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Regularity estimates for measure data problems

(측도데이터를 가지는 방정식에 대한 정규화 추정값)

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박정태

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Regularity estimates for measure data problems

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Abstract

Regularity estimates for measure data problems

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We establish global Calderón-Zygmund type estimates for nonlinear elliptic and parabolic equations in nonsmooth bounded domains when the right-hand side is a finite signed Radon measure.

We first investigate a quasilinear elliptic equation with variable growth. We obtain an optimal global Calderón-Zygmund type estimate for such a measure data problem, by proving that the gradient of a very weak solution to the problem is as globally integrable as the first order maximal function of the associated measure, up to a correct power, under minimal regularity requirements on the nonlinearity, the variable exponent and the boundary of the domain.

Secondly, we study a nonlinear elliptic equation with measurable nonlinearity. A global Calderón-Zygmund type estimate in variable exponent spaces is established under optimal regularity assumptions on the nonlinearity and the Reifenberg flatness of the boundary.

We finally consider a nonlinear parabolic equation with measurable nonlinearity. Under minimal regularity requirements on the nonlinearity and the boundary of the domain, we prove a global Calderón-Zygmund type estimate in weighted Orlicz spaces. As an application we obtain such an estimate in variable exponent spaces, which gives an alternative proof for this new result in the literature.

Key words: measure data, regularity, Calderón-Zygmund estimate, Reifenberg domain, variable exponent, extrapolation
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Contents

Chapter 1

Introduction

The aim of this dissertation is to provide global Calderón-Zygmund type estimates for measure data problems. More precisely, we establish global gradient estimates for solutions of the divergence type nonlinear elliptic and parabolic equations with measure data on the right-hand side, under optimal regularity assumptions on both the nonlinearity and the boundary of the domain.

1.1 Measure data problems

We first outline the existence and uniqueness of a solution for measure data problems, see [78, 81] and the references given there for details.

Let us consider the Dirichlet problem with measure data

$$\begin{cases} -\operatorname{div} \mathbf{a}(Du, x) & = & \mu \quad \text{in } \Omega, \\ u & = & 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 2$, and μ is a signed Radon measure on Ω with finite total variation $|\mu|(\Omega) < \infty$. We assume that μ is defined in $\mathbb{R}^n$ by letting the zero extension to $\mathbb{R}^n$. The nonlinearity $\mathbf{a} = \mathbf{a}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable in ξ and measurable in x, and it satisfies the following growth and ellipticity conditions:

$$\begin{cases} |\xi||D_\xi \mathbf{a}(\xi, x)| + |\mathbf{a}(\xi, x)| \leq \Lambda |\xi|^{p-1}, \\ \lambda |\xi|^{p-2} |\eta|^2 \leq \langle D_\xi \mathbf{a}(\xi, x)\eta, \eta \rangle, \end{cases}$$

whenever $x, \eta, \xi \in \mathbb{R}^n$, where $0 < \lambda \leq \Lambda$ and $p > 2 - \frac{1}{n}$. Here $D_\xi \mathbf{a}$ is the Jacobian matrix of the nonlinearity $\mathbf{a}$ with respect to ξ, and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n \times \mathbb{R}^n$. We remark that when investigating higher regularity, for instance differentiability or integrability on Du, we need additional assumptions on the map $x \mapsto \mathbf{a}(\cdot, x)$.

We now introduce the notion of a distributional solution as follows:

Definition 1.1.1. *A function $u \in W_0^{1,1}(\Omega)$ called a* very weak solution *to the problem* (1.1.1) *if $\mathbf{a}(Du, x) \in L^1(\Omega)$ and*

$$\int_\Omega \langle \mathbf{a}(Du, x), D\varphi \rangle \; dx = \int_\Omega \varphi \, d\mu \quad \textit{for all} \;\; \varphi \in C_c^\infty(\Omega).$$

Note that a very weak solution need not generally be a weak (energy) solution in $W_0^{1,p}(\Omega)$. A wide notion of very weak solutions does not guarantee the uniqueness of such solutions even for uniformly elliptic linear equations of the type $\operatorname{div}(A(x)Du) = 0$, see a counterexample of Serrin [94].

To avoid this difficulty, there exists a proper notion of solutions such as entropy solution [10], renormalized solution [47], SOLA [12, 48], etc. Indeed, from the point of regularity, there is little difference between SOLA and the other solutions, as all are based on approximation arguments. Hereafter we adopt the notion of *SOLA (Solution Obtained by Limits of Approximations)*:

Definition 1.1.2. *$u \in W_0^{1,1}(\Omega)$ is a* SOLA *to the problem* (1.1.1) *if u is a very weak solution of* (1.1.1)*, and there exists a sequence of weak solutions $\{u_h\}_{h \geq 1} \subset W_0^{1,p}(\Omega)$ of the regularized problems*

$$\begin{cases} -\operatorname{div} \mathbf{a}(Du_h, x) &= \mu_h \quad \textit{in } \Omega, \\ \qquad\qquad u_h &= 0 \quad\;\; \textit{on } \partial\Omega \end{cases}$$

such that

$$u_h \to u \quad \textit{in} \;\; W_0^{1,\max\{1,p-1\}}(\Omega) \quad \textit{as } h \to \infty,$$

where $\mu_h \in L^\infty(\Omega)$ converges weakly to μ in the sense of measure and satisfies for each open set $V \subset \mathbb{R}^n$,

$$\limsup_{h \to \infty} |\mu_h|(V) \leq |\mu|(\overline{V}),$$

with μ_h defined in $\mathbb{R}^n$ by considering the zero extension to $\mathbb{R}^n$.

The existence of SOLA was first introduced by Boccardo and Gallouët in [12] with the relation

$$u_h \to u \quad \text{in} \quad W_0^{1,q}(\Omega) \qquad \text{for all} \quad q < \min\left\{p, \frac{n(p-1)}{n-1}\right\}, \tag{1.1.2}$$

who proved a priori $W^{1,q}$ estimate of solutions for regularized problems with a proper approximation scheme. Indeed, the result in (1.1.2) is almost optimal. Consider the problem

$$\begin{cases} -\operatorname{div}(|Du|^{p-2}Du) &= \delta_0 \quad \text{in } B_1, \\ u &= 0 \quad \text{on } \partial B_1, \end{cases} \tag{1.1.3}$$

where δ_0 is the Dirac delta function charging the origin. Then the fundamental solution

$$u(x) = c(n,p)\begin{cases} |x|^{\frac{p-n}{p-1}} - 1 & \text{if } \; 1 < p \neq n, \\ \log|x| & \text{if } \; p = n \end{cases}$$

of (1.1.3) implies that u belongs to $W_0^{1,q}(B_1)$ for $q < \frac{n(p-1)}{n-1}$. The result in (1.1.2) also gives us that $u \in W_0^{1,1}(\Omega)$ if and only if $p > 2 - \frac{1}{n}$.

When $p > n$, it follows from Morrey's inequality that the measure μ belongs to the dual space $W^{-1,p'}(\Omega)$; then a SOLA to the problem (1.1.1) becomes a weak solution in $W_0^{1,p}(\Omega)$. Indeed, a weak solution is unique from the monotone operator theory, see for example [95]. On the other hand, in the case $p \leq n$, it is well known that $\mu \in W^{-1,p'}(\Omega)$ if the measure μ satisfies the following conditions: (i) $\mu \in L^\gamma$ with $\gamma \geq (p^*)' = \frac{np}{np-n+p} > 1$ (by Sobolev's inequality), or (ii) $|\mu|(B_r) \lesssim r^{n-p+\epsilon}$ for some $\epsilon > 0$ (by Adams's trace theorem in [4]).

The uniqueness of a SOLA generally remains unsolved except when $\mu \in L^1$ or when considering linear problems $\mathbf{a}(\xi, x) = A(x)\xi$ in (1.1.1), see [47] and the references therein. Recently, however, it was proved that a SOLA u is unique under the assumption that the nonlinearity $\mathbf{a}$ with the growth $p = 2$ is *strongly asymptotically Uhlenbeck*, see [18, Corollary 2.2]. We also refer to [10, 12, 14, 47, 48] for a further discussion regarding the existence and uniqueness of measure data problems.

We briefly present as well two applications of measure data problems.

a. The flow pattern of blood in the heart, see [84, 96].

The following systems of motion are considered as a model for blood flow in the heart by Peskin in [84]:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \Delta \mathbf{u} + \nabla p = \boldsymbol{\mu}, \\ \operatorname{div} \mathbf{u} = 0, \\ \dfrac{dx_k}{dt} = \mathbf{u}(x_k, t), \quad \boldsymbol{\mu} = \lim\limits_{N \to \infty} \dfrac{1}{N} \sum\limits_{k=1}^{N} f_k \delta_{x_k}, \end{cases} \tag{1.1.4}$$

considered in the cylindrical domain $\Omega \times (0, T]$, with $\Omega \subset \mathbb{R}^3$. Here $\mathbf{u}$ is the velocity of the blood fluid, x_k is the position in space of the material sample point k of the immersed boundary (the moving boundary which interacts the fluid), f_k is the intensity of the boundary force at x_k, and δ_{x_k} is the Dirac delta function charging x_k.

The motion of the blood flow in the heart is closely related to the performance of the heart valves, and it therefore has practical application in the design of artificial valves and artificial hearts. The problem (1.1.4) was at first treated numerically in [84], and Ton in [96] later considered the existence of a solution to (1.1.4) for two dimensional case.

b. State-constrained optimal control problems, see [39–41, 73].

Let Ω be a Lipschitz domain in $\mathbb{R}^n$ with $n \le 3$. Consider

$$\begin{cases} -\operatorname{div}(A(x)Dv) &= \alpha \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $A(x)$ is the $n \times n$ matrix satisfying uniformly ellipticity. Then for each $\alpha \in L^2(\Omega)$, there is a unique weak solution $v_\alpha \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$. We consider the following optimal control problem:

$$\begin{cases} \text{Minimize } J(\alpha) = \frac{1}{2} \int_\Omega (v_\alpha - v_0)^2 \, dx + \frac{r}{2} \int_\Omega \alpha^2 \, dx \\ \text{subject to } \alpha \in K \quad \text{and} \quad |v_\alpha(x)| \le 1 \ \ \forall x \in \Omega, \end{cases} \tag{1.1.5}$$

where $v_0 \in L^2(\Omega)$, $r \ge 0$ is the constant, and K is a nonempty, convex and closed subset of $L^2(\Omega)$. We know that the solution v_α is continuous by the Sobolev-Morrey embedding theorem. Indeed, Lagrange multipliers in the optimality conditions become measures in the case of pointwise state constraints.

According to [39, Theorem 2], under Slater condition, $\beta \in K$ is a solution of the problem (1.1.5) if and only if there exist $v \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$, $u \in L^2(\Omega)$, and a measure μ such that

$$\begin{cases} -\operatorname{div}(A(x)Dv) = \beta \quad \text{in } \Omega, \\ -\operatorname{div}\left(A^T(x)Du\right) = v - v_0 + \mu \quad \text{in } \Omega, \\ \displaystyle\int_\Omega v \, d\mu = \sup_{w \in B} \int_\Omega w \, d\mu, \quad v \in B, \\ \displaystyle\int_\Omega (u + r\beta)(\gamma - \beta) \, dx \geq 0 \quad \text{for all } \gamma \in K, \end{cases} \tag{1.1.6}$$

where $B := \left\{ w \in C_0(\Omega) : ||w||_{L^\infty(\Omega)} \leq 1 \right\}$ and $A^T(x)$ is the transpose matrix of $A(x)$. Thus, finding a solution to the optimal control problems (1.1.5) is deeply connected with the existence of a solution to the corresponding measure data problems in (1.1.6).

1.2 Calderón-Zygmund theory

We start presenting the basic historical development of Calderón-Zygmund theory as well as some technical ideas. We refer to [75] and the references therein for a further discussion on Calderón-Zygmund theory.

Calderón and Zygmund [38] at first proved the integrability of the gradient of the solution to the Poisson equation; let us consider

$$\Delta u = \operatorname{div} Du = \operatorname{div} F \quad \text{in } \Omega \subset \mathbb{R}^n$$

for which the result becomes

$$||Du||_{L^q_{loc}(\Omega)} \leq c \, ||F||_{L^q_{loc}(\Omega)} \qquad \text{for all } 1 < q < \infty, \tag{1.2.1}$$

where $c > 0$ is the constant independent on u and F, that is,

$$|F| \in L^q_{loc}(\Omega) \implies |Du| \in L^q_{loc}(\Omega) \qquad \text{for all } 1 < q < \infty,$$

by using a representation formula and singular integrals. Estimates such as (1.2.1) are called *Calderón-Zygmund estimates*. Iwaniec [60] later established local Calderón-Zygmund estimates for p-Laplace equations. More precisely, if $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution to the problem

$$\operatorname{div}\left(|Du|^{p-2}Du\right) = \operatorname{div}\left(|F|^{p-2}F\right) \quad \text{in } \Omega \subset \mathbb{R}^n,$$

then there holds

$$|F|^p \in L^q_{loc}(\Omega) \implies |Du|^p \in L^q_{loc}(\Omega) \qquad \text{for all } 1 < q < \infty.$$

The approach in [60] uses sharp maximal operators and a priori Lipschitz estimates for the solution w to homogeneous equation $\operatorname{div}(|Dw|^{p-2}Dw) = 0$. See also [50] for the case of p-Laplace systems.

After these pioneering works, Caffarelli and Peral [37] found a systematic and useful approach to obtain Calderón-Zygmund type estimates for elliptic problems having divergence form. This approach will be certain so-called *maximal function technique*, which uses the standard estimates for the problem, Hardy-Littlewood maximal functions and Calderón-Zygmund decompositions. A significant advantage of this approach is that it can completely avoid the use of explicit kernels and singular integrals. This practical method has been widely developed later, see [2, 23, 34, 37, 79, 98]. It is also worth noting that so-called *maximal function free technique*, based only upon PDE estimates without using maximal functions, was first introduced in [3] in the setting of parabolic problems with the constant p-growth. We refer to [7, 15, 22, 27–29, 93] for Calderón-Zygmund estimates using the maximal function free technique.

We now discuss Calderón-Zygmund theory for measure data problems. Mingione [77] derived the differentiability of Du to the problem (1.1.1) for the case $p \geq 2$. In [77, Lemma 4.1], the author in particular proved the difference estimate in L^q space with $q < \min\left\{p, \frac{n(p-1)}{n-1}\right\}$ comparing (1.1.1) with its homogeneous problem, which plays an important role in obtaining Calderón-Zygmund type estimates, see also [54, Lemma 4.2] for the case $2 - \frac{1}{n} < p \leq 2$. Mingione [79] later developed local Calderón-Zygmund type estimates for a SOLA u to the problem (1.1.1). Phuc [88] extended the local estimates up to the nonsmooth boundary as follows:

$$\int_\Omega |Du|^q \, dx \leq c \int_\Omega \mathcal{M}_1(\mu)^{\frac{q}{p-1}} \, dx \qquad \text{for all } 0 < q < \infty,$$

where $c > 0$ is the constant independent on u and μ. Here $\mathcal{M}_1(\mu)$ is the fractional maximal function of order 1 for μ defined by

$$\mathcal{M}_1(\mu)(x) := \sup_{r>0} \frac{r|\mu|(B_r(x))}{|B_r(x)|} \quad \text{for } x \in \mathbb{R}^n. \tag{1.2.2}$$

For various regularity results regarding measure data problems, we refer to [8, 16, 54, 55, 64–66, 68, 77, 80, 81, 83, 86, 87].

We end this chapter with a summary of the main results of this dissertation. The first contribution of this dissertation is to establish a global Calderón-Zygmund type estimate for a SOLA u to the problem (1.1.1) with variable growth $p(\cdot)$, see Chapter 3. Specifically, we prove that for all $q > 0$,

$$\int_\Omega |Du|^q \, dx \le c \left\{ \int_\Omega [\mathcal{M}_1(\mu)]^{\frac{q}{p(x)-1}} \, dx + 1 \right\} \tag{1.2.3}$$

under minimal conditions on $p(\cdot)$, $\mathbf{a}$ and Ω. The main difficulty in carrying out our result (1.2.3) is to establish comparison L^1-estimates and higher integrability for the variable exponent case, see Chapter 3.2. Moreover, unlike the constant exponent case, the problem (1.1.1) with variable growth $p(\cdot)$ has no normalization property, and so it needs a delicate analysis and a very careful computation to obtain the standard L^1-estimates for measure data problems, see Remark 3.4.1 and Remark 3.4.2. The desired estimate (1.2.3) is obtained via the maximal function technique in [37, 98]

The second part is devoted to deriving a global Calderón-Zygmund type estimate to the problem (1.1.1) with $p = 2$ in the variable exponent spaces, under the condition that the nonlinearity $\mathbf{a}(\xi, x)$ is merely measurable with respect to one variable of x, see Chapter 4. The condition on the nonlinearity $\mathbf{a}$ is a possibly optimal assumption for the estimate. In other words, if $\mathbf{a}(\xi, \cdot)$ has two or more measurable coefficients, then this estimate is not generally satisfied even in the constant exponent case, see [74]. Our result generalizes that of [20] in two aspects. For one thing, we consider measure data problems. Since the measure data is not in general regular enough, we need a new notion of a suitable solution (see Definition 4.1.1) and a systematic investigation for uniform regularity estimates (see Chapter 4.2 and Chapter 4.3). For the other, we obtain the Calderón-Zygmund type estimates in the variable exponent context. Unlike the constant exponent case, it is important to study how the variable exponent function changes as a point varies, and so one needs the log-Hölder continuity (see Chapter 2.1.2) in order to control the rate of decrease or increase of function values. We refer to [21–24,51,52] for regularity results on variable exponent spaces.

The last contribution is to prove the parabolic counterpart of the second result in the framework of weighted Orlicz spaces, see Chapter 5. Moreover, our result is to validate an immediate and very useful application of the extrapolation theorem [44–46] to the setting of variable exponent spaces. Indeed, the extrapolation and our result yield a Calderón-Zygmund type estimate in the variable exponent spaces, see Chapter 5.3.

We note that Chapter 3 is based on joint work with Sun-Sig Byun and Jihoon Ok [26]. Chapter 4 and Chapter 5 are parts of the submitted papers [33] and [32] respectively, joint work with Sun-Sig Byun.

Chapter 2

Preliminaries

2.1 Elliptic equations

2.1.1 Notation

We start with some standard notation, which will be used throughout this dissertation.

(1) $x = (x', x_n) \in \mathbb{R}^n$ for $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$.

(2) $B'_r(x') = \{y' \in \mathbb{R}^{n-1} : |x' - y'| < r\}$, $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$, $B'_r = B'_r(0)$, $B_r = B_r(0)$, and $B_r^+ = B_r \cap \{x_n > 0\}$.

(3) $\mathcal{B}_r(x) = B'_r(x') \times (x_n - r, x_n + r)$, $\mathcal{B}_r = \mathcal{B}_r(0)$, and $\mathcal{B}_r^+ = \mathcal{B}_r \cap \{x_n > 0\}$.

(4) Ω is a bounded domain of $\mathbb{R}^n, n \geq 2$, and $\partial\Omega$ is the boundary of Ω.

(5) $\Omega_r(x) = \Omega \cap B_r(x)$ and $\Omega_r = \Omega \cap B_r$ (only used in Chapter 3); $\mathit{\Omega}_r(x) = \Omega \cap \mathcal{B}_r(x)$ and $\mathit{\Omega}_r = \Omega \cap \mathcal{B}_r$ (only used in Chapter 4).

(6) $dist(x, U) = \inf\{|x - y| : y \in U\}$ is the distance from x to a set U.

(7) For each set $U \subset \mathbb{R}^n$, $|U|$ is the n-dimensional Lebesgue measure of U, and $diam(U)$ is the diameter of U.

(8) For $f \in L^1_{loc}(\mathbb{R}^n)$, $\bar{f}_U$ stands for the integral average of f over a bounded open set $U \subset \mathbb{R}^n$, that is,

$$\bar{f}_U = \fint_U f(x)\, dx = \frac{1}{|U|} \int_U f(x)\, dx.$$

(9) For each $\xi \in \mathbb{R}^n$ and $x_n \in \mathbb{R}$, the integral average of the nonlinearity $\mathbf{a}(\xi, \cdot, x_n)$ over a bounded open set $U' \subset \mathbb{R}^{n-1}$ is denoted by

$$\bar{\mathbf{a}}_{U'}(\xi, x_n) = \fint_{U'} \mathbf{a}(\xi, z', x_n)\, dz' = \frac{1}{|U'|} \int_{U'} \mathbf{a}(\xi, z', x_n)\, dz'.$$

(10) We denote by c to mean a universal constant greater than one that can be computed in terms of known quantities, and so may be different from line to line.

2.1.2 Variable exponent spaces

We here recall a brief overview of variable exponent Lebesgue and Sobolev spaces. Let $p(\cdot)$ be a measurable function defined on Ω with

$$1 < \gamma_1 \leq p(\cdot) \leq \gamma_2 < \infty \tag{2.1.1}$$

for appropriate constants γ_1 and γ_2.

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $f : \Omega \to \mathbb{R}$ such that the modular

$$\rho_{p(\cdot)}(f) := \int_\Omega |f(x)|^{p(x)}\, dx$$

is finite. If $f \in L^{p(\cdot)}(\Omega)$, then we define its norm to be

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

Then there is a close relationship between the norm and the modular:

$$\min\left\{ \rho_{p(\cdot)}(f)^{\frac{1}{\gamma_1}}, \rho_{p(\cdot)}(f)^{\frac{1}{\gamma_2}} \right\} \leq \|f\|_{L^{p(\cdot)}(\Omega)} \leq \max\left\{ \rho_{p(\cdot)}(f)^{\frac{1}{\gamma_1}}, \rho_{p(\cdot)}(f)^{\frac{1}{\gamma_2}} \right\}. \tag{2.1.2}$$

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of all functions $f \in L^{p(\cdot)}(\Omega)$ whose gradient Df exists in the weak sense and belongs to $L^{p(\cdot)}(\Omega)$, equipped with the norm

$$\|f\|_{W^{1,p(\cdot)}(\Omega)} := \|f\|_{L^{p(\cdot)}(\Omega)} + \|Df\|_{L^{p(\cdot)}(\Omega,\mathbb{R}^n)}.$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ and $W^{-1,p'(\cdot)}(\Omega)$ the dual space of $W_0^{1,p(\cdot)}(\Omega)$. They are all separable reflexive Banach spaces.

We next introduce the log-Hölder continuity which is the correct condition for regularly varying exponents. In particular, this condition plays an essential role in a systematic analysis of variable exponent Lebesgue and Sobolev spaces and PDEs with variable exponents, such as the boundedness of the Hardy-Littlewood maximal operator, Sobolev's inequality, Poincaré's inequality, etc; see the monographs [44, 51]. Given a function $p(\cdot)$ satisfying (2.1.1), we say that $p(\cdot)$ is *log-Hölder continuous* in Ω if there exists a constant $H > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$,

$$|p(x) - p(y)| \leq \frac{H}{-\log|x-y|}.$$

We remark that $p(\cdot)$ is log-Hölder continuous in Ω if and only if $p(\cdot)$ has a modulus of continuity, that is, there exists a nondecreasing concave function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ and

$$|p(x) - p(y)| \leq \omega(|x - y|) \quad \text{for } x, y \in \Omega, \tag{2.1.3}$$

and moreover

$$\sup_{0<r\leq\frac{1}{2}} \omega(r) \log\left(\frac{1}{r}\right) \leq L \tag{2.1.4}$$

for some constant $L > 0$.

2.1.3 Reifenberg flat domains

Let $\delta \in \left(0, \frac{1}{8}\right)$ and $R > 0$ be constants. The domain Ω is called (δ, R)-*Reifenberg flat* if for each $x_0 \in \partial\Omega$ and each $r \in (0, R]$, there exists a coordinate system $\{y_1, \cdots, y_n\}$ such that in this new coordinate system, the origin is x_0 and

$$B_r \cap \{y_n > \delta r\} \ \subset \ B_r \cap \Omega \ \subset \ B_r \cap \{y_n > -\delta r\}. \tag{2.1.5}$$

The boundary of this domain can be locally approximated by two hyperplanes in the new coordinate system under the scale chosen. This domain may have a very rough boundary including C^1 domain or Lipschitz domain with a small Lipschitz constant. We refer to [34, 69, 70, 90, 97] and the references therein for a further discussion on Reifenberg flat domains.

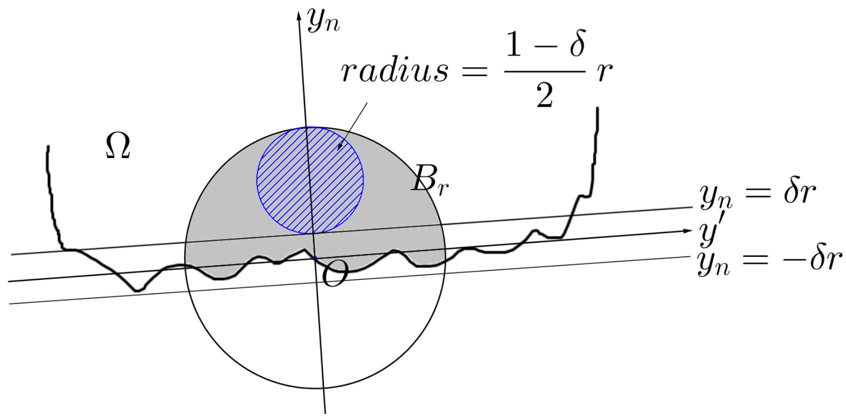

Figure 2.1: (δ, R)-Reifenberg flat domain.

We also use an interior and exterior measure density condition of Reifenberg flat domains, which can be found in [34]:

Lemma 2.1.1. *If Ω is (δ, R)-Reifenberg flat, then we have*

$$\sup_{0<r\leq R} \sup_{y\in\Omega} \frac{|B_r(y)|}{|\Omega \cap B_r(y)|} \leq \left(\frac{2}{1-\delta}\right)^n \leq \left(\frac{16}{7}\right)^n, \tag{2.1.6}$$

and

$$\inf_{0<r\leq R} \inf_{y\in\partial\Omega} \frac{|\Omega^c \cap B_r(y)|}{|B_r(y)|} \geq \left(\frac{1-\delta}{2}\right)^n \geq \left(\frac{7}{16}\right)^n. \tag{2.1.7}$$

2.1.4 Auxiliary results

We now recall some analytic and geometric properties, which are used in Chapter 3–4.

We begin with the *Hardy-Littlewood maximal function*. For $f \in L^1_{loc}(\mathbb{R}^n)$, we define

$$\mathcal{M}f(y) = \mathcal{M}(f)(y) := \sup_{r>0} \fint_{B_r(y)} |f(x)|\, dx \tag{2.1.8}$$

and

$$\mathcal{M}_U f := \mathcal{M}(\chi_U f)$$

if f is not defined outside a bounded open set $U \subset \mathbb{R}^n$. Here χ_U is the characteristic function of U. For simplicity, we drop the index U if $U = \Omega$.

We will use the following *weak* $(1,1)$ *estimates* and *strong* (p,p) *estimates*:

$$|\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \alpha\}| \leq \frac{c(n)}{\alpha} \int_{\mathbb{R}^n} |f| \, dx \quad \text{for all } \alpha > 0, \tag{2.1.9}$$

and for $1 < p \leq \infty$,

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \leq c(n,p) \|f\|_{L^p(\mathbb{R}^n)} \,. \tag{2.1.10}$$

The next lemma is a *Vitali type covering lemma* whose proof is similar to that for Theorem 2.8 in [34].

Lemma 2.1.2. *Suppose Ω is (δ, R)-Reifenberg flat. Consider the domain $\Omega_{R_0}(x_0)$ with $0 < R_0 \leq R$ and $x_0 \in \Omega$. Let $0 < \epsilon < 1$ and $\mathfrak{C} \subset \mathfrak{D} \subset \Omega_{R_0}(x_0)$ be two measurable sets such that*

(i) $|\mathfrak{C}| < \left(\frac{1}{1000}\right)^n \epsilon |B_{R_0}|$, *and*

(ii) for any $y \in \mathfrak{C}$ and any $r_0 \in \left(0, \frac{R_0}{1000}\right]$ with $|\mathfrak{C} \cap B_{r_0}(y)| \geq \epsilon |B_{r_0}(y)|$, $B_{r_0}(y) \cap \Omega_{R_0}(x_0) \subset \mathfrak{D}$.

Then we have

$$|\mathfrak{C}| \leq \left(\frac{10}{1-\delta}\right)^n \epsilon |\mathfrak{D}| \leq \left(\frac{80}{7}\right)^n \epsilon |\mathfrak{D}|.$$

Remark 2.1.3. *Lemma 2.1.2 holds with $\Omega_{R_0}(x_0)$ replaced by Ω.*

We will also use the following measure theoretic property.

Lemma 2.1.4 (See [36, Lemma 7.3])**.** *Let f be a measurable function in a bounded open set $U \subset \mathbb{R}^n$. Let $\theta > 0$ and $N > 1$ be constants. Then, for $0 < q < \infty$, there holds*

$$f \in L^q(U) \iff S := \sum_{k \geq 1} N^{qk} \left|\left\{x \in U : |f(x)| > \theta N^k\right\}\right| < \infty$$

with the estimate

$$c^{-1}\theta^q S \leq \int_U |f|^q \, dx \leq c\theta^q \left(|U| + S\right) \tag{2.1.11}$$

for some constant $c = c(N,q) > 0$.

2.2 Parabolic equations

2.2.1 Notation

We start with some notation, which will be used throughout Chapter 5.

(1) $Q_r(x,t) = \mathcal{B}_r(x) \times (t-r^2, t+r^2)$, $Q_r = Q_r(0,0)$, and $Q_r^+ = Q_r \cap \{x_n > 0\}$.

(2) $Q'_r(x',t) = B'_r(x') \times (t-r^2, t+r^2)$ and $Q'_r = Q'_r(0,0)$.

(3) $C_r(x,t) = B_r(x) \times (t-r^2, t+r^2)$ and $C_r = C_r(0,0)$.

(4) $\Omega_T = \Omega \times (0,T],\ T > 0$, is the space-time domain, and
$\partial_p \Omega_T = (\partial\Omega \times [0,T]) \cup (\Omega \times \{0\})$ is the parabolic boundary of Ω_T.

(5) $K_r(x,t) = \Omega_T \cap Q_r(x,t)$ and $K_r = \Omega_T \cap Q_r$.

