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State Estimation and Tracking Control for Hybrid Systems by Gluing the Domains

상태변수 영역 접합을 통한 하이브리드 시스템의 상태변수 추정 및 추종 제어

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ABSTRACT

State Estimation and Tracking Control for Hybrid Systems by Gluing the Domains

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In this dissertation, we propose a new observer and tracking controller design approach for a class of hybrid dynamical systems with state jumps. The hybrid dynamical system exhibits characteristics typical of both continuous-time dynamical system and discrete-time dynamical system. Therefore, it can be modeled as differential equation of the continuous-time dynamics, difference equation of the discrete-time dynamics, the interaction between them. Since the interaction of continuous-time and discrete-time dynamics in a hybrid system leads to rich dynamical behavior and unfamiliar phenomena, several challenges are encountered when we deal with this system.

The observer design considered in this dissertation is to construct a dynamical system called an observer that estimates the state of a given hybrid dynamical system (without any input), from an output of the given system. In addition, the tracking controller design is to construct a dynamical system called a tracking controller that makes an input for a given hybrid dynamical system (with an input) such that the state of the given system tracks a given reference. There many results of the observer and tracking controller designs for the continuous-time and discrete-time dynamical systems, but the results for the hybrid dynamical systems are insufficient. Moreover, the results are applied to some classes of hybrid systems (switched systems, hormone systems, powertrain systems, and so on) rather than general hybrid dynamical systems.

The proposed idea dealing with the hybrid dynamical system is to "glue" the jump set (a part of the domain where the jumps take place) onto its image. Then, on the "glued" domain, the hybrid dynamical system becomes a continuous-time dynamical system without any jumps. Especially, for some class of the system, the continuous-time dynamical system has a smooth vector field via some notion, "smoothing". Furthermore, we specify this concept of gluing as a map and investigate the essential conditions of the map.

By this map, we obtain the "glued" hybrid dynamical system (which is a continuous-time dynamical system) and it may be possible to construct an observer and/or a tracking controller through conventional methods for continuous-time dynamical systems. From these constructions, we obtain the observer and tracking controller for the hybrid system. Especially, the proposed observer does not require any detection of the state jumps while many previous results does. Furthermore, the proposed tracking controller does not need to make the state jump whenever the jumps of the reference happen. Simulation results for examples including mechanical system with impacts and ripple generator in AC/DC converter illustrate the effectiveness of the proposed approach.

Keywords: hybrid dynamical system, differentiable manifold, gluing, smoothing, state estimation, nonlinear observer design, system immersion, tracking control
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Notation

$\mathbb{R}_{\geq 0}$	set of non-negative real numbers
$\mathbb{R}_{+}$	set of positive real numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^n$	space of n-tuples $x = (x_1, \cdots, x_n)$ with each $x_k \in \mathbb{R}$
$\mathbb{R}^n_+$	subset of $\mathbb{R}^n$ with $x_1 \geq 0$
$\mathbb{R}^{m\times n}$	space of $m \times n$ matrices with real entries
0_n	$n \times 1$ column vector of all zeros
$\mathcal{B}_x(\epsilon)$	open ball at $x \in \mathbb{R}^n$ with radius $\epsilon > 0$
$\tilde{C}$	open neighborhood of subset $C \in \mathbb{R}^n$
$\text{int}(C)$	set of interior points of subset $C \in \mathbb{R}^n$
I_n	$n \times n$ identity matrix
$0_{m\times n}$	$m \times n$ zero matrix
A^{-1}	inverse of square matrix A
$A^\top$	transpose of $A \in \mathbb{R}^{m\times n}$
$(A)_{ij}$	the (i, j)-th entry of $A \in \mathbb{R}^{m\times n}$
$\text{diag}(\alpha_1, \ldots, \alpha_n)$	diagonal matrix with its i-th diagonal entry $\alpha_i \in \mathbb{R}$
$\text{rank}(A)$	rank of the matrix A
$\text{sgn}(x)$	signum function of $x \in \mathbb{R}$; i.e., -1 if $x < 0$ and 1 if $x \geq 0$

δ_{ij}	$:= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ (Kronecker delta)
(x, y)	$:= [x^\top \ y^\top]^\top$ with column vectors x and y
C^r	(of manifold) r-differentiable
C^∞	(of manifold) smooth
∂	boundary of a manifold with boundary
$D_v f(a)$	directional derivative of f at a
$T_p\mathcal{M}$	tangent space at $p \in \mathcal{M}$ to $\mathcal{M}$
$T\mathcal{M}$	tangent bundle of $\mathcal{M}$
$\partial/\partial x^i$	coordinate vector field
F_*	pushforward of F
$f\|_{\mathcal{C}}$	restriction of f to $\mathcal{C}$
C^r	(of Euclidean space) having r continuous derivatives
C^∞	(of Euclidean space) infinitely differentiable
$\mathrm{d}\psi$	Jacobian matrix with C^1 function ψ
$\mathcal{K}$	set of class-$\mathcal{K}$ functions
d_K	distance of x to $K \subset \mathbb{R}^n$
$T_K(x)$	contingent cone to $K \subset \mathbb{R}^n$
$\sum_{i=m}^n x_i$	summation of the sequence x_i; i.e., $x_m + x_{m+1} + \cdots + x_{n-1} + x_n$ if $m < n$, x_m if $m = n$, and 0 if $m > n$
$\prod_{i=m}^n x_i$	product of the sequence x_i; i.e., $x_m \cdot x_{m+1} \cdot \cdots \cdot x_{n-1} \cdot x_n$ if $m < n$, x_m if $m = n$, and 1 if $m > n$
$\coprod_{a \in A} S_a$	disjoint union of a collection of sets $\{S_a\}_{a \in A}$; i.e., $\bigcup_{a \in A}(S_a \times \{a\})$
$\min\{a_1, \cdots, a_k\}$	minimum value among $a_1, a_2, \cdots, a_k$
$\max\{a_1, \cdots, a_k\}$	maximum value among $a_1, a_2, \cdots, a_k$

$\lvert \cdot \rvert$	Euclidean norm
$:=$	defined as
$=:$	denoted by
$\exists$	there exists
$\forall$	for all
$\wedge$	logical conjunction
$\square$	end of proof, definition, theorem, remark, and so on

Chapter 1

Introduction

1.1 Research Background

A dynamical system which exhibits both continuous and discrete dynamic behaviors is called a hybrid system. There are many kinds of hybrid systems such as robotic systems with impacts [SK95, PY04, RLS06], electric circuits with switchings, tank systems [SJLS05], hormone systems [KS99, CMS09, CMS12], and several cyber-physical systems.

The modeling frameworks that involve the continuous-time dynamics, the discrete events, and the interaction between them have been studied (see, e.g.,[Hen00, LJS+03, GST09]). Under these frameworks, many attempts on the state estimation and control problems have been made for some classes of hybrid systems such as switched systems [GMP12, ST14, BMDB12, BBBSV02, SS05, BPU11], powertrain systems [BBBSV01], mechanical systems with impacts [BPU11, BNMM00, MT01, TBP16, MT16], polyhedral billiards with impacts [FTZ13], ripple disturbance on AC/DC converters [BZLC17] and so on [BvdWHN13, SvdWN14, SBvdWH14, RS11]. However, general results for these problems still need to be explored.

For many existing observers, it is required that the jump times of the observer state coincide with those of the plant state. Similarly, in many existing reference tracking controllers, the jump times of the plant state and the reference trajectory should coincide. If they do not coincide, then jump time mismatches occur and the estimation or tracking errors may be large on the time intervals caused by the jump time mismatches.

For the coincidence of jump times, in the case of state estimation problem, most proposed observers require the state jump time information. However, this information is usually not given in practice. Although the observers may be able to detect the jumps of the plant state from an output, the jump detection is another challenge. Even if it is possible, the detection delay yields additional errors in the state estimation. In the case of the tracking control, the reference jump time information may be given. However, in the case when the state jumps are triggered on particular state values, the jump time mismatch may be unavoidable because the controller cannot instantaneously change the state to have the particular value in general. Therefore, the coincidence of jump times is an unrealistic condition in the state estimation and tracking control problems.

In [FTZ13, SvdWN14, BvdWHN13, MT16], novel methods dealing with the jump time mismatches are suggested for the tracking control or state estimation problems. In [FTZ13], a translating mass in a polyhedral billiard is remodeled as a switched system by adding another reference. In [BvdWHN13], authors propose a new non-Euclidean distance measure which redefines the distance between two points and takes the distance between the starting and end points of the jump as zero. As a result, the new distance measure determines that they are "close" when they are related by the jump even though they are not actually close. Actually, this concept begins from the geometric approach studied in [SJLS05]. The result gives the idea that a hybrid system with jumps of the state may be considered as a (piecewise) continuous-time dynamical system without any jump. Furthermore, in [BRS15], the authors propose a class of hybrid systems, which can be changed into a dynamical system defined on some smooth manifold with boundary having some smooth vector field.

Inspired by it, for a class of hybrid systems with the state jumps, we propose a novel observer and controller design approach. The idea is to "glue" the jump set, where the jumps take place, onto its image, the destinations of the jumps. Then, the system after "gluing" becomes a continuous-time dynamical system without any jumps of the state. Therefore, conventional observer or controller design techniques for continuous-time systems may be applied. While many observers for hybrid systems require some assumptions or detection of the time instants when discrete events occur, our approach does not require them (although more stringent structural conditions on the hybrid systems may be assumed). In addition, the tracking controller also does not need to make the state jump whenever the reference jumps occur. Finally, our concept of gluing, similar to a transformation, is more intuitive and specific than other results. To delineate the proposed approach, let us consider a toy system given by

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix},$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x =: Ax \quad \text{when } x \in \{x \in \mathbb{R}^2 : (|\rho| > 0) \wedge (|\theta| \leq \tfrac{\pi}{4})\} =: \mathcal{C}, \tag{1.1.1a}$$

$$x^+ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x =: Fx \quad \text{when } x \in \{x \in \mathcal{C} : \theta = -\tfrac{\pi}{4}\} =: \mathcal{D}, \tag{1.1.1b}$$

$$y = \rho\cos 4\theta,$$

where (1.1.1a) is the continuous-time dynamics, (1.1.1b) is the discrete-time dynamics involving jumps of the state, and x^+ indicates the value of x after the jump. Note that, in this case, there is no change in the output y when the state jumps happen because (1.1.1b) implies $(\rho, \theta) \mapsto (\rho, -\theta)$ at $\mathcal{D}$, so that $y = \rho\cos 4\theta = \rho\cos(-4\theta) = y^+$ for $\theta = -\pi/4 \in \mathcal{D}$. Thus, if the available information is only the current output, then we cannot perceive when the discrete events arise (recall that both ρ and θ are unknown). For this reason, it seems difficult to apply the observer design methods that are based on the detection of jumps. However, let us consider a transformation $\zeta = (\zeta_1, \zeta_2) = \psi(x) = \bar{\psi}(\rho, \theta) = (\rho\cos 4\theta, \rho\sin 4\theta)$. Then, the system is transformed into

$$\begin{aligned} \dot{\zeta} &= 4A\zeta =: \bar{A}\zeta \qquad \text{for } \|\zeta\| \neq 0, \\ y &= \zeta_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \zeta =: \bar{C}\zeta. \end{aligned} \tag{1.1.2}$$

As we can see, the system (1.1.2) is an LTI system without any jumps of the

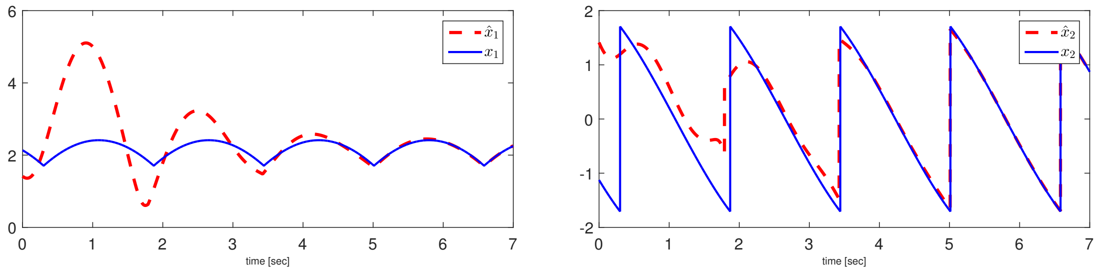

Figure 1.1: True and estimated states of toy system (1.1.1)

state. This is due to the fact that the representations of two points, the starting point and its destination of each jumps, are the same in the ζ-coordinates, i.e., $\psi(x) = \psi(x^+)$ for $x \in \mathcal{D}$. Furthermore, since the pair $(\bar{A}, \bar{C})$ is observable, we can design a Luenberger observer

$$\dot{\hat{\zeta}} = \bar{A}\hat{\zeta} + L(\bar{C}\hat{\zeta} - y),$$

where $\bar{A} + L\bar{C}$ is Hurwitz. Then, $e_\zeta := \hat{\zeta} - \zeta$ converges to zero, and we can reconstruct $\hat{x} = (\hat{x}_1, \hat{x}_2) = (\hat{\rho}\cos\hat{\theta}, \hat{\rho}\sin\hat{\theta})$ that is an estimate of x, by a particular inverse transformation of $\psi(x)$:

$$\hat{\rho} = \|\hat{\zeta}\|,$$
$$\hat{\theta} = \begin{cases} \frac{1}{4}\arctan\frac{\hat{\zeta}_2}{\hat{\zeta}_1} & \text{if } \hat{\zeta}_1 > 0, \\ \frac{1}{4}\left(\arctan\frac{\hat{\zeta}_2}{\hat{\zeta}_1} + \pi\mathrm{sgn}(\hat{\zeta}_2)\right) & \text{if } \hat{\zeta}_1 < 0, \\ \frac{\pi}{8} & \text{if } \hat{\zeta}_1 = 0 \text{ and } \hat{\zeta}_2 > 0, \\ -\frac{\pi}{8} & \text{if } \hat{\zeta}_1 = 0 \text{ and } \hat{\zeta}_2 < 0, \end{cases}$$

where $\mathrm{sgn}(a)$ is 1 if $a \geq 0$ and -1 if $a < 0$. Figure 1.1 is a simulation result showing the effectiveness of the proposed estimation strategy.

1.2 Organization and Contributions of the Dissertation

The following outlines this dissertation and briefly presents the contributions of each chapter.

Chapter 2. Mathematical Preliminaries

As mathematical preliminaries of the dissertation, this chapter establishes the concepts and basic facts needed to understand later chapters. The first section reviews in some detail the differential calculus in $\mathbb{R}^n$. The second section recalls some notions in differential geometry and important mathematical tools on them such as manifold with boundary, differential structure, tangent space, vector field, push-forward, Inverse Function Theorem, Whitney Embedding Theorem, Boundary Flowout Theorem, and so on. The final section is devoted to the viability theory in ordinary differential equation.

Chapter 3. Reviews of Related Previous Works

This chapter reviews some previous results on the hybrid system. The first section deals with gluing the domain of the hybrid systems [SJLS05] and smoothing the vector field after gluing [BRS15]. Next section presents the viability theory of hybrid system [ALQ$^+$02]. In the third section, some results of the state estimation for the hybrid systems [FTZ13, MT01, MT16, BZLC17] and the common assumption required by most conventional observers are introduced. The final section presents some result of the tracking control for the hybrid systems [FTZ13, SvdWN14, BvdWHN13, SBvdWH14] and their ideas.

Chapter 4. Gluing Domain of Hybrid System

This chapter introduces a framework to deal with the main results of this dissertation and proposes a class of the hybrid systems whose the boundaries of the domain can be glued. The key to glue the boundaries is the quotient map. However, since the quotient map and its image are so abstract, a more concrete map, which is a "gluing function", is proposed to glue the boundaries. On the framework, the map gluing the boundaries is obtained as a map defined between two Euclidean spaces.

Chapter 5. State Estimation Strategy

This chapter deals with the state estimation problem of hybrid dynamical systems with state-triggered jumps using a gluing function. By the gluing function, we obtain some continuous-time dynamical system without any state jumps. Then, as the toy system given in Section 1.1, we may obtain the state observer from the conventional observer design methods for the continuous-time dynamical systems (e.g. Luenberger observers for linear systems, high-gain observer for nonlinear systems [MT16], nonlinear observers for Lipschitz systems [KS99], and so on). Parts of this chapter are based on [KCS+14]. The following is a list of the contributions of the chapter.

- We present the conditions of the hybrid dynamical system with an output guaranteeing that it can be considered as a continuous-time dynamical system with an output. In addition, from the observer of this continuous-time dynamical system, we can obtain the state estimation of the hybrid dynamical system in a graphical sense.
- We investigate some conditions such that, under the conditions, the glued system is a linear system up to output injection or a Lipschitz continuous system. Then, we may utilize the convention observer design methods proposed in [BS04] or [KS99].
- Comparing to most previous observer design methods for the hybrid dynamical system, the proposed observer design technique neither requires nor depends on the information of the state jump time instant.
- As a case study, we apply the result to bouncing ball system, mechanical system with impacts, and rippler generator system.

Chapter 6. Tracking control Strategy

This chapter addresses the tracking control problem of hybrid dynamical systems with state-triggered jumps using a gluing function. Similar to the state estimation problem, we obtain some continuous-time dynamical system without any state jumps via the gluing function. Then, we may construct the tracking controller from the conventional controller design techniques for the continuous-time dynamical systems. Parts of this chapter are based on [KSS16]. The following is a list of the contributions of the chapter.

- The conditions implying that the hybrid dynamical system with an input can be changed into a continuous-time dynamical system with an input are proposed. Furthermore, from the controller of the continuous-time dynamical system, we may design a tracking controller for the hybrid dynamical system.

- The conventional controllers of the hybrid dynamical systems require the coincidence of the reference and state jumps times, while the proposed controller does not.

- As a case study, we apply the result to mechanical system with impacts and some academic examples.

Chapter 7. Conclusions

This chapter concludes this dissertation with some concluding remarks and further issues for future research.

Chapter 2

Mathematical Preliminaries

This chapter provides some brief mathematical background. For a full understanding of the chapter, the reader is referred to the books [War71, Boo75, Spi65, Spi99, Mun97, Mun00, Lee12, Wal16, Hir76, Joy09, JdM82, Aub09].

2.1 Calculus in $\mathbb{R}^n$

In this section, we review some facts about partial derivatives from advanced calculus.

We assume that $U \subset \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}^m$ is a real-valued function. In addition, let $x := (x_1, \ldots x_n) \subset U$ and $f(x) := (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$. Now we can define the "partial derivative".

Definition 2.1.1 (Partial derivative)**.** The *partial derivative* $(\partial f_i/\partial x_j)_p$ *of* f_i *with respect to* x_j is the following limit, if it exists:

$$\left(\frac{\partial f_i}{\partial x_j}\right)_p = \lim_{h \to 0} \frac{f_i(p_1, \ldots, p_j + h, \ldots, p_n) - f_i(p_1, \ldots, p_j, \ldots, p_n)}{h}.$$

If $(\partial f_i/\partial x_j) : U \to \mathbb{R}$ is defined by $(\partial f_i/\partial x_j)(p) := (\partial f_i/\partial x_j)_p$, that is, the limit above exists at each point of U for $1 \leq j \leq n$. □

Definition 2.1.2 (Differentiable)**.** We say that f is *differentiable* at $p \in U$ if there is a matrix $T \in \mathbb{R}^{m \times n}$ such that

$$\lim_{v \to 0_n} \frac{|f(p+v) - f(p) - Tv|}{|v|} = 0.$$

We say that the linear matrix T is the *Jacobian of f at p* and write it as $\mathrm{d}f(p)$. □

If $\mathrm{d}f$ is continuous we say that f *is of class* C^1. It is well known that f is C^1 if and only if the partial derivatives $(\partial f_i/\partial x_j) : U \to \mathbb{R}$ exist and are continuous. We define inductively the notion of a function *of class* C^r: f is of class of C^r if all its partial derivatives are of class C^{r-1}. We say that f *is of class* C^∞ or *smooth* if f is of class C^r for all r. Next we define a diffeomorphism.

Definition 2.1.3 (Diffeomorphism)**.** Let A and B be open sets of $\mathbb{R}^n$; let $g : A \to B$ be a one-to-one function. We say that g is a C^r *diffeomorphism* if g and g^{-1} are of class C^r. Furthermore, we say that A is C^r *diffeomorphic* to B. Trivially, B is C^r diffeomorphic to A also when A is C^r diffeomorphic to B. □

The next theorem, known as the "Inverse Function Theorem", is also a convenient tool to investigate a local property.

Theorem 2.1.1 (Inverse Function Theorem)**.** Let U be an open set in $\mathbb{R}^n$ and $\Phi : U \to \mathbb{R}^n$ be a C^r function. If $\mathrm{rank}(\mathrm{d}\Phi(p)) = n$ for a point $p \in U$, i.e., the Jacobian is nonsingular at each point $p \in U$, then there exists an open neighborhood $V \subset U$ of p such that $\Phi|_V : V \to \Phi(V)$ is a C^r diffeomorphism, where $\Phi|_V$ is the restriction of Φ to V. □

Now we generalize the notion of differentiability to functions that are defined on arbitrary subsets of $\mathbb{R}^n$.

Definition 2.1.4 (Class C^r)**.** Let S be a subset of $\mathbb{R}^n$. Consider a function $f : S \to \mathbb{R}^m$. We say that f *is class* C^r if f may be extended to a function $\tilde{f} : U \to \mathbb{R}^n$ that is of class C^r on a open set U of $\mathbb{R}^n$ containing S. □

The following lemma shows that f is of class C^r if it is local of class C^r:

Proposition 2.1.2. Let S be a subset of $\mathbb{R}^n$. Consider a function $f : S \to \mathbb{R}^m$. If for each $x \in S$, there is an open neighborhood U_x of x and a function $g_x : U_x \to \mathbb{R}^m$ of class C^r that agrees with f on $U_x \cap S$, then f is of class C^r on S. □

We shall be particularly interested in functions defined on set that are open in $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$ but not open in $\mathbb{R}^n$. In this situation, we have the following useful result:

Proposition 2.1.3. Let S be open in $\mathbb{R}^n_+$ but not in $\mathbb{R}^n$; let $\alpha : S \to \mathbb{R}^m$ be of class C^r. Let $\tilde{\alpha} : \tilde{S} \to \mathbb{R}^m$ be a C^r extension of α. Then for $x \in S$, the Jacobian $\mathrm{d}\tilde{\alpha}(x)$ expends only on the function α and is independent of the extension $\tilde{\alpha}$. We denote the Jacobian $\mathrm{d}\tilde{\alpha}$ on S by $\mathrm{d}\alpha$ without ambiguity. □

2.2 Differential Geometry

In order to define the notion of topological manifold, we need the concepts of topology, topological space, Hausdorff space, homeomorphism, and so on.

Definition 2.2.1 (Topology and Topological space)**.** A *topology* on a set $\mathcal{M}$ is a collection $\mathcal{T}$ of subsets of $\mathcal{M}$, which are called *open* sets satisfying the following three axioms:

(a) The empty set and $\mathcal{M}$ itself are open.

(b) The union of any number of open sets is open.

(c) The intersection of any finite number of open sets is open.

A set $\mathcal{M}$ together with a topology $\mathcal{T}$ on $\mathcal{M}$ is called a *topological space*. □

A *basis* of a topology $\mathcal{T}$ on $\mathcal{M}$ is a subcollection $\mathcal{B} \subset \mathcal{T}$ such that every open subset of $\mathcal{M}$ can be represented as a union of elements of $\mathcal{B}$. A topological space is said to be *second countable* if there is a countable basis of its topology. A *neighborhood* of a point $p \in \mathcal{M}$ is an open subset of $\mathcal{M}$ containing p. A *Hausdorff space* is a topological space in which any two distinct points have disjoint neighborhoods.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be topological spaces. A map $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ is said to be *continuous* if the inverse image of any open subset of $\mathcal{M}_2$ under Φ is also an open subset of $\mathcal{M}_1$. The map Φ is called a *homeomorphism*, if it is bijective and both the map Φ and its inverse map Φ^{-1} are continuous. We say that $\mathcal{M}_1$ is *homeomorphic* to $\mathcal{M}_2$, if there exists a homeomorphism from $\mathcal{M}_1$ onto $\mathcal{M}_2$. Furthermore, if $\mathcal{M}_1$ is homeomorphic to $\mathcal{M}_2$, then $\mathcal{M}_2$ is homeomorphic to $\mathcal{M}_1$ also, because $\Phi^{-1} : \mathcal{M}_2 \to \mathcal{M}_1$ is clearly a homeomorphism when $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a homeomorphism.

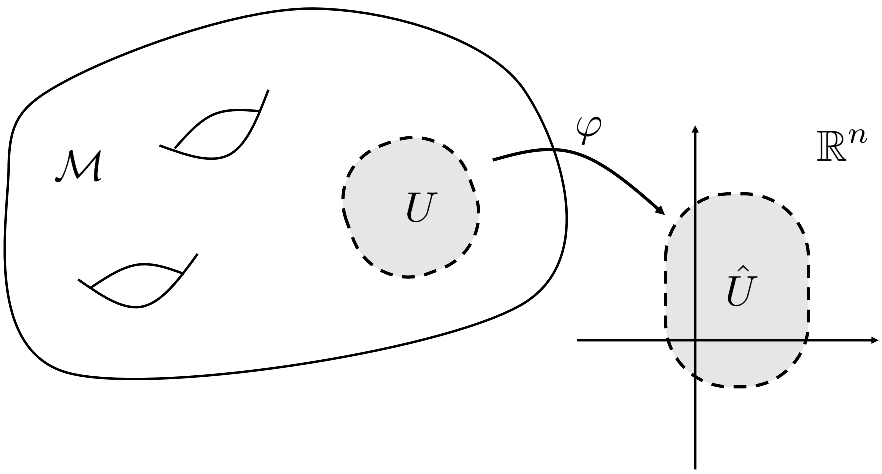

Figure 2.1: Manifold and coordinate chart.

Now, we introduce the notion of topological manifold.

Definition 2.2.2 (Topological manifold)**.** A second countable Hausdorff space $\mathcal{M}$ is called a *topological manifold* of dimension n if each point of $\mathcal{M}$ has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$. □

Roughly speaking, a manifold is a topological space that locally resembles a real Euclidean space. For this reason, we can identify each point of a topological manifold with a point of a real Euclidean space as follows.

Definition 2.2.3 (Coordinate chart)**.** For a topological manifold $\mathcal{M}$ of dimension n, a *coordinate chart* of $\mathcal{M}$ is a pair (U, φ), where U is an open subset of $\mathcal{M}$ and $\varphi : U \to \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$. By definition of a topological manifold, each point is contained in the domain of some chart (U, φ). Given a chart (U, φ), we call the set U a *coordinate domain* or a *coordinate neighborhood* of each of its points. The map φ is called a (*local*) *coordinate map*, and the component functions $(x^1, \cdots x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \cdots x^n(p))$, are called *local coordinates* on U. □

Let (U, φ) and (V, ϕ) be coordinate systems of $\mathcal{M}$ with $U \cap V \neq \varnothing$. Then, the homeomorphism $\phi \circ \varphi^{-1} : \varphi(U \cap V) \to \phi(U \cap V)$ is called a *transition map* from φ to ϕ on $U \cap V$. Two coordinate charts (U, φ) and (V, ϕ) are said to be C^r *related* or C^r *compatible* if both the maps $\phi \circ \varphi^{-1} : \varphi(U \cap V) \to \phi(U \cap V)$ and $\varphi \circ \phi^{-1} : \phi(U \cap V) \to \varphi(U \cap V)$ are C^r. Here r can be a natural number or ∞. A collection $\mathcal{A} = \{(U^i, \varphi^i) : i \in I\}$ (I is an index set) of mutually C^r related coordinate charts of $\mathcal{M}$ with $\bigcup_{i \in I} U^i = M$ is called a C^r *atlas* for $\mathcal{M}$.

Lemma 2.2.1. If $\mathcal{A}$ is an C^r atlas for $\mathcal{M}$, then $\mathcal{A}$ is contained in a unique maximal C^r atlas for $\mathcal{M}$. □

From the lemma, the concept of smooth manifold can be defined as follows.

Definition 2.2.4 (Differentiable manifold)**.** A topological manifold $\mathcal{M}$ together with a maximal C^r atlas for $\mathcal{M}$ is called a C^r *manifold.* When $r = \infty$, we say that $\mathcal{M}$ is a *smooth manifold.* □

A maximal C^r atlas on $\mathcal{M}$ gives a C^r differentiable structure. Thus, we can define the C^r differentiability of maps between C^r manifolds. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be C^r manifolds. A map $F : \mathcal{M}_1 \to \mathcal{M}_2$ is said to be *differentiable of* C^r if, for each $p \in \mathcal{M}_1$, there exist two coordinate systems (U, φ) of $\mathcal{M}_1$ with $x \in U$ and (V, ϕ) of $\mathcal{M}_2$ with $F(x) \in V$ such that $\phi \circ F \circ \varphi^{-1}$ is of C^r. From the concept of differentiability for maps between differentiable manifolds, we introduce the notion of diffeomorphism.

Definition 2.2.5 (Diffeomorphism)**.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be C^r manifolds of the same dimension. A map $F : \mathcal{M}_1 \to \mathcal{M}_2$ is called a C^r *diffeomorphism*, if it is bijective and both F and F^{-1} are differentiable of C^r. If there exists a diffeomorphism from $\mathcal{M}_1$ onto $\mathcal{M}_2$, then we say that $\mathcal{M}_1$ is C^r *diffeomorphic* to $\mathcal{M}_2$. Trivially, $\mathcal{M}_2$ is C^r diffeomorphic to $\mathcal{M}_1$ also when $\mathcal{M}_1$ is C^r diffeomorphic to $\mathcal{M}_2$. □

In many important applications of the differentiable manifolds, we will encounter space that would be differentiable manifold except that they have a "boundary" of some sort. Simple examples of such spaces include closed intervals in $\mathbb{R}$ and closed balls in $\mathbb{R}^n$. To accommodate such space, we need to extend our definition of manifolds.

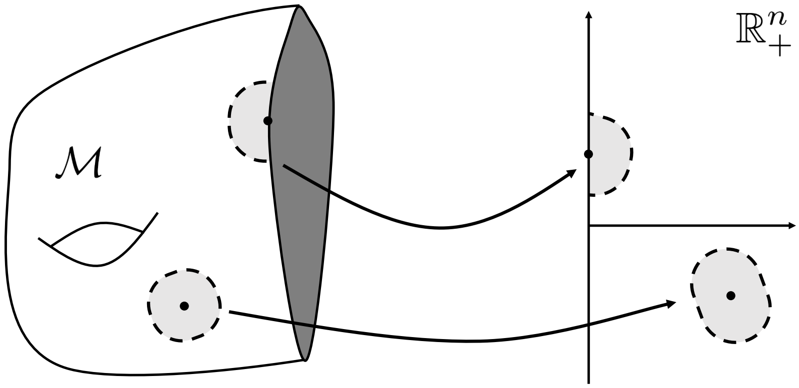

Figure 2.2: A manifold with boundary.

Definition 2.2.6 (Topological manifold with boundary)**.** A second countable Hausdorff space $\mathcal{M}$ is called a *topological manifold with boundary* of dimension n if each point of $\mathcal{M}$ has a neighborhood homeomorphic to an open set in $\mathbb{R}^n_+$. □

Definition 2.2.7 (Differentiable manifold with boundary)**.** A topological manifold with boundary $\mathcal{M}$ is a C^r *n-dimensional manifold with boundary* if it satisfies all the defining conditions of a C^r manifold, with the exception that we allow coordinate domains to map onto open sets in $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$. □

Note that a map from an arbitrary subset $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^k$ is said to be C^r if in an open neighborhood of each point of A it admits an extension to a C^r map defined on an open subset of $\mathbb{R}^n$. If U is an open subset in $\mathbb{R}^n_+$, a map $F : U \to \mathbb{R}^k$ is C^r when, for each $x \in U$, there exists an open subset $\tilde{U} \subseteq \mathbb{R}^n$ containing x and a C^r map $\tilde{F} : \tilde{U} \to \mathbb{R}^k$ that agrees with F on $\tilde{U} \cap \mathbb{R}^n_+$. Therefore, we can define the atlas of the manifold with boundary and the differentiability of functions between the manifolds with boundary in the sense just described.

