이학박사학위논문

A study on the competition graphs of d-partial orders

(d-반순서의 경쟁그래프의 연구)

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최 지 훈

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이 논문을 이학 박사 학위논문으로 제출함

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A study on the competition graphs of d-partial orders

A dissertation
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by

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Abstract

A study on the competition graphs of d-partial orders

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The *competition graph* $C(D)$ of a digraph D is defined to be a graph whose vertex set is the same as D and which has an edge joining two distinct vertices x and y if and only if there are arcs (x, z) and (y, z) for some vertex z in D. Competition graphs have been extensively studied for more than four decades.

Cohen [13, 14, 15] empirically observed that most competition graphs of acyclic digraphs representing food webs are interval graphs. Roberts [51] asked whether or not Cohen's observation was just an artifact of the construction, and then concluded that it was not by showing that if G is an arbitrary graph, then G together with additional isolated vertices as many as the number of edges of G is the competition graph of some acyclic digraph. Then he asked for a characterization of acyclic digraphs whose competition graphs are interval graphs. Since then, the problem has remained elusive and it has been one of the basic open problems in the study of competition graphs. There have been a lot of efforts to settle the problem and some progress has been made. While Cho and Kim [8] tried to answer his question, they could show that the competition graphs of doubly partial orders are interval graphs. They also showed that an interval graph together with sufficiently many isolated vertices is the competition graph of a doubly partial order.

In this thesis, we study the competition graphs of d-partial orders some of which generalize the results on the competition graphs of doubly partial orders.

For a positive integer d, a digraph D is called a *d-partial order* if $V(D) \subset \mathbb{R}^d$ and there is an arc from a vertex $\mathbf{x}$ to a vertex $\mathbf{y}$ if and only if $\mathbf{x}$ is componentwise greater than $\mathbf{y}$. A doubly partial order is a 2-partial order.

We show that every graph G is the competition graph of a d-partial order for some nonnegative integer d, call the smallest such d the *partial order competition dimension* of G, and denote it by $\dim_{\text{poc}}(G)$. This notion extends the statement that the competition graph of a doubly partial order is interval and the statement that any interval graph can be the competition graph of a doubly partial order as long as sufficiently many isolated vertices are added, which were proven by Cho and Kim [8]. Then we study the partial order competition dimensions of some interesting families of graphs. We also study the m-step competition graphs and the competition hypergraph of d-partial orders.

Key words: competition graphs, d-partial orders, partial order competition dimension, homothetic regular simplices, order types
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Contents

Chapter 1

Introduction

1.1 Basic notions in graph theory

In this section, we introduce some basic notions in graph theory, which shall be frequently used in this thesis. For undefined terms, readers may refer to [4].

A *graph* G is defined as an ordered pair (V, E) where V is a set and E is a family of unordered pairs of elements in V. If V is a finite set, then G is called a *finite* graph. Otherwise, G is called an *infinite* graph. An element of V and an element of E are called a *vertex* and an *edge* of G, respectively. If $e = \{u, v\}$ is an edge, then we simply write it by uv for convenience when there is no confusion. The set of vertices and the set of edges of a graph G are called the *vertex set* and the *edge set* of G, respectively, and denoted by $V(G)$ and $E(G)$, respectively.

Let G be a graph and $e = \{u, v\}$ be an edge of G. Then we say that e joins (or connects) u and v, u and v are the *end vertices* (or *ends*) of e, and each of u and v is *incident* to e. In addition, we write $u \sim_G v$ and say that u and v are *adjacent* in G.

Let G be a graph and $\{u, v\}$ be an edge of G. If $u = v$, then the edge $\{u, v\}$ is called a *loop*. If $u \neq v$ and there are more than one edge connecting

u and v, then $\{u, v\}$ is called a *multiple edge* or a *parallel edge*. A graph having no loops and no multiple edges is said to be *simple*.

Let G be a graph and u be a vertex of G. A vertex of G which is adjacent to u is called a *neighbor* of u. The set of all neighbors of u is called the *(open) neighborhood* of u and is denoted by $N_G(u)$. The *degree* of the vertex u is defined to be the number of edges incident to u and is denoted by $d_G(u)$ or $\deg_G(u)$. A vertex with degree 0 is called an isolated vertex. For a positive integer k, the set of k isolated vertices is denoted by I_k. When there is no possibility of confusion, we sometimes omit the subscript G in the notations defined above.

Two graphs G and H are said to be *isomorphic* if there exist bijections $f_V : V(G) \to V(H)$ and $f_E : E(G) \to E(H)$ such that for every edge $e \in E(G)$, e connects vertices u and v in G if and only if $f_E(e)$ connects vertices $f_V(u)$ and $f_V(v)$ in H. If G and H are isomorphic with bijections f_V and f_E described above, then we write $G \cong H$ and call (f_V, f_E) a *graph isomorphism* from G to H.

Let G be a graph. A graph H is called a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. If H is a subgraph G, then G is called a *supergraph* of H. For a nonempty subset S of $V(G)$, the *subgraph of G induced by S*, denoted by $G[S]$, is the simple graph defined by $V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G) \mid u, v \in S\}$. For a nonempty proper subset S of $V(G)$, $G - S$ denotes the subgraph of G induced by $V(G) \setminus S$. For notational simplicity, we write $G - v$ instead of $G - \{v\}$ for a vertex v of G. An *induced subgraph* of G is a graph induced by some nonempty subset of $V(G)$. We say that G is H-free if no induced subgraph of G is isomorphic to H.

Given a simple graph G, the *complement* $\overline{G}$ of G is defined to be a simple graph obtained by reversing the adjacency of G, i.e., $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv \mid uv \notin E(G)\}$.

A *complete graph* K_n is a graph with n vertices in which every pair of vertices are adjacent. A vertex subset S of $V(G)$ is called a *clique* if the induced subgraph $G[S]$ is complete. We sometimes call a complete subgraph a *clique*. A *walk* from a vertex v_1 to a vertex v_{k+1} is an alternating sequence

$$v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$$

of vertices and edges where each v_i $(i = 1, \ldots, k+1)$ is a vertex of G and each e_j $(j = 1, \ldots, k)$ is an edge of G joining v_i and v_{i+1}. The *length* $\ell(W)$ of a walk W is defined to be the number of edges belonging to it. If there exists a walk starting from a vertex v to another vertex w, then we say that v and w are *connected* by a walk. If any two vertices are connected by a walk, then we say that the graph G is *connected*. Otherwise, G is said to be *disconnected*. A maximally connected subgraph of G is called a *(connected) component* of G. It is obvious that G is connected if and only if G has only one connected component.

A walk

$$v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$$

is called a *path* if $v_1, \ldots, v_{k+1}$ are all distinct, and called a *cycle* if $v_1 = v_{k+1}$ and $v_1, \ldots, v_k$ are all distinct. We denote a path on n vertices by P_n, and a cycle on n vertices by C_n. If no subgraph of G is a cycle, then G is called *acyclic*. A connected acyclic graph is called a *tree*. A graph is said to be *chordal* if it does not contain a cycle of length at least four an as induced subgraph.

A *digraph* (or *directed graph*) D is defined as an ordered pair $(V(D), A(D))$ where $V(D)$ is a set and $A(D)$ is a family of ordered pairs of elements in $V(D)$. If $V(D)$ is a finite set, then D is said to be *finite*. Otherwise, D is said to be *infinite*. An element of $V(D)$ and an element of $A(D)$ are called a *vertex* and an *arc* (or *directed edge*) of D, respectively. The subdigraphs and induced subdigraphs of a digraph are similarly defined as the subgraphs and induced subgraphs of a graph. If $(u, v) \in A(D)$, then we say that u and v are the tail and the head of (u, v), respectively, so that the arc (u, v) goes from the tail u to the head v.

Let D be a digraph and u be a vertex of D. A vertex v is called an *out-neighbor* (resp. *in-neighbor*) of u if (u, v) (resp. (v, u)) is an arc in D. The set of all out-neighbors (resp. in-neighbors) of u is called the *out-neighborhood* (resp. *in-neighborhood*) of u in D and denoted by $N_D^+(u)$ (resp. $N_D^-(u)$). The *outdegree* $d_D^+(u)$ is the number of arcs with tail u and the *indegree* $d_D^-(u)$ is the number of arcs with head u.

A *directed walk* from a vertex v_1 to a vertex v_{k+1} is an alternating sequence

$$v_1, a_1, v_2, a_2, \ldots, v_k, a_k, v_{k+1}$$

of vertices and arcs where each v_i $(i = 1, \ldots, k+1)$ is a vertex and each a_j $(j = 1, \ldots, k)$ is an arc from v_i to v_{i+1}. The *length* $\ell(W)$ of a directed walk W is defined to be the number of arcs belonging to it. A directed walk

$$v_1, a_1, v_2, a_2, \ldots, v_k, a_k, v_{k+1}$$

is called a *directed path* if $v_1, \ldots, v_{k+1}$ are all distinct, and called a *directed cycle* if $v_1 = v_{k+1}$ and $v_1, \ldots, v_k$ are all distinct. If no subdigraph of D is a directed cycle, then G is said to be *acyclic*.

Let G be a digraph. If we assign an orientation to each edges of G, then the resulting digraph is called an *orientation* of G. If an orientation D of G satisfies the property that $(u, v), (v, w) \in A(D)$ imply $(u, w) \in A(D)$, then the orientation is said to be *transitive*.

In this paper, all the graphs and digraphs are assumed to be finite and simple unless otherwise stated.

Mathematical Notation

$\mathbb{N}$	:	The set of positive integers
$\mathbb{Z}_{\geq 0}$	:	The set of nonnegative integers
$\mathbb{Z}$	:	The set of integers
$\mathbb{R}$	:	The set of real numbers
$\mathbb{R}^d$	:	The d-dimension Euclidean sapce $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (d times)
$\mathcal{H}^d$	:	The hyperplane $\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 + x_2 + \cdots + x_d = 0\}$
$\mathcal{H}^d_+$	:	The upper half-space $\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 + x_2 + \cdots + x_d > 0\}$
$V(G)$	:	The vertex set of a graph (or a digraph) G
$E(G)$	:	The edge set of a graph G
$A(D)$	:	The arc set of a digraph D
uv in G	:	The edge between a vertex u and a vertex v in a graph G
(u, v) in D	:	The arc from a vertex u to a vertex v in a digraph D
$G[S]$	:	The subgraph of a graphs G induced by a vertex subset S
$G - S$	:	The subgraph of a graph G induced by $V(G) \setminus S$
$G - v$	:	The subgraph of a graph G induced by $V(G) \setminus \{v\}$
$\overline{G}$	:	The complement of a graph G
$d_G(u)$	:	The degree of a vertex u in a graph G
$d^-_D(u)$	:	The indegree of a vertex u in a digraph D
$d^+_D(u)$	:	The outdegree of a vertex u in a digraph D
$N_G(u)$	:	The neighborhood of a vertex u in a graph G
$N^-_D(u)$	:	The in-neighborhood of a vertex u in a digraph D
$N^+_D(u)$	:	The out-neighborhood of a vertex u in a digraph D
$\theta_v(G)$	:	The vertex clique cover number of a graph G
$\theta_e(G)$	:	The edge clique cover number of a graph G
I_k	:	The set of k isolated vertices
K_n	:	A complete graph of n vertices
P_n	:	A path of length n
C_n	:	A cycle of length n

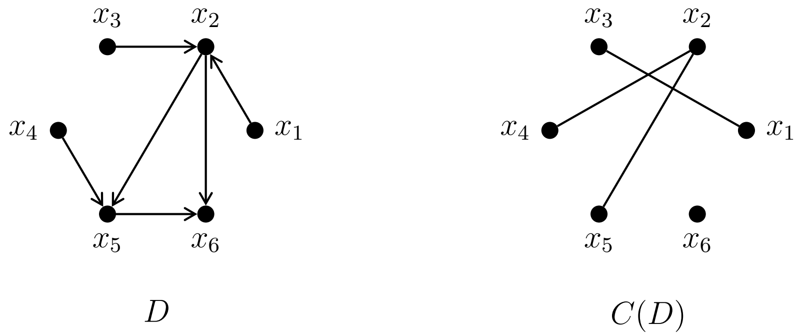

Figure 1.1: A digraph D and its competition graph $C(D)$

1.2 Competition graphs

1.2.1 A brief history of competition graphs

The notion of competition graph arose in the work of Cohen [13] in connection with an application in ecology. Let D be a digraph which represents a food web in an ecosystem which is obtained by drawing an arc from a predator to a prey. The vertex set $V(D)$ represents the set of species in the ecosystem and an arc $(x, y) \in A(D)$ means that a species x preys on a species y. One important assumption in ecology is that two species compete if they have a common prey. Hence the rivalry between species in a food web, which is an important subject in ecology, can be represented by the competition graph of D. The competition graph of a digraph is defined as follows (see Figure 1.1 for an example).

Definition 1.2.1. Given a digraph D, the *competition graph* $C(D)$ of D is a simple graph having the same vertex set as D and there is an edge $\{x, y\}$ in $C(D)$ if and only if $(x, z), (y, z) \in A(D)$ for some $z \in V(D)$.

The ecological application of competition graphs was a primary motivation for the paper [14], the book [15], and many papers on this topic.

The notion of competition graph also arises in a variety of other non-biological contexts. Suppose that the vertex set of a digraph D can be divided into two classes, A and B, and all arcs are from vertices of A to vertices of B. Then we sometimes seek a restriction of the competition graph to the set A. For instance, in communication network models, A is a set of transmitters, B is a set of receivers, and there is an arc from $u \in A$ to $v \in B$ if a message sent at u can be received at v. We then note that x and y in A interfere if signals sent at x and y can be received at the same place, i.e., if and only if x and y are adjacent in the competition graph. The problem of channel assignments in communication networks can be looked at as the problem of coloring the competition graph of a digraph (see [41, 43, 50, 55]). Competition graphs also arise in studies of the structure of models of complex system arising in modeling of energy and economic systems. In such models, we often use matrices and set up linear programs. Let A be the set of rows and B be the set of columns of a matrix M. Let D be a digraph such that $A \cup B$ is the vertex set and there is an arc from a vertex $u \in A$ to a vertex $v \in B$ if and only if the (u, v)-entry of M is nonzero. Then in the corresponding linear program, the constraints corresponding to rows x and y involve a common variable with nonzero coefficients if and only if x and y are adjacent in the competition graph of D. In the literature, the competition graph is called the row graph of matrix M. The row graph is useful in understanding the structure of linear programs. The properties of row graphs are studied in [21, 22, 23, 24]. For further applications on competition graphs, the readers may refer to the survey articles by Kim [32] and Lundgren [38].

1.2.2 Competition numbers

Roberts [51] showed that if G is an arbitrary graph, then G together with additional isolated vertices as many as the number of edges of G is the competition graph of some acyclic digraph. To show this, he constructed an acyclic digraph which has the vertex set $V(G) \cup \{v_{xy} \mid xy \in E(G)\}$ and the arc set $\bigcup_{xy\in E(G)}\{(x,v_{xy}),(y,v_{xy})\}$ and showed that its competition graph is G together with the set $\{v_{xy} \mid xy \in E(G)\}$ of additional isolated vertices. Then he defined the competition number $k(G)$ of G as follows.

Definition 1.2.2. For a graph G, the *competition number* $k(G)$ of G is defined to be the smallest nonnegative integer k such that $G \cup I_k$ is a competition graph of some acyclic digraph.

In Definition 1.2.2, the acyclicity assumption on a digraph is natural in a sense that the origination of competition graphs came from the study of the competition relation in a food web.

Determining whether an arbitrary graph is the competition graph of some acyclic digraph has been one of the most important research problems in the field of competition graphs. Roberts [51] suggested to characterize the competition graphs of acyclic digraphs by computing the competition numbers. Yet, computing the competition number of a graph is in general not easy as Opsut [45] has shown that computing the competition number is an NP-hard problem. Nonetheless, there has been much effort to compute the exact value or a bound of the competition numbers of some interesting graphs.

For a graph G, a set $\mathcal{C}$ of cliques of G is called an *vertex clique cover* (resp. *edge clique cover*) of G if every vertex (resp. every edge) of G belongs to at least one clique in $\mathcal{C}$. The minimum cardinality of a vertex clique cover (resp. edge clique cover) is called the *vertex clique cover number* (resp. *edge clique cover number*) of G and denoted by $\theta_v(G)$ (resp. $\theta_e(G)$).

Dutton and Brigham [16] characterized the competition graphs of acyclic digraphs in terms of edge clique covers.

Theorem 1.2.3 ([16]). *A graph G on n vertices is the competition graph of an acyclic digraph if and only if G has an edge clique cover $\{C_1, \ldots, C_n\}$ and a labeling $v_1, \ldots, v_n$ of the vertices of G such that if $v_i \in C_j$, then $i > j$.*

Lundgren and Maybee [39] and Kim [31] characterized the graphs whose competition numbers are at most m.

Theorem 1.2.4 ([31, 39]). *Let G be a graph on n vertices and m be a nonnegative integer with $m \leq n$. Then $k(G) \leq m$ if and only if G has an edge clique cover $\{C_1, C_2, \ldots, C_{n+m-2}\}$ and a labeling $v_1, \ldots, v_n$ of the vertices of G such that if $v_i \in C_j$, then $i \geq j - m + 1$.*

The competition numbers of some interesting families of graphs were given by Roberts [51].

Theorem 1.2.5 ([51]). *The competition number of a chordal graph is at most* 1.

Theorem 1.2.6 ([51]). *If a graph G is nontrivial, connected, and triangle-free, then $k(G) = |E(G)| - |V(G)| + 2$.*

Opsut [45] gave nice bounds for the competition number of an arbitrary graph as follows.

Theorem 1.2.7 ([45]). *For a graph G, $\theta_e(G) - |V(G)| + 2 \leq k(G) \leq \theta_e(G)$.*

Theorem 1.2.8 ([45]). *For a graph G, $k(G) \geq min_{v \in V(G)} \theta(N_G(v))$.*

Opsut [45] also computed the competition number of a line graph. The line graph of a graph is defined as follows.

Definition 1.2.9. For a graph G, the *line graph* $L(G)$ of G is defined as a graph with the vertex set $E(G)$ and two vertices are adjacent in $L(G)$ if and only if they are adjacent in G as edges. A graph is called a *line graph* if it is the line graph of a graph.

A graph is said to be *locally cobipartite* if the neighborhood of each vertex can be covered by at most two cliques. It is easy to see that every line graph is locally cobipartite.

The following theorem, which had been called Opsut's conjecture, was posed by Opsut [45] in 1982, partially solved by many researchers ([6, 46, 61, 62, 63, 64]), and was at last completely solved by McKay *et al.* [44] in 2014.

Theorem 1.2.10 ([44]). *For a locally cobipartite graph G, $k(G) \leq 2$ and the equality holds if and only if $\theta(N_G(v)) = 2$ for any vertex v of G.*

1.2.3 Interval competition graphs

If two sets A and B have a nonempty intersection, then we say that A and B *overlap* or *intersect*. A graph G is called the *intersection graph* of a family $\mathcal{F}$ of sets if there exists a bijection $\phi : V(G) \to \mathcal{F}$ such that two vertices x and y are adjacent in G if and only if $\phi(x) \cap \phi(y) \neq \emptyset$. A graph is said to be *interval* if it is the intersection graph of a family of intervals on the real line.

The notion of interval graph was introduced independently by G. Hajós [26] and S. Benzer [3]. Since the introduction of an interval graph, it has been extensively studied due to its important role in various fields such as scheduling theory, chemistry, biology, and genetics.

For a simple example, consider a problem of scheduling the courses in a university. Let $\{c_i \mid i \in I\}$ be the set of courses which are to be offered in the next semester at a university and let T_i be the time interval for c_i $(i \in I)$. We should assign each course to a lecture room in such a way that different lecture rooms are assigned to different courses whenever their time intervals intersect. To solve this problem, we define a graph G so that $\{c_i \mid i \in I\}$ is the vertex set and $c_i c_j$ is an edge if and only if $T_i \cap T_j \neq \emptyset$. By the way in which G is defined, G is obviously an interval graph. Then the desired assignment of lecture rooms can be obtained by coloring the vertices of the interval graph G so that adjacent vertices receive different colors where each color represents a lecture room. For another example, let $c_1, \ldots, c_n$ be chemical compounds each of which must be kept at a particular temperature. For each $i = 1, \ldots, n$, suppose that the compound c_i must be kept at a temperature in an interval T_i. Then we ask how many warmers or refrigerators we need to store all the compounds at an appropriate temperature. To solve the problem, we define an interval graph G in the same way as the previous example, i.e., the vertex set of G is $\{c_1, \ldots, c_n\}$ and $c_i c_j$ is an edge if and only if $T_i \cap T_j \neq \emptyset$. It is well-known that every interval graph satisfies the Helly property, that is, if K is a clique in an interval graph, then the intervals corresponding to the vertices in K have a nonempty intersection. Therefore, for every clique K of G, we

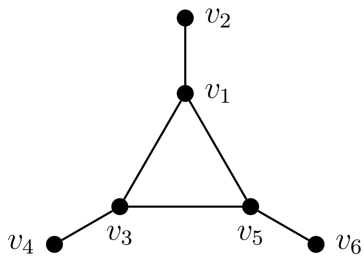

Figure 1.2: The asteroidal triple

can find a temperature which is suitable for every compound belonging to K. Thus we need warmers or refrigerators as many as the number of cliques which covers all the vertices of G. Hence we need warmers or refrigerators at least as many as the vertex clique cover number $\theta(G)$. More applications of interval graphs can be found in [20].

Due to their attractive properties, interval graphs have been extensively studied both in pure mathematics and applied sciences. One of the interesting and important problems on interval graphs was to characterize them. Among many characterizations having been presented, we take a look at the famous two results given by Lekkerkerker and Boland [37] and Gilmore and Hofman [19].

The *asteroidal triple* (AT for short) is defined as a graph with the vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and the edge set $\{v_1v_2, v_3v_4, v_5v_6, v_1v_3, v_3v_5, v_5v_1\}$ (see Figure 1.2).

Theorem 1.2.11 ([37])**.** *A graph is an interval graph if and only if it is chordal and AT-free.*

Theorem 1.2.12 ([19])**.** *A graph is an interval graph if and only if it is C_4-free and its complement has a transitive orientation.*

Interval graphs have a connection with food webs in ecology. Cohen [13, 14, 15] empirically observed that most competition graphs of acyclic digraphs representing food webs are interval graphs. Roberts [51] asked whether or not Cohen's observation was just an artifact of the construction, and then concluded that it was not by showing that if G is an arbitrary graph, then G together with additional isolated vertices as many as the number of edges of G is the competition graph of some acyclic digraph. Then, he asked for a characterization of acyclic digraphs whose competition graphs are interval graphs. Since then the problem has remained elusive and it has been one of the basic open problems in the study of competition graphs. There have been a lot of efforts to settle the problem and some progress has been made. Cohen [15] approached the problem from a statistical point of view, trying to build statistical models for the construction of a random acyclic digraph D such that $C(D)$ becomes an interval graph. Steif [60] showed that it is impossible to give a forbidden subdigraph characterization of acyclic digraphs whose competition graphs are interval. Lundgren and Maybee [40] gave some results which characterize such digraphs. However, their results essentially boiled down to calculating $C(D)$ and using one of the well-known characterizations of an interval graph. While their results solve the problem, we are still more interested in characterization in terms of properties of D. Since the general problem of characterizing acyclic digraphs whose competition graphs are interval seems difficult, Hefner *et al.* [27] attacked this problem by putting a constraint on the indegrees and outdegrees of the vertices on D. The study on acyclic digraphs whose competition graphs are interval led to several new problems and applications in the field of competition graphs. One of them is to characterize competition graphs of an interesting family of digraphs. There have been a number of papers about competition graphs of specific classes of digraphs. For instance, competition graphs of acyclic digraphs have been studied in [16, 53], of arbitrary digraphs with or without loops in [16, 38], of strongly connected digraphs in [18], of Hamiltonian digraphs in [18, 25], of interval digraphs in [36], for various classes of symmetric digraphs in [42, 43, 50], of semiorders, of acyclic digraphs satisfying property

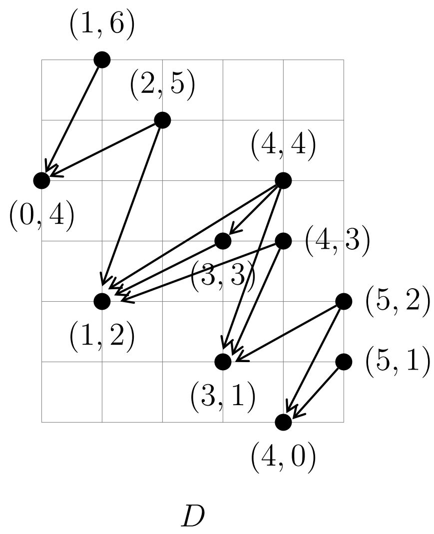

Figure 1.3: A doubly partial order D

$C(p)$, and of acyclic digraphs satisfying property $C^*(p)$ in [35]. By means of extending the results obtained in [35], Roberts (pers. comm.) proposed a problem of characterizing competition graphs of doubly partial orders, which is defined as follows.

Definition 1.2.13. The relation $\prec$ on a subset S of $\mathbb{R}^2$ defined by

$$(x_1, x_2) \prec (y_1, y_2) \Leftrightarrow x_1 < y_1 \text{ and } x_2 < y_2$$

is called a *doubly partial order*. A digraph D is called a *doubly partial order* if D is isomorphic to a doubly partial order relation $(S, \prec)$ for a subset S of $\mathbb{R}^2$. See Figure 1.3 for an example.

While Cho and Kim [8] tried to answer his question, they could show that the competition graphs of doubly partial orders are interval graphs. They also showed that an interval graph together with sufficiently many isolated vertices is the competition graph of a doubly partial order.

Theorem 1.2.14 ([8]). *The competition graph of a doubly partial order is an interval graph.*

Theorem 1.2.15 ([8]). *Every interval graph can be made into the competition graph of a doubly partial order by adding sufficiently many isolated vertices.*

Based on the observation that adding isolated vertices to a graph does not destroy the structure of the original graph, the results of Cho and Kim [8] can be summarized that the family of interval graphs and the family of competition graphs of doubly partial orders are essentially the same in the viewpoint of graph structure.