(6) For two points (x,t) and $(y,s) \in \mathbb{R}^n \times \mathbb{R}$, the standard parabolic distance d_p between (x,t) and (y,s) is defined by

$$d_p((x,t),(y,s)) = \max\left\{|x-y|, \sqrt{|t-s|}\right\}.$$

(7) For $f \in L^1_{loc}(\mathbb{R}^n \times \mathbb{R})$, $\bar{f}_U$ stands for the integral average of f over a bounded open set $U \subset \mathbb{R}^n \times \mathbb{R}$, that is,

$$\bar{f}_U = \fint_U f(x,t)\,dxdt = \frac{1}{|U|}\int_U f(x,t)\,dxdt.$$

(8) For each $\xi \in \mathbb{R}^n$ and $x_n \in \mathbb{R}$, the integral average of the nonlinearity $\mathbf{a}(\xi,\cdot,x_n,\cdot)$ over a bounded open set $U' \subset \mathbb{R}^{n-1} \times \mathbb{R}$ is denoted by

$$\bar{\mathbf{a}}_{U'}(\xi,x_n) = \fint_{U'} \mathbf{a}(\xi,z',x_n,t)\,dz'dt = \frac{1}{|U'|}\int_{U'} \mathbf{a}(\xi,z',x_n,t)\,dz'dt.$$

2.2.2 Muckenhoupt weights

We now present some properties of Muckenhoupt weights from [58, Chapter 7].

We say that a positive locally integrable function $w(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$ (called a weight function) is an A_p *weight*, $1 \le p < \infty$ if there hold

$$[w]_{A_p} := \sup_{Q_r(x,t)} \left(\fint_{Q_r(x,t)} w(y,s)\, dyds \right) \left(\fint_{Q_r(x,t)} w(y,s)^{-\frac{1}{p-1}}\, dyds \right)^{p-1} < \infty$$

when $p > 1$, and

$$[w]_{A_1} := \sup_{Q_r(x,t)} \left(\fint_{Q_r(x,t)} w(y,s)\, dyds \right) \left\| w^{-1} \right\|_{L^\infty(Q_r(x,t))} < \infty.$$

The $[w]_{A_p}$ is called the A_p constant of w.

Also, a weight function w is called an A_∞ *weight* if there are two constants c_0 and α such that

$$w(E) \le c_0 \left(\frac{|E|}{|C|} \right)^\alpha w(C), \tag{2.2.1}$$

for every parabolic cylinder $C \subset \mathbb{R}^{n+1}$ and every measurable subset E of C. Here $w(E) := \int_E w(x,t)\, dxdt$. The pair (c_0, α) is called the A_∞ constants of w and is denoted by $[w]_{A_\infty}$.

It is well known that $A_p \subset A_\infty$ for all $p \ge 1$ and that $A_\infty = \bigcup_{1 \le p < \infty} A_p$.

2.2.3 Weighted Orlicz spaces

We recall the definition and some properties of the weighted Orlicz spaces, see [25] and the references therein for details.

A nonnegative and increasing convex function Φ on $[0,\infty)$ is called a *Young function* if it satisfies the following properties:

$$\Phi(0) = 0, \quad \lim_{\tau \to \infty} \Phi(\tau) = \infty, \quad \lim_{\tau \to 0^+} \frac{\Phi(\tau)}{\tau} = 0, \quad \lim_{\tau \to \infty} \frac{\Phi(\tau)}{\tau} = \infty.$$

A Young function Φ is said to satisfy the Δ_2-*condition* (or ∇_2-*condition*), denoted by $\Phi \in \Delta_2$ (or $\Phi \in \nabla_2$), if there exists a constant $\tau_0 > 1$ (or $\tau_1 > 1$) such that for any $\tau > 0$, we have $\Phi(2\tau) \le \tau_0 \Phi(\tau)$ (or $2\tau_1 \Phi(\tau) \le \Phi(\tau_1 \tau)$). A typical example of a Young function with the $\Delta_2 \cap \nabla_2$-condition is $\Phi(\tau) = \tau^p$ $(1 < p < \infty)$.

For a Young function $\Phi \in \Delta_2 \cap \nabla_2$ and an A_∞ weight $w(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$, the *weighted Orlicz space* $L^\Phi_w(\Omega_T)$ consists of all measurable functions $f : \Omega_T \to \mathbb{R}$

such that the modular

$$\rho_w^\Phi(f) := \int_{\Omega_T} \Phi(|f(x,t)|)w(x,t)\,dxdt$$

is finite. If $f \in L_w^\Phi(\Omega_T)$, then we define its norm as

$$\|f\|_{L_w^\Phi(\Omega_T)} = \inf\left\{\lambda > 0 : \rho_w^\Phi\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

Then there is a close relationship between the norm and the modular:

$$c^{-1}\,\|f\|_{L_w^\Phi(\Omega_T)}^{\gamma_1} \le \rho_w^\Phi(f) \le c\,\|f\|_{L_w^\Phi(\Omega_T)}^{\gamma_2} \tag{2.2.2}$$

for some constants $c, \gamma_1, \gamma_2 > 1$, independent of f.

2.2.4 Auxiliary results

We begin with the *Hardy-Littlewood maximal function*. For $f \in L^1_{loc}(\mathbb{R}^n \times \mathbb{R})$, we define

$$\mathcal{M}f(y,s) = \mathcal{M}(f)(y,s) := \sup_{r>0} \fint_{C_r(y,s)} |f(x,t)|\,dxdt.$$

If $f \in L^1(U)$ for a bounded domain $U \subset \mathbb{R}^{n+1}$, then we set $\mathcal{M}f = \mathcal{M}\bar{f}$, where $\bar{f}$ is the zero extension of f from U to $\mathbb{R}^{n+1}$. We will use the following weak type estimate:

$$|\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \mathcal{M}f(x,t) > \alpha\}| \le \frac{c(n)}{\alpha}\int_{\mathbb{R}^n\times\mathbb{R}} |f|\,dxdt \tag{2.2.3}$$

for all $\alpha > 0$.

The next lemma is a Vitali type covering lemma.

Lemma 2.2.1 (See [25, Lemma 4.2]). *Suppose that Ω is a (δ, R)-Reifenberg flat domain with $0 < \delta < \frac{1}{8}$ and that w is an A_∞ weight. Let $0 < \epsilon < 1$ and let $\mathfrak{C} \subset \mathfrak{D} \subset \Omega_T$ be two measurable sets such that*

(i) for any $(y,s) \in \Omega_T$, $w(\mathfrak{C} \cap Q_{R/100}(y,s)) < \epsilon w(Q_{R/100}(y,s))$, and

(ii) for any $(y,s) \in \Omega_T$ and any $r \in (0, R/100]$ with $w(\mathfrak{C} \cap Q_r(y,s)) \ge \epsilon w(Q_r(y,s))$, $Q_r(y,s) \cap \Omega_T \subset \mathfrak{D}$.

Then we have

$$w(\mathfrak{C}) \leq \epsilon c_1 w(\mathfrak{D})$$

for some constant c_1 *depending only on* n *and* $[w]_{A_\infty}$.

We will also use the following measure theoretic property.

Lemma 2.2.2 (See [25, Lemma 4.6])**.** *Let* f *be a measurable function in* Ω_T, *and let* $\theta > 0$ *and* $N > 1$ *be two constants. Then for any pair* $(w, \Phi) \in (A_\infty, \Delta_2 \cap \nabla_2)$, *there holds*

$$f \in L^{\Phi}_w(\Omega_T) \iff S := \sum_{k \geq 1} \Phi(N^k) w \left(\left\{ (x,t) \in \Omega_T : |f(x,t)| > \theta N^k \right\} \right) < \infty$$

with the estimate

$$c^{-1} S \leq \int_{\Omega_T} \Phi(|f|) w(x,t)\, dxdt \leq c \left(w(\Omega_T) + S \right) \tag{2.2.4}$$

for some constant $c = c(\theta, N, \Phi) > 0$.

Chapter 3

Regularity estimates for elliptic measure data problems with variable growth

There have been considerable theoretical advances in partial differential equations (PDEs) with variable exponent growth in recent years. The study of these problems has also become an important research field, and it represents various phenomena in applied sciences: for instance, electrorheological fluids [91], elasticity [99], flows in porous media [6], image restoration [43], thermo-rheological fluids [5], and magnetostatics [42].

In this chapter, we consider the Dirichlet problem with measure data:

$$\begin{cases} -\operatorname{div} \mathbf{a}(Du, x) & = & \mu \quad \text{in } \Omega, \\ u & = & 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.0.1}$$

where μ is a signed Radon measure on Ω with finite mass. We can assume, by extending μ by zero to $\mathbb{R}^n$, that μ is defined in $\mathbb{R}^n$ with $|\mu|(\Omega) = |\mu|(\mathbb{R}^n) < \infty$. The nonlinearity $\mathbf{a} = \mathbf{a}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable in ξ and measurable in x, and it satisfies the following variable exponent growth and uniformly ellipticity conditions:

$$|\xi||D_\xi \mathbf{a}(\xi, x)| + |\mathbf{a}(\xi, x)| \leq \Lambda |\xi|^{p(x)-1}, \tag{3.0.2}$$

$$\lambda |\xi|^{p(x)-2} |\eta|^2 \leq \langle D_\xi \mathbf{a}(\xi, x)\eta, \eta \rangle \,, \tag{3.0.3}$$

for almost every $x \in \mathbb{R}^n$, every $\eta \in \mathbb{R}^n$, every $\xi \in \mathbb{R}^n \setminus \{0\}$, and appropriate constants $\lambda,\ \Lambda$. Here $D_\xi \mathbf{a}(\xi,x)$ is the Jacobian matrix of $\mathbf{a}$ with respect to ξ, and $p(\cdot)$ is a given continuous function in Ω satisfying

$$2-\frac{1}{n}<\gamma_1\le p(\cdot)\le\gamma_2<\infty. \tag{3.0.4}$$

Note that (3.0.2) implies that $\mathbf{a}(0,x)=0$ for $x\in\mathbb{R}^n$, and (3.0.3) yields the following monotonicity condition:

$$\langle \mathbf{a}(\xi_1,x)-\mathbf{a}(\xi_2,x),\xi_1-\xi_2\rangle \ge \begin{cases} \tilde\lambda\,|\xi_1-\xi_2|^{p(x)} & \text{if} \quad p(x)\ge 2,\\ \tilde\lambda\left(|\xi_1|^2+|\xi_2|^2\right)^{\frac{p(x)-2}{2}}|\xi_1-\xi_2|^2 & \text{if} \quad 1<p(x)<2 \end{cases} \tag{3.0.5}$$

for all $x,\xi_1,\xi_2\in\mathbb{R}^n$ and for some constant $\tilde\lambda=\tilde\lambda(n,\lambda,\gamma_1,\gamma_2)>0$.

If $\gamma_1>n$, then the measure μ belongs to the dual space of $W_0^{1,p(\cdot)}(\Omega)$ as a consequence of Morrey's inequality and a duality argument, and so the existence and uniqueness of a weak solution u to (3.0.1) are well understood from the monotone operator theory, see for instance [95]. In this case, regularity estimates for (3.0.1) have been extensively studied, see for example [1, 2, 23, 28, 56, 59, 76]. For this reason, we only consider the case that $\gamma_1\le n$ for which a solution u of (3.0.1) in the distributional sense does not necessarily become a weak solution in $W_0^{1,p(\cdot)}(\Omega)$. In this respect, we need to consider a more general class of solutions below the duality exponent.

Definition 3.0.1. $u\in W_0^{1,1}(\Omega)$ *is a* SOLA *to the problem* (3.0.1) *under the assumptions* (3.0.2)–(3.0.4) *if the nonlinearity* $\mathbf{a}(Du,x)\in L^1(\Omega,\mathbb{R}^n)$,

$$\int_\Omega \langle \mathbf{a}(Du,x),D\varphi\rangle\ dx=\int_\Omega \varphi\, d\mu$$

holds for all $\varphi\in C_c^\infty(\Omega)$, *and moreover there exists a sequence of weak solutions* $\{u_h\}_{h\ge1}\subset W_0^{1,p(\cdot)}(\Omega)$ *of the Dirichlet problems*

$$\begin{cases} -\operatorname{div}\mathbf{a}(Du_h,x) &= \mu_h \quad \textit{in } \Omega,\\ \hfill u_h &= 0 \quad\ \textit{on } \partial\Omega \end{cases} \tag{3.0.6}$$

such that

$$u_h\to u \quad \textit{in} \ \ W_0^{1,\max\{1,p(\cdot)-1\}}(\Omega) \quad \textit{as } h\to\infty, \tag{3.0.7}$$

where $\mu_h\in L^\infty(\Omega)$ *converges weakly to* μ *in the sense of measure and satisfies* *for each open set* $V \subset \mathbb{R}^n$,

$$\limsup_{h\to\infty} |\mu_h|(V) \le |\mu|(\overline{V}), \tag{3.0.8}$$

with μ_h *defined in* $\mathbb{R}^n$ *by considering the zero extension to* $\mathbb{R}^n$.

Throughout this chapter, we consider $\mu_h := \mu * \phi_h$, where ϕ_h is the usual mollifier, and then $\mu_h \in C^\infty(\Omega)$ converges weakly to μ in the sense of measure satisfying (3.0.8) and the following uniform L^1-estimate:

$$||\mu_h||_{L^1(\Omega)} \le |\mu|(\Omega). \tag{3.0.9}$$

With such μ_h, there exists a SOLA u of (3.0.1) belonging to $W_0^{1,q(\cdot)}(\Omega)$ for all $q(\cdot)$ with

$$1 \le q(\cdot) < \min\left\{\frac{n(p(\cdot)-1)}{n-1}, p(\cdot)\right\}.$$

This existence follows from a priori $L^{q(\cdot)}$ estimate of the gradient of solutions for the regularized problem of $p(\cdot)$-Laplace type and a proper approximation procedure, see [9, 16] and the references therein. Moreover, the condition $p(\cdot) > 2 - \frac{1}{n}$ in (3.0.4) implies $\frac{n(p(\cdot)-1)}{n-1} > 1$, which ensures $u \in W_0^{1,1}(\Omega)$. On the other hand, if $p(\cdot) \le 2 - \frac{1}{n}$, then a solution does not belong to $W_0^{1,1}(\Omega)$, and so a new concept of solutions is needed. We refer to [10, 92] for details, and we will no longer treat the case $p(\cdot) \le 2 - \frac{1}{n}$ here. It is worthwhile to mention that the existence of a solution of (3.0.1) can also be obtained by introducing the notion of renormalized solutions, see [9,47] and the references given there.

The uniqueness of a SOLA remains still an open problem except when linear problems with $p(\cdot) \equiv 2$, see [89, 94] for counterexamples. We also refer to [9, 10, 12–14, 16, 47, 48, 72, 92] for a thorough discussion regarding the existence and uniqueness of measure data problems.

3.1 Main results

We now introduce the main regularity assumptions on $p(\cdot)$, $\mathbf{a}$ and Ω.

Definition 3.1.1. *Let* $R > 0$ *and* $\delta \in (0, \frac{1}{8})$. *We say* $(p(\cdot), \mathbf{a}, \Omega)$ *is* (δ, R)-vanishing *if the following* **(AP)**, **(AA)** *and* **(AΩ)** *hold:*

(AP) *A function $p(\cdot)$ has a modulus of continuity $\omega : [0,\infty) \to [0,\infty)$, and it satisfies that*

$$\sup_{0<r\le R} \omega(r) \log\left(\frac{1}{r}\right) \le \delta. \tag{3.1.1}$$

(AA) *For a bounded open set $U \subset \mathbb{R}^n$, write*

$$\theta(\mathbf{a}, U)(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \left| \frac{\mathbf{a}(\xi, x)}{|\xi|^{p(x)-1}} - \overline{\left(\frac{\mathbf{a}(\xi,\cdot)}{|\xi|^{p(\cdot)-1}}\right)_U} \right|. \tag{3.1.2}$$

Then, the nonlinearity $\mathbf{a}$ satisfies

$$\sup_{0<r\le R} \sup_{y\in\mathbb{R}^n} \fint_{B_r(y)} \theta(\mathbf{a}, B_r(y))(x)\, dx \le \delta. \tag{3.1.3}$$

(AΩ) *The domain Ω is (δ, R)-Reifenberg flat, see Chapter 2.1.3.*

We are ready to state our main result in Chapter 3.

Theorem 3.1.2. *Assume that* (3.0.2)–(3.0.4) *hold. Let $0 < q < \infty$ and let $\gamma_1 \le n$. Then there exists a small constant $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, q) > 0$ such that the following holds: if $(p(\cdot), \mathbf{a}, \Omega)$ is (δ, R)-vanishing for some $R \in (0,1)$, then for any SOLA u of the problem* (3.0.1) *there exists a constant $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), q, R, \Omega) > 0$ such that*

$$\|Du\|_{L^q(\Omega)} \le cK_s \left\{ \left\| \mathcal{M}_1(\mu)^{\frac{1}{p(\cdot)-1}} \right\|_{L^q(\Omega)} + 1 \right\} \tag{3.1.4}$$

for every constant s with $0 < s \le \frac{1}{2}\left(\frac{n}{n-1} - \frac{1}{\gamma_1 - 1}\right) < 1$ depending only on n and γ_1, where

$$K_s := \left(|\mu|(\Omega) + |\mu|(\Omega)^{\frac{1}{(\gamma_1-1)(1-s)}} + 1\right)^{n+1}.$$

Here $\mathcal{M}_1(\mu)$ is given in (1.2.2).

Remark 3.1.3. *We point out that the term K_s in the estimate* (3.1.4) *reflects a deficiency of the normalization property of the problem* (3.0.1) *from the presence of variable exponent $p(\cdot)$. On the other hand, in the constant exponent case, that is, $p(\cdot) \equiv p$, we can derive a more clean estimate than*

(3.1.4) *by employing the normalization property. We would also like to note that the constant c goes to* $+\infty$ *when* $s \searrow 0$*, as we will see later in Remark 3.4.1 and Remark 3.4.2.*

3.2 Comparison estimates in L^1 for regular problems

In this section, we assume that μ in the problem (3.0.1) is regular, which means that

$$\mu \in L^1(\Omega) \cap W^{-1,p'(\cdot)}(\Omega). \tag{3.2.1}$$

Then we derive comparison L^1-estimates for the gradient of the weak solution u to (3.0.1) in localized boundary and interior regions. Note that by (3.2.1), this weak solution u is well defined, that is, there exists a unique $u \in W_0^{1,p(\cdot)}(\Omega)$ satisfying

$$\int_\Omega \langle \mathbf{a}(Du, x), D\varphi \rangle \; dx = \int_\Omega \mu\varphi \, dx \quad \text{for all} \;\; \varphi \in W_0^{1,p(\cdot)}(\Omega). \tag{3.2.2}$$

We denote, for a measurable set $E \subset \mathbb{R}^n$,

$$|\mu|(E) := \int_E |\mu(x)| \, dx.$$

Throughout this section, we assume that $(p(\cdot), \mathbf{a}, \Omega)$ is (δ, R)-vanishing.

3.2.1 Boundary comparisons

Let $0 < r \leq \frac{R_0}{8}$ for small $R_0 > 0$, to be selected later. Assume that the following geometric setting:

$$B_{8r}^+ \subset \Omega_{8r} \subset B_{8r} \cap \{x_n > -16\delta r\}. \tag{3.2.3}$$

Let $w \in u + W_0^{1,p(\cdot)}(\Omega_{8r})$ be the weak solution of

$$\begin{cases} \operatorname{div} \mathbf{a}(Dw, x) &= 0 \quad \text{in } \Omega_{8r}, \\ w &= u \quad \text{on } \partial\Omega_{8r}. \end{cases} \tag{3.2.4}$$

In order to get the comparison result between the problems (3.0.1) and

(3.2.4), it is helpful to define a new measure ν by

$$\nu(E) = |\mu|(E) + |E \cap \Omega| \tag{3.2.5}$$

for a measurable set $E \subset \mathbb{R}^n$, see Remark 3.2.2 for details. Hereafter in this subsection, we write

$$p_0 := p(0), \quad p_1 := \inf_{x \in \Omega_{8r}} p(x), \quad p_2 := \sup_{x \in \Omega_{8r}} p(x),$$

and

$$\chi_{\{p_0<2\}} := \begin{cases} 0 & \text{if} \quad p_0 \geq 2, \\ 1 & \text{if} \quad p_0 < 2. \end{cases}$$

Lemma 3.2.1. *Suppose that $R_0 > 0$ satisfies*

$$R_0 \leq \min\left\{ \frac{R}{2}, \frac{1}{\nu(\Omega)+1}, \frac{1}{\int_\Omega |Du|\, dx + 1} \right\}. \tag{3.2.6}$$

Let $0 < r \leq \frac{R_0}{8}$ and assume that Ω_{8r} satisfies (3.2.3)*. If $w \in u + W_0^{1,p(\cdot)}(\Omega_{8r})$ is the weak solution of* (3.2.4)*, then there is a constant $c = c(n, \lambda, \gamma_1, \gamma_2) > 0$ such that*

$$\fint_{\Omega_{8r}} |Du - Dw|\, dx \leq c \left\{ \left[\frac{\nu(\Omega_{8r})}{r^{n-1}} \right]^{\frac{1}{p_0-1}} + \chi_{\{p_0<2\}} \left[\frac{\nu(\Omega_{8r})}{r^{n-1}} \right] \left(\fint_{\Omega_{8r}} |Du|\, dx \right)^{2-p_0} \right\}.$$

Proof. Since it has already been proved in the case $p_1 \geq 2$, see [16, Lemma 3.1], we only focus on the case $p_1 < 2$.

Step 1. Dimensionless estimates We first consider the case that $8r = 1$. We then claim that

$$\fint_{\Omega_1} |Du - Dw|\, dx \leq c, \tag{3.2.7}$$

under the assumption that

$$|\mu|(\Omega_1) + |\mu|(\Omega_1) \left(\int_{\Omega_1} |Du|\, dx \right)^{2-p_1} \leq c, \tag{3.2.8}$$

where two constants c depend on n, λ, γ_1, and γ_2. We will transfer back to the general case in *Step 2*.

Let us denote $\Omega_1^+ = \{x \in \Omega_1 : p(x) \geq 2\}$, $\Omega_1^- = \{x \in \Omega_1 : p(x) < 2\}$,

$C_k^{\pm} = \{x \in \Omega_1^{\pm} : k < |u(x) - w(x)| \le k+1\}$, and $D_k^{\pm} = \{x \in \Omega_1^{\pm} : |u(x) - w(x)| \le k\}$ for $k \in \mathbb{N} \cup \{0\}$. We define the truncation operators

$$T_k(t) := \max\{-k, \min\{k,t\}\}, \quad \Phi_k(t) := T_1(t - T_k(t)) \quad \text{for } t \in \mathbb{R}. \tag{3.2.9}$$

Since u and w are the weak solutions of (3.0.1) and (3.2.4), respectively, it follows that

$$\int_{\Omega_1} \langle \mathbf{a}(Du,x) - \mathbf{a}(Dw,x), D\varphi \rangle \; dx = \int_{\Omega_1} \mu\varphi \, dx \tag{3.2.10}$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega_1)$.

First, substituting the test function $\varphi = T_k(u-w)$ in (3.2.10), and using (3.0.5) and (3.2.8), we obtain

$$\begin{aligned} \tilde{\lambda} \int_{D_k^+} |Du - Dw|^{p(x)} \, dx &\le \int_{\Omega_1} \langle \mathbf{a}(Du,x) - \mathbf{a}(Dw,x), Du - Dw \rangle \; dx \\ &\le k|\mu|(\Omega_1) \le ck. \end{aligned}$$

Then, for $k \in \mathbb{N}$, we have

$$\int_{D_k^+} |Du - Dw| \, dx \le \int_{D_k^+} (|Du - Dw| + 1)^{p(x)} \; dx \le c(k+1). \tag{3.2.11}$$

Similarly, it follows that, for $k \in \mathbb{N}$,

$$\int_{C_k^+} |Du - Dw|^{p(x)} \, dx \le c$$

by putting the test function $\varphi = \Phi_k(u-w)$ in (3.2.10). Since $p(x) \ge 2$ for $x \in C_k^+$, it follows from Hölder's inequality that

$$\int_{C_k^+} |Du - Dw| \, dx \le |C_k^+|^{\frac{1}{2}} \left(\int_{C_k^+} (|Du - Dw| + 1)^{p(x)} \; dx \right)^{\frac{1}{2}} \le c|C_k^+|^{\frac{1}{2}}.$$

From the definition of C_k^+, we find

$$|C_k^+| = \int_{C_k^+} 1 \, dx \le \int_{C_k^+} \left(\frac{|u-w|}{k} \right)^{\frac{n}{n-1}} \; dx = k^{-\frac{n}{n-1}} \int_{C_k^+} |u-w|^{\frac{n}{n-1}} \, dx.$$

Therefore, we have

$$\int_{C_k^+} |Du - Dw| \, dx \le ck^{-\frac{n}{2(n-1)}} \left(\int_{C_k^+} |u - w|^{\frac{n}{n-1}} \, dx \right)^{\frac{1}{2}}. \tag{3.2.12}$$

Then, by (3.2.11), (3.2.12), Hölder's inequality and Sobolev's inequality, we discover that for $k_0 \in \mathbb{N}$,

$$\begin{aligned} \int_{\Omega_1^+} |Du - Dw| \, dx &= \int_{D_{k_0}^+} |Du - Dw| \, dx + \sum_{k=k_0}^{\infty} \int_{C_k^+} |Du - Dw| \, dx \\ &\le c(k_0 + 1) + c \sum_{k=k_0}^{\infty} k^{-\frac{n}{2(n-1)}} \left(\int_{C_k^+} |u - w|^{\frac{n}{n-1}} \, dx \right)^{\frac{1}{2}} \\ &\le c(k_0 + 1) + c \left[\sum_{k=k_0}^{\infty} k^{-\frac{n}{n-1}} \right]^{\frac{1}{2}} \left(\sum_{k=k_0}^{\infty} \int_{C_k^+} |u - w|^{\frac{n}{n-1}} \, dx \right)^{\frac{1}{2}} \\ &\le c(k_0 + 1) + cH(k_0) \left(\int_{\Omega_1} |Du - Dw| \, dx \right)^{\frac{n}{2(n-1)}}, \end{aligned} \tag{3.2.13}$$

where $H(k_0) := \left[\sum_{k=k_0}^{\infty} k^{-\frac{n}{n-1}} \right]^{\frac{1}{2}}$.

