1}{8} \\ \frac{1}{3} & \cdot & \cdot & \frac{1}{8} \\ 0 & \cdot & \cdot & \frac{1}{12} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

Profile 8′. $R^{8'}$: $bP_1aP_1cP_1d$. $aP_2cP_2bP_2d$. $aP_3bP_3cP_3d$. $aP_4dP_4cP_4b$. By the same logic from Profile 1 to Profile 8, we can derive $\varphi(R^{8'})$.

$$\varphi(R^{8'}) = \begin{pmatrix} 0 & \cdot & \cdot & \frac{1}{12} \\ \frac{1}{3} & \cdot & \cdot & \frac{1}{8} \\ \frac{1}{3} & \cdot & \cdot & \frac{1}{8} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

Profile 9. R^9 : For $i = 1, 3$, $aP_ibP_icP_id$. $aP_2cP_2bP_2d$. $aP_4dP_4cP_4b$.
By Fact 1, $\varphi_{4b}(R^9) = \varphi_{4c}(R^9) = 0$. By lower invariance from Profile 8, $\varphi_{3d}(R^9) = \frac{1}{12}$. By lower invariance from Profile 8′, $\varphi_{1d}(R^9) = \frac{1}{12}$. By strong equal treatment of equals, $\varphi_{1a}(R^9) = \varphi_{2a}(R^9) = \varphi_{3a}(R^9) = \varphi_{4a}(R^9) = \frac{1}{4}$. Then, $\varphi_{4d}(R^9) = \frac{3}{4}$ and $\varphi_{2d}(R^9) = \frac{1}{12}$. By Fact 1(b and c), $\varphi_{2b}(R^9) = 0$. By strong equal treatment of equals, $\varphi_{1b}(R^9) = \varphi_{3b}(R^9) = \frac{1}{2}$. We can easily derive other elements.

$$\varphi(R^9) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{4} & 0 & \frac{1}{3} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}$$

Profile 10. R^{10} : For $i = 1, 3$, $bP_iaP_icP_id$. $aP_2cP_2bP_2d$. $aP_4dP_4cP_4b$.
By Fact 1, $\varphi_{4b}(R^{10}) = \varphi_{4c}(R^{10}) = 0$. By lower invariance from Profile 8, $\varphi_{1d}(R^{10}) = \frac{1}{8}$. By lower invariance from Profile 8′, $\varphi_{3d}(R^{10}) = \frac{1}{8}$. By lower invariance from Profile 5′, $\varphi_{2d}(R^{10}) = \frac{5}{24}$. Then, $\varphi_{4d}(R^{10}) = \frac{13}{24}$. Then, $\varphi_{4a}(R^{10}) = \frac{11}{24}$. By strong equal treatment of equals, $\varphi_{2a}(R^{10}) = \frac{11}{24}$. Then, $\varphi_{1a}(R^{10}) = \varphi_{3a}(R^{10}) = \frac{1}{24}$. By Fact 1(b and c), $\varphi_{2b}(R^{10}) = 0$. Then, $\varphi_{2c}(R^{10}) = \frac{1}{3}$. We can easily derive other elements.

$$\varphi(R^{10}) = \begin{pmatrix} \frac{1}{24} & \frac{1}{2} & \frac{1}{3} & \frac{1}{8} \\ \frac{11}{24} & 0 & \frac{1}{3} & \frac{5}{24} \\ \frac{1}{24} & \frac{1}{2} & \frac{1}{3} & \frac{1}{8} \\ \frac{11}{24} & 0 & 0 & \frac{13}{24} \end{pmatrix}$$

Profile 8. R^8 : $aP_1bP_1cP_1d$. $aP_2cP_2bP_2d$. $bP_3aP_3cP_3d$. $aP_4dP_4cP_4b$.
By lower invariance from Profile 9, $\varphi_{3c}(R^8) = \frac{1}{6}$. By lower invariance from Profile 10, $\varphi_{1c}(R^8) = \frac{1}{3}$. Then, $\varphi_{3b}(R^8) = \frac{3}{4}$ and $\varphi_{1b}(R^8) = \frac{5}{24}$. Also, $\varphi_{2b}(R^8) = \frac{1}{24}$ and $\varphi_{2c}(R^8) = \frac{1}{2}$. However, by Fact 1(b and c), (1) $\varphi_{2b}(R^8)$ and $\varphi_{4b}(R^8)$ must be 0 or (2) $\varphi_{1c}(R^8)$ and $\varphi_{3c}(R^8)$ must be 0. If not, we can find assignment which stochastically dominates this assignment. Therefore, contradiction.