A point $p \in \mathcal{M}$ is called an *interior point* of $\mathcal{M}$ if there is a coordinate chart (U, φ) such that $\varphi(p) \in \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0\}$. Otherwise, it called a *boundary point* of $\mathcal{M}$. The *boundary of* $\mathcal{M}$ (the set of all its boundary points) is denoted by $\partial\mathcal{M}$.

Throughout the rest of this section, $\mathcal{M}$ is a smooth n-dimensional manifold with boundary (unless otherwise noted) and, for a point $p \in \mathcal{M}$, $C^\infty(p)$ denotes the set of all smooth real-valued functions that can be defined on an open neighborhood of p.

Definition 2.2.8 (Tangent vector and Tangent space)**.** A *tangent vector* v_p to $\mathcal{M}$ at a point $p \in M$ is a linear derivation from $C^\infty(p)$ into $\mathbb{R}$. In other words, for all $\phi, \psi \in C^\infty(p)$ and $\alpha, \beta \in \mathbb{R}$, it holds that

(a) $v_p(\alpha\phi + \beta\psi) = \alpha v_p(\phi) + \beta v_p(\psi)$.

(b) $v_p(\phi\psi) = \phi(p)v_p(\psi) + \psi(p)v_p(\phi)$.

The *tangent space* to $\mathcal{M}$ at $p \in \mathcal{M}$ is the set of all tangent vectors to $\mathcal{M}$ at p and denoted by $T_p\mathcal{M}$. □

We can observe that $T_p\mathcal{M}$ is a vector space over the field $\mathbb{R}$ with the vector addition and the scalar multiplication defined as

$$(v_p + w_p)(\phi) := v_p(\phi) + w_p(\phi),$$
$$(\alpha v_p)(\phi) := \alpha v_p(\phi),$$

where $v_p, w_p \in T_p\mathcal{M}$, $\phi \in C^\infty(p)$, and $\alpha \in \mathbb{R}$. Moreover, the dimension of $T_p\mathcal{M}$ is equal to that of $\mathcal{M}$.

Let us consider $p \in \mathcal{M}$. Then there exists a smooth coordinate chart (U, φ) such that $p \in U$. It is well-known that $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p \in T_p\mathcal{M}$ defined by $\partial/\partial x^i|_p(\phi) := \frac{\partial(\phi\circ\varphi^{-1})}{\partial x_i}|_{\varphi(p)}$ (for $\phi \in C^\infty(p)$) form a basis for $T_p\mathcal{M}$. Thus, it holds for $v \in T_p\mathcal{M}$ that $v = \sum_{i=1}^n v(\varphi_i(p))\partial/\partial x^i|_p$, which is a coordinate dependent representation.

Definition 2.2.9 (Differential)**.** If $\mathcal{M}$ and $\mathcal{N}$are smooth manifolds (with boundary) and $F : \mathcal{M} \to \mathcal{N}$ is a smooth map, for each $p \in \mathcal{M}$ we define a map

$$\mathrm{d}F_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N},$$

called the *differential of F at p*, as follows. Given $v \in T_p\mathcal{M}$, we let $\mathrm{d}F_p(v)$ be the derivation at $F(p)$ that acts on $\psi \in C^\infty(\mathcal{N})$ by the rule

$$\mathrm{d}F_p(v)(\psi) = v(\psi \circ F).$$

□

Based on the concept of tangent space, a tangent bundle on a smooth manifold is defined as follows.

Definition 2.2.10 (Vector bundle)**.** Given a smooth manifold $\mathcal{M}$ (with boundary) the *tangent bundle of* $\mathcal{M}$, denoted by $T\mathcal{M}$, is the disjoint union of the tangent spaces at all points of $\mathcal{M}$:

$$T\mathcal{M} = \coprod_{p \in \mathcal{M}} T_p\mathcal{M}.$$

□

The tangent bundle comes equipped with a natural projection map $\pi : T\mathcal{M} \to \mathcal{M}$, which sends each vector in $T_p\mathcal{M}$ to the point p. A section of π is a continuous right inverse for π, i.e., a continuous map $\delta : \mathcal{M} \to T\mathcal{M}$ such that $\pi \circ \delta = \mathrm{Id}_{\mathcal{M}}$.

The tangent bundle $T\mathcal{M}$ has a natural topology and smooth structure that make in into a $2n$-dimensional smooth manifold. With respect to this structure, the projection map $\pi : T\mathcal{M} \to \mathcal{M}$ is smooth.

Definition 2.2.11 (Vector field)**.** A *vector field f on $\mathcal{M}$* is a section of the projection map $\pi : T\mathcal{M} \to \mathcal{M}$. More concretely, a vector field is a continuous map $f : \mathcal{M} \to T\mathcal{M}$, usually written $p \mapsto f_p$, with the property that $\pi \circ f = \mathrm{Id}_{\mathcal{M}}$ or equivalently, $f_p \in T_p\mathcal{M}$ for each $p \in \mathcal{M}$. The vector field f is said to be *smooth* if $f : \mathcal{M} \to T\mathcal{M}$ is smooth. □

Using a coordinate chart (U, φ) on M, the smooth vector field f on U can be represented as

$$f(p) = \sum_{i=1}^{n} f_i(p) \frac{\partial}{\partial x_i}\bigg|_p \qquad \text{for all } p \in U,$$

where n real-valued functions $f_1, \ldots, f_n$ are smooth on U. Note that we can also define the differential of F when $\mathcal{M}$ and $\mathcal{N}$ are C^r manifolds and F is C^r map.

Suppose $F : \mathcal{M} \to \mathcal{N}$ is smooth and v is a vector field on $\mathcal{M}$, and suppose there happens to be a vector field w on $\mathcal{N}$ such that, for each $p \in \mathcal{M}$,

$$\mathrm{d}F_p(v_p) = w_{F(p)}.$$

In this case, we say the vector fields v and w are *F-related.*

Definition 2.2.12 (Pushforward)**.** Suppose $\mathcal{M}$ and $\mathcal{N}$ are smooth manifolds (with boundary), and $F : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism. For every smooth vector field v a vector field on $\mathcal{M}$, there exists a unique smooth vector field on $\mathcal{N}$ that is F-related to v. We denote it by $F_* v$ and call it the *pushforward of v by F.* □

We omit the proof of the existence of $F_* v$. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are differentiable manifold (with boundary). Given a C^1 map $F : \mathcal{M} \to \mathcal{N}$ and a point $p \in \mathcal{M}$, we define the *rank of F at p* to be the rank of the linear map $\mathrm{d}F_p : T_p\mathcal{M} \to T_{F(p)N}$; it is the rank of the Jacobian matrix of F in any smooth chart.

Definition 2.2.13 (Immersion)**.** Suppose that $F : \mathcal{M} \to \mathcal{N}$ is a C^1 map. If its differential is injective at each $p \in \mathcal{M}$ (rank F = dim $\mathcal{M}$), we say that F is immersion. When F is smooth, it is called a smooth immersion. □

Now we introduce the Inverse Function Theorem for differentiable manifolds.

Theorem 2.2.2. (Inverse Function Theorem for manifolds)**.** Suppose $\mathcal{M}$ and $\mathcal{N}$ are differentiable manifolds, and $F : \mathcal{M} \to \mathcal{N}$ is a C^r map. If $p \in \mathcal{M}$ is a point such that $\mathrm{d}F_p$ is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a C^r diffeomorphism. □

The most important fact about immersion is the following consequence of the inverse function theorem. It says that an immersion can be placed locally into a particularly simple canonical form by a change of coordinates. Actually, it can be applied to constant-rank maps, but we only deal with the case when F is immersion in this section.

Theorem 2.2.3. (Rank Theorem)**.** Suppose $\mathcal{M}$ and $\mathcal{N}$ are differentiable manifolds of dimensions m and n, respectively, and $F : \mathcal{M} \to \mathcal{N}$ is a C^r immersion. For each $p \in \mathcal{M}$, there exists C^r charts (U, φ) for $\mathcal{M}$ centered at p and (V, ϕ) for $\mathcal{N}$ centered at $F(p) \in \mathcal{N}$ such that $F(U) \subset V$, in which F has a coordinate representation of the form

$$\hat{F}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

□

Now we consider the special kind of immersion. To deal with this, we define some property.

Definition 2.2.14 (Embedding)**.** If $\mathcal{M}$ and $\mathcal{N}$ are differentiable manifolds (with boundary), a C^r *embedding of* $\mathcal{M}$ *into* $\mathcal{N}$ is a C^r immersion $F : \mathcal{M} \to \mathcal{N}$ that is also a topological embedding, i.e., a homeomorphism onto its image $F(\mathcal{M}) \subset \mathcal{N}$ in the subspace topology. □

Theorem 2.2.4. (Local embedding theorem)**.** Suppose $\mathcal{M}$ and $\mathcal{N}$ are differentiable manifolds (with boundary), and $F : \mathcal{M} \to \mathcal{N}$ is a C^r map. Then F is a immersion if and only if each point in $\mathcal{M}$ has a open neighborhood $U \subset \mathcal{M}$ such that $F|_U : U \to \mathcal{N}$ is a C^r embedding. □

Many familiar manifolds are subsets of other manifold such as a subset of $\mathbb{R}^n$. Therefore, we define some manifolds, which are subsets of other smooth manifolds.

Definition 2.2.15 (Embedded submanifold)**.** An *(embedded) submanifold* of the differentiable manifold $\mathcal{M}$ (with boundary) is a subset $\mathcal{S} \subset \mathcal{M}$ that is a manifold in the subspace topology, endowed with a differential structure with respect to which the inclusion map $\mathcal{S} \hookrightarrow \mathcal{N}$ is an embedding. □

Definition 2.2.16 (Properly embedded)**.** An embedded submanifold $\mathcal{S} \subset \mathcal{M}$ is said to be *properly embedded* if the inclusion $\iota : \mathcal{S} \hookrightarrow \mathcal{N}$ satisfies that, for every compact set $Y \subset \mathcal{M}$, the preimage $\iota^{-1}(Y)$ is compact. □

Next proposition says embedded submanifolds are modeled locally on some standard local embedding.

Proposition 2.2.5. Let $\mathcal{M}$ be a differential n-manifold (with boundary) and $\mathcal{S} \subset \mathcal{M}$ be an embedded k-dimensional submanifold. Then, for each $p \in \mathcal{S}$, there exists a differential chart (U, φ) for $\mathcal{M}$ such that

$$\varphi(\mathcal{S} \cap U) = \{(x_1, \ldots, x_n) \in \varphi(U) : x_{k+1} = c_{k+1}, \ldots, x_n = c_n\},$$

for some constants $c_{k+1}, \ldots, c_n$. □

If $\mathcal{M}$ be a differential n-manifold with boundary, then with the subspace topology, $\partial\mathcal{M}$ is a topological $(n-1)$-dimensional manifold without boundary and has a differential structure such that it is a properly embedded submanifold of $\mathcal{M}$. In addition, the differential structure on $\partial\mathcal{M}$ is unique.

In addition, $\partial\mathcal{M}$ may be a union of $(n-1)$-submanifold of $\mathcal{M}$. To consider some part of the boundary, which consists of the connected components of the boundary $\partial\mathcal{M}$, we define the following.

Definition 2.2.17 (Smooth part of boundary)**.** For a smooth n-manifold with boundary $\mathcal{M}$, Q is a *smooth part* of the boundary if Q is a union of connected components of $\partial\mathcal{M}$. □

Note that since $\partial\mathcal{M}$ is a smooth $(n-1)$-submanifold, Q is also a smooth $(n-1)$-submanifold of $\mathcal{M}$. In addition, Q is open in $\partial\mathcal{M}$ and closed in $\mathcal{M}$.

In fact, we can consider an abstract smooth manifold with boundary as a submanifold of Euclidean space. The following theorem means that every smooth n-manifold with boundary is diffeomorphic to a properly embedded submanifold (with or without boundary) of $\mathbb{R}^{2n+1}$.

Theorem 2.2.6. (Whitney Embedding Theorem)**.** Every smooth n-manifold with or without boundary admits a proper smooth embedding into $\mathbb{R}^{2n+1}$. □

When $\mathcal{S}$ is submanifold of $\mathcal{M}$, the tangent space to $\mathcal{S}$ can be viewed as a subspace of the tangent space to $\mathcal{M}$. To make appropriate identifications, consider the inclusion map $\iota : \mathcal{S} \to \mathcal{M}$. Since the inclusion map $\iota : \mathcal{S} \to \mathcal{M}$ is a immersion, at each point $p \in \mathcal{S}$ we have an injective linear map $d\iota_p : T_p\mathcal{S} \to T_p\mathcal{M}$. In view of derivations, this injection works in the following way: for any vector $v \in T_p\mathcal{S}$, the image vector $\tilde{v} = \mathrm{d}\iota_p(v) \in T_p\mathcal{M}$ acts on differential function on $\mathcal{M}$ by

$$\tilde{v}(\psi) = \mathrm{d}\iota_p(v)(\psi) = v(\psi \circ \iota) = v(\psi|_{\mathcal{S}}),$$

where ψ is a differential map on $\mathcal{M}$. By adopting the convention of identifying $T_p\mathcal{S}$ with its image under this map, we can consider $T_p\mathcal{S}$ as a certain linear subspace of $T_p\mathcal{M}$.

Intuitively, the tangent vector $v \in T_p\mathcal{M}$ can be separated into three classes: it tangent to the boundary, pointing inward, and pointing outward.

Definition 2.2.18 (Inward-pointing and Outward-pointing)**.** Suppose that $\mathcal{M}$ is a differentiable manifold with boundary. If $p \in \partial\mathcal{M}$, a tangent vector $v \in T_p\mathcal{M} \backslash T_p\partial\mathcal{M}$ is said to be *inward-pointing* if for some $\epsilon > 0$ there exists a C^r function $\gamma : [0, \epsilon) \to \mathcal{M}$ such that $\gamma(0) = p$ and, for $\psi \in C^\infty(p)$, $\lim_{t \searrow 0} \frac{\psi \circ \gamma(t)}{t} = v(\psi)$, and it is outward-pointing if there exists a C^r function $\gamma : (-\epsilon, 0] \to \mathcal{M}$ such that $\gamma(0) = p$ and, for $\psi \in C^\infty(p)$, $\lim_{t \nearrow 0} \frac{\psi \circ \gamma(t)}{t} = v(\psi)$. □

In fact, the notion of the function γ is a "curve", which is defined as follows.

Definition 2.2.19 (Curve)**.** If f is a vector field on a differentiable manifold (with boundary) $\mathcal{M}$, we define a *integral curve* of f on $\mathcal{M}$ to be a differential map $\gamma : J \to \mathcal{M}$ and

$$\mathrm{d}\gamma \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t_0} \right) = f_{\gamma(t_0)} \text{ for all } t_0 \in J$$

where $J \subset \mathbb{R}$ is an interval and $\mathrm{d}/\mathrm{d}t|_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$. □

Note that J is usually an open interval, but J may have one or two endpoints. Then, we can interpret derivatives as one-sided derivatives.

Theorem 2.2.7. (Boundary Flowout Theorem)**.** Let $\mathcal{M}$ be a smooth manifold with nonempty boundary, and let f be a smooth vector field on $\mathcal{M}$ that is inward-pointing at each point of $p \in \mathcal{M}$. There exist a smooth function $\delta : \partial\mathcal{M} \to \mathbb{R}^+$ and a smooth embedding $\Phi : \mathcal{P}_\delta \to \mathcal{M}$ where $\mathcal{P}_\delta := \{(t, p) : (p \in \partial\mathcal{M}) \wedge (0 \leq t \leq \delta(p))\} \subset \mathbb{R} \times \partial\mathcal{M}$, such that $\Phi(\mathcal{P}_\delta)$ is a neighborhood of $\partial\mathcal{M}$, and for each $p \in \partial\mathcal{M}$ the map $t \mapsto \Phi(t, p)$ is an integral curve of $\mathcal{M}$ starting at p. □

Now we introduce a quotient topology on some topology space and a quotient map. In addition, using this concept, we attach manifolds along their boundaries.

Definition 2.2.20 (Quotient topology and Quotient Map)**.** If X is a topological space, Y is a set, and $\pi : X \to Y$ is a surjective map, the *quotient topology on* Y *determined by* π is defined by declaring a subset $U \subset Y$ to be open if and only if $\pi^{-1}(U)$ is open in X. If X and Y are topological spaces, a map $\pi : X \to Y$ is called a *quotient map* if it is surjective and continuous and Y has the quotient topology determined by π. □

Definition 2.2.21. (Saturated set and Fiber) If $\pi : X \to Y$ is a map, a subset $U \subset X$ is said to be *saturated with respect to* π if U is the entire preimage of its image: $U = \pi^{-1}(\pi(U))$. For $y \in Y$, the *fiber of over* y is the preimage $\pi^{-1}(y)$. □

Note that a subset of X is saturated if and only if X is a union of fibers.

Definition 2.2.22 (Quotient space)**.** Suppose that X is a topological space and $\sim$ is an equivalence relation on X. Let $X/\sim$ denote the set of equivalence class in X, and let $\pi : X \to X/\sim$ be the natural projection sending each point to its equivalence class. Endowed with the quotient topology determined by π, the space $X/\sim$ is called the *quotient space of* X *determined by* $\sim$. □

For example, suppose that X and Y are topological spaces; $A \subset Y$ is a closed subset; and $g : A \to X$ is a continuous map. The relation $a \sim g(a)$ for all $a \in A$ generates an equivalence relation on $X \coprod Y$, whose quotient space is denoted by $X \cup_g Y$ and called an *adjunction space.* It is said to be formed by attaching Y to X along g. Now we are ready to glue the boundaries of the manifolds.

Theorem 2.2.8. (Attaching smooth manifolds along their boundaries)**.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be smooth n-manifolds with nonempty boundaries, and suppose $g : \partial\mathcal{M}_2 \to \partial\mathcal{M}_1$ is a diffeomorphism. Then $\mathcal{M}_1 \cup_g \mathcal{M}_2$ is a topological manifold (without boundary), and has a smooth structure such that there are properly embedded n-submanifolds with boundary $\mathcal{M}'_1, \mathcal{M}'_2 \subset \mathcal{M}_1 \cup_g \mathcal{M}_2$ diffeomorphic to $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, and satisfying

$$\mathcal{M}'_1 \cup \mathcal{M}'_2 = \mathcal{M}_1 \cup_g \mathcal{M}_2, \qquad \mathcal{M}'_1 \cap \mathcal{M}'_2 = \partial\mathcal{M}'_1 = \partial\mathcal{M}'_2.$$

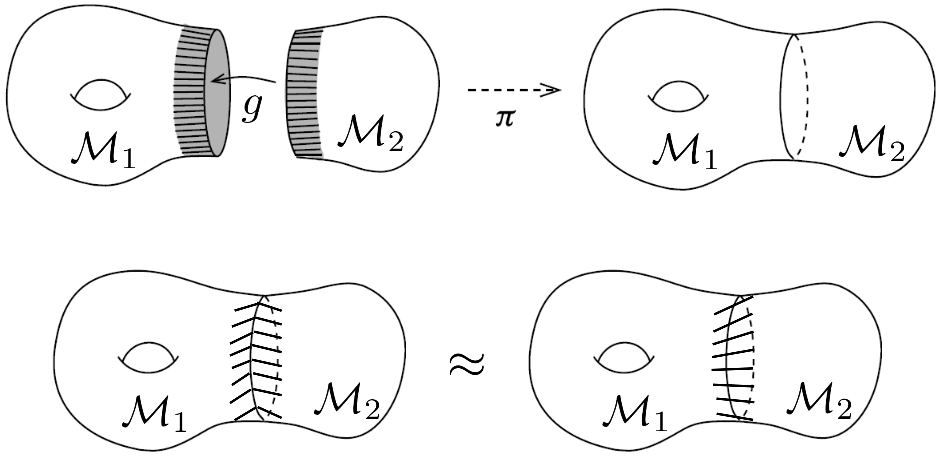

Figure 2.3: Gluing and differential structures.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are both compact, then $\mathcal{M}_1 \cup_g \mathcal{M}_2$ is compact. □

By the theorem, we can obtain a new smooth manifold by gluing the boundaries of the disjoint manifolds.

Next theorem deals with the case when the parts of the boundaries are glued.

Theorem 2.2.9. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be smooth n-manifolds with nonempty boundaries, $S_1 \subset \partial\mathcal{M}_1$ and $S_2 \subset \partial\mathcal{M}_2$ are smooth parts of the respective boundaries, and $g : S_2 \to S_1$ is a diffeomorphism. Then $\mathcal{M}_1 \cup_g \mathcal{M}_2$ is a topological manifold with boundary and can be given a smooth structure such that $\mathcal{M}_1$ and $\mathcal{M}_2$ are diffeomorphic to some n-dimensional submanifolds of $\mathcal{M}_1 \cup_g \mathcal{M}_2$. □

In fact, there are many smooth structures on $\mathcal{M}_1 \cup_g \mathcal{M}_2$. However, they are diffeomorphic by the following theorem. The theorem says that the glued manifold consists of two smooth manifolds and all possible endowed structures of the glued domain are diffeomorphic.

Theorem 2.2.10. (Determined up to diffeomorphism). Let $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ be n-manifolds without boundary. Suppose that $\bar{\mathcal{M}}$ is the union of two closed n-submanifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ and $\bar{\mathcal{N}}$ is the union of two closed n-submanifolds $\mathcal{N}_1$

and $\mathcal{N}_2$ such that

$$\mathcal{M}_1 \cap \mathcal{M}_2 = \partial\mathcal{M}_1 = \partial\mathcal{M}_2 \text{ and } \mathcal{N}_1 \cap \mathcal{N}_2 = \partial\mathcal{N}_1 = \partial\mathcal{N}_2.$$

If $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ are homeomorphic and, for $i = 1, 2$, $\mathcal{M}_i$ and $\mathcal{N}_i$ are diffeomorphic, then $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ are diffeomorphic. □

2.3 Viability Theorems for Ordinary Differential Equations

In this section, we introduce the basic theorems of viability theory in the simple framework of ordinary differential equation $\dot{x} = f(x)$ on a subset K of a finite dimensional vector space. To guarantee the existence of the solutions, it is required that, for any state $x \in K$, the velocity is tangent in some sense to K at x. For this, if K is smooth manifold, we can adopt the notion of the tangent space of smooth manifold. However, on the boundary of the manifold with boundary, the element of the tangent space may point outward from K. Therefore, we need to characterize more generalized the set of the tangent direction.

Definition 2.3.1 (Viable function)**.** Let K be a subset of a finite dimensional vector space X. We shall say that a function $x(\cdot)$ from $[0, T]$ to X is *viable in* K *on* $[0, T]$ if

$$x(t) \in K \quad \text{for all } t \in [0, T].$$

□

Let us describe the dynamics of the system by a map f from some open subset Ω of X to X. We consider the initial value problem associated with the following differential equation:

$$\dot{x}(t) = f(x(t)) \quad \forall t \in [0, T] \tag{2.3.1}$$

with the initial condition $x(0) = x_0$.

Definition 2.3.2 (Viability and Invariance)**.** Let K be a subset of Ω. We shall say that K *is locally viable under* f if for any initial state x_0 of K, there exists $T > 0$ and a viable solution on $[0, T]$ to differential equation (2.3.1) starting at x_0. It is said to be *(globally) viable under* f if we can always take $T = \infty$. The subset K is said to be *invariant under* f if for any initial state x_0 of K, all solution to differential equation (2.3.1) are viable in K. □

Definition 2.3.3 (Contingent cone)**.** Let X be a normed space, K be a nonempty subset of X and x belong to K. The *contingent cone to* K *at* x is the set

$$T_K(x) := \left\{ v \in X : \liminf_{h \to 0+} \frac{d_K(x + hv)}{h} \right\}$$

where $d_K(y)$ denotes the *distance of* y *to* K, defined by

$$d_K(y) := \inf_{z \in K} |y - z|.$$

□

In other worlds, v belongs to $T_K(x)$ if and only if there exists a sequence of $h_n > 0$ converging to $0+$ and a sequence of $v_n \in X$ converging to v such that

$$x + h_n v_n \in K \text{ for all } n \geq 0.$$

Note that when K is a differentiable manifold, the contingent cone $T_K(x)$ coincides with the tangent space to K at x. In addition, when K is a differentiable manifold with boundary and x is boundary point, the contingent cone $T_K(x)$ is a subset of the tangent space to K at x.

The following lemma shows right away why this cone will play a crucial role.

Lemma 2.3.1. Let $x : [0, T] \to K$ be a differentiable viable function. Then, it follows that

$$\dot{x}(t) \in T_K(x(t)) \quad \forall t \in [0, T)$$

□

Definition 2.3.4. Let K be a subset of Ω. We say that K is a *viability domain* *of the map* $f : \Omega \to X$ if

$$f(x) \in T_K(x) \quad \forall x \in K.$$

$\square$

Finally, we introduce some viability theorems.

Theorem 2.3.2. (Nagumo)**.** Let us assume that

- K is locally compact;
- f is continuous from K to X.

Then K is locally viable under f if and only if K is a viability domain of f. $\square$

Theorem 2.3.3. Let us consider a subset K of a finite dimensional vector space X and a map f from K to X. We assume that

- the map f is continuous from K to X;
- there is $c > 0$ such that $|f(x)| \leq c(|x| + 1)$ for all $x \in K$;
- K is a closed viability domain of f.

Then K is globally viable under f (i.e., for every initial state $x \in K$, there exists a viable solution on $[0, \infty]$ to differential equation (2.3.1) starting at x_0.) $\square$

Chapter 3

Reviews of Related Previous Works

In this section, related previous works are introduced. The considered previous works consist of three major topics. First of all, gluing the domain and smoothing the vector field of the hybrid system are considered in Section 3.1. Secondly, the viability theory for the hybrid system is introduced in Section 3.2. Finally, as applications, previous results of state estimation and tracking control are presented in Section 3.3 and Section 3.4, respectively.

Note that there are many ways to model the hybrid dynamical systems ([LJS$^+$03, SJLS05, GST09, Bro00, ALQ$^+$02]). In this chapter, to clearly represent the previous results, each section adopts a different framework. Therefore, to avoid confusion, the reader should be aware that each section is processed with an independent notation.

3.1 Gluing Manifolds and Vector Fields

This section consists of two parts. The first part focuses on gluing manifolds. In this part, we deal with a general manifold and glue the manifolds in a topological sense. In the second part, we constructs the differential structure and the smooth vector field on glued manifold. For this, more restrictive conditions are required.

At first, we introduce a notion of hybridfold proposed in [SJLS05]. To study this, at first we define a framework.

Definition 3.1.1. An n-dimensional hybrid system is a 6-tuple $H = (Q, E, D, X, G, r)$ where:

- $Q = \{1, \ldots, k\}$ is a finite set of *discrete states*, where $k \geq 1$ is an integer;
- $E \subset Q \times Q$ is a collection of *edges* (according to which jumps occur);
- $D = \{D_i \subset \{i\} \times \mathbb{R}^d : i \in Q\}$ is a collection of *domains* (in which continuous state evolutions take place);
- $X = \{X_i : i \in Q\}$ is a collection of *vector fields* (determining the dynamics in these domains), such that X_i is Lipschitz on D_i;
- $G = \{G_e : e \in E\}$ is a collection of *guards* (hitting which is triggering the jumps) where $G_e \subset D_i$ for each $e = (i, j) \in E$;
- $r = \{r_e \subset G_e \times D_j : e = (i, j) \in E\}$ is a collection of *reset relations*; a reset relation r_e is a map $G(e) \to D_j$, with $e = (i, j) \in E$ and we write $y = r_e(x)$ instead of $(x, y) \in r_e$.

□

This definition clearly allows the hybrid system to be a wild object.

Definition 3.1.2. A (forward) *hybrid time trajectory* is finite or infinite sequence of intervals $\tau = \{I_i\}_{i=0}^N$ (N may be ∞) such that

- $I_i = [\tau_i, \tau_i']$ with $\tau_i \leq \tau_i' = \tau_{i+1}$ for all $0 \leq i < N$, in particular, $\tau_0 = 0$;
- when $N < \infty$, either $I_N = [\tau_N, \tau_N']$ or $I_N = [\tau_N, \tau_N')$.

For $\tau = \{I_i\}_{i=0}^N$, let $\langle \tau \rangle := \{0, 1, \ldots, N\}$ (possibly $N = \infty$), and $|\tau| := \sum_{i \in \langle \tau \rangle} (\tau_i' - \tau_i)$. We say that $\tau = \{I_i\}_{i=0}^N$ is a *prefix* of $\tilde{\tau} = \{\tilde{I}_i\}_{i=0}^{\tilde{N}}$ and write $\tau \sqsubseteq \tilde{\tau}$, if either they are identical; or N is finite, $N \leq \tilde{N}$, $I_i = \tilde{I}_i$ for all $0 \leq i < N$, and $I_N \subseteq \tilde{I}_N$. □

Next, we define a solution (execution) of the hybrid system.

Definition 3.1.3. An *execution* of H is a triple $\xi = (\tau, q, x)$ where τ is a hybrid time trajectory; $q : \langle \tau \rangle \to Q$ is a map; $\xi = \{\xi^i : i \in \langle \tau \rangle\}$ is a collection of C^1 maps such that $\xi^i : I_i \to D_{q(i)}$ and

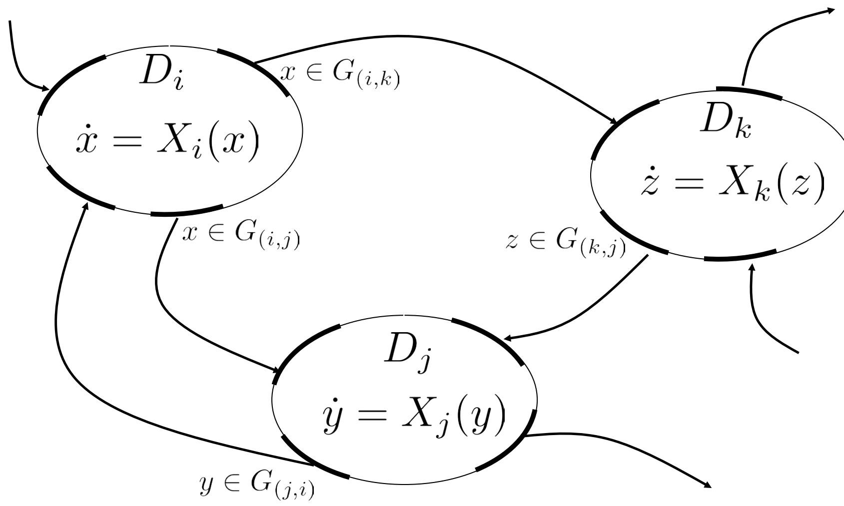

Figure 3.1: Hybrid system defined in Definition 3.1.1

- $\dot{\xi}^i(t) = X_{q(i)}(\xi^i(t))$ for all $i \in \langle\tau\rangle$ and for all $t \in I_i$;
- $(q(j), q(j+1)) \in E$ for all $i \in \langle\tau\rangle$ such that $i < N$;
- $(\xi^i(\tau_i'), \xi^{i+1}(\tau_{i+1})) \in r_{(q(i),q(i+1))}$ for all $i \in \langle\tau\rangle$ such that $i < N$. □

Definition 3.1.4. A hybrid system is called *deterministic* if for every $p \in D$ there exists at most one maximal execution starting from p. It is called *non-blocking* if for every $p \in D$ there is at least one infinite execution starting from p. □

On the definitions, we impose the following assumptions.