For obtaining a comparison estimate in Ω_1^-, we now substitute the test function $\varphi = \Phi_k(u - w)$ in (3.2.10) and use (3.0.5), to find that

$$\int_{C_k^-} \left(|Du|^2 + |Dw|^2 \right)^{\frac{p(x)-2}{2}} |Du - Dw|^2 \, dx \le c|\mu|(\Omega_1). \tag{3.2.14}$$

On the other hand, from the fact that $p_1 \ge \gamma_1 > 2 - \frac{1}{n}$, we can determine $\gamma = \gamma(n, \gamma_1) \in (0, 1)$ such that $p_1 \ge \gamma_1 > 2 - \frac{\gamma}{n}$, and so

$$\frac{n(p_1 - 1)}{n - \gamma} \ge \frac{n(\gamma_1 - 1)}{n - \gamma} > 1. \tag{3.2.15}$$

From Hölder's inequality and (3.2.14), we see that for $k \in \mathbb{N} \cup \{0\}$,

$$\int_{C_k^-} \left(\left(|Du|^2 + |Dw|^2 \right)^{\frac{p(x)-2}{2}} |Du - Dw|^2 \right)^{\frac{1}{p_1}} dx \le c \left[|\mu|(\Omega_1) \right]^{\frac{1}{p_1}} |C_k^-|^{\frac{p_1-1}{p_1}}.$$

For $k \in \mathbb{N}$, it follows from the definition of C_k^- that

$$\begin{aligned}
&\int_{C_k^-} \left(\left(|Du|^2 + |Dw|^2 \right)^{\frac{p(x)-2}{2}} |Du - Dw|^2 \right)^{\frac{1}{p_1}} dx \\
&\quad \leq c \left[|\mu|(\Omega_1) \right]^{\frac{1}{p_1}} \left(\int_{C_k^-} \left(\frac{|u-w|}{k} \right)^{\frac{n}{n-\gamma}} dx \right)^{\frac{p_1-1}{p_1}} \\
&\quad \leq c \left[|\mu|(\Omega_1) \right]^{\frac{1}{p_1}} \frac{1}{k^{\frac{n(p_1-1)}{p_1(n-\gamma)}}} \left(\int_{C_k^-} |u-w|^{\frac{n}{n-\gamma}} dx \right)^{\frac{p_1-1}{p_1}} .
\end{aligned}$$

On the other hand, for $k = 0$, we have that

$$\begin{aligned}
\int_{C_0^-} \left(\left(|Du|^2 + |Dw|^2 \right)^{\frac{p(x)-2}{2}} |Du - Dw|^2 \right)^{\frac{1}{p_1}} dx &\leq c \left[|\mu|(\Omega_1) \right]^{\frac{1}{p_1}} |B_1|^{\frac{\gamma_2-1}{\gamma_1}} \\
&\leq c \left[|\mu|(\Omega_1) \right]^{\frac{1}{p_1}} .
\end{aligned}$$

From the two estimates above, Hölder's inequality, Sobolev's inequality, and

(3.2.15), we discover that

$$
\begin{aligned}
I &:= \int_{\Omega_1^-} \left(\left(|Du|^2 + |Dw|^2\right)^{\frac{p(x)-2}{2}} |Du - Dw|^2 \right)^{\frac{1}{p_1}} \, dx \\
&= \int_{C_0^-} \left(\left(|Du|^2 + |Dw|^2\right)^{\frac{p(x)-2}{2}} |Du - Dw|^2 \right)^{\frac{1}{p_1}} \, dx \\
&\qquad + \sum_{k=1}^{\infty} \int_{C_k^-} \left(\left(|Du|^2 + |Dw|^2\right)^{\frac{p(x)-2}{2}} |Du - Dw|^2 \right)^{\frac{1}{p_1}} \, dx \\
&\leq c \left[|\mu|(\Omega_1)\right]^{\frac{1}{p_1}} \left\{ \sum_{k=1}^{\infty} \frac{1}{k^{\frac{n(p_1-1)}{p_1(n-\gamma)}}} \left(\int_{C_k^-} |u - w|^{\frac{n}{n-\gamma}} \, dx \right)^{\frac{p_1-1}{p_1}} + 1 \right\} \\
&\leq c \left[|\mu|(\Omega_1)\right]^{\frac{1}{p_1}} \left\{ \left[\sum_{k=1}^{\infty} \frac{1}{k^{\frac{n(p_1-1)}{n-\gamma}}} \right]^{\frac{1}{p_1}} \left(\sum_{k=1}^{\infty} \int_{C_k^-} |u - w|^{\frac{n}{n-\gamma}} \, dx \right)^{\frac{p_1-1}{p_1}} + 1 \right\} \\
&\leq c \left[|\mu|(\Omega_1)\right]^{\frac{1}{p_1}} \left\{ \left(\int_{\Omega_1} |Du - Dw| \, dx \right)^{\frac{n(p_1-1)}{p_1(n-\gamma)}} + 1 \right\}.
\end{aligned}
\tag{3.2.16}
$$

For $x \in \Omega_1^-$, we use Young's inequality to find that

$$
\begin{aligned}
|Du - Dw| &= \left(|Du|^2 + |Dw|^2\right)^{\frac{p(x)-2}{4}} |Du - Dw| \left(|Du|^2 + |Dw|^2\right)^{\frac{2-p(x)}{4}} \\
&\leq \left(|Du|^2 + |Dw|^2\right)^{\frac{p(x)-2}{4}} |Du - Dw| \left(|Du|^2 + |Dw|^2 + 1\right)^{\frac{2-p_1}{4}} \\
&\leq c \left(|Du|^2 + |Dw|^2\right)^{\frac{p(x)-2}{4}} |Du - Dw| |Du - Dw|^{\frac{2-p_1}{2}} \\
&\qquad + c \left(|Du|^2 + |Dw|^2\right)^{\frac{p(x)-2}{4}} |Du - Dw| \left(|Du|^2 + 1\right)^{\frac{2-p_1}{4}} \\
&\leq \frac{1}{2}|Du - Dw| + c \left(\left(|Du|^2 + |Dw|^2\right)^{\frac{p(x)-2}{4}} |Du - Dw| \right)^{\frac{2}{p_1}} \\
&\qquad + c \left(|Du|^2 + |Dw|^2\right)^{\frac{p(x)-2}{4}} |Du - Dw| \left(|Du| + 1\right)^{\frac{2-p_1}{2}}.
\end{aligned}
$$

Then it follows from Hölder's inequality, (3.2.16), (3.2.8), and Young's inequality that

$$\begin{aligned}\int_{\Omega_1^-} |Du - Dw|\, dx &\le cI + cI^{\frac{p_1}{2}} \left(\int_{\Omega_1} |Du| + 1\, dx\right)^{\frac{2-p_1}{2}} \\ &\le c + c\left(\int_{\Omega_1} |Du - Dw|\, dx\right)^{\frac{n(p_1-1)}{p_1(n-\gamma)}} + c\left(\int_{\Omega_1} |Du - Dw|\, dx\right)^{\frac{n(p_1-1)}{2(n-\gamma)}} \\ &\le c + c\left(\int_{\Omega_1} |Du - Dw|\, dx\right)^{\frac{n(p_1-1)}{p_1(n-\gamma)}}.\end{aligned} \tag{3.2.17}$$

Combining (3.2.13) with (3.2.17), we have

$$\begin{aligned}\int_{\Omega_1} |Du - Dw|\, dx \le c + c\left(\int_{\Omega_1} |Du - Dw|\, dx\right)^{\frac{n(p_1-1)}{p_1(n-\gamma)}} \\ + ck_0 + cH(k_0)\left(\int_{\Omega_1} |Du - Dw|\, dx\right)^{\frac{n}{2(n-1)}}.\end{aligned}$$

Recall that $\frac{n(p_1-1)}{p_1(n-\gamma)} < \frac{p_1(n-1)}{p_1(n-\gamma)} < 1$. We then use Young's inequality to find

$$\int_{\Omega_1} |Du - Dw|\, dx \le c + ck_0 + cH(k_0)\left(\int_{\Omega_1} |Du - Dw|\, dx\right)^{\frac{n}{2(n-1)}}.$$

For $n > 2$, we know $0 < \frac{n}{2(n-1)} < 1$. Then there holds

$$\int_{\Omega_1} |Du - Dw|\, dx \le c \tag{3.2.18}$$

from Young's inequality and by taking $k_0 = 1$. For $n = 2$, we select $k_0 > 1$ sufficiently large in order to satisfy that $cH(k_0) < 1$. Then the desired estimate (3.2.18) follows, and the claim (3.2.7) is now proved.

Step 2. Scaling and Normalization Let us define

$$\tilde{u}(y) = \frac{u(8ry)}{8Ar}, \quad \tilde{w}(y) = \frac{w(8ry)}{8Ar}, \quad \tilde{\mu}(y) = \frac{8r\mu(8ry)}{A^{p_0-1}}, \quad \tilde{p}(y) := p(8ry),$$

and

$$\tilde{\mathbf{a}}(\xi, y) = \frac{\mathbf{a}(A\xi, 8ry)}{A^{p_0-1}} \tag{3.2.19}$$

for $y \in \Omega_1$, $\xi \in \mathbb{R}^n$ and for some positive constant A, being determined below. We readily check that $\tilde{u}$ and $\tilde{w}$ are the weak solutions of

$$-\operatorname{div} \tilde{\mathbf{a}}(D\tilde{u}, y) = \tilde{\mu} \quad \text{in } \tilde{\Omega}_1 := \tilde{\Omega} \cap B_1, \tag{3.2.20}$$

where $\tilde{\Omega} := \{y \in \mathbb{R}^n : 8ry \in \Omega\}$, and

$$\begin{cases} -\operatorname{div} \tilde{\mathbf{a}}(D\tilde{w}, y) &= 0 \quad \text{in } \tilde{\Omega}_1 \\ \tilde{w} &= \tilde{u} \quad \text{on } \partial\tilde{\Omega}_1, \end{cases} \tag{3.2.21}$$

respectively. Moreover, we see that $p_1 \le \tilde{p}(y) \le p_2$ for $y \in \Omega_1$.

We next claim that $\tilde{\mathbf{a}}$ satisfies the corresponding growth condition (3.0.2) and uniformly ellipticity (3.0.3).

Indeed, if we set

$$A := \left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right]^{\frac{1}{p_0-1}} + \chi_{\{p_0<2\}} \left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right] \left(\fint_{\Omega_{8r}} |Du|\, dx\right)^{2-p_0}, \tag{3.2.22}$$

and denote $\bar{\xi} := A\xi$ and $x := 8ry$, then we discover that

$$\begin{aligned} |\xi||D_\xi \tilde{\mathbf{a}}(\xi, y)| + |\tilde{\mathbf{a}}(\xi, y)| &\le A^{1-p_0} \left\{|\bar{\xi}||D_{\bar{\xi}} \mathbf{a}(\bar{\xi}, x)| + |\mathbf{a}(\bar{\xi}, x)|\right\} \\ &\le A^{p(x)-p_0} \Lambda |\bar{\xi}|^{\tilde{p}(y)-1}, \end{aligned} \tag{3.2.23}$$

and

$$\begin{aligned} \langle D_\xi \tilde{\mathbf{a}}(\xi, y)\eta, \eta\rangle &= A^{2-p_0} \left\langle D_{\bar{\xi}} \mathbf{a}(\bar{\xi}, x)\eta, \eta\right\rangle \ge A^{2-p_0} \lambda \left|\bar{\xi}\right|^{p(x)-2} |\eta|^2 \\ &\ge A^{p(x)-p_0} \lambda \left|\xi\right|^{p(x)-2} |\eta|^2 = A^{p(x)-p_0} \lambda \left|\xi\right|^{\tilde{p}(y)-2} |\eta|^2. \end{aligned} \tag{3.2.24}$$

In addition, it follows from (3.2.5) and (2.1.6) that

$$A \ge \left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right]^{\frac{1}{p_0-1}} \ge \left[\frac{8^n r |B_1||\Omega_{8r}|}{|B_{8r}|}\right]^{\frac{1}{p_0-1}} \ge \frac{1}{c} r^{\frac{1}{p_0-1}}. \tag{3.2.25}$$

On the other hand, we see from (2.1.6) and (3.2.6) that

$$\begin{aligned} A &\leq [\nu(\Omega)+1]^{\frac{1}{\gamma_1-1}} r^{-\frac{n-1}{\gamma_1-1}} \\ &\quad + c\,[\nu(\Omega)+1]\left(\int_\Omega |Du|\,dx+1\right)^{2-\gamma_1} r^{-(n-1)-n(2-\gamma_1)} \\ &\leq r^{-\frac{n}{\gamma_1-1}} + cr^{-n(3-\gamma_1)-(2-\gamma_1)} \\ &\leq cr^{-\tilde{c}} \end{aligned} \tag{3.2.26}$$

for some $\tilde{c}=\tilde{c}(n,\gamma_1)>1$. In the case that $p(x)-p_0\geq 0$, we find from (3.1.1) and (3.2.25) that

$$A^{p(x)-p_0} \geq \left(\frac{1}{c}\right)^{\gamma_2-\gamma_1} r^{\frac{p(x)-p_0}{p_0-1}} \geq \frac{1}{c} r^{\frac{\omega(16r)}{\gamma_1-1}} \geq \frac{1}{c}, \tag{3.2.27}$$

and using (3.1.1) and (3.2.26), we discover

$$A^{p(x)-p_0} \leq c\left(\frac{1}{r}\right)^{\omega(16r)\tilde{c}} \leq c. \tag{3.2.28}$$

Similarly, in the case that $p(x)-p_0<0$, then we infer from (3.1.1), (3.2.25), and (3.2.26) that

$$\frac{1}{c} \leq A^{p(x)-p_0} \leq c \tag{3.2.29}$$

for some constant $c=c(n,\gamma_1,\gamma_2)>0$. In light of (3.2.23), (3.2.24), (3.2.27), (3.2.28), and (3.2.29), we thus deduce

$$|\xi||D_\xi \tilde{\mathbf{a}}(\xi,y)| + |\tilde{\mathbf{a}}(\xi,y)| \leq c\Lambda|\bar{\xi}|^{\tilde{p}(y)-1},$$

and

$$\langle D_\xi \tilde{\mathbf{a}}(\xi,y)\eta,\eta\rangle \geq \frac{\lambda}{c}|\xi|^{\tilde{p}(y)-2}|\eta|^2$$

for some constant $c=c(n,\gamma_1,\gamma_2)>0$.

We next prove that (3.2.8) holds for $\tilde{u}$ and $\tilde{\mu}$, instead of u and μ, respectively. We recall (3.2.22) to see

$$|\tilde{\mu}|(\tilde{\Omega}_1) = A^{1-p_0}\frac{|\mu|(\Omega_{8r})}{(8r)^{n-1}} \leq 1. \tag{3.2.30}$$

Moreover we note that

$$\begin{aligned}|\tilde{\mu}|(\tilde{\Omega}_1)\left(\int_{\tilde{\Omega}_1}|D\tilde{u}|\,dy\right)^{2-p_1} &\leq cA^{p_1-p_0-1}\frac{|\mu|(\Omega_{8r})}{r^{n-1}}\left(\fint_{\Omega_{8r}}|Du|\,dx\right)^{2-p_1}\\ &\leq cA^{-1}\frac{|\mu|(\Omega_{8r})}{r^{n-1}}\left(\fint_{\Omega_{8r}}|Du|\,dx\right)^{2-p_1},\end{aligned} \tag{3.2.31}$$

as $A^{p_1-p_0} \leq c$ by (3.2.25). But then we use (2.1.6), (3.2.6), and (3.1.1) to discover that

$$\begin{aligned}\left(\fint_{\Omega_{8r}}|Du|\,dx\right)^{2-p_1} &= \left(\fint_{\Omega_{8r}}|Du|\,dx\right)^{(2-p_0)+(p_0-p_1)}\\ &\leq c\left(\fint_{\Omega_{8r}}|Du|\,dx\right)^{2-p_0}\left(\left(\frac{16}{7}\right)^n\frac{1}{|B_{8r}|}\int_{\Omega}|Du|\,dx\right)^{p_0-p_1}\\ &\leq c\left(\frac{1}{r}\right)^{\omega(16r)(n+1)}\left(\fint_{\Omega_{8r}}|Du|\,dx\right)^{2-p_0}\\ &\leq c\left(\fint_{\Omega_{8r}}|Du|\,dx\right)^{2-p_0}.\end{aligned} \tag{3.2.32}$$

Combining (3.2.31) with (3.2.32), we find that, for $p_0 < 2$,

$$|\tilde{\mu}|(\tilde{\Omega}_1)\left(\int_{\tilde{\Omega}_1}|D\tilde{u}|\,dy\right)^{2-p_1} \leq cA^{-1}\frac{|\mu|(\Omega_{8r})}{r^{n-1}}\left(\fint_{\Omega_{8r}}|Du|\,dx\right)^{2-p_0} \leq c.$$

On the other hand, for $p_0 \geq 2$, it follows from (2.1.6), (3.2.6), and (3.1.1) that

$$\left(\fint_{\Omega_{8r}}|Du|\,dx\right)^{2-p_1} \leq \left(\fint_{\Omega_{8r}}|Du|\,dx+1\right)^{(2-p_0)+(p_0-p_1)} \leq c. \tag{3.2.33}$$

Then, from (3.2.31) and (3.2.33), we deduce that, for $p_0 \geq 2$,

$$|\tilde{\mu}|(\tilde{\Omega}_1)\left(\int_{\tilde{\Omega}_1}|D\tilde{u}|\,dy\right)^{2-p_1} \leq cA^{p_1-p_0-1}\frac{|\mu|(\Omega_{8r})}{r^{n-1}} \leq cA^{p_1-2}.$$

If $A > 1$, then $A^{p_1-2} \leq 1$. If $A \leq 1$, then $A^{p_0-1} \leq A$, and so we have that $A^{p_1-2} = A^{p_1-p_0-1}A^{p_0-1} \leq A^{p_1-p_0} \leq c$. Thus, the property (3.2.8) holds for $\tilde{u}$

and $\tilde{\mu}$.

From the estimate (3.2.7) in Step 1, we obtain

$$\fint_{\tilde{\Omega}_1} |D\tilde{u} - D\tilde{w}|\,dx = \fint_{\Omega_{8r}} \frac{|Du - Dw|}{A}\,dx \le c$$

for some constant $c = c(n, \lambda, \gamma_1, \gamma_2) > 0$, which completes the proof. □

Remark 3.2.2. *If $p(\cdot)$ is a constant, then in step 2 of Lemma 3.2.1, the nonlinearity $\tilde{\mathbf{a}}$ directly satisfies the growth condition* (3.0.2) *and uniformly ellipticity* (3.0.3)*, and we can readily derive the condition* (3.2.8)*. We refer to [54, 55] for details. In this case, we can prove Lemma 3.2.1 without introducing the measure ν. However, if $p(\cdot)$ is not a constant, then the log-Hölder continuity of $p(\cdot)$ and the property of ν are crucial to proving* (3.0.2), (3.0.3) *and* (3.2.8) *in step 2, see* (3.2.25) *and* (3.2.26)*.*

The following lemma yields some self-improving property for the homogeneous problem (3.2.4):

Lemma 3.2.3. *Let $M_1 > 1$. Suppose that $R_0 > 0$ satisfies*

$$R_0 \le \min\left\{\frac{R}{2}, \frac{1}{4}, \frac{1}{2M_1}\right\} \quad \textit{and} \quad \omega(2R_0) \le \frac{1}{2n} < 1. \tag{3.2.34}$$

Then there exists a constant $\sigma_0 = \sigma_0(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$ such that the following holds: for any $r \in \left(0, \frac{R_0}{8}\right]$, if w is the weak solution of (3.2.4) *with*

$$\int_{\Omega_{8r}} |Dw|\,dx + 1 \le M_1, \tag{3.2.35}$$

then there is a constant $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, t) > 0$ such that for every $t \in (0, 1]$,

$$\left(\fint_{\Omega_{\tilde{r}}(\tilde{x}_0)} |Dw|^{p(x)(1+\sigma)}\,dx\right)^{\frac{1}{1+\sigma}} \le c\left\{\left(\fint_{\Omega_{2\tilde{r}}(\tilde{x}_0)} |Dw|^{p(x)t}\,dx\right)^{\frac{1}{t}} + 1\right\},$$

whenever $0 < \sigma \le \sigma_0$ and $\Omega_{2\tilde{r}}(\tilde{x}_0) \subset \Omega_{8r}$ with $\tilde{r} \le 4r$.

Proof. To simplify notation, we write $B_{\tilde{r}} \equiv B_{\tilde{r}}(\tilde{x}_0)$, $B_{2\tilde{r}} \equiv B_{2\tilde{r}}(\tilde{x}_0)$, $\Omega_{\tilde{r}} \equiv \Omega_{\tilde{r}}(\tilde{x}_0)$, $\Omega_{2\tilde{r}} \equiv \Omega_{2\tilde{r}}(\tilde{x}_0)$, $\bar{p}_1 := \inf_{x\in\Omega_{2\tilde{r}}} p(x)$, and $\bar{p}_2 := \sup_{x\in\Omega_{2\tilde{r}}} p(x)$.

We first consider the interior case, that is, $\Omega_{2\tilde{r}} = B_{2\tilde{r}}$. We take $\eta^{\bar{p}_2}(w - \bar{w}_{B_{2\tilde{r}}})$ as a test function in (3.2.4), where $\eta \in C_0^\infty(B_{2\tilde{r}})$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{\tilde{r}}$, and $|D\eta| \le \frac{2}{\tilde{r}}$. Then it follows from (3.0.5) and Young's inequality that

$$\fint_{B_{\tilde{r}}} |Dw|^{p(x)}\, dx \le c\left\{\fint_{B_{2\tilde{r}}} \left|\frac{w - \bar{w}_{B_{2\tilde{r}}}}{\tilde{r}}\right|^{\bar{p}_2} dx + 1\right\}. \tag{3.2.36}$$

Using Sobolev-Poincaré's inequality, we have

$$\left(\fint_{B_{2\tilde{r}}} \left|\frac{w - \bar{w}_{B_{2\tilde{r}}}}{\tilde{r}}\right|^{\bar{p}_2} dx\right)^{\frac{1}{\bar{p}_2}} \le c\left(\fint_{B_{2\tilde{r}}} |Dw|^{\frac{n\bar{p}_2}{n+\bar{p}_2}}\, dx\right)^{\frac{n+\bar{p}_2}{n\bar{p}_2}}. \tag{3.2.37}$$

From (3.1.1) and (3.2.34), we note that $\bar{p}_2 - \bar{p}_1 \le \omega(4\tilde{r}) \le \omega(2R_0) \le \frac{1}{2n}$. By setting $s := 1 + \frac{1}{2n}$, we find that

$$\frac{\bar{p}_2}{\bar{p}_2 - \bar{p}_1 + s} \ge \frac{n\bar{p}_2}{n+1} \ge \frac{n\bar{p}_2}{n+\bar{p}_2}.$$

Then, by (3.2.36), (3.2.37) and Hölder's inequality, we discover

$$\begin{aligned}\fint_{B_{\tilde{r}}} |Dw|^{p(x)}\, dx &\le c\left\{\left(\fint_{B_{2\tilde{r}}} |Dw|^{\frac{n\bar{p}_2}{n+\bar{p}_2}}\, dx\right)^{\frac{n+\bar{p}_2}{n}} + 1\right\}\\ &\le c\left\{\left(\fint_{B_{2\tilde{r}}} |Dw|^{\frac{\bar{p}_2}{\bar{p}_2-\bar{p}_1+s}}\, dx\right)^{\bar{p}_2-\bar{p}_1+s} + 1\right\}.\end{aligned}$$

But then the interpolation inequality yields

$$\left(\fint_{B_{2\tilde{r}}} |Dw|^{\frac{\bar{p}_2}{\bar{p}_2-\bar{p}_1+s}}\, dx\right)^{\frac{\bar{p}_2-\bar{p}_1+s}{\bar{p}_2}} \le \left(\fint_{B_{2\tilde{r}}} |Dw|^{\frac{\bar{p}_1}{s}}\, dx\right)^{\frac{s}{\bar{p}_2}} \left(\fint_{B_{2\tilde{r}}} |Dw|\, dx\right)^{\frac{\bar{p}_2-\bar{p}_1}{\bar{p}_2}}.$$

Moreover, it follows from (3.2.34), (3.2.35) and (3.1.1) that

$$\begin{aligned}\left(\fint_{B_{2\tilde{r}}} |Dw|\, dx\right)^{\bar{p}_2-\bar{p}_1} &\le \left(\int_{\Omega_{8r}} |Dw|\, dx\right)^{\bar{p}_2-\bar{p}_1} |B_{2\tilde{r}}|^{-n(\bar{p}_2-\bar{p}_1)}\\ &\le cM_1^{\omega(4\tilde{r})}(2\tilde{r})^{-n\omega(4\tilde{r})}\\ &\le c\left(\frac{1}{2R_0}\right)^{\omega(2R_0)}\left(\frac{1}{4\tilde{r}}\right)^{\omega(4\tilde{r})n} \le c.\end{aligned}$$

Consequently, we have

$$\begin{aligned} \fint_{B_{\tilde{r}}} |Dw|^{p(x)}\,dx &\le c\left\{\left(\fint_{B_{2\tilde{r}}} |Dw|^{\frac{\bar{p}_1}{s}}\,dx\right)^s + 1\right\} \\ &\le c\left\{\left(\fint_{B_{2\tilde{r}}} |Dw|^{\frac{p(x)}{s}}\,dx\right)^s + 1\right\}. \end{aligned} \tag{3.2.38}$$

We next consider the boundary case, that is, $\Omega_{2\tilde{r}} \neq B_{2\tilde{r}}$. Without loss of generality, one can assume that $\tilde{x}_0 \in \partial\Omega \cap B_{8r}(0)$. Taking a test function $\eta^{\bar{p}_2}u$ to (3.2.4), and using Sobolev-Poincaré's inequality along with the measure density condition (2.1.7), we have

$$\fint_{\Omega_{\tilde{r}}} |Dw|^{p(x)}\,dx \le c\left\{\left(\fint_{\Omega_{2\tilde{r}}} |Dw|^{\frac{\bar{p}_1}{s}}\,dx\right)^s \left(\fint_{\Omega_{2\tilde{r}}} |Dw|\,dx\right)^{\bar{p}_2-\bar{p}_1} + 1\right\}.$$

Now it follows from (3.2.34), (3.2.35), (2.1.6) and (3.1.1) that

$$\begin{aligned} \left(\fint_{\Omega_{2\tilde{r}}} |Dw|\,dx\right)^{\bar{p}_2-\bar{p}_1} &\le \left(\int_{\Omega_{8r}} |Dw|\,dx\right)^{\bar{p}_2-\bar{p}_1} |\Omega_{2\tilde{r}}|^{-n(\bar{p}_2-\bar{p}_1)} \\ &\le c{M_1}^{\omega(4\tilde{r})}\left(\left(\frac{7}{16}\right)^n |B_{2\tilde{r}}|\right)^{-n\omega(4\tilde{r})} \\ &\le c. \end{aligned}$$

Therefore, we have

$$\fint_{\Omega_{\tilde{r}}} |Dw|^{p(x)}\,dx \le c\left\{\left(\fint_{\Omega_{2\tilde{r}}} |Dw|^{\frac{p(x)}{s}}\,dx\right)^s + 1\right\}. \tag{3.2.39}$$

Applying the modified version of Gehring's lemma [57, Remark 6.12] to the estimates (3.2.38) and (3.2.39), we finally reach the desired conclusion. □

Corollary 3.2.4. *Under the same assumptions as in Lemma 3.2.3, we have*

$$\fint_{\Omega_{\tilde{r}}(\tilde{x}_0)} |Dw|^{p(x)}\,dx \le c\left\{\left(\fint_{\Omega_{2\tilde{r}}(\tilde{x}_0)} |Dw|\,dx\right)^{\bar{p}_2} + 1\right\} \tag{3.2.40}$$

for some constant $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$.

Proof. From Hölder's inequality and Lemma 3.2.3, we have

$$\begin{aligned}\fint_{\Omega_{\tilde r}(\tilde x_0)} |Dw|^{p(x)}\,dx &\le c\left\{\left(\fint_{\Omega_{2\tilde r}(\tilde x_0)} (|Dw|+1)^{\bar p_2 t}\,dx\right)^{\frac{1}{t}}+1\right\}\\ &\le c\left\{\left(\fint_{\Omega_{2\tilde r}(\tilde x_0)} (|Dw|+1)^{\frac{\bar p_2}{\gamma_2}}\,dx\right)^{\gamma_2}+1\right\}\\ &\le c\left\{\left(\fint_{\Omega_{2\tilde r}(\tilde x_0)} (|Dw|+1)\,dx\right)^{\bar p_2}+1\right\},\end{aligned}$$

by taking $t=\frac{1}{\gamma_2}$ for the second inequality. Thus, this corollary follows. □

Now we will specifically derive the universal constant M_1 given in Lemma 3.2.3. Suppose that $R_0>0$ satisfies (3.2.6). Then from Lemma 3.2.1 and the measure density condition (2.1.6), we calculate

$$\begin{aligned}\int_{\Omega_{8r}} |Dw|\,dx &\le \int_{\Omega_{8r}} |Du|\,dx + c|\Omega_{8r}|\left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right]^{\frac{1}{p_0-1}}\\ &\qquad + c\chi_{\{p_0<2\}}|\Omega_{8r}|\left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right]\left(\fint_{\Omega_{8r}} |Du|\,dx\right)^{2-p_0}\\ &\le \int_{\Omega} |Du|\,dx + cr^{\alpha}\left[\nu(\Omega)\right]^{\frac{1}{p_0-1}} + c\chi_{\{p_0<2\}}r^{\beta}\nu(\Omega)\left(\int_{\Omega}|Du|\,dx\right)^{2-p_0}\\ &\le \int_{\Omega} |Du|\,dx + c\ diam(\Omega)^{\alpha}\left[\nu(\Omega)+1\right]^{\frac{1}{\gamma_1-1}}\\ &\qquad + c\chi_{\{p_0<2\}}\ diam(\Omega)^{\beta}\left[\nu(\Omega)+1\right]\left(\int_{\Omega}|Du|\,dx+1\right)^{2-\gamma_1}\end{aligned}$$

for some $c=c(n,\lambda,\gamma_1,\gamma_2)>0$, where $\alpha:=n-\frac{n-1}{\gamma_1-1}>0$, $\beta:=\alpha(\gamma_1-1)>0$. We define

$$M := \int_{\Omega} |Du|\,dx + c\ diam(\Omega)^{\alpha}\left[\nu(\Omega)+1\right]^{\frac{1}{\gamma_1-1}}+1.$$

Then we conclude that

$$\int_{\Omega_{8r}} |Dw|\,dx + 1 \le (c\chi_{\{p_0<2\}}+1)M \le c_0 M =: M_1 \tag{3.2.41}$$

for some $c_0 = c_0(n, \lambda, \gamma_1, \gamma_2) > 0$.

With this M_1, we obtain the following higher integrability result which is used later:

Lemma 3.2.5. *Suppose that $R_0 > 0$ satisfies* (3.2.6), (3.2.34) *with M_1 given in* (3.2.41). *Let w be the weak solution of* (3.2.4) *satisfying* (3.2.3). *Then we have $w \in W^{1,p_2}(\Omega_{3r})$ and the estimate*

$$\fint_{\Omega_{3r}} |Dw|^{p_2}\, dx \le c\left\{\left(\fint_{\Omega_{8r}} |Dw|\, dx\right)^{p_2} + 1\right\} \tag{3.2.42}$$

for some constant $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$.

Proof. We deduce from Lemma 3.2.3, (3.1.1), and [23, Section 3.3] that

$$\fint_{\Omega_{3r}} |Dw|^{p_2}\, dx \le c\left\{\fint_{\Omega_{4r}} |Dw|^{p(x)}\, dx + 1\right\}.$$

Applying Corollary 3.2.4 with $\tilde{r}$ and $\tilde{x}_0$ replaced by $4r$ and 0, we obtain the desired estimate (3.2.42). □

We next consider a new operator $\mathbf{b} = \mathbf{b}(\xi, x) : \mathbb{R}^n \times \Omega_{8r} \to \mathbb{R}^n$ by

$$\mathbf{b}(\xi, x) = \mathbf{a}(\xi, x)|\xi|^{p_2 - p(x)}.$$

Then it satisfies the following growth and ellipticity conditions:

$$|\xi||D_\xi \mathbf{b}(\xi, x)| + |\mathbf{b}(\xi, x)| \le 3\Lambda|\xi|^{p_2-1}, \tag{3.2.43}$$

$$\frac{\lambda}{2}|\xi|^{p_2-2}|\eta|^2 \le \langle D_\xi \mathbf{b}(\xi, x)\eta, \eta\rangle \tag{3.2.44}$$

for all $\eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and $x \in \Omega_{8r}$ provided that

$$p_2 - p_1 \le \omega(16r) \le \omega(2R_0) \le \min\left\{1, \frac{\lambda}{2\Lambda}\right\}, \tag{3.2.45}$$

see [28] for details. Here λ and Λ are the constants given in (3.0.2) and (3.0.3), respectively. We denote by $\bar{\mathbf{b}} = \bar{\mathbf{b}}(\xi) : \mathbb{R}^n \to \mathbb{R}^n$ the integral average of $\mathbf{b}(\xi, \cdot)$ on B_{8r}^+, as

$$\bar{\mathbf{b}}(\xi) = \fint_{B_{8r}^+} \mathbf{b}(\xi, x)\, dx.$$

Then $\bar{\mathbf{b}}$ also satisfies (3.2.43) and (3.2.44) with $\mathbf{b}(\xi,\cdot)$ replaced by $\bar{\mathbf{b}}(\xi)$. Moreover, we observe that

$$\sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{\left|\mathbf{b}(\xi,\cdot)-\bar{\mathbf{b}}(\xi)\right|}{|\xi|^{p_2-1}} = \theta(\mathbf{a}, B^+_{8r})(x),$$

where θ is defined in (3.1.2). Then we recall (3.1.3) to discover that

$$\sup_{0<r\le R} \fint_{B^+_r} \theta(\mathbf{a}, B^+_r)(x)\,dx \le 4\delta.$$

We next let $v \in w + W^{1,p_2}_0(\Omega_{3r})$ be the weak solution of the homogeneous frozen problem

$$\begin{cases} \operatorname{div} \bar{\mathbf{b}}(Dv) &= 0 \quad \text{in } \Omega_{3r}, \\ \qquad\quad v &= w \quad \text{on } \partial\Omega_{3r}, \end{cases} \tag{3.2.46}$$

where w is the weak solution of (3.2.4), which belongs to $W^{1,p_2}(\Omega_{3r})$ from Lemma 3.2.5. By putting the test function $v-w$ into (3.2.46), we derive the standard energy estimate as follows:

$$\fint_{\Omega_{3r}} |Dv|^{p_2}\,dx \le c \fint_{\Omega_{3r}} |Dw|^{p_2}\,dx. \tag{3.2.47}$$

From Corollary 3.2.4, Lemma 3.2.5, and [23, Lemma 3.7], we obtain the comparison estimate between (3.2.4) and (3.2.46), as we now state

Lemma 3.2.6. *Suppose that $R_0 > 0$ satisfies* (3.2.6)*,* (3.2.34)*,* (3.2.45) *with M_1 given in* (3.2.41)*, and*

$$p_2 - p_1 \le \omega(16r) \le \omega(2R_0) \le \frac{\sigma_0}{4}, \tag{3.2.48}$$

where σ_0 is given in Lemma 3.2.3. Let w be the weak solution of (3.2.4) *satisfying* (3.2.3)*, and let v be as in* (3.2.46)*. Then there is a constant $c = c(n,\lambda,\Lambda,\gamma_1,\gamma_2) > 0$ such that*

$$\fint_{\Omega_{3r}} |Dw-Dv|^{p_2}\,dx \le c\delta^{\frac{\sigma_0}{4+\sigma_0}} \left\{ \left(\fint_{\Omega_{8r}} |Dw|\,dx \right)^{p_2} + 1 \right\}. \tag{3.2.49}$$

We now consider a weak solution $\bar{v} \in W^{1,p_2}(B^+_{2r})$ of the reference problem

$$\begin{cases} \operatorname{div} \bar{\mathbf{b}}(D\bar{v}) &= 0 \quad \text{in } B^+_{2r}, \\ \bar{v} &= 0 \quad \text{on } B_{2r} \cap \{x_n = 0\}. \end{cases} \tag{3.2.50}$$

Then we have the following Lipschitz regularity of $\bar{v}$ up to the flat boundary:

Lemma 3.2.7 (See [71]). *For any weak solution $\bar{v} \in W^{1,p_2}(B^+_{2r})$ of (3.2.50), we have $D\bar{v} \in L^\infty(B^+_r)$ and*

$$\|D\bar{v}\|_{L^\infty(B^+_r)} \leq c \fint_{B^+_{2r}} |D\bar{v}| \, dx \tag{3.2.51}$$

for some $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$.