1.3 Variants of competition graphs

Since the introduction of competition graph, various variations of competition graphs have been introduced and studied by many researchers. For example, common enemy graph by Lundgren and and Maybee [40], competition-common enemy graph by Scott [54], niche graph by Cable *et al.* [5], p-competition graph by Kim *et al.* [34], phylogeny graph by Roberts and Sheng [52], m-step competition graph by Cho *et al.* [9], competition hypergraph by Sonntag and Teichert [56], and (i,j)-competition graph by Factor and Merz [17] are variations of competition graphs which have been the most extensively studied.

In this section, we will introduce the notions of m-step competition graph and competition hypergraph, which will be dealt with in the thesis. For more information on variants of competition graphs, readers may refer to the survey articles by Kim [32] and Lundgren [38].

1.3.1 m-step competition graphs

Cho *et al.* [9] introduced the notion of m-step competition graph of a digraph as a generalization of the competition graph of D. Let D be a digraph and m be a positive integer. A vertex y is called an *m-step prey* of a vertex x in D if and only if there exists a directed walk from x to y of length m.

Definition 1.3.1. The *m-step competition graph* of a digraph D, denoted by $C^m(D)$, is defined to be the graph having the same vertex set as D and having an edge xy if and only if there exists an m-step common prey of x and y in D.

A relationship between the m-step competition graph and the ordinary competition graph was given by Cho *et al.* [9] as follows.

Theorem 1.3.2 ([9]). *For a digraph D and a positive integer m, $C^m(D) = C(D^m)$.*

The notion of m-step competition graph is one of the important variants of competition graph. Since its introduction, it has been extensively studied. For example, the structural properties of m-step competition graphs were given in [9, 29, 47], the characterizations of paths or cycles which is the m-step competition graph of a digraph were studied in [2, 28, 66], and the matrix sequence $\{C^m(D)\}_{m=1}^{\infty}$ for a digraph D were studied in [7, 11, 30, 49].

The m-step competition graphs have a connection with the matrix theory. Let $\mathcal{B} = \{0, 1\}$ denote the two-element Boolean algebra with addition $(+)$ and multiplication $(\cdot)$ defined by

$$
\begin{array}{cccc}
0+0=0, & 0+1=1, & 1+0=1, & 1+1=1, \\
0\cdot 0=0, & 0\cdot 1=0, & 1\cdot 0=0, & 1\cdot 1=1.
\end{array}
$$

Let $\mathcal{B}_n$ be the set of all $n \times n$ matrices over $\mathcal{B}$. We define a matrix operator

$$\begin{tikzcd}
A^m \arrow[rr, "\Gamma"] \arrow[d] & & \Gamma(A^m) \arrow[d] \\
& \circlearrowright & \\
D(A^m) \arrow[rr, "C"'] & & C(D^m(A))
\end{tikzcd}$$

Figure 1.4: A commutative diagram

$\Gamma : \mathcal{B}_n \to \mathcal{B}_n$ by $\Gamma(A) = (\gamma_{ij})$ where

$$\gamma_{ij} = \begin{cases} 0 & \text{if } i = j; \\ 0 & \text{if } i \neq j \text{ and the dot product of row } i \text{ and row } j \text{ of } A \text{ is } 0; \\ 1 & \text{if } i \neq j \text{ and the dot product of row } i \text{ and row } j \text{ of } A \text{ is not } 0. \end{cases}$$

Given a matrix $A \in \mathcal{B}_n$, there exists a unique digraph whose adjacency matrix is A. We call such a digraph the *digraph of* A and denote it by $D(A)$. Then, for a matrix $A \in \mathcal{B}_n$ and a positive integer m, the relationships between A^m, $\Gamma(A^m)$, $D(A^m)$, and $C(D^m(A))$ can be described by the commutative diagram in Figure 1.4.

1.3.2 Competition hypergraphs

A hypergraph is a generalized notion of a graph. Formally, a *hypergraph* H is defined as an ordered pair $(V(H), E(H))$ where $V(H)$ is a set and $E(H)$ is a family of nonempty subsets of V. For a hypergraph $H = (V(H), E(H))$, the elements in $V(H)$ is called the *vertices* and the elements in $E(H)$ is called the *hyperedges* of H. A difference between a graph and a hypergraph is that every edge of a graph consists of one or two vertices while a hyperedge of a hypergraph may consists of any number of vertices. A hypergraph is said to be *k-uniform* for some positive integer k if every hyperedge consists of

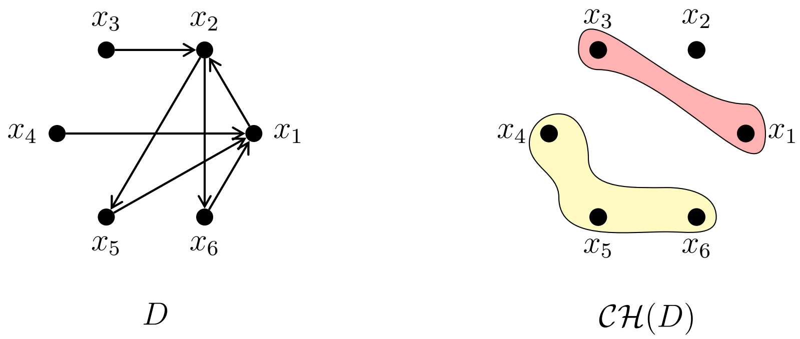

Figure 1.5: A digraph D and its competition hypergraph $\mathcal{CH}(D)$ which has two hyperedges

exactly k vertices. We note that a graph may be regarded as a hypergraph by regarding its edges as hyperedges consisting of one or two vertices.

The notion of competition hypergraph was introduced by Sonntag and Teichert [56].

Definition 1.3.3. Given a digraph D, the *competition hypergraph* of D, denoted by $\mathcal{CH}(D)$, is a hypergraph defined so that the vertex set of $\mathcal{CH}(D)$ is $V(D)$ and a subset e of $V(D)$ is a hyperedge in $\mathcal{CH}(D)$ if and only if $|e| \geq 2$ and e coincides with the in-neighborhood of some vertex in D (see Figure 1.5 for an example).

The notion of competition hypergraph is one of the important variants of competition graphs. For significant results on this topic, readers may refer to [48, 56, 57, 58, 59].

The first research results on the competition hypergraphs of d-partial orders were given by Kim *et al.* [33]. They studied the competition graphs of 2-partial orders and showed that, contrary to the fact that the competition graph of a 2-partial order is always an interval graph, the competition hypergraph of a 2-partial order may not be an interval hypergraph, which will be defined in Section 6.3. Then they characterized 2-partial orders whose competition hypergraphs are interval.

1.4 A preview of the thesis

In Chapter 2, we introduce the notion of d-partial order as an extension of doubly partial order and study the competition graphs of d-partial orders. Especially, we show that every graph together with sufficiently many isolated vertices is the competition graph of a d-partial order for some positive integer d, and then introduce a notion of partial order competition dimension of a graph. In addition, we characterize the competition graphs of d-partial orders by using homothetic regular simplices.

In Chapter 3, we present upper or lower bounds for the partial order competition dimensions of some chordal graphs. Especially, we present a sufficient condition for a chordal graph G to satisfy $\dim_{\mathrm{poc}}(G) \leq 3$, and present a family of chordal graphs G with $\dim_{\mathrm{poc}}(G) > 3$.

In Chapter 4, we introduce a notion of order type and then compute the exact value of partial order competition dimension of all the complete bipartite graphs. In addition, by utilizing the notion of order type, we present some graphs with partial order competition dimensions greater than three. Furthermore, we give an upper bound for the partial order competition dimension of a graph in terms of its chromatic number and give an upper bound for the partial order competition dimensions of planar graphs.

In Chapter 5, we study the m-step competition graphs of d-partial orders. We show that, for an arbitrary positive integer m, every graph together with some additional isolated vertices is the m-step competition graph of a d-partial order for some positive integer d, and then introduce a notion of partial order m-step competition dimension of a graph. Then we study it in the aspect of the partial order competition dimension.

In Chapter 6, we study the competition hypergraph of d-partial orders. We show that every hypergraph together with some additional isolated vertices is the competition hypergraph of a d-partial order for some positive integer d, and then introduce a notion of partial order competition hyper-dimension of a graph. Then we show that, for any positive integer d, there exists a hypergraph whose partial order competition hyper-dimension is greater than d and that every interval hypergraph has the partial order competition hyper-dimension at most three.

Chapter 2

On the competition graphs of d-partial orders[1]

2.1 The notion of d-partial order

In this section, we define a d-partial order which is an extension of a doubly partial order to a general dimension. Let d be a positive integer. For $x = (x_1, x_2, \ldots, x_d)$, $y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$, we write $x \prec y$ (resp. $x \preceq y$) if $x_i < y_i$ (resp. $x_i \leq y_i$) for each $i = 1, \ldots, d$. If $x \preceq y$ or $y \preceq x$, then we say that x and y are *comparable* in $\mathbb{R}^d$. Otherwise, we say that x and y are *incomparable* in $\mathbb{R}^d$. For a finite subset S of $\mathbb{R}^d$, let D_S be the digraph defined by $V(D_S) = S$ and $A(D_S) = \{(\mathbf{x}, \mathbf{v}) \mid \mathbf{v}, \mathbf{x} \in S, \mathbf{v} \prec \mathbf{x}\}$. A digraph D is called a *d-partial order* if there exists a finite subset S of $\mathbb{R}^d$ such that D is isomorphic to the digraph D_S. By convention, the zero-dimensional Euclidean space $\mathbb{R}^0$ consists of a single point 0. In this context, we define a digraph with exactly one vertex as a *0-partial order*. Note that a doubly

[1] The material in the section is from the paper "On the competition graphs of d-partial orders" by Jihoon Choi, Kyeong Seok Kim, Suh-Ryung Kim, Jung Yeun Lee, and Yoshio Sano which appears in Discrete Applied Mathematics 204 (2016) 29–37 ([10]). The author thanks Kyeong Seok Kim, Prof. Suh-Ryung Kim, Dr. Jung Yeun Lee, and Prof. Yoshio Sano for allowing him to use its contents for his thesis.

partial order is just a 2-partial order.

In this chapter, we study the competition graphs of d-partial orders. We obtain their characterization which nicely extends results given by Cho and Kim [8]. We also show that any graph can be made into the competition graph of a d-partial order for some positive integer d as long as adding isolated vertices is allowed. We then introduce the notion of the partial order competition dimension of a graph. Especially, we study graphs whose partial order competition dimensions are at most three.

2.2 The competition graphs of d-partial orders

In this section, we use the following notation. We use a bold faced letter to represent a point in $\mathbb{R}^d$ ($d \geq 2$). For $\mathbf{x} \in \mathbb{R}^d$, let x_i denote the ith component of $\mathbf{x}$ for each $i = 1, \ldots, d$. Let $\mathbf{e}_i \in \mathbb{R}^d$ be the standard unit vector whose ith component is 1, i.e., $\mathbf{e}_1 := (1, 0, \ldots, 0)$, ..., $\mathbf{e}_d := (0, \ldots, 0, 1)$. Let $\mathbf{1}$ be the all-one vector $(1, \ldots, 1)$ in $\mathbb{R}^d$. Note that, for $\mathbf{x} \in \mathbb{R}^d$, the standard inner product of $\mathbf{x}$ and $\mathbf{1}$ is

$$\mathbf{x} \cdot \mathbf{1} = \sum_{i=1}^{d} x_i.$$

For $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^d$, let $\mathrm{Conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ denote the *convex hull* of $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^d$, i.e.,

$$\mathrm{Conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mid \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, 1 \leq i \leq n \right\}.$$

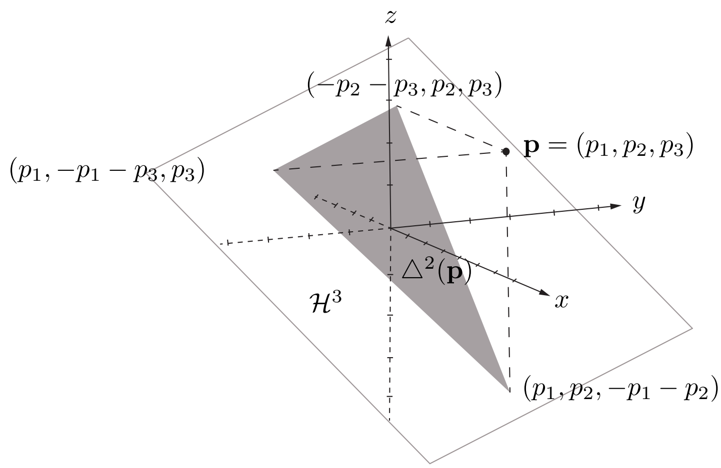

Figure 2.1: A point $\mathbf{p} \in \mathcal{H}^3_+$ and the triangle $\triangle^2(\mathbf{p})$

2.2.1 The regular $(d-1)$-dimensional simplex $\triangle^{d-1}(\mathbf{p})$

Let $\mathcal{H}^d$ be the hyperplane in $\mathbb{R}^d$ defined by the equation $\mathbf{x} \cdot \mathbf{1} = 0$, and let $\mathcal{H}^d_+$ be the open half space in $\mathbb{R}^d$ defined by the inequality $\mathbf{x} \cdot \mathbf{1} > 0$, i.e.,

$$\mathcal{H}^d := \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} \cdot \mathbf{1} = 0\}, \qquad \mathcal{H}^d_+ := \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} \cdot \mathbf{1} > 0\}.$$

We fix a point $\mathbf{p}$ in $\mathcal{H}^d_+$. Let $\triangle^{d-1}(\mathbf{p})$ be the intersection of the hyperplane $\mathcal{H}^d$ and the closed cone

$$\{\mathbf{x} \in \mathbb{R}^d \mid x_i \leq p_i \ (i = 1, \ldots, d)\}.$$

Lemma 2.2.1. *For $\mathbf{p} \in \mathcal{H}^d_+$, the set $\triangle^{d-1}(\mathbf{p})$ is the convex hull $\mathrm{Conv}(\mathbf{v}_1, \ldots, \mathbf{v}_d)$ of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$ defined by*

$$\mathbf{v}_i = \mathbf{p} - (\mathbf{p} \cdot \mathbf{1})\mathbf{e}_i \quad (i = 1, \ldots, d).$$

Moreover, $\triangle^{d-1}(\mathbf{p})$ is a regular $(d-1)$-simplex.

Proof. Let $\mathbf{v}_1$, …, $\mathbf{v}_d \in \mathbb{R}^d$ be the intersections of the hyperplane $\mathcal{H}^d$ and the lines going through $\mathbf{p}$ with directional vectors $\mathbf{e}_1, \ldots, \mathbf{e}_d$, respectively. Then $\mathbf{v}_i = \mathbf{p} - (\mathbf{p} \cdot \mathbf{1})\mathbf{e}_i$ for $i = 1, \ldots, d$. By definition, we have $\triangle^{d-1}(\mathbf{p}) = \mathrm{Conv}(\mathbf{v}_1, \ldots, \mathbf{v}_d)$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_d$ are linearly independent, the set $\triangle^{d-1}(\mathbf{p})$ is a $(d-1)$-simplex. Moreover, since the length of each edge of $\triangle^{d-1}(\mathbf{p})$ is equal to $\sqrt{2}(\mathbf{p} \cdot \mathbf{1})$, the simplex $\triangle^{d-1}(\mathbf{p})$ is regular. □

Note that the distance between $\mathbf{p}$ and each vertex of $\triangle^{d-1}(\mathbf{p})$ is equal to $\mathbf{p}\cdot\mathbf{1}$. Moreover, the directional vector for the line passing through the vertices $\mathbf{v}_i$ and $\mathbf{v}_j$ is $\mathbf{e}_j - \mathbf{e}_i$ for distinct i, j in $\{1, \ldots, d\}$. The center of $\triangle^{d-1}(\mathbf{p})$ is $\frac{1}{d}\sum_{i=1}^{d} \mathbf{v}_i = \mathbf{p} - \frac{1}{d}(\mathbf{p} \cdot \mathbf{1})\mathbf{1}$. Therefore, the directional vector from this center to the point $\mathbf{p}$ is parallel to the all-one vector $\mathbf{1}$, and the distance between this center and the point $\mathbf{p}$ is $\frac{1}{\sqrt{d}}(\mathbf{p} \cdot \mathbf{1})$ which is $\frac{1}{\sqrt{2d}}$ times the edge length of $\triangle^{d-1}(\mathbf{p})$.

We say that two geometric figures in $\mathbb{R}^d$ are *homothetic* if they are related by a geometric contraction or expansion. From the above observation, we can conclude the following:

Proposition 2.2.2. *If $\mathbf{p}, \mathbf{q} \in \mathcal{H}^d_+$, then $\triangle^{d-1}(\mathbf{p})$ and $\triangle^{d-1}(\mathbf{q})$ are homothetic.*

2.2.2 A bijection from $\mathcal{H}^d_+$ to a set of regular $(d-1)$-simplices

Lemma 2.2.3. *The vertices of $\triangle^{d-1}(\mathbf{1})$ may be labeled as $\mathbf{w}_1, \ldots, \mathbf{w}_d$ so that $\mathbf{w}_j - \mathbf{w}_i$ is a positive scalar multiple of $\mathbf{e}_i - \mathbf{e}_j$ for any distinct i, j in $\{1, \ldots, d\}$.*

Proof. By Lemma 2.2.1, the vertices of $\triangle^{d-1}(\mathbf{1})$ are $\mathbf{1} - d\mathbf{e}_i$ for $i = 1, \ldots, d$. We denote $\mathbf{1} - d\mathbf{e}_i$ by $\mathbf{w}_i$ to obtain the desired labeling. □

Lemma 2.2.4. *Let d be an integer with $d \geq 2$. Suppose that Λ is a regular $(d-1)$-simplex contained in the hyperplane $\mathcal{H}^d$ homothetic to $\triangle^{d-1}(\mathbf{1})$. Then, there exists $\mathbf{p} \in \mathcal{H}^d_+$ such that $\Lambda = \triangle^{d-1}(\mathbf{p})$.*

Proof. Since Λ is homothetic to $\triangle^{d-1}(\mathbf{1})$, there exists $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{R}^d$ which are linearly independent such that $\Lambda = \mathrm{Conv}(\mathbf{v}_1, \ldots, \mathbf{v}_d)$ and $\mathbf{v}_j - \mathbf{v}_i$ is a positive scalar multiple of $\mathbf{e}_i - \mathbf{e}_j$ for any distinct i and j in $\{1, \ldots, d\}$ by Lemma 2.2.3. Moreover, $\mathbf{v}_i \cdot \mathbf{e}_i < \mathbf{v}_j \cdot \mathbf{e}_i$ for any distinct i and j in $\{1, \ldots, d\}$. Let $\mathbf{p} \in \mathbb{R}^d$ be a vector defined by $p_i : \max\{\mathbf{v}_k \cdot \mathbf{e}_i \mid 1 \leq k \leq d\}$ $(i = 1, \ldots, d)$. Then, $\Lambda = \triangle^{d-1}(\mathbf{p})$. Since $\mathbf{v}_1 \in \mathcal{H}^d$, we have $\mathbf{v}_1 \cdot \mathbf{1} = 0$. Since $p_i \geq \mathbf{v}_1 \cdot \mathbf{e}_i$ for any $i = 1, \ldots, d$, we have $\mathbf{p} \cdot \mathbf{1} \geq \mathbf{v}_1 \cdot \mathbf{1} = 0$. If $\mathbf{p} \cdot \mathbf{1} = 0$, then we obtain $\mathbf{p} = \mathbf{v}_1$ and $p_1 = \mathbf{v}_1 \cdot \mathbf{e}_1 < \mathbf{v}_2 \cdot \mathbf{e}_1$, which is a contradiction to the definition of p_1. Therefore $\mathbf{p} \cdot \mathbf{1} > 0$, i.e., $\mathbf{p} \in \mathcal{H}^d_+$. Thus the lemma holds. ☐

Let d be an integer with $d \geq 2$. Let $\mathcal{F}^{d-1}_*$ be the set of the regular $(d-1)$-simplices in the hyperplane $\mathcal{H}^d$ which are homothetic to $\triangle^{d-1}(\mathbf{1})$. Let f_* be a map from $\mathcal{H}^d_+$ to $\mathcal{F}^{d-1}_*$ defined by $f_*(\mathbf{p}) = \triangle^{d-1}(\mathbf{p})$. By Lemma 2.2.1 and Proposition 2.2.2, $\triangle^{d-1}(\mathbf{p}) \in \mathcal{F}^{d-1}_*$ and therefore the map f_* is well-defined.

Proposition 2.2.5. *For each integer $d \geq 2$, the map $f_* : \mathcal{H}^d_+ \to \mathcal{F}^{d-1}_*$ is a bijection.*

Proof. By Lemma 2.2.4, the map f_* is surjective. Suppose that $\triangle^{d-1}(\mathbf{p}) = \triangle^{d-1}(\mathbf{q})$. Since the centers of $\triangle^{d-1}(\mathbf{p})$ and $\triangle^{d-1}(\mathbf{q})$ are the same, we have $\mathbf{p} - \frac{1}{d}(\mathbf{p} \cdot \mathbf{1})\mathbf{1} = \mathbf{q} - \frac{1}{d}(\mathbf{q} \cdot \mathbf{1})\mathbf{1}$. Since the lengths of edges of $\triangle^{d-1}(\mathbf{p})$ and $\triangle^{d-1}(\mathbf{q})$ are the same, we have $\sqrt{2}(\mathbf{p} \cdot \mathbf{1}) = \sqrt{2}(\mathbf{q} \cdot \mathbf{1})$. Therefore, we have $\mathbf{p} = \mathbf{q}$. Thus the map f_* is injective. Hence the map f_* is a bijection. ☐

Let $\mathcal{F}^{d-1}$ be the set of the interiors of the regular $(d-1)$-simplices in the hyperplane $\mathcal{H}^d$ which are homothetic to $\triangle^{d-1}(\mathbf{1})$. Then there is a clear bijection $\varphi : \mathcal{F}^{d-1}_* \to \mathcal{F}^{d-1}$ such that for each element in $\mathcal{F}^{d-1}_*$, its φ-value is its interior. Therefore we obtain the following corollary.

Corollary 2.2.6. *For each integer $d \geq 2$, the map $\varphi \circ f_* : \mathcal{H}^d_+ \to \mathcal{F}^{d-1}$ is a bijection.*

2.2.3 A characterization of the competition graphs of d-partial orders

Let $A^{d-1}(\mathbf{p})$ be the interior of the regular simplex $\triangle^{d-1}(\mathbf{p})$, i.e., $A^{d-1}(\mathbf{p}) := \mathrm{int}(\triangle^{d-1}(\mathbf{p}))$. Then

$$A^{d-1}(\mathbf{p}) = \{\mathbf{x} \in \mathcal{H}^d \mid \mathbf{x} \prec \mathbf{p}\}.$$

Proposition 2.2.7. *For $\mathbf{p}, \mathbf{q} \in \mathcal{H}^d_+$, $\triangle^{d-1}(\mathbf{p})$ is contained in $A^{d-1}(\mathbf{q})$ if and only if $\mathbf{p} \prec \mathbf{q}$.*

Proof. Suppose that $\mathbf{p} \prec \mathbf{q}$. Take a point $\mathbf{a}$ in $\triangle^{d-1}(\mathbf{p})$. Then $a_k \leq p_k$ for each $k = 1, \ldots, d$. By the assumption that $\mathbf{p} \prec \mathbf{q}$, $a_k < q_k$ for each $k = 1, \ldots, d$, that is, $\mathbf{a} \prec \mathbf{q}$. Thus $\triangle^{d-1}(\mathbf{p})$ is contained in $A^{d-1}(\mathbf{q})$.

Suppose that $\triangle^{d-1}(\mathbf{p})$ is contained in $A^{d-1}(\mathbf{q})$. Then, by Lemma 2.2.1, $\mathbf{p} - (\mathbf{p} \cdot \mathbf{1})\mathbf{e}_i$ $(i = 1, \ldots, d)$ are points in $A^{d-1}(\mathbf{q})$. By the definition of $A^{d-1}(\mathbf{q})$, we have $\mathbf{p} - (\mathbf{p} \cdot \mathbf{1})\mathbf{e}_i \prec \mathbf{q}$ $(i = 1, \ldots, d)$, which implies $\mathbf{p} \prec \mathbf{q}$. $\square$

Lemma 2.2.8. *Let d be a positive integer and let D be a d-partial order. Then, two vertices $\mathbf{v}$ and $\mathbf{w}$ of D are adjacent in the competition graph of D if and only if there exists a vertex $\mathbf{a}$ in D such that $\triangle^{d-1}(\mathbf{a}) \subseteq A^{d-1}(\mathbf{v}) \cap A^{d-1}(\mathbf{w})$.*

Proof. By definition, two vertices $\mathbf{v}$ and $\mathbf{w}$ are adjacent in the competition graph of D if and only if there exists a vertex $\mathbf{a}$ in D such that $\mathbf{a} \prec \mathbf{v}$ and $\mathbf{a} \prec \mathbf{w}$. By Proposition 2.2.7, $\mathbf{a} \prec \mathbf{v}$ and $\mathbf{a} \prec \mathbf{w}$ holds if and only if $\triangle^{d-1}(\mathbf{a}) \subseteq A^{d-1}(\mathbf{v})$ and $\triangle^{d-1}(\mathbf{a}) \subseteq A^{d-1}(\mathbf{w})$, that is, $\triangle^{d-1}(\mathbf{a}) \subseteq A^{d-1}(\mathbf{v}) \cap A^{d-1}(\mathbf{w})$. Thus the lemma holds. $\square$

The following result extends Theorems 1.2.14 and 1.2.15.

Theorem 2.2.9. *Let G be a graph and let d be an integer with $d \geq 2$. Then, G is the competition graph of a d-partial order if and only if there exists a family $\mathcal{F}$ of the interiors of regular $(d-1)$-simplices in $\mathbb{R}^d$ which are* *contained in the hyperplane $\mathcal{H}^d$ and homothetic to $A^{d-1}(\mathbf{1})$ and there exists a one-to-one correspondence between the vertex set of G and $\mathcal{F}$ such that*

($\star$) *two vertices v and w are adjacent in G if and only if two elements in $\mathcal{F}$ corresponding to v and w have the intersection containing the closure of another element in $\mathcal{F}$.*

Proof. First we show the "only if" part. Let D be a d-partial order and let G be the competition graph of D. Without loss of generality, we may assume that $V(D) \subseteq \mathcal{H}^d_+$ by translating each of the vertices of D in the same direction and by the same amount since the competition graph of D is determined only by the adjacency among vertices of D. Consequently $A^{d-1}(\mathbf{v}) \neq \emptyset$ for each vertex $\mathbf{v}$ of D. Let $\mathcal{F} = \{A^{d-1}(\mathbf{v}) \mid \mathbf{v} \in V(D)\}$ and let $f : V(G) \to \mathcal{F}$ be the map defined by $f(\mathbf{v}) = A^{d-1}(\mathbf{v})$ for $\mathbf{v} \in V(D)$. Note that $\mathcal{F} \subseteq \mathcal{F}^{d-1}$. Since the map $f : V(G) \to \mathcal{F}$ is a restriction of the map $\varphi \circ f_* : V(G) \to \mathcal{F}^{d-1}$, it follows from Corollary 2.2.6 that f is a bijection. By Lemma 2.2.8, the condition ($\star$) holds.