Assumption 3.1.1. A hybrid system $H = (Q, E, D, X, G, r)$ satisfies that

(A1) H is deterministic and non-blocking;

(A2) each domain D_i is a connected smooth n-submanifold of $\mathbb{R}^d$ for $i \in Q$ with piecewise smooth boundary, and the angle between any two intersecting smooth components of the boundary is nonzero;

(A3) each guard $G_{(i,j)}$ is a smooth $(n-1)$-submanifold of the boundary of corresponding domain D_i; The boundary of each guard is piecewise smooth (or possibly empty);

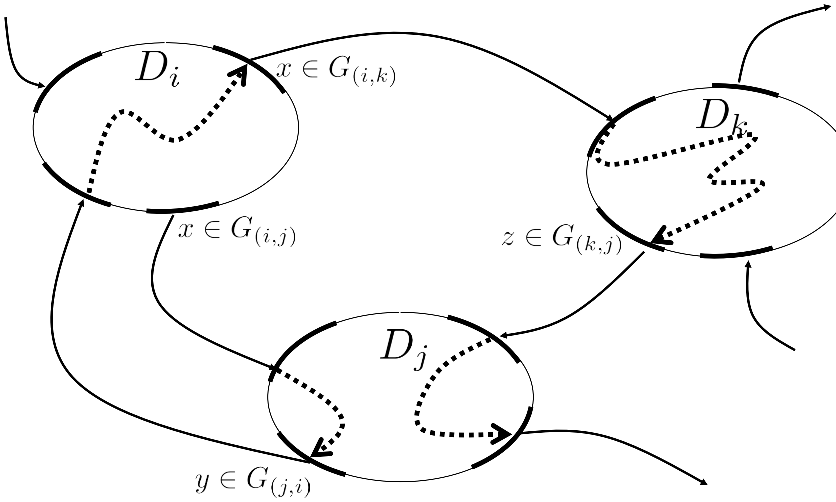

Figure 3.2: An example of execution

(A4) each reset $r_{(i,j)}$ is a diffeomorphism from its domain $G_{(i,j)}$ onto its image, which is on a boundary of corresponding domain D_j;

(A5) The closures of guards and their reset images can intersect only along their boundaries, and moreover points from such intersections can be only of one of the four following types (and furthermore, resets preserve types of the points):

Type I: point belongs to only one set and is internal point for it

Type II: point belongs to only one set and is boundary point for it

Type III: point belongs to exactly two sets and is internal point for them

Type IV: point belongs to exactly two sets and is boundary point for them

(A6) for each edge $e = (i, j) \in E$, on the interior of the guard G_e on domain D_i, vector field X_i points outside of D_i and, on the interior of the reset image $r_e(G_e)$, vector field X_j points inside D_j;

(A7) each vector field X_i is a restriction to D_i of some smooth vector field (also denoted by X_i) defined in the neighborhood of D_i in $\mathbb{R}^d$; each reset $r_{e=(i,j)}$

extends to a map $\tilde{r}_e$ defined in the neighborhood of the guard G_e in D_i, such that $\tilde{r}_e$ is a diffeomorphism onto its image, which is the neighborhood of $r_e(G_e)$ in D_j; furthermore special consistency condition of maps $f_m \circ \cdots \circ f_1 = \mathrm{id}$, for every collection of maps $f_1, \cdots, f_m, f_{m+1} = f_1$, such that for each j there exists $\exists e_j \in E$ with the property that either $f_j = \tilde{r}_{e_j}$ or $f_j = \tilde{r}_{e_j}^{-1}$, and image of f_j meets the domain of f_{j+1};

(A8) In vector field points inside corresponding domain on some boundary point, then this boundary point is i the image of some reset.

A hybrid system H satisfying the assumptions is called *regular*. □

This assumption makes sure that everything remains satisfied when time is reversed.

Note that a manifold is piecewise smooth if, intuitively speaking, it is the union of finitely many smooth smooth manifold. Since each domain is embedded into $\mathbb{R}^d$, it inherits from it the standard Riemann structure so the notion of angle is defined. The non-zero angle requirement eliminates, for instance, "cusps" in dimension two, but does not eliminate "corners". Thus domains of hybrid system can be disks, half-space, rectangles, cubes, etc. In fact, the non zero angle assumption can be easily relaxed for most of the results.

Given H, define a mp Φ^H on a subset of $\mathbb{R} \times D$ as follows. Let $p \in D$ be arbitrary. By (A1), there exists a unique infinite execution (τ, q, ξ) starting at p. We will denote it by $\chi(p)$. Set $\Phi^H(0, p) = p$. Assume that $|\tau| > 0$. For any $0 < t < |\tau|$ there exists a unique $j(p,t) \in \langle\tau\rangle$ such that $t \in [\tau_j, \tau_j')$ (even though there may be multiple $j \in \langle\tau\rangle$ for which $t \in [\tau_j, \tau_j']$). Then define

$$\Phi^H(t, p) = \xi^j(t)$$

The function $\Phi^H : (t, p) \mapsto \xi^{j(p,t)}(t)$ is called *flow*. Denote by $\Omega_0 \subset \mathbb{R} \times D$ maximal set on which flow $\Phi^H(t, p)$ is defined.

On the assumption, we define equivalence relation $\sim$ on domain $D = \bigcup_{i \in Q} D_i$ generated by relation $p \sim \tilde{r}_e(p)$ for all edges $e \in E$ and points $p \in \mathrm{cl}(G_e)$. Then denote by M^H the quotient space (with quotient topology), obtained by collapsing each equivalent class into point, i.e.,

$$M^H = D/\sim = \cup_{\tilde{r}} D_i$$

Denote by $\pi : D \to M^H$ the natural projection that maps each point into its equivalent class.

Definition 3.1.5. M^H is called the *hybridfold* of the regular hybrid system H.□

Theorem 3.1.1. Under Assumption 3.1.1, the following statements hold:

(a) M^H is a topological manifold with boundary.

(b) Both M^H and its boundary is piecewise smooth.

(c) The restriction $\pi_{\text{int}(D)} : \text{int}(D) \to \pi(\text{int}(D))$ is a diffeomorphism.

□

In a viewpoint of gluing manifolds, the above theorem consider more general case than Theorem 2.2.8 because D_i is the smooth manifold with piecewise smooth boundary. However, it just guarantees that, the set after gluing can be not a smooth manifold but a topological manifold. The idea to make a topological manifold is based on the following lemma.

Proof. Proof can be founded in [SJLS05]. □

Lemma 3.1.2. (Gluing homeomorphisms)**.** Suppose $h_+ : A_+ \to \mathbb{R}^n_+$ and $h_- : A_- \to \mathbb{R}^n_-$ are homeomorphisms, where A_+ and A_- are disjoint topological spaces. Let $H_s = h_s^{-1}(\{0\} \times \mathbb{R}^{n-1})$, for $s \in \{-, +\}$, and assume that there exists a homeomorphism $g : H_+ \to H_-$such that $h_+|_{H_+} = h_- \circ g$. Let $A = (A_+ \cup A_-)/\sim$ be the quotient space, where $\sim$ is the smallest equivalence class of $x \in A_+ \cup A_-$. Then the map $h : A \to \mathbb{R}^n$ defined by

$$h(x/\sim) = \begin{cases} h_+(x) & \text{if } x \in A_+ \\ h_-(x) & \text{if } x \in A_- \end{cases}$$

is a homeomorphism. □

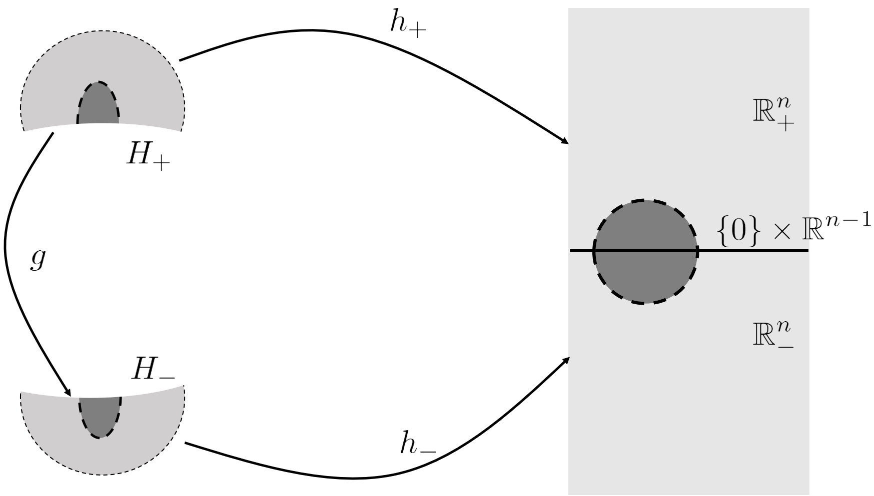

Figure 3.3: Gluing homeomorphisms along their "common" boundary.

Define the *hybridflow* Ψ^H by

$$\Psi^H(t, \pi(p)) := \pi(\Phi^H(t, p)).$$

Note that this possibly set-valued map is defined on $\Omega = \{(t, \pi(p)) : (t, p) \in \Omega_0\}$. The flow $\Psi_t^H(x) = \Psi^H(t, x)$ can be set-valued. This occurs when, due to gluing, the projections of tow disjoint executions of H overlap in the hybrid fold. This can happen at $x = \pi(p) \in M^H$, where p is a point of type III or IV; since p is glued to two or more other points, the orbit through x could branch.To deal with this situation, introduce the following notions.

Definition 3.1.6. Let X be a smooth vector field on a smooth manifold M, with flow $\phi_t(p)$. We say that $q \in \mathcal{M}$ is *X-reachable from a point* $p \in \mathcal{M}$ if $q = \phi_t(p)$, for some $t > 0$. A set S is X-reachable from a point p if there exists a point $q \in S$ such that q is X-reachable from p. □

Definition 3.1.7. A regular hybrid system H is said to be *without branching* if for every point $p \in \partial D$ of thy III or IV with $p/\sim = \{p_1, \ldots, p_m\}$, where $p_j \in D_{i_j}$, there exists at most on k and at most one l, $1 \leq k \neq l \leq m$, such that p_k is X_{i_k}-reachable from D_{i_k} and D_{i_l} is X_{i_l}-reachable from p_l. □

Let us define

$$M(t) := \{y \in M^H : \Psi^H(t,y) \text{ is defined}\}$$
$$J(x) := \{s \in \mathbb{R} : \Psi^H(s,x) \text{ is defined}\}.$$

Then the following theorem represents the properties of the hybridflow.

Theorem 3.1.3. Suppose hybrid system H is regular and without branching. Then:

(a) For each $(t,x) \in \Omega$, $\Psi_t^H(x)$ is a single point.

(b) For each $x \in M^H$, the map $t \mapsto \Psi_t(x)$ is continuous. Moreover, if $J(x)$ is not a single point, the map is smooth except at (at most) countably many points in $J(x)$.

(c) Each map Ψ_t^H is one-to-one.

(d) Whenever bot sides are defined,

$$\Psi_t^H \Psi_s^H(x) = \Psi_{t+s}^H(x).$$

(e) For each $t \in \mathbb{R}$, there exists an open and dense subset of $M(t)$ on which Ψ_t^H is continuous.

□

Two continuous-time dynamical systems can be smoothly attached to one another along their boundaries to obtain a new continuous-time system (Theorem 8.2.1 in [Hir76]). Distinct hybrid domains were attached to one another using this construction in [BRS15]. In this case, impose the addition assumptions on the domains and the vector fields.

For this purposes, it is expedient to define hybrid dynamical system over a finite disjoint union $\mathcal{M} = \coprod_{j \in J} \mathcal{M}_j$ where $\mathcal{M}_j$ is a connected manifold with boundary for each $j \in J$; $\mathcal{M}$ is endowed with natural (piecewise-defined) topology and smooth structure. This is called *smooth hybrid manifolds.* Note that the dimensions of the constituent manifolds are not required to be equal. Several differential-geometric constructions naturally generalize to such space. For instance, the *hybrid tangent bundle* is the disjoint union of the tangent bundles $T\mathcal{M}_j$, and the hybrid boundary $\partial\mathcal{M}$ is the disjoint union of the boundaries $\partial\mathcal{M}_j$.

Definition 3.1.8. A hybrid dynamical system is specified by a tuple $H = (\mathcal{M}, F, G, R)$ where:

- $\mathcal{M} = \coprod_{j\in J} \mathcal{M}_j$ is a smooth hybrid manifold;
- $F : \mathcal{M} \to T\mathcal{M}$ is a smooth vector field;
- $G \subset \partial\mathcal{M}$ is an open subset of $\partial\mathcal{M}$;
- $R : G \to \mathcal{M}$ is a smooth map and $R(G)$ is an open subset of $\partial\mathcal{M}$.

□

As in the before section, R and G is called the *reset map* and the *guard*, respectively.

Assumption 3.1.2. F is outward-pointing on G and inward-pointing on $R(G)$. □

Assumption 3.1.3. G is a diffeomorphism between disjoint portions of the boundary. □

Under these conditions, we can globally smooth the hybrid transitions using techniques from differential topology to obtain a single continuous-time dynamical system. This provides a smooth n-dimensional generalization of the *hybridfold* construction in [SJLS05].

Theorem 3.1.4. (Smoothing). Let $H = (\mathcal{M}, F, G, R)$ be a hybrid dynamical system with $\mathcal{M} = \coprod_{j\in J} \mathcal{M}_j$ satisfying Assumptions 3.1.2–3.1.3. Suppose $\dim(\mathcal{M}_j) = n$ for all $j \in J$ and $\partial\mathcal{M} = G \coprod R(G)$. Then the topological quotient $\overline{\mathcal{M}} = \cup_R \mathcal{M}_j$ may be endowed with the structure of a smooth manifold such that:

- the quotient projection $\pi : \mathcal{M} \to \overline{\mathcal{M}}$ restricts to a smooth embedding $\pi_{\mathcal{M}_j} : \mathcal{M}_j \to \overline{\mathcal{M}}$ for each $j \in J$;

- there is a smooth vector field $\overline{F}$ on $\overline{\mathcal{M}}$ such that any execution $x : T \to \mathcal{M}$ of H descends to an integral curve of $\overline{F}$ on $\overline{\mathcal{M}}$ via $\pi : \mathcal{M} \to \overline{\mathcal{M}}$

$$\forall t \in T : \frac{\mathrm{d}}{\mathrm{d}t}\pi \circ x(t) = \overline{F}(\pi \circ x(t)).$$

Note that the execution $x : T \to \mathcal{M}$ is $\Phi(t, p)$ with some $p \in \mathcal{M}$. □

Proof. Let $S \subset G \cap \mathcal{M}_i$ be a connected component in some domain $i \in J$, and $k \in J$ be the index for which $R(S) \subset \mathcal{M}_k$. The assumptions of this Theorem ensure that Theorem 2.2.8 may be applied to attach $\mathcal{M}_i$ to $\mathcal{M}_k$ to yield a new smooth manifold $\overline{\mathcal{M}}_{ik}$. The hybrid system defined over the domain $\coprod \overline{\mathcal{M}}_{ik} \cup \{\mathcal{M}_j : j \in J \backslash \{i, k\}\}$ and guard $G \backslash S$ satisfies the hypotheses of Theorem 2.2.8, hence we may inductively attach domains on each connected component that remains in $G \backslash S$. This yields a smooth manifold $\overline{\mathcal{M}}$ and vector field $\overline{F}$ with the required properties. □

3.2 Viability Condition

As shown in Section 2.3, the viability condition guarantees the existence of the system solution on the considered domain. The viability for the hybrid dynamical system also was proposed in [ALQ$^+$02]. This section, we introduce some viability conditions proposed in [ALQ$^+$02]. In [ALQ$^+$02], the authors consider the "impulse differential inclusion", whose the flow map and reset map are set-valued maps, but we consider a case of single-valued maps in this section.

Definition 3.2.1 (Impulse Differential equation)**.** An *impulse differential equation* is a collection $H = (X, f, J, r)$, consisting of

- a finite dimensional vector space X, regarded as a domain,
- a map $f : X \to X$, regarded as a differential equation $\dot{x} = f(x)$,
- a set $J \subset X$ regarded as a guard set,
- a map $r : X \to X$, regarded as a impulse equation $x^+ = r(x)$.

□

For this equation, we can define an execution similar to the case of the hybrid dynamical system of Definition 3.1.1.

Definition 3.2.2 (Run of an impulse differential equation)**.** A *run of an impulse differential equation*, $H = (X, f, J, r)$, is a pair, (τ, ξ), consisting of a hybrid time trajectory τ (defined in Definition 3.1.2) and a collection of C^1 functions $\xi = \{\xi^i : i \in \langle\tau\rangle\}$ with $\xi^i : I_i \to X$, that satisfies

- Discrete evolution: $\xi^{i+1}(\tau_{i+1}) = r(\xi^i(\tau_i'))$ for all $i \in \langle\tau\rangle\backslash\{N\}$,
- Continuous evolution: if $\tau_i < \tau_i'$, $\xi^i(\cdot)$ is a solution to the differential equation $\dot{x} = f(x)$ over the interval $[\tau_i, \tau_i']$ starting at $x(\tau_i)$, with $\xi^i(t) \notin J$ for all $t \in [\tau_i, \tau_i')$.

□

On the framework, we introduce the notion of viable run.

Definition 3.2.3 (Viable run)**.** A run (r, ξ) of an impulse differential equation $H = (X, f, J, r)$ is called *viable in a set* $K \subset X$ if $\xi^i(t) \in K$ for all $i \in \langle\tau\rangle$ and for all $t \in I_i$. □

Notice that the definition of a viable run requires the state to remain in the set K throughout the run, along continuous evolution up until and including the state before discrete transitions. Based on the notion of a viable run, one can define a class of sets.

Definition 3.2.4 (Viable set)**.** A set $K \subset X$ is called viable under an impulse differential equation $H = (X, f, J, R)$, if for all $x_0 \in K$ there exists an infinite run viable in K. □

Note that an infinite run viable means that its hybrid time trajectory is infinite. The conditions characterizing viable sets depend on whether the set J is open or closed. In this thesis, we just consider the case where J is closed. In this case, we have the following.

Theorem 3.2.1. Consider an impulse differential equation $H = (X, f, J, r)$ such that f is locally Lipschitz and J is closed. A closed set $K \subset X$ is viable under H if and only if

- $K \cap J \subset r^{-1}(K)$,
- $f(x) \in T_K(x) \quad \forall x \in K \backslash r^{-1}(K)$.

□

Proof. Detailed proof can be founded in [ALQ+02]. □

In words, the conditions of the theorem require that for any state $x \in K$, whenever a discrete transition has to take place ($x \in K \cap J$), a transition back into K is possible ($r(x) \in K$), and, whenever a discrete transition to another point in K is not possible ($r(x) \notin K$), continuous evolution that remains in K has to be possible (encoded by the local viability condition $f(x) \in T_K(x)$ for ordinary differential equation).

Ideally, one would like all runs to be non-Zeno. A condition for a simple case is given below.

Proposition 3.2.2. Consider an impulse differential equation $H = (X, f, J, r)$ such that f is locally Lipschitz. Assume that J is closed set and $J \subset r^{-1}(X)$. In addition, $r^{-1}(X) \cap r(X) = \varnothing$ and $r(X)$ is compact. Then, for any $x_0 \in K$, there is non-Zeno and infinite runes of H. □

Proof. Detailed proof can be founded in [ALQ+02]. □

More general condition is proposed in [ALQ+02] and it is the topic of on-going research.

3.3 State Estimation

There are some observer design techniques for the hybrid dynamical systems. Each result has advantages in different classes of the hybrid systems such as switched systems ([AC01, BBBSV02, Pet05, BPU11, ST14]), mechanical systems with impacts ([TBP16, MT01, MT16]), polyhedral billiards with impacts ([FTZ13]), powertrain systems ([BBBSV01]), hormone systems ([CMS12]), ripple disturbance system ([BZLC17]) and so on.

A common part of most existing observer design methods is that they assume that the knowledge of the time instants when discrete events occur is obtainable. These observers are usually composed of the continuous observer and the location observer (the discrete mode observer). The continuous observer reconstructs the continuous state of the hybrid system. The location observer detects discrete events, and uses this information to switch and/or reinitialize the continuous observer whenever discrete events happen. Therefore, all the listed observer design techniques need an assumption or detection of the time instants when discrete events arise. For switched systems, it seems reasonable to switch the continuous observer whenever discrete mode is changed. However, in the case, where discrete events just involve instantaneous jumps of the continuous state and the rule of jumps depends only on the continuous state so that jumps of the state cannot be detected from only the output information, it would be difficult to apply those observer design methods without any assumption for the time instants of jumps. Moreover, even if those hybrid observers can be constructed, delayed detection of discrete jumps may increase the state estimation error during the delayed time. Hence, this dissertation made an objective to develop a new approach of the state observation, which does not require any detection of jumps. To compare with our approach proposed in Chapter 5, we introduce some results, which are related to examples considered in Chapter 5.

As mentioned before, many previous observer design approaches require knowledge of jump time instant. For the hybrid system having state jumps, these observers make jump in its estimate value ([MT01, TBP16, CMS12, BZLC17]) or change their observer dynamics ([FTZ13]) whenever the state jump happens. These actions maintain or reduce the estimation error even if the discrete events happen. Of course, it is required that the estimation error decreases when the system state equation is governed by the continuous-time evolution.

Let us consider a mechanical system with impacts. If its constraints are independent of time, its impacts happen with losing a fraction of its energy or with maintaining its energy in general. This kind of passivity helps to construct an observer (for the velocity) if there is an observer for the system without the impact constraints. Let us consider the following simple system:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ k(x_1, x_2) \end{bmatrix} & \text{when } x_1 \geq 0 \\ \begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} x_1 \\ -\gamma x_2 \end{bmatrix} & \text{when } (x_1 = 0) \wedge (x_2 \leq 0) \\ y \quad = x_1, & \end{cases} \tag{3.3.1}$$

where x_1 is a position; x_2 is a velocity; y is an output; $\gamma \in [0, 1]$ is a restitution coefficient; k is global Lipschitz. Consider an observer with a state $\hat{x}$ satisfying that there exists a positive definite function $V : \mathbb{R} \to \mathbb{R}$ such that with $e := \hat{x} - x$ and $k > 0$

$$\dot{V}(e) < -kV(e) \quad \text{for all } e \neq 0$$

when the system (without impacts) is given as

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ k(x_1, x_2) \end{bmatrix} \\ y \quad = x_1. \end{cases}$$

In this case, we can adopt this observer for the continuous-time evolution of the hybrid system and update its estimate as $\hat{x}^+ = (\hat{x}_1, -\gamma\hat{x}_2)$ whenever the state jump occurs. Note that this is possible when we have the knowledge of the jump time instant. Then, since $|e^+| \leq \gamma|e| \leq |e|$, the estimation error monotonically decreases when the state flows or jump. In addition, if the state is defined for all continuous time $t \geq 0$, the estimation error goes to zero as time goes to infinity.

This idea can be also applied to the ripple disturbance system in [BZLC17]:

$$\begin{cases} \begin{bmatrix} \dot{r} \\ \dot{\theta} \\ \dot{b} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \text{when } -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6} \\ \begin{bmatrix} r^+ \\ \theta^+ \\ b^+ \end{bmatrix} = \begin{bmatrix} r \\ -\theta \\ b \end{bmatrix} & \text{when } \theta = \frac{\pi}{6} \\ y = r\cos\theta + b. & \end{cases}$$

The details are in [BZLC17]. Note that the estimation algorithm without the knowledge of the jump time instant for the above system is also proposed in [BZLC17], which estimates the jump time instant and uses the estimate of jump time instead of the real jump time information.

When impacts of mechanical systems only depend on the position, the jump time information is available because the output is the position. However, the jump time information cannot be obtained directly from the output in general. To overcome the problem, there some results ([BZLC17, MT16, KCS$^+$14]) are proposed.

The first idea is trivially to estimate the jump time instants using the output ([BZLC17]). Then we can utilize the observer requiring the jump time information.

The second idea is to change the hybrid dynamics system into the continuous dynamical system without any discrete events ([SJLS05, BRS15, MT16, KCS$^+$14]). The observer construction idea is proposed in [KCS$^+$14]. Motivated this, in [MT16], the observer design technique is proposed for a class of the mechanical systems with impacts. Let us consider the simple mechanical system with impacts in (3.3.1) with $\gamma = 1$ and $k(x_1, x_2) = k_1 x_1 + k_2 x_2$. In this case, by taking an auxiliary output $y^* := y^2 = x_1^2$, we can change the system into the following system with $(\zeta_1, \zeta_2, \zeta_3) := (y^*, \dot{y}^* \ddot{y}^*)$

$$\dot{\zeta} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \zeta + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} k^*(\zeta)$$
$$y^* = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \zeta$$

where $k^*(\zeta)$ is a linear commutation of ζ_1, ζ_2, and ζ_3. Since k^* is a global Lipschitz, we can design a high gain observer for the above system. Then, we obtain an estimate $\hat{\zeta}$ for ζ and, from $\hat{\zeta}$, construct an estimate $\hat{x}$ for x. However, this results is restrictive and the procedure to obtain the estimate $\hat{x}$ is ambiguous. In addition, the estimation error $\hat{x} - x$ does not converge to zero asymptotically because of the jump time mismatches between the state and estimate. These problems were already considered in [KCS$^+$14] and Chapters 4–5 of this dissertation deal with them in detail.

3.4 Tracking Control

There are some tracking controller design techniques for the hybrid dynamical systems. Each result has advantages in different classes of the hybrid systems such as switched system ([SG12, dBMS13, SS05]) mechanical systems with impacts ([BNMM00, BNO97]), polyhedral billiards with impacts ([FTZ13]), hormone systems ([CMS09]) and so on ([RS11, SBvdWH14, BvdWHN13, SvdWN14, GMP12]).

In the case of the tracking control, a reference, which a plant state should track, may have jumps. Similar to the state estimation results, many existing tracking control design techniques require that the plant state jump times and the reference jump times are coincide; otherwise, jump time mismatches occur and the tracking errors may be large on the time intervals caused by the jump time mismatches. To compare with the estimation problem, we can easily obtain the reference jump time instants because the reference trajectory is given in general. However, the problem is how to make jump in the plant state. In the estimation problem, we can make jump in the value of the estimate when we want because the estimate depends on the observer construction. However, in the tracking control problem, the plant state may not be able to jump anytime since the state depends on the given plant dynamics.

Let us consider a following hybrid system

$$\begin{cases} \dot{x} &= f_p(x,u) \qquad \text{when } (x,u) \in \mathcal{C}_p \\ x^+ &= g_p(x,u) \qquad \text{when } (x,u) \in \mathcal{D}_p. \end{cases} \tag{3.4.1}$$

The details of this framework are given in [SBvdWH14]. In this case, a input u may make a jump in a state x anytime since its jump constraint depends on u. However, its jump constraint may depend on the state x only as the following system

$$\begin{cases} \dot{x} &= f_p(x,u) \qquad \text{when } x \in \mathcal{C}_p \\ x^+ &= g_p(x) \qquad \text{when } x \in \mathcal{D}_p. \end{cases} \tag{3.4.2}$$

In this case, the mismatch of the jump times of the reference and state is unavoidable. To overcome the this limitation, some results ([FTZ13, SvdWN14, BvdWHN13, KSS16]) permit the tracking error may be large the vicinity of the reference jump time instants, but the error decrease except for the region.

In [FTZ13], authors consider a tracking control problem of a translating mass in a polyhedral billiard. Main idea is to consider the translating mass system with impacts as a switched system through a novel concept of mirrored images of the target mass. For that switched system, a tracking controller is proposed, which is also modeled as a switched system and changes this switching mode whenever the switching modes of the reference system or plant are changed.

In [SvdWN14], a new notion of error is proposed to clarify in which sense the approximate trajectory is, at each instant of time, a first-order approximation of the perturbed trajectory. This notion of error is well-defined even if the trajectory is not continuous. Therefore, this notion of error naturally is applied to the (local) tracking problem of hybrid systems with a time varying reference trajectory.

In [BvdWHN13], A new definition of the tracking error which is not sensitive to jumps of the plant and the reference trajectory is proposed to deal with the jump time mismatches i.e., $d(r,x) = d(g(r),x)$ for $r \in \mathcal{D}$ and $d(r,x) = d(r,g(x))$

for $x \in \mathcal{D}$ where d is proposed error function; g is a jump map; $\mathcal{D}$ is a set where state jumps should happen. For details of the definition, refer to [BvdWHN13]. The proposed tracking error is a non-Euclidean distance between the plant and reference states, where convergence of this distance measure corresponds to the desired notion of tracking. Since this distance measure incorporates information on the "closeness" of the reference state and plant state at each time instant, the tracking problem can be formulated based on the time evolution of the distance measure evaluated along trajectories of the closed-loop system. This idea is similar to the proposed idea using "gluing function" in Chapter 4 and Chapter 6, but, by a geometrical sense, the proposed approach is more intuitive and, by a "glued system", gives a more concrete way to construct a tracking controller.

Chapter 4

Gluing Domain of Hybrid System

This chapter introduces a framework to deal with the main results of this thesis and proposes the conditions guaranteeing that the smooth manifolds with boundary are glued along some part of the boundary. The key to glue the manifold is the quotient map. However, since the quotient map and its image are so abstract, in the final section of this chapter, a more concrete framework is proposed. On the framework, the quotient map is obtained as a map defined between two Euclidean spaces.

4.1 Frameworks

As shown in Chapter 3, there are many frameworks for the hybrid dynamical systems. In this section, we introduce frameworks to define a solution of the system, which are motivated by [SJLS05, BRS15, GST09].

Definition 4.1.1. A *hybrid system* is a 4-tuple $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$ such that

$$\mathcal{H}\begin{cases} \dot{x} &= f(t,x) \qquad \text{when } x \in \mathcal{C} \\ x^+ &= g(x) \qquad \text{when } x \in \mathcal{D} \end{cases}$$

with time $t \in \mathbb{R}_{\geq 0}$ and a state x where

- $\mathcal{C}$ is *flow set*, which is a smooth k-manifold with boundary;
- $f : \mathbb{R}_{\geq 0} \times \mathcal{C} \to T\mathcal{C}$ is *flow map*, which is a time varying vector field on $\mathcal{C}$;

- $\mathcal{D}$ is *jump set*;
- $g : \mathcal{D} \to \mathcal{C} \cup \mathcal{D}$ is *jump map*.

□

Notice that $\mathcal{C}$ may consist of many connected smooth manifolds with boundary. The differential equation, governing the continuous-time evolution when the state x remains in the *flow set* $\mathcal{C}$, and the difference equation, determining the discrete event when x is in the *jump set* $\mathcal{D}$, are given by the *flow map* $f : \mathbb{R}_{\geq 0} \times \mathcal{C} \to T\mathcal{M}$ and the *jump map* $g : \mathcal{D} \to \partial\mathcal{C}$, respectively. Note that the continuous-time dynamics may depend on the time and state. On the other hand, the discrete events depend only on the state, which are called as state-triggered jumps in [BvdWHN13].

Next, we recall some notions for the solutions of the hybrid systems. The following definitions come from [LJS+03, SJLS05, BRS15].

Definition 4.1.2 (Hybrid time trajectory)**.** A *hybrid time trajectory* is finite or infinite sequence of intervals $\tau = \{I_i\}_{i=0}^N$ (N may be ∞) such that

- $I_i = [\tau_i, \tau_i']$ with $\tau_i \leq \tau_i' = \tau_{i+1}$ for all $0 \leq i < N$, in particular, $\tau_0 = 0$,
- when $N < \infty$, either $I_N = [\tau_N, \tau_N']$ or $I_N = [\tau_N, \tau_N')$.

For $\tau = \{I_i\}_{i=0}^N$, let $\langle\tau\rangle := \{0, 1, \ldots, N\}$ (possibly $N = \infty$), and $|\tau| := \sum_{i\in\langle\tau\rangle}(\tau_i' - \tau_i)$. We say that $\tau = \{I_i\}_{i=0}^N$ is a *prefix* of $\tilde{\tau} = \{\tilde{I}_i\}_{i=0}^{\tilde{N}}$ and write $\tau \sqsubseteq \tilde{\tau}$, if either they are identical; or N is finite, $N \leq \tilde{N}$, $I_i = \tilde{I}_i$ for all $0 \leq i < N$, and $I_N \subseteq \tilde{I}_N$.

□

In the framework, each τ_i' for $i \in \langle\tau\rangle$ indicates the time instant of the $(i+1)$-th discrete event.