Note that the constant c given in (3.2.51) actually depends only on n, λ, Λ, and p_2; however, since $\gamma_1 \leq p_2 \leq \gamma_2$, we can choose c depending only on $n, \lambda, \Lambda, \gamma_1$, and γ_2.

We can now state the comparison estimate between (3.2.46) and (3.2.50).

Lemma 3.2.8 (See [28]). *For any $0 < \epsilon < 1$, there is $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \epsilon) > 0$ such that if $v \in W^{1,p_2}(\Omega_{3r})$ is the weak solution of (3.2.46) with (3.2.3), then there exists a weak solution $\bar{v} \in W^{1,p_2}(B^+_{2r})$ of (3.2.50) such that*

$$\fint_{\Omega_{2r}} |Dv - D\bar{v}|^{p_2} \, dx \leq \epsilon^{p_2} \fint_{\Omega_{3r}} |Dv|^{p_2} \, dx, \tag{3.2.52}$$

where $\bar{v}$ is extended by zero from B^+_{2r} to Ω_{2r}.

We finally summarize the comparison L^1-estimates near a boundary region.

Lemma 3.2.9. *Suppose that $R_0 > 0$ satisfies (3.2.6), (3.2.34), (3.2.45), and (3.2.48) with M_1 given in (3.2.41). Let $\rho > 1$ and $0 < r \leq \frac{R_0}{8}$. Then, for any $0 < \epsilon < 1$, there exists a small constant $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \epsilon) > 0$ such that if $(p(\cdot), \mathbf{a}, \Omega)$ is (δ, R)-vanishing, $u \in W^{1,p(\cdot)}_0(\Omega)$, $w \in u + W^{1,p(\cdot)}_0(\Omega_{8r})$, and $v \in w + W^{1,p_2}_0(\Omega_{3r})$ are the weak solutions of (3.0.1), (3.2.4), and (3.2.46), respectively, with (3.2.3),*

$$\fint_{\Omega_{8r}} |Du| \, dx \leq \rho \quad \text{and} \quad \left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right]^{\frac{1}{p_0 - 1}} \leq \delta\rho, \tag{3.2.53}$$

where ν *is given in* (3.2.5), *then there exists a weak solution* $\bar{v} \in W^{1,p_2}(B_{2r}^+)$ *of* (3.2.50) *such that*

$$\fint_{\Omega_{2r}} |Du - D\bar{v}| \, dx \le \epsilon\rho \quad \text{and} \quad \|D\bar{v}\|_{L^\infty(\Omega_r)} \le c\rho \tag{3.2.54}$$

for some $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$. *Here* $\bar{v}$ *is extended by zero from* B_{2r}^+ *to* Ω_{2r}.

Proof. From Lemma 3.2.1, we have

$$\fint_{\Omega_{8r}} |Du - Dw| \, dx \le c\delta^{\min\{1,\gamma_1 - 1\}}\rho \;\text{ and }\; \fint_{\Omega_{8r}} |Dw| \, dx \le c\rho. \tag{3.2.55}$$

According to Hölder's inequality and Lemma 3.2.6, we observe

$$\fint_{\Omega_{3r}} |Dw - Dv| \, dx \le \left(\fint_{\Omega_{3r}} |Dw - Dv|^{p_2} \, dx \right)^{\frac{1}{p_2}} \le c\delta^{\frac{\sigma_0}{\gamma_2(4+\sigma_0)}}\rho, \tag{3.2.56}$$

and

$$\fint_{\Omega_{3r}} |Dv| \, dx \le c\rho. \tag{3.2.57}$$

At this point that by Lemma 3.2.8 with ϵ replaced by $\tilde{\epsilon}$, there is a weak solution $\bar{v} \in W^{1,p_2}(B_{2r}^+)$ of (3.2.50) such that

$$\fint_{\Omega_{2r}} |Dv - D\bar{v}|^{p_2} \, dx \le \tilde{\epsilon}^{p_2} \fint_{\Omega_{3r}} |Dv|^{p_2} \, dx.$$

We also discover from Hölder's inequality, (3.2.47), and Lemma 3.2.5 that

$$\fint_{\Omega_{2r}} |Dv - D\bar{v}| \, dx \le c\tilde{\epsilon} \left\{ \fint_{\Omega_{8r}} |Dw| \, dx + 1 \right\} \le 2c\tilde{\epsilon}\rho \le \frac{\epsilon}{3}\rho \tag{3.2.58}$$

by choosing small $\tilde{\epsilon}$ such that $0 < \tilde{\epsilon} \le \frac{\epsilon}{6c}$, and it follows from (3.2.57) and (3.2.58) that

$$\fint_{\Omega_{2r}} |D\bar{v}| \, dx \le c\rho. \tag{3.2.59}$$

Then we combine (3.2.55), (3.2.56) and (3.2.58), to discover

$$\begin{aligned}\fint_{\Omega_{2r}} |Du - D\bar{v}|\,dx &\le \fint_{\Omega_{2r}} |Du - Dw| + |Dw - Dv| + |Dv - D\bar{v}|\,dx \\ &\le c\delta^{\min\{1,p_0-1\}}\rho + c\delta^{\frac{\sigma_0}{\gamma_2(4+\sigma_0)}}\rho + \frac{\epsilon}{3}\rho \\ &\le \epsilon\rho,\end{aligned}$$

by selecting δ sufficiently small.

On the other hand, according to Lemma 3.2.7, (3.2.58) and (3.2.59), we obtain

$$\|D\bar{v}\|_{L^\infty(\Omega_r)} \le c\rho,$$

which completes the proof. □

3.2.2 Interior comparisons

With the same spirit as in the boundary case, one can derive a comparison estimate in L^1 for the interior case, and we just sketch it here for the sake of simplicity. Let $0 < r \le \frac{R_0}{8}$ with $B_{8r}(x_0) \subset\subset \Omega$, where R_0 is selected so small that it satisfies (3.2.6), (3.2.34), (3.2.45), and (3.2.48) with M_1 given in (3.2.41). In this subsection, we denote

$$p_0 := p(x_0), \quad p_1 := \inf_{x\in B_{8r}(x_0)} p(x), \quad p_2 := \sup_{x\in B_{8r}(x_0)} p(x),$$

and

$$B_{kr} \equiv B_{kr}(x_0) \quad (k \in \mathbb{N}).$$

With the weak solution $u \in W_0^{1,p(\cdot)}(\Omega)$ of (3.0.1), let $w \in u + W_0^{1,p(\cdot)}(B_{8r})$ be the weak solution of

$$\begin{cases} \operatorname{div} \mathbf{a}(Dw, x) = 0 & \text{in } B_{8r}, \\ w = u & \text{on } \partial B_{8r}. \end{cases} \tag{3.2.60}$$

Then from the same argument for the boundary case, we have $w \in W^{1,p_2}(B_{3r})$.

Let $v \in w + W_0^{1,p_2}(B_{3r})$ be the weak solution of

$$\begin{cases} \operatorname{div} \bar{\mathbf{b}}(Dv) = 0 & \text{in } B_{3r}, \\ v = w & \text{on } \partial B_{3r}, \end{cases} \tag{3.2.61}$$

where $\bar{\mathbf{b}} = \bar{\mathbf{b}}(\xi) : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\bar{\mathbf{b}}(\xi) = \fint_{B_{8r}} \mathbf{b}(\xi, x)\, dx = \fint_{B_{8r}} \mathbf{a}(\xi, x)|\xi|^{p_2 - p(x)}\, dx.$$

Then we have $Dv \in L^\infty(B_{2r})$ and

$$\|Dv\|_{L^\infty(B_{2r})} \le c \fint_{B_{3r}} |Dv|\, dx$$

for some $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$, see [49] for details.

We now state the comparison L^1-estimates for the interior case.

Lemma 3.2.10. *Suppose that $R_0 > 0$ satisfies* (3.2.6), (3.2.34), (3.2.45), *and* (3.2.48) *with M_1 given in* (3.2.41). *Let $\rho > 1$ and $0 < r \le \frac{R_0}{8}$. Then, for any $0 < \epsilon < 1$, there exists a small constant $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \epsilon) > 0$ such that if $p(\cdot)$ and $\mathbf{a}(\xi, x)$ satisfy the assumptions* **(AP)** *and* **(AA)** *in Definition 3.1.1, respectively, and if $u \in W_0^{1,p(\cdot)}(\Omega)$, $w \in u + W_0^{1,p(\cdot)}(B_{8r})$, and $v \in w + W_0^{1,p_2}(B_{3r})$ are the weak solutions of* (3.0.1), (3.2.60), *and* (3.2.61), *respectively, with*

$$\fint_{B_{8r}} |Du|\, dx \le \rho \quad \textit{and} \quad \left[\frac{\nu(B_{8r})}{r^{n-1}}\right]^{\frac{1}{p_0 - 1}} \le \delta\rho, \tag{3.2.62}$$

where ν is given in (3.2.5), *then*

$$\fint_{B_{3r}} |Du - Dv|\, dx \le \epsilon\rho \quad \textit{and} \quad \|Dv\|_{L^\infty(B_{2r})} \le c\rho \tag{3.2.63}$$

for some $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$.

3.3 Covering arguments

Now, we consider a SOLA u of (3.0.1) and the weak solutions u_h, $h \in \mathbb{N}$, of (3.0.6), where $\mu_h = \mu * \phi_h$ with ϕ_h the usual mollifier. Suppose that $(p(\cdot), \mathbf{a}, \Omega)$ is (δ, R)-vanishing. Since $\mu_h \in C^\infty(\Omega)$, one can apply all the results obtained in Chapter 3.2 to $u = u_h$ and $\mu = \mu_h$. In this case, we denote by w_h, v_h, and $\bar{v}_h$ the weak solutions to (3.2.4) or (3.2.60), (3.2.46) or (3.2.61), and (3.2.50), respectively. Moreover, we assume that $R_0 > 0$ satisfies

$$R_0 \leq \min\left\{\frac{R}{2}, \frac{1}{6M_1}, \frac{1}{\int_\Omega |Du|\,dx + 2}, \frac{1}{\nu(\Omega)+2}\right\}, \tag{3.3.1}$$

$$\omega(2R_0) \leq \min\left\{\frac{\lambda}{2\Lambda}, \frac{1}{2n}, \frac{\sigma_0}{4}\right\}, \tag{3.3.2}$$

where ν, M_1, and σ_0 are given in (3.2.5), (3.2.41), and Lemma 3.2.3, respectively. Then, thanks to (3.0.7) and (3.0.8), we see that R_0 satisfies (3.2.6), (3.2.34), (3.2.45), and (3.2.48) with (u,μ) replaced by (u_h,μ_h) for sufficiently large h.

For any fixed $\epsilon \in (0,1)$ and $N > 1$, we define

$$\lambda_0 := \frac{1}{\epsilon|B_{R_0}|}\left\{\int_\Omega |Du|\,dx + 1\right\} > 1 \tag{3.3.3}$$

and upper-level sets: for $k \in \mathbb{N}\cup\{0\}$,

$$\mathfrak{C}_k := \left\{x\in\Omega : \mathcal{M}(|Du|)(x) > N^{k+1}\lambda_0\right\},$$

$$\mathfrak{D}_k := \left\{x\in\Omega : \mathcal{M}(|Du|)(x) > N^{k}\lambda_0\right\}\cup\left\{x\in\Omega : [\mathcal{M}_1(\nu)(x)]^{\frac{1}{p(x)-1}} > \delta N^{k}\lambda_0\right\}.$$

Note that ϵ and N will be determined later as universal constants depending only on $n, \lambda, \Lambda, \gamma_1, \gamma_2$, and q.

We now verify two assumptions of the Vitali type covering lemma (Lemma 2.1.2).

Lemma 3.3.1. *There exists a constant $N_1 = N_1(n) > 1$ such that for any fixed $N \geq N_1$ and $k \in \mathbb{N}\cup\{0\}$,*

$$|\mathfrak{C}_k| \leq \frac{\epsilon}{(1000)^n}|B_{R_0}|. \tag{3.3.4}$$

Proof. For each $k \in \mathbb{N}\cup\{0\}$, $|\mathfrak{C}_k| \leq |\mathfrak{C}_0|$. Thus, we only need to show that (3.3.4) holds for $k = 0$. It follows from (2.1.9) and (3.3.3) that

$$\begin{aligned}|\mathfrak{C}_0| &= |\{x\in\Omega : \mathcal{M}(|Du|)(x) > N\lambda_0\}|\\ &\leq \frac{c}{N\lambda_0}\int_\Omega |Du|\,dx \leq \frac{c\epsilon}{N}|B_{R_0}| \leq \frac{\epsilon}{(1000)^n}|B_{R_0}|,\end{aligned}$$

by selecting $N \geq N_1 = c(1000)^n > 1$. □

Lemma 3.3.2. *There is a constant $N_2 = N_2(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 1$ so that for any $\epsilon > 0$, there exists a small constant $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \epsilon) > 0$ such that for any fixed $N \geq N_2$, $k \in \mathbb{N} \cup \{0\}$, $y_0 \in \Omega$ and $r_0 \leq \frac{R_0}{1000}$, if*

$$|\mathfrak{C}_k \cap B_{r_0}(y_0)| \geq \epsilon |B_{r_0}(y_0)|, \tag{3.3.5}$$

then $\Omega_{r_0}(y_0) \subset \mathfrak{D}_k$.

Proof. We simply write $\lambda_k := N^k \lambda_0 > 1$, where $N \geq N_2 > 1$. We argue by contradiction. Suppose that there exists $y_1 \in \Omega_{r_0}(y_0)$ such that $y_1 \notin \mathfrak{D}_k$. Then we have

$$\frac{1}{|B_r(y_1)|} \int_{\Omega_r(y_1)} |Du|\, dx \leq \lambda_k \quad \text{and} \quad \left[\frac{\nu(B_r(y_1))}{r^{n-1}}\right]^{\frac{1}{p(y_1)-1}} \leq c(n, \gamma_1)\delta\lambda_k \tag{3.3.6}$$

for all $r > 0$.

We divide the proof into two cases: an interior and a boundary case.

Case 1. The interior case $B_{10r_0}(y_1) \subset \Omega$.

Since $y_1 \in \Omega_{r_0}(y_0)$, we see that $\overline{B_{8r_0}(y_0)} \subset B_{10r_0}(y_1)$. We set

$$p_1 := \inf_{x \in B_{8r_0}(y_0)} p(x) \quad \text{and} \quad p_2 := \sup_{x \in B_{8r_0}(y_0)} p(x).$$

Then it follows that $p_2 - p_1 \leq \omega(16r_0)$.

From (3.3.6), we have

$$\fint_{B_{8r_0}(y_0)} |Du|\, dx \leq \frac{|B_{10r_0}(y_1)|}{|B_{8r_0}(y_0)|} \fint_{B_{10r_0}(y_1)} |Du|\, dx \leq \left(\frac{5}{4}\right)^n \lambda_k,$$

and it follows from (3.0.7) that for any $\tilde{\epsilon} \in (0, 1)$,

$$\fint_{B_{8r_0}(y_0)} |Du - Du_h|\, dx \leq \tilde{\epsilon}\lambda_k \tag{3.3.7}$$

for h large enough. Then we discover

$$\begin{aligned}\fint_{B_{8r_0}(y_0)} |Du_h|\,dx &\le \fint_{B_{8r_0}(y_0)} |Du|\,dx + \fint_{B_{8r_0}(y_0)} |Du - Du_h|\,dx \\ &\le \left(\left(\frac{5}{4}\right)^n + \tilde{\epsilon}\right)\lambda_k.\end{aligned}$$

We next claim that

$$\left[\frac{\nu(\overline{B_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{p(y_1)-p(y_0)} \le c \tag{3.3.8}$$

for some constant $c = c(n) > 0$.

If $p(y_1) > p(y_0)$, then $p(y_1) - p(y_0) \le \omega(16r_0)$, and so we see from (3.3.1) and (3.1.1) that

$$\begin{aligned}\left[\frac{\nu(\overline{B_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{p(y_1)-p(y_0)} &\le \left(\frac{1}{r_0}\right)^{(n-1)\omega(16r_0)} (\nu(\Omega)+1)^{\omega(16r_0)} \\ &\le c\left(\frac{1}{r_0}\right)^{n\omega(16r_0)} \le ce^{\delta n} \le c.\end{aligned}$$

If $p(y_1) < p(y_0)$, then $p(y_0) - p(y_1) \le \omega(16r_0)$, and so we find from (3.1.1) and (3.2.5) that

$$\begin{aligned}\left[\frac{\nu(\overline{B_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{p(y_1)-p(y_0)} &= \left[\frac{8^n r_0 |B_1| \nu(\overline{B_{8r_0}(y_0)})}{|B_{8r_0}|}\right]^{p(y_1)-p(y_0)} \\ &\le [8^n r_0 |B_1|]^{p(y_1)-p(y_0)} \le c\left(\frac{1}{16r_0}\right)^{\omega(16r_0)} \\ &\le ce^{\delta} \le c.\end{aligned}$$

In any case, we obtain the inequality (3.3.8). We therefore have from (3.3.6) and (3.3.8) that

$$
\begin{aligned}
\left[\frac{\nu(\overline{B_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{\frac{1}{p(y_0)-1}} &= \left[\frac{\nu(\overline{B_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{\frac{1}{p(y_1)-1}+\frac{p(y_1)-p(y_0)}{(p(y_0)-1)(p(y_1)-1)}} \\
&\le \left[\frac{\nu(B_{10r_0}(y_1))}{r_0^{n-1}}\right]^{\frac{1}{p(y_1)-1}} \left[\frac{\nu(\overline{B_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{\frac{p(y_1)-p(y_0)}{(p(y_0)-1)(p(y_1)-1)}} \\
&\le c\delta\lambda_k.
\end{aligned}
$$

In addition, it follows from (3.0.8) that

$$
\left[\frac{\nu_h(B_{8r_0}(y_0))}{r_0^{n-1}}\right]^{\frac{1}{p(y_0)-1}} \le \left[\frac{\nu(\overline{B_{8r_0}(y_0)})+\bar{\epsilon}}{r_0^{n-1}}\right]^{\frac{1}{p(y_0)-1}} \le c_1\delta\lambda_k,
$$

by selecting $\bar{\epsilon}$ small enough, the constant c_1 depending only on n, γ_1, and γ_2. Here ν_h is given in (3.2.5) with μ replaced by μ_h.

Consequently, we obtain

$$
\fint_{B_{8r_0}(y_0)} |Du_h|\,dx \le c_2\lambda_k \quad \text{and} \quad \left[\frac{\nu_h(B_{8r_0}(y_0))}{r_0^{n-1}}\right]^{\frac{1}{p(y_0)-1}} \le c_2\delta\lambda_k, \tag{3.3.9}
$$

where $c_2 := \max\left\{\left(\frac{5}{4}\right)^n + \tilde{\epsilon}, c_1\right\}$. Applying Lemma 3.2.10 with x_0, ρ, r, and ϵ replaced by y_0, $c_2\lambda_k$, r_0, and η, respectively, we can find $\delta = \delta(n,\lambda,\Lambda,\gamma_1,\gamma_2,\eta)$ such that

$$
\begin{aligned}
&\fint_{B_{3r_0}(y_0)} |Du_h - Dv_h|\,dx \le c_2\eta\lambda_k, \\
&\|Dv_h\|_{L^\infty(B_{2r_0}(y_0))} \le cc_2\lambda_k =: c_3\lambda_k
\end{aligned} \tag{3.3.10}
$$

for some $c_3 = c_3(n,\lambda,\Lambda,\gamma_1,\gamma_2) > 0$. Thus, we have from (3.3.7) and (3.3.10) that

$$
\fint_{B_{2r_0}(y_0)} |Du - Dv_h|\,dx \le \left(4^n + \left(\frac{3}{2}\right)^n\right) c_2\eta\lambda_k =: c_4\eta\lambda_k \tag{3.3.11}
$$

by choosing sufficiently small $\tilde{\epsilon}$ with $\tilde{\epsilon} \le c_2\eta$.

Now we claim that

$$\begin{aligned}\mathfrak{C}_k \cap B_{r_0}(y_0) &= \{x \in B_{r_0}(y_0) : \mathcal{M}(|Du|)(x) > N\lambda_k\} \\ &\subset \left\{x \in B_{r_0}(y_0) : \mathcal{M}_{B_{2r_0}(y_0)}(|Du - Dv_h|)(x) > \lambda_k\right\} =: Q,\end{aligned} \tag{3.3.12}$$

provided $N \geq N_2 \geq \max\{3^n, 1 + c_3\}$.

Let $y \notin Q$. If $y \notin B_{r_0}(y_0)$, then it is done. Suppose $y \in B_{r_0}(y_0)$. If $\tilde{r} < r_0$, then $B_{\tilde{r}}(y) \subset B_{2r_0}(y_0)$. We have from (3.3.10) that

$$\begin{aligned}\fint_{B_{\tilde{r}}(y)} |Du|\, dx &\leq \fint_{B_{\tilde{r}}(y)} \chi_{B_{2r_0}(y_0)}|Du - Dv_h|\, dx + \fint_{B_{\tilde{r}}(y)} |Dv_h|\, dx \\ &\leq \mathcal{M}_{B_{2r_0}(y_0)}(|Du - Dv_h|)(y) + c_3\lambda_k \\ &\leq (1 + c_3)\lambda_k.\end{aligned}$$

If $\tilde{r} \geq r_0$, then $B_{\tilde{r}}(y) \subset B_{2\tilde{r}}(y_0) \subset B_{3\tilde{r}}(y_1)$. We have from (3.3.6) that

$$\frac{1}{|B_{\tilde{r}}(y)|}\int_{\Omega_{\tilde{r}}(y)} |Du|\, dx \leq \frac{3^n}{|B_{3\tilde{r}}(y)|}\int_{\Omega_{3\tilde{r}}(y_1)} |Du|\, dx \leq 3^n\lambda_k.$$

Consequently, we have

$$\mathcal{M}(|Du|)(y) \leq \max\{(1 + c_3)\lambda_k, 3^n\lambda_k\}.$$

Choosing $N_2 \geq \max\{1 + c_3, 3^n\}$, we have $y \notin \mathfrak{C}_k \cap B_{r_0}(y_0)$, that is, the claim (3.3.12) holds.

Using (3.3.12), (2.1.9), and (3.3.11), we discover

$$\begin{aligned}|\mathfrak{C}_k \cap B_{r_0}(y_0)| &\leq \left|\left\{x \in B_{r_0}(y_0) : \mathcal{M}_{B_{2r_0}(y_0)}(|Du - Dv_h|)(x) > \lambda_k\right\}\right| \\ &\leq \frac{c}{\lambda_k}\int_{B_{2r_0}(y_0)} |Du - Dv_h|\, dx \leq cc_4\eta|B_{r_0}(y_0)| < \epsilon|B_{r_0}(y_0)|,\end{aligned}$$

by selecting η and δ that satisfy the last inequality above, which is a contradiction to (3.3.5).

Case 2. The boundary case $B_{10r_0}(y_1) \not\subset \Omega$.

At first we find a boundary point $\tilde{y}_1 \in \partial\Omega \cap B_{10r_0}(y_1)$. Since $640r_0 \leq R_0 < \frac{R}{2}$ and the domain Ω is (δ, R)-Reifenberg flat, there exists a coordinate system, which we still denote by $x = (x_1, \cdots, x_n)$, with the origin at $\tilde{y_1}$, such that

$$B_{640r_0} \cap \{x_n > 640\delta r_0\} \subset \Omega_{640r_0} \subset B_{640r_0} \cap \{x_n > -640\delta r_0\}.$$

We select δ so small with $0 < \delta < \frac{1}{16}$. Then we see that $B_{480r_0}(640\delta r_0 e_n) \subset B_{640r_0}$, where $e_n = (0, \cdots, 0, 1)$. Translating this coordinate system to the x_n-direction $640\delta r_0$, still say x-coordinate, we observe

$$B^+_{480r_0} \subset \Omega_{480r_0} \subset B_{480r_0} \cap \{x_n > -1280\delta r_0\}. \tag{3.3.13}$$

Since $|y_1| \leq |y_1 - \tilde{y_1}| + |\tilde{y_1}| \leq 10r_0 + 640\delta r_0 \leq 50r_0$ in the new coordinate, we have

$$\Omega_{2r_0}(y_0) \subset \Omega_{3r_0}(y_1) \subset \Omega_{60r_0} \quad \text{and} \quad \Omega_{480r_0} \subset \Omega_{640r_0}(y_1). \tag{3.3.14}$$

We denote

$$p_1 := \inf_{x \in \Omega_{480r_0}} p(x) \quad \text{and} \quad p_2 := \sup_{x \in \Omega_{480r_0}} p(x).$$

Then it follows that $p_2 - p_1 \leq \omega(960r_0)$.

To obtain the corresponding estimates (3.3.9) in the boundary case, we deduce from (3.1.1), (3.3.6), (3.3.13), and (3.3.14) that

$$⨍_{\Omega_{480r_0}} |Du|\, dx \leq c_5\lambda_k \quad \text{and} \quad \left[\frac{\nu(\overline{\Omega_{480r_0}})}{r_0^{n-1}}\right]^{\frac{1}{p(0)-1}} \leq c_5\delta\lambda_k \tag{3.3.15}$$

for some constant $c_5 = c_5(n, \gamma_1) > 0$. Moreover, it follows from (3.0.7), (3.0.8) and (3.3.15) that, for any $\tilde{\epsilon} \in (0, 1)$,

$$⨍_{\Omega_{480r_0}} |Du_h|\, dx \leq (c_5 + \tilde{\epsilon})\lambda_k =: c_6\lambda_k \quad \text{and} \quad \left[\frac{\nu_h(\Omega_{480r_0})}{r_0^{n-1}}\right]^{\frac{1}{p(0)-1}} \leq c_6\delta\lambda_k$$

for h large enough. Applying Lemma 3.2.9 with ρ, r, and ϵ replaced by $c_6\lambda_k$, $60r_0$, and η, respectively, we can find $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \eta)$ such that

$$\begin{aligned} &⨍_{\Omega_{120r_0}} |Du - D\bar{v}_h|\, dx \leq c_6\eta\lambda_k, \\ &||D\bar{v}_h||_{L^\infty(\Omega_{60r_0})} \leq cc_6\lambda_k =: c_7\lambda_k \end{aligned} \tag{3.3.16}$$

for some $c_7 = c_7(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$. Here we have chosen sufficiently small $\tilde{\epsilon}$ such that $\tilde{\epsilon} \leq \frac{c_6 \eta}{2}$.

Proceeding as in *Case 1*, we infer

$$\begin{aligned} \mathfrak{C}_k \cap B_{r_0}(y_0) &= \{x \in \Omega_{r_0}(y_0) : \mathcal{M}(|Du|)(x) > N\lambda_k\} \\ &\subset \left\{x \in \Omega_{r_0}(y_0) : \mathcal{M}_{\Omega_{2r_0}(y_0)}(|Du - D\bar{v}_h|)(x) > \lambda_k\right\} \end{aligned} \tag{3.3.17}$$

provided $N \geq N_2 \geq \max\{3^n, 1 + c_7\}$.

Thus, we have from (3.3.17), (2.1.9), (3.3.14) and (3.3.16) that

$$\begin{aligned} |\mathfrak{C}_k \cap B_{r_0}(y_0)| &\leq \left|\left\{x \in \Omega_{r_0}(y_0) : \mathcal{M}_{\Omega_{2r_0}(y_0)}(|Du - D\bar{v}_h|)(x) > \lambda_k\right\}\right| \\ &\leq \frac{c}{\lambda_k} \int_{\Omega_{2r_0}(y_0)} |Du - D\bar{v}_h| \, dx \\ &\leq \frac{c|\Omega_{120r_0}|}{\lambda_k} \fint_{\Omega_{120r_0}} |Du - D\bar{v}_h| \, dx \\ &\leq cc_6\eta |B_{r_0}(y_0)| < \epsilon |B_{r_0}(y_0)| \end{aligned}$$

by taking η sufficiently small, as a consequence $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \epsilon)$ is also determined. This is a contradiction to (3.3.5). □

3.4 Global Calderón-Zygmund type estimates

We are now ready to prove Theorem 3.1.2.