$$\varphi(R^8) = \begin{pmatrix} \frac{1}{3} & \cdot & \cdot & \frac{1}{8} \\ \frac{1}{3} & \cdot & \cdot & \frac{1}{8} \\ 0 & \cdot & \cdot & \frac{1}{12} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix} \Rightarrow \varphi(R^8) = \begin{pmatrix} \frac{1}{3} & \frac{5}{24} & \frac{1}{3} & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{24} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{3}{4} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

If $n > 4$, we can construct a preference profile by making the agent k who is newly added prefer the object l which is newly added to any other objects. Also, we can make the other agents's worst object be l. Then, by sd-efficiency, $\varphi_{kl}(R) = 1$. Therefore, the assignment problem is reduced to the first four agents. Hence, it is enough to consider the case $n = 4$. □

Serial dictatorship satisfies sd-efficiency and lower invariance. The probabilistic serial rule satisfies sd-efficiency and strong equal treatment of equals. Random serial dictatorship satisfies strong equal treatment of equals and lower invariance.

4. Discussion

As other papers considering variations in the random assignment problem, we discuss whether our main impossibility results will be changed by extentions in the problem.

4.1. Indifference over objects

In this case, we add new indifference relation to strict binary relation. So, we can prove our main results using the same preference profile in this case. Also, as Katta and Sethuraman (2006), a possibility result in original domain can be changed into an impossibility result when we permit indifference over objects.

4.2. Multi-unit demands

In this case, we consider each agent has multi-unit demands. Each agent is supposed to receive $q \in \mathbb{Z}$ objects($q \geq 1$). Also, we assume that preferences have additive representations, which means that the utility of each agent can be determined by the sum of each objects they receive. We can also show that the impossibility results still hold. As Kojima (2009)'s result, for $q \geq 2$, we can add new objects of which cardinality is the same with the number of the agents. Also, each agent has the same preferences about the new objects and always prefers the orignial objects to the new objects. By making each agent receive equal probabilities of these new objects at the preference list, we can derive the same impossibility results. Furthermore, when $n \geq 2$ and $q \geq 2$, Aziz and Kasajima (2017) showed that there is no rule meeting the three following axioms: sd-efficiency, equal treatment of equals, and sd-strategy-proofness. The proof of this result still holds when we replace sd-strategy-proofness with upper-contour strategy-proofness. Therefore, when each agent receive at least two objects, Theorem 1 holds with just two agents.

4.3. Different number of agents and objects, opting out

If the number of agents is smaller than the number of objects, there exist objects which no longer are allocated to exactly one agent. In this case, we can make all agents prefer $|A| - |I|$ objects least. Then, by sd-efficiency, the probability to get these objects is zero and impossibility results still hold.

In some examples, an agent may prefer null object, say ϕ, to some of the objects. In this case, if the number of objects is smaller than the number of agents, we can make $|I| - |A|$ agents prefer null object to any other objects. Then, by sd-efficiency, they must not get positive probability to get one of all objects. Therefore, impossibility results in this paper still hold.

First, we define an unacceptable object. a is unacceptable if for all $i \in I$, $P_i \in \mathcal{R}$, $\phi P_i a$. Given $P_i \in \mathcal{R}$, let $Un(P_i)$ denote the set of unacceptable objects in P_i. Given $P_i \in \mathcal{R}$, let m denote a rank of the least acceptable object, which means $max_{k \in A \setminus Un(P_i)} rank(P_i, k)$. Then, we can redefine the stochastic dominance relation in case of opting out. Given $P_i \in \mathcal{R}$ and lotteries $\lambda, \lambda' \in \Delta(A)$, λ *stochastically dominates* λ' according to P_i, denoted $\lambda R_i^{sd} \lambda'$, if
$\sum_{l=1}^{k} \lambda_{r_l(P_i)} \geq \sum_{l=1}^{k} \lambda'_{r_l(P_i)}$ for all $1 \leq k \leq m$,
$\sum_{l=m}^{k} \lambda_{r_l(P_i)} \leq \sum_{l=m}^{k} \lambda'_{r_l(P_i)}$ for all $m \leq k \leq |A|$.