Second, we show the "if" part. Suppose that there exist a family $\mathcal{F} \subseteq \mathcal{F}^{d-1}$ and a bijection $f : V(G) \to \mathcal{F}$ such that the condition ($\star$) holds. By Corollary 2.2.6, each element in $\mathcal{F}$ can be represented as $A^{d-1}(\mathbf{p})$ for some $\mathbf{p} \in \mathcal{H}^d_+$. Let D be a digraph with vertex set $V(D) = \{\mathbf{p} \in \mathbb{R}^d \mid A^{d-1}(\mathbf{p}) \in \mathcal{F}\}$ and arc set $A(D) = \{(\mathbf{p}, \mathbf{q}) \mid \mathbf{p}, \mathbf{q} \in V(D), \mathbf{p} \neq \mathbf{q}, \triangle^{d-1}(\mathbf{q}) \subseteq A^{d-1}(\mathbf{p})\}$. By Proposition 2.2.7, $(\mathbf{p}, \mathbf{q}) \in A(D)$ if and only if $\mathbf{q} \prec \mathbf{p}$, so D is a d-partial order. Now, take two vertices v and w in G. Then, by the hypothesis and above argument, v and w correspond to some points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^d$, respectively, so that v and w are adjacent if and only if both $A^{d-1}(\mathbf{p})$ and $A^{d-1}(\mathbf{q})$ contain the closure of an element in $\mathcal{F}$, that is, $\triangle^{d-1}(\mathbf{r})$ for some $\mathbf{r} \in \mathbb{R}^d$. By the definition of D, $(\mathbf{p}, \mathbf{r}) \in A(D)$ and $(\mathbf{q}, \mathbf{r}) \in A(D)$. Consequently, v and w are adjacent if and only if the corresponding vertices $\mathbf{p}$ and $\mathbf{q}$ have a common out-neighbor in D. Hence G is the competition graph of the d-partial order D. $\square$

2.2.4 Intersection graphs and competition graphs of d-partial orders

Theorem 2.2.10. *If G is the intersection graph of a finite family of homothetic open regular $(d-1)$-simplices, then G together with sufficiently many new isolated vertices is the competition graph of a d-partial order.*

Proof. Let $\mathcal{A} = \{A_1, \dots A_n\}$ be a finite family of homothetic open regular $(d-1)$-simplices, and let G be the intersection graph of $\mathcal{A}$ with bijection $\phi : \mathcal{A} \to V(G)$. For each distinct pair of i and j in $\{1, \dots, n\}$ such that $A_i \cap A_j \neq \emptyset$, let B_{ij} be an open regular $(d-1)$-simplex homothetic A_1 such that the closure of B_{ij} is contained in $A_i \cap A_j$. We can take such B_{ij} so that $B_{ij} \cap B_{i'j'} = \emptyset$ for distinct pairs $\{i, j\}$ and $\{i', j'\}$. Let $\mathcal{B} = \{B_{ij} \mid i, j \in \{1, \dots, n\}, i \neq j, A_i \cap A_j \neq \emptyset\}$. Then the family $\mathcal{F} := \mathcal{A} \cup \mathcal{B}$ and a map $f : \mathcal{F} \to V(G) \cup \{z_1, \dots, z_{|\mathcal{B}|}\}$ such that $f|_{\mathcal{A}} = \phi$ and $f(\mathcal{B}) = \{z_1, \dots, z_{|\mathcal{B}|}\}$ satisfy the condition $(\star)$ in Theorem 2.2.9. Thus $G \cup \{z_1, \dots, z_{|\mathcal{B}|}\}$ is the competition graph of a d-partial order. $\square$

Lemma 2.2.11. *If G is the intersection graph of a finite family $\mathcal{F}$ of homothetic closed regular $(d-1)$-simplices, then G is the intersection graph of a finite family of homothetic open regular $(d-1)$-simplices.*

Proof. Let $\mathcal{D} = \{\triangle_1, \dots \triangle_n\}$ be a finite family of homothetic closed regular $(d-1)$-simplices, and let G be the intersection graph of $\mathcal{D}$ with bijection $\phi : \mathcal{D} \to V(G)$. Let

$$\varepsilon = \frac{1}{3} \min\{d(\triangle_i, \triangle_j) \mid i, j \in \{1, \dots, n\}, \triangle_i \cap \triangle_j = \emptyset\}$$

where $d(\triangle_i, \triangle_j) = \inf\{d(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \triangle_i, \mathbf{y} \in \triangle_j\}$.

We make each simplex in $\mathcal{D}$ $(1+\varepsilon)$ times bigger while the center of each simplex is fixed. Then we take the interiors of these closed simplices. By the choice of ε, the graph G is the intersection graph of the family of newly obtained open simplices. Hence the lemma holds. $\square$

Theorem 2.2.12. *If G is the intersection graph of a finite family $\mathcal{F}$ of homothetic closed regular $(d-1)$-simplices, then G together with sufficiently many new isolated vertices is the competition graph of a d-partial order.*

Proof. The theorem follows from Lemma 2.2.11 and Theorem 2.2.10. $\square$

Remark 2.2.13. In the case where $d = 2$, Theorem 2.2.12 is the same as Theorem 1.2.15. Due to Theorem 1.2.14, the converse of Theorem 2.2.12 is true for $d = 2$. In fact, we can show that the converse of Theorem 2.2.10 is also true for $d = 2$.

The following example shows that the converses of Theorems 2.2.10 and 2.2.12 are not true for $d = 3$.

Example 2.2.14. Let G be a subdivision of K_5 given in Figure 2.2. Then, by Theorem 2.2.9, the family of homothetic equilateral triangles given in the figure makes G together with 9 isolated vertices into the competition graph of a 3-partial order. However, G is not the intersection graph of any family of homothetic equilateral closed triangles. By Lemma 2.2.11, G is not the intersection graph of any family of homothetic equilateral open triangles, either.

Proof. Suppose that there exists a family $\mathcal{F} := \{\triangle(v) \mid v \in V(G)\}$ of homothetic equilateral closed triangles such that G is the intersection graph of $\mathcal{F}$. Since $v_1v_2v_3v_4v_1$ is an induced cycle in G, the triangles $\triangle(v_1)$, $\triangle(v_2)$, $\triangle(v_3)$, and $\triangle(v_4)$ are uniquely located as in Figure 2.3 up to the sizes of triangles. Since the vertices v_1, v_3, and v_4 are neighbors of both v_5 and v_7 in G whereas v_2 is not, and the vertices v_5 and v_7 are not adjacent in G, we may conclude that the locations of $\triangle(v_5)$ and $\triangle(v_7)$ should be those for the triangles I and II given in Figure 2.3. Since the triangle II cannot have intersections with $\triangle(v_2)$ and the triangle I, all of its sides are surrounded by $\triangle(v_1)$, $\triangle(v_2)$, $\triangle(v_3)$, and $\triangle(v_4)$. Now, since v_6 is adjacent to v_5 and v_7, $\triangle(v_6)$ must have intersections with both $\triangle(v_5)$ and $\triangle(v_7)$. However, it cannot be done without

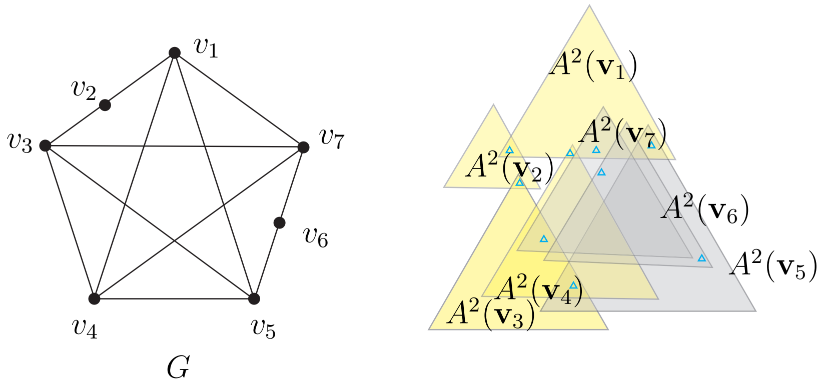

Figure 2.2: A subdivision G of K_5 and a family of homothetic equilateral triangles making G together with 9 isolated vertices into the competition graph of a 3-partial order

having an intersection with one of $\triangle(v_1)$, $\triangle(v_2)$, $\triangle(v_3)$, and $\triangle(v_4)$, which is a contradiction to the fact that none of v_1, v_2, v_3, and v_4 is adjacent to v_6 in G. □

2.3 The partial order competition dimension of a graph

Proposition 2.3.1. *Let d be a positive integer. If G is the competition graph of a d-partial order, then G is the competition graph of a $(d+1)$-partial order.*

Proof. Let D be a d-partial order such that G is the competition graph of D. For each $\mathbf{v} \in V(D) \subseteq \mathbb{R}^d$, we define $\tilde{\mathbf{v}} \in \mathbb{R}^{d+1}$ by

$$\tilde{\mathbf{v}} = \left(v_1, \ldots, v_d, \sum_{i=1}^{d} v_i \right).$$

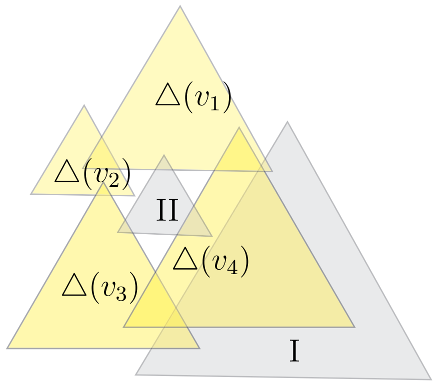

Figure 2.3: An assignment of homothetic equilateral triangles to vertices v_1, v_2, v_3, v_4, v_5, v_7 of G given in Figure 2.2

Then $\{\tilde{\mathbf{v}} \mid \mathbf{v} \in V(D)\}$ defines a $(d+1)$-partial order $\tilde{D}$. Since

$$\begin{aligned}
\tilde{\mathbf{v}} \prec \tilde{\mathbf{w}} \quad &\Leftrightarrow \quad v_i < w_i \ (i = 1, \ldots, d) \ \text{ and } \ \sum_{i=1}^{d} v_i < \sum_{i=1}^{d} w_i \\
&\Leftrightarrow \quad v_i < w_i \ (i = 1, \ldots, d) \\
&\Leftrightarrow \quad \mathbf{v} \prec \mathbf{w},
\end{aligned}$$

the $(d+1)$-partial order $\tilde{D}$ is the same digraph as the d-partial order D. Hence G is the competition graph of the $(d+1)$-partial order $\tilde{D}$. □

Proposition 2.3.2 can be shown by using Theorem 8 in [65]. We present a new proof from which Proposition 2.3.5 also follows.

Proposition 2.3.2. *For any graph G, there exists positive integers d and k such that G together with k isolated vertices is the competition graph of a d-partial order.*

Proof. Let $n = |V(G)|$ and label the vertices of G as $v_1, \ldots, v_n$. Fix four real numbers r_1, r_2, r_3, and r_4 such that $r_1 < r_2 < r_3 < r_4$. We define a map

$\phi : V(G) \to \mathbb{R}^n$ by

$$\phi(v_i)_j = \begin{cases} r_2 & \text{if } j = i; \\ r_4 & \text{if } j \neq i. \end{cases}$$

We define a map $\psi : E(G) \to \mathbb{R}^n$ by

$$\psi(e)_k = \begin{cases} r_1 & \text{if } v_k \in e; \\ r_3 & \text{if } v_k \notin e. \end{cases}$$

Let $V = \{\phi(v_i) \mid v_i \in V(G)\} \cup \{\psi(e) \mid e \in E(G)\} \subseteq \mathbb{R}^n$. Then V defines an n-partial order D. By definition, the in-neighborhood of the vertex $\psi(e) \in V$ is $\{\phi(v_i), \phi(v_j)\}$ for an edge $e = \{v_i, v_j\}$ of G and the in-neighborhood of the vertex $\phi(v) \in V$ is the empty set for a vertex v. Thus the competition graph of D is G together with isolated vertices as many as $|E(G)|$. Hence, by taking $d = n$ and $k = |E(G)|$, we complete the proof. $\square$

Now we may introduce the following notion. Recall that for a finite subset S of $\mathbb{R}^d$, D_S is the digraph defined by $V(D_S) = S$ and $A(D_S) = \{(\mathbf{x}, \mathbf{v}) \mid \mathbf{v}, \mathbf{x} \in S, \mathbf{v} \prec \mathbf{x}\}$.

Definition 2.3.3. For a graph G, we define the *partial order competition dimension* $\dim_{\text{poc}}(G)$ of G as the smallest nonnegative integer d such that G together with k isolated vertices is the competition graph of D for some d-partial order D and some nonnegative integer k, i.e.,

$$\dim_{\text{poc}}(G) := \min\{d \in \mathbb{Z}_{\geq 0} \mid \exists k \in \mathbb{Z}_{\geq 0}, \exists S \subseteq \mathbb{R}^d, \text{ s.t. } G \cup I_k = C(D_S)\},$$

where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers and I_k is a set of k isolated vertices.

Remark 2.3.4. Wu and Lu [65] introduced the notion of the *dimension-d poset competition number* of a graph G, denoted by $\mathcal{PK}^d(G)$, which is defined to be the smallest nonnegative integer p such that G together p additional isolated vertices is isomorphic to the competition graph of a poset of dimension at most d if such a poset exists, and to be ∞ otherwise. By using $\mathcal{PK}^d(G)$, the partial order competition dimension of G can be represented as $\dim_{\text{poc}}(G) = \min\{d \mid \mathcal{PK}^d(G) < \infty\}$. Therefore $\mathcal{PK}^d(G) < \infty$ implies that $\dim_{\text{poc}}(G) \leq d$. In this respect, Proposition 3.2 and the "if" part of Proposition 3.10 may follow from their result presenting the dimension-d poset competition numbers of a complete graph with or without isolated vertices, which are also shown to be trivially true in this paper.

Proposition 2.3.5. *For any graph G, we have* $\dim_{\text{poc}}(G) \leq |V(G)|$.

Proof. The proposition follows from the construction of a d-partial order in the proof of Proposition 2.3.2. □

For a graph G, the partial order competition dimension of an induced subgraph of G is less than or equal to that of G. To show this, we need the following lemmas.

Lemma 2.3.6. *Let D be a digraph and let G be the competition graph of D. Let S be a set of vertices. The competition graph of $D[S]$ is a subgraph of $G[S]$, where $D[S]$ and $G[S]$ mean the subdigraph of D and the subgraph of G, respectively, induced by S.*

Proof. Let H be the competition graph of $D[S]$. Obviously, $V(H) = S$. Take an edge $\{u, v\}$ of H. By definition, there exists a vertex w in $D[S]$ such that (u, w) and (v, w) are arcs of $D[S]$. Consequently, (u, w) and (v, w) are arcs of D and so $\{u, v\}$ is an edge of G. Since $u, v \in S$, $\{u, v\}$ is an edge of $G[S]$. Hence H is a subgraph of $G[S]$. □

Lemma 2.3.7. *Let D be a transitive acyclic digraph and let G be the competition graph of D. For any non-isolated vertex u of G, there exists an isolated vertex v of G such that (u, v) is an arc of D.*

Proof. Take a non-isolated vertex u of G. Since u has a neighbor w in G, u and w have a common out-neighbor in D. Take a longest directed path in D originating from u. We denote by v the terminal vertex of the directed path. Since D is acyclic, the out-degree of v in D is zero and so v is isolated in G. By the hypothesis that D is transitive, (u, v) is an arc of D. $\square$

Proposition 2.3.8. *Let G be a graph and let H be an induced subgraph of G. Then* $\dim_{\text{poc}}(H) \leq \dim_{\text{poc}}(G)$.

Proof. Let $d = \dim_{\text{poc}}(G)$. Then, there exists a d-partial order D whose competition graph is the disjoint union of G and a set J of isolated vertices. Let I be the set of isolated vertices in G. Let $S = V(H) \cup I \cup J \subseteq \mathbb{R}^d$. Then the digraph D_S is a d-partial order. By Lemma 2.3.6, the competition graph of D_S is a subgraph of $H \cup (I \setminus V(H)) \cup J$.

Now take two adjacent vertices $\mathbf{x}$ and $\mathbf{y}$ in H. Then, since they are adjacent in G, there exists a vertex $\mathbf{v} \in V(D)$ such that $\mathbf{v} \prec \mathbf{x}$ and $\mathbf{v} \prec \mathbf{y}$. If $\mathbf{v}$ is isolated in G or $\mathbf{v} \in J$, then $(\mathbf{x}, \mathbf{v})$ and $(\mathbf{y}, \mathbf{v})$ belong to $A(D_S)$ by definition. Suppose that $\mathbf{v} \notin I \cup J$. Then, by Lemma 2.3.7, there exists a vertex $\mathbf{w}$ in $I \cup J$ such that $\mathbf{w} \prec \mathbf{v}$. Then $\mathbf{w} \prec \mathbf{x}$ and $\mathbf{w} \prec \mathbf{y}$ and so $(\mathbf{x}, \mathbf{w})$ and $(\mathbf{y}, \mathbf{w})$ belong to $A(D_S)$. Thus $H \cup (I \setminus V(H)) \cup J$ is a subgraph of the competition graph of D_S and we have shown that it is the competition graph of D_S. Hence $\dim_{\text{poc}}(H) \leq d$ and the proposition holds. $\square$

It does not seem to be easy to compute the partial order competition dimension of a graph in general. In this context, we first characterize graphs having small partial order competition dimensions. In such a way, we wish to have a better idea to settle the problem.

Let K_n denote the complete graph with n vertices.

Proposition 2.3.9. *Let G be a graph. Then,* $\dim_{\text{poc}}(G) = 0$ *if and only if $G = K_1$.*

Proof. The proposition immediately follows from the definition of 0-partial order. $\square$

Proposition 2.3.10. *Let G be a graph. Then,* $\dim_{\mathrm{poc}}(G) = 1$ *if and only if* $G = K_{t+1}$ *or* $G = K_t \cup K_1$ *for some positive integer* t.

Proof. First we remark that if D is a 1-partial order with $V(D) \subseteq \mathbb{R}^1$ and $v^* \in V(D)$ is the minimum among $V(D)$, then the competition graph of D is the disjoint union of a clique $V(D) \setminus \{v^*\}$ and an isolated vertex v^*. Therefore, if $\dim_{\mathrm{poc}}(G) = 1$, then we obtain $G = K_{t+1}$ or $G = K_t \cup K_1$ for some nonnegative integer t. By Proposition 2.3.9, $G \neq K_1$ and thus t is a positive integer.

If $G = K_{t+1}$ or $G = K_t \cup K_1$ for some positive integer t, then we obtain $\dim_{\mathrm{poc}}(G) \leq 1$. By Proposition 2.3.9, since $G \neq K_1$, we have $\dim_{\mathrm{poc}}(G) = 1$. □

Lemma 2.3.11. *Let G be a graph such that* $\dim_{\mathrm{poc}}(G) \geq 2$ *and let G' be a graph obtained from G by adding isolated vertices. Then* $\dim_{\mathrm{poc}}(G) = \dim_{\mathrm{poc}}(G')$.

Proof. Let $a_1, \ldots, a_k$ be the isolated vertices added to G to obtain G'. Let $d = \dim_{\mathrm{poc}}(G)$. Then G can be made into the competition graph a d-partial order D by adding sufficiently many isolated vertices. Since $d \geq 2$, we can locate k points $\mathbf{a}_1, \ldots, \mathbf{a}_k$ in $\mathbb{R}^d$ corresponding to $a_1, \ldots, a_k$ so that no two points in $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ are related by $\prec$ and that no point in $V(D)$ and no point in $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ are related by $\prec$. Indeed, we can do this in the following way: for $i = 1, \ldots, k$, let $\mathbf{a}_i$ be a point in $\mathbb{R}^d$ defined by

$$(\mathbf{a}_i)_1 = r_1 + i; \quad (\mathbf{a}_i)_2 = r_2 - i; \quad (\mathbf{a}_i)_j = 0 \ (j = 3, \ldots, d),$$

where $r_1 := \max\{(\mathbf{v})_1 \mid \mathbf{v} \in V(D)\}$ and $r_2 := \min\{(\mathbf{v})_2 \mid \mathbf{v} \in V(D)\}$. Then G' is the competition graph of D together with $\mathbf{a}_1, \ldots, \mathbf{a}_k$ and thus $\dim_{\mathrm{poc}}(G') \leq \dim_{\mathrm{poc}}(G)$.

Since G is an induced subgraph of G', by Proposition 2.3.8, $\dim_{\mathrm{poc}}(G) \leq \dim_{\mathrm{poc}}(G')$. Hence $\dim_{\mathrm{poc}}(G) = \dim_{\mathrm{poc}}(G')$. □

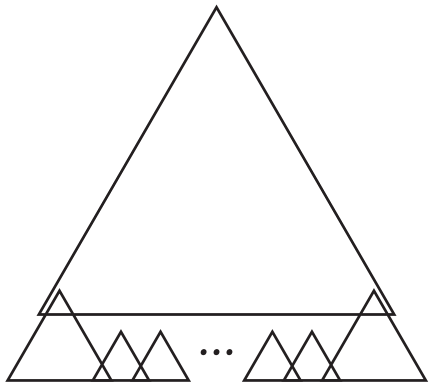

Figure 2.4: A family of homothetic equilateral closed triangles

Proposition 2.3.12. *Let G be a graph. Then,* $\dim_{\text{poc}}(G) = 2$ *if and only if G is an interval graph which is neither K_s nor $K_t \cup K_1$ for any positive intergers s and t.*

Proof. Suppose that $\dim_{\text{poc}}(G) = 2$. By Theorem 1.2.14, G is an interval graph. By Propositions 2.3.9 and 2.3.10, G is neither K_s nor $K_t \cup K_1$ for any positive intergers s and t.

Suppose that G is an interval graph which is neither K_s nor $K_t \cup K_1$ for any positive intergers s and t. By Theorem 1.2.15, $\dim_{\text{poc}}(G) \leq 2$. By Propositions 2.3.9 and 2.3.10 $\dim_{\text{poc}}(G) \geq 2$. Thus, $\dim_{\text{poc}}(G) = 2$. □

Proposition 2.3.13. *If G is a cycle of length at least four, then* $\dim_{\text{poc}}(G) = 3$.

Proof. Let G be a cycle of length n with $n \geq 4$. Note that G is not an interval graph. By Propositions 2.3.9, 2.3.10, and 2.3.12, we have $\dim_{\text{poc}}(G) \geq 3$. Let $\mathcal{F}$ be the family of n closed triangles given in Figure 2.4. Then the intersection graph of $\mathcal{F}$ is the cycle of length n. By Theorem 2.2.12 with $d = 3$, G together with sufficiently many isolated vertices is the competition graph of a 3-partial order. Thus $\dim_{\text{poc}}(G) \leq 3$. Hence $\dim_{\text{poc}}(G) = 3$. □

Theorem 2.3.14. *If a graph G contains an induced cycle of length at least four, then* $\dim_{\mathrm{poc}}(G) \geq 3$.

Proof. The theorem follows from Propositions 2.3.8 and 2.3.13. $\square$

Theorem 2.3.15. *Let T be a tree. Then* $\dim_{\mathrm{poc}}(T) \leq 3$*, and the equality holds if and only if T is not a caterpillar.*

Proof. By Theorem 2.2.12 with $d = 3$, we need to show that there exists a family of homothetic equilateral closed triangles in $\mathbb{R}^2$ whose intersection graph is T. As a matter of fact, it is sufficient to find such a family in the xy-plane with the base of each triangle parallel to the x-axis. We call the vertex of a triangle which is opposite to the base the *apex* of the triangle. We show the following stronger statement by induction on the number of vertices:

> For a tree T and a vertex v of T, there exists a family $\mathcal{F}_v^T := \{\triangle(x) \mid x \in V(T)\}$ of homothetic equilateral closed triangles whose intersection graph is T such that, for any vertex x distinct from v, the apex and the base of $\triangle(x)$ are below the apex and the base of $\triangle(v)$, respectively.

We call the family $\mathcal{F}_v^T$ in the above statement a *good family* for T and v.

If T is the tree having exactly one vertex, then the statement is vacuously true. Assume that the statement holds for any tree on $n-1$ vertices, where $n \geq 2$. Let T be a tree with n vertices. We fix a vertex v of T as a root. Let $T_1, \ldots, T_k$ ($k \geq 1$) be the connected components of $T - v$. Then $T_1, \ldots, T_k$ are trees. For each $i = 1, \ldots, k$, T_i has exactly one vertex, say w_i, which is a neighbor of v in T. We take w_i as a root of T_i. By the induction hypothesis, there exists a good family $\mathcal{F}_{w_i}^{T_i}$ for T_i and w_i for each $i = 1, \ldots, k$. Preserving the intersection or the non-intersection of two triangles in $\mathcal{F}_{w_i}^{T_i}$ for each $i = 1, \ldots, k$, we may translate the triangles in $\mathcal{F}_{w_1}^{T_1} \cup \cdots \cup \mathcal{F}_{w_k}^{T_k}$ so that the apexes of $\triangle(w_1)$, …, $\triangle(w_k)$ are on the x-axis and any two triangles from distinct families do not intersect. Let δ_i be the distance between the apex of $\triangle(w_i)$ and the apex of a triangle which is the second highest among the apexes of the triangles in $\mathcal{F}_{w_i}^{T_i}$. Now we draw a triangle $\triangle(v)$ in such a way that the base of $\triangle(v)$ is a part of the line $y = -\frac{1}{2}\min\{\delta_1, \ldots, \delta_k\}$ and long enough to intersect all of the triangles $\triangle(w_1), \ldots, \triangle(w_k)$. Then the family $\mathcal{F}_v^T :=$ $\mathcal{F}_{w_1}^{T_1} \cup \cdots \cup \mathcal{F}_{w_k}^{T_k} \cup \{\triangle(v)\}$ is a good family for T and v and thus the statement holds. Hence, $\dim_{\mathrm{poc}}(T) \leq 3$ for a tree T.