Proof of Theorem 3.1.2. Choosing $N = \max\{N_1, N_2\}$ from Lemma 3.3.1 and Lemma 3.3.2, we can apply Lemma 2.1.2 to obtain

$$|\mathfrak{C}_k| \leq \left(\frac{80}{7}\right)^n \epsilon |\mathfrak{D}_k| =: \epsilon_1 |\mathfrak{D}_k| \quad \text{for } k \in \mathbb{N} \cup \{0\}.$$

As a consequence above and its iteration argument, we deduce the power decay estimates for the level sets of $\mathcal{M}(|Du|)$ to increasing levels, as follows:

$$\begin{aligned} &\left|\left\{x \in \Omega : \mathcal{M}(|Du|)(x) > N^k \lambda_0\right\}\right| \\ &\quad \leq \epsilon_1^k \left|\{x \in \Omega : \mathcal{M}(|Du|)(x) > \lambda_0\}\right| \\ &\qquad + \sum_{i=1}^{k} \epsilon_1^i \left|\left\{x \in \Omega : [\mathcal{M}_1(\nu)(x)]^{\frac{1}{p(x)-1}} > \delta N^{k-i} \lambda_0\right\}\right|. \end{aligned} \tag{3.4.1}$$

Now we write

$$S := \sum_{k=1}^{\infty} N^{qk} \left| \left\{ x \in \Omega : \mathcal{M}(|Du|)(x) > N^k \lambda_0 \right\} \right|.$$

Then we have from (3.4.1) and (2.1.11) that

$$\begin{aligned} S &\leq \sum_{k=1}^{\infty} N^{qk} \epsilon_1^k \left| \{ x \in \Omega : \mathcal{M}(|Du|)(x) > \lambda_0 \} \right| \\ &\quad + \sum_{k=1}^{\infty} N^{qk} \sum_{i=1}^{k} \epsilon_1^i \left| \left\{ x \in \Omega : [\mathcal{M}_1(\nu)(x)]^{\frac{1}{p(x)-1}} > \delta N^{k-i} \lambda_0 \right\} \right| \\ &\leq |\Omega| \sum_{k=1}^{\infty} (N^q \epsilon_1)^k \\ &\quad + \sum_{i=1}^{\infty} (N^q \epsilon_1)^i \sum_{k=i}^{\infty} N^{q(k-i)} \left| \left\{ x \in \Omega : [\mathcal{M}_1(\nu)(x)]^{\frac{1}{p(x)-1}} > \delta N^{k-i} \lambda_0 \right\} \right| \\ &\leq \sum_{i=1}^{\infty} (N^q \epsilon_1)^i \left\{ 2|\Omega| + \frac{c}{(\delta\lambda_0)^q} \int_{\Omega} \mathcal{M}_1(\nu)^{\frac{q}{p(x)-1}} \, dx \right\}. \end{aligned}$$

Now we select ϵ_1 with $N^q \epsilon_1 = \frac{1}{2}$, and then we can take ϵ and a corresponding $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, q) > 0$. Consequently, we find

$$S \leq 2|\Omega| + \frac{c}{\lambda_0^q} \int_{\Omega} \mathcal{M}_1(\nu)^{\frac{q}{p(x)-1}} \, dx. \tag{3.4.2}$$

According to (2.1.11) and (3.4.2), we have

$$\begin{aligned} \int_{\Omega} |Du|^q \, dx &\leq \int_{\Omega} \mathcal{M}(|Du|)^q \, dx \leq c\lambda_0^q (|\Omega| + S) \\ &\leq c \left\{ |\Omega| \lambda_0^q + \int_{\Omega} \mathcal{M}_1(\nu)^{\frac{q}{p(x)-1}} \, dx \right\} \\ &\leq c \left\{ \frac{|\Omega|}{|B_{R_0}|^q} \left(\int_{\Omega} |Du| \, dx + 1 \right)^q + \int_{\Omega} \mathcal{M}_1(\nu)^{\frac{q}{p(x)-1}} \, dx \right\} \end{aligned} \tag{3.4.3}$$

for some $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, q) > 0$.

On the other hand, it follows from the estimate (3.4.7) in Remark 3.4.1

that

$$\int_{\Omega} |Du| \, dx \leq c(n, \lambda, \gamma_1, \gamma_2, s, \Omega) \left\{ |\mu|(\Omega) + [|\mu|(\Omega)]^{\frac{1}{(\gamma_1 - 1)(1-s)}} \right\}.$$

Since R_0 satisfies (3.3.1) and (3.3.2) with M_1 given in (3.2.41), we see from the estimate above that

$$\frac{1}{R_0} \leq \frac{c}{R} \left\{ |\mu|(\Omega) + [|\mu|(\Omega)]^{\frac{1}{(\gamma_1 - 1)(1-s)}} + 1 \right\}$$

for some constant $c = c(n, \lambda, \gamma_1, \gamma_2, \omega(\cdot), s, \Omega) > 0$ and for some $R < 1$. Thus, it follows that

$$\int_{\Omega} |Du|^q \, dx \leq c K_s^q \left\{ \int_{\Omega} \mathcal{M}_1(\nu)^{\frac{q}{p(x)-1}} \, dx + 1 \right\} \tag{3.4.4}$$

for some $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), q, R, s, \Omega) > 0$. Here we define

$$K_s := \left(|\mu|(\Omega) + [|\mu|(\Omega)]^{\frac{1}{(\gamma_1 - 1)(1-s)}} + 1 \right)^{n+1}.$$

Recalling the definition (3.2.5), we have that for $x \in \Omega$,

$$\begin{aligned} \mathcal{M}_1(\nu)(x) &:= \sup_{r>0} \frac{r\nu(B_r(x))}{|B_r(x)|} \leq \sup_{r>0} \frac{r|\mu|(B_r(x))}{|B_r(x)|} + \sup_{r>0} \frac{r|B_r(x) \cap \Omega|}{|B_r(x)|} \\ &\leq \mathcal{M}_1(\mu)(x) + c(n) diam(\Omega) \leq \mathcal{M}_1(\mu)(x) + c(n)|\Omega|^{\frac{1}{n}}. \end{aligned}$$

Then we have

$$\int_{\Omega} \mathcal{M}_1(\nu)^{\frac{q}{p(x)-1}} \, dx \leq \int_{\Omega} \mathcal{M}_1(\mu)^{\frac{q}{p(x)-1}} \, dx + c|\Omega|^{\frac{q}{n(\gamma_1 - 1)} + 1}. \tag{3.4.5}$$

The estimates (3.4.4) and (3.4.5) yield the desired estimate (3.1.4). This completes the proof. □

Remark 3.4.1. *We derive a standard estimate for measure data. Consider the regularized problem* (3.0.6)*, we denote, for* $k \in \mathbb{N}$*,*

$$D_k := \{x \in \Omega : |u_h(x)| \leq k\} \quad \text{and} \quad C_k := \{x \in \Omega : k < |u_h(x)| \leq k+1\}.$$

Then from (3.2.2), (3.0.5), *and* (3.0.9), *we have*

$$\int_{D_k} |Du_h|^{p(x)}\, dx \le ck|\mu|(\Omega) \quad \text{and} \quad \int_{C_k} |Du_h|^{p(x)}\, dx \le c|\mu|(\Omega)$$

by substituting test functions $\varphi = T_k(u_h)$ *and* $\varphi = \Phi_k(u_h)$ *in* (3.2.2), *respectively. Here the functions* T_k *and* Φ_k *are defined as in* (3.2.9). *Then we discover*

$$\int_{D_k} |Du_h|\, dx \le \int_{D_k} (|Du_h|+1)^{p(x)}\ dx \le ck\nu(\Omega),$$

where ν *is given in* (3.2.5). *If* $\frac{1}{\gamma_1 - 1} < t < \frac{n}{n-1}$, *then it follows that*

$$\begin{aligned}
\int_{C_k} |Du_h|\, dx &\le \left(\int_{C_k} |Du_h|^{\gamma_1}\, dx\right)^{\frac{1}{\gamma_1}} |C_k|^{\frac{1}{\gamma_1{}'}} \\
&\le c\,[\nu(\Omega)]^{\frac{1}{\gamma_1}} \left(\int_{C_k} \left(\frac{|u_h|}{k}\right)^t dx\right)^{\frac{1}{\gamma_1{}'}} \\
&\le c\,[\nu(\Omega)]^{\frac{1}{\gamma_1}} \left(\frac{1}{k}\right)^{\frac{t}{\gamma_1{}'}} \left(\int_{C_k} |u_h|^t\, dx\right)^{\frac{1}{\gamma_1{}'}},
\end{aligned}$$

where $\gamma_1{}'$ *is the Hölder conjugate of* γ_1. *Then we see that*

$$\int_{\Omega} |Du_h|\, dx \le c\nu(\Omega) + c\,[\nu(\Omega)]^{\frac{1}{\gamma_1}} \underbrace{\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{\frac{t}{\gamma_1{}'}} \left(\int_{C_k} |u_h|^t\, dx\right)^{\frac{1}{\gamma_1{}'}}}_{(*)}.$$

Applying Hölder's inequality and Sobolev's inequality to $(*)$, *we find that*

$$\begin{aligned}
(*) &\le \left(\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{\frac{t\gamma_1}{\gamma_1{}'}}\right)^{\frac{1}{\gamma_1}} \left(\sum_{k=1}^{\infty} \int_{C_k} |u_h|^t\, dx\right)^{\frac{1}{\gamma_1{}'}} \\
&\le c \left(\int_{\Omega} |u_h|^{\frac{n}{n-1}}\, dx\right)^{\frac{(n-1)t}{n\gamma_1{}'}} |\Omega|^{\left(1-\frac{(n-1)t}{n}\right)\frac{1}{\gamma_1{}'}} \qquad (3.4.6) \\
&\le c \left(\int_{\Omega} |Du_h|\, dx\right)^{\frac{t}{\gamma_1{}'}} |\Omega|^{\left(1-\frac{(n-1)t}{n}\right)\frac{1}{\gamma_1{}'}}
\end{aligned}$$

for some constant $c = c(n, \gamma_1, t) > 0$*. Thus, we have*

$$\int_\Omega |Du_h|\,dx \le c\left\{\nu(\Omega) + [\nu(\Omega)]^{\frac{1}{\gamma_1}}\left(\int_\Omega |Du_h|\,dx\right)^{\frac{t}{\gamma_1{}'}} |\Omega|^{\left(1-\frac{(n-1)t}{n}\right)\frac{1}{\gamma_1{}'}}\right\}$$

for some constant $c = c(n, \lambda, \gamma_1, \gamma_2, t) > 0$*. From the fact that* $\gamma_1 \le n$*, on the other hand, we see that* $t < \frac{n}{n-1} \le \gamma_1{}'$*. Then it follows from Young's inequality that*

$$\int_\Omega |Du_h|\,dx \le c\nu(\Omega) + \frac{1}{2}\int_\Omega |Du_h|\,dx + c\,[\nu(\Omega)]^{\frac{1}{\gamma_1-(\gamma_1-1)t}} |\Omega|^{\left(1-\frac{(n-1)t}{n}\right)\frac{1}{\gamma_1{}'-t}}.$$

Then we have from (3.0.7) *that*

$$\int_\Omega |Du|\,dx \le c\left\{\nu(\Omega) + [\nu(\Omega)]^{\frac{1}{\gamma_1-(\gamma_1-1)t}} |\Omega|^{\left(1-\frac{(n-1)t}{n}\right)\frac{1}{\gamma_1{}'-t}}\right\}.$$

Therefore we obtain from (3.2.5) *that*

$$\int_\Omega |Du|\,dx \le c\left\{|\mu|(\Omega) + [|\mu|(\Omega)]^{\frac{1}{(\gamma_1-1)(1-s)}}\right\} \tag{3.4.7}$$

for some constant $c = c(n, \lambda, \gamma_1, \gamma_2, s, \Omega) > 0$*, where we have selected* $t := \frac{1}{\gamma_1-1} + s$ *for small* s *with* $0 < s \le \frac{1}{2}\left(\frac{n}{n-1} - \frac{1}{\gamma_1-1}\right) < 1$.

We clearly point out that this constant c *goes to* $+\infty$ *as* $s \searrow 0$*, since the exponent* $\frac{t\gamma_1}{\gamma_1{}'} = 1$ *in the first inequality of* (3.4.6).

Remark 3.4.2. *If* $p(\cdot)$ *is a constant, then we infer from Remark 3.2.2 and* (3.4.3) *that*

$$\int_\Omega |Du|^q\,dx \le c\left\{\frac{|\Omega|}{R^{nq}}\left(\int_\Omega |Du|\,dx\right)^q + \int_\Omega \mathcal{M}_1(\mu)^{\frac{q}{p-1}}\,dx\right\} \tag{3.4.8}$$

for some $c = c(n, \lambda, \Lambda, p, q) > 0$*. On the other hand, a standard estimate for measure data can be obtained by the normalization property for the problem* (3.0.1)*, that is,*

$$\int_\Omega |Du|\,dx \le c(n, \lambda, p)\int_\Omega \mathcal{M}_1(\mu)^{\frac{1}{p-1}}\,dx. \tag{3.4.9}$$

Indeed, the proof of (3.4.9) *is similar to that of Lemma 3.2.1. Using* (3.4.8) *and* (3.4.9)*, we derive*

$$\int_\Omega |Du|^q \, dx \le c \int_\Omega \mathcal{M}_1(\mu)^{\frac{q}{p-1}} \, dx$$

for some constant $c = c(n, \lambda, \Lambda, p, q, R, \Omega) > 0$*. This is the main estimate in [88]. However, in the case that* $p(\cdot)$ *is not a constant, the normalization property of* (3.0.1) *does not hold, and so* (3.4.9) *is no longer satisfied. See also Remark 3.4.1.*

Chapter 4

Optimal regularity for elliptic measure data problems in variable exponent spaces

Consider the Dirichlet problem with measure data

$$\begin{cases} -\operatorname{div} \mathbf{a}(Du, x) & = & \mu \quad \text{in } \Omega, \\ u & = & 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.0.1}$$

where μ be a signed Radon measure on Ω with finite total variation $|\mu|(\Omega) < \infty$. Here we assume that μ is defined in $\mathbb{R}^n$ by considering the zero extension to $\mathbb{R}^n$, and the nonlinearity $\mathbf{a} = \mathbf{a}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable in ξ and measurable in x, and it satisfies the following conditions:

$$|\xi||D_\xi \mathbf{a}(\xi, x)| + |\mathbf{a}(\xi, x)| \leq \Lambda |\xi|, \tag{4.0.2}$$

$$\lambda |\eta|^2 \leq \langle D_\xi \mathbf{a}(\xi, x)\eta, \eta \rangle , \tag{4.0.3}$$

for every $x, \eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$, and some constants λ, Λ. Note that (4.0.2) implies that $\mathbf{a}(0, x) = 0$ for $x \in \mathbb{R}^n$, and (4.0.3) yields the following monotonicity condition:

$$\langle \mathbf{a}(\xi_1, x) - \mathbf{a}(\xi_2, x), \xi_1 - \xi_2 \rangle \geq \tilde{\lambda} \, |\xi_1 - \xi_2|^2$$

for all $x, \xi_1, \xi_2 \in \mathbb{R}^n$ and some constant $\tilde{\lambda} = \tilde{\lambda}(n, \lambda) > 0$.

4.1 Main results

A solution u of (4.0.1) will be treated in the sense of distribution which does not generally belong to a weak solution in $W_0^{1,2}(\Omega)$ (consider Laplace's equation with the Dirac measure). For this reason, it is necessary to generalize a class of solutions below the natural exponent.

Definition 4.1.1. *$u \in W_0^{1,1}(\Omega)$ is a* SOLA *to the problem* (4.0.1) *under the assumptions* (4.0.2) *and* (4.0.3) *if the nonlinearity $\mathbf{a}(Du, x) \in L^1(\Omega, \mathbb{R}^n)$,*

$$\int_\Omega \langle \mathbf{a}(Du, x), D\varphi \rangle \; dx = \int_\Omega \varphi \, d\mu$$

holds for all $\varphi \in C_c^\infty(\Omega)$, and moreover there exists a sequence of weak solutions $\{u_h\}_{h \geq 1} \subset W_0^{1,2}(\Omega)$ of the Dirichlet problems

$$\begin{cases} -\operatorname{div} \mathbf{a}(Du_h, x) &= \mu_h \quad \text{in } \Omega, \\ u_h &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{4.1.1}$$

such that

$$u_h \to u \quad \text{in} \;\; W_0^{1,1}(\Omega) \quad \text{as } h \to \infty,$$

where $\mu_h \in L^\infty(\Omega)$ converges weakly to μ in the sense of measure and satisfies for each open set $V \subset \mathbb{R}^n$,

$$\limsup_{h \to \infty} |\mu_h|(V) \leq |\mu|(\overline{V}), \tag{4.1.2}$$

with μ_h defined in $\mathbb{R}^n$ by the zero extension of μ_h to $\mathbb{R}^n$.

Here we consider $\mu_h := \mu * \phi_h$, where ϕ_h is the standard mollifier, and then $\mu_h \in C^\infty(\Omega)$ converges weakly to μ in the sense of measure, the following uniform L^1-estimate holds:

$$||\mu_h||_{L^1(\Omega)} \leq |\mu|(\Omega), \tag{4.1.3}$$

and such a SOLA u of (4.0.1) belongs to $W_0^{1,q}(\Omega)$ such that

$$u_h \to u \quad \text{in} \;\; W_0^{1,q}(\Omega) \quad \text{for all} \;\; q \in \left[1, \frac{n}{n-1}\right). \tag{4.1.4}$$

We now state the main assumption on $\mathbf{a}$ and Ω in Chapter 4.

Definition 4.1.2. *We say* $(\mathbf{a}, \Omega)$ *is* (δ, R)-vanishing of codimension 1 *if for every point* $y \in \Omega$ *and number* $r \in \left(0, \frac{R}{3}\right]$*, the following conditions hold.*

(i) If $dist(y, \partial\Omega) > r\sqrt{2}$*, then there exists a new coordinate system depending only on* y *and* r*, still denoted by* $\{x_1, \cdots, x_n\}$*, in which the origin is* y *and*

$$\fint_{\mathcal{B}_{r\sqrt{2}}} \left|\theta(\mathbf{a}, \mathcal{B}_{r\sqrt{2}})(x)\right| \, dx \leq \delta,$$

where

$$\theta(\mathbf{a}, \mathcal{B}_r)(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\left|\mathbf{a}(\xi, x', x_n) - \bar{\mathbf{a}}_{B'_r}(\xi, x_n)\right|}{|\xi|},$$

and $\bar{\mathbf{a}}_{B'_r}(\xi, x_n)$ *is the integral average of* $\mathbf{a}(\xi, \cdot, x_n)$ *over* $B'_r \subset \mathbb{R}^{n-1}$*.*

(ii) If $dist(y, \partial\Omega) = |y - y_0| \leq r\sqrt{2}$ *for some* $y_0 \in \partial\Omega$*, then there is a new coordinate system depending only on* y *and* r*, still denoted by* $\{x_1, \cdots, x_n\}$*, in which the origin is* $y_0 + 3\delta r e_n$*, where* $e_n := (0, \cdots, 0, 1)$*,*

$$\mathcal{B}^+_{3r} \subset \Omega_{3r} \subset \mathcal{B}_{3r} \cap \{(x', x_n) : x_n > -6\delta r\}, \tag{4.1.5}$$

and

$$\fint_{\mathcal{B}_{3r}} |\theta(\mathbf{a}, \mathcal{B}_{3r})(x)| \, dx \leq \delta.$$

Remark 4.1.3. *(i) The number* δ *is a sufficiently small universal constant with* $\delta \in \left(0, \frac{1}{8}\right)$*, as determined later in the proof of Theorem 4.1.4. This number is invariant under the dilation scaling for the problem* (4.0.1)*. On the other hand, the number* R *is given arbitrary.*

(ii) The numbers $r\sqrt{2}$ *and* $3r$ *above are selected so that the size is large enough so that rotation in any direction is allowed.*

(iii) If $(\mathbf{a}, \Omega)$ *is* (δ, R)*-vanishing of codimension* 1*, then for each point and sufficiently small scale, there is a coordinate system for which the nonlinearity* $\mathbf{a}(\xi, \cdot)$ *is merely measurable in the* x_n *variable and of small BMO in the other variables* x'*. Moreover, the domain* Ω *with* (4.1.5) *is called a* (δ, R)-Reifenberg flat domain*, see also Chapter 2.1.3.*

(iv) If (4.1.5) *holds, then there is the following measure density condition:*

$$\sup_{0<r\leq \frac{R}{3}} \sup_{y\in\Omega} \frac{|\mathcal{B}_r(y)|}{|\Omega\cap\mathcal{B}_r(y)|} \leq \left(\frac{2\sqrt{2}}{1-\delta}\right)^n \leq \left(\frac{16\sqrt{2}}{7}\right)^n, \tag{4.1.6}$$

which can be found in [29].

We are ready to present our main results in Chapter 4.

Theorem 4.1.4. *Assume that* (4.0.2) *and* (4.0.3) *are hold, and that* u *is a SOLA of the problem* (4.0.1). *Let* $0 < R < 1$ *and let* $p(\cdot)$ *be log-Hölder continuous satisfying* (2.1.1). *Then there is a small constant* $\delta = \delta(n,\lambda,\Lambda,\gamma_1,\gamma_2,L) \in \left(0,\frac{1}{8}\right)$ *such that if* $(\mathbf{a},\Omega)$ *is* (δ,R)*-vanishing of codimension 1, then there exists a constant* $c_0 = c_0(n,\lambda,\Lambda,\gamma_1,\omega(\cdot),R,\Omega) > 1$ *so that for any* $x_0\in\Omega$ *and* $R_0 \in \left(0, \frac{1}{c_0(|\mu|(\Omega)+1)}\right]$, *we have*

$$\begin{aligned} \fint_{\Omega_{R_0}(x_0)} |Du|^{p(x)}\,dx \leq c\Bigg\{ & \left(\fint_{\Omega_{4R_0}(x_0)} |Du|^{\frac{p(x)}{p_-}}\,dx\right)^{p_-} \\ & + \fint_{\Omega_{4R_0}(x_0)} \mathcal{M}_1(\mu)^{p(x)}\,dx + 1\Bigg\} \end{aligned} \tag{4.1.7}$$

for some constant $c = c(n,\lambda,\Lambda,\gamma_1,\gamma_2,L) > 0$, *where* $p_- := \inf_{x\in\Omega_{4R_0}(x_0)} p(x)$. *Moreover, we have*

$$\int_\Omega |Du|^{p(x)}\,dx \leq c\left\{\left(\int_\Omega \mathcal{M}_1(\mu)^{p(x)}\,dx\right)^{\frac{n(\gamma_2-1)+\gamma_2}{\gamma_1}} + 1\right\} \tag{4.1.8}$$

and

$$\|Du\|_{L^{p(\cdot)}(\Omega)} \leq c\,\|\mathcal{M}_1(\mu)\|_{L^{p(\cdot)}(\Omega)}, \tag{4.1.9}$$

where the constants c *depend only on* n, λ, Λ, γ_1, γ_2, $\omega(\cdot)$, L, R, *and* Ω. *Here* $\mathcal{M}_1(\mu)$ *is given in* (1.2.2).

Remark 4.1.5. *We know from* (4.1.4) *that* $u \in W_0^{1,q}(\Omega)$ *for all* $1 \leq q < \frac{n}{n-1}$, *and then the first term of the right-hand side in* (4.1.7) *is well defined by selecting* c_0 *sufficiently large with* $\frac{p(x)}{p_-} < \frac{n}{n-1}$ *for* $x \in \Omega_{4R_0}(x_0)$, *see Chapter 4.4.1 for details.*

Remark 4.1.6. *The condition on the above nonlinearity* $\mathbf{a}$ *is a possibly optimal assumption for the estimates* (4.1.7)–(4.1.9). *In other words, if* $\mathbf{a}(\xi, \cdot)$ *has two or more measurable coefficients, then these estimates are not generally satisfied even in the constant exponent case* $p(\cdot) \equiv p$*, see [74]. For the measurability in one variable, there have been regularity results for linear elliptic equations, see [29, 30, 35, 53]. Recently, Byun and Kim [20] considered nonlinear elliptic equations, without measure data, to obtain global* L^p *estimates for the gradient of a weak solution under the assumptions* (4.0.2), (4.0.3)*, and Definition 4.1.2. They obtained the desired results by proving Lipschitz regularity for limiting problems. It is worth noting that we refer to [61] for the case of problems having p-growth under the same condition (Definition 4.1.2).*

A main ingredient in our proof is to derive a power decay estimate of the upper-level sets of $|Du|^{\frac{p(x)}{p_-}}$ for a SOLA u on a small ball B with $p_- = \inf_{x \in B} p(x)$. We employ some properties of the SOLA, comparison estimates along with higher integrability of homogeneous problems and the log-Hölder continuity of $p(\cdot)$, and then the so-called *maximal function technique* which was introduced in [37, 98]. The difficulty in the present work comes from the measure data μ and the presence of the variable exponent $p(\cdot)$, and so more complicated and finer analysis than that previously made in [20, 23] has to be carefully carried out in the whole process.

4.2 Comparison estimates for regular problems

In Chapter 4.2, we consider the regular problem (4.1.1), where $\mu_h = \mu * \phi_h$ with ϕ_h the standard mollifier. We as always assume that $(\mathbf{a}, \Omega)$ is (δ, R)-vanishing of codimension 1. This section develops the comparison L^1-estimates for the gradient of the weak solution u_h to (4.1.1) in localized boundary and interior regions. We denote, for a measurable set $E \subset \mathbb{R}^n$,

$$|\mu_h|(E) := \int_E |\mu_h(x)| \, dx.$$

4.2.1 Boundary comparisons

Let $0 < r \le \frac{R}{8}$. Assume that $(\mathbf{a}, \Omega)$ is (δ, R)-vanishing of codimension 1 such that

$$\mathcal{B}_{8r}^{+} \subset \Omega_{8r} \subset \mathcal{B}_{8r} \cap \{x_n > -16\delta r\}, \tag{4.2.1}$$

and

$$\fint_{\mathcal{B}_{8r}} |\theta(\mathbf{a}, \mathcal{B}_{8r})(x)| \, dx \le \delta, \tag{4.2.2}$$

where δ is to be determined later in a universal way.

Let $w_h \in u_h + W_0^{1,2}(\Omega_{8r})$ be the weak solution of the homogeneous problem

$$\begin{cases} \operatorname{div} \mathbf{a}(Dw_h, x) &=& 0 & \text{in } \Omega_{8r}, \\ w_h &=& u_h & \text{on } \partial\Omega_{8r}. \end{cases} \tag{4.2.3}$$

Using the measure density condition (4.1.6), we can extend the comparison result in [66, Lemma 2] up to the boundary.

Lemma 4.2.1. *If $w_h \in u_h + W_0^{1,2}(\Omega_{8r})$ is the weak solution of (4.2.3) satisfying (4.2.1), then there exists a constant $c = c(n, \lambda, q) > 0$ such that*

$$\fint_{\Omega_{8r}} |Du_h - Dw_h|^q \, dx \le c \left[\frac{|\mu_h|(\Omega_{8r})}{r^{n-1}} \right]^q \quad \text{for all} \quad q \in \left(0, \frac{n}{n-1}\right). \tag{4.2.4}$$

Applying Gehring's lemma to the weak solution w_h of (4.2.3), we discover some higher integrability result, see [57, Remark 6.12], as we now state.

Lemma 4.2.2. *There exists a constant $\sigma_0 = \sigma_0(n, \lambda, \Lambda) > 0$ such that the following holds: for any $r \in \left(0, \frac{R}{8}\right]$, if w_h is the weak solution of (4.2.3) satisfying (4.2.1), then for any $0 < \sigma \le \sigma_0$ and $\Omega_{2\tilde{r}}(\tilde{x}_0) \subset \Omega_{8r}$ with $\tilde{r} \le 4r$, there is a constant $c = c(n, \lambda, \Lambda, t) > 0$ such that*

$$\left(\fint_{\Omega_{\tilde{r}}(\tilde{x}_0)} |Dw_h|^{2(1+\sigma)} \, dx \right)^{\frac{1}{1+\sigma}} \le c \left(\fint_{\Omega_{2\tilde{r}}(\tilde{x}_0)} |Dw_h|^{2t} \, dx \right)^{\frac{1}{t}}$$

for all $t \in (0, 1]$.

From Hölder's inequality and Lemma 4.2.2, we can directly obtain the following estimate.

Corollary 4.2.3. *Under the same assumptions as in Lemma 4.2.2, we have*

$$\fint_{\Omega_{\tilde{r}}(\tilde{x}_0)} |Dw_h|^2\,dx \le c\left(\fint_{\Omega_{2\tilde{r}}(\tilde{x}_0)} |Dw_h|\,dx\right)^2$$

for some constant $c = c(n, \lambda, \Lambda) > 0$*.*

We next consider the homogeneous frozen problem

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{B'_{4r}}(Dv_h, x_n) &= 0 \quad \text{in } \Omega_{4r}, \\ v_h &= w_h \quad \text{on } \partial\Omega_{4r}, \end{cases} \tag{4.2.5}$$

where w_h is the weak solution of (4.2.3). Then $v_h \in w_h + W_0^{1,2}(\Omega_{3r})$ is the weak solution of (4.2.5), and the operator $\bar{\mathbf{a}}_{B'_{4r}}$ satisfies (4.0.2) and (4.0.3) with $\mathbf{a}(\xi, \cdot, x_n)$ replaced by $\bar{\mathbf{a}}_{B'_{4r}}(\xi, x_n)$. Moreover, we derive the standard energy estimate

$$\fint_{\Omega_{4r}} |Dv_h|^2\,dx \le c \fint_{\Omega_{4r}} |Dw_h|^2\,dx, \tag{4.2.6}$$

by substituting the test function $v_h - w_h$ into the weak formulation of (4.2.5).

The following lemma demonstrates some comparison result between two problems (4.2.3) and (4.2.5).