Furthermore, we can get another impossibility result with weak fairness axiom, weak sd-envyfree. A rule φ is **weak sd-envyfree** if for all $i, j \in I, R \in \mathcal{R}^n$, if $\varphi_j(R) R_i^{sd} \varphi_i(R)$, then $\varphi_i(R) = \varphi_j(R)$.

Theorem 3. *Assume $|I| \geq 4$ and $|A| \geq 3$. Then there is no rule meeting the three following axioms: sd-efficiency, weak sd-envyfreeness, and upper-contour strategy-proofness.*

Proof.

Suppose φ is sd-efficient, weak sd-envyfree, and upper-contour strategyproof. Consider the following subset of the full preference domain: agent $n > 4$ prefer null obects to any other objects and $n \in \{1, 2, 3, 4\}$ think $x \notin \{a, b, c\}$ unacceptable. Then, by sd-efficiency, we can make the problem reduced to a problem with 4 agents and 3 objects. Thus, it is enough to think only the case where $|A| = 3$ and $|I| = 4$.

First, for all $R \in \mathcal{R}^{|I|}$, for all $k \in A$, and for all $i \in I$, $\varphi_{ik}(R) = 0$ if $k \in Un(P_i)$ because of sd-efficiency.

$$\varphi(R^{11}) = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix} \qquad \varphi(R^{12}) = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix} \qquad \varphi(R^{13}) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix}$$

$$\varphi(R^{14}) = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} \qquad \varphi(R^{15}) = \varphi(R^{17}) \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} \qquad \varphi(R^{16}) = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix}$$

$$\varphi(R^{18}) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} \qquad \varphi(R^{19}) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix} \qquad \varphi(R^{20}) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix}$$

Profile 1-1. R^{11} : $aP_i\phi$ for all i.
First, by weak sd-envyfreeness, $\varphi_{1a}(R^{11}) = \varphi_{2a}(R^{11}) = \varphi_{3a}(R^{11}) = \varphi_{4a}(R^{11})$. If not, assume that $\varphi_{1a}(R^{11}) > \varphi_{2a}(R^{11})$. Then, $\varphi_1(R^{11})$ stochastically dominates $\varphi_2(R^{11})$ because $\varphi_{1k}(R^{11}) = \varphi_{2k}(R^{11}) = 0$ for $k \in \{b, c, d\}$. Therefore by weak sd-envyfreeness, $\varphi_{1k}(R^{11}) = \varphi_{2k}(R^{11})$ for all k. Contradiction. We can apply this logic for all pairs of agents. Also, by sd-efficiency, $\varphi_{1a}(R^{11}) + \varphi_{2a}(R^{11}) + \varphi_{3a}(R^{11}) + \varphi_{4a}(R^{11}) = 1$. Therefore, By weak sd-envyfreeness and sd-efficiency, $\varphi_{1a}(R^{11}) = \varphi_{2a}(R^{11}) = \varphi_{3a}(R^{11}) = \varphi_{4a}(R^{11}) = \frac{1}{4}$.
Profile 1-2. R^{12} : $aP_1bP_1\phi$. $aP_i\phi$ for $i = 2, 3, 4$.
By upper-contour strategy-proofness from Profile 1-1, $\varphi_{1a}(R^{12}) = \frac{1}{4}$. Using the same logic in Profile 1-1, by weak sd-envyfreeness and sd-efficiency, $\varphi_{2a}(R^{12}) = \varphi_{3a}(R^{12}) = \varphi_{4a}(R^{12}) = \frac{1}{4}$. By sd-efficiency, $\varphi_{1b}(R^{12}) = \frac{3}{4}$.
Profile 1-3. R^{13} : $aP_ibP_i\phi$ for $i = 1, 2$. $aP_i\phi$ for $i = 3, 4$.
By upper-contour strategy-proofness from Profile 1-2, $\varphi_{2a}(R^{13}) = \frac{1}{4}$. By permutating the agent 1 and agent 2 in Profile 1-2 and upper-contour strategy-proofness from this profile, $\varphi_{1a}(R^{13}) = \frac{1}{4}$. By weak sd-envyfreeness and sd-efficiency, $\varphi_{1b}(R^{13}) = \varphi_{2b}(R^{13}) = \frac{1}{2}$. By weak sd-envyfreeness, $\varphi_{3a}(R^{13}) = \varphi_{4a}(R^{13}) = \frac{1}{4}$.
Profile 1-4. R^{14} : $aP_i\phi$ for $i = 1, 2, 3$. aP_4cP_4b.