Definition 4.1.3 (Execution)**.** An *execution* of $\mathcal{H}$ excited by an initial condition $x_0 \in \mathcal{C} \cup \mathcal{D}$ is a pair $\chi = (\tau, \xi)$ where τ is a hybrid time trajectory and $\xi = \{\xi^i : i \in \langle\tau\rangle\}$ is a collection of absolutely continuous maps $\xi^i : I_i \to \mathcal{C}$ such that

- $\xi^0(0) = x_0$,

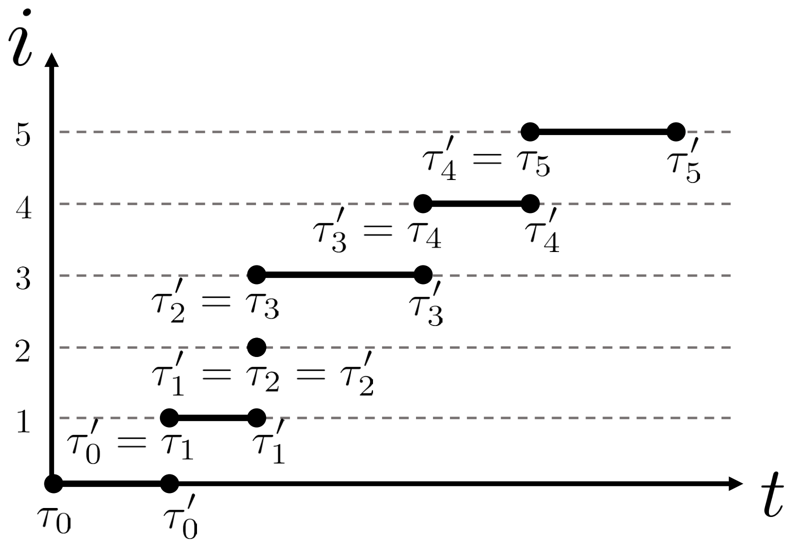

Figure 4.1: An example of hybrid time trajectory $\tau = \{I_i\}_{i=0}^{5}$

- $\mathrm{d}\xi^i(\frac{\mathrm{d}}{\mathrm{d}t}|_s) = f(s, \xi^i(s))$ for all $i \in \langle\tau\rangle$ and for almost all $s \in (\tau_i, \tau_i')$,
- $g(\xi^i(\tau_i')) = \xi^{i+1}(\tau_{i+1})$ for all $i \in \langle\tau\rangle\backslash\{N\}$.

We say that an execution $\chi = (\tau, \xi)$ of $\mathcal{H}$ is a *prefix* of another execution $\tilde{\chi} = (\tilde{\tau}, \tilde{\xi})$ of $\mathcal{H}$ and write $\chi \sqsubseteq \tilde{\chi}$, if $\tau \sqsubseteq \tilde{\tau}$ and $\xi^i(t) = \tilde{\xi}^i(t)$ for all $i \in \langle\tau\rangle$ and $t \in I_i$. We say that χ is a *strict prefix* of $\tilde{\chi}$, if $\chi \sqsubseteq \tilde{\chi}$ and $\chi \neq \tilde{\chi}$. An execution is called *maximal* if it is not a strict prefix of any other executions. An execution is called *infinite* if either $N = \infty$ or $|\tau|$ is not finite. Otherwise, it is called *finite*. An execution is called *Zeno* if it is infinite but $|\tau| < \infty$. Especially, we say an execution is *infinite in t-direction* when it is non-Zeno infinite execution, which means that $|\tau|$ is not finite. For each maximal execution, a *state trajectory* $x : [0, |\tau|) \to \mathcal{C}$ is given by

$$x(t) := \xi^{i(t)}(t) \qquad \text{for each } t \in [0, |\tau|)$$

where $i : [0, |\tau|) \to \langle\tau\rangle$ satisfies $t \in [\tau_{i(t)}, \tau_{i(t)}')$. Note that $i(t)$ is uniquely defined for each $t \in [0, |\tau|)$. □

Example 4.1.1. (Bouncing ball system)**.** Let us consider a bouncing ball system in Figure 4.2. The bouncing ball system is well-known as an example of the hybrid

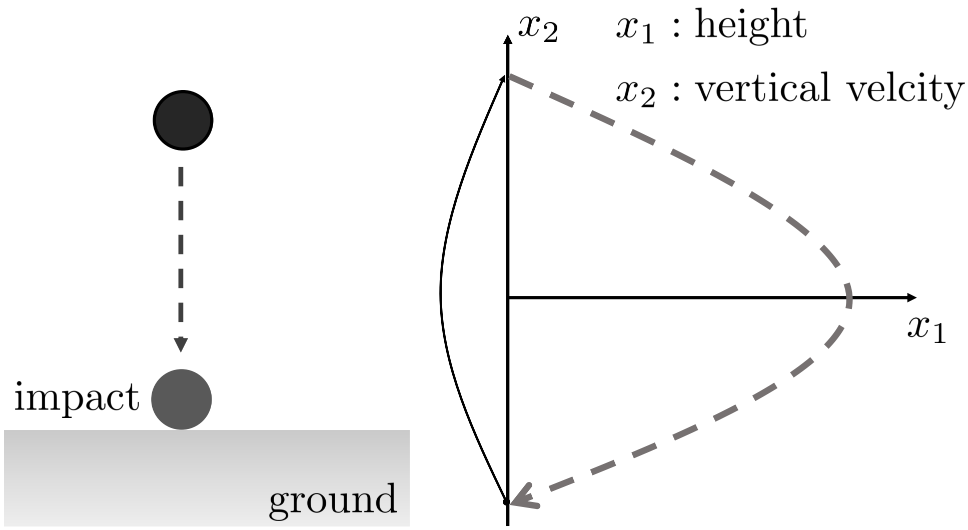

Figure 4.2: Bouncing ball system (restitution coefficient= 1).

dynamical system. The ball is accelerated by the gravity and external force. The gravitational constant is $\rho > 0$. The acceleration by the external force is given as $u(t)$. The height and velocity of the ball are x_1 and x_2, respectively. Suppose that the coefficient of restitution is 1. Then, the bouncing ball is modeled as $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$ such that

$$\begin{aligned} \dot{x} &= f(x,t) := \begin{bmatrix} x_2 \\ -\rho + u(t) \end{bmatrix} && \text{when } x \in \mathcal{C} := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}, \\ x^+ &= g(x) := -x && \text{when } x \in \mathcal{D} := \{(x_1, x_2) \in \mathcal{C} : (x_1 = 0) \wedge (x_2 \leq 0)\}. \end{aligned} \tag{4.1.1}$$

□

4.2 Gluing and Smoothing

For all $x \in \mathcal{D}$, the state jump happens as $x \mapsto x^+ = g(x)$. Let $[x] := \pi(x)$ where $\pi : \mathcal{C} \cup \mathcal{D} \to (\mathcal{C} \cup \mathcal{D})/\sim$ is a natural projection and $x \sim g(x)$ for all $x \in \mathcal{D}$. Note that $\sim$ is an equivalence relation on $\mathcal{C} \cup \mathcal{D}$ and $(\mathcal{C} \cup \mathcal{D})/\sim$ is the set of equivalence classes in $\mathcal{C}$. Then, there is not any discrete event in $[x]$ because $[x^+] = [g(x)] = [x]$. Furthermore, for some class of hybrid system, $[x]$ can be modeled as a solution of continuous-time dynamical system defined on a smooth manifold $(\mathcal{C} \cup \mathcal{D})/\sim$ for a smooth structure.

Assumption 4.2.1. A hybrid system $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$ satisfies the followings:

- $\mathcal{D}$ and $\mathcal{G} := g(\mathcal{D})$ are smooth parts of $\partial\mathcal{C}$;
- g is a diffeomorphism from $\mathcal{D}$ to its image $\mathcal{G}$;
- $\mathcal{D} \cap \mathcal{G} = \varnothing$.

□

Note that the jump set $\mathcal{D}$ and its image $\mathcal{G}$ are included in the boundary of $\mathcal{C}$, $\partial\mathcal{C}$. In addition, since $\mathcal{D} \subset \partial\mathcal{C} \subset \mathcal{C}$, the set of the equivalence classes $(\mathcal{C} \cup \mathcal{D})/\sim$ is obtained as $\mathcal{C}/\sim$.

Example 4.2.1. (Bouncing ball system without origin)**.** Consider the bouncing ball system (4.1.1) without the origin. The system is modeled as $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$ such that

$$\begin{aligned} \dot{x} &= f(x,t) := \begin{bmatrix} x_2 \\ -\rho + u(t) \end{bmatrix} && \text{when } x \in \mathcal{C} := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\} \backslash \{0_2\}, \\ x^+ &= g(x) := -x && \text{when } x \in \mathcal{D} := \{(x_1, x_2) \in \mathcal{C} : (x_1 = 0) \wedge (x_2 < 0)\}. \end{aligned} \tag{4.2.1}$$

Then, $\mathcal{C}/\sim$ is a smooth manifold. □

In the case of the bouncing ball system of Example 4.1.1 satisfying the first and second assumptions, by the result of [SJLS05], its image $\pi(\mathcal{C}) = \mathcal{C}/\sim$ can be topological manifold. However, it cannot be endowed with the structure of $\mathcal{C}$ because of the origin. In general, at $[x] \in \mathcal{C}/\sim$, we construct its structure by gluing the structures of $\pi^{-1}([x])$. However, at the origin of the bouncing ball domain, we can obtain just one structure from $\{0_2\} = \pi^{-1}([0_2])$, which gives the half structure at $[0_2]$ although it is an interior point of the topological manifold $\mathcal{C}/\sim$. To avoid this case, we impose the third assumption.

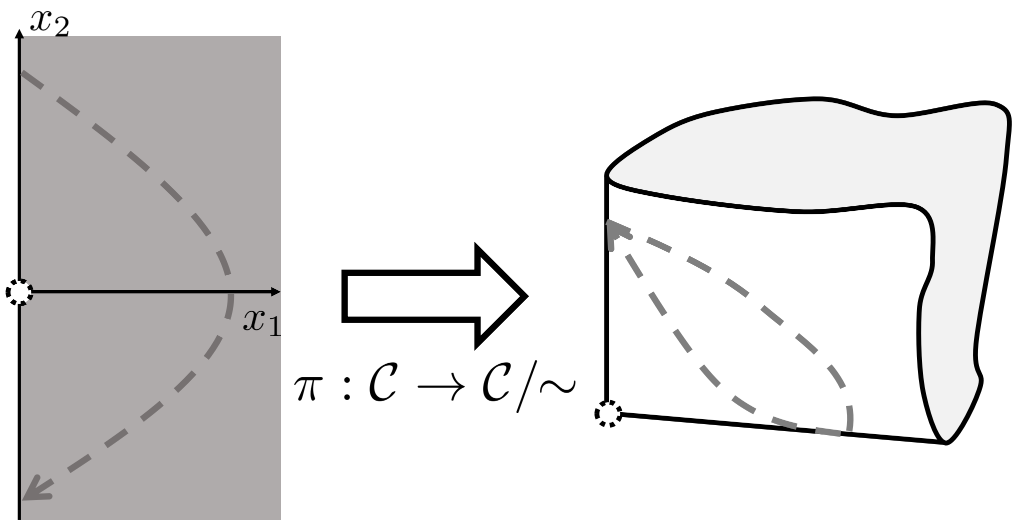

Figure 4.3: Gluing the domain of the bouncing ball system without the origin.

Under Assumption 4.2.1, since the conditions of Theorem 2.2.9 are satisfied, there exists a smooth structure $\mathcal{A}$ on $\mathcal{C}/\sim$ so that $\mathcal{C}/\sim$ with $\mathcal{A}$ is a smooth k-manifold. However, its vector field may not be smooth in x. Note that there are many smooth structures guaranteeing this and they are diffeomorphic by Theorem 2.2.10. Using this property, among them, we may find a smooth structure $\mathcal{A}^*$ on $\mathcal{C}/\sim$ guaranteeing that the glued vector field is smooth. The following assumption on the flow map f guarantees the existence of the smooth structure $\mathcal{A}^*$.

Assumption 4.2.2. A hybrid system $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$ satisfies the followings:

- f is independent of t and f is smooth;
- f is outward-pointing on $\mathcal{D}$ and inward-pointing on $\mathcal{G}$.

□

Under the above assumption, we can apply Theorem 2.2.10 to the considered case.

Theorem 4.2.1. (Gluing and Smoothing)**.** Suppose that hybrid system $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$ defined in Theorem 4.1.1 satisfies Assumptions 4.2.1–4.2.2. Then there

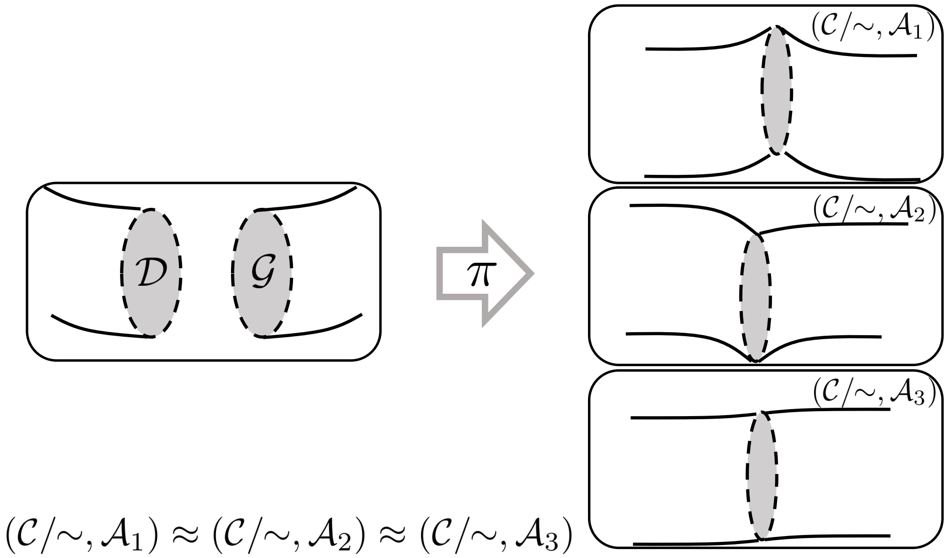

Figure 4.4: The glued manifold is determined up to diffeomorphism.

exist a smooth structure $\mathcal{A}^*$ on $\mathcal{C}/\sim$ and a vector field $f^\pi : \mathcal{C}/\sim \to T(\mathcal{C}/\sim)$ on $\mathcal{C}/\sim$ with $\mathcal{A}^*$ such that $\mathcal{C}/\sim$ with $\mathcal{A}^*$ is a smooth k-smooth manifold (with boundary) and

$$\mathrm{d}\pi_x(f(x)) = f^\pi(\pi(x)) \text{ for all } x \in \mathcal{C}. \tag{4.2.2}$$

□

The idea of the theorem is to glue the charts of two points in $\pi^{-1}([x])$, which generated by f.

Proof of Theorem 4.2.1. By Theorem 2.2.7, for each $i \in \{\mathcal{D}, \mathcal{G}\}$, we can find a smooth function $\delta_i : i \to \mathbb{R}^+$ and a smooth embedding $\Phi_i : \mathcal{P}_{\delta^i} \to \mathcal{C}$ where $\mathcal{P}_{\delta^i} := \{(t, x) : (x \in i) \wedge (0 \leq t \leq \delta^i(x))\} \subset \mathbb{R} \times i$ such that $\Phi_i(\mathcal{P}_{\delta^i})$ is a neighborhood of i. In addition, for each $x \in \mathcal{D}$ the map $t \mapsto \Phi_{\mathcal{D}}(t, x)$ is an integral curve of $-f$ starting at x and for each $x \in \mathcal{G}$ the map $t \mapsto \Phi_{\mathcal{G}}(t, x)$ is an integral curve of f starting at x.

At first, show that $\mathcal{C}/\sim$ is a topological k-manifold. To show that it locally resembles a real Euclidean space, assume $y \in \mathcal{C}/\sim$. Then, there exists $x \in \mathcal{C}$ such that $y = \pi(x)$.

If x is not identified with any other points, $x \in \mathcal{C}\backslash(\mathcal{D} \cup \mathcal{G})$ because $\mathcal{D}$ and $\mathcal{G}$ are closed sets in $\mathcal{C}$. In this case, we can take a coordinate chart (U, φ) such that $U \cap (\mathcal{D} \cup \mathcal{G}) = \varnothing$, so that $\pi|_U$ is a homeomorphism. Then, $\varphi \circ \pi|U^{-1}$ is a homeomorphism from an open neighborhood $\pi(U)$ of y to an open neighborhood $\varphi(U)$ of $\mathbb{R}^k$.

If $x \in \mathcal{D} \cup \mathcal{G}$, without loss of generality, we can take $x_1 \in \mathcal{D}$ and $x_2 \in \mathcal{G}$ such that $x_2 = g(x_1)$ and

$$\pi(x_1) = \pi(x_2) = y.$$

Since $\mathcal{D}$ and $\mathcal{G}$ are closed sets in $\mathcal{C}$ and $\mathcal{C}$ is Hausdorff space, we can find coordinate charts (U_1, φ_1) and (U_2, φ_2) at x_1 and x_2, respectively, satisfying that

- $U_1 \cap \mathcal{G} = \varnothing$ and $U_2 \cap \mathcal{D} = \varnothing$;
- $U_1 \cap U_2 = \varnothing$;
- $U_1 \subset \Phi_{\mathcal{D}}(\mathcal{P}_{\delta^{\mathcal{D}}})$ and $U_2 \subset \Phi_{\mathcal{G}}(\mathcal{P}_{\delta^{\mathcal{G}}})$.

Then, we define $\phi : U_1 \cup U_2 \to \mathbb{R} \times (\mathcal{D} \cup \mathcal{G})$ by

$$\phi(x) := \begin{cases} (-t, g(x_{\mathcal{D}})) & \text{when } x = \Phi_{\mathcal{D}}(t, x_{\mathcal{D}}) \in U_1, \\ (t, x_{\mathcal{G}}) & \text{when } x = \Phi_{\mathcal{G}}(t, x_{\mathcal{G}}) \in U_2. \end{cases}$$

Then the restriction of ϕ to U_1 or U_2 is a topological embedding with closed image, from which it follows easily that ϕ is a closed map. In addition, since $\pi(U_1 \cup U_2)$ is an open and $\mathcal{D}$ and $\mathcal{G}$ are smooth $(k-1)$-manifolds, $\pi(U_1 \cup U_2)$ is a homeomorphic to an open set in $\mathbb{R}^k$. Therefore, $\mathcal{C}/\sim$ is a locally Euclidean space.

Since $\mathcal{C}/\sim$ is the union of the second-countable open subsets $\pi(\mathcal{C}\backslash(\mathcal{D} \cup \mathcal{G}))$ and $\pi(\Phi_{\mathcal{D}}(\mathcal{P}_{\delta^{\mathcal{D}}}) \cup \Phi_{\mathcal{G}}(\mathcal{P}_{\delta^{\mathcal{G}}}))$, it is second countable. In addition, any two fibers in $\mathcal{C}$ can be separated by saturated open subsets, so $\mathcal{C}/\sim$ is Hausdorff. Therefore, it is a topological k-manifold.

Now construct charts for a smooth structure guaranteeing (4.2.2) as follows:

- $(\pi(U), \varphi \circ \pi^{-1}|_{\pi(U)})$ for $y \in \pi(\mathcal{C}\backslash(\mathcal{D} \cup \mathcal{G})) \subset \mathcal{C}/\sim$ where (U, ϕ) is a chart of $\mathcal{C}\backslash(\mathcal{D} \cup \mathcal{G})$ at $\pi^{-1}(y)$;

- $(\pi(U_1 \cup U_2), \varphi' \circ \phi \circ \pi^{-1}|_{\pi(U_1 \cup U_2)})$ for $y \in \pi(\mathcal{D} \cup \mathcal{G}) \subset \mathcal{C}/\sim$ where $(\phi(U_1 \cup U_2), \varphi')$ is a chart of $\mathbb{R} \times \mathcal{G}$ at $(0, x_2)$ (note that U_1, U_2, ϕ, x_2 are obtained as above).

It is straightforward to check that they are all smoothly compatible and thus define a smooth structure on $\mathcal{C}/\sim$. Let $f^\pi(y) := \mathrm{d}\pi_x(f(x))$ where $y = \pi(x)$ and $x \notin \mathcal{D}$. Then, it is trivially smooth on $y \in \pi(\mathcal{C} \backslash (\mathcal{D} \cup \mathcal{G}))$. In addition, for $y \in \pi(\mathcal{D} \cup \mathcal{G})$ with a coordinate $(y_1, \ldots, y_k)$ it holds that

$$\mathrm{d}\pi_{x_\mathcal{D}} f(x_\mathcal{D}) = \left.\frac{\partial}{\partial y_1}\right|_y = \mathrm{d}\pi_{x_\mathcal{G}} f(x_\mathcal{G})$$

where $(x_\mathcal{D}, x_\mathcal{G}) \in \mathcal{D} \times \mathcal{G}$ and $\pi(x_\mathcal{D}) = \pi(x_\mathcal{G}) = y$, which implies f^π is a smooth vector field on $\mathcal{C}/\sim$. □

By this theorem, for a hybrid dynamical system $\mathcal{H}$ satisfying Assumptions 4.2.1–4.2.2, we can obtain a continuous-time dynamical system on $\mathcal{C}/\sim$ with a smooth vector field f^π i.e.,

$$\dot{y} = f^\pi(y) \text{ on } \mathcal{C}/\sim \tag{4.2.3}$$

such that, for a state trajectory $x(t)$ of $\mathcal{H}$ starting x_0,

$$[x(t)] = y(t) \text{ for all } t \in [0, |\tau|)$$

where $y(t)$ is a solution to (4.2.3) starting $[x_0]$ (which is a integral curve of f^π on $\mathcal{C}/\sim$).

4.3 Frameworks in $\mathbb{R}^n$ and Gluing Function

To deal with state estimation and tracking control problems in next two chapters, we need to a metric to measure the estimation and tracking errors. As a simple expedient, we consider the system whose the domain is embedded in $\mathbb{R}^n$, so that we adopt the norm of Euclidean space as a metric. The following assumption includes this and the essential parts of Assumption 4.2.1–4.2.2.

Assumption 4.3.1. The hybrid system $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$ satisfies that

- the flow set $\mathcal{C}$ is a smooth k-submanifold of $\mathbb{R}^n$ with boundary;
- the jump set $\mathcal{D}$ and its image $\mathcal{G} := g(\mathcal{D})$ are smooth parts of $\partial\mathcal{C}$;
- g is a diffeomorphism from $\mathcal{D}$ to its image $\mathcal{G}$;
- $\mathcal{D} \cap \mathcal{G} = \varnothing$;
- the flow map f is independent of t;
- f is outward-pointing on $\mathcal{D}$ and inward-pointing on $\mathcal{G}$.

□

By the definition of the submanifold, we say that $\mathcal{C} \subset \mathbb{R}^n$ is a smooth k-submanifold of $\mathbb{R}^n$ with boundary $(0 < k \leq n)$ if, for each $x \in \mathcal{C}$, there are an open neighborhood U of x in $\mathbb{R}^n$, an open subset V of $\mathbb{R}^n$, and a diffeomorphism $\alpha : U \to V$ such that

$$\alpha(U \cap \mathcal{C}) = \{(v_1, \ldots, v_n) \in V : v_1 \geq 0, v_{k+1} = \cdots = v_n = 0\}.$$

Trivially, we say that $\mathcal{D} \subset \mathbb{R}^n$ is a smooth 0-submanifold of $\mathbb{R}^n$ with boundary if, for each $x \in \mathcal{D}$, there exists an open neighborhood U of x in $\mathbb{R}^n$ such that $U \cap \mathcal{D} = \{x\}$.

Note that since $\mathcal{C}$ is a smooth k-submanifold of $\mathbb{R}^n$ with boundary, the vector field f can be specified as a vector of $\mathbb{R}^n$.

When f is smooth, by Theorem 4.2.1, we obtain a system (4.2.3). However, the system is abstract. Therefore, to specify the system, we also embed the domain $\mathcal{C}/\sim$ of the system into Euclidean space via Theorem 2.2.6.

Theorem 4.3.1. Suppose that the hybrid system $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$ defined in Theorem 4.1.1 satisfies Assumption 4.3.1 and that f is smooth. Then there exist a smooth function $\Psi : \mathcal{C} \to \mathbb{R}^{2k+1}$ such that $\Psi(\mathcal{C})$ is a smooth k-submanifold of $\mathbb{R}^{2k+1}$ and diffeomorphic to $\mathcal{C}/\sim$. In addition, there exists a smooth vector field

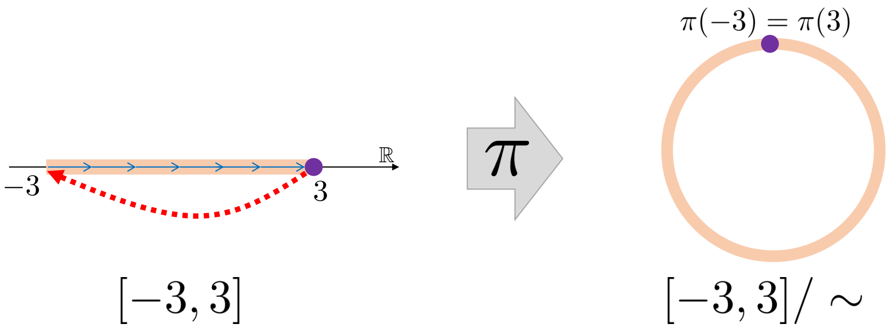

Figure 4.5: Gluing the domain in Example 4.3.1.

$f^\Psi : \Psi(\mathcal{C}) \to \mathbb{R}^{2k+1}$ on $\Psi(\mathcal{C})$ such that

$$\mathrm{d}\Psi_x(f(x)) = f^\Psi(\Psi(x)) \text{ for all } x \in \mathcal{C}. \tag{4.3.1}$$

□

Proof of Theorem 4.3.1. The proof is trivial by Theorem 4.2.1 and 2.2.6. By the theorem, we obtain a proper smooth embedding $s : \mathcal{C}/\sim \to \mathbb{R}^{2k+1}$. Then, by using $\Psi := s \circ \pi$, we can obtain (4.2.3) as a system in $\mathbb{R}^{2k+1}$. Notice that f^Ψ is a pushforward of f^π by a smooth embedding s. □

Example 4.3.1. Let us consider the following simple hybrid system $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$

$$\mathcal{H} \begin{cases} \dot{x} = 1 =: f(x) & \text{when } x \in \{x \in \mathbb{R} : |x| \le 3\} =: \mathcal{C} \\ x^+ = -x =: g(x) & \text{when } x \in \{3\} =: \mathcal{D}. \end{cases}$$

It is easy to check that the hybrid system $\mathcal{H}$ satisfies Assumption 4.3.1.

Via an equivalence relation $\sim$ generated by $3 \sim -3$, we can define the canonical projection $\pi : [-3, 3] \to [-3, 3]/\sim$ such that $\pi(x) = [x]$ for all $x \in [-3, 3]$ where $[x]$ is the equivalence class of x and $[-3, 3]/\sim$ is the set of all equivalent classes (i.e., $[-3, 3]/\sim := \{[x] : x \in [-3, 3]\}$). Since π is surjective map, we can construct the quotient topology on $[-3, 3]/\sim$ determined by π. Note that, by the definition of the quotient topology, a subset $U \subset [-3, 3]/\sim$ is open if and only if $\pi^{-1}(U)$ is open in $[-3, 3]$. Consider two open sets U_1 and U_2 in $[-3, 3]/\sim$ defined

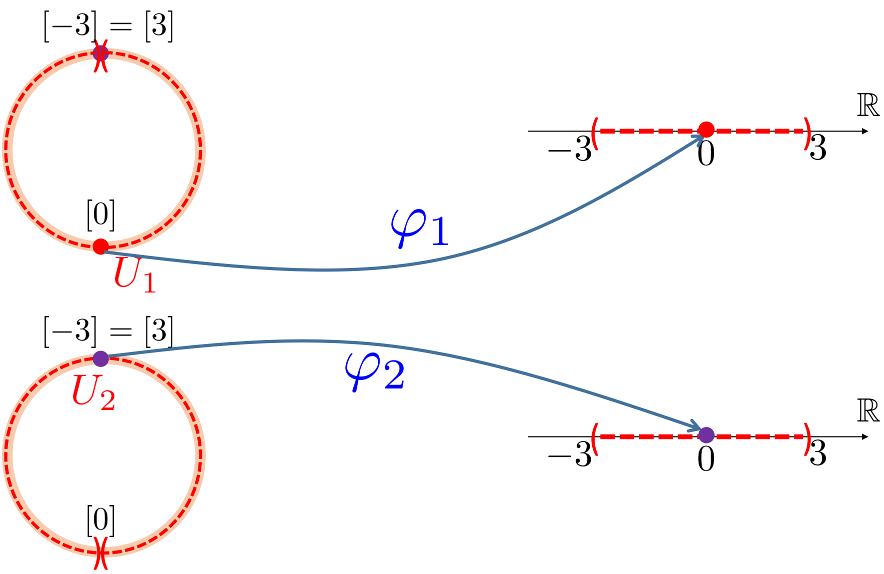

Figure 4.6: Coordinate charts for the glued domain in Example 4.3.1.

as

$$U_1 := \{[x] \in [-3,3]/\sim: 0 \le |x| < 3\},$$
$$U_2 := \{[x] \in [-3,3]/\sim: 0 < |x| \le 3\}.$$

Then, for each $i \in \{1,2\}$, we can construct a map $\varphi_i : U_i \to \mathbb{R}$ on U_i as

$$\varphi_1([x]) = x,$$
$$\varphi_2([x]) = x - 3\frac{x}{|x|}.$$

such that $\mathcal{A} = \{(U_1,\varphi_1),(U_2,\varphi_2)\}$ is a C^∞ atlas for $[-3,3]/\sim$. Therefore, $[-3,3]/\sim$ with $\mathcal{A}$ is a smooth 1-dimensional manifold. In addition, we can take a vector field f^π on $[-3,3]/\sim$ as

$$f^\pi([x]) = \frac{\partial}{\partial t_i}\bigg|_{[x]} \quad \text{for all } [x] \in U_i \subset [-3,3]/\sim \text{ and for all } i \in \{1,2\}$$

where $\frac{\partial}{\partial t_i}\Big|_{[x]}(\phi) := \frac{\partial(\phi\circ\varphi_i^{-1})}{\partial t_i}\Big|_{\varphi_i([x])}$ for all $\phi := C^\infty([x])$.

Note that $f^\pi : [-3,3]/\sim\to T([-3,3]/\sim)$ is well-defined and it is a smooth vector field satisfying that

$$\mathrm{d}\pi_x(f(x)) = f^\pi([3]) \text{ for all } x \in [-3,3].$$

Now we construct a smooth embedding $s : [-3,3]/\sim\to \mathbb{R}^4$. Let consider functions $\rho_1 : [-3,3]/\sim\to [0,1]$ and $\rho_2 : [-3,3]/\sim\to [0,1]$ such that

- ρ_1 and ρ_2 are smooth;
- $\rho_1([x]) = 0 \Leftrightarrow [x] \notin U_1$;
- $\rho_2([x]) = 0 \Leftrightarrow [x] \notin U_2$;
- $\rho_1([x]) = 1 \Leftrightarrow [x] \in \{[x] \in [-3,3]/\sim: 0 \leq |x| \leq 2\}$;
- $\rho_2([x]) = 1 \Leftrightarrow [x] \in \{[x] \in [-3,3]/\sim: 1 \leq |x| \leq 3\}$.