Since trees which are interval graphs are caterpillars, the latter part of the theorem follows from Propositions 2.3.9, 2.3.10, and 2.3.12. $\square$

Remark 2.3.16. In this chapter, we studied the competition graphs of d-partial orders and gave a characterization by using homothetic open simplices. Since any graph can be made into the competition graph of a d-partial order for some positive integer d by adding isolated vertices, we introduced the notion of the partial order competition dimension of a graph. We gave characterizations of graphs having partial order competition dimension 0, 1, and 2. We also showed that cycles and trees have partial order competition dimension at most 3. It would be an interesting research problem to characterize graphs G having partial order competition dimension 3.

Chapter 3

On the partial order competition dimensions of chordal graphs[1]

In this chapter, we study the partial order competition dimensions of chordal graphs. We thought that most likely candidates for the family of graphs having partial order competition dimension at most three are chordal graphs since both trees and interval graphs, which are chordal graphs, have partial order competition dimensions at most three by Theorems 1.2.15 and 2.3.15. In fact, we can show that chordal graphs have partial order competition dimensions at most three if the graphs are diamond-free. However, contrary to our presumption, we could show the existence of chordal graphs with partial order competition dimensions greater than three.

[1]The material in the section is from the paper "On the partial order competition dimensions of chordal graphs" by Jihoon Choi, Suh-Ryung Kim, Jung Yeun Lee, and Yoshio Sano which appears in Discrete Applied Mathematics 222 (2017) 89–96 ([12]). The author thanks Prof. Suh-Ryung Kim, Dr. Jung Yeun Lee, and Prof. Yoshio Sano for allowing him to use its contents for his thesis.

3.1 Basic properties on the competition graphs of 3-partial orders

The following key theorem is a special case of Theorem 2.2.9 for the case where $d = 3$.

Theorem 3.1.1. *A graph G is the competition graph of a 3-partial order if and only if there exists a family $\mathcal{F}$ of homothetic open equilateral triangles contained in the plane $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ and there exists a one-to-one correspondence $A : V(G) \to \mathcal{F}$ such that*

($\star$) *two vertices v and w are adjacent in G if and only if two elements $A(v)$ and $A(w)$ have the intersection containing the closure $\triangle(x)$ of an element $A(x)$ in $\mathcal{F}$.*

The following sufficient condition for a graph being the competition graph of a 3-partial order is an immediate consequence of Lemma 2.2.11 when $d = 3$.

Theorem 3.1.2. *If G is the intersection graph of a finite family of homothetic closed equilateral triangles, then G together with sufficiently many new isolated vertices is the competition graph of a 3-partial order.*

By the definition of the partial order competition dimension of a graph, we have the following:

Corollary 3.1.3. *If G is the intersection graph of a finite family of homothetic closed equilateral triangles, then $\dim_{\text{poc}}(G) \leq 3$.*

Note that the converse of Corollary 3.1.3 is not true by Example 2.2.14. In this context, one can guess that it is not so easy to show that a graph has partial order competition dimension greater than three.

The correspondence A in Theorem 2.2.9 is the map A^{d-1} defined in Section 2.2 with $d = 3$. To illustrate it when $d = 3$, let $\mathcal{H} := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ and $\mathcal{H}_+ := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 > 0\}$. For a point $v = (v_1, v_2, v_3) \in \mathcal{H}_+$, let $p_1^{(v)}$, $p_2^{(v)}$, and $p_3^{(v)}$ be points in $\mathbb{R}^3$ defined by $p_1^{(v)} := (-v_2-v_3, v_2, v_3)$, $p_2^{(v)} := (v_1, -v_1-v_3, v_3)$, and $p_3^{(v)} := (v_1, v_2, -v_1-v_2)$, and let $\triangle(v)$ be the convex hull of the points $p_1^{(v)}$, $p_2^{(v)}$, and $p_3^{(v)}$, i.e., $\triangle(v) := \mathrm{Conv}(p_1^{(v)}, p_2^{(v)}, p_3^{(v)}) = \left\{\sum_{i=1}^3 \lambda_i p_i^{(v)} \mid \sum_{i=1}^3 \lambda_i = 1, \lambda_i \geq 0 \ (i = 1, 2, 3)\right\}$. Then it is easy to check that $\triangle(v)$ is an closed equilateral triangle which is contained in the plane $\mathcal{H}$. Let $A(v)$ be the relative interior of the closed triangle $\triangle(v)$, i.e., $A(v) := \mathrm{rel.int}(\triangle(v)) = \left\{\sum_{i=1}^3 \lambda_i p_i^{(v)} \mid \sum_{i=1}^3 \lambda_i = 1, \lambda_i > 0 \ (i = 1, 2, 3)\right\}$. Then $A(v)$ and $A(w)$ are homothetic for any $v, w \in \mathcal{H}_+$.

For $v \in \mathcal{H}_+$ and $(i, j) \in \{(1, 2), (2, 3), (1, 3)\}$, let $l_{ij}^{(v)}$ denote the line through the two points $p_i^{(v)}$ and $p_j^{(v)}$, i.e., $l_{ij}^{(v)} := \{x \in \mathbb{R}^3 \mid x = \alpha p_i^{(v)} + (1 - \alpha)p_j^{(v)}, \alpha \in \mathbb{R}\}$, and let $R_{ij}(v)$ denote the following region:

$$R_{ij}(v) := \{x \in \mathbb{R}^3 \mid x = (1-\alpha-\beta)p_k^{(v)} + \alpha p_i^{(v)} + \beta p_j^{(v)}, 0 \leq \alpha \in \mathbb{R}, 0 \leq \beta \in \mathbb{R}, \alpha+\beta \geq 1\},$$

where k is the element in $\{1, 2, 3\} \setminus \{i, j\}$; for $k \in \{1, 2, 3\}$, let $R_k(v)$ denote the following region:

$$R_k(v) := \{x \in \mathbb{R}^3 \mid x = (1+\alpha+\beta)p_k^{(v)} - \alpha p_i^{(v)} - \beta p_j^{(v)}, 0 \leq \alpha \in \mathbb{R}, 0 \leq \beta \in \mathbb{R}\},$$

where i and j are elements such that $\{i, j, k\} = \{1, 2, 3\}$. (See Figure 3.1 for an illustration.)

If a graph G satisfies $\dim_{\mathrm{poc}}(G) \leq 3$, then, by Theroem 2.2.9, we may assume that $V(G) \subseteq \mathcal{H}_+$ by translating each of the vertices of G in the same direction and by the same amount.

Lemma 3.1.4. *Let D be a 3-partial order and let G be the competition graph of D. Suppose that G contains an induced path uvw of length two. Then neither $A(u) \cap A(v) \subseteq A(w)$ nor $A(v) \cap A(w) \subseteq A(u)$.*

Proof. We show by contradiction. Suppose that $A(u) \cap A(v) \subseteq A(w)$ or $A(v) \cap A(w) \subseteq A(u)$. By symmetry, we may assume without loss of generality that $A(u) \cap A(v) \subseteq A(w)$. Since u and v are adjacent in G, there exists a vertex

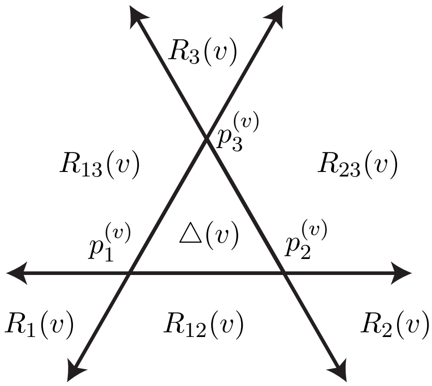

Figure 3.1: The regions determined by v. By our assumption, for any vertex u of a graph considered in this paper, $p_1^{(u)}$, $p_2^{(u)}$, $p_3^{(u)}$ correspond to $p_1^{(v)}$, $p_2^{(v)}$, $p_3^{(v)}$ respectively.

$a \in V(G)$ such that $\triangle(a) \subseteq A(u) \cap A(v)$ by Theorem 2.2.9. Therefore $\triangle(a) \subseteq A(w)$. Since $\triangle(a) \subseteq A(u)$, u and w are adjacent in G by Theorem 2.2.9, which is a contradiction to the assumption that u and w are not adjacent in G. Hence the lemma holds. $\square$

Definition 3.1.5. For $v, w \in \mathcal{H}_+$, we say that v and w are *crossing* if $A(v) \cap A(w) \neq \emptyset$, $A(v) \setminus A(w) \neq \emptyset$, and $A(w) \setminus A(v) \neq \emptyset$.

Lemma 3.1.6. *Let D be a 3-partial order and let G be the competition graph of D. Suppose that G contains an induced path $xuvw$ of length three. Then u and v are crossing.*

Proof. Since u and v are adjacent in G, there exists a vertex $a \in V(G)$ such that $\triangle(a) \subseteq A(u) \cap A(v)$ by Theorem 2.2.9. Therefore $A(u) \cap A(v) \neq \emptyset$. If $A(v) \subseteq A(u)$, then $A(v) \cap A(w) \subseteq A(u)$, which contradicts Lemma 3.1.4. Thus $A(v) \setminus A(u) \neq \emptyset$. If $A(u) \subseteq A(v)$, then $A(x) \cap A(u) \subseteq A(v)$, which contradicts Lemma 3.1.4. Thus $A(u) \setminus A(v) \neq \emptyset$. Hence u and v are crossing. $\square$

Lemma 3.1.7. *If v and w in $\mathcal{H}_+$ are crossing, then $p_k^{(x)} \in \triangle(y)$ for some $k \in \{1, 2, 3\}$ where $\{x, y\} = \{v, w\}$.*

Proof. Since v and w are crossing, we have $A(v) \cap A(w) \neq \emptyset$, $A(v) \setminus A(w) \neq \emptyset$, and $A(w) \setminus A(v) \neq \emptyset$. Then one of the vertices of the triangles $\triangle(v)$ and $\triangle(w)$ is contained in the other triangle, thus the lemma holds. $\square$

Definition 3.1.8. For $k \in \{1, 2, 3\}$, we define a binary relation $\stackrel{k}{\rightarrow}$ on $\mathcal{H}_+$ by

$$x \stackrel{k}{\rightarrow} y \quad \Leftrightarrow \quad x \text{ and } y \text{ are crossing, and } p_k^{(y)} \in \triangle(x)$$

for any $x, y \in \mathcal{H}_+$.

Lemma 3.1.9. *Let $x, y, z \in \mathcal{H}_+$. Suppose that $x \stackrel{k}{\rightarrow} y$ and $y \stackrel{k}{\rightarrow} z$ for some $k \in \{1, 2, 3\}$ and that x and z are crossing. Then $x \stackrel{k}{\rightarrow} z$.*

Proof. Since $x \stackrel{k}{\rightarrow} y$, $p_l^{(x)} \notin R_i(y) \cup R_{ij}(y) \cup R_j(y)$ for each $l \in \{1, 2, 3\}$, where $\{i, j, k\} = \{1, 2, 3\}$ Since $y \stackrel{k}{\rightarrow} z$, $p_l^{(z)} \in R_i(y) \cup R_{ij}(y) \cup R_j(y)$ for each $l \in \{i, j\}$. Since x and z are crossing, $p_k^{(z)} \in \triangle(x)$. $\square$

Definition 3.1.10. For $k \in \{1, 2, 3\}$, a sequence $(v_1, \ldots, v_m)$ of m points in $\mathcal{H}_+$, where $m \geq 2$, is said to be *consecutively tail-biting in Type k* if $v_i \stackrel{k}{\rightarrow} v_j$ for any $i < j$ (see Figure 3.2). A finite set V of points in $\mathcal{H}_+$ is said to be *consecutively tail-biting* if there is an ordering $(v_1, \ldots, v_m)$ of V such that $(v_1, \ldots, v_m)$ is consecutively tail-biting.

3.2 The partial order competition dimensions of diamond-free chordal graphs

A *diamond* of a graph G is an induced subgraph G isomorphic to the complete tripartite graph $K_{1,1,2}$. We call the edge connecting the partite sets of size 1 the *diagonal* of the diamond. A graph G is said to be *diamond-free* if G does not contain a diamond.

In this section, we show that a chordal graph has partial order competition dimension at most three if it is diamond-free.

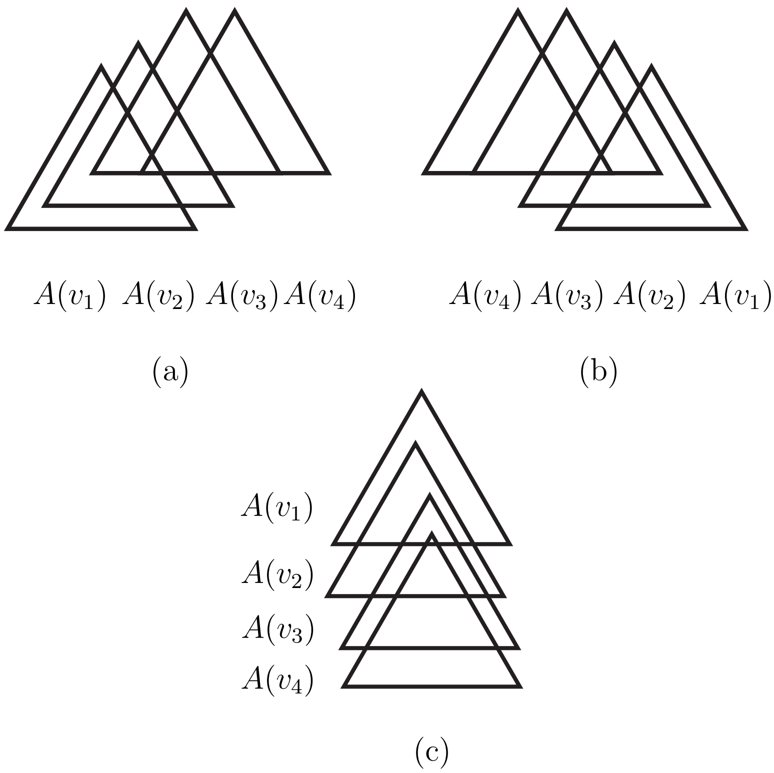

Figure 3.2: The sequences (v_1, v_2, v_3, v_4) in (a), (b), (c) are consecutively tail-biting of Type 1, 2, 3, respectively.

A *block graph* is a graph such that each of its maximal 2-connected subgraphs is a complete graph. The following is well-known.

Lemma 3.2.1 ([1, Proposition 1]). *A graph is a block graph if and only if the graph is a diamond-free chordal graph.*

Note that a block graph having no cut vertex is a disjoint union of complete graphs. For block graphs having cut vertices, the following lemma holds.

Lemma 3.2.2. *Let G be a block graph having at least one cut vertex. Then G has a maximal clique that contains exactly one cut vertex.*

Proof. Let H be the subgraph induced by the cut vertices of G. By definition, H is obviously a block graph, so H is chordal and there is a simplicial vertex v in H. Since v is a cut vertex of G, v belongs to at least two maximal cliques of G. Suppose that each maximal clique containing v contains another cut vertex of G. Take two maximal cliques X_1 and X_2 of G containing v and let x and y be cut vertices of G belonging to X_1 and X_2, respectively. Then both x and y are adjacent to v in H. Since G is a block graph, $X_1 \setminus \{v\}$ and $X_2 \setminus \{v\}$ are contained in distinct connected components of $G - v$. This implies that x and y are not adjacent in H, which contradicts the choice of v. Therefore there is a maximal clique X containing v without any other cut vertex of G. $\square$

Lemma 3.2.3. *Every block graph G is the intersection graph of a family $\mathcal{F}$ of homothetic closed equilateral triangles in which every clique of G is consecutively tail-biting.*

Proof. We show by induction on the number of cut vertices of G. If a block graph has no cut vertex, then it is a disjoint union of complete graphs and the statement is trivially true as the vertices of each complete subgraph can be formed as a sequence which is consecutively tail-biting (refer to Figure 3.2).

Assume that the statement is true for any block graph G with m cut vertices where $m \geq 0$. Now we take a block graph G with $m+1$ cut vertices.

By Lemma 3.2.2, there is a maximal clique X that contains exactly one cut vertex, say w. By definition, the vertices of X other than w are simplicial vertices.

Deleting the vertices of X other than w and the edges adjacent to them, we obtain a block graph G^* with m cut vertices. Then, by the induction hypothesis, G^* is the intersection graph of a family $\mathcal{F}^*$ of homothetic closed equilateral triangles satisfying the statement. We consider the triangles corresponding to w. Let C and C' be two maximal cliques of G^* containing w. By the induction hypothesis, the vertices of C and C' can be ordered as $v_1, v_2, \ldots, v_l$ and $v'_1, v'_2, \ldots, v'_{l'}$, respectively, so that $v_i \stackrel{k}{\to} v_j$ if $i < j$, for some $k \in \{1, 2, 3\}$ and that $v'_{i'} \stackrel{k'}{\to} v'_{j'}$ if $i' < j'$, for some $k' \in \{1, 2, 3\}$.

Suppose that $\triangle(v_i) \cap \triangle(v'_j) \neq \emptyset$ for v_i and v'_j which are distinct from w. Then v_i and v'_j are adjacent in G^*, which implies the existence of a diamond in G since maximal cliques have size at least two. We have reached a contradiction to Lemma 3.2.1 and so $\triangle(v_i) \cap \triangle(v'_j) = \emptyset$ for any i, j. Therefore there is a segment of a side on $\triangle(w)$ (with a positive length) that does not intersect with the triangle assigned to any vertex in G^* other than w since there are finitely many maximal cliques in G^* that contain w. If the side belongs to $l_{ij}^{(w)}$ for $i, j \in \{1, 2, 3\}$, then we may order the deleted vertices and assign the homothetic closed equilateral triangles with sufficiently small sizes to them so that the closed neighborhood of v is consecutively tail-biting in Type k for $k \in \{1, 2, 3\} \setminus \{i, j\}$ and none of the triangles intersects with the triangle corresponding to any vertex other than w in G^*. It is not difficult to see that the set of the triangles in $\mathcal{F}^*$ together with the triangles just obtained is the one desired for $\mathcal{F}$. □

As block graphs are not necessarily interval graphs, the following result extends a known family of graphs with partial order competition dimension three.

Theorem 3.2.4. *For any diamond-free chordal graph G, $\dim_{\text{poc}}(G) \leq 3$.*

Proof. The theorem follows from Corollary 3.1.3 and Lemma 3.2.3. □

3.3 Chordal graphs having partial order competition dimension greater than three

In this section, we present infinitely many chordal graphs G with $\dim_{\mathrm{poc}}(G) > 3$.

We first show two lemmas which will be repeatedly used in the proof of the theorem in this section.

Lemma 3.3.1. *Let D be a 3-partial order and let G be the competition graph of D. Suppose that G contains a diamond $K_4 - e$ as an induced subgraph, where u, v, w, x are the vertices of the diamond and $e = vx$. If the sequence (u, v, w) is consecutively tail-biting in Type k for some $k \in \{1, 2, 3\}$, then $p_i^{(x)} \in R_i(v)$ and $p_j^{(x)} \notin R_j(v)$ hold or $p_i^{(x)} \notin R_i(v)$ and $p_j^{(x)} \in R_j(v)$ hold where $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. Without loss of generality, we may assume that $k = 3$. We first claim that $p_1^{(x)} \in R_1(v) \cup R_2(v) \cup R_{12}(v)$. Suppose not. Then $p_1^{(x)} \in R := \mathcal{H} \setminus (R_1(v) \cup R_2(v) \cup R_{12}(v))$. Since $A(x)$ and $A(v)$ are homothetic, $A(x) \subseteq R$. Thus $A(w) \cap A(x) \subseteq A(w) \cap R$. Since (u, v, w) is consecutively tail-biting in Type 3, $A(w) \cap R \subseteq A(v)$. Therefore $A(w) \cap A(x) \subseteq A(v)$, which contradicts Lemma 3.1.4. Thus $p_1^{(x)} \in R_1(v) \cup R_2(v) \cup R_{12}(v)$. By symmetry, $p_2^{(x)} \in R_1(v) \cup R_2(v) \cup R_{12}(v)$.

Suppose that both $p_1^{(x)}$ and $p_2^{(x)}$ are in $R_{12}(v)$. Since $A(x)$ and $A(v)$ are homothetic, $A(x) \cap R \subseteq A(v)$. By the hypothesis that (u, v, w) is consecutively tail-biting in Type 3, we have $A(u) \subseteq R$. Therefore $A(x) \cap A(u) \subseteq A(x) \cap R$. Thus $A(x) \cap A(u) \subseteq A(v)$, which contradicts Lemma 3.1.4. Therefore $p_1^{(x)} \in R_1(v) \cup R_2(v)$ or $p_2^{(x)} \in R_1(v) \cup R_2(v)$. Since $p_1^{(x)} \in R_2(v)$ (resp. $p_2^{(x)} \in R_1(v)$) implies $p_2^{(x)} \in R_2(v)$ (resp. $p_1^{(x)} \in R_1(v)$), which is impossible, we have $p_1^{(x)} \in R_1(v)$ or $p_2^{(x)} \in R_2(v)$.

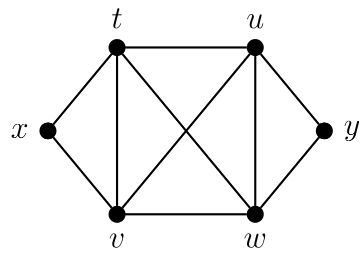

Figure 3.3: The graph $\overline{\mathrm{H}}$

Suppose that both $p_1^{(x)} \in R_1(v)$ and $p_2^{(x)} \in R_2(v)$ hold. Then $A(v) \subseteq A(x)$ since $A(v)$ and $A(x)$ are homothetic. Then $A(u) \cap A(v) \subseteq A(x)$, which contradicts Lemma 3.1.4. Hence $p_1^{(x)} \in R_1(v)$ and $p_2^{(x)} \notin R_2(v)$ hold or $p_1^{(x)} \notin R_1(v)$ and $p_2^{(x)} \in R_2(v)$ hold. $\square$

Let $\overline{\mathrm{H}}$ be the graph on vertex set $\{t, u, v, w, x, y\}$ such that $\{t, u, v, w\}$ forms a complete graph K_4, x is adjacent to only t and v, and y is adjacent to only u and w in $\overline{\mathrm{H}}$ (see Figure 3.3 for an illustration).

Lemma 3.3.2. *Let D be a 3-partial order and let G be the competition graph of D. Suppose that G contains the graph $\overline{\mathrm{H}}$ as an induced subgraph and (t, u, v, w) is consecutively tail-biting in Type k for some $k \in \{1, 2, 3\}$. Then, for i, j with $\{i, j, k\} = \{1, 2, 3\}$, $p_i^{(x)} \in R_i(u)$ implies $p_j^{(y)} \in R_j(v)$.*

Proof. Without loss of generality, we may assume that $k = 3$. It is sufficient to show that $p_1^{(x)} \in R_1(u)$ implies $p_2^{(y)} \in R_2(v)$. Now suppose that $p_1^{(x)} \in R_1(u)$. Since (t, u, v, w) is a tail-biting sequence of Type 3, (t, u, v) and (u, v, w) are tail-biting sequences of Type 3. Since $\{t, u, v, x\}$ induces a diamond and (t, u, v) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 3.3.1 that $p_1^{(x)} \in R_1(u)$ and $p_2^{(x)} \notin R_2(u)$ hold or $p_1^{(x)} \notin R_1(u)$ and $p_2^{(x)} \in R_2(u)$ hold. Since $p_1^{(x)} \in R_1(u)$, it must hold that $p_1^{(x)} \in R_1(u)$ and $p_2^{(x)} \notin R_2(u)$. Since $A(u)$ and $A(x)$ are homothetic and $p_1^{(x)} \in R_1(u)$, we have $A(u) \subseteq A(x) \cup R_{23}(x)$.

Since $\{u, v, w, y\}$ induces a diamond and (u, v, w) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 3.3.1 that $p_1^{(y)} \in R_1(v)$ and $p_2^{(y)} \notin R_2(v)$ hold or $p_1^{(y)} \notin R_1(v)$ and $p_2^{(y)} \in R_2(v)$ hold. We will claim that the latter is true as it implies $p_2^{(y)} \in R_2(v)$. To reach a contradiction, suppose the former, that is, $p_1^{(y)} \in R_1(v)$ and $p_2^{(y)} \notin R_2(v)$. Since $A(v)$ and $A(y)$ are homothetic and $p_1^{(y)} \in R_1(v)$, we have $A(v) \subseteq A(y) \cup R_{23}(y)$. We now show that $A(x) \cap A(v) \subseteq A(y)$. Take any $a \in A(x) \cap A(v)$. Since $A(v) \subseteq A(y) \cup R_{23}(y)$, we have $a \in A(y) \cup R_{23}(y)$. Suppose that $a \notin A(y)$. Then $a \in R_{23}(y)$. This together with the fact that $a \in A(x)$ implies $A(y) \cap R_{23}(x) = \emptyset$. Since $A(u) \subseteq A(x) \cup R_{23}(x)$, we have

$$\begin{aligned} A(u) \cap A(y) &\subseteq (A(x) \cup R_{23}(x)) \cap A(y) \\ &= (A(x) \cap A(y)) \cup (R_{23}(x)) \cap A(y)) \\ &= (A(x) \cap A(y)) \cup \emptyset \\ &= A(x) \cap A(y) \subseteq A(x). \end{aligned}$$

Therefore $A(u) \cap A(y) \subseteq A(u) \cap A(x)$. Since u and y are adjacent in G, there exists $b \in V(G)$ such that $\triangle(b) \subseteq A(u) \cap A(y)$. Then $\triangle(b) \subseteq A(u) \cap A(x)$, which is a contradiction to the fact that u and x are not adjacent in G. Thus $a \notin R_{23}(y)$ and so $a \in A(y)$. Hence we have shown that $A(x) \cap A(v) \subseteq A(y)$. Since x and v are adjacent in G, there exists $c \in V(G)$ such that $\triangle(c) \subseteq A(x) \cap A(v)$. Then $\triangle(c) \subseteq A(v) \cap A(y)$, which is a contradiction to the fact that v and y are not adjacent in G. Thus we have $p_1^{(y)} \notin R_1(v)$ and $p_2^{(y)} \in R_2(v)$. Hence the lemma holds. $\square$

Definition 3.3.3. For a positive integer n, let G_n be the graph obtained from the complete graph K_n by adding a path of length 2 for each pair of vertices of K_n, i.e., $V(G_n) = \{v_i \mid 1 \leq i \leq n\} \cup \{v_{ij} \mid 1 \leq i < j \leq n\}$ and $E(G_n) = \{v_i v_j \mid 1 \leq i < j \leq n\} \cup \{v_i v_{ij} \mid 1 \leq i < j \leq n\} \cup \{v_j v_{ij} \mid 1 \leq i < j \leq n\}$.