By upper-contour strategy-proofness from Profile 1-1, $\varphi_{4a}(R^{14}) = \frac{1}{4}$. By weak sd-envyfreeness, $\varphi_{1a}(R^{14}) = \varphi_{2a}(R^{14}) = \varphi_{3a}(R^{14}) = \frac{1}{4}$. By sd-efficiency, $\varphi_{3c}(R^{14}) = \frac{3}{4}$.

Profile 1-5. $R^{15} : aP_1bP_1\phi$. $aP_i\phi$ for $i = 2, 3$. aP_4cP_4b.

By upper-contour strategy-proofness from Profile 1-4, $\varphi_{1a}(R^{15}) = \frac{1}{4}$. By upper-contour strategy-proofness from Profile 1-2, $\varphi_{4a}(R^{15}) = \frac{1}{4}$. By weak sd-envyfreeness, $\varphi_{2a}(R^{15}) = \varphi_{3a}(R^{15}) = \frac{1}{4}$. By sd-efficiency, $\varphi_{3c}(R^{15}) = \frac{3}{4}$. By sd-efficiency, $\varphi_{1b}(R^{15}) = \frac{3}{4}$.

Profile 1-6. $R^{16} : aP_1bP_1c$. $aP_i\phi$ for $i = 2, 3, 4$.

By upper-contour strategy-proofness from Profile 1-2, $\varphi_{1a}(R^{16}) = \frac{1}{4}$. By weak sd-envyfreeness, $\varphi_{2a}(R^{16}) = \varphi_{3a}(R^{16}) = \varphi_{4a}(R^{16}) = \frac{1}{4}$. By sd-efficiency, $\varphi_{1b}(R^{16}) = \frac{3}{4}$.

Profile 1-7. $R^{17} : aP_1bP_1c$. $aP_i\phi$ for $i = 2, 3$. aP_4cP_4b.

By upper-contour strategy-proofness from Profile 1-5, $\varphi_{1a}(R^{17}) = \frac{1}{4}$ and $\varphi_{1b}(R^{17}) = \frac{3}{4}$. By upper-contour strategy-proofness from Profile 1-6, $\varphi_{4a}(R^{17}) = \frac{1}{4}$. By weak sd-envyfreeness, $\varphi_{2a}(R^{17}) = \varphi_{3a}(R^{17}) = \frac{1}{4}$. By sd-efficiency, $\varphi_{3c}(R^{17}) = \frac{3}{4}$.

Profile 1-8. $R^{18} : aP_ibP_i\phi$ for $i = 1, 2$. $aP_3\phi$. aP_4cP_4b.

By upper-contour strategy-proofness from Profile 1-5, $\varphi_{1a}(R^{18}) = \frac{1}{4}$. By permutating the agent 1 and the agent 2 in Profile 1-5 and upper-contour strategy-proofness from this profile, $\varphi_{2a}(R^{18}) = \frac{1}{4}$. By upper-contour strategy-proofness from Profile 1-3, $\varphi_{4a}(R^{18}) = \frac{1}{4}$. By sd-efficiency, $\varphi_{3a}(R^{18}) = \frac{1}{4}$. By sd-efficiency, $\varphi_{3c}(R^{18}) = \frac{3}{4}$. By sd-efficiency and weak sd-envyfreeness, $\varphi_{2b}(R^{18}) = \varphi_{3b}(R^{18}) = \frac{1}{2}$.

Profile 1-9. $R^{19} : aP_1bP_1c$. $aP_2bP_2\phi$. $aP_i\phi$ for $i = 3, 4$.

By upper-contour strategy-proofness from Profile 1-3 and the same logic from Profile 1-3, $\varphi_{1a}(R^{19}) = \varphi_{2a}(R^{19}) = \frac{1}{4}$ and $\varphi_{1b}(R^{19}) = \varphi_{2b}(R^{19}) = \frac{1}{2}$. By sd-efficiency, $\varphi_{1c}(R^{19}) = \frac{1}{4}$. By weak sd-envyfreeness, $\varphi_{3a}(R^{19}) = \varphi_{4a}(R^{19}) = \frac{1}{4}$.

Profile 1-10. $R^{20} : aP_1bP_1c$. $aP_2bP_2\phi$. $aP_3\phi$. aP_4cP_4b.