Note that these functions alway exist. For example, we can construct as

$$\rho_1([x]) := \begin{cases} 1 & \text{if } 0 \leq |x| \leq 2, \\ \frac{e^{-1/(|x|-3)^2}}{e^{-1/(|x|-3)^2}+e^{-1/(|x|-2)^2}} & \text{if } 2 < |x| < 3, \\ 0 & \text{if } |x| = 3, \end{cases}$$

$$\rho_2([x]) := \begin{cases} 0 & \text{if } |x| = 0, \\ \frac{e^{-1/(|x|)^2}}{e^{-1/(|x|)^2}+e^{-1/(|x|-1)^2}} & \text{if } 0 < |x| < 1, \\ 1 & \text{if } 1 \leq |x| \leq 3, \end{cases}$$

which are general bump functions (see Figure 4.7). Define a function $s : [-3,3]/\sim\to \mathbb{R}^4$ as

$$s([x]) := (\rho_1([x])\varphi_1([x]), \rho_2([x])\varphi_2([x]), \rho_1([x]), \rho_2([x])).$$

Since this map is an injective immersion and $[-3,3]/\sim$ is compact, $s : [-3,3]/\sim\to \mathbb{R}^4$ is a smooth embedding. In addition, by Lemma 6.13 in [Lee12], we find a projection $\pi_v : (x_1,x_2,x_3,x_4) \mapsto (x_1,x_2,x_3)$ is an injective immersion of $s([-3,3]/\sim)$ into $\mathbb{R}^3$. Therefore, we can take $\Psi := \pi_v \circ s \circ \pi$ proposed in Theorem 4.3.1 and

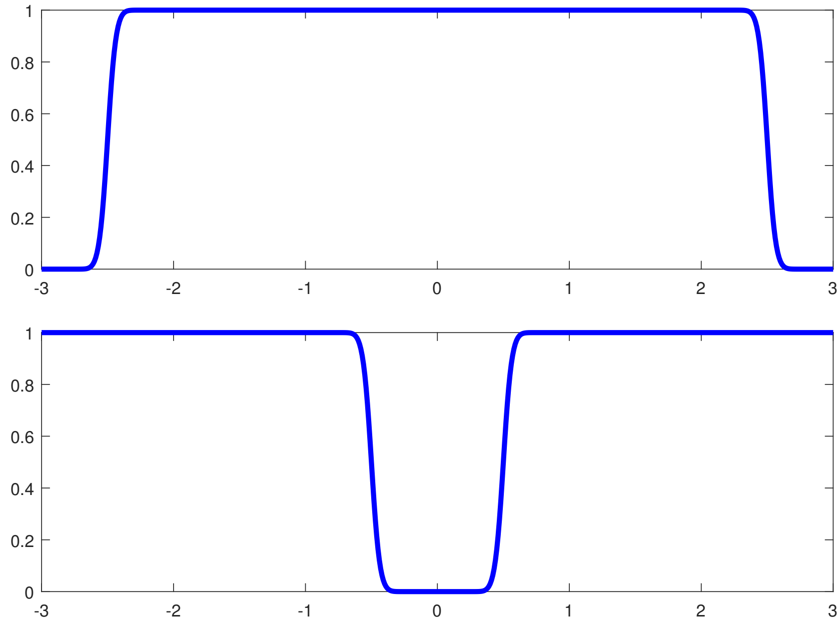

Figure 4.7: Examples of ρ_1 (above) and ρ_2 (below).

its explicit form is obtained as

$$\Psi(x) = \begin{cases} \begin{bmatrix} x \\ \frac{e^{-1/(|x|)^2}}{e^{-1/(|x|)^2}+e^{-1/(|x|-1)^2}}(x - 3\frac{x}{|x|}) \\ 1 \end{bmatrix} & \text{if } 0 < |x| < 1, \\ \begin{bmatrix} x \\ x - 3\frac{x}{|x|} \\ 1 \end{bmatrix} & \text{if } 1 < |x| < 2, \\ \begin{bmatrix} \frac{e^{-1/(|x|-3)^2}}{e^{-1/(|x|-3)^2}+e^{-1/(|x|-2)^2}}x \\ x - 3\frac{x}{|x|} \\ \frac{e^{-1/(|x|-3)^2}}{e^{-1/(|x|-3)^2}+e^{-1/(|x|-2)^2}} \end{bmatrix} & \text{if } 2 < |x| < 3, \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \text{if } |x| = 0, \\ \begin{bmatrix} x \\ x - 3\frac{x}{|x|} \\ 1 \end{bmatrix} & \text{if } |x| = 1 \text{ or } |x| = 2, \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \text{if } |x| = 3. \end{cases}$$

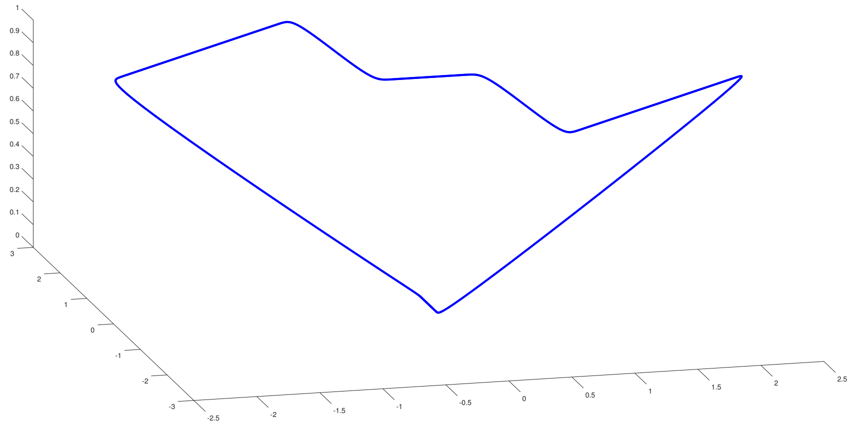

Figure 4.8: The glued domain $\Psi([-3,3])$ in $\mathbb{R}^3$

□

By Theorem 4.3.1, we obtain a continuous-time dynamical system on $\Psi(\mathcal{C}) \subset \mathbb{R}^{2k+1}$

$$\dot{\zeta} = f^{\Psi}(\zeta) \tag{4.3.2}$$

where

- $\Psi(\mathcal{C})$ is a smooth k-submanifold with boundary of $\mathbb{R}^{2k+1}$;
- $f^{\Psi} : \Psi(\mathcal{C}) \to \mathbb{R}^{2k+1}$ is a smooth vector field on $\Psi(\mathcal{C})$;
- $\mathrm{d}\Psi_x(f(x)) = f^{\Psi}(\Psi(x))$ for all $x \in \mathcal{C}$;
- $\Psi(x(t)) = \zeta(t)$ for all $t \in [0, |\tau|)$ where $x(t)$ is a state trajectory of $\mathcal{H}$ and $\zeta(t)$ is a solution of (4.3.2) with $\Psi(x(0)) = \zeta(0)$.

Therefore, if we have the system (4.3.2) and its solution, we can obtain the execution of $\mathcal{H}$ through preimage of Ψ. Thus, the property of the hybrid system $\mathcal{H}$ may be investigated through system (4.3.2) such as viability and stability. Furthermore, this function can be utilized as a transformation to construct state observer and tracking controller.

Although the map Ψ exists under the proposed assumptions, obtaining it is not easy because the procedure is abstract and there is no systematic method to construct it. Therefore, we propose a relaxed condition to glue the hybrid dynamics.

Definition 4.3.1. A function $\psi : \mathcal{C} \to \mathbb{R}^m$ $(m \geq k)$ is called a *gluing function* of the system $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$ if it satisfies the following conditions:

(G1) $\psi(x) = \psi(g(x))$ for all $x \in \mathcal{D}$;

(G2) $\psi|_{\mathcal{C}\setminus\mathcal{D}}$ is injective;

(G3) ψ is of class C^1;

(G4) ψ is a immersion;

(G5) for all $U \subset \psi(\mathcal{C})$, if $\psi^{-1}(U)$ is open, U is open in $\psi(\mathcal{C})$.

We call $\psi(\mathcal{C})$ a *glued domain* and denote it by $\mathcal{C}^\psi$. □

Notice that the gluing function is one of the quotient maps and local C^1 whose image is a subset of $\mathbb{R}^m$.

The gluing function has the essential properties (G1) and (G2) of the natural projection. In addition, by (G3) and (G4), the gluing function is a local C^1 embedding in Euclidean space. Note that, except on the glued region, its C^1 structures is preserved. The condition (G5) preserves the topology of $\mathcal{C}/\sim$ so that a pathological case like Figure 4.9 is excluded.

Suppose that when $k < n$ there exists there exists a smooth map $\varphi : \mathbb{R}^n \to \mathbb{R}^{n-k}$ such that $\varphi(x) = 0_{n-k}$ for all $x \in \mathcal{C}$ where 0_{n-k} is a regular value[1] of φ. Then, a tangent space at x can be characterized as $\ker(\mathrm{d}r_{\mathcal{C}}(x))$. Therefore, since f is a vector field on $\mathcal{C}$, if $k < n$, $\mathrm{d}r_{\mathcal{C}}(x)f(t,x) = 0$ for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathcal{C}$. In addition the condition (G4) can be replaced as the following condition:

- $\mathrm{rank}(\mathrm{d}\psi(x)) = n$ for any $x \in \mathcal{C}$ if $k = n$;

[1] Since 0_{n-k} is a regular value of φ, all points x in pre-image $\varphi^{-1}(0_{n-k})$ are regular points. This means that, for $x \in \varphi^{-1}(0_{n-k})$, $\mathrm{d}\varphi(x)$ always has full row rank.

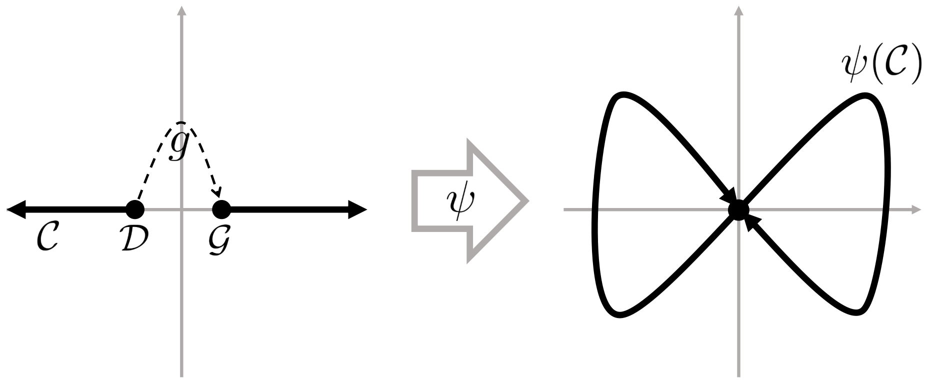

Figure 4.9: An example of pathological cases where $\mathcal{C} = \{(x_1, x_2) \in \mathbb{R}^2 : (|x_1| \geq 1) \wedge (x_2 = 0)\}$, $\mathcal{D} = \{(-1, 0)\}$, $g(x) = -x$, and $\mathcal{G} = \{(1, 0)\}$.

- $\mathrm{d}\psi(x)v \neq 0_m$ for each $x \in \mathcal{C}$ and for each non-zero vector $v \in \ker(\mathrm{d}r_{\mathcal{C}}(x))$ if $k < n$.

The above condition is independent of the local charts to check whether ψ is immersion or not.

If there exists a gluing function ψ of $\mathcal{H}$, then the state trajectory $x(t)$ can be expressed on the glued domain $\mathcal{C}^{\psi}$ as

$$\zeta(t) := \psi(x(t)) \in \mathcal{C}^{\psi}. \tag{4.3.3}$$

Since all the discontinuities of $x(t)$ are "glued" by (G1), $\zeta(t)$ is continuous with respect to t. In this sense, we call $\zeta(t)$ the *glued trajectory* of $x(t)$ by ψ.

We now define an *inverse gluing function* induced from ψ. Actually, ψ^{-1} on $\psi(\mathcal{C})$ cannot be defined due to (G1). However, $\psi|_{\mathcal{C}\backslash\mathcal{D}}$ is bijective onto its image $\psi(\mathcal{C}\backslash\mathcal{D})$ by (G2), and we have that $\mathcal{C}^{\psi} = \psi(\mathcal{C}) = \psi(\mathcal{C}\backslash\mathcal{D}) \cup \psi(\mathcal{D}) = \psi(\mathcal{C}\backslash\mathcal{D})$ because $\psi(\mathcal{D}) = \psi(\mathcal{G})$ by (G1) and $\psi(\mathcal{G}) \subset \psi(\mathcal{C}\backslash\mathcal{D})$ by $\mathcal{D} \cap \mathcal{G} = \varnothing$. Hence, we obtain the inverse gluing function from $\mathcal{C}^{\psi}$ to $\mathcal{C}$ by (we abuse notation by writing it as ψ^{-1})

$$\psi^{-1}(\zeta) := \psi|_{\mathcal{C}\setminus\mathcal{D}}^{-1}(\zeta) \quad \text{for all } \zeta \in \mathcal{C}^{\psi}. \tag{4.3.4}$$

Example 4.3.2. (Gluing for bouncing ball system)**.** Consider the bouncing ball system in Example 4.2.1. Suppose that there is no external force (i.e.,$u(t) \equiv 0$) and the mechanical energy of the ball $E(x) := m\rho x_1 + \frac{1}{2}mx_2^2$ remains in $(\underline{\delta}, \overline{\delta})$ where m is the mass of the ball and $\overline{\delta} > \underline{\delta} > 0$. Note that we restrict the domain in this example because we see the change of the domain in the following figures. Then, the bouncing ball is modeled as

$$\begin{aligned} \dot{x} &= f(x) := \begin{bmatrix} x_2 \\ -\rho \end{bmatrix} && \text{when } x \in \mathcal{C} := \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 \geq 0) \wedge (\underline{\delta} < E(x) < \delta)\}, \\ x^+ &= g(x) := -x && \text{when } x \in \mathcal{D} := \{(x_1, x_2) \in \mathcal{C} : (x_1 = 0) \wedge (x_2 < 0)\}, \end{aligned} \tag{4.3.5}$$

which satisfies Assumption 4.3.1.

There may be many gluing functions of (4.2.1), but there is no systematic way to design them yet. However, the geometrical properties of the domain may help to find them. We introduce two gluing functions.

a) Doubling in $\mathbb{R}^2$

The first idea is to make the corresponding angle double in the polar coordinates proposed in [Sha09, KCS+14]. Let $\psi_1 : \mathcal{C} \to \mathbb{R}^2$, $(x_1, x_2) = (\rho\cos\theta, \rho\sin\theta) \mapsto (\zeta_1, \zeta_2) = (\rho\cos 2\theta, \rho\sin 2\theta)$. By straightforward calculations, we have

$$\begin{aligned} \psi_1(x) &= \begin{bmatrix} \frac{x_1^2 - x_2^2}{|x|} \\ \frac{2x_1x_2}{|x|} \end{bmatrix}, \\ \mathrm{d}\psi_1(x) &= \begin{bmatrix} \frac{x_1^3 + 3x_1x_2^2}{|x|^3} & \frac{-3x_1^2x_2 - x_2^3}{|x|^3} \\ \frac{2x_2^3}{|x|^3} & \frac{2x_1^3}{|x|^3} \end{bmatrix}. \end{aligned} \tag{4.3.6}$$

Since $\psi_1(x) = (x_2, 0) = \psi_1(g(x))$ for all $x \in \mathcal{D}$, (G1) holds. The condition (G2)

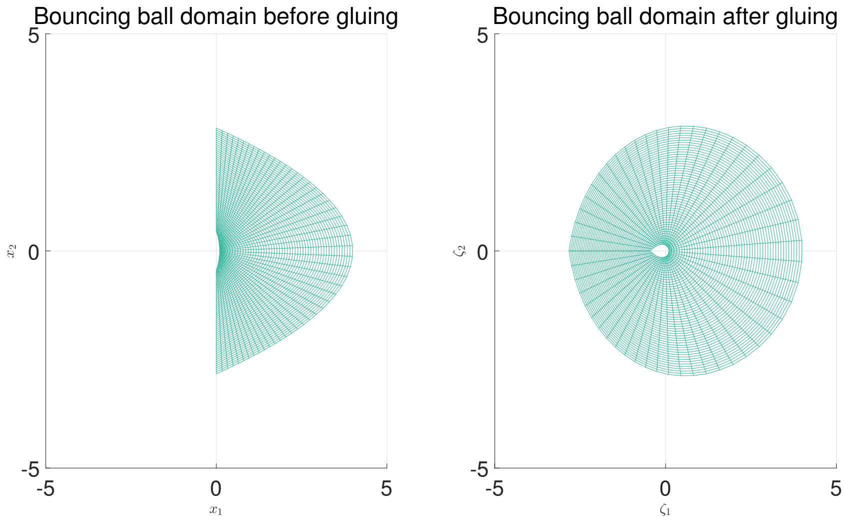

Figure 4.10: Domains of the bouncing ball system before gluing $\mathcal{C}$ (left) and after gluing $\mathcal{C}^{\psi_1}$ (right) when $\underline{\delta}= 0.1$, $\overline{\delta} = 4$, and $\rho = 1$.

and (G3) are trivially satisfied. In addition, we have that, for all $x \in \mathcal{C}$,

$$\begin{aligned}\det(\mathrm{d}\psi_1(x)) &= \frac{2(x_1^6 + x_2^6) + 6x_1^2x_2^2(x_1^2 + x_2^2)}{|x|^6} \\ &= \frac{2(x_1^4 + x_2^4 - x_1^2x_2^2) + 6x_1^2x_2^2}{|x|^4} = \frac{2(x_1^2 + x_2^2)^2}{|x|^4} = 2,\end{aligned}$$

which guarantees (G4). Therefore, ψ_1 is a gluing function. Thus, in a similar fashion to (4.3.4), we can also define $\psi_1^{-1} : \mathcal{C}^{\psi_1} \to \mathcal{C}$ as

$$\psi_1^{-1}(\zeta) := \begin{bmatrix} \sqrt{\frac{1}{2}|\zeta|(|\zeta| + \zeta_1)} \\ \operatorname{sgn}(\zeta_2)\sqrt{\frac{1}{2}|\zeta|(|\zeta| - \zeta_1)} \end{bmatrix},$$

which is the function halving the corresponding angle where $\operatorname{sgn}(\zeta_2)$ is 1 if $\zeta_2 \geq 0$ and -1 if $\zeta_2 < 0$.

b) Dragging in $\mathbb{R}^3$

The gluing function ψ_1 is intuitive, but its expression is complex. The second gluing function has simple form. The idea is to embed the domain in $\mathbb{R}^3$ and to glue $\mathcal{D}$ and $\mathcal{G}$ together by dragging them onto the vertical axis. As one of

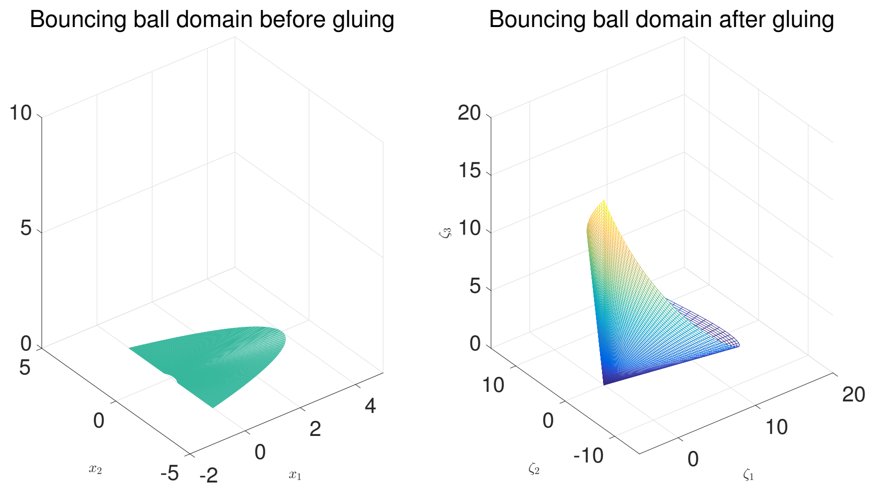

Figure 4.11: Domains of the bouncing ball system before gluing $\mathcal{C}$ (left) and after gluing $\mathcal{C}^{\psi_2}$ (right) when $\underline{\delta}= 0.1$, $\overline{\delta} = 4$, and $\rho = 1$.

the functions realizing it, we suggest a function $\psi_2 : \mathcal{C} \to \mathbb{R}^3$, $x = (x_1, x_2) \mapsto \zeta = (\zeta_1, \zeta_2, \zeta_3) = (x_1^2, 2x_1x_2, 2x_2^2)$, which is proposed as a state immersion for a mechanical system with impacts in [MT16]. Since $\psi_2(x) = (0, 0, 2x_2^2) = \psi_2(g(x))$ for all $x \in \mathcal{D}$, (G1) is satisfied. It can be easily checked that (G2) and (G3) also hold. Finally, we have that

$$\operatorname{rank}(\mathrm{d}\psi(x)) = \operatorname{rank}\left(\begin{bmatrix} 2x_1 & 0 \\ 2x_2 & 2x_1 \\ 0 & 4x_2 \end{bmatrix}\right) = 2 \qquad \text{for all } x \in \mathcal{C}.$$

and (G4) holds. Accordingly, ψ_2 is a gluing function. Figure 4.11 illustrates the domain $\mathcal{C}$ and the glued domain $\mathcal{C}^{\psi_2}$. The inverse gluing function of ψ_2 is obtained as

$$\psi_2^{-1}(\zeta) = \begin{bmatrix} \sqrt{\zeta_1} \\ \operatorname{sgn}(\zeta_2)\sqrt{\frac{\zeta_3}{2}} \end{bmatrix} \qquad \text{for all } \zeta \in \mathcal{C}^{\psi_2}.$$

□

The above example, we glue the domain of the bouncing ball system. However, it may be not the realization of the quotient map π we want. Let us consider the pushforward of f by ψ_2. In fact, since ψ_2 is not a diffeomorphism, we cannot define the pushforward in Definition 2.2.12 strictly. However, in a local sense, we can consider the pushforward and, for $x \in \mathcal{D}$, the vector field $f(x)$ gives two candidates $\mathrm{d}\psi_2(x)f(x)$ and $\mathrm{d}\psi_2(g(x))f(g(x))$ for the vector field at $\psi_2(x)$. Since

$$\mathrm{d}\psi_2(x)f(x) = \begin{bmatrix} 2x_1 & 0 \\ 2x_2 & 2x_1 \\ 0 & 4x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\rho \end{bmatrix} = \begin{bmatrix} 2x_1x_2 \\ 2x_2^2 - 2\rho x_1 \\ -4\rho x_2 \end{bmatrix}$$

and

$$\mathrm{d}\psi_2(g(x))f(g(x)) = \begin{bmatrix} -2x_1 & 0 \\ -2x_2 & -2x_1 \\ 0 & -4x_2 \end{bmatrix} \begin{bmatrix} -x_2 \\ -\rho \end{bmatrix} = \begin{bmatrix} 2x_1x_2 \\ 2x_2^2 + 2\rho x_1 \\ 4\rho x_2 \end{bmatrix},$$

it follows that

$$\mathrm{d}\psi_2(x)f(x) \neq \mathrm{d}\psi_2(g(x))f(g(x)),$$

which implies that the vector field on $\mathcal{C}^{\psi_2}$ is not continuous.

In fact, the proposed gluing function just glues the domain in a topological sense. Therefore, the vector field on the glued region may be not differentiable and even it may be discontinuous. At least to guarantee the continuity of the vector field on the glued domain, the gluing function ψ naturally should satisfy that

$$\mathrm{d}\psi(x)f(x) = \mathrm{d}\psi(g(x))f(g(x)) \quad \text{for all } x \in \mathcal{D}, \tag{4.3.7}$$

which is denoted by the *vector field matching condition.* Note that there exists a gluing function $\psi : \mathcal{C} \to \mathbb{R}^{2k+1}$ satisfying the vector field matching condition (4.3.7) under some assumptions by Theorem 4.3.1.

Example 4.3.3. (Vector field matching condition of bouncing ball system)**.** Consider the gluing functions ψ_1 and ψ_2 in Example 4.3.2. These two gluing functions do not satisfy the vector field matching condition (4.3.7). However, for each case, we can find another gluing function satisfying the vector field (4.3.7).

a) Doubling in $\mathbb{R}^2$

Suppose that $g = 1$. Then, we propose another gluing function $\psi_3 := \psi_1 \circ \phi$ where

$$\phi(x) := \begin{bmatrix} x_1 \\ \frac{1}{3}x_2^3 + x_1 x_2 \end{bmatrix},$$

which is obtain the following preprocessor.

At first we have that, for all $x = (0, x_2) \in \mathcal{D}$,

$$\mathrm{d}\psi_1(x) = \mathrm{d}\psi_1(\phi(x)) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

and

$$\mathrm{d}\psi_1(g(x)) = \mathrm{d}\psi_1(\phi(g(x))) = -\mathrm{d}\psi_1(\phi(x)),$$

which are verified by computation. Then, since

$$\mathrm{d}(\psi_1 \circ \phi)(x) f(x) = \mathrm{d}\psi_1(\phi(x)) \mathrm{d}\phi(x) f(x),$$

and

$$\begin{aligned} \mathrm{d}(\psi_1 \circ \phi)(g(x)) f(g(x)) &= \mathrm{d}\psi_1(\phi(g(x))) \mathrm{d}\phi(g(x)) f(g(x)) \\ &= -\mathrm{d}\psi_1(\phi(x)) \mathrm{d}\phi(g(x)) f(g(x)), \end{aligned}$$

in order to make these two the same so that the vector matching condition (4.3.7) holds, it is enough to have that $\mathrm{d}\phi(x) f(x) = -\mathrm{d}\phi(g(x)) f(g(x))$. Motivated by this fact, we propose a partial differential equation for the function ϕ as

$$\mathrm{d}\phi(x) f(x) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \text{ for all } x \in \mathcal{C}, \tag{4.3.8}$$

and its solution is given by

$$\phi(x) = \begin{bmatrix} x_1 \\ \frac{1}{3}x_2^3 + x_1 x_2 \end{bmatrix}, \tag{4.3.9}$$

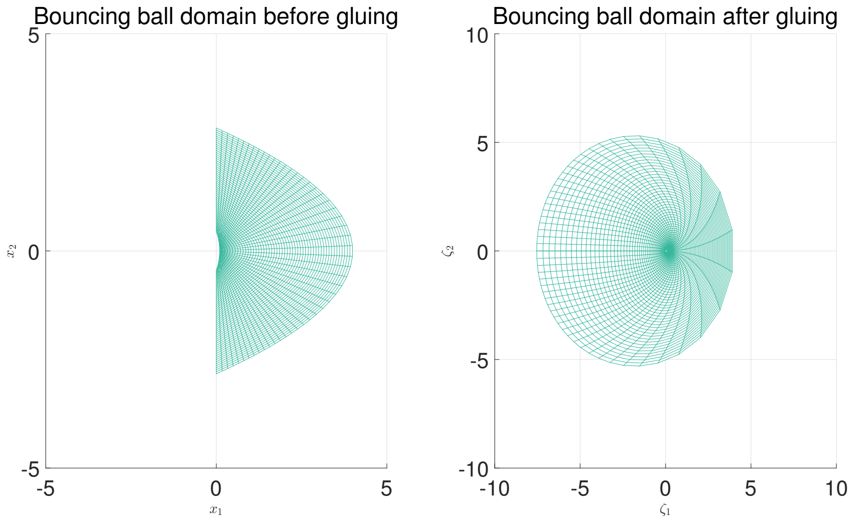

Figure 4.12: Domains of the bouncing ball system before gluing $\mathcal{C}$ (left) and after gluing $\mathcal{C}^{\psi_3}$ (right) when $\underline{\delta}$= 0.1, $\overline{\delta} = 4$, and $\rho = 1$.

whose inverse $\phi^{-1} : \phi(\mathcal{C}) \to \mathcal{C}$ is also given by Cardano's formula

$$\phi^{-1}(\bar{x}) = \left[\frac{\bar{x}_1}{\sqrt[3]{\frac{3}{2}\bar{x}_2 + \sqrt{d(\bar{x})}} + \sqrt[3]{\frac{3}{2}\bar{x}_2 - \sqrt{d(\bar{x})}}} \right],$$

where $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \phi(\mathcal{C})$ and $d(\bar{x}) := \bar{x}_1^3 + (\frac{3}{2}\bar{x}_2)^2$. Finally, it can be shown that $\psi_3 = \psi_1 \circ \phi$ and $\psi_3^{-1} = \phi^{-1} \circ \tilde{\psi}_1^{-1}$ are well-defined on $\mathcal{C}$ and on $\psi_3(\mathcal{C}) = \mathcal{C}^{\psi_3}$, respectively, and ψ_3 is a gluing function satisfying the vector field matching condition (4.3.7). (Note that it satisfies the condition (G1) since $g(\phi(x)) = \phi(g(x))$ on $\mathcal{D}$ so that $\psi_3(g(x)) = \psi_1(\phi(g(x))) = \psi_1(g(\phi(x))) = \psi_1(\phi(x)) = \psi(x), \forall x \in \mathcal{D}$.) Figure 4.12 illustrates the domain $\mathcal{C}$ and the glued domain $\mathcal{C}^{\psi_3}$.

b) Dragging in $\mathbb{R}^3$

As shown after Example 4.3.2, ψ_2 does not satisfy the vector field matching condition. However, motivated by this function, we can a function $\psi_4 : \mathbb{R}^2 \to \mathbb{R}^3$, $x = (x_1, x_2) \to \zeta = (\zeta_1, \zeta_2, \zeta_3) = (x_1^2, 2x_1x_2, 2x_2^2 + 4\rho x_1)$. Since $\psi_4(x) = (0, 0, 2x_2^2) = \psi_4(g(x))$ for all $x \in \mathcal{D}$, (G1) is satisfied. It can be easily checked that (G2) and (G3) also hold. Furthermore, we obtain that

$$\operatorname{rank}(\mathrm{d}\psi_4(x)) = \operatorname{rank}\left(\begin{bmatrix} 2x_1 & 0 \\ 2x_2 & 2x_1 \\ 4\rho & 4x_2 \end{bmatrix}\right) = 2 \qquad \text{for all } x \in \mathcal{C},$$

which implies that (G4) holds. Finally, for all $x \in \mathcal{D}$, since

$$\mathrm{d}\psi_4(x)f(x) = \begin{bmatrix} 2x_1x_2 \\ 2x_2^2 - 2\rho x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_2^2 \\ 0 \end{bmatrix}$$

and

$$\mathrm{d}\psi_4(g(x))f(g(x)) = \begin{bmatrix} 2x_1x_2 \\ 2x_2^2 + 2\rho x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_2^2 \\ 0 \end{bmatrix},$$

we obtain that

$$\mathrm{d}\psi_4(x)f(x) = \mathrm{d}\psi_4(g(x))f(g(x))$$

and ψ_4 is a gluing function satisfying the vector field matching condition (4.3.7). Figure 4.13 illustrates the domain $\mathcal{C}$ and the glued domain $\mathcal{C}^{\psi_4}$. The inverse gluing function is given as

$$\psi_4{}^{-1}(\zeta) = \begin{bmatrix} \sqrt{\zeta_1} \\ \operatorname{sgn}(\zeta_2)\sqrt{\frac{\zeta_3 - 4\rho\sqrt{\zeta_1}}{2}} \end{bmatrix} \quad \text{for all } \zeta \in \mathcal{C}^{\psi_4}.$$

□

Now we come back the proposed frameworks. The following lemma is useful in further chapters dealing with the state estimation and tracking control problems of the hybrid system.