Definition 3.3.4. For a positive integer m, the *Ramsey number* $r(m, m, m)$

is the smallest positive integer r such that any 3-edge-colored complete graph K_r of order r contains a monochromatic complete graph K_m of order m.

Lemma 3.3.5. *Let m be a positive integer at least 3 and let n be an integer greater than or equal to the Ramsey number $r(m,m,m)$. If $\dim_{\mathrm{poc}}(G_n) \leq 3$, then there exists a sequence $(x_1, \ldots, x_m)$ of vertices of G_n such that $\{x_1, \ldots, x_m\}$ is a clique of G_n and that any subsequence $(x_{i_1}, \ldots, x_{i_l})$ of $(x_1, \ldots, x_m)$ is consecutively tail-biting, where $2 \leq l \leq m$ and $1 \leq i_1 < \cdots < i_l \leq m$.*

Proof. Since the vertices v_i and v_j of G_n are internal vertices of an induced path of length three by the definition of G_n, it follows from Lemma 3.1.6 that the vertices v_i and v_j of G_n are crossing. By Lemma 3.1.7, for any $1 \leq i < j \leq n$, there exists $k \in \{1,2,3\}$ such that $v_i \stackrel{k}{\to} v_j$ or $v_j \stackrel{k}{\to} v_i$. Now we define an edge-coloring $c : \{v_iv_j \mid 1 \leq i < j \leq n\} \to \{1,2,3\}$ as follows: For $1 \leq i < j \leq n$, we let $c(v_iv_j) = k$ so that $v_i \stackrel{k}{\to} v_j$ or $v_j \stackrel{k}{\to} v_i$. Then, by the definition of $r(m,m,m)$, K_n contains a monochromatic complete subgraph K with m vertices.

Suppose that the edges of K have color k, where $k \in \{1,2,3\}$. We assign an orientation to each edge xy of K so that x goes toward y if $x \stackrel{k}{\to} y$. In that way, we obtain a tournament $\overrightarrow{K}$ with m vertices. It is well-known that every tournament has a directed Hamiltonian path. Therefore, $\overrightarrow{K}$ has a directed Hamiltonian path. Let $x_1 \to x_2 \to \cdots \to x_m$ be a directed Hamiltonian path of $\overrightarrow{K}$. Then, by Lemma 3.1.9, $x_i \stackrel{k}{\to} x_j$ for any $i < j$. Thus any subsequence $(x_{i_1}, \ldots, x_{i_l})$ of $(x_1, \ldots, x_m)$ is consecutively tail-biting, where $2 \leq l \leq m$ and $1 \leq i_1 < \cdots < i_l \leq m$. $\square$

Since the graph G_n is chordal, the following theorem shows the existence of chordal graphs with partial order competition dimensions greater than three. Given a graph G and a set X consisting of six vertices in G, we say that X *induces an* $\overline{\mathrm{H}}$ if it induces a subgraph of G isomorphic to $\overline{\mathrm{H}}$.

Theorem 3.3.6. *For $n \geq r(5,5,5)$, $\dim_{\mathrm{poc}}(G_n) > 3$.*

Proof. We prove by contradiction. Suppose that $\dim_{\mathrm{poc}}(G_n) \leq 3$ for some $n \geq r(5,5,5)$. By Lemma 3.3.5, G_n contains a consecutively tail-biting sequence $(v_1, \ldots, v_5)$ of five vertices in Type k such that $\{v_1, \ldots, v_5\}$ is a clique of G_n and that $(v_{i_1}, v_{i_2}, v_{i_3})$ is a consecutively tail-biting sequence for any $1 \leq i_1 < i_2 < i_3 \leq 5$ and $(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4})$ is a consecutively tail-biting sequence for any $1 \leq i_1 < i_2 < i_3 < i_4 \leq 5$. Without loss of generality, we may assume that $k = 3$.

Since $\{v_1, v_2, v_3, v_{13}\}$ induces a diamond and (v_1, v_2, v_3) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 3.3.1 that $p_1^{(v_{13})} \in R_1(v_2)$ and $p_2^{(v_{13})} \notin R_2(v_2)$ hold or $p_1^{(v_{13})} \notin R_1(v_2)$ and $p_2^{(v_{13})} \in R_2(v_2)$ hold.

We first suppose that $p_1^{(v_{13})} \in R_1(v_2)$ and $p_2^{(v_{13})} \notin R_2(v_2)$. Since $\{v_1, v_2, v_3, v_4, v_{13}, v_{24}\}$ induces an $\overline{\mathrm{H}}$ and (v_1, v_2, v_3, v_4) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 3.3.2 and $p_1^{(v_{13})} \in R_1(v_2)$ that $p_2^{(v_{24})} \in R_2(v_3)$. Since $\{v_1, v_2, v_3, v_5, v_{13}, v_{25}\}$ induces an $\overline{\mathrm{H}}$ and (v_1, v_2, v_3, v_5) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 3.3.2 and $p_1^{(v_{13})} \in R_1(v_2)$ that

$$p_2^{(v_{25})} \in R_2(v_3). \tag{3.3.1}$$

Since $\{v_2, v_3, v_4, v_5, v_{24}, v_{35}\}$ induces an $\overline{\mathrm{H}}$ and (v_2, v_3, v_4, v_5) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 3.3.2 and $p_2^{(v_{24})} \in R_2(v_3)$ that

$$p_1^{(v_{35})} \in R_1(v_4). \tag{3.3.2}$$

Since $\{v_1, v_3, v_4, v_{14}\}$ induces a diamond and (v_1, v_3, v_4) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 3.3.1 that $p_1^{(v_{14})} \in R_1(v_3)$ and $p_2^{(v_{14})} \notin R_2(v_3)$ hold or $p_1^{(v_{14})} \notin R_1(v_3)$ and $p_2^{(v_{14})} \in R_2(v_3)$ hold. Suppose that $p_1^{(v_{14})} \in R_1(v_3)$ and $p_2^{(v_{14})} \notin R_2(v_3)$. Since $\{v_1, v_3, v_4, v_5, v_{14}, v_{35}\}$ induces an $\overline{\mathrm{H}}$ and (v_1, v_3, v_4, v_5) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 3.3.2 and $p_1^{(v_{14})} \in R_1(v_3)$ that

$$p_2^{(v_{35})} \in R_2(v_4). \tag{3.3.3}$$

Since $\{v_3, v_4, v_5, v_{35}\}$ induces a diamond and (v_3, v_4, v_5) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 3.3.1 that $p_1^{(v_{35})} \in R_1(v_4)$ and $p_2^{(v_{35})} \notin R_2(v_4)$ hold or $p_1^{(v_{35})} \notin R_1(v_4)$ and $p_2^{(v_{35})} \in R_2(v_4)$ hold, which is a contradiction to the fact that both (3.3.2) and (3.3.3) hold. Thus

$$p_1^{(v_{14})} \notin R_1(v_3) \text{ and } p_2^{(v_{14})} \in R_2(v_3). \tag{3.3.4}$$

Since $\{v_1, v_2, v_4, v_{14}\}$ induces a diamond and (v_1, v_2, v_4) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 3.3.1 that $p_1^{(v_{14})} \in R_1(v_2)$ and $p_2^{(v_{14})} \notin R_2(v_2)$ hold or $p_1^{(v_{14})} \notin R_1(v_2)$ and $p_2^{(v_{14})} \in R_2(v_2)$ hold. Suppose that $p_1^{(v_{14})} \notin R_1(v_2)$ and $p_2^{(v_{14})} \in R_2(v_2)$. Since $\{v_1, v_2, v_4, v_5, v_{14}, v_{25}\}$ induces an $\overline{\mathrm{H}}$ and (v_1, v_2, v_4, v_5) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 3.3.2 and $p_2^{(v_{14})} \in R_2(v_2)$ that

$$p_1^{(v_{25})} \in R_1(v_4). \tag{3.3.5}$$

By (3.3.1) and (3.3.5), since $A(v_4)$ and $A(v_{25})$ are homothetic, we have

$$p_2^{(v_{25})} \in R_2(v_4). \tag{3.3.6}$$

Since $\{v_2, v_4, v_5, v_{25}\}$ induces a diamond and (v_2, v_4, v_5) is a consecutively tail-biting sequence of Type 3, it follows from Lemma 3.3.1 that $p_1^{(v_{25})} \in R_1(v_4)$ and $p_2^{(v_{25})} \notin R_2(v_4)$ hold or $p_1^{(v_{25})} \notin R_1(v_4)$ and $p_2^{(v_{25})} \in R_2(v_4)$ hold, which is a contradiction to the fact that both (3.3.5) and (3.3.6) hold. Thus $p_1^{(v_{14})} \in R_1(v_2)$ and $p_2^{(v_{14})} \notin R_2(v_2)$.

Since $A(v_3)$ and $A(v_{14})$ are homothetic, we have

$$p_1^{(v_{14})} \in R_1(v_3), \tag{3.3.7}$$

contradicting (3.3.4).

In the case where $p_1^{(v_{13})} \notin R_1(v_2)$ and $p_2^{(v_{13})} \in R_2(v_2)$, we also reach a contradiction by applying a similar argument.

Hence, $\dim_{\text{poc}}(G_n) > 3$ holds for any $n \geq r(5,5,5)$. $\square$

Remark 3.3.7. We have shown that there are chordal graphs with partial order competition dimension greater than three and that there are chordal graphs with partial order competition dimension at most three such as chordal diamond-free graphs. In this vein, it would be interesting to characterize the chordal graphs with partial order competition dimension at most three.

Chapter 4

The partial order competition dimensions of bipartite graphs[1]

In this chapter, we compute the partial order competition dimensions of all the complete bipartite graphs. We show that every bipartite graph has partial order competition dimension at most four, and it is equal to four when the bipartite graph contains $K_{3,3}$ as a subgraph. To do so, we introduce a useful notion "the order type for two points in $\mathbb{R}^3$" and give an upper bound of the POC of a graph in terms of its chromatic number. We also utilize the upper bound to show that the partial order competition dimension of every planar graph is at most 4.

4.1 Order types of two points in $\mathbb{R}^3$

Take two distinct points $u := (u_1, u_2, u_3)$ and $v := (v_1, v_2, v_3)$ in $\mathbb{R}^3$. For a nonempty proper subset S of $\{1, 2, 3\}$, we write $u \preceq_S v$ if $u_i \leq v_i$ for each $i \in S$.

[1]The material in the section is from the paper "The partial order competition dimensions of bipartite graphs" by Jihoon Choi, Soogang Eoh, Suh-Ryung Kim, Yeun Lee, and Prof. Yoshio Sano. The author thanks Soogang Eoh, Prof. Suh-Ryung Kim, Dr. Jung Yeun Lee, and Prof. Yoshio Sano for allowing him to use its contents for his thesis.

Suppose that u and v are incomparable in $\mathbb{R}^3$. Then there exists a partition $\{S_1, S_2\}$ of the set $\{1, 2, 3\}$ such that either $u \preceq_{S_1} v$ and $v \preceq_{S_2} u$ or $u \preceq_{S_2} v$ and $v \preceq_{S_1} u$ (the equalities cannot hold at the same time in each case). We call such a partition $\{S_1, S_2\}$ an *order type* for $\{u, v\}$.

The following lemma is obvious by the definition of order types for a pair of points in $\mathbb{R}^3$.

Lemma 4.1.1. *Let u and v be distinct incomparable points in $\mathbb{R}^3$. Then $\{u, v\}$ has order types $\{\{1\}, \{2, 3\}\}$ or $\{\{2\}, \{1, 3\}\}$ or $\{\{3\}, \{1, 2\}\}$.*

Note that order types of $\{u, v\}$ are not unique, for example, for the two points $u = (1, 2, 5)$ and $v = (1, 3, 4)$ in $\mathbb{R}^3$, $\{u, v\}$ has order types $\{\{1, 2\}, \{3\}\}$ and $\{\{2\}, \{1, 3\}\}$.

The order types of two points in $\mathbb{R}^3$ have the following geometric interpretation. Two sets in $\mathbb{R}^d$ are said to be *homothetic* if the two sets are related by a geometric contraction or expansion.

For $k \in \{1, 2, 3\}$ and $v \in \mathcal{H}_+$, let $\angle_k(v)$ be the following region:

$$\angle_k(v) := \{x \in \mathbb{R}^3 \mid x = p_k^{(v)} + \alpha(p_i^{(v)} - p_k^{(v)}) + \beta(p_j^{(v)} - p_k^{(v)}), 0 \leq \alpha \in \mathbb{R}, 0 \leq \beta \in \mathbb{R}\},$$

where i and j are elements such that $\{i, j, k\} = \{1, 2, 3\}$ (see Figure 4.1).

By Lemma 4.1.1, a pair $\{u, v\}$ of incomparable points in $\mathbb{R}^3$ has an order type $\{\{k\}, \{1, 2, 3\} \setminus \{k\}\}$ for some $k \in \{1, 2, 3\}$. Now we are ready to give a geometric interpretation for a pair $\{u, v\}$ having an order type $\{\{k\}, \{1, 2, 3\} \setminus \{k\}\}$.

Lemma 4.1.2. *For two incomparable points u and v in $\mathcal{H}_+$ and for an integer $k \in \{1, 2, 3\}$, $v \preceq_{\{k\}} u$ and $u \preceq_{\{1,2,3\} \setminus \{k\}} v$ if and only if $\angle_k(u) \subsetneq \angle_k(v)$.*

Proof. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$. Since $v \in \mathcal{H}_+$, $v_1 + v_2 + v_3 > 0$.

Without loss of generality, we may assume $k = 1$. Now

$$\begin{aligned}
p_1^{(u)} &= (-u_2 - u_3, u_2, u_3) \\
&= (-v_2 - v_3, v_2, v_3) + ((v_2 - u_2) + (v_3 - u_3), u_2 - v_2, u_3 - v_3) \\
&= (-v_2 - v_3, v_2, v_3) + (v_2 - u_2, u_2 - v_2, 0) + (v_3 - u_3, 0, u_3 - v_3) \\
&= p_1^{(v)} + \alpha(v_1 + v_2 + v_3, -v_1 - v_2 - v_3, 0) + \beta(v_1 + v_2 + v_3, 0, -v_1 - v_2 - v_3) \\
&= p_1^{(v)} + \alpha(p_2^{(v)} - p_1^{(v)}) + \beta(p_3^{(v)} - p_1^{(v)})
\end{aligned}$$

where $\alpha = \dfrac{v_2 - u_2}{v_1 + v_2 + v_3}$ and $\beta = \dfrac{v_3 - u_3}{v_1 + v_2 + v_3}$. Since $p_2^{(v)} - p_1^{(v)}$ and $p_3^{(v)} - p_1^{(v)}$ form a basis of $\mathcal{H}$, α and β are the unique coefficients satisfying the above equalities. Then

$$\begin{aligned}
v \preceq_{\{1\}} u,\ u \preceq_{\{2,3\}} v &\Leftrightarrow u_2 \leq v_2,\ u_3 \leq v_3 && (u \text{ and } v \text{ are incomparable}) \\
&\Leftrightarrow \alpha \geq 0 \text{ and } \beta \geq 0 && (v_1 + v_2 + v_3 > 0) \\
&\Leftrightarrow p_1^{(u)} \in \angle_1(v) \\
&\Leftrightarrow \angle_1(u) \subset \angle_1(v) && (A(u) \text{ and } A(v) \text{ are homothetic}) \\
&\Leftrightarrow \angle_1(u) \subsetneq \angle_1(v) && (u \text{ and } v \text{ are distinct})
\end{aligned}$$

and so the lemma follows. □

Theorem 4.1.3. *For two incomparable points u and v in $\mathcal{H}_+$ and for an integer $k \in \{1, 2, 3\}$, the pair $\{u, v\}$ has an order type $\{\{k\}, \{1, 2, 3\} \setminus \{k\}\}$ if and only if $\angle_k(u) \subsetneq \angle_k(v)$ or $\angle_k(v) \subsetneq \angle_k(u)$.*

Proof. By Lemma 4.1.2, $v \preceq_{\{k\}} u$ and $u \preceq_{\{1,2,3\}\setminus\{k\}} v$ if and only if $\angle_k(u) \subsetneq \angle_k(v)$, and $u \preceq_{\{k\}} v$ and $v \preceq_{\{1,2,3\}\setminus\{k\}} u$ if and only if $\angle_k(v) \subsetneq \angle_k(u)$ for any integer $k \in \{1, 2, 3\}$. Therefore the theorem follows. □

Let D be a 3-partial order. Since D can be embedded in $\mathbb{R}^3$, we may identify a vertex in D with a point in $\mathbb{R}^3$. In this vein, for two vertices x and y in D, we use both expressions "$x \prec y$" and "(y, x) is an arc in a 3-partial order" without distinction.

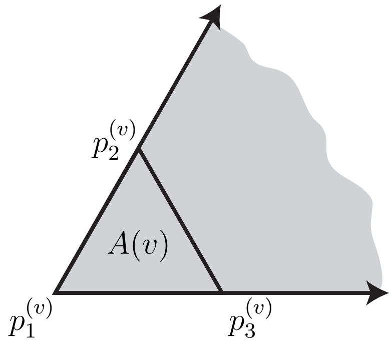

Figure 4.1: The shaded region represents $\angle_1(v)$.

Lemma 4.1.4. *Suppose that there is a common order type* $\{S_1, S_2\}$ *for each pair of the three points* $x_1, x_2, x_3 \in \mathbb{R}^3$. *Then, for some permutation* σ *on* $\{1, 2, 3\}$, $x_{\sigma(1)} \preceq_{S_1} x_{\sigma(2)} \preceq_{S_1} x_{\sigma(3)}$ *and* $x_{\sigma(1)} \succeq_{S_2} x_{\sigma(2)} \succeq_{S_2} x_{\sigma(3)}$.

Proof. By the hypothesis, $\preceq_{S_1}$ is a total order on $\{x_1, x_2, x_3\}$. Therefore there exists a permutation σ on $\{1, 2, 3\}$ such that $x_{\sigma(1)} \preceq_{S_1} x_{\sigma(2)} \preceq_{S_1} x_{\sigma(3)}$. Since $y_1 \preceq_{S_1} y_2$ if and only if $y_1 \succeq_{S_2} y_2$ for any pair $\{y_1, y_2\}$ of points in $\mathbb{R}^d$ with the order type $\{S_1, S_2\}$, the lemma immediately follows. □

Lemma 4.1.5. *Let* D *be a* 3*-partial order and let* x *and* y *be non-isolated vertices in* $C(D)$. *If* x *and* y *are nonadjacent in* $C(D)$, *then* x *and* y *are incomparable in* $\mathbb{R}^3$.

Proof. By contradiction. Suppose $x \preceq y$ or $y \preceq x$. By symmetry, we may assume $x \preceq y$. Since x is non-isolated, x has a neighbor z in $C(D)$. Then there exists $w \in V(D)$ such that w is a common out-neighbor of x and z in D, i.e., $w \prec x$ and $w \prec z$. By the assumption $x \preceq y$, $w \prec x$ implies $w \prec y$. Therefore w is a common out-neighbor of x and y in D. Thus x and y are adjacent in $C(D)$, which is a contradiction. □

Theorem 4.1.6. *Suppose that* $xyzwx$ *is an induced cycle of length* 4 *in the competition graph of a* 3*-partial order* D. *Then* $\{x, z\}$ *and* $\{y, w\}$ *do not share an order type in common.*

Proof. By identifying the vertices of D with points in $\mathbb{R}^3$, we may assign coordinates in $\mathbb{R}^3$ to x, y, z, and w so that $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3)$, and $w = (w_1, w_2, w_3)$. By Lemma 4.1.5, x and z are incomparable, and so are y and w. Therefore the order types for $\{x, z\}$ and the order types for $\{y, w\}$ are well-defined. Suppose to the contrary that $\{x, z\}$ and $\{y, w\}$ share a common order type. Without loss of generality, we may assume that they share $\{\{1\}, \{2, 3\}\}$ as a common order type and that $x \preceq_{\{1\}} z$ and $y \preceq_{\{1\}} w$, i.e.,

$$x_1 \leq z_1, \quad z_2 \leq x_2, \quad z_3 \leq x_3; \qquad y_1 \leq w_1, \quad w_2 \leq y_2, \quad w_3 \leq y_3. \tag{4.1.1}$$

Since yz is an edge in $C(D)$, there exists a vertex $a = (a_1, a_2, a_3)$ in D such that $a \prec y$ and $a \prec z$, from which the inequalities $a_2 < z_2$, $a_3 < z_3$, and $a_1 < y_1$ follow. Then $a_2 < z_2$ and $z_2 \leq x_2$ in (4.1.1) give $a_2 < x_2$. Similarly, $a_3 < z_3$ and $z_3 \leq x_3$ give $a_3 < x_3$. If $y_1 \leq x_1$, then $a_1 < x_1$ by the inequality $a_1 < y_1$, so we have $a \prec x$, which contradicts the fact that x and z are not adjacent in $C(D)$. Thus $y_1 > x_1$.

Since xw is an edge in $C(D)$, there exists a vertex $b = (b_1, b_2, b_3)$ in D such that $b \prec x$ and $b \prec w$. Then we apply an argument parallel to the one given in the previous paragraph to deduce $x_1 > y_1$. Now we have $y_1 > x_1$ and $x_1 > y_1$, which is impossible. Hence the theorem holds. ☐

4.2 An upper bound for the the partial order competition dimension of a graph

In this section, we derive an upper bound for the partial order competition dimension of a graph in terms of its chromatic number. We utilize it to show that any graph with chromatic number 4 has the partial order competition dimension 4 if it contains the cocktail-party graph as a subgraph. with four partite sets as an induced subgraph. In addition, we present some graphs with partial order competition dimension greater than three.

Theorem 4.2.1. *For an integer $n \geq 4$ and a graph G with $\chi(G) \leq n \left\lfloor \frac{n-1}{2} \right\rfloor$, $\dim_{\text{poc}}(G) \leq n$.*

Proof. From a graph with chromatic number less than k, we may construct a graph with chromatic number k by adding sufficient many vertices so that the added vertices are adjacent to each other and to each vertex of the given graph. Therefore, by Proposition 2.3.8, it is sufficient to prove the statement for a graph with chromatic number $n \left\lfloor \frac{n-1}{2} \right\rfloor$. Let G be a graph with chromatic number $n \left\lfloor \frac{n-1}{2} \right\rfloor$. Then the vertex set of G is partitioned into $n \left\lfloor \frac{n-1}{2} \right\rfloor$ independent sets. By the same proposition, we may assume that the independent sets have the same size. Let q be the size of each independent set. Now we group the independent sets into n groups $V_1, \ldots, V_n$ of the same size and denote the independent sets belonging to V_j by $V_{1,j}, V_{2,j}, \ldots, V_{\left\lfloor \frac{n-1}{2} \right\rfloor, j}$ for each $j = 1, \ldots, n$.

To have the vertices of G embedded in $\mathbb{R}^n$, we will assign coordinates in $\mathbb{R}^n$ to each vertex of G as follows. We start with a square matrix N of order n such that the (n, j)-entry is j, the $(n-1, j)$-entry is $j-1$, the $(1, j)$-entry is $j-2$, the $(2, j)$-entry is $j-3$, (we identify 0, -1, and -2 with n, $n-1$, and $n-2$, respectively) for each $j = 1, \ldots, n$, and the remaining entries of N may be any integers in $\{1, \ldots, n\}$ as long as no same number appears twice in each column (while a row may contain identical elements). See Figure 4.2 for an example.

For each $j = 1, \ldots, n$, we carry out the following procedure. We let $\mathbf{c}_j$ be the $q \left\lfloor \frac{n-1}{2} \right\rfloor \times 1$ matrix which contains $V_{1,j}, V_{2,j}, \ldots, V_{\left\lfloor \frac{n-1}{2} \right\rfloor, j}$ in this order as blocks and in which the block corresponding to $V_{i,j}$ is a $q \times 1$ matrix whose entries are the vertices of $V_{i,j}$ for $i = 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor$. Then we denote by $\mathbf{c}_j^{-1}$ the $q \left\lfloor \frac{n-1}{2} \right\rfloor \times 1$ matrix obtained by reversing the order of the blocks in $\mathbf{c}_j$ and the order of the vertices in each block. Now, for $i = 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor$, we cyclically permute the blocks corresponding to $V_{i,j}$ in $\mathbf{c}_j$ and $\mathbf{c}_j^{-1}$ l times and denote the resulting matrix by $\mathbf{c}_{j,l}$ and $\mathbf{c}_{j,l}^{-1}$ for each $l = 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor - 1$.

$$N = \begin{bmatrix} 6 & 7 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 & 1 & 2 & 2 \\ 4 & 4 & 5 & 6 & 7 & 1 & 3 \\ 3 & 3 & 4 & 5 & 6 & 7 & 1 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix}$$

Figure 4.2: An example of N when $n = 7$

$$\mathbf{c}_j = \begin{bmatrix} V_{1,j} \\ V_{2,j} \\ V_{3,j} \end{bmatrix}, \quad \mathbf{c}_j^{-1} = \begin{bmatrix} V_{3,j}^{-1} \\ V_{2,j}^{-1} \\ V_{1,j}^{-1} \end{bmatrix},$$

$$\mathcal{C}_j = \left\{ \mathbf{c}_{j,1} := \begin{bmatrix} V_{3,j} \\ V_{1,j} \\ V_{2,j} \end{bmatrix}, \mathbf{c}_{j,2} := \begin{bmatrix} V_{2,j} \\ V_{3,j} \\ V_{1,j} \end{bmatrix}, \mathbf{c}_{j,1}^{-1} := \begin{bmatrix} V_{2,j}^{-1} \\ V_{1,j}^{-1} \\ V_{3,j}^{-1} \end{bmatrix}, \mathbf{c}_{j,2}^{-1} := \begin{bmatrix} V_{1,j}^{-1} \\ V_{3,j}^{-1} \\ V_{2,j}^{-1} \end{bmatrix} \right\}.$$

Figure 4.3: Examples of $\mathbf{c}_j$, $\mathbf{c}_j^{-1}$, and $\mathcal{C}_j$ for $j = 1, \ldots, 7$ when $n = 7$. We regard $V_{i,j}$ in each $3q \times 1$ matrix as a $q \times 1$ block with the entries in the set $V_{i,j}$. In addition, $V_{i,j}^{-1}$ means the $q \times 1$ matrix obtained by turning $V_{i,j}$ upside down.