Lemma 4.2.4 (See [20, Lemma 5.6])**.** *Suppose that* Ω_{8r} *satisfies* (4.2.1) *and* (4.2.2)*. If* w_h *and* v_h *are the weak solutions of* (4.2.3) *and* (4.2.5)*, respectively, then there is a constant* $c = c(n, \lambda, \Lambda) > 0$ *such that*

$$\fint_{\Omega_{4r}} |Dw_h - Dv_h|^2\,dx \le c\delta^{\frac{\sigma_0}{1+\sigma_0}} \left(\fint_{\Omega_{8r}} |Dw_h|\,dx\right)^2,$$

where σ_0 *is given in Lemma 4.2.2.*

Let us assume now $\bar{v}_h \in W^{1,2}(\mathcal{B}^+_{3r})$ is a weak solution of the reference problem

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{B'_{4r}}(D\bar{v}_h, x_n) &= 0 \quad \text{in } \mathcal{B}^+_{3r}, \\ \bar{v}_h &= 0 \quad \text{on } \mathcal{B}_{3r} \cap \{x_n = 0\}. \end{cases} \tag{4.2.7}$$

We can now state some comparison estimate and Lipschitz regularity result.

Lemma 4.2.5 (See [20, Lemma 5.8])**.** *For any* $\epsilon \in (0, 1)$*, there is* $\delta = \delta(n, \lambda, \Lambda, \epsilon) > 0$ *such that if* $v_h \in w_h + W_0^{1,2}(\Omega_{4r})$ *is the weak solution* *of* (4.2.5) *with* (4.2.1) *and* (4.2.2), *then there exists a weak solution* $\bar{v}_h \in W^{1,2}(\mathcal{B}_{3r}^+)$ *of* (4.2.7) *such that*

$$\fint_{\Omega_{3r}} |Dv_h - D\bar{v}_h|^2 \, dx \le \epsilon^2 \fint_{\Omega_{4r}} |Dv_h|^2 \, dx,$$

and

$$\|D\bar{v}_h\|_{L^\infty(\Omega_{2r})} \le c \fint_{\Omega_{3r}} |D\bar{v}_h| \, dx$$

for some constant $c = c(n, \lambda, \Lambda) > 0$. *Here* $\bar{v}_h$ *is extended by zero from* $\mathcal{B}_{3r}^+$ *to* Ω_{3r}.

We finally summarize the comparison L^1-estimates near a boundary region.

Lemma 4.2.6. *Let* $\rho > 1$ *and* $0 < r \le \frac{R}{8}$. *Then for any* $0 < \epsilon < 1$, *there exists a small constant* $\delta = \delta(n, \lambda, \Lambda, \epsilon) > 0$ *such that if* $u_h \in W_0^{1,2}(\Omega)$, $w_h \in u_h + W_0^{1,2}(\Omega_{8r})$, *and* $v_h \in w_h + W_0^{1,2}(\Omega_{4r})$ *are the weak solutions of* (4.1.1), (4.2.3), *and* (4.2.5), *respectively, with* (4.2.1), (4.2.2) *and*

$$\fint_{\Omega_{8r}} |Du_h| \, dx \le \rho \quad \text{and} \quad \frac{|\mu_h|(\Omega_{8r})}{r^{n-1}} \le \delta\rho,$$

then there is a weak solution $\bar{v}_h \in W^{1,2}(\mathcal{B}_{3r}^+)$ *of* (4.2.7) *such that*

$$\fint_{\Omega_{3r}} |Du_h - D\bar{v}_h| \, dx \le \epsilon\rho \quad \text{and} \quad \|D\bar{v}_h\|_{L^\infty(\Omega_{2r})} \le c\rho$$

for some constant $c = c(n, \lambda, \Lambda) > 0$. *Here* $\bar{v}_h$ *is extended by zero from* $\mathcal{B}_{3r}^+$ *to* Ω_{3r}.

Proof. We first have from Lemma 4.2.1 ($q = 1$) that

$$\fint_{\Omega_{8r}} |Du_h - Dw_h| \, dx \le c\delta\rho \quad \text{and} \quad \fint_{\Omega_{8r}} |Dw_h| \, dx \le c\rho. \tag{4.2.8}$$

Hölder's inequality and Lemma 4.2.4 yield

$$\fint_{\Omega_{4r}} |Dw_h - Dv_h| \, dx \le \left(\fint_{\Omega_{4r}} |Dw_h - Dv_h|^2 \, dx \right)^{\frac{1}{2}} \le c\delta^{\frac{\sigma_0}{2(1+\sigma_0)}}\rho, \tag{4.2.9}$$

and
$$\fint_{\Omega_{4r}} |Dv_h| \, dx \le c\rho. \tag{4.2.10}$$

According to Lemma 4.2.5 with ϵ replaced by $\tilde{\epsilon}$, there exists a weak solution $\bar{v}_h \in W^{1,2}(\mathcal{B}_{3r}^+)$ of (4.2.7) such that
$$\fint_{\Omega_{3r}} |Dv_h - D\bar{v}_h|^2 \, dx \le \tilde{\epsilon}^2 \fint_{\Omega_{4r}} |Dv_h|^2 \, dx.$$

Then we see from this estimate, Hölder's inequality, (4.2.6), Corollary 4.2.3, and (4.2.8) that
$$\fint_{\Omega_{3r}} |Dv_h - D\bar{v}_h| \, dx \le c\tilde{\epsilon} \fint_{\Omega_{8r}} |Dw_h| \, dx \le c\tilde{\epsilon}\rho \le \frac{\epsilon}{3}\rho \tag{4.2.11}$$

by choosing $\tilde{\epsilon}$ sufficiently small, and it follows from (4.2.10) and (4.2.11) that
$$\fint_{\Omega_{3r}} |D\bar{v}_h| \, dx \le c\rho. \tag{4.2.12}$$

Finally, we combine (4.2.8), (4.2.9) and (4.2.11), to obtain
$$\begin{aligned}\fint_{\Omega_{3r}} |Du_h - D\bar{v}_h| \, dx &\le \fint_{\Omega_{3r}} |Du_h - Dw_h| + |Dw_h - Dv_h| + |Dv_h - D\bar{v}_h| \, dx \\ &\le c\delta\rho + c\delta^{\frac{\sigma_0}{2(1+\sigma_0)}}\rho + \frac{\epsilon}{3}\rho \\ &\le \epsilon\rho,\end{aligned}$$

by selecting δ small enough.

On the other hand, in light of Lemma 4.2.5 and (4.2.12), we obtain
$$||D\bar{v}_h||_{L^\infty(\Omega_{2r})} \le c\rho,$$

which completes the proof. □

4.2.2 Interior comparisons

In this subsection we derive comparison L^1-estimates for the interior case in a similar way that we derived their counterparts in the previous subsection. We just outline it for the sake of completeness.

Let $0 < r \le \frac{R}{8}$ with $\mathcal{B}_{8r}(x_0) \subset\subset \Omega$. With the weak solution $u_h \in W_0^{1,2}(\Omega)$ of (4.1.1), we consider the weak solution $w_h \in u_h + W_0^{1,2}(\mathcal{B}_{8r}(x_0))$ of the homogeneous problem

$$\begin{cases} \operatorname{div} \mathbf{a}(Dw_h, x) = 0 & \text{in } \mathcal{B}_{8r}(x_0), \\ w_h = u_h & \text{on } \partial\mathcal{B}_{8r}(x_0). \end{cases} \tag{4.2.13}$$

Next let $v_h \in w_h + W_0^{1,2}(\mathcal{B}_{4r}(x_0))$ be the weak solution of the homogeneous frozen problem

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{B'_{4r}}(Dv_h, x_n) = 0 & \text{in } \mathcal{B}_{4r}(x_0), \\ v_h = w_h & \text{on } \partial\mathcal{B}_{4r}(x_0). \end{cases} \tag{4.2.14}$$

Then we have $Dv_h \in L^\infty(\mathcal{B}_{2r}(x_0))$ with the estimate

$$\|Dv_h\|_{L^\infty(\mathcal{B}_{2r}(x_0))} \le c \fint_{\mathcal{B}_{4r}(x_0)} |Dv_h| \, dx$$

for some constant $c = c(n, \lambda, \Lambda) > 0$, see [49] for details.

We now state the comparison L^1-estimates in an interior region.

Lemma 4.2.7. *Let $\rho > 1$ and $0 < r \le \frac{R}{8}$. Then, for any $\epsilon \in (0,1)$, there is a small constant $\delta = \delta(n, \lambda, \Lambda, \epsilon) > 0$ such that if $(\mathbf{a}, \Omega)$ is (δ, R)-vanishing of codimension 1, and if $u_h \in W_0^{1,2}(\Omega)$, $w_h \in u_h + W_0^{1,2}(\mathcal{B}_{8r}(x_0))$, and $v_h \in w_h + W_0^{1,2}(\mathcal{B}_{4r}(x_0))$ are the weak solutions* (4.1.1), (4.2.13), *and* (4.2.14), *respectively, with*

$$\fint_{\mathcal{B}_{8r}(x_0)} |Du_h| \, dx \le \rho \quad \text{and} \quad \frac{|\mu_h|(\mathcal{B}_{8r}(x_0))}{r^{n-1}} \le \delta\rho,$$

then we have

$$\fint_{\mathcal{B}_{4r}(x_0)} |Du_h - Dv_h| \, dx \le \epsilon\rho \quad \text{and} \quad \|Dv_h\|_{L^\infty(\mathcal{B}_{2r}(x_0))} \le c\rho$$

for some constant $c = c(n, \lambda, \Lambda) > 0$.

4.3 Covering arguments

Now, we consider a SOLA u of (4.0.1) and suppose that $(\mathbf{a}, \Omega)$ is (δ, R)-vanishing of codimension 1. Moreover, we assume that $R_0 > 0$ satisfies

$$R_0 \leq \min\left\{\frac{1}{16\sqrt{2}}, \frac{R}{8\sqrt{2}}, \frac{1}{\int_\Omega |Du|\,dx + 1}, \frac{1}{|\mu|(\Omega) + 1}\right\}, \tag{4.3.1}$$

$$\omega(8R_0\sqrt{2}) \leq \min\left\{1, \frac{\sigma_0\gamma_1}{2}, \frac{\gamma_1}{4(n-1)}\right\}, \tag{4.3.2}$$

where σ_0 is given in Lemma 4.2.2. Fix any $x_0 \in \Omega$ and consider $\Omega_{4R_0}(x_0)$. In this section we omit the center x_0 for simplicity. We set

$$p_- := \inf_{x\in\Omega_{4R_0}} p(x) \quad \text{and} \quad p_+ := \sup_{x\in\Omega_{4R_0}} p(x).$$

For any fixed $\epsilon \in (0,1)$ and $N > 1$, we define

$$\lambda_0 := \frac{1}{\epsilon}\left\{\fint_{\Omega_{4R_0}} |Du|^{\frac{p(x)}{p_-}}\,dx + 1\right\} > 1 \tag{4.3.3}$$

and upper-level sets: for $k \in \mathbb{N} \cup \{0\}$,

$$\mathfrak{C}_k := \left\{x \in \Omega_{R_0} : \mathcal{M}\left(|Du|^{\frac{p(\cdot)}{p_-}}\chi_{\Omega_{4R_0}}\right)(x) > N^{k+1}\lambda_0\right\},$$

$$\mathfrak{D}_k := \left\{x \in \Omega_{R_0} : \mathcal{M}\left(|Du|^{\frac{p(\cdot)}{p_-}}\chi_{\Omega_{4R_0}}\right)(x) > N^{k}\lambda_0\right\} \cup \left\{x \in \Omega_{R_0} : [\mathcal{M}_1(\mu)(x)]^{\frac{p(x)}{p_-}} > \delta N^k\lambda_0\right\},$$

where $\mathcal{M}$ and $\mathcal{M}_1$ are given in (2.1.8) and (1.2.2), respectively, while χ is the standard characteristic function. Note that ϵ and N are to be chosen later depending only on $n, \lambda, \Lambda, \gamma_1, \gamma_2$, and L.

We now verify two assumptions of the Vitali type covering lemma (Lemma 2.1.2).

Lemma 4.3.1. *There exists a constant $N_1 = N_1(n) > 1$ such that for any* *fixed* $N \geq N_1$ *and* $k \in \mathbb{N} \cup \{0\}$,

$$|\mathfrak{C}_k| < \frac{\epsilon}{(1000)^n} |B_{R_0}|. \tag{4.3.4}$$

Proof. For each $k \in \mathbb{N} \cup \{0\}$, $|\mathfrak{C}_k| \leq |\mathfrak{C}_0|$. Thus, it suffices to prove that (4.3.4) holds for $k = 0$. We have from (2.1.9) and (4.3.3) that

$$\begin{aligned} |\mathfrak{C}_0| &= \left| \left\{ x \in \Omega_{R_0} : \mathcal{M}\left(|Du|^{\frac{p(\cdot)}{p_-}} \chi_{\Omega_{4R_0}} \right)(x) > N\lambda_0 \right\} \right| \\ &\leq \frac{c}{N\lambda_0} \int_{\Omega_{4R_0}} |Du|^{\frac{p(x)}{p_-}} \, dx \leq \frac{c\epsilon}{N} |B_{R_0}| < \frac{\epsilon}{(1000)^n} |B_{R_0}|, \end{aligned}$$

by selecting N_1 large enough. □

Lemma 4.3.2. *There is a constant* $N_2 = N_2(n, \lambda, \Lambda, \gamma_1, \gamma_2, L) > 1$ *so that for any* $\epsilon > 0$, *there exists a small constant* $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, L, \epsilon) > 0$ *such that for any fixed* $N \geq N_2$, $k \in \mathbb{N} \cup \{0\}$, $y_0 \in \mathfrak{C}_k$ *and* $r_0 \leq \frac{R_0}{1000}$, *if*

$$|\mathfrak{C}_k \cap B_{r_0}(y_0)| \geq \epsilon |B_{r_0}(y_0)|, \tag{4.3.5}$$

then $B_{r_0}(y_0) \cap \Omega_{R_0} \subset \mathfrak{D}_k$.

Proof. We write $\lambda_k := N^k \lambda_0 > 1$, where $N \geq N_2 > 1$. The proof is by contradiction. Were $B_{r_0}(y_0) \cap \Omega_{R_0} \subset \mathfrak{D}_k$ false, there exists $y_1 \in B_{r_0}(y_0) \cap \Omega_{R_0}$ such that $y_1 \notin \mathfrak{D}_k$. Then we have

$$\begin{aligned} &\frac{1}{|B_r(y_1)|} \int_{B_r(y_1) \cap \Omega_{4R_0}} |Du|^{\frac{p(x)}{p_-}} \, dx \leq \lambda_k, \quad \text{and} \\ &\left[\frac{|\mu|(B_r(y_1))}{r^{n-1}} \right]^{\frac{p(y_1)}{p_-}} \leq c(n, \gamma_1, \gamma_2) \delta \lambda_k, \end{aligned} \tag{4.3.6}$$

for all $r > 0$.

Before proving this lemma, we outline the plan of the proof.

(i) We first divide the proof into two cases: $\mathcal{B}_{10r_0\sqrt{2}}(y_1) \subset \Omega$ and $\mathcal{B}_{10r_0\sqrt{2}}(y_1) \not\subset \Omega$.

(ii) We transfer the exponent powers in (4.3.6) from $\frac{p(\cdot)}{p_-}$ to 1, see (4.3.8) and (4.3.15).

(iii) We apply Lemma 4.2.7 (Lemma 4.2.6) to obtain the L^1 comparison estimates for the interior (boundary) case, see (4.3.9) and (4.3.16).

(iv) We transfer the exponent powers in the comparison estimates from 1 to $\frac{p(\cdot)}{p_-}$, see (4.3.11) and (4.3.17).

(v) We arrive at a contradiction by using standard technique of the covering argument mentioned in [37, 98].

Note that the log-Hölder continuity, from (ii) and (iv), is an essential ingredient in correcting the exponent powers.

Case 1. The interior case $\mathcal{B}_{10r_0\sqrt{2}}(y_1) \subset \Omega$.
Since $y_1 \in B_{r_0}(y_0) \cap \Omega_{R_0}$, we see that

$$\overline{\mathcal{B}_{8r_0}(y_0)} \subset B_{10r_0\sqrt{2}}(y_1) \subset \mathcal{B}_{10r_0\sqrt{2}}(y_1) \subset \Omega_{4R_0}.$$

We set

$$p_1 := \inf_{x \in \mathcal{B}_{8r_0}(y_0)} p(x) \quad \text{and} \quad p_2 := \sup_{x \in \mathcal{B}_{8r_0}(y_0)} p(x).$$

Then it follows that $p_2 - p_1 \le \omega(16r_0\sqrt{2})$, and for $x \in \mathcal{B}_{8r_0}(y_0)$,

$$1 < \gamma_1 \le p_- \le p_1 \le p(x) \le p_2 \le p_+ \le \gamma_2 < \infty.$$

Using Hölder's inequality, (4.3.1), (4.3.6), and (2.1.4), we have

$$\begin{aligned}
\fint_{\mathcal{B}_{8r_0}(y_0)} |Du|\, dx &= \left(\fint_{\mathcal{B}_{8r_0}(y_0)} |Du|\, dx \right)^{\frac{p_2-p_1}{p_2} + \frac{p_1}{p_2}} \\
&\le \left(\fint_{\mathcal{B}_{8r_0}(y_0)} |Du|\, dx + 1 \right)^{\frac{\omega(16r_0\sqrt{2})}{\gamma_1}} \left(\fint_{\mathcal{B}_{8r_0}(y_0)} |Du|^{\frac{p_1}{p_-}}\, dx \right)^{\frac{p_-}{p_2}} \\
&\le c \left(\frac{1}{r_0} \right)^{\frac{(n+1)\omega(16r_0\sqrt{2})}{\gamma_1}} \left(\fint_{B_{10r_0\sqrt{2}}(y_1)} |Du|^{\frac{p(x)}{p_-}}\, dx + 1 \right)^{\frac{p_-}{p_2}} \\
&\le c(n, \gamma_1, \gamma_2, L) \lambda_k^{\frac{p_-}{p_2}},
\end{aligned}$$

and it follows from (4.1.4) that for any $\tilde{\epsilon}_h \in (0,1)$ and $q \in \left[1, \frac{n}{n-1}\right)$,

$$\fint_{\mathcal{B}_{8r_0}(y_0)} |Du - Du_h|^q \, dx \le \tilde{\epsilon}_h \tag{4.3.7}$$

for h large enough. Then these estimates imply

$$\fint_{\mathcal{B}_{8r_0}(y_0)} |Du_h| \, dx \le c_1 \lambda_k^{\frac{p_-}{p_2}}$$

for some constant $c_1 = c_1(n, \gamma_1, \gamma_2, L) > 0$.

On the other hand, we compute from (4.3.1), (4.3.6), and (2.1.4) that

$$\begin{aligned}
\frac{|\mu|(\overline{\mathcal{B}_{8r_0}(y_0)})}{r_0^{n-1}} &= \left[\frac{|\mu|(\overline{\mathcal{B}_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{\frac{p_2 - p(y_1)}{p_2}} \left[\frac{|\mu|(\overline{\mathcal{B}_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{\frac{p(y_1)}{p_2}} \\
&\le \left(\frac{1}{r_0}\right)^{\frac{(n-1)\omega(16r_0\sqrt{2})}{\gamma_1}} (|\mu|(\Omega)+1)^{\frac{\omega(16r_0\sqrt{2})}{\gamma_1}} \left[\frac{|\mu|(B_{10r_0\sqrt{2}}(y_1))}{r_0^{n-1}}\right]^{\frac{p(y_1)}{p_2}} \\
&\le \left(\frac{1}{r_0}\right)^{\frac{n\omega(16r_0\sqrt{2})}{\gamma_1}} c\delta^{\frac{p_-}{p_2}} \lambda_k^{\frac{p_-}{p_2}} \\
&\le c(n, \gamma_1, \gamma_2, L) \delta^{\frac{\gamma_1}{\gamma_2}} \lambda_k^{\frac{p_-}{p_2}},
\end{aligned}$$

and so, we have from (4.1.2) that

$$\frac{|\mu_h|(\mathcal{B}_{8r_0}(y_0))}{r_0^{n-1}} \le \frac{|\mu|(\overline{\mathcal{B}_{8r_0}(y_0)})}{r_0^{n-1}} + \frac{\epsilon_h}{r_0^{n-1}} \le c\delta^{\frac{\gamma_1}{\gamma_2}} \lambda_k^{\frac{p_-}{p_2}} + \delta^{\frac{\gamma_1}{\gamma_2}} \le c_2 \delta^{\frac{\gamma_1}{\gamma_2}} \lambda_k^{\frac{p_-}{p_2}}$$

for some constant $c_2 = c_2(n, \gamma_1, \gamma_2, L) > 0$, by selecting ϵ_h sufficiently small with $\epsilon_h \le r_0^{n-1} \delta^{\frac{\gamma_1}{\gamma_2}}$.

Consequently, we obtain

$$\fint_{\mathcal{B}_{8r_0}(y_0)} |Du_h| \, dx \le c_3 \lambda_k^{\frac{p_-}{p_2}} \quad \text{and} \quad \frac{|\mu_h|(\mathcal{B}_{8r_0}(y_0))}{r_0^{n-1}} \le c_3 \delta^{\frac{\gamma_1}{\gamma_2}} \lambda_k^{\frac{p_-}{p_2}}, \tag{4.3.8}$$

where $c_3 := \max\{c_1, c_2\}$. Applying Lemma 4.2.7 with x_0, ρ, r, δ, and ϵ replaced by y_0, $c_3 \lambda_k^{\frac{p_-}{p_2}}$, r_0, $\delta^{\frac{\gamma_1}{\gamma_2}}$, and η, respectively, we find that there is

$\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \eta) > 0$ such that

$$\begin{aligned} &\fint_{\mathcal{B}_{4r_0}(y_0)} |Du_h - Dv_h| \, dx \le c_3 \eta \lambda_k^{\frac{p_-}{p_2}} \quad \text{and} \\ &\|Dv_h\|_{L^\infty(\mathcal{B}_{2r_0}(y_0))} \le c c_3 \lambda_k^{\frac{p_-}{p_2}} =: c_4 \lambda_k^{\frac{p_-}{p_2}} \end{aligned} \tag{4.3.9}$$

for some constant $c_4 = c_4(n, \lambda, \Lambda, \gamma_1, \gamma_2, L) > 0$. Then (4.3.7) and (4.3.9) imply

$$\fint_{\mathcal{B}_{2r_0}(y_0)} |Du - Dv_h| \, dx \le (4^n + 2^n) \, c_3 \eta \lambda_k^{\frac{p_-}{p_2}} =: c_5 \eta \lambda_k^{\frac{p_-}{p_2}} \tag{4.3.10}$$

by choosing $\tilde{\epsilon}_h$ sufficiently small with $\tilde{\epsilon}_h \le c_3 \eta$.

We next claim that

$$\begin{aligned} \fint_{\mathcal{B}_{2r_0}(y_0)} |Du - Dv_h|^{\frac{p(x)}{p_-}} \, dx &\le c_6 \eta^{\frac{1}{2}} \lambda_k, \\ \left\| |Dv_h|^{\frac{p(\cdot)}{p_-}} \right\|_{L^\infty(\mathcal{B}_{2r_0}(y_0))} &\le c_6 \lambda_k \end{aligned} \tag{4.3.11}$$

for some constant $c_6 = c_6(n, \lambda, \Lambda, \gamma_1, \gamma_2, L) > 0$.

Clearly, we compute from (4.3.9) that

$$\begin{aligned} \left\| |Dv_h|^{\frac{p(\cdot)}{p_-}} \right\|_{L^\infty(\mathcal{B}_{2r_0}(y_0))} &\le \sup_{x \in \mathcal{B}_{2r_0}(y_0)} (|Dv_h(x)| + 1)^{\frac{p_2}{p_-}} \\ &\le c \left(\|Dv_h\|_{L^\infty(\mathcal{B}_{2r_0}(y_0))}^{\frac{p_2}{p_-}} + 1 \right) \\ &\le c \lambda_k. \end{aligned}$$

Returning to (4.3.11), we have from Hölder's inequality and (4.3.10) that

$$\begin{aligned}
\fint_{\mathcal{B}_{2r_0}(y_0)} |Du - Dv_h|^{\frac{p(x)}{p_-}}\, dx &= \fint_{\mathcal{B}_{2r_0}(y_0)} |Du - Dv_h|^{\frac{1}{2}+\left(\frac{p(x)}{p_-}-\frac{1}{2}\right)}\, dx \\
&\le \left(\fint_{\mathcal{B}_{2r_0}(y_0)} |Du - Dv_h|\, dx\right)^{\frac{1}{2}} \left(\fint_{\mathcal{B}_{2r_0}(y_0)} |Du - Dv_h|^{2\frac{p(x)}{p_-}-1}\, dx\right)^{\frac{1}{2}} \\
&\le c\eta^{\frac{1}{2}}\lambda_k^{\frac{p_-}{2p_2}} \left(\fint_{\mathcal{B}_{2r_0}(y_0)} |Du - Dv_h|^{2\frac{p(x)}{p_-}-1}\, dx\right)^{\frac{1}{2}} =: c\eta^{\frac{1}{2}}\lambda_k^{\frac{p_-}{2p_2}} I_1.
\end{aligned}$$

It follows from (4.3.9) that

$$\begin{aligned}
I_1^2 &\le \fint_{\mathcal{B}_{2r_0}(y_0)} (|Du| + |Dv_h| + 1)^{2\frac{p_2}{p_-}-1}\, dx \\
&\le c\left\{\fint_{\mathcal{B}_{2r_0}(y_0)} |Du|^{2\frac{p_2}{p_-}-1}\, dx + \fint_{\mathcal{B}_{2r_0}(y_0)} |Dv_h|^{2\frac{p_2}{p_-}-1}\, dx + 1\right\} \\
&\le c\left\{\fint_{\mathcal{B}_{2r_0}(y_0)} |Du|^{2\frac{p_2}{p_-}-1}\, dx + \lambda_k^{2-\frac{p_-}{p_2}}\right\} =: c\left(I_2 + \lambda_k^{2-\frac{p_-}{p_2}}\right).
\end{aligned}$$

Continuously, we discover from (4.3.7), Lemma 4.2.1, Lemma 4.2.2, and (4.3.8) that

$$\begin{aligned}
I_2 &\le c\fint_{\mathcal{B}_{2r_0}(y_0)} |Du - Du_h|^{2\frac{p_2}{p_-}-1}\, dx + c\fint_{\mathcal{B}_{2r_0}(y_0)} |Du_h - Dw_h|^{2\frac{p_2}{p_-}-1}\, dx \\
&\quad + c\fint_{\mathcal{B}_{2r_0}(y_0)} |Dw_h|^{2\frac{p_2}{p_-}-1}\, dx \\
&\le c\left\{\tilde{\epsilon}_h\lambda_k^{2-\frac{p_-}{p_2}} + \left[\frac{|\mu_h|(\mathcal{B}_{2r_0}(y_0))}{r_0^{n-1}}\right]^{2\frac{p_2}{p_-}-1} + \left(\fint_{\mathcal{B}_{4r_0}(y_0)} |Dw_h|\, dx\right)^{2\frac{p_2}{p_-}-1}\right\} \\
&\le c\lambda_k^{2-\frac{p_-}{p_2}},
\end{aligned}$$

since $2\frac{p_2}{p_-} - 1 \le 1 + 2\frac{p_+ - p_-}{p_-} \le 1 + 2\frac{\omega(8R_0\sqrt{2})}{\gamma_1} \le 1 + \min\left\{\frac{1}{2(n-1)}, \sigma_0\right\}$, according to (4.3.2). Then we combine these estimates above, to obtain

$$\fint_{\mathcal{B}_{2r_0}(y_0)} |Du - Dv_h|^{\frac{p(x)}{p_-}}\, dx \le c_6 \eta^{\frac{1}{2}} \lambda_k$$

for some $c_6 = c_6(n, \lambda, \Lambda, \gamma_1, \gamma_2, L) > 0$. We thereby establish the claim (4.3.11).

Next we claim that

$$\begin{aligned} \mathfrak{C}_k \cap B_{r_0}(y_0) &= \left\{ x \in B_{r_0}(y_0) : \mathcal{M}\left(|Du|^{\frac{p(\cdot)}{p_-}} \chi_{\Omega_{4R_0}} \right)(x) > N\lambda_k \right\} \\ &\subset \left\{ x \in B_{r_0}(y_0) : \mathcal{M}\left(|Du - Dv_h|^{\frac{p(\cdot)}{p_-}} \chi_{B_{2r_0}(y_0)} \right)(x) > \lambda_k \right\} =: J_k, \end{aligned} \tag{4.3.12}$$

provided $N \ge N_2 \ge \max\left\{ 2^{\frac{\gamma_2}{\gamma_1}-1}(1 + c_6), 3^n \right\}$.

Let $y \notin J_k$. If $y \notin B_{r_0}(y_0)$, then (4.3.12) is done. Suppose $y \in B_{r_0}(y_0)$. If $\tilde{r} < r_0$, then $B_{\tilde{r}}(y) \subset B_{2r_0}(y_0) \subset \Omega_{4R_0}$. It follows from (4.3.11) that

$$\begin{aligned} |Du(x)|^{\frac{p(x)}{p_-}} &\le 2^{\frac{\gamma_2}{\gamma_1}-1} \left(|Du(x) - Dv_h(x)|^{\frac{p(x)}{p_-}} + |Dv_h(x)|^{\frac{p(x)}{p_-}} \right) \\ &\le 2^{\frac{\gamma_2}{\gamma_1}-1} \left(|Du(x) - Dv_h(x)|^{\frac{p(x)}{p_-}} + c_6 \lambda_k \right) \end{aligned}$$

for almost every $x \in B_{\tilde{r}}(y)$. Integrating over $B_{\tilde{r}}(y)$ gives

$$\begin{aligned} \fint_{B_{\tilde{r}}(y)} |Du|^{\frac{p(x)}{p_-}}\, dx &\le 2^{\frac{\gamma_2}{\gamma_1}-1} \left\{ \mathcal{M}\left(|Du - Dv_h|^{\frac{p(\cdot)}{p_-}} \chi_{B_{2r_0}(y_0)} \right)(y) + c_6 \lambda_k \right\} \\ &\le 2^{\frac{\gamma_2}{\gamma_1}-1}(1 + c_6)\lambda_k. \end{aligned}$$

If $\tilde{r} \ge r_0$, then $B_{\tilde{r}}(y) \subset B_{2\tilde{r}}(y_0) \subset B_{3\tilde{r}}(y_1)$. We have from (4.3.6) that

$$\frac{1}{|B_{\tilde{r}}|} \int_{B_{\tilde{r}}(y) \cap \Omega_{4R_0}} |Du|^{\frac{p(x)}{p_-}}\, dx \le \frac{3^n}{|B_{3\tilde{r}}|} \int_{B_{3\tilde{r}}(y_1) \cap \Omega_{4R_0}} |Du|^{\frac{p(x)}{p_-}}\, dx \le 3^n \lambda_k.$$

Consequently, we obtain

$$\mathcal{M}\left(|Du|^{\frac{p(\cdot)}{p_-}} \chi_{\Omega_{4R_0}} \right)(y) \le \max\left\{ 2^{\frac{\gamma_2}{\gamma_1}-1}(1 + c_6)\lambda_k, 3^n \lambda_k \right\}.$$

Choosing $N_2 \geq \max\left\{2^{\frac{\gamma_2}{\gamma_1}-1}(1+c_6), 3^n\right\}$, we have $y \notin \mathfrak{C}_k \cap B_{r_0}(y_0)$, that is, the claim (4.3.12) holds.