Lemma 4.3.2. Suppose there exists a gluing function ψ of $\mathcal{H}$. Then, for any compact set $\mathcal{M} \subset \mathcal{C}$ satisfying that $\mathcal{M} \cap \mathcal{D} = \varnothing$ or $\mathcal{M} \cap \mathcal{G} = \varnothing$, it holds that ψ

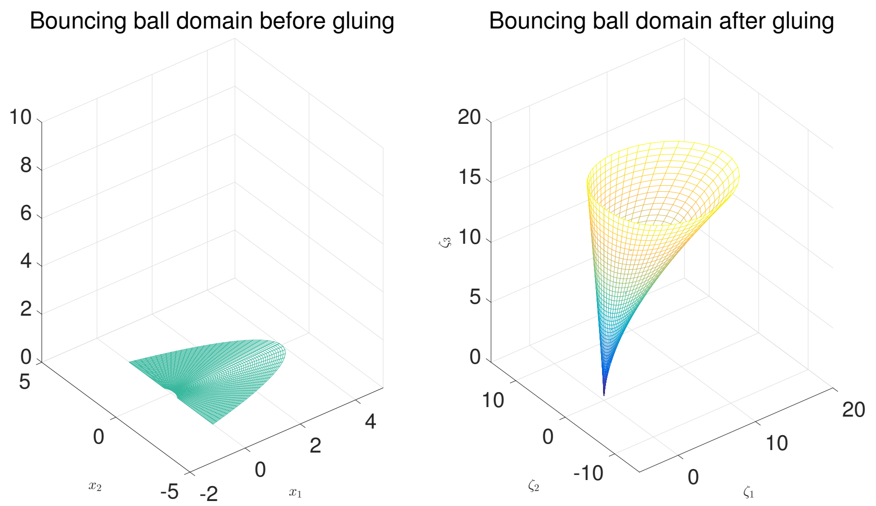

Figure 4.13: Domains of the bouncing ball system before gluing $\mathcal{C}$ (left) and after gluing $\mathcal{C}^{\psi_4}$ (right) when $\underline{\delta}= 0.1$, $\overline{\delta} = 4$, and $\rho = 1$.

is injective on $\mathcal{M}$ and, for all $(x, y) \in \mathcal{M} \times \mathcal{M}$, there exists $L > 0$ such that

$$|x - y| \leq L|\psi(x) - \psi(y)| \tag{4.3.10}$$

□

Proof of Lemma 4.3.2. If $\mathcal{M} \cap \mathcal{D} = \varnothing$, then $\mathcal{M} \subset \mathcal{C} \backslash \mathcal{D}$ and, by (G2), ψ is injective on $\mathcal{M}$. Next we consider the case when $\mathcal{M} \cap \mathcal{G} = \varnothing$. Suppose that there are $x, y \in \mathcal{M} \subset \mathcal{C} \backslash \mathcal{G}$ such that $x \neq y$ and $\psi(x) = \psi(y)$. Due to (G2), at least one should be in $\mathcal{D}$. Without loss of generality, we have that $x \in \mathcal{D} \cap (\mathcal{C} \backslash \mathcal{G})$ and $y \in \mathcal{C} \backslash \mathcal{G}$. We first consider the case when y is also included in $\mathcal{D}$. Then, by (G1), it holds that $\psi(g(x)) = \psi(g(y))$, which implies that $x = y$ because g and ψ are injective on $\mathcal{D}$ and $g(\mathcal{D}) = \mathcal{G} \subset \mathcal{C} \backslash \mathcal{D}$, respectively. This is a contradiction. Secondly, consider the case when y is not included in $\mathcal{D}$. Then, it follows from (G1) that $\psi(x) = \psi(x') = \psi(y)$ where $x' := g(x) \in \mathcal{G}$ and $x' \neq y \in \mathcal{C} \backslash (\mathcal{D} \cup \mathcal{G})$, which is also a contradiction because $x', y \in \mathcal{C} \backslash \mathcal{D}$ and, by (G2), ψ is injective on $\mathcal{C} \backslash \mathcal{D}$. Therefore, ψ is injective on $\mathcal{M}$ whenever $\mathcal{M} \cap \mathcal{D} = \varnothing$ or $\mathcal{M} \cap \mathcal{G} = \varnothing$.

Next, to show (4.3.10), it is enough to check that

$$\inf_{\substack{x \neq y \\ x,y \in \mathcal{M}}} \frac{|\psi(x) - \psi(y)|}{|x - y|} > 0.$$

Suppose that there exist sequences $\{x_i\}$ and $\{y_i\}$ in $\mathcal{M}$ such that

$$\lim_{i \to \infty} \frac{|\psi(x_i) - \psi(y_i)|}{|x_i - y_i|} = 0. \tag{4.3.11}$$

Since $\mathcal{M} \subset \mathbb{R}^n$ is compact, by Bolzano-Weierstrass theorem, without loss of generality, we may assume that $\{x_i\}$ and $\{y_i\}$ converge to some points x' and y' in $\mathcal{M}$, respectively. If $x' \neq y'$, then $\psi(x') = \psi(y')$, which contradicts to injectivity of ψ on $\mathcal{M}$. Therefore, $x' = y'$. Since $\mathcal{C}$ is a smooth manifold with boundary, there exists a coordinate chart (U, φ) at $x' \in \mathcal{M}$. Then, there exists $k^* > 0$ such that $x_i, y_i \in U$ for all $i > k^*$. Since ψ is C^1, $\psi_\varphi := \psi \circ \varphi^{-1}$ is C^1. Let $\mathsf{x_i} := \varphi^{-1}(x_{k^*+i})$ and $\mathsf{y_i} := \varphi^{-1}(y_{k^*+i})$. By the definition of the Jacobian,

$$\lim_{i \to \infty} \frac{|\psi_\varphi(\mathsf{x}_i) - \psi_\varphi(\mathsf{y}_i) + \mathrm{d}\psi_\varphi(\varphi^{-1}(x'))(\mathsf{x}_i - \mathsf{y}_i)|}{|\mathsf{x}_i - \mathsf{y}_i|} = \lim_{i \to \infty} |\mathrm{d}\psi_\varphi(\varphi^{-1}(x'))\mathsf{w}_i| = 0,$$

where $\mathsf{w}_i := \frac{\mathsf{x}_i - \mathsf{y}_i}{|\mathsf{x}_i - \mathsf{y}_i|} \in \mathbb{R}^k$, which can be a tangent vector of at x'. Since w_i is bounded in $\mathbb{R}^k$, we can take a subsequence converging to $\mathsf{w} \in \mathbb{R}^k$. Then, it follows that $\mathrm{d}\psi_\varphi(\varphi^{-1}(x'))\mathsf{w}$, which is a contradiction to the fact that ψ is immersion by (G4). $\square$

Since the gluing function $\psi : \mathcal{C} \to \mathbb{R}^m$ is a local C^1 embedding, we can utilize the Inverse Function Theorem. Then for $x \in \mathcal{C}$ there exists an open neighborhood $U \subset \mathcal{C}$ such that $\psi|_U$ is injective and $\psi|_U^{-1} : \psi(U) \to \mathcal{C}$ is C^1.

Chapter 5

State Estimation Strategy

In this chapter, we suggest a new state estimation strategy using the gluing function, which does not require any detection of the time instants when the state jumps.

5.1 Standing Assumptions

To deal with the state estimation problem, we need to define an output because we should estimate the state from the output. For example, an output of the bouncing ball system in Example 4.2.1 may be either the height x_1 or the velocity x_2.

Definition 5.1.1. A *hybrid system* $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$ *with an output* $y = h(x)$ is denoted by $\mathcal{H}^h := (\mathcal{C}, f, \mathcal{D}, g, h)$ where $h : \mathcal{C} \to \mathbb{R}^q$ is an *output map.* □

We impose the following additional assumptions on $\mathcal{H}^h$ under consideration.

Assumption 5.1.1. The system $\mathcal{H}^h = (\mathcal{C}, f, \mathcal{D}, g, h)$ satisfies that

(E1) f is locally Lipschitz;

(E2) h is continuous and

$$h(x) = h(g(x)) \text{ for all } x \in \mathcal{D}. \tag{5.1.1}$$

□

We impose (E1) to guarantee the existence and uniqueness of the local flow starting at each point in $\mathcal{C}\backslash\mathcal{D}$. Under (E3), since there is no jump in the output value when the state jump occurs, it may be difficult to detect the state jumps from the mere observation of the output. Therefore, the previous observers, requiring the jump time information, may not be applied. We call (5.1.1) an *output matching condition.* Note that the bouncing ball system with the output $y = h(x) = x_1$ satisfies the assumptions. Another example is a ripple disturbance arising in AC/DC converters proposed in [BZLC17].

To consider the state estimation problem, we restrict an interested region where the state trajectory remains as a compact set $K \subset \mathcal{C}$. The following assumptions are imposed on K.

Assumption 5.1.2. For $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$, there exists a compact subset $K \subset \mathcal{C}$ such that

(E3) $g(K \cap \mathcal{D}) = K \cap \mathcal{G}$;

(E4) $f(x) \in T_K(x)$ for $x \in K\backslash\mathcal{D}$;

(E5) there are smooth maps $r_{\mathcal{D}} : \mathbb{R}^n \to \mathbb{R}$ and $r_{\mathcal{G}} : \mathbb{R}^n \to \mathbb{R}$ such that $\mathcal{D}^K := K \cap \mathcal{D} = \{x \in K : r_{\mathcal{D}}(x) = 0\}$, $\mathcal{G}^K := K \cap \mathcal{G} = \{x \in K : r_{\mathcal{G}}(x) = 0\}$, $K \subset \{x \in \mathbb{R}^n : (r_{\mathcal{D}}(x) \le 0) \wedge (r_{\mathcal{G}}(x) \ge 0)\}$, and

$$\begin{cases} \nabla r_{\mathcal{D}}(x) \cdot f(x) > 0 & \text{for all } x \in \mathcal{D}^K; \\ \nabla r_{\mathcal{G}}(x) \cdot f(x) > 0 & \text{for all } x \in \mathcal{G}^K; \end{cases}$$

where 0 is a regular value of $r_{\mathcal{D}}$ and $r_{\mathcal{G}}$. □

We impose (E3–4) to make K as an invariant set. Note that, $\mathcal{D}^K$ and $\mathcal{G}^K$ are compact because K is compact and $\mathcal{D}$ and $\mathcal{G}$ are closed in $\mathcal{C}$. In addition, (E5) implies that there are two hypersurfaces in $\mathbb{R}^n$ such that the intersections of them and K are $\mathcal{D}^K$ and $\mathcal{G}^K$, respectively.

Example 5.1.1. (Invariant set of bouncing ball system)**.** Consider the bouncing ball system satisfying Assumption 4.3.1 in Example 4.2.1. Suppose that $u(t) \equiv 0$ and let $E(x) := m\rho x_1 + \frac{1}{2}m x_2^2$. Then, we can take a compact set $K := \{x \in \mathcal{C} : E(x) \in [\underline{\delta}, \overline{\delta}]\}$ satisfying Assumption 5.1.2 where $0 < \underline{\delta} < \overline{\delta}$. □

The following example is a ripple generation model of a three-phase system suggested as a practical example of hybrid system in [BZLC17].

Example 5.1.2. (Ripple disturbance)**.** Let us consider a ripple disturbance in AC/DC converters in Section 3.3. The disturbance is modeled as a system $\mathcal{H}^h = (\mathcal{C}, f, \mathcal{D}, g, h)$ with $x := (r, \theta, b)$ such as

$$\begin{cases} \begin{bmatrix} \dot{r} \\ \dot{\theta} \\ \dot{b} \end{bmatrix} = f(x) := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \text{when } x \in \mathcal{C} := \{x = (r, \theta, b) \in \mathbb{R}^3 : (r > 0) \wedge (-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6})\} \\ \begin{bmatrix} r^+ \\ \theta^+ \\ b^+ \end{bmatrix} = g(x) := \begin{bmatrix} r \\ -\theta \\ b \end{bmatrix} & \text{when } x \in \mathcal{D} := \{x = (r, \theta, b) \in \mathcal{C} : \theta = \frac{\pi}{6}\} \\ y \quad = h(x) =: r\cos\theta + b. & \end{cases}$$

Note that $\mathcal{C}$ is a smooth 3-manifold manifold with boundary and $\mathcal{D}$ and $\mathcal{G} = g(\mathcal{D})$ are smooth parts of $\partial\mathcal{C}$. Since, for each $x \in \mathcal{D}$, $h(x) = r\cos(\frac{\pi}{6}) + b = \frac{\sqrt{3}}{2}r + b$ and $h(g(x)) = r\cos(-\frac{\pi}{6}) + b = \frac{\sqrt{3}}{2}r + b$, the output matching condition (E2) holds. In addition, the other conditions in Assumption 4.3.1 and Assumption 5.1.1 are satisfied. Finally, for $\overline{r} > \underline{r} > 0$ and $\overline{b} > \underline{b} > 0$, the set $K := \{x = (r, \theta, b) \in \mathcal{C} : (\underline{r} \leq r \leq \overline{r}) \wedge (\underline{b} \leq b \leq \overline{b})\}$ satisfies Assumption 5.1.2. □

Lemma 5.1.1. Under Assumption 4.3.1, (E1), and (E3–4), the execution starting K is infinite in t-direction and unique. □

Proof of Lemma 5.1.1. It follows from Proposition 3 in [ALQ$^+$02] that each maximal execution is non-blocking in t-direction and remains in K. Next, since g is injective on $\mathcal{D}^K$ and f is locally Lipschitz, its jump and flow are defined uniquely. In addition, by assumption, we have that $f(x) \notin T_{\mathcal{C}}(x)$ for all $x \in \mathcal{D}^K$, which implies that the state must jump on $\mathcal{D}^K$. From this, we obtain that either the continuous-time evolution or discrete event happens at each point but not both. Therefore, it follows that there is at most one maximal execution for each initial condition. □

The condition (E5) is imposed to show that the state cannot stay in $\mathcal{D}$ and

$\mathcal{G}$ by flowing. Actually, this fact also comes from the final condition in Assumption 4.3.1. Nevertheless, we adopt (E5) for a brief proof.

Lemma 5.1.2. Consider $\mathcal{H}$ satisfying Assumption 4.3.1, (E1), and (E3–5). Let $\mathcal{O}_{\mathcal{G}}(\epsilon) := \{x \in K : d_{\mathcal{G}}(x) < \epsilon\}$ and $\mathcal{O}_{\mathcal{D}}(\epsilon) := \{x \in K : d_{\mathcal{D}}(x) < \epsilon\}$. Then, there exists a class-$\mathcal{K}$ function α such that, for sufficiently small $\epsilon > 0$, each state trajectory $x(t)$ starting on K satisfies that

$$x(t) \notin \mathcal{O}_{\mathcal{G}}(\epsilon) \cup \mathcal{O}_{\mathcal{D}}(\epsilon) \quad \text{for all } t \in \tau_\alpha(\epsilon) := \mathbb{R}_{\geq 0} \backslash \bigcup_{i=0}^{N} \mathcal{B}_{\tau_i}(\alpha(\epsilon))$$

where $\tau = \{I_i\}_{i=0}^{N}$ is the hybrid time trajectory corresponding to $x(t)$ such that $I_i = [\tau_i, \tau_i']$ and $x(t) \in \mathcal{G}$ implies that $t \in \{\tau_i\}_{i=0}^{N}$. □

Proof of Lemma 5.1.2. Define $w(x) := \nabla r_{\mathcal{G}}(x) \cdot f(x)$ for $x \in K$. Then, by (E5), $w(x) > 0$ for all $x \in \mathcal{G}^K$. Moreover, $w(\cdot)$ is uniformly continuous on K, because $r_{\mathcal{G}}$ is smooth, f is locally Lipschitz, and K is compact. Consequently, $\mu := \inf_{x \in \mathcal{G}^K} w(x)$ is positive by the compactness of $\mathcal{G}^K$ and there exists $\epsilon_1 > 0$ such that if $(x_1, x_2) \in K \times K$ and $|x_1 - x_2| < \epsilon_1$, then

$$|w(x_1) - w(x_2)| < \frac{\mu}{2}.$$

If $x \in \mathcal{O}_{\mathcal{G}}(\epsilon_1)$, by the definition, there exists $x_{\mathcal{G}} \in \mathcal{G}^K$ such that $|x - x_{\mathcal{G}}| < \epsilon_1$ and, from the above equation, we have that

$$-\frac{\mu}{2} < w(x) - w(x_{\mathcal{G}}) < \frac{\mu}{2}.$$

Then, by the definition of μ, it follows that $w(x) > \frac{\mu}{2}$ for all $x \in \mathcal{O}_{\mathcal{G}}(\epsilon_1)$.

Next, since $r_{\mathcal{G}}$ is smooth, it is Lipschitz on the compact set $\mathcal{C}$ with a Lipschitz constant $L > 0$. Moreover, by (E5), it is satisfied that $r_{\mathcal{G}}(x) > 0$ if $x \in K \backslash \mathcal{G}^K$ and $r_{\mathcal{G}}(x) = 0$ if $x \in \mathcal{G}^K$. From this fact, we obtain that

$$r_{\mathcal{G}}(x) = |r_{\mathcal{G}}(x)| = |r_{\mathcal{G}}(x) - r_{\mathcal{G}}(x_{\mathcal{G}})| \leq L|x - x_{\mathcal{G}}| \quad \text{for all } (x, x_{\mathcal{G}}) \in K \times \mathcal{G}^K.$$

Therefore, if $x \in \mathcal{O}_{\mathcal{G}}(\epsilon)$, then $r_{\mathcal{G}}(x) \leq L\epsilon$. By contraposition, if $r_{\mathcal{G}}(x) > L\epsilon$, then $x \notin \mathcal{O}_{\mathcal{G}}(\epsilon)$.

For a state trajectory $x(t)$, let τ be its hybrid time trajectory. Then, we have that $x(\tau_i) \in \mathcal{G}^K$ and $r_{\mathcal{G}}(x(\tau_i)) = 0$ for $i = 1, \ldots, N$. Let $\alpha_1(\epsilon) := \frac{2L}{\mu}\epsilon$. We first claim that, for all $\epsilon < \epsilon_1$, $x(t)$ escapes from $\mathcal{O}_{\mathcal{G}}(\epsilon)$ at least once before $t = \tau_i + \alpha_1(\epsilon)$. Suppose that $x(t)$ remains in $\mathcal{O}_{\mathcal{G}}(\epsilon)$ for all $t \in [\tau_i, t_\epsilon]$ where $t_\epsilon \in [\tau_i + \alpha_1(\epsilon), \tau_i')$. Then, since $x(t) \in \mathcal{O}_{\mathcal{G}}(\epsilon) \subset \mathcal{O}_{\mathcal{G}}(\epsilon_1)$ for $t \in [\tau_i, t_\epsilon]$, it holds that $w(x(t)) > \frac{\mu}{2}$ and

$$r_{\mathcal{G}}(x(t_\epsilon)) = r_{\mathcal{G}}(x(t_\epsilon)) - r_{\mathcal{G}}(x(\tau_i)) = \int_{\tau_i}^{t_\epsilon} w(x(s))ds > \frac{\mu}{2}|t_\epsilon - \tau_i| \geq \frac{\mu}{2}\alpha_1(\epsilon) = L\epsilon.$$

This implies that $x(t_\epsilon) \notin \mathcal{O}_{\mathcal{G}}(\epsilon)$, which is a contraction.

Since $r_{\mathcal{G}}(x(t))$ increases when $x(t) \in \mathcal{O}_{\mathcal{G}}(\epsilon_1)$, $x(t)$ cannot return to $\mathcal{O}_{\mathcal{G}}(\epsilon)$ again. Therefore, we obtain that $x(t) \notin \mathcal{O}_{\mathcal{G}}(\epsilon)$ for all $t \in [\tau_i + \alpha_1(\epsilon), \tau_i')$ and for all $i = 1, \ldots, N$. Since the same result can be obtained for the case when $i = 0$, it follows that

$$x(t) \notin \mathcal{O}_{\mathcal{G}}(\epsilon) \quad \text{for all } t \in \bigcup_{i=0}^{N} [\tau_i + \alpha_1(\epsilon), \tau_i').$$

In a similar way, there exist $\epsilon_2 > 0$ and class-$\mathcal{K}$ function α_2 such that, for $\epsilon < \epsilon_2$,

$$x(t) \notin \mathcal{O}_{\mathcal{D}}(\epsilon) \qquad \text{for all } t \in \begin{cases} \bigcup_{i=0}^{N-1} (\tau_i, \tau_i' - \alpha_2(\epsilon)] \cup (\tau_N, \infty) & \text{if } N < \infty, \\ \bigcup_{i=0}^{N} (\tau_i, \tau_i' - \alpha_2(\epsilon)] & \text{if } N = \infty. \end{cases}$$

Finally, take $\alpha(\cdot) := \max(\alpha_1(\cdot), \alpha_2(\cdot))$ and consider $\epsilon < \min(\epsilon_1, \epsilon_2)$. Then, $x(t) \notin \mathcal{O}_{\mathcal{G}}(\epsilon) \cup \mathcal{O}_{\mathcal{D}}(\epsilon)$ for all $t \in \tau_\alpha(\epsilon)$. □

From this lemma, it follows that for any K the state remains near $\mathcal{D}^K \cup \mathcal{G}^K$ only at the time in the vicinity of each jump time instant.

5.2 State Estimation

For a given $\mathcal{H}^h$, suppose that Assumption 4.3.1 and Assumption 5.1.1 hold and that there exists a gluing function ψ of $\mathcal{H}$. Under the vector field matching condition (4.3.7), the vector field at $\zeta \in \psi(\mathcal{D})$ is also uniquely defined. Then, the vector field at $\zeta \in \mathcal{C}^\psi$ can be represented as $\mathrm{d}\psi(x)f(x)$ where $\psi(x) = \zeta$. By (G2), for each $\zeta \in \mathcal{C}^\psi$, we can find x through the inverse gluing function ψ^{-1} in (4.3.4).

Accordingly, for all $\zeta \in \mathcal{C}^\psi$, the vector field is obtained as

$$f^\psi(\zeta) := \mathrm{d}\psi(\psi^{-1}(\zeta)) f(\psi^{-1}(\zeta)), \tag{5.2.1}$$

which is a function of ζ. By the Inverse Function Theorem, ψ^{-1} is C^1 on $\mathcal{C}^\psi \backslash \psi(\mathcal{D})$. Therefore, f^ψ is also continuous under the vector field matching condition (4.3.7), because f is continuous and, by (G3), $\mathrm{d}\psi$ is continuous. Similarly, we obtain the output map on $\mathcal{C}^\psi$ as

$$h^\psi(\zeta) := h(\psi^{-1}(\zeta)) \qquad \zeta \in \mathcal{C}^\psi, \tag{5.2.2}$$

which is also continuous because of the continuity of h and output matching condition (5.1.1) in (E2). Then, ψ changes $\mathcal{H}^h$ into the following continuous-time system:

$$\dot{\zeta} = f^\psi(\zeta), \qquad \zeta \in \mathcal{C}^\psi \subset \mathbb{R}^m, \tag{5.2.3a}$$

$$y = h^\psi(\zeta). \tag{5.2.3b}$$

Let us consider a solution to (5.2.3a) starting from $\zeta_0 \in \psi(K)$. We claim that it is unique and identical to the glued trajectory $\zeta(t) = \psi(x(t))$ in (4.3.3) where $x(t)$ is the state trajectory of $\mathcal{H}$ with the initial condition $x_0 := \psi^{-1}(\zeta_0) \in K$. We first have that $\zeta(t)$ is a solution to (5.2.3a), because it is satisfied that $\dot{\zeta}(t) = \mathrm{d}\psi(x(t)) f(x(t)) = \mathrm{d}\psi(\psi^{-1}(\zeta(t))) f(\psi^{-1}(\zeta(t)))$ for all $t \geq 0$. In addition, if there is another solution $\zeta'(t)$ to (5.2.3a), it follows from the Inverse Function Theorem that $\psi^{-1}(\zeta'(t))$ provides another state trajectory of $\mathcal{H}$, which contradicts the fact that the state trajectory starting at a point of K is unique from Lemma 5.1.1. Note that, since the solution to (5.2.3a) starting K is unique and equal to $\zeta(t)$, we denote it by $\zeta(t)$ for convenience. Lastly, since each execution starting at a point of K is non-blocking in t-direction and remains in K, $x(t)$ is defined on K for all $t \geq 0$, which implies that $\psi(K)$ is invariant under f^ψ.

The system (5.2.3) is a continuous-time system without any discrete events and we call it a *glued system* of $\mathcal{H}^h$ by ψ. If the glued system is of the form that admits a conventional observer design method for continuous-time systems, then we can design an observer and obtain an estimate $\hat{\zeta}(t)$ for $\zeta(t) = \psi(x(t))$. Through the inverse gluing function $\psi^{-1} : \mathcal{C}^\psi \to \mathcal{C}$ in (4.3.4), we may also obtain an estimate for $x(t)$. However, since the estimate $\hat{\zeta}(t)$ can be defined on $\mathbb{R}^m$, it may be out of the domain $\mathcal{C}^\psi$ of the inverse gluing function ψ^{-1} (i.e., $\hat{\zeta}(t) \in \mathbb{R}^m \backslash \mathcal{C}^\psi$). To deal with this, we introduce a projection map $\Pi_{\psi(K)} : \mathbb{R}^m \to K \subset \mathcal{C}$ satisfying that

$$\Pi_{\psi(K)}(\zeta) \in \arg\min_{\zeta' \in \psi(K)} |\zeta - \zeta'| \text{ for all } \zeta \in \mathbb{R}^m. \tag{5.2.4}$$

Finally, let

$$\hat{x}(t) := \psi^{-1}(\Pi_{\psi(K)}(\hat{\zeta}(t))). \tag{5.2.5}$$

Then, $\hat{x}(t)$ becomes an estimate for the state trajectory $x(t)$ on K. Obviously, the proposed idea does not require any detection of the jumps of $x(t)$. The following theorem justifies the idea.

Remark 5.2.1. There may be many projection maps satisfying (5.2.4). Therefore, it is still a matter of choice. However, in this paper, we suppose that $\hat{\zeta}(t)$ is an asymptotic estimate of $\zeta(t)$. In this case, whatever we choose, $\Pi_{\psi(K)}(\hat{\zeta}(t))$ is another asymptotic estimate of $\zeta(t)$. Therefore, we can take any of them. □

Theorem 5.2.1. Suppose that Assumption 4.3.1 and Assumptions 5.1.1–5.1.2 hold and that there exist a gluing function ψ satisfying (4.3.7). Also suppose that an asymptotic observer of (5.2.3) for all $\zeta(0) \in \psi(K)$ exists in the sense that, for each $\gamma > 0$, there is $T_\zeta \geq 0$ such that

$$|\zeta(t) - \hat{\zeta}(t)| < \gamma \qquad \text{for all } t > T_\zeta.$$

Then, there exists an observer of $\mathcal{H}^h$ for the state trajectory starting on K in the sense that, for sufficiently small $\epsilon > 0$, there is $T \geq 0$ such that

$$|x(t) - \hat{x}(t)| < \epsilon \qquad \text{for all } t \in \tau_\alpha(\epsilon) \cap (T, \infty),$$

where $x(0) \in K$ and $\hat{x}(t)$ is in (5.2.5). □

Proof of Theorem 5.2.1. We obtain that $K\backslash\mathcal{O}_{\mathcal{D}}(\epsilon)$ is compact for all $\epsilon > 0$, because K is compact and $\mathcal{O}_{\mathcal{D}}(\epsilon)$ is open relative to K. From Lemma 4.3.2, for each $(x, x') \in K\backslash\mathcal{O}_{\mathcal{D}}(\epsilon) \times K\backslash\mathcal{O}_{\mathcal{D}}(\epsilon)$, there exists $L_1(\epsilon) > 0$ such that $|x - x'| \leq L_1|\psi(x) - \psi(x')|$. Similarly, for each $(x, x') \in K\backslash\mathcal{O}_{\mathcal{G}}(\epsilon) \times K\backslash\mathcal{O}_{\mathcal{G}}(\epsilon)$, there exists $L_2(\epsilon) > 0$ such that $|x - x'| \leq L_2|\psi(x) - \psi(x')|$. Finally, take $L := \max(L_1, L_2)$.

Consider a sufficiently small $\epsilon > 0$ satisfying $\mathcal{O}_{\mathcal{D}}(\epsilon) \cap \mathcal{O}_{\mathcal{G}}(\epsilon) = \varnothing$ and the condition given in Lemma 5.1.2. By the asymptotic observer of (5.2.3), for $\gamma = \frac{\epsilon}{2L} > 0$, there is $T_\zeta \geq 0$ such that

$$|\zeta(t) - \Pi_{\psi(K)}(\hat{\zeta}(t))| \leq |\zeta(t) - \hat{\zeta}(t)| + |\hat{\zeta}(t) - \Pi_{\psi(K)}(\hat{\zeta}(t))| < 2\gamma = \frac{\epsilon}{L}$$

for all $t > T_\zeta$. Note that, from the definition of $\Pi_{\psi(K)}$, it holds that $|\hat{\zeta}(t) - \Pi_{\psi(K)}(\hat{\zeta}(t))| \leq |\hat{\zeta}(t) - \zeta(t)|$.

For all $t \in \tau_\alpha(\epsilon)$, since $x(t) \in K\backslash(\mathcal{O}_{\mathcal{D}}(\epsilon) \cup \mathcal{O}_{\mathcal{G}}(\epsilon)) = K\backslash\mathcal{O}_{\mathcal{D}}(\epsilon) \cap K\backslash\mathcal{O}_{\mathcal{G}}(\epsilon)$ and $\hat{x}(t) \in K = K\backslash\mathcal{O}_{\mathcal{D}}(\epsilon) \cup K\backslash\mathcal{O}_{\mathcal{G}}(\epsilon)$, it follows that $\big(x(t), \hat{x}(t)\big) \in \big(K\backslash\mathcal{O}_{\mathcal{D}}(\epsilon) \times K\backslash\mathcal{O}_{\mathcal{D}}(\epsilon)\big) \cup \big(K\backslash\mathcal{O}_{\mathcal{G}}(\epsilon) \times K\backslash\mathcal{O}_{\mathcal{G}}(\epsilon)\big)$. As a result, we obtain that $|x(t) - \hat{x}(t)| \leq L_1|\zeta(t) - \Pi_{\psi(K)}(\hat{\zeta}(t))| < L_1\frac{\epsilon}{L} \leq \epsilon$ for the case when $\big(x(t), \hat{x}(t)\big) \in K\backslash\mathcal{O}_{\mathcal{D}}(\epsilon) \times K\backslash\mathcal{O}_{\mathcal{D}}(\epsilon)$ and that $|x(t) - \hat{x}(t)| \leq L_2|\zeta(t) - \Pi_{\psi(K)}(\hat{\zeta}(t))| < L_2\frac{\epsilon}{L} \leq \epsilon$ for the case when $\big(x(t), \hat{x}(t)\big) \in K\backslash\mathcal{O}_{\mathcal{G}}(\epsilon) \times K\backslash\mathcal{O}_{\mathcal{G}}(\epsilon)$. Therefore, by taking $T := T_\zeta$, we obtain that $|x(t) - \hat{x}(t)| < \epsilon$ for all $t \in \tau_\alpha(\epsilon) \cap (T, \infty)$. □

Example 5.2.1. (Observer design for the bouncing ball system) Consider the bouncing ball system in Example 5.1.1 with the output $y = x_1$, which means that the output is the height and the ball is accelerated only by gravity. Then, the system is described by

$$\begin{aligned} \dot{x} &= f(x) = \begin{bmatrix} x_2 \\ -\rho \end{bmatrix} && \text{when } x \in \mathcal{C}, \\ x^+ &= g(x) = -x && \text{when } x \in \mathcal{D}, \\ y &= h(x) = x_1, \end{aligned} \tag{5.2.6}$$

where $\mathcal{C}$ and $\mathcal{D}$ are given in (4.2.1). Take $K := \{x \in \mathcal{C} : \underline{\delta} \leq E(x) \leq \overline{\delta}\}$ for some $0 < \underline{\delta} \leq \overline{\delta}$. Note that $E(x) := m\rho x_1 + \frac{1}{2}mx_2^2$ and suppose that the mass m is

1. It is trivial that the system satisfies Assumption 4.3.1, Assumption 5.1.1, and Assumption 5.1.2. Let us take the gluing function as $\psi := \psi_4$ in Example 4.3.3, which satisfies the vector field matching condition (4.3.7). Recall that the inverse gluing function is given as

$$\psi^{-1}(\zeta) = \begin{bmatrix} \sqrt{\zeta_1} \\ \mathrm{sgn}(\zeta_2)\sqrt{\frac{\zeta_3 - 4\rho\sqrt{\zeta_1}}{2}} \end{bmatrix} \text{ for all } \zeta \in \mathcal{C}^{\psi}.$$

Through the gluing function, we obtain the functions

$$\begin{aligned} f^{\psi}(\zeta) &= \begin{bmatrix} \zeta_2 \\ \zeta_3 - 6\rho\sqrt{\zeta_1} \\ 0 \end{bmatrix}, \\ h^{\psi}(\zeta) &= \sqrt{\zeta_1}, \end{aligned}$$

and the following continuous-time dynamical system,

$$\begin{aligned} \dot{\zeta} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \zeta + \begin{bmatrix} 0 \\ -6\rho \\ 0 \end{bmatrix} \sqrt{\zeta_1} =: A\zeta + B\sqrt{\zeta_1}, \\ y &= \sqrt{\zeta_1}, \end{aligned} \tag{5.2.7}$$

for $\zeta \in \psi(K)$. We propose the following observer for (5.2.7),

$$\dot{\hat{\zeta}} = A\hat{\zeta} + L(C\hat{\zeta} - y^2) + By, \tag{5.2.8}$$

where $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $A + LC$ is Hurwitz. Let $\zeta_e := \hat{\zeta} - \zeta$. Then, the dynamics of ζ_e results in

$$\dot{\zeta}_e = (A + LC)\zeta_e,$$

because $y^2 = C\zeta$. Therefore, ζ_e converges to zero exponentially and, by Theorem

5.2.1, we obtain an estimate

$$\hat{x} = \psi^{-1}(\Pi_{\psi(K)}(\hat{\zeta})) =: \psi^{-1}(\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3) = \begin{bmatrix} \sqrt{\bar{\zeta}_1} \\ \mathrm{sgn}(\bar{\zeta}_2)\sqrt{\frac{\bar{\zeta}_3 - 4\rho\sqrt{\bar{\zeta}_1}}{2}} \end{bmatrix}. \tag{5.2.9}$$

A simulation result is reported in Figure 5.1. □

Remark 5.2.2. In Example 5.2.1, if $\rho = 0$, the proposed gluing function is the same as the state immersion given in [MT16] and the observer designs are similar. However, the proposed observer in Example 5.2.1 is still valid even if $\rho > 0$. □

Remark 5.2.3. Most conventional observers for hybrid system employ constraint sets in which the estimate may jump. This is necessary to catch up the state jump. The proposed observer may have the constraint set $\mathcal{D} \cup \mathcal{G}$, because the inverse gluing function ψ^{-1} is not continuous on $\psi(\mathcal{D} \cup \mathcal{G})$. Specifically, the proposed estimate can jump from $\mathcal{D}$ to $\mathcal{G}$ via g or reversely jump via g^{-1} from $\mathcal{G}$ to $\mathcal{D}$ to reduce the estimation error. The former is the common property that most existing observers have, but the latter is novel one. □

Corollary 5.2.2. (Estimation in a graphical sense)**.** The following statement reinterprets Theorem 5.2.1 in a graphical sense: under the assumptions of Theorem 5.2.1, for sufficiently small $\epsilon^* > 0$, there exists $T^* > 0$ such that

(a) for each $t > T^*$, there exists $s > 0$ satisfying that $|(t, x(t)) - (s, \hat{x}(s))| < \epsilon^*$,

(b) for each $t > T^*$, there exists $s > 0$ satisfying that $|(s, x(s)) - (t, \hat{x}(t))| < \epsilon^*$.