Let $\mathcal{C}_j$ be the set of $\mathbf{c}_{j,l}$ and $\mathbf{c}_{j,l}^{-1}$ for each $l = 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor - 1$. Obviously $|\mathcal{C}_j| = 2\left\lfloor \frac{n-1}{2} \right\rfloor - 2$. Refer to Figure 4.3 for an illustration.

We will replace each entry of N with a $q\left\lfloor \frac{n-1}{2} \right\rfloor \times 1$ matrix with entries in $V(G)$ to obtain a matrix M in the following way. For each $j = 1, \ldots, n$, we replace j in the nth row of N with $\mathbf{c}_j$, and j in the second row of N and j in the $(n-1)$st row of N with $\mathbf{c}_j^{-1}$ (refer to Figure 4.4 for an illustration). Now we wish to locate each matrix in $\bigcup_{j=1}^{n} \mathcal{C}_j$ exactly once in the $n-3$ remaining rows of N so that j in each row is replaced with an $q\left\lfloor \frac{n-1}{2} \right\rfloor \times 1$ matrix in $\mathcal{C}_j$ for each $j = 1, \ldots, n$. If n is odd, then the desired process works to give M in which an $q\left\lfloor \frac{n-1}{2} \right\rfloor \times 1$ matrix in $\mathcal{C}_j$ appears in each column of M exactly once for each $j = 1, \ldots, n$. If n is even, then there is one j in N which has not yet

$$M = \begin{bmatrix} \mathbf{c}_{6,1} & \mathbf{c}_{7,1} & \mathbf{c}_{1,1} & \mathbf{c}_{2,1} & \mathbf{c}_{3,1} & \mathbf{c}_{4,1} & \mathbf{c}_{5,1} \\ \mathbf{c}_5^{-1} & \mathbf{c}_6^{-1} & \mathbf{c}_7^{-1} & \mathbf{c}_1^{-1} & \mathbf{c}_2^{-1} & \mathbf{c}_3^{-1} & \mathbf{c}_4^{-1} \\ \mathbf{c}_{2,2} & \mathbf{c}_{5,2} & \mathbf{c}_{6,2} & \mathbf{c}_{7,2} & \mathbf{c}_{1,2} & \mathbf{c}_{2,1}^{-1} & \mathbf{c}_{2,2}^{-1} \\ \mathbf{c}_{4,2} & \mathbf{c}_{4,1}^{-1} & \mathbf{c}_{5,1}^{-1} & \mathbf{c}_{6,1}^{-1} & \mathbf{c}_{7,1}^{-1} & \mathbf{c}_{1,1}^{-1} & \mathbf{c}_{3,2} \\ \mathbf{c}_{3,1}^{-1} & \mathbf{c}_{3,2}^{-1} & \mathbf{c}_{4,2}^{-1} & \mathbf{c}_{5,2}^{-1} & \mathbf{c}_{6,2}^{-1} & \mathbf{c}_{7,2}^{-1} & \mathbf{c}_{1,2}^{-1} \\ \mathbf{c}_7^{-1} & \mathbf{c}_1^{-1} & \mathbf{c}_2^{-1} & \mathbf{c}_3^{-1} & \mathbf{c}_4^{-1} & \mathbf{c}_5^{-1} & \mathbf{c}_6^{-1} \\ \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 & \mathbf{c}_5 & \mathbf{c}_6 & \mathbf{c}_7 \end{bmatrix}$$

Figure 4.4: An example of M when $n = 7$

been replaced and we replace it with a matrix arbitrarily chosen from $\mathcal{C}_j$ for each $j = 1, \ldots, n$.

We are ready to assign a coordinate $[v]$ to each vertex v of G. Fix $j \in \{1, \ldots, n\}$. We determine the jth component of $[v]$ as follows. By definition, v appears exactly once in each column of M. Suppose that v appears in the ith row from the bottom in the jth column for some $i \in \{1, \ldots, nq \left\lfloor \frac{n-1}{2} \right\rfloor\}$. Then we let the jth component of $[v]$ be $2i$. We denote the jth component of $[v]$ by $[v]_j$ for each $j = 1, \ldots, n$.

For two adjacent vertices v and w in G, we add a vertex x_{vw} to G satisfying $[x_{vw}]_j = \min\{[v]_j, [w]_j\} - 1$ for each $j = 1, \ldots, n$. Then, for two adjacent vertices v and w, $x_{vw} \prec v$ and $x_{vw} \prec w$ in $\mathbb{R}^n$. Let I be the set of those added vertices and D be the n-partial order induced by the points in $V(G) \cup I$. In order to show that the competition graph of D is G together with I, we need to check the following:

(i) There is no arc between any two vertices of G in D.

(ii) For any adjacent vertices v and w in G, $N^+(x_{vw}) = \emptyset$ and $N^-(x_{vw}) = \{v, w\}$ where $N^+(x_{vw})$ and $N^-(x_{vw})$ denote the out-neighborhood and the in-neighborhood of x_{vw}, respectively, in D.

To show (i), we take two vertices v and w in G. Then $v \in V_i$ and $w \in V_j$

for some $i, j \in \{1, \ldots, n\}$. If $i = j$, then there exist a column in which v is above w and a column in which w is above v since there exist a column containing $\mathbf{c}_i$ and a column containing $\mathbf{c}_i^{-1}$. If $i \neq j$, then either $\mathbf{c}_i$ or $\mathbf{c}_i^{-1}$ is above $\mathbf{c}_j$ in the jth column while either $\mathbf{c}_j$ or $\mathbf{c}_j^{-1}$ is above $\mathbf{c}_i$ in the ith column, and so v is above w in the jth column while w is above v in the ith column. Therefore,

(§) there exists a column in which v is above w for any two vertices v and w in G

and so any two vertices of G are not adjacent in D.

To show (ii), we take two adjacent vertices v and w and a vertex u in G distinct from v and w. In addition, we take a vertex in I distinct from x_{vw}. Then it is in the form of $x_{v'w'}$ for some adjacent vertices v' and w' such that $\{v, w\} \neq \{v', w'\}$. Without loss of generality, we may assume v and v' are vertices not in $\{v, w\} \cap \{v', w'\}$.

Then $u \in V_{p,i}$, $v \in V_{q,j}$, and $w \in V_{r,k}$ for some $i, j, k \in \{1, \ldots, n\}$ and $p, q, r \in \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$. If $i \neq j$ and $i \neq k$, then there exists a column of M in which u is below v and w since one of $\mathbf{c}_j$, $\mathbf{c}_j^{-1}$ and one of $\mathbf{c}_k$, $\mathbf{c}_k^{-1}$ are above $\mathbf{c}_i$ in the ith column. To show that such a column exists even for the case $i = j$ or $i = k$, suppose $i = j$ or $i = k$. Without loss of generality, we may assume $i = j$. We first consider the case $i = k$. Then $j = k$, which implies $q \neq r$ as v and w are adjacent. Then $p \neq q$ or $p \neq r$. Since $i = j = k$, u, v, and w belong to $\mathbf{c}_i$ and $\mathbf{c}_i^{-1}$ and we may assume $p \neq q$. Assume that u is below w in $\mathbf{c}_i$ (resp. $\mathbf{c}_i^{-1}$). If v is below u in $\mathbf{c}_i$ (resp. $\mathbf{c}_i^{-1}$), then there exists an $q \lfloor \frac{n-1}{2} \rfloor \times 1$ matrix in $\mathcal{C}_i$ which was obtained by cyclically permuting the blocks in $\mathbf{c}_i$ (resp. $\mathbf{c}_i^{-1}$) and in which the block corresponding to $V_{p,i}$ is at the bottom. Since u is below w in $\mathbf{c}_i$ or $\mathbf{c}_i^{-1}$, we may conclude that there is a column of M in which u is below v and w in this case. Finally consider the case $i \neq k$. If u is below v in $\mathbf{c}_i$, then u is below v and w in the ith column. Suppose that u is above v in $\mathbf{c}_i$. Then u is below v in $\mathbf{c}_i^{-1}$. Thus, if $k \neq i + 1$ (resp. $k = i + 1$), then u is below v and w in the $(i + 1)$st (resp.

$(i+3)$rd) column of M (we identify $n+1$, $n+2$, and $n+3$ with 1, 2, and 3, respectively). Hence we have shown that, for adjacent vertices v and w and a vertex u distinct from v and w, there exists a column of M in which u is below v and w, that is,

$$[u]_l \leq \min\{[v]_l, [w]_l\} - 2 \tag{4.2.1}$$

for some $l \in \{1, \ldots, n\}$. Therefore u cannot be an out-neighbor of x_{vw}. On the other hand, there exists a column of M in which u is above v by (§), so, by the definition of x_{vw},

$$[x_{vw}]_{l'} \leq [v]_{l'} < [u]_{l'}$$

for some $l' \in \{1, \ldots, n\}$. Thus u cannot be an in-neighbor of x_{vw}. Hence

$$N^+(x_{vw}) \cap V(G) = \emptyset \qquad \text{and} \qquad N^-(x_{vw}) \cap V(G) = \{v, w\}. \tag{4.2.2}$$

Suppose to the contrary that x_{vw} and $x_{v'w'}$ are adjacent in D. Without loss of generality, we may assume $x_{v'w'} \prec x_{vw}$. By the definition of x_{vw}, $x_{vw} \prec v$ and $x_{vw} \prec w$. Then, by the transitivity of $\prec$, $x_{v'w'} \prec v$, which contradicts (4.2.2) as v and w were arbitrarily chosen adjacent vertices. Therefore

$$N^+(x_{vw}) \cap I = N^-(x_{vw}) \cap I = \emptyset. \tag{4.2.3}$$

Thus, by (4.2.2) and (4.2.3), (ii) holds. Hence we have shown that the competition graph of D is G together with $|E(G)|$ isolated vertices and $\dim_{\text{poc}}(G) \leq n$. □

As a planar graph has chromatic number at most 4 by the four color theorem, we have the following corollary immediately.

Corollary 4.2.2. *For every planar graph G, $\dim_{\text{poc}}(G) \leq 4$.*

Recall that, for a positive integer n, the cocktail-party graph $K_{2\times n}$ is the complete multipartite graph with n partite sets all of which are of size two, i.e., $K_{2\times n}$ is the graph defined by

$$V(K_{2\times n}) = \bigcup_{i=1}^{n}\{x_i, y_i\}, \qquad E(K_{2\times n}) = \bigcup_{i=1}^{n}\bigcup_{j=i+1}^{n}\{x_ix_j, x_iy_j, x_jy_i, y_iy_j\}.$$

Theorem 4.2.3. *The partial order competition dimension of $K_{2\times 4}$ is 4.*

Proof. Since $\chi(K_{2\times 4}) = 4$, $\dim_{\mathrm{poc}}(K_{2\times 4}) \leq 4$ by Theorem 4.2.1.

Suppose to the contrary that $\dim_{\mathrm{poc}}(K_{2\times 4}) \leq 3$. Then $K_{2\times 4}$ together with some isolated vertices is the competition graph of a 3-partial order. Let $\{x_i, y_i\}$ ($i = 1, \ldots, 4$) be the 4 partite sets of $K_{2\times 4}$. Then x_i and y_i are incomparable in $\mathbb{R}^3$ by Lemma 4.1.5. Thus, by Lemma 4.1.1, order types for $\{x_i, y_i\}$ exist for each $i = 1, \ldots, 4$. Since there are only three possible order types by the same lemma, there exist at least two partite sets $\{x_i, y_i\}$ and $\{x_j, y_j\}$ which have a common order type by the Pigeonhole principle. However, $x_ix_jy_iy_jx_i$ is an induced 4-cycle of $K_{2\times 4}$ and we reach a contradiction to Theorem 4.1.6. Thus $\dim_{\mathrm{poc}}(K_{2\times 4}) \geq 4$ and the theorem is true. $\square$

By Proposition 2.3.8 and Theorems 4.2.1 and 4.2.3, the following corollary is immediately true:

Corollary 4.2.4. *If a graph G contains $K_{2\times 4}$ as an induced subgraph and has chromatic number 4, then $\dim_{\mathrm{poc}}(G) = 4$.*

Since $K_{2\times n}$, $K_{m_1,\ldots,m_n}$, and the complement of the cycle C_l contains the cocktail-party graph $K_{2\times 4}$ as an induced subgraph for $n \geq 4$, $m_i \geq 2$ for each $i = 1, \ldots, n$, and $l \geq 12$, we obtain the following corollaries.

Corollary 4.2.5. *For any integer $n \geq 4$, $\dim_{\mathrm{poc}}(K_{2\times n}) > 3$.*

Corollary 4.2.6. *Let $K_{m_1,\ldots,m_n}$ be a complete multipartite graph with at least four partite sets of size at least two. Then $\dim_{\mathrm{poc}}(K_{m_1,\ldots,m_n}) > 3$.*

Let $\overline{C_n}$ denote the complement of a cycle of length n.

Corollary 4.2.7. *For any integer $n \geq 12$,* $\dim_{\text{poc}}(G) > 3$*.*

We can slightly improve Corollary 4.2.7 as follows.

Theorem 4.2.8. *For any integer $n \geq 10$,* $\dim_{\text{poc}}(\overline{C_n}) > 3$*.*

Proof. Take an integer $n \geq 10$ and let $C_n = v_1v_2 \ldots v_nv_1$. Suppose to the contrary that $\dim_{\text{poc}}(\overline{C_n}) \leq 3$. Then, by definition, $\overline{C_n}$ together with some isolated vertices is the competition graph of a 3-partial order. By Lemma 4.1.5, for $i = 1, \ldots, n$, v_i and v_{i+1} are incomparable in $\mathbb{R}^3$ and so, by Lemma 4.1.1, the order types for $\{v_i, v_{i+1}\}$ exist (we identify v_{n+1} with v_1). Without loss of generality, we may assume that $\{\{1\}, \{2, 3\}\}$ is an order type for $\{v_1, v_2\}$. For each $i \in \{4, \ldots, n-2\}$, $v_1v_iv_2v_{i+1}v_1$ is an induced 4-cycle of $\overline{C_n}$, so, by Theorem 4.1.6, $\{v_i, v_{i+1}\}$ does not have the order type $\{\{1\}, \{2, 3\}\}$ for any $i \in \{4, \ldots, n-2\}$. Therefore $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$, $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$ are the only possible consecutive pairs of vertices on C_n that might have the order type $\{\{1\}, \{2, 3\}\}$. Note that $v_{n-1}v_2v_nv_3v_{n-1}$ and $v_nv_3v_1v_4v_n$ are induced 4-cycles of $\overline{C_n}$. By Theorem 4.1.6, neither $\{v_2, v_3\}$ and $\{v_{n-1}, v_n\}$ nor $\{v_1, v_n\}$ and $\{v_3, v_4\}$ share a common order type. Thus at most three consecutive pairs of vertices on C_n have the order type $\{\{1\}, \{2, 3\}\}$. By a similar argument, at most three consecutive pairs have the order type $\{\{2\}, \{1, 3\}\}$ and at most three consecutive pairs have the order type $\{\{3\}, \{1, 2\}\}$. Hence $n \leq 9$ and we reach a contradiction. □

4.3 Partial order competition dimensions of bipartite graphs

In this section, we show that the partial order competition dimension of the graphs containing $K_{3,3}$ as an induced subgraph is 4 and compute partial order competition dimensions of complete bipartite graphs.

Theorem 4.3.1. *For any graph G containing $K_{3,3}$ as an induced subgraph,* $\dim_{\text{poc}}(G) > 3$*.*

Proof. By Proposition 2.3.8, it suffices to show $\dim_{\mathrm{poc}}(K_{3,3}) > 3$. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ be the partite sets of $K_{3,3}$. Suppose to the contrary that $\dim_{\mathrm{poc}}(K_{3,3}) \leq 3$. Then $K_{3,3}$ together with some isolated vertices becomes the competition graph of a 3-partial order D.

Since X is an independent set in $K_{3,3}$, any two vertices in X are incomparable by Lemma 4.1.5. Similarly, any two vertices in Y are incomparable. If there exist $x \in X$ and $y \in Y$ satisfying $x \preceq y$ or $y \preceq x$, $N(x) \subset N(y)$ or $N(y) \subset N(x)$ in $K_{3,3}$, which is impossible. Therefore x and y are incomparable for any $x \in X$ and $y \in Y$. This observation together with the fact that each of the nine edges in $K_{3,3}$ is a maximal clique imply that we need at least nine additional isolated vertices inducing the nine edges of $K_{3,3}$.

We denote the isolated vertex inducing the edge x_iy_j by z_{ij} for $i, j \in \{1, 2, 3\}$. We may assign coordinates to x_i, y_j, z_{ij} in $\mathbb{R}^3$ so that $x_i = (x_i^{(1)}, x_i^{(2)}, x_i^{(3)})$, $y_j = (y_j^{(1)}, y_j^{(2)}, y_j^{(3)})$, and $z_{ij} = (z_{ij}^{(1)}, z_{ij}^{(2)}, z_{ij}^{(3)})$ for all $i, j \in \{1, 2, 3\}$.

Since any two vertices in X (resp. Y) are incomparable, and any vertex in X and any vertex in Y are incomparable by the above observation, the order types for $\{x_i, x_j\}$ (resp. $\{y_i, y_j\}$) for distinct i and j in $\{1, 2, 3\}$, and the order types for $\{x, y\}$ for $x \in X$ and $y \in Y$ are well-defined.

Take a pair $\{x_i, x_j\}$ from X and a pair $\{y_k, y_l\}$ from Y. Then $x_iy_kx_jy_lx_i$ is an induced 4-cycle in $K_{3,3}$ and so $\{x_i, x_j\}$ and $\{y_k, y_l\}$ do not share a common order type by Theorem 4.1.6. Since the pairs $\{x_i, x_j\}$ and $\{y_k, y_l\}$ were arbitrarily chosen, we can conclude that every pair from X and every pair from Y do not share a common order type.

If two pairs from X have distinct order types and two pairs from Y have distinct order types, then one of the two pairs from X and one of the two pairs from Y must share a common order type by the Pigeonhole principle since there are only three order types, which contradicts the previous observation. Thus either the three pairs from X or the three pairs from Y share a common order type. Without loss of generality, we may assume that $\{x_1, x_2\}$, $\{x_2, x_3\}$, and $\{x_1, x_3\}$ share an order type $\{\{1\}, \{2, 3\}\}$. Then, by Lemma 4.1.4, we may assume that $x_1 \preceq_{\{1\}} x_2 \preceq_{\{1\}} x_3$, i.e.,

$$x_1^{(1)} \leq x_2^{(1)} \leq x_3^{(1)}, \quad x_1^{(2)} \geq x_2^{(2)} \geq x_3^{(2)}, \quad x_1^{(3)} \geq x_2^{(3)} \geq x_3^{(3)}. \tag{4.3.1}$$

Suppose that there exists $i \in \{1, 2, 3\}$ such that $y_i^{(1)} \leq x_2^{(1)}$. Since z_{3i} is the isolated vertex inducing the edge x_3y_i, we have $z_{3i} \prec x_3$ and $z_{3i} \prec y_i$. Therefore we have three inequalities $z_{3i}^{(2)} < x_3^{(2)}$, $z_{3i}^{(3)} < x_3^{(3)}$, and $z_{3i}^{(1)} < y_i^{(1)}$. Then the first inequality $z_{3i}^{(2)} < x_3^{(2)}$ and the second inequality $z_{3i}^{(3)} < x_3^{(3)}$ and (4.3.1) give $z_{3i}^{(2)} < x_2^{(2)}$ and $z_{3i}^{(3)} < x_2^{(3)}$. In addition, the third inequality $z_{3i}^{(1)} < y_i^{(1)}$ and the assumption $y_i^{(1)} \leq x_2^{(1)}$ give $z_{3i}^{(1)} < x_2^{(1)}$. Therefore we have shown $z_{3i} \prec x_2$. Then x_2 and x_3 are adjacent in $K_{3,3}$ and reach a contradiction. Thus $y_i^{(1)} > x_2^{(1)}$ for each $i = 1, 2, 3$.

To show that $\{x_2, y_i\}$ does not have an order type $\{\{1\}, \{2, 3\}\}$ for any $i = 1, 2, 3$, suppose to the contrary that there exists $i \in \{1, 2, 3\}$ such that $x_2^{(2)} \geq y_i^{(2)}$ and $x_2^{(3)} \geq y_i^{(3)}$. Since z_{1i} is the isolated vertex inducing the edge x_1y_i, we have $z_{1i} \prec x_1$ and $z_{1i} \prec y_i$. Therefore we have three inequalities $z_{1i}^{(1)} < x_1^{(1)}$, $z_{1i}^{(2)} < y_i^{(2)}$, and $z_{1i}^{(3)} < y_i^{(3)}$. Then $z_{1i}^{(1)} < x_1^{(1)}$ and the first inequality in (4.3.1) give $z_{1i}^{(1)} < x_2^{(1)}$. In addition, $z_{1i}^{(2)} < y_i^{(2)}$, $z_{1i}^{(3)} < y_i^{(3)}$ together with our assumption that $x_2^{(2)} \geq y_i^{(2)}$ and $x_2^{(3)} \geq y_i^{(3)}$ give $z_{1i}^{(2)} < x_2^{(2)}$ and $z_{1i}^{(3)} < x_2^{(3)}$. Therefore we have shown $z_{1i} \prec x_2$. Then x_1 and x_2 are adjacent in $K_{3,3}$, which is impossible. Thus $\{x_2, y_i\}$ does not have an order type $\{\{1\}, \{2, 3\}\}$ for any $i = 1, 2, 3$. Hence $\{x_2, y_i\}$ has an order type $\{\{2\}, \{1, 3\}\}$ or $\{\{3\}, \{1, 2\}\}$ for each $i = 1, 2, 3$. Then, by the Pigeonhole principle, there exists two vertices $y_j, y_k \in Y$ such that $\{x_2, y_j\}$ and $\{x_2, y_k\}$ share a common order type. Without loss of generality, we may assume that they have $\{\{2\}, \{1, 3\}\}$ as a common order type. As we have shown that $y_j^{(1)} > x_2^{(1)}$ and $y_k^{(1)} > x_2^{(1)}$, we have

$$y_j^{(1)} > x_2^{(1)}, \quad y_j^{(2)} \leq x_2^{(2)}, \quad y_j^{(3)} \geq x_2^{(3)}; \quad y_k^{(1)} > x_2^{(1)}, \quad y_k^{(2)} \leq x_2^{(2)}, \quad y_k^{(3)} \geq x_2^{(3)}. \tag{4.3.2}$$

Without loss of generality, we may assume $y_j^{(2)} \leq y_k^{(2)}$. Since z_{2i} is the isolated vertex inducing the edge x_2y_j, we have $z_{2i} \prec x_2$ and and $z_{2i} \prec y_j$. Therefore we have three inequalities $z_{2i}^{(1)} < x_2^{(1)}$, $z_{2i}^{(3)} < x_2^{(3)}$, and $z_{2i}^{(2)} < y_j^{(2)}$. Then the first and the second inequalities together with the fourth and the sixth inequalities in (4.3.2) give $z_{2i}^{(1)} < y_k^{(1)}$ and $z_{2i}^{(3)} < y_k^{(3)}$. In addition, the third inequality and the assumption $y_j^{(2)} \leq y_k^{(2)}$ give $z_{2i}^{(2)} < y_k^{(2)}$. Therefore we have shown $z_{2i} \prec y_k$. Then y_j and y_k are adjacent in $K_{3,3}$, which is impossible. Hence $\dim_{\text{poc}}(K_{3,3}) > 3$. □

Since any n-partite graph has chromatic number at most 4 for $n = 2, 3, 4$, by Theorems 4.2.1 and 4.3.1, the following corollaries are immediately true:

Corollary 4.3.2. *For any n-partite graph containing $K_{3,3}$ as an induced subgraph for $n = 2, 3, 4$, $\dim_{\text{poc}}(G) = 4$.*

Corollary 4.3.3. *For any positive integers $m, n \geq 3$, $\dim_{\text{poc}}(K_{m,n}) = 4$.*

Remark 4.3.4. By Proposition 2.3.10, $\dim_{\text{poc}}(K_{1,1}) = 1$. By Proposition 2.3.12, $\dim_{\text{poc}}(K_{1,n}) = \dim_{\text{poc}}(K_{n,1}) = 2$ for $n \geq 2$.

By Theorem 2.2.9, the family of homothetic equilateral triangles given in Figure 4.5 makes $K_{2,n}$ together with $2n$ isolated vertices into the competition graph of a 3-partial order. Thus $\dim_{\text{poc}}(K_{2,n}) = \dim_{\text{poc}}(K_{n,2}) \leq 3$. By Propositions 2.3.13 and 2.3.8, $\dim_{\text{poc}}(K_{2,n}) = \dim_{\text{poc}}(K_{n,2}) \geq 3$ for $n \geq 2$, so $\dim_{\text{poc}}(K_{2,n}) = \dim_{\text{poc}}(K_{n,2}) = 3$ for $n \geq 2$.

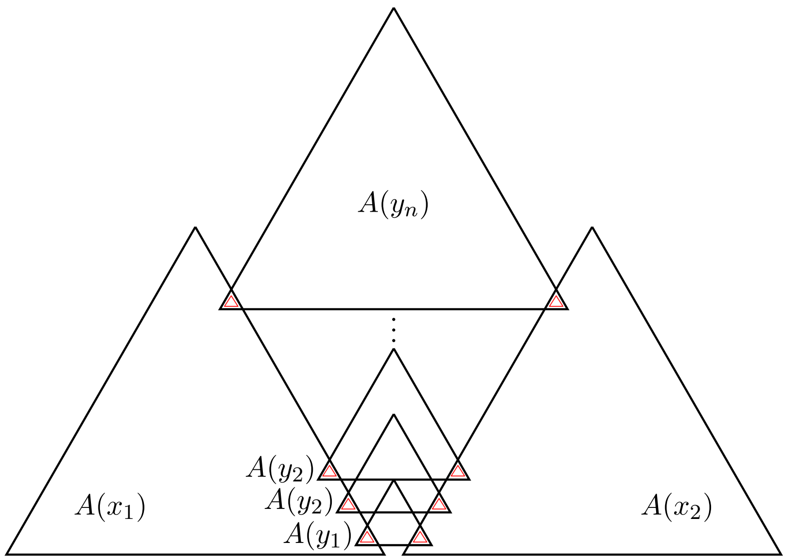

Figure 4.5: A family of homothetic equilateral triangles on the hyperplane $x + y + z = 0$ making $K_{2,n}$ with bipartition $(\{x_1, x_2\}, \{y_1, \ldots, y_n\})$ together with $2n$ isolated vertices into the competition graph of a 3-partial order.