We finally conclude, using (4.3.12), (2.1.9), and (4.3.11), that

$$\begin{aligned}|\mathfrak{C}_k \cap B_{r_0}(y_0)| &\leq \left|\left\{x \in B_{r_0}(y_0) : \mathcal{M}\left(|Du - Dv_h|^{\frac{p(\cdot)}{p_-}} \chi_{B_{2r_0}(y_0)}\right)(x) > \lambda_k\right\}\right| \\ &\leq \frac{c}{\lambda_k} \int_{B_{2r_0}(y_0)} |Du - Dv_h|^{\frac{p(x)}{p_-}} \, dx \leq cc_6 \eta^{\frac{1}{2}} |B_{r_0}(y_0)| < \epsilon |B_{r_0}(y_0)|,\end{aligned}$$

by selecting η and δ that satisfy the last inequality above, which is a contradiction to (4.3.5).

Case 2. The boundary case $\mathcal{B}_{10r_0\sqrt{2}}(y_1) \not\subset \Omega$.
We find a boundary point $\tilde{y}_1 \in \partial\Omega \cap \mathcal{B}_{10r_0\sqrt{2}}(y_1)$. Since $540r_0 \leq R_0 \leq \frac{R}{8\sqrt{2}}$ and the domain Ω is (δ, R)-Reifenberg flat of codimension 1, there exists a coordinate system, which we still denote $x = (x_1, \cdots, x_n)$, with the origin at $\tilde{y}_1 + 480\delta r_0 e_n$, such that

$$\mathcal{B}^+_{480r_0}(0) \subset \Omega_{480r_0}(0) \subset \mathcal{B}_{480r_0}(0) \cap \{x_n > -960\delta r_0\}, \tag{4.3.13}$$

and

$$\fint_{\mathcal{B}_{480r_0}} |\theta(\mathbf{a}, \mathcal{B}_{480r_0})(x)| \, dx \leq \delta.$$

We select δ so small with $0 < \delta < \frac{1}{24}$. Then we have

$$\begin{aligned}&\Omega_{2r_0}(y_0) \subset \Omega_{3r_0}(y_1) \subset \Omega_{120r_0}(0), \quad \text{and} \\ &\Omega_{480r_0}(0) \subset \Omega_{540r_0}(y_1) \subset \Omega_{R_0}(y_0) \subset \Omega_{4R_0},\end{aligned} \tag{4.3.14}$$

since $|y_1| \leq |y_1 - \tilde{y}_1| + |\tilde{y}_1| \leq 20r_0 + 480\delta r_0 \leq 40r_0$ in the new coordinate. We denote

$$p_1 := \inf_{x \in \Omega_{480r_0}(0)} p(x) \quad \text{and} \quad p_2 := \sup_{x \in \Omega_{480r_0}(0)} p(x).$$

Then $p_2 - p_1 \leq \omega(960r_0)$.

We next obtain the estimates in the boundary case corresponding to

(4.3.8). From (2.1.4), (4.3.1), (4.3.6), (4.3.13), and (4.3.14), we have

$$\fint_{\Omega_{480r_0}(0)} |Du|\,dx \le c_7\lambda_k^{\frac{p_-}{p_2}} \quad \text{and} \quad \frac{|\mu|(\overline{\Omega_{480r_0}(0)})}{r_0^{n-1}} \le c_7\delta^{\frac{\gamma_1}{\gamma_2}}\lambda_k^{\frac{p_-}{p_2}} \tag{4.3.15}$$

for some constant $c_7 = c_7(n,\gamma_1,\gamma_2,L) > 0$. Furthermore, it follows from (4.1.4), (4.1.2) and (4.3.15) that

$$\fint_{\Omega_{480r_0}(0)} |Du_h|\,dx \le c_8\lambda_k^{\frac{p_-}{p_2}} \quad \text{and} \quad \frac{|\mu_h|(\Omega_{480r_0}(0))}{r_0^{n-1}} \le c_8\delta^{\frac{\gamma_1}{\gamma_2}}\lambda_k^{\frac{p_-}{p_2}}$$

for h large enough and some constant $c_8 = c_8(n,\gamma_1,\gamma_2,L) > 0$. Applying Lemma 4.2.6 with ρ, r, δ, and ϵ replaced by $c_8\lambda_k^{\frac{p_-}{p_2}}$, $60r_0$, $\delta^{\frac{\gamma_1}{\gamma_2}}$, and η, respectively, we deduce that there exists $\delta = \delta(n,\lambda,\Lambda,\gamma_1,\gamma_2,\eta) > 0$ such that

$$\begin{aligned} &\fint_{\Omega_{180r_0}(0)} |Du_h - D\bar{v}_h|\,dx \le c_8\eta\lambda_k^{\frac{p_-}{p_2}} \quad \text{and} \\ &\|D\bar{v}_h\|_{L^\infty(\Omega_{120r_0}(0))} \le cc_8\lambda_k^{\frac{p_-}{p_2}} =: c_9\lambda_k^{\frac{p_-}{p_2}} \end{aligned} \tag{4.3.16}$$

for some constant $c_9 = c_9(n,\lambda,\Lambda,\gamma_1,\gamma_2,L) > 0$. Then (4.1.4) and (4.3.16) imply

$$\fint_{\Omega_{120r_0}(0)} |Du - D\bar{v}_h|\,dx \le c_{10}\eta\lambda_k^{\frac{p_-}{p_2}}.$$

Proceeding as in *Case 1*, we infer

$$\begin{aligned} &\fint_{\Omega_{120r_0}(0)} |Du - D\bar{v}_h|^{\frac{p(x)}{p_-}}\,dx \le c_{11}\eta^{\frac{1}{2}}\lambda_k \quad \text{and} \\ &\left\| |D\bar{v}_h|^{\frac{p(\cdot)}{p_-}} \right\|_{L^\infty(\Omega_{120r_0}(0))} \le c_{11}\lambda_k \end{aligned} \tag{4.3.17}$$

for some constant $c_{11} = c_{11}(n,\lambda,\Lambda,\gamma_1,\gamma_2,L) > 0$, and moreover

$$\begin{aligned} \mathfrak{C}_k \cap B_{r_0}(y_0) &= \left\{ x \in \Omega_{r_0}(y_0) : \mathcal{M}\left(|Du|^{\frac{p(\cdot)}{p_-}}\chi_{\Omega_{4R_0}} \right)(x) > N\lambda_k \right\} \\ &\subset \left\{ x \in \Omega_{r_0}(y_0) : \mathcal{M}\left(|Du - D\bar{v}_h|^{\frac{p(\cdot)}{p_-}}\chi_{\Omega_{2r_0}(y_0)} \right)(x) > \lambda_k \right\} \end{aligned} \tag{4.3.18}$$

provided $N \geq N_2 \geq \max\left\{2^{\frac{\gamma_2}{\gamma_1}-1}(1+c_{11}), 3^n\right\}$.

Finally, we conclude from (4.3.18), (2.1.9), (4.3.14) and (4.3.17) that

$$\begin{aligned}|\mathfrak{C}_k \cap B_{r_0}(y_0)| &\leq \left|\left\{x \in \Omega_{r_0}(y_0) : \mathcal{M}\left(|Du - D\bar{v}_h|^{\frac{p(\cdot)}{p_-}}\chi_{\Omega_{2r_0}(y_0)}\right)(x) > \lambda_k\right\}\right| \\ &\leq \frac{c}{\lambda_k}\int_{\Omega_{2r_0}(y_0)} |Du - D\bar{v}_h|^{\frac{p(x)}{p_-}}\,dx \\ &\leq \frac{c|\Omega_{120r_0}(0)|}{\lambda_k} \fint_{\Omega_{120r_0}(0)} |Du - D\bar{v}_h|^{\frac{p(x)}{p_-}}\,dx \\ &\leq c c_{11}\eta^{\frac{1}{2}}|B_{r_0}(y_0)| < \epsilon|B_{r_0}(y_0)|\end{aligned}$$

by taking η sufficiently small. As a consequence $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, L, \epsilon)$ is also determined. This is a contradiction to (4.3.5). $\square$

Choosing $N = \max\{N_1, N_2\}$ from Lemma 4.3.1 and Lemma 4.3.2, we can apply Lemma 2.1.2 to derive the following power decay estimates:

Corollary 4.3.3. *Under the same assumptions as in Lemma 4.3.1 and Lemma 4.3.2, we have*

$$|\mathfrak{C}_k| \leq \left(\frac{80}{7}\right)^n \epsilon|\mathfrak{D}_k| =: \epsilon_1|\mathfrak{D}_k| \quad \textit{for} \quad k \in \mathbb{N} \cup \{0\}.$$

In addition, by iteration, we obtain

$$\begin{aligned}&\left|\left\{x \in \Omega_{R_0} : \mathcal{M}\left(|Du|^{\frac{p(\cdot)}{p_-}}\chi_{\Omega_{4R_0}}\right)(x) > N^k\lambda_0\right\}\right| \\ &\quad\leq \epsilon_1^k\left|\left\{x \in \Omega_{R_0} : \mathcal{M}\left(|Du|^{\frac{p(\cdot)}{p_-}}\chi_{\Omega_{4R_0}}\right)(x) > \lambda_0\right\}\right| \\ &\qquad+ \sum_{i=1}^{k}\epsilon_1^i\left|\left\{x \in \Omega_{R_0} : [\mathcal{M}_1(\mu)(x)]^{\frac{p(x)}{p_-}} > \delta N^{k-i}\lambda_0\right\}\right|.\end{aligned} \tag{4.3.19}$$

4.4 Calderón-Zygmund type estimates

We first obtain standard energy type estimate for the problem (4.0.1), which will be used to prove Theorem 4.1.4.

Lemma 4.4.1. *Assume that* (4.0.2) *and* (4.0.3), *and let* $1 \le q < \frac{n}{n-1}$. *If* u *is a SOLA of* (4.0.1), *then there exist positive constants* $\bar{c}_1$ *and* $\bar{c}_2$, *depending only on* n, λ, q, *and* Ω, *such that*

$$\int_\Omega |Du|^q \, dx \le \bar{c}_1 |\mu|(\Omega) \le \bar{c}_2 \int_\Omega \mathcal{M}_1(\mu) \, dx. \tag{4.4.1}$$

Proof. Since $|\mu|(\Omega) \le diam(\Omega)^{n-1}\mathcal{M}_1(\mu)(x)$ for all $x \in \Omega$, the second inequality of (4.4.1) holds. We set

$$\tilde{u}(y) = \frac{u(x_0 + ry)}{Ar}, \quad \tilde{\mu}(y) = \frac{r\mu(x_0 + ry)}{A}, \quad \text{and} \quad \tilde{\mathbf{a}}(A\xi, y) = \frac{\mathbf{a}(A\xi, x_0 + ry)}{A},$$

where $r := diam(\Omega)$, $x_0 \in \Omega$,

$$A := \frac{|\mu|(\Omega)}{r^{n-1}}, \quad \text{and} \quad \tilde{\Omega} := \{y \in \mathbb{R}^n : x_0 + ry \in \Omega\} \subset B_1.$$

Here we extend u and μ by zero to $\mathbb{R}^n$. Then we see that the operator $\tilde{\mathbf{a}}$ satisfies (4.0.2) and (4.0.3), and $\tilde{u}$ is a SOLA of the following problem

$$\begin{cases} -\operatorname{div} \tilde{\mathbf{a}}(D\tilde{u}, y) = \tilde{\mu} & \text{in } \tilde{\Omega}, \\ \tilde{u} = 0 & \text{on } \partial\tilde{\Omega}. \end{cases} \tag{4.4.2}$$

Fix $q \in [1, \frac{n}{n-1})$. If $\int_{\tilde{\Omega}} |D\tilde{u}| \, dy = \int_{B_1} |D\tilde{u}| \, dy \le c$, then $\int_\Omega |Du| \, dx \le cr^n A$, that is, the first inequality of (4.4.1) holds. Thus, it suffices to show that $\int_{B_1} |D\tilde{u}| \, dy \le c$.

Consider the regularized problem (4.1.1) with u_h replaced by $\tilde{u}_h$. We denote, for $k \in \mathbb{N}$,

$$D_k := \{y \in B_1 : |\tilde{u}_h(y)| \le k\} \quad \text{and} \quad C_k := \{y \in B_1 : k < |\tilde{u}_h(y)| \le k+1\}.$$

Then (4.1.3) implies

$$\int_{D_k} |D\tilde{u}_h|^2 \, dy \le ck \quad \text{and} \quad \int_{C_k} |D\tilde{u}_h|^2 \, dy \le c$$

by substituting test functions $T_k(\tilde{u}_h)$ and $\Phi_k(\tilde{u}_h)$, respectively, into the weak formulation of (4.4.2). Here the functions T_k and Φ_k are given in (3.2.9). We discover

$$\int_{D_k} |D\tilde{u}_h|^q \, dy \le \int_{D_k} \left(|D\tilde{u}_h| + 1\right)^2 \, dy \le c(k+1).$$

From the definition of C_k, we see

$$|C_k| = \int_{C_k} 1 \, dy \le \int_{C_k} \left(\frac{|\tilde{u}_h|}{k}\right)^{\frac{nq}{n-q}} \, dy = k^{-\frac{nq}{n-q}} \int_{C_k} |\tilde{u}_h|^{\frac{nq}{n-q}} \, dy.$$

It therefore follows from Hölder's inequality that

$$\int_{C_k} |D\tilde{u}_h|^q \, dy \le ck^{-\frac{nq(2-q)}{2(n-q)}} \left(\int_{C_k} |\tilde{u}_h|^{\frac{nq}{n-q}} \, dy\right)^{\frac{2-q}{2}}.$$

Then we discover from Hölder's and Sobolev's inequality that for $k_0 \in \mathbb{N}$,

$$\begin{aligned}
\int_{B_1} |D\tilde{u}_h|^q \, dy &\le c(k_0+1) + c\sum_{k=k_0}^{\infty} k^{-\frac{nq(2-q)}{2(n-q)}} \left(\int_{C_k} |\tilde{u}_h|^{\frac{nq}{n-q}} \, dy\right)^{\frac{2-q}{2}} \\
&\le c(k_0+1) + c\left[\sum_{k=k_0}^{\infty} k^{-\frac{n(2-q)}{n-q}}\right]^{\frac{q}{2}} \left(\sum_{k=k_0}^{\infty} \int_{C_k} |\tilde{u}_h|^{\frac{nq}{n-q}} \, dy\right)^{\frac{2-q}{2}} \\
&\le c(k_0+1) + cH(k_0) \left(\int_{B_1} |D\tilde{u}_h|^q \, dy\right)^{\frac{n(2-q)}{2(n-q)}},
\end{aligned}$$

where $H(k_0) := \left[\sum_{k=k_0}^{\infty} k^{-\frac{n(2-q)}{n-q}}\right]^{\frac{q}{2}}$. Note that $\frac{n(2-q)}{n-q} > 1$, since $q < \frac{n}{n-1}$.

For $n > 2$, we know $0 < \frac{n(2-q)}{2(n-q)} < 1$, and then the above estimate and Young's inequality yield

$$\int_{B_1} |D\tilde{u}_h|^q \, dy \le c(n, \lambda, q, \Omega) \tag{4.4.3}$$

by putting $k_0 = 1$. For $n = 2$, we know $\frac{n(2-q)}{2(n-q)} = 1$. We take an integer $k_0 > 1$ so that $cH(k_0) < \frac{1}{2}$. Then (4.4.3) also holds. Using this estimate and letting h go to zero, we conclude from (4.1.4) that $\int_{B_1} |D\tilde{u}|^q \, dy \le c$, which completes the proof. $\square$

4.4.1 Local estimates

We first obtain local estimates for the problem (4.0.1).

Proof of (4.1.7). We first recall (4.3.1) and (4.3.2). Fix any $R_0 \in \left(0, \frac{1}{c_0(|\mu|(\Omega)+1)}\right]$ with

$$\frac{1}{c_0(n,\lambda,\Lambda,\gamma_1,\omega(\cdot),R,\Omega)} := \min\left\{\frac{1}{16\sqrt{2}}, \frac{R}{8\sqrt{2}}, \frac{1}{\bar{c}_1}, \frac{\omega^{-1}(d)}{8\sqrt{2}}\right\}, \tag{4.4.4}$$

where the constant $\bar{c}_1$ is given in Lemma 4.4.1 ($q = 1$),

$$d := \min\left\{1, \frac{\sigma_0\gamma_1}{2}, \frac{\gamma_1}{4(n-1)}\right\}, \text{ and } \omega^{-1}(t) := \sup\{r \in (0,1) : \omega(r) \le t\}$$

for $t > 0$. Note that the function ω^{-1} is well defined by the definition of ω. Then we see from Lemma 4.4.1 that this R_0 above satisfies (4.3.1) and (4.3.2), and one can apply all the results obtained in Chapter 4.3 as follows.

Set

$$S := \sum_{k=1}^{\infty} N^{kp_-} \left|\left\{x \in \Omega_{R_0}(x_0) : \mathcal{M}\left(|Du|^{\frac{p(\cdot)}{p_-}} \chi_{\Omega_{4R_0}(x_0)}\right)(x) > N^k\lambda_0\right\}\right|,$$

where λ_0 and N are given in (4.3.3) and (4.3.19), respectively, and $p_- := \inf_{x\in\Omega_{4R_0}(x_0)} p(x)$. Then we deduce from (4.3.19), Fubini's theorem, and Lemma 2.1.4 that

$$S \le \sum_{i=1}^{\infty} (N^{\gamma_2}\epsilon_1)^i \left\{2|\Omega_{R_0}(x_0)| + \frac{c}{(\delta\lambda_0)^{p_-}} \int_{\Omega_{R_0}(x_0)} \mathcal{M}_1(\mu)^{\frac{p(x)}{p_-}}\, dx\right\}.$$

Now we select ϵ_1 with $N^{\gamma_2}\epsilon_1 = \frac{1}{2}$, and then we can take ϵ and a corresponding $\delta = \delta(n,\lambda,\Lambda,\gamma_1,\gamma_2,L) > 0$. Consequently, we obtain

$$S \le 2|\Omega_{R_0}(x_0)| + \frac{c}{\lambda_0^{p_-}} \int_{\Omega_{R_0}(x_0)} \mathcal{M}_1(\mu)^{\frac{p(x)}{p_-}}\, dx. \tag{4.4.5}$$

Finally, according to Lemma 2.1.4, (4.4.5), and (4.3.3), we conclude

$$\begin{aligned}
\fint_{\Omega_{R_0}(x_0)} |Du|^{p(x)}\,dx &\le \fint_{\Omega_{R_0}(x_0)} \mathcal{M}\left(|Du|^{\frac{p(\cdot)}{p_-}}\chi_{\Omega_{4R_0}(x_0)}\right)^{p_-} dx \\
&\le c\lambda_0^{p_-}\left(1+\frac{S}{|\Omega_{R_0}(x_0)|}\right) \le c\left\{\lambda_0^{p_-} + \fint_{\Omega_{R_0}(x_0)} \mathcal{M}_1(\mu)^{\frac{p(x)}{p_-}}\,dx\right\} \\
&\le c\left\{\left(\fint_{\Omega_{4R_0}(x_0)} |Du|^{\frac{p(x)}{p_-}}\,dx + 1\right)^{p_-} + \fint_{\Omega_{4R_0}(x_0)} \mathcal{M}_1(\mu)^{p(x)}\,dx + 1\right\}
\end{aligned}$$

for some constant $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, L) > 0$. This completes the proof. □

4.4.2 Global estimates

Now we extend the local estimates (4.1.7) up to the boundary by a standard covering argument.

Proof of (4.1.8). Let $R_0 = \frac{1}{c_0(|\mu|(\Omega)+1)}$, where c_0 is given in (4.4.4). Since $\bar{\Omega}$ is compact, we can cover $\bar{\Omega}$ by a collection of finitely many balls, each of which has radius $\frac{R_0}{3}$ and center in Ω. By the Vitali covering lemma, there exists a finite family of disjoint open balls $\left\{B_{\frac{R_0}{3}}(y_i)\right\}_{i=1}^{m}$, $y_i \in \Omega$, such that $\bar{\Omega} \subset \bigcup_{i=1}^{m} B_{R_0}(y_i)$. Note that there is a constant c depending only on the dimension n so that

$$\sum_{i=1}^{m} \int_{\Omega_{4R_0}(y_i)} f\,dx \le c\int_{\Omega} f\,dx.$$

Then our applying the estimate (4.1.7) with y_i ($i \in \mathbb{N}$) in place of x_0 yields

$$\begin{aligned}
\int_{\Omega} |Du|^{p(x)}\,dx &\le \sum_{i=1}^{m}\int_{\Omega_{R_0}(y_i)} |Du|^{p(x)}\,dx \\
&\le c\sum_{i=1}^{m}\left\{R_0^n\left(\fint_{\Omega_{4R_0}(y_i)} [|Du|+1]^{\frac{p_{i+}}{p_{i-}}}\,dx\right)^{p_{i-}} + \int_{\Omega_{4R_0}(y_i)} \left[\mathcal{M}_1(\mu)^{p(x)}+1\right]\,dx\right\} \\
&\le c\left\{R_0^{n(1-\gamma_2)}\left(\int_{\Omega} [|Du|+1]^{1+\frac{1}{4(n-1)}}\,dx\right)^{\gamma_2} + \int_{\Omega} \left[\mathcal{M}_1(\mu)^{p(x)}+1\right]\,dx\right\},
\end{aligned}$$

since

$$\frac{p_{i+}}{p_{i-}} = 1 + \frac{p_{i+} - p_{i-}}{p_{i-}} \le 1 + \frac{\omega(8R_0\sqrt{2})}{\gamma_1} \le 1 + \frac{1}{4(n-1)} < \frac{n}{n-1},$$

where $p_{i-} := \inf_{x\in\Omega_{4R_0}(y_i)} p(x)$, and $p_{i+} := \sup_{x\in\Omega_{4R_0}(y_i)} p(x)$.

Finally, we obtain, using Lemma 4.4.1 and Hölder's inequality, that

$$\begin{aligned}
&\int_\Omega |Du|^{p(x)}\,dx \\
&\quad \le c\left\{\left(\int_\Omega \mathcal{M}_1(\mu)\,dx\right)^{n(\gamma_2-1)}\left(\int_\Omega \mathcal{M}_1(\mu)\,dx\right)^{\gamma_2} + \int_\Omega \mathcal{M}_1(\mu)^{p(x)}\,dx + 1\right\} \\
&\quad \le c\left\{\left(\int_\Omega \mathcal{M}_1(\mu)^{\gamma_1}\,dx\right)^{\frac{n(\gamma_2-1)+\gamma_2}{\gamma_1}} + \int_\Omega \mathcal{M}_1(\mu)^{p(x)}\,dx + 1\right\} \\
&\quad \le c\left\{\left(\int_\Omega [\mathcal{M}_1(\mu)+1]^{p(x)}\,dx\right)^{\frac{n(\gamma_2-1)+\gamma_2}{\gamma_1}} + 1\right\}
\end{aligned}$$

for some constant $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), L, R, \Omega) > 0$. □

We are now in a position to prove the desired estimate (4.1.9).

Proof of (4.1.9). First let us define

$$\bar{u} = \frac{u}{A}, \quad \bar{\mu} = \frac{\mu}{A}, \quad \text{and} \quad \bar{\mathbf{a}}(\xi, x) = \frac{\mathbf{a}(A\xi, x)}{A},$$

for some positive constant $A > 0$. Then it readily check that $\bar{\mathbf{a}}$ satisfies (4.0.2) and (4.0.3), $(\bar{\mathbf{a}}, \Omega)$ is (δ, R)-vanishing of codimension 1, and $\bar{u}$ is a SOLA of the following problem

$$\begin{cases} -\operatorname{div} \bar{\mathbf{a}}(D\bar{u}, x) = \bar{\mu} & \text{in } \Omega, \\ \bar{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Set $A := ||\mathcal{M}_1(\mu)||_{L^{p(\cdot)}(\Omega)}$. We may as well assume $||\mathcal{M}_1(\mu)||_{L^{p(\cdot)}(\Omega)} > 0$. Then we know $||\mathcal{M}_1(\bar{\mu})||_{L^{p(\cdot)}(\Omega)} = 1$.

On the other hand, (2.1.2) implies that $\int_\Omega \mathcal{M}_1(\bar{\mu})^{p(x)}\,dx = 1$. Furthermore, in light of (2.1.2) and (4.1.8), we have $||D\bar{u}||_{L^{p(\cdot)}(\Omega)} \le c^{\frac{1}{\gamma_1}}$. Consequently we conclude that

$$\|Du\|_{L^{p(\cdot)}(\Omega)} \leq c\,\|\mathcal{M}_1(\mu)\|_{L^{p(\cdot)}(\Omega)}$$

for some constant $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), L, R, \Omega) > 0$. $\square$

Chapter 5

Global weighted Orlicz estimates for parabolic measure data problems: Application to estimates in variable exponent spaces

In this chapter we study the Cauchy-Dirichlet problem with measure data

$$\begin{cases} u_t - \operatorname{div} \mathbf{a}(Du, x, t) &= \mu \quad \text{in } \Omega_T, \\ u &= 0 \quad \text{on } \partial_p \Omega_T, \end{cases} \tag{5.0.1}$$

where the nonhomogeneous term μ is a signed Radon measure on Ω_T with finite total variation $|\mu|(\Omega_T) < \infty$. We assume that μ is defined in $\mathbb{R}^{n+1}$ by considering the zero extension to $\mathbb{R}^{n+1}$, and the nonlinearity $\mathbf{a} = \mathbf{a}(\xi, x, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is assumed to be measurable in x and t and satisfies the following structure conditions:

$$\begin{cases} |\xi||D_\xi \mathbf{a}(\xi, x, t)| + |\mathbf{a}(\xi, x, t)| \leq \Lambda|\xi|, \\ \lambda|\eta|^2 \leq \langle D_\xi \mathbf{a}(\xi, x, t)\eta, \eta\rangle, \end{cases} \tag{5.0.2}$$

for every $x, \eta \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n \setminus \{0\}$, and some positive constants λ, Λ.

The aim of Chapter 5 is to develop a global Calderón-Zygmund type estimate for a solution to the problem (5.0.1) in weighted Orlicz spaces. More precisely, we deduce a weighted norm inequality for the spatial gradient of a solution to (5.0.1) in the setting of Orlicz spaces by essentially proving that for any pair $(w, \Phi) \in (A_\infty, \Delta_2 \cap \nabla_2)$, there holds

$$\int_{\Omega_T} \Phi(|Du|)w \, dxdt \leq c \int_{\Omega_T} \Phi(\mathcal{M}_1(\mu))w \, dxdt$$

for some positive constant c, being independent of u and μ, under possibly optimal regularity assumptions on both $\mathbf{a}$ and Ω, see Chapter 5.1 for details. Here $\mathcal{M}_1(\mu)$ is the fractional maximal function of order 1 for μ defined by

$$\mathcal{M}_1(\mu)(x,t) := \sup_{r>0} \frac{|\mu|(Q_r(x,t))}{r^{n+1}} \quad \text{for } (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{5.0.3}$$

The present work improves the previous works [25, 31], where weak solutions are considered, to a very general extent. When dealing with a measure μ on the right-hand side of (5.0.1), it does not generally belong to the dual of the energy space $C([0,T]; L^2(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega))$, and so we need a very general notion of solution beyond that of weak solution, as we will see in Chapter 5.1.

The nonlinearity $\mathbf{a}(\xi, \cdot, \cdot)$ is allowed to be merely measurable with respect to one of the spatial variables and of small bounded mean oscillation (BMO) in the other spatial variables and time variable. This makes it different from the very interesting measure data problems [82, 83] in which the nonlinearity $\mathbf{a}$ has a small BMO in the all spatial variables.

Our proof consists in showing a decay estimate of the upper-level sets of the maximal function of $|Du|$ by means of $\mathcal{M}_1(\mu)$ in the frame of weighted Orlicz spaces. In the process we make difference estimates by comparing the problem (5.0.1) with an associated homogeneous problem, see Chapter 5.2.

A main point in Chapter 5 is to validate a practical application of the extrapolation theorem [44–46] to the setting of variable exponent spaces. This application confirms that obtaining weighted L^p estimates for the problem (5.0.1) essentially leads to obtaining $L^{p(\cdot)}$ estimates. Indeed, the application of the extrapolation and our result (Theorem 5.1.4) yield the $L^{p(\cdot)}$ estimate of the gradient of a solution to (5.0.1), see Chapter 5.3.

5.1 Main results

A solution u of (5.0.1) will be treated in the sense of distribution. This does not in general belong to a weak solution in $C([0,T];L^2(\Omega))\cap L^2(0,T;W_0^{1,2}(\Omega))$ (consider the heat equation with the Dirac measure). For this reason, it is necessary to extend a class of solutions below the natural exponent.

Definition 5.1.1. *$u \in L^1(0,T;W_0^{1,1}(\Omega))$ is a* SOLA *to* (5.0.1) *if the nonlinearity $\mathbf{a} \in L^1(\Omega_T, \mathbb{R}^n)$,*

$$\int_{\Omega_T} -u\varphi_t + \langle \mathbf{a}(Du,x,t), D\varphi\rangle \; dxdt = \int_{\Omega_T} \varphi \, d\mu$$

holds for all $\varphi \in C_c^\infty(\Omega_T)$, and moreover there exists a sequence of weak solutions $\{u_h\}_{h\geq 1} \subset C([0,T];L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega))$ of the regularized problems

$$\begin{cases} (u_h)_t - \operatorname{div} \mathbf{a}(Du_h,x,t) &= \mu_h \quad in \ \Omega_T, \\ u_h &= 0 \quad on \ \partial_p\Omega_T \end{cases} \tag{5.1.1}$$

such that

$$u_h \to u \quad in \ \ L^1(0,T;W_0^{1,1}(\Omega)) \quad as \ h \to \infty,$$

where $\mu_h \in L^\infty(\Omega_T)$ converges weakly to μ in the sense of measure and satisfies that for each open set $Q \subset \mathbb{R}^{n+1}$,

$$\limsup_{h\to\infty} |\mu_h|(Q) \leq |\mu|(\bar{Q}), \tag{5.1.2}$$

with μ_h defined in $\mathbb{R}^{n+1}$ by the zero extension of μ_h to $\mathbb{R}^{n+1}\setminus\Omega_T$.