By upper-contour strategy-proofness from Profile 1-7, $\varphi_{2a}(R^{20}) = \frac{1}{4}$. By upper-contour strategy-proofness from Profile 1-5, $\varphi_{1a}(R^{20}) = \frac{1}{4}$ and $\varphi_{1b}(R^{20}) = \frac{1}{2}$. By upper-contour strategy-proofness from Profile 1-6, $\varphi_{4a}(R^{20}) = \frac{1}{4}$. By sd-efficiency, $\varphi_{1c}(R^{20}) = \frac{1}{4}$. By sd-efficiency, $\varphi_{2c}(R^{20}) = \frac{1}{4}$ and $\varphi_{4c}(R^{20}) = \frac{3}{4}$.

$$\varphi(R^{21}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \varphi(R^{22}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \varphi(R^{23}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\varphi(R^{24}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \varphi(R^{25}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \varphi(R^{26}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\varphi(R^{27}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \varphi(R^{28}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \varphi(R^{29}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Profile 2-1. R^{21} : $aP_i\phi$ for $i = 1, 3$. $cP_i\phi$ for $i = 2, 4$.
By weak sd-envyfreeness and sd-efficiency, $\varphi_{1a}(R^{21}) = \varphi_{3a}(R^{21}) = \frac{1}{2}$ and $\varphi_{2c}(R^{21}) = \varphi_{4c}(R^{21}) = \frac{1}{2}$.

Profile 2-2. R^{22} : $aP_1bP_1\phi$. $aP_3\phi$. $cP_i\phi$ for $i = 2, 4$.
By upper-contour strategy-proofness from Profile 2-1, $\varphi_{1a}(R^{22}) = \frac{1}{2}$. By weak sd-envyfreeness and sd-efficiency, $\varphi_{3a}(R^{22}) = \frac{1}{2}$ and $\varphi_{2c}(R^{22}) = \varphi_{4c}(R^{22}) = \frac{1}{2}$. By sd-efficiency, $\varphi_{1b}(R^{22}) = \frac{1}{2}$.

Profile 2-3. R^{23} : aP_1bP_1c. $aP_3\phi$. $cP_i\phi$ for $i = 2, 4$.
By upper-contour strategy-proofness from Profile 2-2, $\varphi_{1a}(R^{23}) = \varphi_{1b}(R^{23}) = \frac{1}{2}$. By weak sd-envyfreeness and sd-efficiency, $\varphi_{3a}(R^{23}) = \frac{1}{2}$ and $\varphi_{2c}(R^{23}) = \varphi_{4c}(R^{23}) = \frac{1}{2}$.

Profile 2-4. R^{24} : $aP_i\phi$ for $i = 1, 3$. cP_2b. $cP_4\phi$.
By upper-contour strategy-proofness from Profile 2-1, $\varphi_{2c}(R^{24}) = \frac{1}{2}$. By weak sd-envyfreeness and sd-efficiency, $\varphi_{4c}(R^{24}) = \frac{1}{2}$ and $\varphi_{1a}(R^{24}) = \varphi_{3a}(R^{24}) = \frac{1}{2}$. By sd-efficiency, $\varphi_{2b}(R^{24}) = \frac{1}{2}$.

Profile 2-5. R^{25} : $aP_1bP_1\phi$. $cP_2bP_2\phi$. $aP_3\phi$. $cP_4\phi$.
By upper-contour strategy-proofness from Profile 2-4, $\varphi_{1a}(R^{25}) = \frac{1}{2}$. By upper-contour strategy-proofness from Profile 2-2, $\varphi_{2c}(R^{25}) = \frac{1}{2}$. By weak sd-envyfreeness, $\varphi_{4c}(R^{25}) = \frac{1}{2}$ and $\varphi_{3a}(R^{25}) = \frac{1}{2}$. By sd-efficiency, $\varphi_{1b}(R^{25}) = \varphi_{2b}(R^{25}) = \frac{1}{2}$.

Profile 2-6. R^{26} : aP_1bP_1c. $cP_2bP_2\phi$. $aP_3\phi$. $cP_4\phi$.
By upper-contour strategy-proofness from Profile 2-5, $\varphi_{1a}(R^{26}) = \varphi_{1b}(R^{26}) = \frac{1}{2}$. By upper-contour strategy-proofness from Profile 2-3, $\varphi_{2c}(R^{26}) = \frac{1}{2}$. By weak sd-envyfreeness, $\varphi_{4c}(R^{26}) = \frac{1}{2}$ and $\varphi_{3a}(R^{26}) = \frac{1}{2}$. By sd-efficiency, $\varphi_{2b}(R^{26}) = \frac{1}{2}$.