Chapter 5

On the m-step competition graphs of d-partial orders[1]

In this chapter, we defined the notion of m-step competition graph which is an important variants of competition graph. Then we study the m-step competition graphs of d-partial orders and generalize some results on the competition graphs of d-partial orders.

For two vertices x and y in a d-partial order D which satisfy $x \prec y$ in $\mathbb{R}^d$, we sometimes write $x \overset{D}{\prec} y$ to emphasize the digraph D. For a graph G and a nonnegative integer k, $G \cup I_k$ is the graph obtained from G by adding k isolated vertices.

5.1 A characterization of the m-step competition graphs of d-partial orders

Recall that, in Section 2.2.2, for a point $\mathbf{p}$ in $\mathcal{H}^d_+$, $\triangle^{d-1}(\mathbf{p})$ is a closure of $A^{d-1}(\mathbf{p})$ with respect to the usual topology in $\mathbb{R}^d$, and that $\mathcal{F}^{d-1}$ is defined to be the set of regular $(d-1)$-simplices in $\mathbb{R}^d$ which are contained in the

[1]The material in the section is from the paper "On the m-step competition graphs of d-partial orders" by Jihoon Choi.

hyperplane $\mathcal{H}^d$ and homothetic to $\triangle^{d-1}(\mathbf{1})$. Then the bijection $\varphi \circ f_* : \mathcal{H}^d_+ \to \mathcal{F}^{d-1}$ defined in 2.2.2 naturally induces the following bijection.

Corollary 5.1.1. *For each integer $d \geq 2$, the function $\triangle^{d-1} : \mathcal{H}^d_+ \to \mathcal{F}^{d-1}$ mapping $\mathbf{p}$ to $\triangle^{d-1}(\mathbf{p})$ is a bijection.*

As an analogue of Theorem 2.2.9 which characterizes the competition graphs of d-partial orders, we present the following theorem which characterizes the m-step competition graphs of d-partial orders.

Theorem 5.1.2. *For positive integers d and m, a graph G is the m-step competition graph of a d-partial order if and only if there exist a subset $\mathcal{F}$ of $\mathcal{F}^{d-1}$ and a bijection $f : V(G) \to \mathcal{F}$ such that*

- ($\star$) *two vertices v and w are adjacent in G if and only if there exist sequences $(v_0, v_1, \ldots, v_m)$ and $(w_0, w_1, \ldots, w_m)$ of vertices of G such that $v_0 = v$, $w_0 = w$, $v_m = w_m$, the interior of $f(v_i)$ (resp. $f(w_i)$) includes $f(v_{i+1})$ (resp. $f(w_{i+1})$) for $i = 0, 1, \ldots, m-1$.*

Proof. ($\Rightarrow$) Suppose that G is the m-step competition graph of a d-partial order D. By translating each vertex of D by $T : v \mapsto v + k\mathbf{1}$ for a sufficiently large real number k, we may obtain a d-partial order all of whose vertices belong to $\mathcal{H}^d_+$ and is isomorphic to D. Therefore we may assume all the vertices of D are located in $\mathcal{H}^d_+$. Let $\mathcal{F} = \{\triangle^{d-1}(v) \mid v \in V(D)\}$. Then $\mathcal{F} \subset \mathcal{F}^{d-1}$. Let $f : V(G) \to \mathcal{F}$ be the function defined by $f(v) = \triangle^{d-1}(v)$. By Corollary 5.1.1, f is a bijection. The condition ($\star$) immediately follows from the definition of m-step competition graph and Proposition 2.2.7.

($\Leftarrow$) Suppose that there exist a subset $\mathcal{F}$ of $\mathcal{F}^{d-1}$ and a bijection $f : V(G) \to \mathcal{F}$ such that the condition ($\star$) holds. By Corollary 5.1.1, each element in $\mathcal{F}$ can be written as $\triangle^{d-1}(\mathbf{p})$ for some $\mathbf{p} \in \mathcal{H}^d_+$. Let $S = \{\mathbf{p} \in \mathbb{R}^d \mid \triangle(\mathbf{p}) \in \mathcal{F}\}$. Then D_S is a d-partial order. Take two vertices v and w in G. Then, by Proposition 2.2.7 and the condition ($\star$), v and w are adjacent in G if and only if v and w have a common m-step prey in D_S. Therefore G is the m-step competition graph of the d-partial order D_S. $\square$

5.2 Partial order m-step competition dimensions of graphs

In this section, we introduce the notion of partial order m-step competition dimension of a graph as a generalization of that of partial order competition dimension of a graph and investigate basic properties of the m-step competition graphs of d-partial orders in terms of it.

Lemma 5.2.1. *Let D be a transitive digraph and m be a positive integer. Then an m-step prey of x is a k-step prey of x for each $k = 1, \ldots, m$.*

Proof. Let y be an m-step prey of x. Then there exists a directed path P from x to y of length m. Since D is transitive, there is an arc from x to every vertex on P. Therefore there exists a directed path from x to y of length k for each $k = 1, \ldots, m$. □

Lemma 5.2.2. *Let D be a d-partial order for a positive integer d. Then there exists $(d+1)$-partial order $\tilde{D}$ which is isomorphic to D.*

Proof. The lemma immediately follows from the proof of Proposition 2.3.1. □

The following theorem is a generalization of Proposition 2.3.1.

Proposition 5.2.3. *If a graph G is the m-step competition graph of a d-partial order for some positive integers m and d, then G is the m-step competition graph of a $(d+1)$-partial order.*

Proof. Let G be the m-step competition graph of a d-partial order D for some positive integers m and d. Then, by Lemma 5.2.2, there exists a $(d+1)$-partial order $\tilde{D}$ which is isomorphic to D. Therefore $G = C^m(D) \cong C^m(\tilde{D})$. Thus G is the m-step competition graph of a $(d+1)$-partial order. □

A vertex subset K of a graph G is called a *clique* if every pair of vertices in K is an edge in G. For a clique K and an edge e of a graph G, we say that K *covers* e (or e *is covered by* K) if and only if K contains the two end points of e.

Theorem 5.2.4. *Let G be a graph and m be a positive integer. Then there exist positive integers d and k such that G together with k isolated vertices is the m-step competition graph of a d-partial order.*

Proof. Let $|V(G)| = n$ and label the vertices of G as $v_1, \ldots, v_n$. We define a map $\phi : V(\mathcal{H}) \to \mathbb{R}^n$ by

$$\phi(v_i)_j = \begin{cases} 2 & \text{if } j = i; \\ 4 & \text{if } j \neq i. \end{cases}$$

Let $\theta = \theta_e(G)$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_\theta\}$ be an edge clique cover of G consisting of maximal cliques in G. For each $i \in \{1, \ldots, \theta\}$, we define a map $\psi_i : \mathcal{C} \to \mathbb{R}^n$ by $\psi_i(C)_k = 1 - \frac{i}{m+1}$ if $v_k \in C$ and $\psi_i(C)_k = 3 - \frac{i}{m+1}$ if $v_k \notin C$.

Let $V = \phi(V(G)) \cup \left(\bigcup_{i=1}^{\theta} \psi(\mathcal{C}) \right) = \{\phi(v_i) \mid v_i \in V(G)\} \cup \left(\bigcup_{i=1}^{\theta} \{\psi(C) \mid C \in \mathcal{C}\} \right) \subseteq \mathbb{R}^n$. Then V defines an n-partial order D. By the construction of D, it easily be checked that Thus $C^m(D) = G \cup I_{m\theta}$. Hence, by taking $d = n$ and $k = m\theta$, we complete the proof. □

By Proposition 5.2.3 and Theorem 5.2.4, we are ready to define the notion partial order competition dimension of a graph.

Definition 5.2.5. For a graph G and a positive integer m, we define the *partial order m-step competition dimension* $\dim_{\text{poc}}(G; m)$ of G as the smallest nonnegative integer d such that G together with k isolated vertices is the m-step competition graph of D for some d-partial order D and some nonnegative integer k, i.e.,

$$\dim_{\text{poc}}(G; m) := \min\{d \in \mathbb{Z}_{\geq 0} \mid \exists k \in \mathbb{Z}_{\geq 0}, \exists S \subseteq \mathbb{R}^d, \text{ s.t. } G \cup I_k = C^m(D_S)\}.$$

It immediately follows from the definition that $\dim_{\mathrm{poc}}(G;1) = \dim_{\mathrm{poc}}(G)$ for every graph G.

Proposition 5.2.6. *For any graph G and any positive integer m,* $\dim_{\mathrm{poc}}(G;m) \leq |V(G)|$.

Proof. The proposition follows from the construction of the d-partial order D in the proof of Theorem 5.2.4. □

As Propositions 2.3.9, 2.3.10, and 2.3.12 characterize graphs with small partial order competition dimensions, it is natural to ask which graphs have small partial order m-step competition dimensions. For a positive integer m, it is easy to characterize graphs G with $\dim_{\mathrm{poc}}(G;m) \leq 1$.

Proposition 5.2.7. *Let G be a graph and m be a positive integer m. Then* $\dim_{\mathrm{poc}}(G;m) = 0$ *if and only if $G = K_1$.*

Proof. It is clear by the definition of a 0-partial order. □

Proposition 5.2.8. *Let G be a graph and m be a positive integer m. Then* $\dim_{\mathrm{poc}}(G;m) \leq 1$ *if and only if $G = K_t \cup I_s$ for some nonnegative integers t and s with $t \geq 1$ and $s \leq m$. Especially,* $\dim_{\mathrm{poc}}(G;m) = 1$ *if and only if $G = K_{t+1}$ or $G = K_t \cup I_s$ for some positive integers t and s with $s \leq m$.*

Proof. Let D be a 1-partial order with $V(D) = \{v_1, \ldots, v_n\}$. We may assume that the vertices of D are labeled so that $v_1 < v_2 < \cdots < v_n$ in $\mathbb{R}$. Then, for each $i \leq m$, v_i has no m-step prey in D, so it is an isolated vertex in $C^m(D)$. If $i > m$, then v_i has v_1 as an m-step prey in D and so the set $\{v_{m+1}, v_{m+2}, \ldots, v_n\}$ is a clique in $C^m(D)$ unless it is empty. Since we may take one of the isolated vertices as a clique of size one if the set $\{v_{m+1}, v_{m+2}, \ldots, v_n\}$ is empty, we may conclude that $G = K_t \cup I_s$ for some nonnegative integers t and s with $t \geq 1$ and $s \leq m$ if $\dim_{\mathrm{poc}}(G;m) \leq 1$.

Conversely, suppose that $G = K_t \cup I_s$ for some nonnegative integers t and s with $t \geq 1$ and $s \leq m$. We denote the vertices of in K_t by $x_1, \ldots, x_t$ and the vertices in I_s by $y_1, \ldots, y_s$ if $s \neq 0$. Then we assign a coordinate in $\mathbb{R}$ to each vertex of G by $y_i = i$ for $i = 1, \ldots, s$ and $x_j = j + s$ for $j = 1, \ldots, t$. Take a set J of $m - s$ points in $\mathbb{R}$ with negative coordinates. Then the set $V(G) \cup J \subset \mathbb{R}$ induces a 1-partial order and its m-step competition graph is $G \cup I_{m-s}$. Thus $\dim_{\mathrm{poc}}(G; m) \leq 1$.

The equality part follows from Propositions 5.2.7. $\square$

Park *et al.* [47] studied the m-step competition graphs of 2-partial orders and presented the following results.

Theorem 5.2.9 ([47])**.** *For any positive integer m, the m-step competition graph of a 2-partial order is an interval graph.*

Theorem 5.2.10 ([47])**.** *For any positive integer m, an interval graph with sufficiently many isolated vertices is the m-step competition graph of a 2-partial order.*

In terms of partial order m-step competition dimension, the results of Park *et al.* [47] can be restated as follows:

Proposition 5.2.11. *Let G be a graph and m be a positive integer. Then* $\dim_{\mathrm{poc}}(G; m) \leq 2$ *if and only if G is an interval graph.*

Proof. The 'if' part immediately follows from Theorem 5.2.10. To show the 'only if' part, suppose $\dim_{\mathrm{poc}}(G; m) \leq 2$. Then there exists a 2-partial order D whose m-step competition graph equals $G \cup I_k$ for some $k \in \mathbb{Z}_{\geq 0}$. By Theorem 5.2.9, $G \cup I_k$ is interval and so G is interval. $\square$

The following proposition implies that deleting some isolated vertices from a graph does not increase the partial order m-step competition dimension.

Proposition 5.2.12. *For a graph G and positive integers k and m,* $\dim_{\mathrm{poc}}(G; m) \leq \dim_{\mathrm{poc}}(G \cup I_k; m)$.

Proof. Let $d = \dim_{\text{poc}}(G \cup I_k; m)$. Then there exists a d-partial order D whose m-step competition graph equals $(G \cup I_k) \cup I_s$ for some $s \in \mathbb{Z}_{\geq 0}$. Since $(G \cup I_k) \cup I_s = G \cup I_{k+s}$, $\dim_{\text{poc}}(G; m) \leq d$. $\square$

As a matter of fact, in Proposition 5.2.12, the equality mostly holds except for some specific graphs.

Proposition 5.2.13. *For a graph G and positive integers m and k, $\dim_{\text{poc}}(G \cup I_k; m) > \dim_{\text{poc}}(G; m)$ if and only if $G = K_1$ or $G = K_t \cup I_s$ for some non-negative integers t and s with $t \geq 1$ and $m - k < s \leq m$.*

Proof. ($\Leftarrow$) By Propositions 5.2.7 and 5.2.8, $\dim_{\text{poc}}(K_1 \cup I_k; m) = 1 > 0 = \dim_{\text{poc}}(K_1; m)$. Suppose that $G = K_t \cup I_s$ for some nonnegative integers t and s with $t \geq 1$ and $m - k < s \leq m$. Since $G \cup I_k = G \cup I_{s+k}$ and $s + k > m$, $\dim_{\text{poc}}(G \cup I_k; m) > 1$ by Proposition 5.2.8. Yet, since $s \leq m$, $\dim_{\text{poc}}(G; m) = 1$ by the same proposition. Therefore $\dim_{\text{poc}}(G \cup I_k; m) > \dim_{\text{poc}}(G; m)$.

($\Rightarrow$) Let $d = \dim_{\text{poc}}(G; m)$. Assume $d \geq 2$. Then there exists a d-partial order D whose m-step competition graph is $G \cup I_s$ for some $s \in \mathbb{Z}_{\geq 0}$. We let

$$\alpha = \max\{v_1 \mid (v_1, v_2, \ldots, v_d) \in V(D)\} \quad \text{and} \quad \beta = \min\{v_2 \mid (v_1, v_2, \ldots, v_d) \in V(D)\},$$

which are well-defined as $d \geq 2$. Let $z_i = (\alpha + i, \beta - i, 0, \ldots, 0) \in \mathbb{R}^d$ for each $i = 1, \ldots, k$ and $S = V(D) \cup \{z_1, \ldots, z_k\} \subset \mathbb{R}^d$. Then D_S is a d-partial order. By definition, no vertex in $\{z_1, \ldots, z_k\}$ is comparable with any vertex of D_S in $\mathbb{R}^d$. Therefore $C^m(D_S) = C^m(D) \cup I_k = (G \cup I_s) \cup I_k = (G \cup I_k) \cup I_s$. Thus $\dim_{\text{poc}}(G \cup I_k; m) \leq d$, which contradicts the hypothesis that $\dim_{\text{poc}}(G \cup I_k; m) > \dim_{\text{poc}}(G; m)$. Hence $d \leq 1$. Then, by Proposition 5.2.8, $G = K_t \cup I_s$ for some nonnegative integers t and s with $t \geq 1$ and $s \leq m$. If $s > m - k$, when we are done. Suppose $s \leq m - k$ or $s + k \leq m$. Then, since $G \cup I_k = G \cup I_{s+k}$,

$$d = \dim_{\text{poc}}(G; m) < \dim_{\text{poc}}(G \cup I_k; m) \leq 1$$

by our assumption and Proposition 5.2.8. Thus $d = 0$. Hence $G = K_1$ by Proposition 5.2.7. $\square$

5.3 $\dim_{\text{poc}}(G; m)$ in the aspect of $\dim_{\text{poc}}(G)$

We recall that $\dim_{\text{poc}}(G) = \dim_{\text{poc}}(G; 1)$ for every graph G. As Cho and Kim [8] showed that the interval graphs are exactly the graphs satisfying $\dim_{\text{poc}}(G) \leq 2$, Proposition 5.2.11 tells us that $\dim_{\text{poc}}(G) \leq 2$ if and only if $\dim_{\text{poc}}(G; m) \leq 2$ for every positive integer m. Now it is natural to ask whether or not $\dim_{\text{poc}}(G) \leq d$ if and only if $\dim_{\text{poc}}(G; m) \leq d$ for a graph G and positive integers m and d with $d \geq 2$. In this section, we shall answer this question.

Definition 5.3.1. Let d be a positive integer. A d-partial order D is said to satisfy the *distinct coordinate property* (*DC-property* for short) provided that, for each $i = 1, \ldots, d$, the i^{th} coordinates of the vertices of D are all distinct.

For a d-partial order D and an ordered pair $(i, k) \in \{1, \ldots, d\} \times \mathbb{R}$, we partition $V(D)$ into three disjoint subsets

$$\begin{aligned}
V_{i,k}(D) &= \{(a_1, \ldots, a_d) \in V(D) \mid a_i = k\}, \\
V^+_{i,k}(D) &= \{(a_1, \ldots, a_d) \in V(D) \mid a_i > k\}, \\
V^-_{i,k}(D) &= \{(a_1, \ldots, a_d) \in V(D) \mid a_i < k\},
\end{aligned}$$

and let $\Gamma(D) = \{(i, k) \in \{1, \ldots, d\} \times \mathbb{R} : |V_{i,k}(D)| \geq 2\}$.

It is clear from the definition that a d-partial order D satisfies the DC-property if and only if $\Gamma(D) = \emptyset$.

Proposition 5.3.2. *Given a positive integer d and a d-partial order D, there exists a d-partial order D' isomorphic to D such that such that D' satisfies the DC-property.*

Proof. If $\Gamma(D) = \emptyset$, then we let $D' = D$ to finish the proof. Suppose $\Gamma(D) \neq \emptyset$. Take an element $(i,k) \in \Gamma(D)$. Let $V_{i,k} = \{v_1, \ldots, v_l\}$ $(l \geq 2)$ and $V^*_{i,k} = \{v^*_1, \ldots, v^*_\ell\}$ where v^*_j is the point in $\mathbb{R}^{d-1}$ obtained from $v_j \in \mathbb{R}^d$ by deleting its i^{th} coordinate. Then $V^*_{i,k}$ defines a $(d-1)$-partial orders D^*. Since D^* is acyclic, we may assume that the labeling of the vertices of $V_{i,k}$ guarantees that $v^*_j \overset{D^*}{\prec} v^*_{j'}$ only if $j > j'$. Now we define a new d-partial order $D_{i,k}$ with vertex set $\{\phi_{i,k}(v) \in \mathbb{R}^d \mid v \in V(D)\}$ with $\phi_{i,k}(v)$ defined by

$$\phi_{i,j}(v) = \begin{cases} v & \text{if } v \in V^-_{i,k}, \\ v + je_j, & \text{if } v \in V_{i,k} \text{ and } v = v_j, \\ v + \ell e_j & \text{if } v \in V^+_{i,k}, \end{cases}$$

where e_j is the j^{th} standard basis vector in $\mathbb{R}^d$. By the way of construction, $D_{i,k}$ is isomorphic to D and $|\Gamma(D_{i,k})| = |\Gamma(D)| - 1$. If $\Gamma(D_{i,k}) = \emptyset$, then we let $D' = D_{i,k}$ to finish the proof. Otherwise, we repeat this process until we obtain a d-partial order D' satisfying $\Gamma(D') = \emptyset$, which is a desired digraph. □

For a directed path P in a digraph, the *length* $\ell(P)$ of P is defined to be the number of arcs in P.

Lemma 5.3.3. *Let G be the m-step competition graph of a d-partial D for some positive integers m and d. If vertices u and v are adjacent in G, then they have an m-step common prey which has outdegree 0 in D.*

Proof. Take two adjacent vertices u and v in G. Then, by the definition $C^m(D)$, u and v have an m-step common prey, say z, in D. Take a longest directed path P starting from z in D. Let w be the terminus of P. Then w is an $(m + \ell(P))$-step common prey of u and v. Therefore w is an m-step common prey of u and v by Lemma 5.2.1. If w had an out-neighbor in D, it would either P would extend to a longer directed path or a directed cycle would be yielded in D, both of which are impossible. Thus w is a desired vertex. $\square$

Theorem 5.3.4. *For a graph G and a positive integer m,* $\dim_{\mathrm{poc}}(G;m) \geq \dim_{\mathrm{poc}}(G;m+1)$.

Proof. If $G = K_1$, then $\dim_{\mathrm{poc}}(G;m) = 0 = \dim_{\mathrm{poc}}(G;m+1)$ by Proposition 5.2.7. Suppose $G \neq K_1$. Then $\dim_{\mathrm{poc}}(G;m) \geq 1$ by Proposition 5.2.7. Let $d = \dim_{\mathrm{poc}}(G;m)$. Then there exists a d-partial order D such that $C^m(D) = G \cup I_k$ for some $k \in \mathbb{Z}_{\geq 0}$. By Proposition 5.3.2, we may assume that D satisfies the DC-property. Then $\delta > 0$ for

$$\delta = \min_i \{|a_i - b_i| : (a_1, \ldots, a_d) \text{ and } (b_1, \ldots, b_d) \text{ are distinct vertices of } D\}.$$

Let Y be the set of vertices of D with outdegree 0. Then $Y \neq \emptyset$ since D is acyclic. Let

$$Z = \left\{ \phi(y) \mid \phi(y) = y - \frac{\delta}{2}(1, \ldots, 1) \in \mathbb{R}^d, y \in Y \right\}.$$

Then the set $S := V(D) \cup Z \subset \mathbb{R}^d$ defines the d-partial order D_S. By the DC-property of D and by choice of δ, it is not difficult to check that $N^-_{D_S}(\phi(y)) = \{y\} \cup N^-_D(y)$ and $N^+_{D_S}(\phi(y)) = \emptyset$ for each $y \in Y$. Furthermore, by the definitions of Y and Z, the set of vertices of outdegree 0 in D_S is Z and the set of vertices of outdegree 1 in D_S is Y.

We claim that $C^m(D)$ and $C^{m+1}(D_S)$ have the same edge sets. Take an edge uv in $C^m(D)$. By Lemma 5.3.3, u and v have an m-step common prey y which has outdegree 0 in D. Since Y is the set of vertices of D with outdegree 0, $y \in Y$. Since $\phi(y)$ is a 1-step prey of y in D_S, $\phi(y)$ is an $(m+1)$-step common prey of u and v in D_S. Hence uv is an edge in $C^{m+1}(D_S)$.

Conversely, take an edge uv in $C^{m+1}(D_S)$. By Lemma 5.3.3, u and v have an $(m+1)$-step common prey z which has outdegree 0 in D_S. Then there exist directed paths

$$P_u : u = u_0 \to u_1 \to \cdots \to u_{m-1} \to u_m \to u_{m+1} = z$$

and

$$P_v : v = v_0 \to v_1 \to \cdots \to v_{m-1} \to v_m \to v_{m+1} = z$$

of length $m+1$ in D_S. Since Z is the set of vertices of D_S with outdegree 0, $z \in Z$ and so $z = \phi(y)$ for some $y \in Y$. Since D_S is transitive and $u_{m-1} \to u_m \to \phi(y)$ in D_S, we have $u_{m-1} \to \phi(y)$. Then $u_{m-1} \in N^-_{D_S}(\phi(y)) = \{y\} \cup N^-_D(y)$. However, $u_{m-1} \neq y$, for otherwise u_{m-1} has outdegree 1 in D_S, which is impossible as $u_{m-1} \to u_m$ and $u_{m-1} \to \phi(y)$. Thus $u_{m-1} \in N^-_D(y)$. Hence the sequence $P'_u : u = u_0 \to u_1 \to \cdots u_{m-1} \to y$ is a directed path in D from u to y of length m. Similarly, $P'_v : v = v_0 \to v_1 \to \cdots v_{m-1} \to y$ is a directed path in D from v to y of length m. Then y is an m-step common prey of u and v in D. Therefore uv is an edge in $C^m(D)$.

Thus we have shown that $C^m(D)$ and $C^{m+1}(D_S)$ have the same edge set. Since $C^m(D) = G \cup I_k$, we have $C^{m+1}(D_S) = (G \cup I_k) \cup I_\ell = G \cup I_{k+\ell}$ where $\ell = |Z|$. Hence $\dim_{\mathrm{poc}}(G; m+1) \leq d$. $\square$

By applying induction on m, we obtain the following corollary from Theorem 5.3.4.

Corollary 5.3.5. *For every graph G and every positive integer m,* $\dim_{\mathrm{poc}}(G) \geq \dim_{\mathrm{poc}}(G; m)$.

5.4 Partial order competition exponents of graphs

In this section, we define an analogue concept of exponent for graphs in the aspect of partial order m-step competition dimensions.

For a $(0,1)$-matrix with Boolean operation, it is well known that the matrix sequence $\{A^m\}_{m=1}^{\infty}$ converges to the all-one matrix J if and only if A is primitive. The smallest positive integer M such that $m \geq M$ implies $A^m = J$ is called the *exponent* of A.

Let G be a graph. Then the integer-valued sequence $\{\dim_{\text{poc}}(G;m)\}_{m=1}^{\infty}$ is bounded by Proposition 5.2.6 and decreasing by Theorem 5.3.4. Therefore there exists a positive integer M such that $\dim_{\text{poc}}(G;m)$ is constant for all $m \geq M$. We call the smallest such M the *partial order competition exponent* of G and denote it by $\exp_{\text{poc}}(G)$.

Proposition 5.4.1. *For any graph G with* $\dim_{\text{poc}}(G;1) = 1$, $\exp_{\text{poc}}(G) = 1$.