□

Proof of Corollary 5.2.2. If $x(t)$ has a finite number of jumps (i.e. $N < \infty$), then it follows from Theorem 5.2.1 for the case where $\epsilon = \epsilon^*$ that (a) and (b) are true by taking $T^* := \max(\tau_N + \alpha(\epsilon^*), T)$ and $s := t$. Now, we prove the case when $N = \infty$.

(a) Let $\epsilon := \min\left(\frac{\epsilon^*}{4}, \alpha^{-1}(\frac{\epsilon^*}{2}), \alpha^{-1}(\frac{\epsilon^*}{4M})\right)$ where $M := \sup_{x \in K} |f(x)|$. If ϵ^* is sufficiently small, by Theorem 5.2.1, there exists $T > 0$ such that $|x(t) - \hat{x}(t)| < \epsilon$ for $t \in \tau_\alpha(\epsilon) \cap (T, \infty)$. Trivially, it holds that $T \in [\tau_j, \tau_j')$ for the unique $j \in \langle\tau\rangle$. We set $T_a := \tau_j'$ and show that (a) holds when $T^* = T_a$. We divide $t > T_a$ into

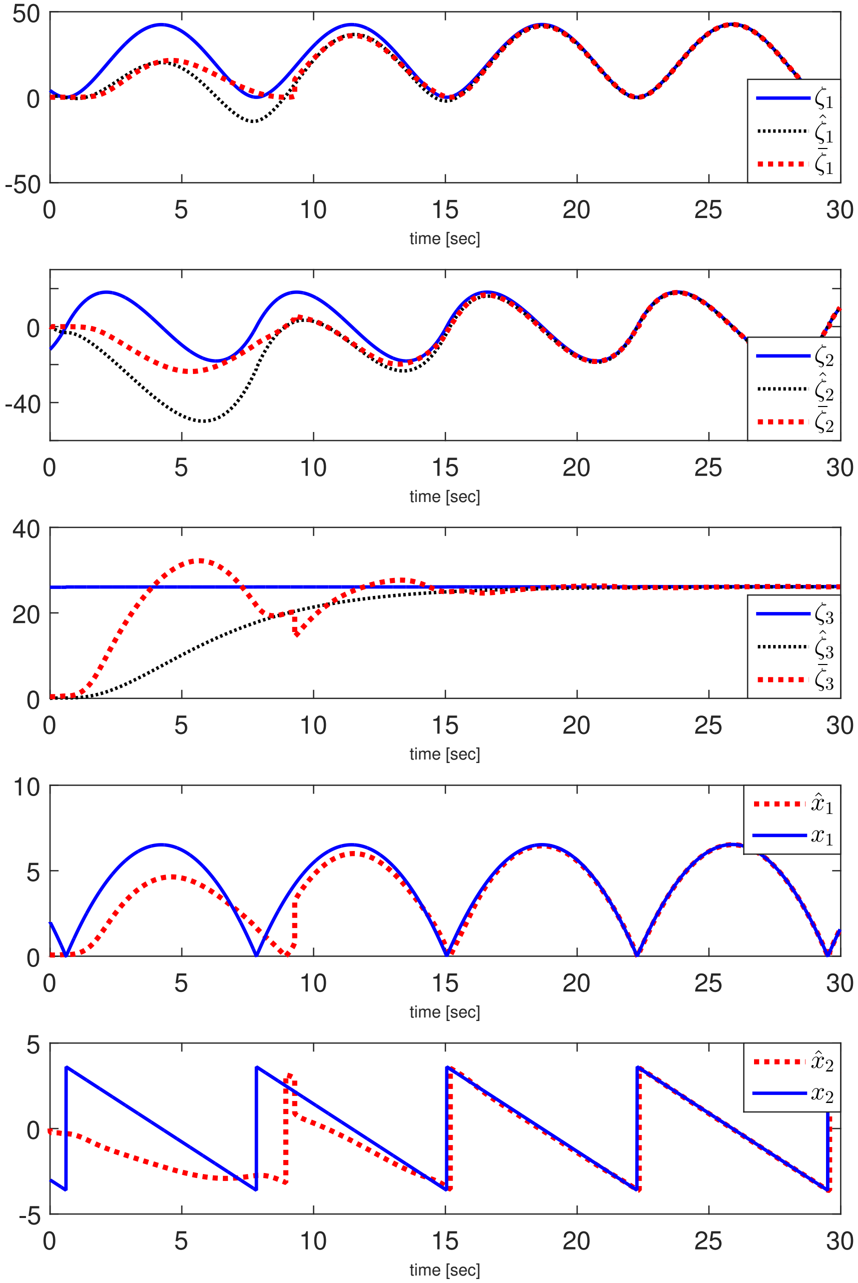

Figure 5.1: A simulation result of Example 5.2.1 when $\rho = 1$, $x_0 = (2, -3)$, and $L = [-1.80 \quad -0.95 \quad -0.15]^\top$. The first three: ζ is a state of (5.2.7) and $\hat{\zeta}$ is an estimate of ζ obtained from (5.2.8). $\bar{\zeta}$ is an approximated projection of $\hat{\zeta}$ to $\psi(K)$. The last two: x is a state of (5.2.6) and $\hat{x}$ is an estimate of x obtained from (5.2.9).

three cases: Case 1: $t \in (T_a, \infty) \cap \tau_\alpha(\epsilon)$, Case 2: $t \in [\tau_i, \tau_i + \alpha(\epsilon))$ for $i > j$, and Case 3: $t \in (\tau_i' - \alpha(\epsilon), \tau_i')$ for $i > j$.

In Case 1, take $s = t$. Then, it holds that $|(t, x(t)) - (s, \hat{x}(s))| = |x(t) - \hat{x}(t)| < \epsilon \le \frac{\epsilon^*}{4} < \epsilon^*$. In Case 2, take $s = \tau_i + \alpha(\epsilon)$. Then, we have that $|s - t| \le \alpha(\epsilon) \le \frac{\epsilon^*}{2}$ and

$$\begin{aligned} |x(t) - \hat{x}(s)| &\le |x(t) - x(\tau_i + \alpha(\epsilon))| + |x(\tau_i + \alpha(\epsilon)) - \hat{x}(\tau_i + \alpha(\epsilon))| \\ &< M\alpha(\epsilon) + \epsilon \\ &\le \frac{\epsilon^*}{2}. \end{aligned}$$

Therefore, it follows that $|(t, x(t)) - (s, \hat{x}(s))| < \epsilon^*$. Similarly, in Case 3, it is satisfied that $|(t, x(t)) - (s, \hat{x}(s))| < \epsilon^*$ when we take $s = \tau_i - \alpha(\epsilon)$.

(b) Consider the sufficiently small ϵ^* satisfying the condition of ϵ given in Theorem 5.2.1. Note that $\mathcal{O}_{\mathcal{D}}(\epsilon^*) \cap \mathcal{O}_{\mathcal{G}}(\epsilon^*) = \varnothing$. As shown in the proof of Theorem 5.2.1, by Lemma 4.3.2, we can take $L_1(\epsilon^*) > 0$ and $L_2(\epsilon^*) > 0$ such that $|x - x'| < L_1|\psi(x) - \psi(x')|$ for all $x, x' \in K \backslash \mathcal{O}_{\mathcal{D}}(\epsilon^*)$ and $|x - x'| < L_2|\psi(x) - \psi(x')|$ for all $x, x' \in K \backslash \mathcal{O}_{\mathcal{G}}(\epsilon^*)$.

Let $\epsilon := \min\left(\epsilon^*, \alpha^{-1}(\frac{\epsilon^*}{4}), \alpha^{-1}(\frac{\epsilon^*}{4LM^\psi}), \alpha^{-1}(\frac{\epsilon^*}{2M})\right)$ where $M^\psi := \max_{x \in K} |\mathrm{d}\psi(x) f(x)|$ and $L := 2 \cdot \max(L_1, L_2)$. Then, by Theorem 5.2.1, there exist $T_\zeta > 0$ and $T > 0$ such that $|\zeta(t) - \Pi_{\psi(K)}(\hat{\zeta}(t))| < 2\gamma = \frac{\epsilon^*}{2L}$ for $t > T_\zeta$ and $|x(t) - \hat{x}(t)| < \epsilon$ for $t \in \tau_\alpha(\epsilon) \cap (T, \infty)$. Trivially, there exists the unique $j \in \langle \tau \rangle$ such that $\max(T_\zeta, T) \in [\tau_j, \tau_j')$. We set $T_b := \tau_j' + \alpha(\epsilon)$ and show that (b) holds when $T^* = T_b$. We divide $t > T_b$ into three cases: Case 1: $t \in (T_b, \infty) \cap \tau_\alpha(\epsilon)$, Case 2: $t \in (T_b, \infty) \backslash \tau_\alpha(\epsilon)$ and $\hat{x}(t) \in K \backslash \mathcal{O}_{\mathcal{D}}(\epsilon^*)$, and Case 3: $t \in (T_b, \infty) \backslash \tau_\alpha(\epsilon)$ and $\hat{x}(t) \in K \backslash \mathcal{O}_{\mathcal{G}}(\epsilon^*)$.

In Case 1, take $s = t$. Then, $|(s, x(s)) - (t, \hat{x}(t))| = |x(t) - \hat{x}(t)| < \epsilon \le \epsilon^*$. In Case 2, let $s = \tau_{i^*} + \alpha(\epsilon)$ where i^* is the positive integer (larger than $j + 1$) such that $t \in (\tau_{i^*} - \alpha(\epsilon), \tau_{i^*} + \alpha(\epsilon))$. Since $|s - t| < 2\alpha(\epsilon) \le \frac{\epsilon^*}{2}$, we just show that

$|x(s) - \hat{x}(t)| \leq \frac{\epsilon^*}{2}$ for the proof of this case. Since $t > T_\zeta$, it is satisfied that

$$\begin{aligned}|\zeta(s) - \Pi_{\psi(K)}(\hat{\zeta}(t))| &\leq |\zeta(\tau_{i^*} + \alpha(\epsilon)) - \zeta(t)| + |\zeta(t) - \Pi_{\psi(K)}(\hat{\zeta}(t))| \\ &< 2M^\psi \alpha(\epsilon) + \frac{\epsilon^*}{2L} \\ &\leq \frac{\epsilon^*}{L}.\end{aligned}$$

In addition, since $|x(s) - x(\tau_{i^*})| = |x(\tau_{i^*} + \alpha(\epsilon)) - x(\tau_{i^*})| \leq M\alpha(\epsilon) \leq \frac{\epsilon^*}{2}$ and $x(\tau_{i^*}) \in \mathcal{G}$, we have that $x(s) \in \mathcal{O}_\mathcal{G}(\epsilon^*) \subset K \backslash \mathcal{O}_\mathcal{D}(\epsilon^*)$. Therefore, it follows that

$$|x(s) - \hat{x}(t)| < L_1|\psi(x(s)) - \psi(\hat{x}(t))| < L_1|\zeta(s) - \Pi_{\psi(K)}(\hat{\zeta}(t))| < \frac{\epsilon^*}{2}.$$

In a similar way, we can prove Case 3 by setting $s := \tau_{i^*} - \alpha(\epsilon)$.

Finally, let $T^* := \max(T_a, T_b)$. Then the proof is complete. □

5.3 Observer with Linearized Error Dynamics

In this section and next section, we deal with the observer design for the glued system. Since the glued system is a continuous-time system, we may apply conventional observer design techniques proposed for the continuous-time systems. However, most of them require the additional properties of f^ψ and h^ψ such as linearity, smoothness, or Lipschitz continuity. Therefore, we investigate the condition to guarantee such properties and propose the observer designs for the glued system.

We first propose the condition for the existence of the gluing function which guarantees that the glued system becomes a linear system with output injection. To simplify the presentation, we only consider the case when the output dimension is 1. The idea is related to the conventional technique proposed in [BS04, MT16]. For the system $\mathcal{H}^h$, suppose that f and h are smooth and that there exists a positive integer $m \geq k$ such that

$$L_f^i h(x) = L_f^i h(x_g)|_{x_g = g(x)} \text{ for all } 0 \leq i \leq m \text{ and for all } x \in \mathcal{D}.$$

This implies that the output trajectory is of class C^m with respect to t. From the above condition, we may obtain the gluing function we desire. In fact, this condition is restrictive, but can be relaxed by introducing an auxiliary output $y^* = h^*(x) := \phi(h(x))$ such that $\phi : h(\mathcal{C}) \to \mathbb{R}$ is injective and smooth.

Assumption 5.3.1. f and h are smooth and the output dimension is 1 ($q = 1$). In addition, there exist an injective smooth function $\phi : h(\mathcal{C}) \to \mathbb{R}$ and an integer m not smaller than the manifold dimension k of $\mathcal{C}$ such that with $h^* = \phi \circ h$

(I1) $L_f^i h^*(x) = L_f^i h^*(x_g)|_{x_g = g(x)}$ for $1 \le i \le m$ and for all $x \in \mathcal{D}$,

(I2) there exists a smooth solution $a_1(h^*), \ldots, a_m(h^*)$ to the differential equation

$$L_f^m h^* = a_m(h^*) + L_f a_{m-1}(h^*) + \cdots + L_f^{m-1} a_1(h^*).$$

□

Theorem 5.3.1. Suppose that Assumptions 4.3.1, 5.1.1, and 5.3.1 hold. If $\psi(x) := (h^*(x), L_f h^*(x) - a_1(y^*)|_{y^*=h^*(x)}, \cdots, L_f^{m-1} h^*(x) - \sum_{i=1}^{m-1} L_f^{m-i-1} a_i(y^*)|_{y^*=h^*(x)})$ satisfies (G2) and (G4), then ψ is a gluing function satisfying (4.3.7). Furthermore, the glued system is

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 + a_1(\zeta_1) \\ &\vdots \\ \dot{\zeta}_{m-1} &= \zeta_m + a_{m-1}(\zeta_1) \\ \dot{\zeta}_m &= a_m(\zeta_1) \\ y^* &= \phi(y) = \zeta_1. \end{aligned}$$

□

Proof of Theorem 5.3.1. Since $a_1, \ldots, a_{m-1}$, f, and h^* are smooth, (G3) holds. In addition, by (E3) and (I1), we obtain that (G1) also holds and that ψ is a gluing function satisfying (4.3.7). By the construction, it follows that the equations for $\dot{\zeta}_1, \ldots, \dot{\zeta}_{m-1}$ hold. Finally, we have that $\dot{\zeta}_m = a_m(\zeta_1)$ using (I2). □ □

Remark 5.3.1. In Theorem 5.3.1, the glued system is a linear systems up to

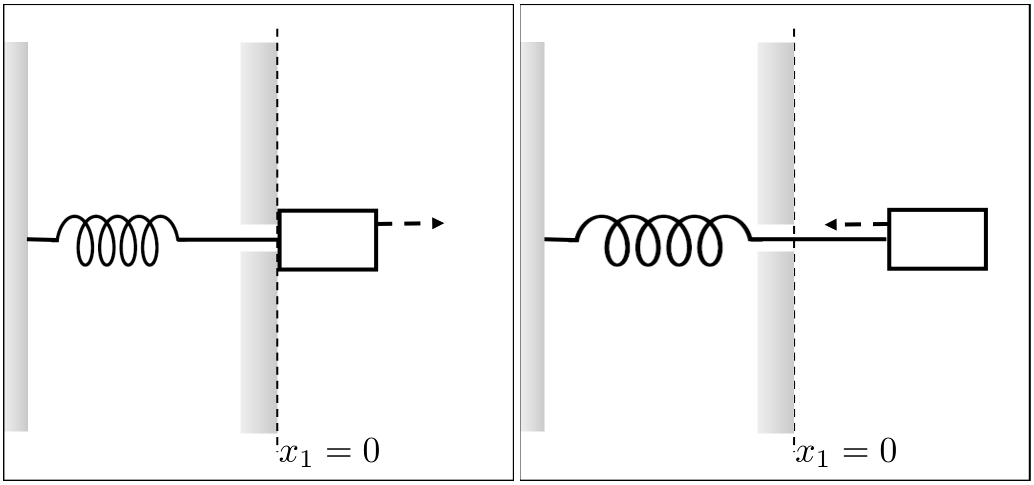

Figure 5.2: One-degree of freedom system having impacts.

output injection. In this case, we can design the observer for $\mathcal{H}^h$ as

$$\dot{\hat{\zeta}} = A\hat{\zeta} + L(C\hat{\zeta} - y^*) + a(y^*)$$

where $A := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$, $C := \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$, $a(y^*) := \begin{bmatrix} a_1(y^*) \\ \vdots \\ a_m(y^*) \end{bmatrix}$, and $A + LC$ is Hurwitz. Notice that we use the auxiliary output $y^* = \phi(y)$ because $h^* = \phi \circ h$. $\square$

Example 5.3.1. (One-degree of freedom system with impact)**.** Consider the simple mechanical system with impact in Figure 5.2. Suppose that the impacts is modeled perfectly elastic and that its initial condition is in some compact set not including the origin. In addition, the output is the position. Then, the system

$\mathcal{H}^h = (\mathcal{C}, f, \mathcal{D}, g, h)$ and interested domain K are obtained as

$$\begin{aligned} \dot{x} &= f(x) := \begin{bmatrix} x_2 \\ -cx_1 \end{bmatrix} && \text{when } x \in \mathcal{C} := \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 \geq 0) \wedge (|x| > 0)\}, \\ x^+ &= g(x) := -x && \text{when } x \in \mathcal{D} := \{(x_1, x_2) \in \mathcal{C} : (x_1 = 0) \wedge (x_2 < 0)\}, \\ y &= h(x) := x_1, \end{aligned} \tag{5.3.1}$$

and $K := \{x \in \mathcal{C} : \underline{\delta} \leq M(x) := \frac{1}{2}cx_1^2 + \frac{1}{2}x_2^2 \leq \overline{\delta}\}$ with $0 < \underline{\delta} < \overline{\delta}$, respectively, where x_1 is the position and x_2 is the velocity.

At first, $\phi(y) := y$. Then, since

$$L_f h(x) = x_2 \text{ and } L_f h(x_g)|_{x_g = g(x)} = -x_2,$$

(I1) is not satisfied. To cope with this, take $\phi(y) := y^2$, which is injective and smooth. Then, with $h^* := \phi \circ h$, since

$$L_f h^*(x) = x_1 x_2 \text{ and } L_f h^*(x_g)|_{x_g = g(x)} = (-x_1)(-x_2),$$

it follows that

$$L_f h^*(x) = L_f h^*(x_g)|_{x_g = g(x)}.$$

In addition, since

$$L_f^2 h^*(x) = -2cx_1^2 + 2x_2^2 \text{ and } L_f^2 h^*(x_g)|_{x_g = g(x)} = -2c(-x_1)^2 + 2(-x_2)^2$$

we obtain that

$$L_f^2 h^*(x) = L_f^2 h^*(x_g)|_{x_g = g(x)}.$$

Finally, since

$$L_f^3 h^*(x) = -4cx_1x_2 - 4cx_1x_2 \text{ and } L_f^3 h^*(x_g)|_{x_g=g(x)} = -4c(-x_1)(-x_2) - 4c(-x_1)(-x_2),$$

(I2) holds. To summarize, we have that

$$\begin{aligned} L_f h^* &= 2x_1x_2, \\ L_f^2 h^* &= -2cx_1^2 + 2x_2^2, \\ L_f^3 h^* &= -8cx_1x_2 \end{aligned}$$

and the equation

$$L_f^3 h^* = -4cL_f h^*,$$

holds, which implies (I2) with $a_1 = a_3 = 0$ and $a_2(y^*) = -4cy^*$. Therefore, by Theorem 5.3.1, we can take $\psi(x) := (x_1^2, 2x_1x_2, 2cx_1^2 + 2x_2^2)$, which satisfies (G2) and (G4), and the vector field matching condition (4.3.7) holds. Figure 5.3 depicts the interested domain K and the glued one $\psi(K)$. The inverse gluing function is obtained as

$$\psi^{-1}(\zeta) = \begin{bmatrix} \sqrt{\zeta_1} \\ \mathrm{sgn}(\zeta_2)\sqrt{\frac{1}{2}\zeta_3 - \zeta_1} \end{bmatrix} \text{ for all } \zeta \in \mathcal{C}^\psi.$$

With $\zeta := \psi(x)$, the glued system is written as

$$\begin{aligned} \dot\zeta_1 &= \zeta_2 \\ \dot\zeta_2 &= \zeta_3 - 4c\zeta_1 \\ \dot\zeta_3 &= 0 \\ y^* &= \zeta_1 = y^2, \end{aligned}$$

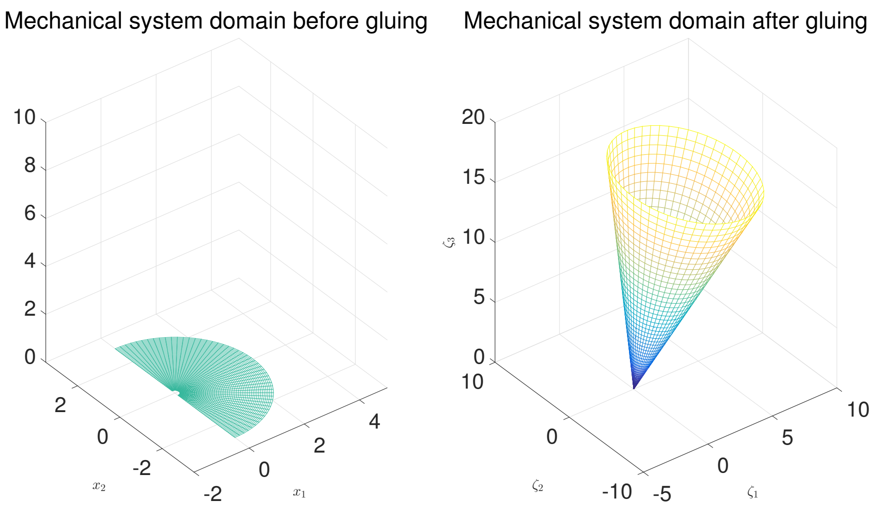

Figure 5.3: Domains of the one-degree of freedom system (5.3.1) before gluing K (left) and after gluing $\psi(K)$ using $\psi = (x_1^2, 2x_1x_2, 2cx_1^2 + 2x_2^2)$ (right) when $\underline{\delta}= 0.01$, $\overline{\delta} = 4$, and $c = 1$.

and the proposed observer is following:

$$\begin{aligned}\dot{\hat{\zeta}} &= A\hat{\zeta} + L(C\hat{\zeta} - y^*) + a(y^*) \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \hat{\zeta} + L\left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \hat{\zeta} - y^2\right) + \begin{bmatrix} 0 \\ -4cy^2 \\ 0 \end{bmatrix}\end{aligned} \tag{5.3.2}$$

where $A + LC$ is Hurwitz. In fact, it needs a projection map (5.2.4) to obtain $\hat{x}$ from $\hat{\zeta}$. The process finding it is omitted here. A simulation result is shown in Figure 5.4. Note that this observer form is also proposed in [MT16]. □

Although we relax the condition by introducing the auxiliary output, it is still restrictive. Therefore, we propose a less restrictive condition.

5.4 Observer for Lipschitz Continuous Systems

If the glued system is Lipschitz continuous, which means f^ψ and h^ψ are Lipschitz continuous on the domain, we may employ an observer proposed in [KE03]. However, f^ψ and h^ψ are complicated functions of ζ in general. In addition, it is

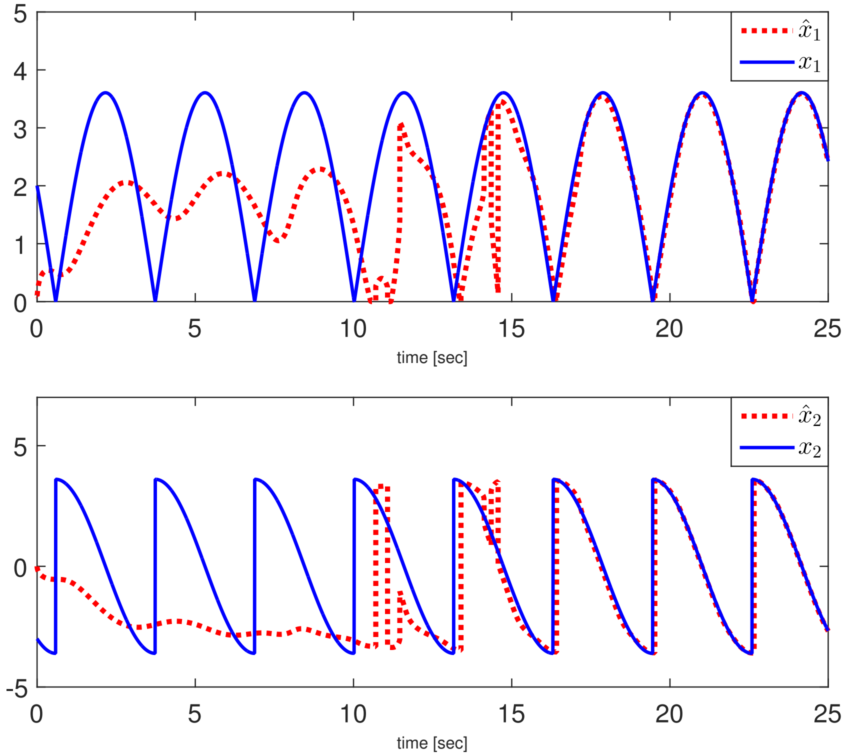

Figure 5.4: A simulation result of Example 5.3.1 when $c = 1$. The state of (5.3.1) is $x = (x_1, x_2)$ and its estimate is $\hat{x} = (\hat{x}_1, \hat{x}_2)$ obtained from (5.3.2) when $L = [-1.8 \quad -0.95 \quad -0.15]^\top$.

tedious tasks to obtain them as the functions of ζ from the equations (5.2.1) and (5.2.2). Therefore, it is difficult to check their Lipschitz continuities through f^ψ and h^ψ and we propose the conditions guaranteeing Lipschitz continuity of the glued system without explicitly obtaining them. We first impose some additional conditions on f, h, and ψ.

Assumption 5.4.1. Suppose that f and h are of class C^1 and that $k = n$. Moreover, under Assumption 4.3.1 and Assumption 5.1.1, there exists a C^2 gluing function $\psi : \mathcal{C} \to \mathbb{R}^n$ satisfying the vector field matching condition (4.3.7). □

Notice that $\mathcal{C}$ is a smooth n-manifold with boundary embedded in $\mathbb{R}^n$ and the gluing function is a local C^2 embedding from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Theorem 5.4.1. Under Assumption 4.3.1, Assumptions 5.1.1–5.1.2, and Assumption 5.4.1, f^ψ and h^ψ defined in (5.2.1) and (5.2.2) are Lipschitz continuous on $\psi(K)$. □

Proof of Theorem 5.4.1. Take $\zeta \in \mathcal{C}$. Then, either $\zeta \in \mathcal{C}\backslash\psi(\mathcal{D})$ or $\zeta \in \mathcal{C} \cap \psi(\mathcal{D})$ is satisfied. At first, consider the case when $\zeta \in \mathcal{C}\backslash\psi(\mathcal{D})$ and take $x := \psi^{-1}(\zeta)$. Since ψ is a local C^2 embedding, there exists an open neighborhood U of x in $\mathcal{C}$ such that $\psi|_U : U \to \mathbb{R}^n$ is a C^2 diffeomorphism. Therefore, since $\mathrm{d}\psi$ and f are C^1, $f^\psi(\zeta) = \mathrm{d}\psi(\psi^{-1}(\zeta))f(\psi^{-1}(\zeta))$ is C^1 on $\zeta(U)$. Notice that C^1 means there exists an open neighborhood of the domain so that its extension is C^1. Therefore, f^ψ is locally Lipschitz at $\zeta \in \mathcal{C}\backslash\psi(\mathcal{D})$.

Secondly, consider the case when $\zeta \in \psi(\mathcal{D})$ and take $y := \psi^{-1}(\zeta)$ and $x := g^{-1}(y)$. Then, there exist U_x and U_y such that $\psi_1 := \psi|_{U_x}$ and $\psi_2 := \psi|_{U_y}$ are C^2 diffeomorphism. Since ζ is interior point of $\mathcal{C}^\psi$, there exists an open ball V such that $V \subset \psi(U_x \cup U_y)$. Let us consider $\zeta_1, \zeta_2 \in V$. Then, without loss of generality, the one of the following three cases holds:

- $\zeta_1, \zeta_2 \in \psi(U_x)$.
- $\zeta_1, \zeta_2 \in \psi(U_y)$.
- $\zeta_1 \in \psi(U_x)$ and $\zeta_2 \in \psi(U_y)$.