Profile 2-7. R^{27} : $aP_i\phi$ for $i = 1, 3$. cP_2bP_2a. $cP_4\phi$.
By upper-contour strategy-proofness from Profile 2-4, $\varphi_{2b}(R^{27}) = \varphi_{2c}(R^{27}) = \frac{1}{2}$. By weak sd-envyfreeness and sd-efficiency, $\varphi_{4c}(R^{27}) = \frac{1}{2}$ and $\varphi_{1a}(R^{27}) = \varphi_{3a}(R^{27}) = \frac{1}{2}$.

Profile 2-8. R^{28} : $aP_1bP_1\phi$. cP_2bP_2a. $aP_3\phi$. $cP_4\phi$.

By upper-contour strategy-proofness from Profile 2-5, $\varphi_{2b}(R^{28}) = \varphi_{2c}(R^{28}) = \frac{1}{2}$. By upper-contour strategy-proofness from Profile 2-7, $\varphi_{1a}(R^{28}) = \frac{1}{2}$. By sd-efficiency, $\varphi_{1b}(R^{28}) = \varphi_{3a}(R^{28}) = \varphi_{4c}(R^{28}) = \frac{1}{2}$.

Profile 2-9. R^{29} : aP_1bP_1c. cP_2bP_2a. $aP_3\phi$. $cP_4\phi$.

By upper-contour strategy-proofness from Profile 2-6, $\varphi_{2b}(R^{29}) = \varphi_{2c}(R^{29}) = \frac{1}{2}$. By upper-contour strategy-proofness from Profile 2-8, $\varphi_{1a}(R^{29}) = \varphi_{1b}(R^{29}) = \frac{1}{2}$. By sd-efficiency, $\varphi_{3a}(R^{29}) = \varphi_{4c}(R^{29}) = \frac{1}{2}$.

$$\varphi(R^{31}) = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \varphi(R^{32}) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \varphi(R^{33}) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\varphi(R^{34}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \varphi(R^{35}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \cdot \\ \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & \cdot \end{pmatrix} \qquad \varphi(R^{36}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}$$

Profile 3-1. R^{31} : $aP_i\phi$ for $i = 1, 2, 3$. $cP_4\phi$.

By weak sd-envyfreeness and sd-efficiency, $\varphi_{1a}(R^{31}) = \varphi_{2a}(R^{31}) = \varphi_{3a}(R^{31}) = \frac{1}{3}$. By sd-efficiency, $\varphi_{4c}(R^{31}) = 1$.

Profile 3-2. R^{32} : $aP_1bP_1\phi$. $aP_i\phi$ for $i = 2, 3$. $cP_4\phi$.

By upper-contour strategy-proofness from Profile 3-1, $\varphi_{1a}(R^{32}) = \frac{1}{3}$. By weak sd-envyfreeness and sd-efficiency, $\varphi_{2a}(R^{32}) = \varphi_{3a}(R^{32}) = \frac{1}{3}$. By sd-efficiency, $\varphi_{1b}(R^{32}) = \frac{2}{3}$ and $\varphi_{4c}(R^{32}) = 1$.

Profile 3-3. R^{33} : aP_1bP_1c. $aP_i\phi$ for $i = 2, 3$. $cP_4\phi$.

By upper-contour strategy-proofness from Profile 3-2, $\varphi_{1a}(R^{33}) = \frac{1}{3}$ and $\varphi_{1b}(R^{33}) = \frac{2}{3}$. By weak sd-envyfreeness and sd-efficiency, $\varphi_{2a}(R^{33}) = \varphi_{3a}(R^{33}) = \frac{1}{3}$. By sd-efficiency, $\varphi_{4c}(R^{33}) = 1$.

Profile 3-4. R^{34} : $aP_ibP_i\phi$ for $i = 1, 2$. $aP_3\phi$. $cP_4\phi$.

By upper-contour strategy-proofness from Profile 3-2, $\varphi_{1a}(R^{34}) = \varphi_{2a}(R^{34}) = \frac{1}{3}$. By weak sd-envyfreeness, $\varphi_{3a}(R^{34}) = \frac{1}{3}$. By sd-efficiency, $\varphi_{4c}(R^{34}) = 1$. By sd-efficiency and weak sd-envyfreeness, $\varphi_{1b}(R^{34}) = \varphi_{2b}(R^{34}) = \frac{1}{2}$.