Proof. Since $\{\dim_{\text{poc}}(G;m)\}_{m=1}^{\infty}$ is decreasing, $1 = \dim_{\text{poc}}(G;1) \geq \dim_{\text{poc}}(G;2) \geq \cdots$ and so $\dim_{\text{poc}}(G;m) = 1$ for all $m \in \mathbb{Z}_{>0}$. Hence $\exp_{\text{poc}}(G) = 1$. $\square$

Proposition 5.4.2. *For any positive integer M, there exists a graph G such that* $\dim_{\text{poc}}(G;1) = 2$ *and* $\exp_{\text{poc}}(G) = M$.

Proof. Take an interval graph G which is not $K_t \cup I_s$ for any $t \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}_{\geq 0}$. Then $\dim_{\text{poc}}(G;m) = 2$ for any $m \in \mathbb{Z}_{>0}$ by Propositions 5.2.8 and 5.2.11. Therefore $\exp_{\text{poc}}(G) = 1$.

Let M be a positive integer at least two. Consider the graph $H = K_t \cup I_M$ where t is an arbitrary positive integer. Then $\dim_{\text{poc}}(H;M-1) = 2$ and $\dim_{\text{poc}}(H;M) = 1$ by Proposition 5.2.8. Therefore $\exp_{\text{poc}}(H) = M$. $\square$

Proposition 5.4.3. *For any graph G with* $\dim_{\text{poc}}(G;1) = 3$, $\exp_{\text{poc}}(G) = 1$.

Proof. Cho and Kim [8] showed that the interval graphs are exactly the graphs satisfying $\dim_{\text{poc}}(G;1) \leq 2$. Therefore G is not an interval graph. Thus $\dim_{\text{poc}}(G;m) > 2$ for any $m \in \mathbb{Z}_{>0}$ by Proposition 5.2.11. On the other hand, by Corollary 5.3.5, $\dim_{\text{poc}}(G;m) \leq \dim_{\text{poc}}(G;1) = 3$ and so $\dim_{\text{poc}}(G;m) = 3$ for any $m \in \mathbb{Z}_{>0}$. Hence $\exp_{\text{poc}}(G) = 1$. $\square$

Chapter 6

On the competition hypergraphs of d-partial orders[1]

In this chapter, we study the competition hypergraphs of d-partial orders and generalize the results given by Kim *et al.* [33].

6.1 A characterization of the competition hypergraphs of d-partial orders

In this section, we characterize the competition hypergraphs of d-partial orders.

Theorem 6.1.1. *A hypergraph H is the competition hypergraph of a d-partial order if and only if there exists a family $\mathcal{F}$ of homothetic open d-regular simplices contained in the hyperplane $\{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 + \cdots + x_d = 0\}$ and there exists a one-to-one correspondence $A : V(G) \to \mathcal{F}$ such that*

($\star$) $\{u_1, \ldots, u_t\} \in E(H)$ *if and only if there exists a vertex x of H such*

[1]The material in the section is from the paper "On the competition hypergraphs of d-partial orders" by Jihoon Choi and Suh-Ryung Kim. The author thanks Prof. Suh-Ryung Kim for allowing him to use its contents for his thesis.

that $\triangle(x) \subset A(u_i)$ *for each* $i = 1, \dots, t$ *but* $\triangle(x) \not\subset A(u)$ *for any* $u \notin \{u_1, \dots, u_t\}$.

Proof. ($\Rightarrow$) Suppose that H is the competition hypergraph of a d-partial order D. By translating each vertex of D by $T : v \mapsto v + k\mathbf{1}$ for a sufficiently large real number k, we may obtain a d-partial order all of whose vertices belong to $\mathcal{H}^d_+$ and is isomorphic to D. Therefore we may assume all the vertices of D are located in $\mathcal{H}^d_+$. Let $\mathcal{F} = \{\triangle^{d-1}(v) \mid v \in V(D)\}$. Then $\mathcal{F} \subset \mathcal{F}^{d-1}$. Let $f : V(H) \to \mathcal{F}$ be the function defined by $f(v) = \triangle^{d-1}(v)$. By Corollary 5.1.1, f is a bijection. The condition ($\star$) immediately follows from the definition of competition hypergraph and Proposition 2.2.7.

($\Leftarrow$) Suppose that there exist a subset $\mathcal{F}$ of $\mathcal{F}^{d-1}$ and a bijection $f : V(H) \to \mathcal{F}$ such that the condition ($\star$) holds. By Corollary 5.1.1, each element in $\mathcal{F}$ can be written as $\triangle^{d-1}(\mathbf{p})$ for some $\mathbf{p} \in \mathcal{H}^d_+$. Let $S = \{\mathbf{p} \in \mathbb{R}^d \mid \triangle(\mathbf{p}) \in \mathcal{F}\}$. Then D_S is a d-partial order. Take a subset $\{u_1, \dots, u_t\}$ of vertices of H. Then, by Proposition 2.2.7 and the condition ($\star$), $\{u_1, \dots, u_t\}$ is an hyperedge in H if and only if there exists a vertex x in D_S with $\{u_1, \dots, u_t\}$ as its in-neighborhood in D_S. Therefore H is the competition hypergraph of the d-partial order D_S. $\square$

6.2 The partial order competition hyper-dimension of a hypergraph

Proposition 6.2.1. *Let d be a positive integer. If H is the competition hypergraph of a d-partial order, then H is the competition hypergraph of a $(d+1)$-partial order.*

Proof. Let H be the competition hypergraph of a d-partial order D for some positive integer d. Then, by Lemma 5.2.2, there exists a $(d+1)$-partial order $\tilde{D}$ which is isomorphic to D. Therefore $H = \mathcal{CH}(D) \cong \mathcal{CH}(\tilde{D})$. Thus H is the competition hypergraph of a $(d+1)$-partial order. $\square$

Theorem 6.2.2. *For any hypergraph H, there exist positive integers d and k such that H together with k isolated vertices is the competition hypergraph of a d-partial order.*

Proof. The proof is exactly parallel to that of Proposition 2.3.2. Let $|V(H)| = n$ and label the vertices of H as $v_1, \dots, v_n$. We define a map $\phi : V(H) \to \mathbb{R}^n$ by

$$\phi(v_i)_j = \begin{cases} 2 & \text{if } j = i; \\ 4 & \text{if } j \neq i. \end{cases}$$

We define a map $\psi : E(H) \to \mathbb{R}^n$ by

$$\psi(e)_k = \begin{cases} 1 & \text{if } v_k \in e; \\ 3 & \text{if } v_k \notin e. \end{cases}$$

Let $V = \{\phi(v_i) \mid v_i \in V(H)\} \cup \{\psi(e) \mid e \in E(H)\} \subseteq \mathbb{R}^n$. Then V defines an n-partial order D. By the construction of D, it easily follows that $\mathcal{CH}(D) = H \cup I_{|E(H)|}$. Hence, by taking $d = n$ and $k = |E(H)|$, we complete the proof. $\square$

By Proposition 6.2.1 and Theorem 6.2.2, we can define a notion of dimension of a hypergraph.

Definition 6.2.3. For a hypergraph H, we define the *partial order competition hyper-dimension* $\dim_{\text{poch}}(H)$ of H as the smallest nonnegative integer d such that H together with k isolated vertices is the competition hypergraph of D for some d-partial order D and some nonnegative integer k, i.e.,

$$\dim_{\text{poch}}(H) := \min\{d \in \mathbb{Z}_{\geq 0} \mid \exists k \in \mathbb{Z}_{\geq 0}, \exists S \subseteq \mathbb{R}^d, \text{ s.t. } H \cup I_k = \mathcal{CH}(D_S)\},$$

where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers and I_k is a set of k isolated vertices.

Proposition 6.2.4. *For any hypergraph H, we have $dim_{\text{poch}}(H) \leq |V(H)|$.*

Proof. The proposition follows from the construction of the d-partial order in the proof of Theorem 6.2.2. $\square$

The notion of partial order competition hyper-dimension is applicable to a graph as long as it is regarded as a hypergraph.

Example 6.2.5. We consider the complete graph K_3 with the vertex set $\{a, b, c\}$. Suppose to the contrary that $\dim_{\text{poch}}(K_3) \leq 2$. Then K_3 together with some isolated vertices becomes the competition hypergraph of a 2-partial order D. By identifying the vertices of D with points in $\mathbb{R}^2$, we may assign coordinates a, b, c so that $a = (a_1, a_2)$, $b = (b_1, b_2)$, and $c = (c_1, c_2)$. Since they are three points in $\mathbb{R}^2$, one of a, b, c has the property that its x-coordinate is greater than or equal to those of the other two vertices; so is its y-coordinate. Without loss of generality, we may assume that $a_1 \geq \max\{b_1, c_1\}$ and $a_2 \geq \max\{b_2, c_2\}$. Since b and c are adjacent in K_3, they have a common out-neighbor, say d, in D. Since $a_1 \geq \max\{b_1, c_1\}$ and $a_2 \geq \max\{b_2, c_2\}$, d is an out-neighbor of a in D. Therefore $\{a, b, c\}$ forms a hyperedge in K_3, which is a contradiction. Thus $\dim_{\text{poch}}(K_3) \geq 3$. By Proposition 6.2.4, $\dim_{\text{poch}}(K_3) \leq |V(K_3)| = 3$. Hence $\dim_{\text{poch}}(K_3) = 3$.

By Proposition 2.3.10, $\dim_{\text{poc}}(K_3) = 1$ and so $\dim_{\text{poc}}(K_3) \neq \dim_{\text{poch}}(K_3)$. Therefore it is natural to ask how different $\dim_{\text{poc}}(G)$ and $\dim_{\text{poch}}(G)$ are.

Proposition 6.2.6. *For a graph G, $\dim_{\text{poc}}(G) \leq \dim_{\text{poch}}(G)$.*

Proof. Let $d = \dim_{\text{poch}}(G)$. Then there exists a d-partial order D such that the hypergraph G together with additional isolated vertices is the competition hypergraph of D. As G is actually a graph, this implies $\dim_{\text{poc}}(G) \leq d$. $\square$

The following proposition gives a sufficient condition for a graph G satisfying $\dim_{\text{poc}}(G) = \dim_{\text{poch}}(G)$.

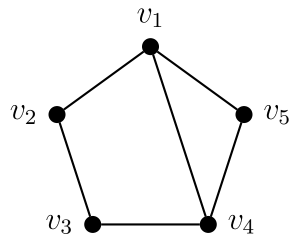

Figure 6.1: A graph G with $\dim_{\text{poc}}(G) = \dim_{\text{poch}}(G) = 3$

Proposition 6.2.7. *If a graph G is triangle-free, then* $\dim_{\text{poc}}(G) = \dim_{\text{poch}}(G)$.

Proof. Let G be a triangle-free graph and let $\dim_{\text{poc}}(G) = d$. Then $G \cup I_k$ is the competition graph of a d-partial order for some nonnegative integer k. Since G is triangle-free, each vertex in D has at most two in-neighbors, and so the the competition hypergraph of D equals $G \cup I_k$. Therefore $\dim_{\text{poc}}(G) \geq \dim_{\text{poch}}(G)$. Thus the proposition follows from Proposition 6.2.6. $\square$

Remark 6.2.8. The converse of Proposition 6.2.7 is not true. For example, it can be easily checked that the graph G with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_1v_4\}$ (see Figure 6.1) satisfies $\dim_{\text{poc}}(G) = \dim_{\text{poch}}(G) = 3$ even if it contains a triangle.

It might be interesting to find out whether or not there exists a hypergraph with arbitrarily large partial order competition hyper-dimension. Proposition 6.2.6 tells us that it is more likely than for We may answer this question by utilizing the notion of Ramsey number.

Definition 6.2.9. For a positive integer m, the *Ramsey number* $r(m; k) = r(m, m, \ldots, m)$ (m appears k times) denotes the smallest positive integer r such that any k-edge-colored complete graph K_r of order r contains a monochromatic complete graph K_m of order m.

Now we extend the notion of order types defined in 4.1. Let d be a positive integer. Take two distinct points $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ in $\mathbb{R}^d$.

For a nonempty proper subset S of $\{1, \ldots, d\}$, we write $u \preceq_S v$ if $u_i \leq v_i$ for each $i \in S$.

Suppose that u and v are incomparable in $\mathbb{R}^d$. Then there exists a partition $\{S_1, S_2\}$ of the set $\{1, \ldots, d\}$ such that $u \preceq_{S_1} v$ and $v \preceq_{S_2} u$. We call such a partition $\{S_1, S_2\}$ an *order type* for $\{u, v\}$.

Conversely, for a partition $\{S_1, S_2\}$ of $\{1, 2, \ldots, d\}$, let $x = (x_1, \ldots, x_d)$ be the point in $\mathbb{R}^d$ defined so that $x_i = 1$ if $i \in S_1$, $x_i = 2$ if $i \in S_2$, and let $y = (3, 3, \ldots, 3) - \mathbf{x}$. Then it is easy to see that x and y are incomparable in $\mathbb{R}^d$, and that $\{S_1, S_2\}$ is an order type of x and y.

Therefore, for two points u and v which are incomparable in $\mathbb{R}^d$, the number of possible order types for $\{u, v\}$ is equal to the Stirling number $S(n, 2)$ of second kind, which is known to be equal to $2^{n-1} - 1$.

The following two lemma is an analogue of Lemma 4.1.4.

Lemma 6.2.10. *Suppose that three points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^d$ satisfy the property that there is a common order type $\{S_1, S_2\}$ for each pair of them. Then, for some permutation σ on $\{1, 2, 3\}$, $\mathbf{x}_{\sigma(1)} \preceq_{S_1} \mathbf{x}_{\sigma(2)} \preceq_{S_1} \mathbf{x}_{\sigma(3)}$ and $\mathbf{x}_{\sigma(1)} \succeq_{S_2} \mathbf{x}_{\sigma(2)} \succeq_{S_2} \mathbf{x}_{\sigma(3)}$.*

Proof. The proof is exactly parallel to that of Lemma 4.1.4. Since any pair of x_1, x_2, x_3 is comparable by $\preceq_{S_1}$ by the hypothesis, $\preceq_{S_1}$ is a total order on $\{x_1, x_2, x_3\}$. Therefore there exists a permutation σ on $\{1, 2, 3\}$ such that $x_{\sigma(1)} \preceq_{S_1} x_{\sigma(2)} \preceq_{S_1} x_{\sigma(3)}$. Since $y_1 \preceq_{S_1} y_2$ if and only if $y_1 \succeq_{S_2} y_2$ for any pair $\{\mathbf{y}_1, \mathbf{y}_2\}$ of points in $\mathbb{R}^d$ of order type $\{S_1, S_2\}$, the lemma immediately follows. ☐

Theorem 6.2.11. *For any positive integers d and n satisfying $n \geq r(3; S(d, 2))$, $\dim_{\mathrm{poch}}(K_n) > d$.*

Proof. We prove this by contradiction. Suppose $\dim_{\mathrm{poch}}(K_n) \leq d$. Then, by definition, there exists a d-partial order D such that $\mathcal{CH}(D)$ equals $K_n \cup I_k$ for some nonnegative integer k. We may identify the vertices of $\mathcal{CH}(D)$ and $K_n \cup I_k$ with points in $\mathbb{R}^d$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ denote the vertices of K_n. Suppose that $\mathbf{v}_i \prec \mathbf{v}_j$ for some distinct i and j. Then every out-neighbor of $\mathbf{v}_i$ is an out-neighbor of $\mathbf{v}_j$ in D. This implies that every hyperedge of K_n containing $\mathbf{v}_i$ must contain $\mathbf{v}_j$, which is not the case for K_n. Therefore $\mathbf{v}_i \not\prec \mathbf{v}_j$ for any distinct i and j.

For a time being, we regard the hypergraph K_n as a graph which happen to be the complete graph with vertex set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then we define an $S(d,2)$-edge-coloring c in the following way. If $\{\mathbf{v}_i, \mathbf{v}_j\}$ is of order type $\{S_1, S_2\}$, then $c(\mathbf{v}_i\mathbf{v}_j) = \{S_1, S_2\}$ (in case where $\{\mathbf{v}_i, \mathbf{v}_j\}$ has more than one order type, we just take any order type and assign). Then, by the definition of $r(3, S(d,2))$, K_n contains a monochromatic complete subgraph K with three vertices, say $\mathbf{v}_i$, $\mathbf{v}_j$, and $\mathbf{v}_k$, that is, for some partition $\{S_1, S_2\}$ of $\{1, \ldots, d\}$, each pair of $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$ has order type $\{S_1, S_2\}$. Therefore, by Lemma 6.2.10, the three vertices can be labeled as $\mathbf{v}_i = (v_1^{(i)}, \ldots, v_d^{(i)})$, $\mathbf{v}_j = (v_1^{(j)}, \ldots, v_d^{(j)})$, and $\mathbf{v}_k = (v_1^{(k)}, \ldots, v_d^{(k)})$ so that $v_l^{(i)} \le v_l^{(j)} \le v_l^{(k)}$ if $l \in S_1$, and $v_l^{(i)} \ge v_l^{(j)} \ge v_l^{(k)}$ if $l \in S_2$. Thus $\min\{v_l^{(i)}, v_l^{(k)}\} \le v_l^{(j)}$ for each $l = 1, \ldots, d$.

By the definition of K_n, $\{\mathbf{v}_i, \mathbf{v}_k\}$ is a hyperedge of K_n. Therefore there exists $\mathbf{w} = (w_1, \ldots, w_d) \in V(D) \setminus \{\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k\}$ such that $N_D^{-1}(\mathbf{w}) = \{\mathbf{v}_i, \mathbf{v}_k\}$. Then $\mathbf{w} \prec \mathbf{v}_i$ and $\mathbf{w} \prec \mathbf{v}_j$ and so $w_i \le \min\{v_l^{(i)}, v_l^{(k)}\}$ for each $l = 1, \ldots, d$. As we have shown that $\min\{v_l^{(i)}, v_l^{(k)}\} \le v_l^{(j)}$, we have $w_l \le v_l^{(j)}$ for each $l = 1, \ldots, d$. Thus $\mathbf{w} \prec \mathbf{v}_j$ and this contradicts the fact that $N_D^{-1}(w) = \{\mathbf{v}_i, \mathbf{v}_k\}$. Hence $\dim_{\text{poch}}(K_n) > d$. $\square$

Remark 6.2.12. Theorem 6.2.11 asserts that there exists a hypergraph with arbitrarily large partial order competition hyper-dimension. On the other hand, $\dim_{\text{poc}}(K_n) = 1$ by Proposition 2.3.10 for $n \ge 2$ and we can conclude that that the difference between $\dim_{\text{poc}}(G)$ and $\dim_{\text{poch}}(G)$ can be arbitrarily large.

6.3 Interval competition hypergraphs

A hypergraph $H = (V, E)$ is said to be *interval* if there exists a injective function $f : V \to \mathbb{R}$ such that, for each $e \in E(H)$, there exists an interval on $\mathbb{R}$ which contains all elements of e, but does not contain the image of any vertex not in e.

Kim *et al.* [33] showed that an interval hypergraph may have partial order competition hyper-dimension greater than 2, and then characterized the interval hypergraphs H with $\dim_{\text{poch}}(H) \leq 2$. By the way, all the interval hypergraphs turn out to have partial order competition hyper-dimension at most three.

Proposition 6.3.1. *Every interval hypergraph H satisfies* $\dim_{\text{poch}}(H) \leq 3$.

Proof. Let H be an interval hypergraph with n vertices. Then the vertices of H can be linearly ordered as $v_1, v_2, \ldots, v_n$ on the real line $\mathbb{R}$ so that each v_i is to the left of v_{i+1} and that every hyperedge is of the form $\{v_i, v_{i+1}, \ldots, v_j\}$.

Now we will construct a 3-partial order whose competition hypergraph is $\mathcal{CH}(D) \cup I_{|E(H)|}$. For each $i = 1, \ldots, n$, we assign $v_i := (i, i, 3n - 2i) \in \mathbb{R}^3$. Figure 6.2 shows how the vertices $v_1, \ldots, v_n$ are located in $\mathbb{R}^3$ and the corresponding $A(v_1), \ldots, A(v_n)$ are located on the hyperplane $x + y + z = 0$. We note that every pair of $v_1, \ldots, v_n$ has an order type $\{\{1, 2\}, \{3\}\}$.

Take a hyperedge e of H. Then it is of the form $e = \{v_j, v_{j+1}, \ldots, v_k\}$ for some j and k with $j < k$. Refer to Figure 6.3 for an arrangement of $A(v_{j-1}), \ldots, A(v_{k+1})$ on the hyperplane $x + y + z = 0$. We observe that the shaded region in Figure 6.3 is included in $A(v_i)$ for each $i = j, j+1, \ldots, k$ but is disjoint with any of $A(v_1), \ldots, A(v_{j-1}), A(v_{k+1}), \ldots, A(v_k)$, and that the shaded regions corresponding to hyperedges e and f are disjoint if $e \neq f$. Based on this observation, we add a new isolated vertex $\psi(e) \in \mathbb{R}^3$ to H for which $\triangle(\psi(e))$ is included in the interior of the shaded region in Figure 6.3. Then $V(H) \cup \{\psi(e) \mid e \in E(H)\}$ defines a 3-partial order D. By the way of constructing D, $\psi(e) \neq \psi(f)$ for distinct hyperedges e and f, and the out-

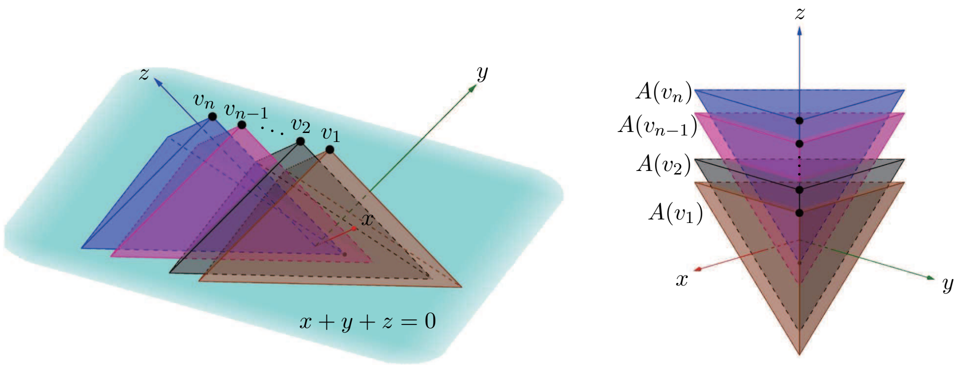

Figure 6.2: The points $v_1, \ldots, v_n$ with $v_i = (i, i, 3n - i) \in \mathbb{R}^3$ shown on the left and $A(v_1), \ldots, A(v_n)$ on the hyperplane $x+y+z=0$ shown on the right.

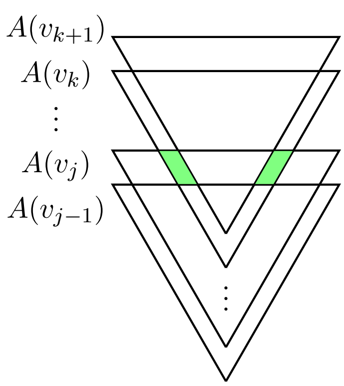

Figure 6.3: The triangles $A(v_{j-1}), \ldots, A(v_{k+1})$ on the hyperplane $x+y+z=0$.

neighborhood of $\psi(e)$ in D is equal to e for each hyperedge e of H. Hence $\mathcal{CH}(D) = \mathcal{CH}(D) \cup I_{|E(H)|}$. $\square$

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국문초록

유향그래프 D의 *경쟁그래프* $C(D)$란 D와 같은 꼭짓점의 집합을 갖고, 두 개의 꼭짓점 x와 y가 인접할 필요충분조건이 D의 적당한 꼭짓점 z에 대하여 (x, z)와 (y, z)가 모두 D의 유향변인 것으로 주어지는 그래프이다. 경쟁그래프는 지난 40년간 아주 활발하게 연구되어 왔다.

Cohen [13, 14, 15]은 먹이사슬을 나타내는 비순환 유향그래프의 경쟁그래프는 대부분 구간그래프임을 경험적으로 관찰하였다. Roberts [51]는 Cohen의 관찰이 인위적인 구성에 의한 결과였는지에 대해 질문을 제기했으며, 비순환 유향그래프의 경쟁그래프가 항상 구간그래프가 되는 것은 아니라는 결론을 얻었다. 이를 증명하는 과정에서 임의로 주어진 그래프 G에 변의 개수만큼의 고립꼭짓점을 추가하면 그 결과물이 적당한 비순환 유향그래프의 경쟁그래프로 나타내어짐을 보였다. 그러고 나서 Roberts는 비순환 유향그래프 D의 경쟁그래프 $C(D)$가 구간그래프가 되도록 하는 D의 특징에 대한 질문을 제기했다. 그 후 이 문제는 경쟁그래프에서 가장 중요하고 근본적인 미해결 문제 중 하나로 남아있다. 많은 연구자들이 이 문제를 해결하기 위해 여러가지 풀이를 시도했으며, 부분적인 해답이 제시되었다. Cho와 Kim [8]은 이 문제에 답하기 위하여 2-반순서의 경쟁그래프는 항상 구간그래프가 되며, 역으로 임의의 구간그래프에 충분한 개수의 고립꼭짓점을 붙여주면 그 결과물이 적당한 2-반순서의 경쟁그래프가 됨을 보였다.

본 학위논문에서는 d-반순서의 경쟁그래프의 구조를 연구하며, 2-반순서의 경쟁그래프의 연구결과의 일부를 일반화한다.

어떤 유향그래프 D가 적당한 자연수 d에 대하여 다음 두 조건을 만족할 때 D를 *d-반순서*라 정의한다: (i) $V(D) \subset \mathbb{R}^d$; (ii) $(\mathbf{x}, \mathbf{y}) \in A(D)$일 필요충분조건은 각 $i = 1, \ldots, d$에 대하여 $\mathbf{x}$의 i번째 성분이 $\mathbf{y}$의 i번째 성분보다 큰 것이다.

임의로 주어진 그래프 G에 충분한 개수의 고립꼭짓점을 붙여주면 그 결과물이 적당한 자연수 d에 대하여 어떤 d-반순서의 경쟁그래프가 됨을

보이고, 그러한 d중에서 가장 작은 값을 G의 *반순서 경쟁 차원*이라 부르며 $\dim_{\text{poc}}(G)$로 표기한다. 이를 통해 Cho와 Kim [8]의 결과를 일반화하며, 구조가 흥미로운 그래프의 반순서 경쟁차원을 계산한다. 또한 d-반순서의 m-step 경쟁그래프 및 경쟁하이퍼그래프도 연구한다.

주요어휘: 경쟁그래프, d-반순서, 반순서 경쟁차원, 닮은 정단체, 순서형
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