Here we regard $\{\mu_h\}$ as a sequence of smooth functions derived from the measure μ via mollification. Then μ_h converges weakly to μ in the sense of measure, and the uniform L^1-estimate, $||\mu_h||_{L^1(\Omega_T)} \leq |\mu|(\Omega_T)$, holds, see also [63, Lemma 5.1].

A method of approximation in [11, 12] gives the existence of a SOLA u to the problem (5.0.1), and the results in [11, 12] imply that such a SOLA u of (5.0.1) belongs to $L^q(0,T;W_0^{1,q}(\Omega))$ such that

$$u_h \to u \quad \text{in} \ \ L^q(0,T;W_0^{1,q}(\Omega)) \quad \text{for all} \ \ q \in \left[1, \frac{n+2}{n+1}\right). \tag{5.1.3}$$

On the other hand, the uniqueness of a SOLA is still a main open problem except for the linear case, $\mathbf{a}(\xi, x, t) = \mathbf{a}(x, t)\xi$, see [85] and the references therein.

We start with the main regularity assumptions on $\mathbf{a}$ and Ω.

Definition 5.1.2. *We say* $(\mathbf{a}, \Omega)$ *is* (δ, R)-vanishing of codimension 1 *if for every point* $(y, s) \in \Omega_T$ *and every number* $r \in \left(0, \frac{R}{3}\right]$, *the following conditions hold.*

(i) If $dist(y, \partial\Omega) > r\sqrt{2}$, *then there exists a new coordinate system depending only on* (y, s) *and* r, *still denoted by* $\{x_1, \cdots, x_n, t\}$, *in which the origin is* (y, s) *and*

$$\fint_{Q_{r\sqrt{2}}} \left|\theta(\mathbf{a}, Q_{r\sqrt{2}})(x, t)\right| \, dxdt \leq \delta,$$

where

$$\theta(\mathbf{a}, Q_r)(x, t) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\left|\mathbf{a}(\xi, x', x_n, t) - \bar{\mathbf{a}}_{Q'_r}(\xi, x_n)\right|}{|\xi|},$$

and $\bar{\mathbf{a}}_{Q'_r}(\xi, x_n)$ *is the integral average of* $\mathbf{a}(\xi, \cdot, x_n, \cdot)$ *over* $Q'_r \subset \mathbb{R}^{n-1} \times \mathbb{R}$.

(ii) If $dist(y, \partial\Omega) = dist(y, y_0) \leq r\sqrt{2}$ *for some* $y_0 \in \partial\Omega$, *then there is a new coordinate system depending only on* (y, s) *and* r, *still denoted by* $\{x_1, \cdots, x_n, t\}$, *in which the origin is* $(y_0, s) + 3\delta r e_n$, *where* $e_n := (0, \cdots, 0, 1)$,

$$\mathcal{B}^+_{3r} \subset \mathcal{B}_{3r} \cap \Omega \subset \mathcal{B}_{3r} \cap \{(x', x_n, t) : x_n > -6\delta r\}, \tag{5.1.4}$$

and

$$\fint_{Q_{3r}} |\theta(\mathbf{a}, Q_{3r})(x, t)| \, dxdt \leq \delta.$$

Remark 5.1.3. *(i) Definition 5.1.2 is the parabolic version of Definition 4.1.2, see also Remark 4.1.3.*

(ii) If $(\mathbf{a}, \Omega)$ *is* (δ, R)*-vanishing of codimension* 1, *then* $\mathbf{a}(\xi, \cdot)$ *is merely measurable in the* x_n *variable and of small BMO in the other variables* x' *and* t. *Moreover, the domain* Ω *with* (5.1.4) *is called a* (δ, R)-Reifenberg flat domain*, see Chapter 2.1.3. This domain also satisfies the following*

measure density condition:

$$\sup_{0<r\leq\frac{R}{3}}\sup_{(y,s)\in\Omega_T}\frac{|Q_r(y,s)|}{|\Omega_T\cap Q_r(y,s)|}\leq\left(\frac{2\sqrt{2}}{1-\delta}\right)^{n+2}\leq\left(\frac{16\sqrt{2}}{7}\right)^{n+2}. \quad (5.1.5)$$

We are now ready to present our main results in Chapter 5.

Theorem 5.1.4. *Let u be a SOLA of the problem* (5.0.1)*. Assume that Φ satisfies $\Delta_2\cap\nabla_2$-condition and that w is an A_∞ weight. Then there is a small constant $\delta=\delta(n,\lambda,\Lambda,\Phi,[w]_{A_\infty})\in\left(0,\frac{1}{8}\right)$ such that if $(\mathbf{a},\Omega)$ is (δ,R)-vanishing of codimension 1 for some $R>0$, then there holds*

$$\|Du\|_{L^\Phi_w(\Omega_T)}\leq c\,\|\mathcal{M}_1(\mu)\|_{L^\Phi_w(\Omega_T)}, \quad (5.1.6)$$

for some constant $c=c\,(n,\lambda,\Lambda,\Phi,[w]_{A_\infty},R,\Omega_T)>0$.

Remark 5.1.5. *The estimate* (5.1.6) *is a generalization of the estimate in the weighted Lebesgue spaces, previously established in [20, 25, 31], from two aspects. When $\Phi(\tau)=\tau^p$ $(p>1)$, it reduces to*

$$\int_{\Omega_T}|Du|^p w(x,t)\,dxdt\leq c\int_{\Omega_T}\mathcal{M}_1(\mu)^p w(x,t)\,dxdt \quad \forall p\in(1,\infty).$$

It also holds for $w\in A_p$ for all $p\geq 1$, as $A_p\subset A_\infty$.

5.2 Proof of Theorem 5.1.4

5.2.1 Comparisons

Let us for simplicity assume that our solution u of (5.0.1) is defined on $\Omega\times\mathbb{R}$ as follows: for $t\geq T$ the solution and the equation can be extended by taking $\mu=0$ so that all the properties in question are preserved. For $t\leq 0$ one can use the zero extension of u. Thus, it suffices to consider only the estimates on the lateral boundary.

This subsection will discuss the comparison L^1-estimates for the spatial gradient of the weak solution u_h to the regularized problem (5.1.1) only in a localized cylinder near the boundary, as analogous estimates in the interior cylinder can be derived in a similar way.

We denote, for a measurable set $E \subset \mathbb{R}^n \times \mathbb{R}$,

$$|\mu_h|(E) := \int_E |\mu_h(x)| \, dxdt.$$

Let $0 < r \leq \frac{R}{8}$. Assume that $(\mathbf{a}, \Omega)$ is (δ, R)-vanishing of codimension 1 such that

$$\mathcal{B}_{8r}^{+} \subset \mathcal{B}_{8r} \cap \Omega \subset \mathcal{B}_{8r} \cap \{x_n > -16\delta r\}, \tag{5.2.1}$$

and

$$\fint_{Q_{8r}} |\theta(\mathbf{a}, Q_{8r})(x,t)| \, dxdt \leq \delta, \tag{5.2.2}$$

where δ is determined later in a universal way.

Let $w_h \in u_h + C([-(8r)^2,(8r)^2]; L^2(\mathcal{B}_{8r} \cap \Omega)) \cap L^2(-(8r)^2,(8r)^2; W_0^{1,2}(\mathcal{B}_{8r} \cap \Omega))$ be the weak solution of the homogeneous problem

$$\begin{cases} (w_h)_t - \operatorname{div} \mathbf{a}(Dw_h, x, t) &= 0 \quad \text{in } K_{8r}, \\ w_h &= u_h \quad \text{on } \partial_p K_{8r}. \end{cases} \tag{5.2.3}$$

Using the measure density condition (5.1.5), we can extend the comparison result in [67, Lemma 4.1] (or [55, Lemma 4.1]) up to the boundary, as we now state.

Lemma 5.2.1. *If w_h is the weak solution of* (5.2.3) *satisfying* (5.2.1)*, then there exists a constant $c = c(n, \lambda, q) > 0$ such that*

$$\fint_{K_{8r}} |Du_h - Dw_h|^q \, dxdt \leq c \left[\frac{|\mu_h|(K_{8r})}{r^{n+1}} \right]^q \quad \textit{for all} \quad q \in \left(0, \frac{n+2}{n+1}\right).$$

We now need a higher integrability result of the weak solution w_h to (5.2.3), as we have seen in [17, 62].

Lemma 5.2.2. *There exists a constant $\sigma_0 = \sigma_0(n, \lambda, \Lambda) > 0$ such that the following holds: for any $r \in \left(0, \frac{R}{8}\right]$, if w_h is the weak solution of* (5.2.3) *satisfying* (5.2.1)*, then for any $0 < \sigma \leq \sigma_0$ and $K_{2\tilde{r}}(\tilde{x}_0, \tilde{t}_0) \subset K_{8r}$ with $\tilde{r} \leq 4r$, there is a constant $c = c(n, \lambda, \Lambda, t) > 0$ such that*

$$\left(\fint_{K_{\tilde{r}}(\tilde{x}_0, \tilde{t}_0)} |Dw_h|^{2(1+\sigma)} \, dxdt \right)^{\frac{1}{1+\sigma}} \leq c \left(\fint_{K_{2\tilde{r}}(\tilde{x}_0, \tilde{t}_0)} |Dw_h|^{2t} \, dxdt \right)^{\frac{1}{t}}$$

for all $t \in (0, 1]$.

From Hölder's inequality and Lemma 5.2.2, we directly deduce

Corollary 5.2.3. *Under the same assumptions as in Lemma 5.2.2, we have*

$$\fint_{K_{\tilde{r}}(\tilde{x}_0,\tilde{t}_0)} |Dw_h|^2 \, dxdt \le c \left(\fint_{K_{2\tilde{r}}(\tilde{x}_0,\tilde{t}_0)} |Dw_h| \, dxdt \right)^2$$

for some constant $c = c(n, \lambda, \Lambda) > 0$.

Let $v_h \in w_h + C([-(4r)^2, (4r)^2]; L^2(\mathcal{B}_{4r} \cap \Omega)) \cap L^2(-(4r)^2, (4r)^2; W_0^{1,2}(\mathcal{B}_{4r} \cap \Omega))$ be the weak solution to the homogeneous frozen problem

$$\begin{cases} (v_h)_t - \operatorname{div} \bar{\mathbf{a}}_{Q'_{4r}}(Dv_h, x_n) = 0 & \text{in } K_{4r}, \\ v_h = w_h & \text{on } \partial_p K_{4r}. \end{cases} \tag{5.2.4}$$

Note that the operator $\bar{\mathbf{a}}_{Q'_{4r}}$ satisfies the structure condition (5.0.2) with $\mathbf{a}(\xi, \cdot, x_n, \cdot)$ replaced by $\bar{\mathbf{a}}_{Q'_{4r}}(\xi, x_n)$. Then we have the following comparison result between (5.2.3) and (5.2.4).

Lemma 5.2.4. *Assume that* K_{8r} *satisfies* (5.2.1) *and* (5.2.2). *Let* w_h *and* v_h *be the weak solutions of* (5.2.3) *and* (5.2.4), *respectively. Then there is a constant* $c = c(n, \lambda, \Lambda) > 0$ *such that*

$$\fint_{K_{4r}} |Dw_h - Dv_h|^2 \, dxdt \le c\delta^{\sigma_1} \left(\fint_{K_{8r}} |Dw_h| \, dxdt \right)^2,$$

where σ_1 *is the constant depending only on* σ_0 *given in Lemma 5.2.2.*

Proof. The proof is similar to that of the elliptic case (see [20, Lemma 5.6]). □

We next consider the limiting problem

$$\begin{cases} (\bar{v}_h)_t - \operatorname{div} \bar{\mathbf{a}}_{Q'_{4r}}(D\bar{v}_h, x_n) = 0 & \text{in } Q^+_{3r}, \\ \bar{v}_h = 0 & \text{on } Q_{3r} \cap \{x_n = 0\}, \end{cases} \tag{5.2.5}$$

to deduce the following comparison estimate between (5.2.4) and (5.2.5), and Lipschitz estimate for (5.2.5).

Lemma 5.2.5. *For any small* $\epsilon \in (0, 1)$, *there is* $\delta = \delta(n, \lambda, \Lambda, \epsilon) > 0$ *such that if* v_h *is the weak solution of* (5.2.4) *with* (5.2.1) *and* (5.2.2), *then there is* *a weak solution* $\bar{v}_h \in C([-(3r)^2, (3r)^2]; L^2(\mathcal{B}_{3r}^+)) \cap L^2(-(3r)^2, (3r)^2; W^{1,2}(\mathcal{B}_{3r}^+))$ *of* (5.2.5) *such that*

$$\fint_{K_{3r}} |Dv_h - D\bar{v}_h|^2 \, dxdt \leq \epsilon^2 \fint_{K_{4r}} |Dv_h|^2 \, dxdt, \tag{5.2.6}$$

and

$$\|D\bar{v}_h\|_{L^\infty(K_{2r})} \leq c \fint_{K_{3r}} |D\bar{v}_h| \, dxdt \tag{5.2.7}$$

for some constant $c = c(n, \lambda, \Lambda) > 0$*. Here* $\bar{v}_h$ *is extended by zero from* Q_{3r}^+ *to* K_{3r}*.*

Proof. The estimate (5.2.6) can be obtained from the compactness argument (for instance [27, Lemma 3.8]). On the other hand, combining the interior Lipschitz estimate (see [19, Section 4]) of the problem (5.2.5) and the reflection argument (see [20, Section 4]) yields the estimate (5.2.7). □

We finally combine Lemma 5.2.1, Lemma 5.2.4 and Lemma 5.2.5 to find the boundary comparison L^1-estimate.

Lemma 5.2.6. *Let* $\rho > 1$ *and* $0 < r \leq \frac{R}{8}$*. Then for any* $0 < \epsilon < 1$*, there exists a small* $\delta = \delta(n, \lambda, \Lambda, \epsilon) > 0$ *such that if* u_h*,* w_h*, and* v_h *are the weak solutions of* (5.1.1), (5.2.3)*, and* (5.2.4)*, respectively, with* (5.2.1), (5.2.2) *and*

$$\fint_{K_{8r}} |Du_h| \, dxdt \leq \rho \quad \text{and} \quad \frac{|\mu_h|(K_{8r})}{r^{n+1}} \leq \delta\rho,$$

then there exists a weak solution $\bar{v}_h$ *of* (5.2.5) *such that*

$$\fint_{K_{3r}} |Du_h - D\bar{v}_h| \, dxdt \leq \epsilon\rho \quad \text{and} \quad \|D\bar{v}_h\|_{L^\infty(K_{2r})} \leq c\rho$$

for some constant $c = c(n, \lambda, \Lambda) > 0$*. Here* $\bar{v}_h$ *is extended by zero from* Q_{3r}^+ *to* K_{3r}*.*

5.2.2 Covering arguments

For any fixed $\epsilon \in (0, 1)$ and $N > 1$, we define upper-level sets as follows:

$$\mathfrak{C} := \{(x,t) \in \Omega_T : \mathcal{M}(|Du|) > N\},$$
$$\mathfrak{D} := \{(x,t) \in \Omega_T : \mathcal{M}(|Du|) > 1\} \cup \{(x,t) \in \Omega_T : \mathcal{M}_1(\mu) > \delta\},$$

where $\mathcal{M}_1$ is given in (5.0.3).

Lemma 5.2.7. *Let w be an A_∞ weight. There is a constant $N = N(n,\lambda,\Lambda) > 1$ so that for any $\epsilon > 0$, there exists a small constant $\delta = \delta(n,\lambda,\Lambda,[w]_{A_\infty},\epsilon) > 0$ such that if u is a SOLA of (5.0.1) with $(\mathbf{a},\Omega)$ is (δ, R)-vanishing of codimension 1, and*

$$w\left(\mathfrak{C} \cap Q_r(y,s)\right) \geq \epsilon w\left(Q_r(y,s)\right) \tag{5.2.8}$$

for some cylinder $Q_r(y,s)$ with $(y,s) \in \Omega_T$ and $100r \leq R$, then $Q_r(y,s) \cap \Omega_T \subset \mathfrak{D}$.

Proof. The proof is similar to that of the elliptic case in [37, Lemma 1.3] (see also [31, Lemma 5.3], [83, Lemma 3.2], [88, Proposition 3.1], or [79, Lemma 10]). For the sake of completeness, we sketch the proof.

Step 1. Assume that $Q_r(y,s) \cap \Omega_T \not\subset \mathfrak{D}$. Then there would exist a point $(y_1, s_1) \in Q_r(y,s) \cap \Omega_T$ such that $(y_1,s_1) \notin \mathfrak{D}$, that is,

$$\fint_{Q_r(y_1,s_1)} |Du|\, dxdt \leq 1 \quad \text{and} \quad \frac{|\mu|(Q_r(y_1,s_1))}{r^{n+1}} \leq \delta \quad \text{for all } r > 0. \tag{5.2.9}$$

Step 2. We divide the proof into two cases: the interior $\left(Q_{10r\sqrt{2}}(y_1,s_1) \subset \Omega_T\right)$ and the boundary $\left(Q_{10r\sqrt{2}}(y_1,s_1) \not\subset \Omega_T\right)$ cases. We only consider the boundary case, as the interior case can be proved in a same way.

Step 3. Let $Q_{10r\sqrt{2}}(y_1,s_1) \not\subset \Omega_T$. Then there exists a boundary point $(\tilde{y}_1, s_1) \in (\partial\Omega \times (0,T]) \cap Q_{10r\sqrt{2}}(y_1,s_1)$. We remark that the boundary point in the bottom of Ω_T can be treated in the same way. Definition 5.1.2 implies that there is a new coordinate system modulo rotation and translation, which we still denote by $\{x_1,\cdots,x_n,t\}$, with the origin is $(\tilde{y}_1,s_1) + 30\delta r e_n$, where $e_n := (0,\cdots,0,1)$,

$$\mathcal{B}^+_{30r} \subset \mathcal{B}_{30r} \cap \Omega \subset \mathcal{B}_{30r} \cap \{(x',x_n,t) : x_n > -60\delta r\}, \quad Q_r(y,s) \subset Q_{6r},$$

and

$$\fint_{Q_{30r}} |\theta(\mathbf{a}, Q_{30r})(x,t)|\, dxdt \leq \delta.$$

Then it follows from (5.2.9), (5.1.3) and (5.1.2) that

$$\fint_{K_{30r}} |Du_h|\, dxdt \leq c \quad \text{and} \quad \frac{|\mu_h|(K_{30r})}{r^{n+1}} \leq c\delta$$

for h sufficiently large.

Step 4. According to Lemma 5.2.6 and Definition 5.1.1, we deduce that for any $0 < \eta < 1$, there exists a constant $\delta = \delta(n, \lambda, \Lambda, \eta) > 0$ such that

$$\fint_{K_{10r}} |Du - D\bar{v}_h| \, dxdt \leq c\eta \quad \text{and} \quad \|D\bar{v}_h\|_{L^\infty(K_{10r})} \leq c. \tag{5.2.10}$$

Step 5. We now have from (5.2.10) that

$$\mathfrak{C} \cap Q_r(y,s) \subset \{(x,t) \in K_r(y,s) : \mathcal{M}(|Du - D\bar{v}_h|) > 1\} \cap K_{6r} \tag{5.2.11}$$

by choosing $N = N(n, \lambda, \Lambda)$ large enough.

Step 6. We employ (5.2.11), (2.2.3) and (5.2.10) to obtain

$$|\mathfrak{C} \cap Q_r(y,s)| \leq c \int_{K_{10r}} |Du - D\bar{v}_h| \, dxdt \leq c_2 \eta |Q_r(y,s)| \tag{5.2.12}$$

for some constant $c_2 = c_2(n, \lambda, \Lambda) > 1$.

Step 7. We finally conclude, using (2.2.1) and (5.2.12), that

$$\begin{aligned} w(\mathfrak{C} \cap Q_r(y,s)) &\leq c_0 \left(\frac{|\mathfrak{C} \cap Q_r(y,s)|}{|Q_r(y,s)|} \right)^\alpha w(Q_r(y,s)) \\ &\leq c_0 (c_2 \eta)^\alpha w(Q_r(y,s)) < \epsilon w(Q_r(y,s)) \end{aligned}$$

by taking η sufficiently small. As a consequence, δ is also determined, which is a contradiction to (5.2.8). □

Lemma 5.2.7, Lemma 2.2.1 and the iterative procedure yield the following power decay estimate.

Lemma 5.2.8. *In addition to the assumptions as in Lemma 5.2.7, suppose that*

$$w\left(\mathfrak{C} \cap Q_{R/100}(y,s)\right) < \epsilon w\left(Q_{R/100}(y,s)\right) \quad \text{for all } (y,s) \in \Omega_T. \tag{5.2.13}$$

Then we have

$$\begin{aligned} w\left(\left\{(x,t) \in \Omega_T : \mathcal{M}\left(|Du|\right) > N^k\right\}\right) \\ \leq \epsilon_1^k w\left(\left\{(x,t) \in \Omega_T : \mathcal{M}\left(|Du|\right) > 1\right\}\right) \\ + \sum_{i=1}^{k} \epsilon_1^i w\left(\left\{(x,t) \in \Omega_T : \mathcal{M}_1(\mu) > \delta N^{k-i}\right\}\right), \end{aligned} \tag{5.2.14}$$

where $\epsilon_1 := \epsilon c_1$ *and the constant* c_1 *is given in Lemma 2.2.1.*

5.2.3 Calderón-Zygmund type estimates

We first derive standard energy type estimate for the problem (5.0.1), which will be used to prove Theorem 5.1.4.

Lemma 5.2.9. *Let* $1 \leq q < \frac{n+2}{n+1}$. *If* u *is a SOLA of* (5.0.1) *satisfying* (5.0.2), *then there exists a constant* $c = c(n, \lambda, q, \Omega_T) > 0$ *such that*

$$\left(\int_{\Omega_T} |Du|^q \, dxdt\right)^{\frac{1}{q}} \leq c \int_{\Omega_T} \mathcal{M}_1(\mu) \, dxdt. \tag{5.2.15}$$

Proof. From [11, Lemma 2.2] and the scaling invariance property of (5.0.1), we deduce the estimate (5.2.15). We refer to Lemma 4.4.1 for the elliptic case. □

We are now in a position to prove the main result in Chapter 5.

Proof of Theorem 5.1.4. In light of the scaling invariance property of (5.0.1), we may take $R = 1$ and assume that

$$\|\mathcal{M}_1(\mu)\|_{L^\Phi_w(\Omega_T)} = \delta \quad \text{and} \quad \int_{\Omega_T} \mathcal{M}_1(\mu) \, dxdt \leq c\delta. \tag{5.2.16}$$

Here the second inequality of (5.2.16) is derived from [25, Lemma 4.1] and the property $A_\infty = \bigcup_{1 \leq p < \infty} A_p$.

We employ (2.2.1), (2.2.3), (5.2.15) and (5.2.16) to obtain the condition

(5.2.13) as follows:

$$\begin{aligned}\frac{w\left(\mathfrak{C}\cap Q_{1/100}(y,s)\right)}{w\left(Q_{1/100}(y,s)\right)} &\le c_0\left(\frac{|\mathfrak{C}\cap Q_{1/100}(y,s)|}{|Q_{1/100}(y,s)|}\right)^{\alpha}\\ &\le c|\mathfrak{C}|^{\alpha}\le c\left(\int_{\Omega_T}|Du|\,dxdt\right)^{\alpha}\le c\left(\int_{\Omega_T}\mathcal{M}_1(\mu)\,dxdt\right)^{\alpha}\le c\delta^{\alpha}<\epsilon\end{aligned}$$

by taking δ sufficiently small.

Set

$$S:=\sum_{k=1}^{\infty}\Phi(N^k)w\left(\left\{(x,t)\in\Omega_T:\mathcal{M}\left(|Du|\right)>N^k\right\}\right),$$

where N is given in (5.2.14).

According to Lemma 2.2.2 and scaling invariance property of (5.0.1), it suffices to show that S is finite. Note that the Δ_2-condition of Φ implies that there exists a constant N_1 depending only on N such that $\Phi(N^k)\le N_1^k\Phi(1)$. Then we deduce from (5.2.14), Fubini's theorem, (2.2.4), (2.2.2) and (5.2.16) that

$$\begin{aligned}S &\le \Phi(1)\sum_{k=1}^{\infty}(N_1\epsilon_1)^k w\left(\{(x,t)\in\Omega_T:\mathcal{M}\left(|Du|\right)>1\}\right)\\ &\quad+\Phi(1)\sum_{i=1}^{\infty}(N_1\epsilon_1)^i\sum_{k=i}^{\infty}\Phi(N^{k-i})w\left(\left\{(x,t)\in\Omega_T:\mathcal{M}_1(\mu)>\delta N^{k-i}\right\}\right)\\ &\le c\left\{w(\Omega_T)+\int_{\Omega_T}\Phi\left(\frac{\mathcal{M}_1(\mu)}{\delta}\right)w(x,t)\,dxdt\right\}\sum_{i=1}^{\infty}(N_1\epsilon_1)^i\\ &\le c\left(n,\lambda,\Lambda,\Phi,[w]_{A_\infty},\Omega_T\right),\end{aligned}$$

by selecting ϵ with $N_1\epsilon_1=N_1c_1\epsilon\le\frac{1}{2}$, and thereby a corresponding δ. This completes the proof. □

5.3 Application

In this section, we obtain the global Calderón-Zygmund type estimate for the problem (5.0.1) in the variable exponent spaces from Theorem 5.1.4 and the extrapolation (Proposition 5.3.1).

We recall a brief overview of variable exponent spaces and log-Hölder continuity, see the monographs [44, 51] for details. Let $p(\cdot)$ be a measurable function defined on Ω_T with

$$1 < \gamma_1 \leq p(\cdot) \leq \gamma_2 < \infty \tag{5.3.1}$$

for some constants γ_1 and γ_2. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega_T)$ consists of all measurable functions $f : \Omega_T \to \mathbb{R}$ satisfying

$$\int_{\Omega_T} |f(x,t)|^{p(x,t)} \, dxdt < \infty,$$

equipped with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(\Omega_T)} = \inf \left\{ \lambda > 0 : \int_{\Omega_T} \left(\frac{|f(x,t)|}{\lambda} \right)^{p(x,t)} dxdt \leq 1 \right\}.$$

We introduce the log-Hölder continuity, which is crucial for proving important properties such as the boundedness of the Hardy-Littlewood maximal operator, Sobolev's inequality, Poincaré's inequality, etc. Given a function $p(\cdot)$ satisfying (5.3.1), we say that $p(\cdot)$ is *log-Hölder continuous* in Ω_T if there exists a constant $L > 0$ such that for all $(x,t), (y,s) \in \Omega_T$ with $d_p((x,t),(y,s)) \leq \frac{1}{2}$,

$$|p(x,t) - p(y,s)| \leq \frac{L}{-\log d_p((x,t),(y,s))},$$

where d_p is the standard parabolic distance, see Chapter 2.2.1.

We now mention the following variable version of the extrapolation theorem introduced by Rubio de Francia:

Proposition 5.3.1 (See [44, Corollary 5.32])**.** *Let U be a bounded domain of $\mathbb{R}^{n+1}$. Suppose that for some $p_0 \geq 1$ the following inequality holds: for all $w \in A_{p_0}$,*

$$\int_U F(x,t)^{p_0} w(x,t) \, dxdt \leq c_{p_0} \int_U G(x,t)^{p_0} w(x,t) \, dxdt,$$

where (F, G) is a pair of nonnegative and measurable functions.

Given a log-Hölder continuous function $p(\cdot)$ satisfying (5.3.1)*, we have*

$$\|F\|_{L^{p(\cdot)}(U)} \leq c\,\|G\|_{L^{p(\cdot)}(U)}\,, \tag{5.3.2}$$

where the constant $c = c(n, \gamma_1, \gamma_2, L, U) > 1$.

Remark 5.3.2. *It was recently proved that there is a more general version of the extrapolation. Indeed, we can extend the estimate* (5.3.2) *to that in the weighted variable exponent Lebesgue space (see [46]) or that in the generalized Orlicz space (see [45]) under some proper conditions.*

From Proposition 5.3.1 and Theorem 5.1.4, we discover

Theorem 5.3.3. *Let $p(\cdot)$ be log-Hölder continuous satisfying* (5.3.1)*. Under the same assumptions as in Theorem 5.1.4, we have*

$$\|Du\|_{L^{p(\cdot)}(\Omega_T)} \leq c\,\|\mathcal{M}_1(\mu)\|_{L^{p(\cdot)}(\Omega_T)}\,, \tag{5.3.3}$$

where the constant $c = c\,(n, \lambda, \Lambda, \gamma_1, \gamma_2, L, R, \Omega_T) > 0$.

Proof. We apply Theorem 5.1.4 when $\Phi(\tau) = \tau^{p_0}$ and w is an A_{p_0} weight with $p_0 > 1$, to discover

$$\int_{\Omega_T} |Du|^{p_0} w(x,t)\,dxdt \leq c \int_{\Omega_T} \mathcal{M}_1(\mu)^{p_0} w(x,t)\,dxdt.$$

According to Proposition 5.3.1 with $F = |Du|$ and $G = \mathcal{M}_1(\mu)$, we obtain the estimate (5.3.3). □

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국문초록

이 학위논문에서는 매끄럽지 않은 경계를 가지는 영역 하에서 측도데이터를 가지는 비선형 타원형 및 포물형 방정식에 대한 대역적인 칼데론-지그문트 유형의 추정값에 대하여 연구한다. 비선형성과 영역의 경계에 대한 최소한의 조건하에서 해의 그래디언트가 주어진 측도데이터의 극대함수와 대역적으로 동등한 적분가능성이 가지는 것을 증명함으로써 최적의 칼데론-지그문트 유형의 추정값을 입증한다. 우리는 변수지수 성장조건을 가지는 비선형 타원형 방정식, 가측 비선형성을 가지는 타원형 및 포물형 방정식에 대하여 각각 대역적인 칼데론-지그문트 유형의 추정값을 제시한다.

주요어휘: 측도데이터, 정칙성, 칼데론-지그문트 추정값, 라이펜버그 영역, 변수지수, 외삽법
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