Profile 3-5. R^{35} : aP_1bP_1c. $aP_2bP_2\phi$. $aP_3\phi$. $cP_4\phi$.

By upper-contour strategy-proofness from Profile 3-4, $\varphi_{1a}(R^{35}) = \varphi_{2a}(R^{35}) = \frac{1}{3}$ and $\varphi_{1b}(R^{35}) =$

$\frac{1}{2}$. By weak sd-envyfreeness, $\varphi_{3a}(R^{35}) = \frac{1}{3}$ and $\varphi_{2b}(R^{35}) = \frac{1}{2}$.

Profile 3-6. R^{36} : aP_ibP_ic for $i = 1, 2$. $aP_3\phi$. $cP_4\phi$.

By upper-contour strategy-proofness from Profile 3-5, $\varphi_{1a}(R^{36}) = \varphi_{2a}(R^{36}) = \frac{1}{3}$ and $\varphi_{1b}(R^{36}) = \varphi_{2b}(R^{36}) = \frac{1}{2}$. By upper-contour strategy-proofness from Profile 2-9, $\varphi_{1c}(R^{36}) = \varphi_{2c}(R^{36}) = \frac{1}{6}$. By weak sd-envyfreeness, $\varphi_{3a}(R^{36}) = \frac{1}{3}$. By sd-efficiency, $\varphi_{4c}(R^{36}) = \frac{2}{3}$.

$$\varphi(R^{41}) = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \varphi(R^{42}) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \varphi(R^{43}) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\varphi(R^{44}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \varphi(R^{45}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \cdot \\ \frac{1}{3} & \frac{1}{2} & \cdot \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & \cdot \end{pmatrix} \qquad \varphi(R^{46}) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}$$

Profile 4-1. R^{41} : $aP_i\phi$ for $i = 1, 2, 3$. cP_4aP_4b.

By upper-contour strategy-proofness from Profile 3-1, $\varphi_{4c}(R^{41}) = 1$. By weak sd-envyfreeness and sd-efficiency, $\varphi_{1a}(R^{41}) = \varphi_{2a}(R^{41}) = \varphi_{3a}(R^{41}) = \frac{1}{3}$.

Profile 4-2. R^{42} : $aP_1bP_1\phi$. $aP_i\phi$ for $i = 2, 3$. cP_4aP_4b.

By upper-contour strategy-proofness from Profile 3-2, $\varphi_{4c}(R^{42}) = 1$. By upper-contour strategy-proofness from Profile 4-1, $\varphi_{1a}(R^{42}) = \frac{1}{3}$. By weak sd-envyfreeness, $\varphi_{2a}(R^{42}) = \varphi_{3a}(R^{42}) = \frac{1}{3}$. By sd-efficiency, $\varphi_{1b}(R^{42}) = \frac{2}{3}$.

Profile 4-3. R^{43} : aP_1bP_1c. $aP_i\phi$ for $i = 2, 3$. cP_4aP_4b.

By upper-contour strategy-proofness from Profile 3-3, $\varphi_{4c}(R^{43}) = 1$. By upper-contour strategy-proofness from Profile 4-2, $\varphi_{1a}(R^{43}) = \frac{1}{3}$ and $\varphi_{1b}(R^{43}) = \frac{2}{3}$. By weak sd-envyfreeness, $\varphi_{2a}(R^{43}) = \varphi_{3a}(R^{43}) = \frac{1}{3}$.

Profile 4-4. R^{44} : $aP_ibP_i\phi$ for $i = 1, 2$. $aP_3\phi$. cP_4aP_4b.

By upper-contour strategy-proofness from Profile 3-4, $\varphi_{4c}(R^{44}) = 1$. By upper-contour strategy-proofness from Profile 4-2, $\varphi_{1a}(R^{44}) = \varphi_{2a}(R^{44}) = \frac{1}{3}$. By weak sd-envyfreeness, $\varphi_{3a}(R^{44}) = \frac{1}{3}$. By sd-efficiency and weak sd-envyfreeness, $\varphi_{1b}(R^{44}) = \varphi_{2b}(R^{44}) = \frac{1}{2}$.

Profile 4-5. R^{45} : aP_1bP_1c. $aP_2bP_2\phi$. $aP_3\phi$. cP_4aP_4b.

By upper-contour strategy-proofness from Profile 4-4, $\varphi_{1a}(R^{45}) = \varphi_{2a}(R^{45}) = \frac{1}{3}$ and $\varphi_{1b}(R^{45}) = \varphi_{2b}(R^{45}) = \frac{1}{2}$. By weak sd-envyfreeness, $\varphi_{3a}(R^{45}) = \frac{1}{3}$.

Profile 4-6. R^{46} : aP_ibP_ic for $i = 1, 2$. $aP_3\phi$. cP_4aP_4b.

By upper-contour strategy-proofness from Profile 3-6, $\varphi_{4c}(R^{46}) = \frac{2}{3}$. By upper-contour strategy-proofness from Profile 4-5, $\varphi_{1a}(R^{46}) = \varphi_{2a}(R^{46}) = \frac{1}{3}$ and $\varphi_{1b}(R^{46}) = \varphi_{2b}(R^{46}) = \frac{1}{2}$. By sd-efficiency and weak sd-envyfreeness, $\varphi_{1c}(R^{46}) = \varphi_{2c}(R^{46}) = \frac{1}{6}$. By weak sd-envyfreeness, $\varphi_{3a}(R^{46}) = \frac{1}{3}$.

$$\varphi(R^{51}) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix} \qquad \varphi(R^{52}) = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{5}{12} \end{pmatrix} \qquad \varphi(R') = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}$$

Profile 5-1. R^{51} : aP_1bP_1c. aP_2bP_2c. $aP_i\phi$ for $i = 3, 4$.
By upper-contour strategy-proofness from Profile 1-9, $\varphi_{1a}(R^{51}) = \varphi_{2a}(R^{51}) = \frac{1}{4}$ and $\varphi_{1b}(R^{51}) = \varphi_{2b}(R^{51}) = \frac{1}{2}$. By sd-efficiency and weak sd-envyfreeness, $\varphi_{1c}(R^{51}) = \varphi_{2c}(R^{51}) = \frac{1}{4}$. By weak sd-envyfreeness, $\varphi_{3a}(R^{51}) = \varphi_{4a}(R^{51}) = \frac{1}{4}$.

Profile 5-2. R^{52} : aP_1bP_1c. aP_2bP_2c. $aP_3\phi$. aP_4cP_4b.
By upper-contour strategy-proofness from Profile 1-10, $\varphi_{1a}(R^{52}) = \varphi_{2a}(R^{52}) = \frac{1}{4}$ and $\varphi_{2a}(R^{52}) = \varphi_{2b}(R^{52}) = \frac{1}{2}$. By upper-contour strategy-proofness from Profile 5-1, $\varphi_{4a}(R^{52}) = \frac{1}{4}$. By weak sd-envyfreeness, $\varphi_{3a}(R^{52}) = \frac{1}{4}$. By upper-contour strategy-proofness from Profile 4-6, $\varphi_{4a}(R^{52}) + \varphi_{4c}(R^{52}) = \frac{2}{3}$. Therefore, $\varphi_{4c}(R^{52}) = \frac{5}{12}$. By sd-efficiency and weak sd-envyfreeness, $\varphi_{1c}(R^{52}) = \varphi_{2c}(R^{52}) = \frac{1}{4}$. However, $\varphi(R^{52})$ is stochastically dominated by $\varphi(R')$. Contradiction.

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국문초록

랜덤 배정 문제에서의 전략 무용성 연구

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불가분재화의 랜덤 배정 문제에서 확률적 지배관계에서의 효율성, 동일한 사람들에 대한 동일한 대우, 확률적 지배관계에서의 전략무용성 이 세 가지를 만족하는 메커니즘이 존재하지 않는다는 것이 알려져 있다. 또한 전략무용성을 약화시킨 기존의 결과를 참고하여 새로운 개념의 전략무용성 개념을 제시하였다. 이 개념은 기존의 확률적 지배관계에서의 전략무용성보다 약한 공리이지만, 기존의 확률적 지배관계에서의 전략무용성을 새로운 개념의 전략무용성 개념으로 대체하더라도 기존의 불가능성 정리는 그대로 성립하게 된다.

주요어 : 랜덤 배정 문제, 불가분재화, 확률적 지배관계, 효율성, 전략무용성
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