For the first and second case, it follow from the before claim that, for some $L > 0$,

$$|f^{\psi}(\zeta_1) - f^{\psi}(\zeta_2)| \le L|\zeta_1 - \zeta_2|. \tag{5.4.1}$$

Since V is divided into two regions $\psi(U_x) \cap V$ and $\psi(U_y) \cap V$ by $\psi(\mathcal{D})$ and the last case implies that ζ_1 and ζ_2 are placed at the different regions, there exists at least one $\zeta^* \in \psi(\mathcal{D}) \cap l(\zeta_1, \zeta_2) \cap V$, where $l(\zeta_1, \zeta_2)$ is the line segment whose end points are ζ_1 and ζ_2. Note that since V is a open ball, ζ^* is always in V. Let $x^* := g^{-1}(\psi^{-1}(\zeta^*))$. Then, for some $L_0(\zeta), L_1(\zeta), L_2(\zeta) > 0$,

$$\begin{aligned}
|f^{\psi}(\zeta_1) - f^{\psi}(\zeta_2)| &\le |f^{\psi}(\zeta_1) - \mathrm{d}\psi(x^*)f(x^*)| + |\mathrm{d}\psi(g(x^*))f(g(x^*)) - f^{\psi}(\zeta_2)| \\
&= |\mathrm{d}\psi(\psi_1^{-1}(\zeta_1))f(\psi_1^{-1}(\zeta_1)) - \mathrm{d}\psi(\psi_1^{-1}(\zeta^*))f(\psi_1^{-1}(\zeta^*))| \\
&\quad + |\mathrm{d}\psi(\psi_2^{-1}(\zeta^*))f(\psi_2^{-1}(\zeta^*)) - \mathrm{d}\psi(\psi_2^{-1}(\zeta_2))f(\psi_2^{-1}(\zeta_2))| \\
&\le L_0(\zeta)\left(|\psi_1^{-1}(\zeta_1) - \psi_1^{-1}(\zeta^*)| + |\psi_2^{-1}(\zeta^*) - \psi_2^{-1}(\zeta_2)|\right) \\
&\le L_0(\zeta)\left(L_1(\zeta)|\zeta_1 - \zeta^*| + L_2(\zeta)|\zeta^* - \zeta_2|\right) \\
&\le L_0(\zeta) \cdot \max(L_1(\zeta), L_2(\zeta))|\zeta_1 - \zeta_2|.
\end{aligned} \tag{5.4.2}$$

Consequently, by (5.4.1)-(5.4.2), f^{ψ} is locally Lipschitz at every $\zeta \in \psi(\mathcal{D})$ on $\mathcal{C}^{\psi}$. Therefore f^{ψ} is locally Lipschitz on $\mathcal{C}^{\psi}$.

Since K is compact and ψ is C^2, $\psi(K)$ is also compact. Therefore, it follows that f^{ψ} is Lipschitz continuous on $\psi(K) \subset \mathcal{C}^{\psi}$. In the similar way, we can show that h^{ψ} is also Lipschitz continuous on $\psi(K)$. □

By using the theorem, we propose another observer design approach for the bouncing ball system in Example 5.2.1 via the gluing function ψ_3 in Example 4.3.3.

Example 5.4.1. Consider the bouncing ball system with an output (5.2.6) in Example 5.2.1 and take a gluing function as $\psi := \psi_3$ in Example 4.3.3. Since ψ satisfies the vector field matching condition (4.3.7), the glued system on $\mathcal{C}^{\psi}$ is obtained as

$$\begin{aligned}\dot{\zeta} &= f^{\psi}(\zeta) := \mathrm{d}\psi\big(\psi^{-1}(\zeta)\big)f\big(\psi^{-1}(\zeta)\big),\\ y &= h^{\psi}(\zeta) := h\big(\psi^{-1}(\zeta)\big) = \sqrt{\tfrac{1}{2}\|\zeta\|(\|\zeta\|+\zeta_1)}.\end{aligned} \tag{5.4.3}$$

Note that we omit the concrete expression of $f^{\psi}(\zeta)$ in (5.4.3) because it is complex. However, since the system trivially satisfies Assumption 5.4.1, by Theorem 5.4.1, we obtain its Lipschitz continuity without the expression.

Now we employ the observer presented in [KE03], which requires Lipschitz continuities of f^{ψ} and h^{ψ} on $\psi(K)$, and a certain observability property. Let us check the observability in the following sense: the system (5.4.3) is said to be observable in $\psi(K)$, if there exists a class-$\mathcal{K}$ function κ such that, for all $(\zeta^1, \zeta^2) \in \psi(K) \times \psi(K)$,

$$\|Y_1(t) - Y_2(t)\|_{(-\infty,0]} \geq \kappa\big(\|\zeta^1 - \zeta^2\|\big),$$

where $\|\cdot\|_{(-\infty,0]}$ is $\mathcal{L}_2$-norm on $(-\infty, 0]$; $Y_i(t) := e^{\sigma t}h^{\psi}\big(\zeta_i(t)\big)$; $\zeta_i(t)$ is a solution starting at ζ^i; $\sigma > 0$ is a constant. Since $\psi(K)$ is compact, according to the results in [KE03], it is enough to check, for any $\zeta^1, \zeta^2 \in \psi(K)$ such that $\zeta^1 \neq \zeta^2$, two functions $Y_1(\cdot)$ and $Y_2(\cdot)$ are not the same on $(-\infty, 0]$. Pick two different ζ^1 and ζ^2 in $\psi(K)$, and let $x^1 = \psi^{-1}(\zeta^1)$ and $x^2 = \psi^{-1}(\zeta^2)$. Then, $x^1 \neq x^2$ in $K \backslash \mathcal{D}$. It is then clear from the behavior of the bouncing ball that, for each $x^i \in K$, its state trajectory $x^i(\cdot)$ is defined on $(-\infty, 0]$ and $h(x^1(\cdot)) \neq h(x^2(\cdot)$ on $(-\infty, 0]$. Thus, at some $t \in (-\infty, 0]$, we have $Y_1(t) = e^{\sigma t}h(x^1(t)) \neq e^{\sigma t}h(x^2(t)) = Y_2(t)$. Hence, the system (5.4.3) is observable in $\psi(K)$.

Following the recipe of [KE03], the observer for the system (5.4.3) is constructed as follows[1]:

1. Choose a positive integer l and construct a controllable matrix pair (F, b) with $F \in \mathbb{R}^{l\times l}$ and $b \in \mathbb{R}^{l\times 1}$ where F is sufficiently stable; for our case, we

[1]See [KE03, Lin99] for more details.

have chosen

$$F = \begin{bmatrix} -1 & 0 & 0 \\ -2\sqrt{2} & -2 & 0 \\ -2\sqrt{2} & -4 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} \sqrt{2} \\ 2 \\ 2 \end{bmatrix}.$$

2. Choose design parameters $T_0 > 0$, $\varepsilon > 0$, and $d > 0$.

3. Consider a grid set (of the interval d) in $\psi(K)$ defined as $\{\zeta = [di, dj]^\top \in \psi(K) : i \in \mathbb{Z}, j \in \mathbb{Z}\}$, and let ζ^i, $i = 1, \cdots, M$, be the elements of the set where M is the cardinality of the set.

4. Compute $q^i := \int_{-T_0}^{0} e^{-Ft} b h^\psi\big(\zeta_i(t)\big) dt$ for $i = 1, \cdots, M$ where $\zeta_i(t)$ is the solution of (5.4.3) with the initial condition ζ^i.

5. Define a function $Q : \mathbb{R}^l \to \mathbb{R}^n$ where n is the dimension of (5.4.3):

$$Q(z) := \frac{\sum_{i=1}^{M} \zeta^i / \big[\varepsilon + \|q^i - z\|\big]^{n+2}}{\sum_{i=1}^{M} 1 / \big[\varepsilon + \|q^i - z\|\big]^{n+2}}.$$

6. The observer is then now constructed as

$$\begin{aligned} \dot{z}(t) &= Fz(t) + by(t), \\ \hat{\zeta}(t) &= Q\big(z(t)\big), \\ \hat{x}(t) &= \psi^{-1}\big(\Pi_{\mathcal{C}^\psi}(\hat{\zeta}(t))\big). \end{aligned}$$

For this, the explicit form of (5.4.3) may not be necessary because one can solve $x_i(t)$ starting at $\psi^{-1}(\zeta^i)$ in $\mathcal{C}$ using (5.2.6) and compute q^i correspondingly, or convert $x_i(t)$ into $\zeta_i(t)$ by ψ.

According to [KE03], the estimate $\hat{\zeta}(t)$ approximates the true glued flow $\zeta(t)$ with the accuracy $\gamma \approx \max_{1 \le i \le M} \|Q(q^i) - \zeta^i\|$ in the sense that

$$\limsup_{t \to \infty} \|\zeta(t) - \hat{\zeta}(t)\| \le \gamma,$$

and the accuracy depends on the design parameters. In fact, γ can be made suffi-

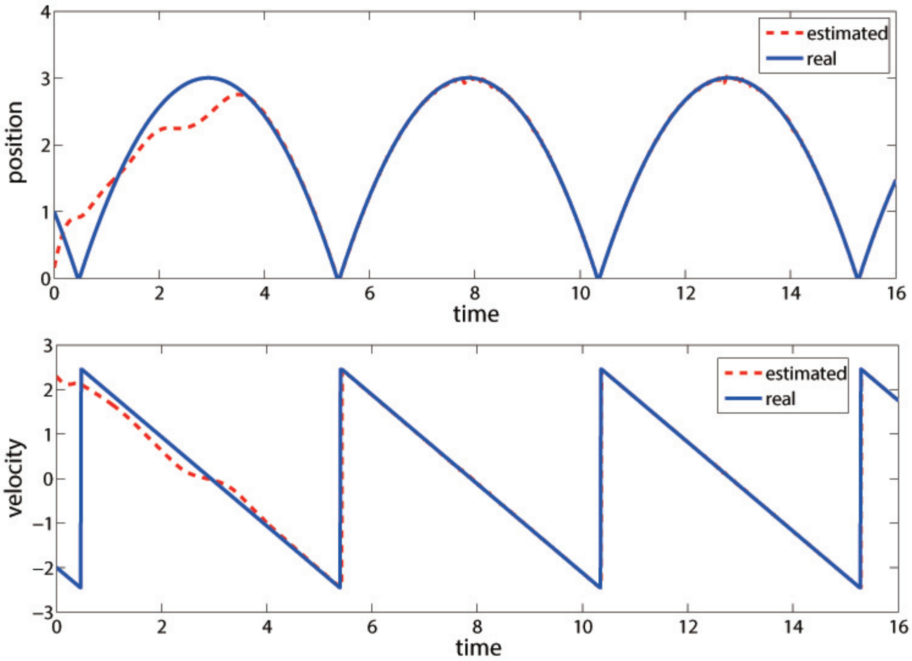

Figure 5.5: Real and estimated states of the bouncing ball system

ciently small by taking ε and d to be small enough. Finally, $\hat{x}(t)$ reconstructs the true flow $x(t)$ on $\mathcal{C}$ with the estimation accuracy $\epsilon \approx \max_{1\leq i\leq M} \|\psi^{-1}(Q(q^i)) - \psi^{-1}(\zeta^i)\|$. For our case of (5.4.3) with $(\underline{\delta}, \overline{\delta}) = (0.5, 5)$, we set $(T_0, \varepsilon, d) = (30, 10^{-6}, 0.1)$ which determines the observer accuracy $\epsilon \approx 7.2 \times 10^{-13}$ and a simulation result is illustrated in Figure 5.5. □

In addition, we also construct an observer for a simplified version of the ripple disturbance introduced in Example 5.1.2.

Example 5.4.2. Consider a hybrid system with linear mappings given by

$$\begin{aligned} \dot{x} &= Ax \qquad \text{when } x \in \mathcal{C}, \\ x^+ &= Jx \qquad \text{when } x \in \mathcal{D}, \\ y &= Hx, \end{aligned} \tag{5.4.4}$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\mathcal{C} = \{x \in \mathbb{R}^2 : (|x| > 0) \wedge (h_1 x \geq 0) \wedge (h_2 x \geq 0)\}$, $\mathcal{D} = \{x \in \mathcal{C} : h_1 x = 0\}$, $h_1 = [\sqrt{3}, 1]$, and

$h_2 = [\sqrt{3}, -1]$. Take $K := \{x \in \mathcal{C} : 1 \leq |x| \leq 3\}$. The output matching condition (E3) holds, because $HJx = x_1 = Hx$. The other conditions in Assumption 4.3.1 and Assumptions 5.1.1–5.1.2 can be easily checked.

Intuitively, we construct a gluing function ψ of the system (5.4.4), which makes the corresponding angle be tripled in polar coordinates. Then, $\psi : \mathcal{C} \to \mathbb{R}^2$, $(x_1, x_2) = (\rho\cos\theta, \rho\sin\theta) \mapsto (\zeta_1, \zeta_2) = (\rho\cos 3\theta, \rho\sin 3\theta)$ such that $0_2 \notin \mathcal{C}$. By straightforward calculations, we have

$$\psi(x) = \begin{bmatrix} \frac{4x_1^3}{|x|^2} - 3x_1 \\ -\frac{4x_2^3}{|x|^2} + 3x_2 \end{bmatrix},$$

$$\mathrm{d}\psi(x) = \begin{bmatrix} \frac{x_1^4+6x_1^2x_2^2-3x_2^4}{|x|^4} & \frac{-8x_1^3x_2}{|x|^4} \\ \frac{8x_1x_2^3}{|x|^4} & \frac{3x_1^4-6x_1^2x_2^2-x_2^4}{|x|^4} \end{bmatrix}.$$

Trivially, (G3) is satisfied. Moreover, since $\mathcal{C}$ does not contain the origin, it is not difficult to check that ψ satisfies (G4). Finally, (G1-2) hold because we can find $\mathcal{C}_e$ and $\mathcal{D}_e$ by taking $r_{\mathcal{D}}(x) := -h_1 x$ and $r_{\mathcal{G}}(x) := h_2 x$.

Furthermore, for all $x \in \mathcal{D}_e$, since

$$\mathrm{d}\psi(x)Ax = \mathrm{d}\psi(x)\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6x_1 \end{bmatrix}$$

and

$$\mathrm{d}\psi(Jx)AJx = \mathrm{d}\psi(Jx)\begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6x_1 \end{bmatrix},$$

it holds that

$$\mathrm{d}\psi(x)Ax = \mathrm{d}\psi(Jx)AJx,$$

which guarantees 4.3.7. Therefore, Assumption 5.4.1 holds.

Then, by Theorem 5.4.1, the glued system of (5.4.4) by ψ is a Lipschitz continuous on $\psi(K)$. In fact, the glued system is obtained as, for all $\zeta \in \mathcal{C}^\psi =$

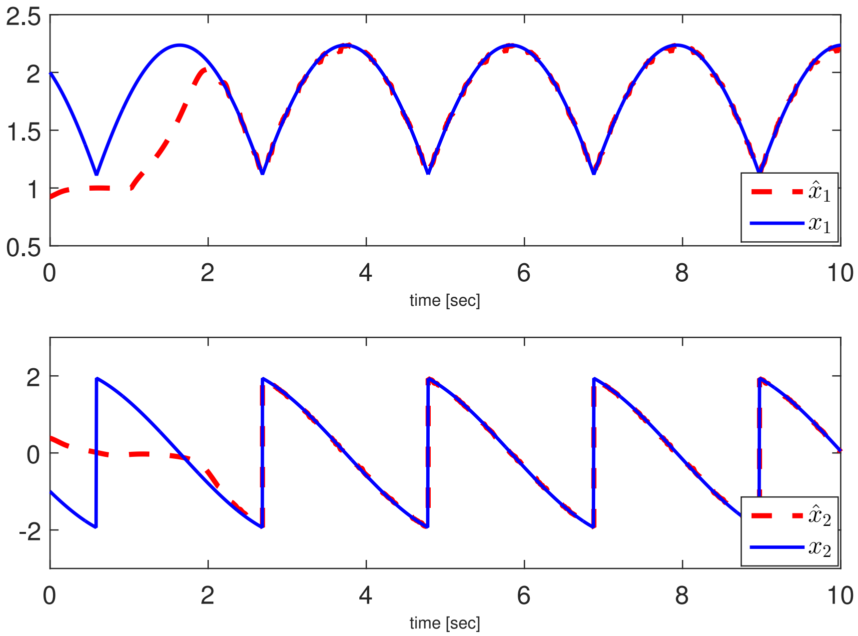

Figure 5.6: Real and estimated states of the system (5.4.4)

$\mathbb{R}^2 \backslash 0_2$,

$$\begin{aligned} \dot{\zeta} &= f^{\psi}(\zeta) = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \zeta, \\ y &= h^{\psi}(\zeta) = |\zeta| \mathrm{Re}\left(\sqrt[3]{\frac{\zeta_1 + i|\zeta_2|}{|\zeta|}} \right), \end{aligned} \tag{5.4.5}$$

where i is the imaginary unit. We construct an observer for the glued system with the interested domain $\psi(K)$ via a similar way of Example 5.4.1. The detail is omitted in this case.

Then, we obtain an estimate $\hat{\zeta}(t)$ for $\zeta(t)$ in the glued domain and an estimate $\hat{x}(t) := \psi^{-1}(\Pi_{\psi(K)}(\hat{\zeta}(t)))$ for $x(t)$ in the original domain. A simulation result is illustrated in Figure 5.6. □

Remark 5.4.1. In Example 5.1.2 with $b \equiv 0$, by taking $(x_1, x_2) = (r\cos\theta, r\sin\theta)$, the ripple generator can be considered as the system (5.4.4) with $h_1 = [1, \sqrt{3}]$ and $h_2 = [1, -\sqrt{3}]$. For this case, we can also develop a similar observer design approach. □

In fact, the observer designs proposed in [KE03] require that the system is Lipschitz continuous and have some additional properties. For example, the numerical observer in [KE03] can be applied, when the glued system is Lipschitz continuous and satisfies an observability condition defined in [KE03]. In this case, the observability condition of the glued system can be easily checked through a modified condition of the hybrid system.

Chapter 6

Tracking Control Strategy

The gluing function is useful in constructing tracking controllers as well as the state observers. In this section, we deal with the state tracking control problem. The goal is to obtain a controller which steers the state trajectory to track a given reference.

6.1 Standing Assumptions

To deal with the tracking control problem, we first need to define input. For example, an input of the bouncing ball system in Example 4.1.1 may be external force.

Definition 6.1.1. A *hybrid system* $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$ *with an input vector field* $\omega(x, u)$ is denoted by $\mathcal{H}_\omega := (\mathcal{C}, f, \mathcal{D}, g, \omega)$ such that

$$\mathcal{H}_\omega \begin{cases} \dot{x} &= f(x) + \omega(x, u) \qquad \text{when } x \in \mathcal{C} \\ x^+ &= g(x) \qquad \text{when } x \in \mathcal{D} \end{cases}$$

where $\omega : \mathcal{C} \times \mathcal{U} \to T\mathcal{C}$ is an *input map* and $\mathcal{U}$ is a subset of $\mathbb{R}^q$. □

Assumption 6.1.1. The hybrid system $\mathcal{H}_\omega = (\mathcal{C}, f, \mathcal{D}, g, \omega)$ satisfies that

(C1) f is locally Lipschitz;

(C2) w is locally Lipschitz.

□

Next a reference the state trajectory should track satisfies the following assumption.

Assumption 6.1.2. For the hybrid system with an input $\mathcal{H}_\omega = (\mathcal{C}, f, \mathcal{D}, g, \omega)$, the reference $r(t)$ satisfies that

(C3) $r(t)$ is a state trajectory of $\mathcal{H}_\omega$ under the input $u = u_{\mathrm{r}}(t)$ such that the execution is infinite in t-direction starting some initial point $r_0 \in \mathcal{C}$ where $u_{\mathrm{r}} : \mathbb{R}_{\geq 0} \to \mathcal{U}$ is piecewise continuous;

(C4) there exists a compact set $\mathcal{R} \subset \mathcal{C}$ such that $r(t) \in \mathcal{R}$ all $t \geq 0$.

□

Since the state remains in $\mathcal{C}$ and may jump on $\mathcal{D}$ via g, it is natural that the reference should be in $\mathcal{C}$ and may jump on $\mathcal{D}$ via g. To guarantees this, (C1) is adopted. Now we consider a closed-loop system under the following dynamic state-feedback controller

$$\begin{aligned} \dot{\eta} &= f_{\mathrm{c}}(t, x, \eta) \\ u &= u_{\mathrm{c}}(t, x, \eta) \end{aligned}$$

where $\eta \in \mathbb{R}^{n_{\mathrm{c}}}$ is the controller state (or $u = u_{\mathrm{c}}(t, x)$ in a static controller case). The closed-loop hybrid system with its state $x_{\mathrm{cl}} := (x, \eta) \in \mathbb{R}^{n+n_{\mathrm{c}}}$ is described as

$$\mathcal{H}_{\mathrm{cl}} \begin{cases} \dot{x}_{\mathrm{cl}} = f_{\mathrm{cl}}(t, x_{\mathrm{cl}}) := \begin{bmatrix} f(x) + \omega(x, u_{\mathrm{c}}(t, x, \eta)) \\ f_{\mathrm{c}}(t, x, \eta) \end{bmatrix} & \text{when } x_{\mathrm{cl}} \in \mathcal{C}_{\mathrm{cl}} := \mathcal{C} \times \mathbb{R}^{n_{\mathrm{c}}}, \\ x_{\mathrm{cl}}^+ = g_{\mathrm{cl}}(x_{\mathrm{cl}}) := \begin{bmatrix} g(x) \\ \eta \end{bmatrix} & \text{when } x_{\mathrm{cl}} \in \mathcal{D}_{\mathrm{cl}} := \mathcal{D} \times \mathbb{R}^{n_{\mathrm{c}}}. \end{cases} \tag{6.1.1}$$

In the static controller case, the closed-loop system is obtained as

$$\mathcal{H}_{\mathrm{cl}} \begin{cases} \dot{x} \;\; = f_{\mathrm{cl}}(t, x) := f(x) + \omega(x, u_{\mathrm{c}}(t, x)) & \text{when } x \in \mathcal{C}, \\ x^+ = g_{\mathrm{cl}}(x) := g(x) & \text{when } x \in \mathcal{D}, \end{cases}$$

because $x_{\mathrm{cl}} = x$. The objective is to construct $(f_{\mathrm{c}}, u_{\mathrm{c}})$ making $x(t)$ track $r(t)$ where $x(t)$ is the plant state part of the state trajectory of $\mathcal{H}_{\mathrm{cl}}$.

6.2 Tracking Control

Suppose that there exists a gluing function of $\mathcal{H}$ satisfying the vector field matching condition (4.3.7) and consider vector fields on $\mathcal{C}^{\psi}$. Then, if the condition

$$\mathrm{d}\psi(x)\omega(x,u) = \mathrm{d}\psi(g(x))\omega(g(x),u) \text{ for all } (x,u) \in \mathcal{D} \times \mathcal{U} \tag{6.2.1}$$

is satisfied, we can take the tangent vector at $\psi(x) \in \mathcal{C}^{\psi}$ as $\mathrm{d}\psi(x)(f(x)+\omega(x,u))$. We call (6.2.1) an *input matching condition*.

In addition, we obtain the glued system as, for all $\zeta \in \mathcal{C}^{\psi} \subset \mathbb{R}^m$,

$$\begin{aligned}\dot{\zeta} &= \mathrm{d}\psi(\psi^{-1}(\zeta))(f(\psi^{-1}(\zeta)) + \omega(\psi^{-1}(\zeta),u)) \\ &=: f^{\psi}(\zeta) + \omega^{\psi}(\zeta,u)\end{aligned} \tag{6.2.2}$$

Note that, f^{ψ} and ω^{ψ} are continuous by (C1), (G3), (4.3.7), (6.2.1), and the Inverse Function Theorem.

Next, we define a glued reference by ψ as

$$\zeta_{\mathrm{r}}(t) := \psi(r(t)) \quad \text{for all } t \geq 0. \tag{6.2.3}$$

Then, $\zeta_{\mathrm{r}}(t)$ is continuous and it is a solution to (6.2.2) when $u = u_{\mathrm{r}}(t)$. For (6.2.2) and (6.2.3), we first suppose that there is a dynamic (or static) feedback controller of the form

$$\begin{aligned}\dot{\eta} &= f_{\mathrm{c}}^{\psi}(t,\zeta_{\mathrm{r}},\zeta,\eta) \\ u &= u_{\mathrm{c}}^{\psi}(t,\zeta_{\mathrm{r}},\zeta,\eta)\end{aligned} \tag{6.2.4}$$

with its state $\eta \in \mathbb{R}^{n_{\mathrm{c}}}$ (or $u = u_{\mathrm{c}}^{\psi}(t,\zeta_{\mathrm{r}},\zeta)$ if it is a static controller). In this case, the closed-loop system becomes

$$\begin{aligned}\dot{\zeta} &= f^{\psi}(\zeta) + \omega^{\psi}(\zeta, u_{\mathrm{c}}^{\psi}(t,\zeta_{\mathrm{r}}(t),\zeta,\eta)) \\ \dot{\eta} &= f_{\mathrm{c}}^{\psi}(t,\zeta_{\mathrm{r}}(t),\zeta,\eta).\end{aligned} \tag{6.2.5}$$

Assumption 6.2.1. For a given $\mathcal{H}_{\omega}$, there exists a gluing function ψ satisfying

(4.3.7) and (6.2.1). In addition, for the given glued reference $\zeta_{\mathrm{r}}(t)$ in (6.2.3), there is a controller (6.2.4) such that

(C5) $f_{\mathrm{c}}^{\psi} : \mathbb{R}_{\geq 0} \times \mathcal{C}^{\psi} \times \mathcal{C}^{\psi} \times \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}}}$ and $u_{\mathrm{c}}^{\psi} : \mathbb{R}_{\geq 0} \times \mathcal{C}^{\psi} \times \mathcal{C}^{\psi} \times \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{p}$ are locally Lipschitz in $(\zeta_{\mathrm{r}}, \zeta, \eta)$ and piecewise continuous in t;

(C6) there exists $\mathcal{V}^{\psi} \times \mathcal{Z}^{\psi} \times \mathcal{E} \subset \mathcal{C}^{\psi} \times \mathcal{C}^{\psi} \times \mathbb{R}^{n_{\mathrm{c}}}$ such that, for each solution $(\zeta(t), \eta(t))$ of the closed-loop system (6.2.5) starting on $\mathcal{V}^{\psi} \times \mathcal{E}$,

$$(\zeta(t), \eta(t)) \in \mathcal{Z}^{\psi} \times \mathbb{R}^{n_{\mathrm{c}}} \text{ and } u_{\mathrm{c}}^{\psi}(t, \zeta_{\mathrm{r}}(t), \zeta(t), \eta(t)) \in \mathcal{U} \qquad \text{for all } t \geq 0;$$

(C7) it holds that

$$\begin{cases} f(x) + \omega(x, u) \notin T_{\mathcal{C}}(x) \text{ for all } (x, u) \in (\mathcal{Z} \cap \mathcal{D}) \times \mathcal{U}, \\ f(x) + \omega(x, u) \in T_{\mathcal{C}}(x) \text{ for all } (x, u) \in (\mathcal{Z} \cap \mathcal{G}) \times \mathcal{U}, \end{cases}$$

where $\mathcal{Z} := \psi^{-1}(\mathcal{Z}^{\psi})$. □

From (C6), the solution is well-defined for all $t \geq 0$. In addition, by (C5), the solution is uniquely defined when it flows. Note that there is a solution trajectory on glued domain such that the trajectory, obtained by detaching through ψ^{-1}, has the jumps from $\mathcal{G}$ to $\mathcal{D}$, which cannot be a solution to the hybrid system. To exclude this case, we impose the condition (C7).

Through the controller in Assumption 6.2.1, we design a controller for $\mathcal{H}_{\omega}$ as

$$\begin{aligned} \dot{\eta} &= f_{\mathrm{c}}(t, x, \eta) := f_{\mathrm{c}}^{\psi}(t, \psi(r(t)), \psi(x), \eta), \\ u &= u_{\mathrm{c}}(t, x, \eta) := u_{\mathrm{c}}^{\psi}(t, \psi(r(t)), \psi(x), \eta). \end{aligned} \tag{6.2.6}$$

Then, the closed-loop hybrid system of $\mathcal{H}_{\omega}$ with (6.2.6) is described as $\mathcal{H}_{\mathrm{cl}}$ in (6.1.1).

Theorem 6.2.1. Under Assumption 4.3.1 and (C1–7), for each $(x_0, \eta_0) \in \psi^{-1}(\mathcal{V}^{\psi}) \times \mathcal{E}$, the maximal execution of $\mathcal{H}_{\mathrm{cl}}$ is infinite in t-direction and unique. Furthermore, suppose that, for each solution to (6.2.5) starting at $(\zeta_0, \eta_0) \in \mathcal{V}^{\psi} \times \mathcal{E}$ and each

$\gamma > 0$, there is $T_\zeta \geq 0$ such that

$$|\zeta(t) - \zeta_{\rm r}(t)| < \gamma \qquad \text{for all } t > T_\zeta. \tag{6.2.7}$$

Then, for each state trajectory $x_{\rm cl}(t) = (x(t), \eta(t))$ of $\mathcal{H}_{\rm cl}$ starting at $(x_0, \eta_0) \in \psi^{-1}(\mathcal{V}^\psi) \times \mathcal{E}$ and for each $\epsilon > 0$, there exists $T > 0$ such that $|x(t) - r(t)| < \epsilon$ for all $t \in \{t > T : \inf_{\theta \in \mathcal{D} \cup \mathcal{G}} |r(t) - \theta| \geq \epsilon\}$. □

Proof of Theorem 6.2.1. We first show the state trajectory $(x(t), \eta(t))$ is infinite in t-direction. Let us take $(x_0, \eta_0) \in \psi^{-1}(\mathcal{V}^\psi) \times \mathcal{E}$. Then, by (C6), there exists a solution $(\zeta(t), \eta(t))$ starting at $(\psi(x_0), \eta_0)$, which is well-defined for all $t \geq 0$. Therefore, it follows from the Inverse Function Theorem of manifolds and (C7) that $(\psi^{-1}(\zeta(t)), \eta(t))$ provides an infinite in t-direction execution of $\mathcal{H}_{\rm cl}$ starting at (x_0, η_0).

Secondly, we show that the execution is unique. Since $\psi(r(t))$ is continuous and ψ is class of C^1, it follows from (C5) that $f_{\rm c}$ and $u_{\rm c}$ of (6.2.6) are locally Lipschitz in (x, η) and piecewise continuous in t. Then, by (C1–2), it follows that $f_{\rm cl}$ of (6.1.1) is piecewise continuous in t and locally Lipschitz in (x, η). Consequently, the flow is uniquely defined before it meets $(\mathcal{Z} \cap \mathcal{D}) \times \mathbb{R}^{n_{\rm c}}$. On this set, by (C7), it will be defined not by flowing but by jumping to the unique point via $g_{\rm cl}$.

Finally, for $\epsilon > 0$, find $T(\epsilon) > 0$ such that $|x(t) - r(t)| < \epsilon$ for all $t \in \{t > T : \inf_{\theta \in \mathcal{D} \cup \mathcal{G}} |r(t) - \theta| \geq \epsilon\}$. Since $\zeta_{\rm r}(t) \in \psi(\mathcal{R})$ for all $t \geq 0$ and $\psi(\mathcal{R})$ is a compact subset of $\mathcal{C}^\psi$, there exists $\gamma > 0$ such that $\mathcal{R}_\gamma^\psi := \{\zeta \in \mathcal{C}^\psi : d_{\psi(\mathcal{R})}(\zeta) \leq \gamma\}$ is a compact subset of $\mathcal{C}^\psi$. Then, by (G5) and the following lemma, $\mathcal{R}_\gamma := \psi^{-1}(\mathcal{R}_\gamma)$ is a compact subset of $\mathcal{C}$.

Lemma 6.2.2. Suppose that under Assumption 4.3.1 there exists a gluing function ψ. Then, ψ is a proper map. □

Proof of Lemma 6.2.2. Let us consider a compact subset $K^\psi \subset \mathcal{C}^\psi$. By (G5), we have that a subset $W \subset \mathcal{C}^\psi$ is closed if and only if $\psi^{-1}(W)$ is closed in $\mathcal{C}$. Since $\mathcal{D} \cup \mathcal{G}$ is closed in $\mathcal{C}$ and $\mathcal{D} \cup \mathcal{G} = \psi^{-1}(\psi(\mathcal{D} \cup \mathcal{G}))$, it follows that $\psi(\mathcal{D} \cup \mathcal{G})$ is closed in $\mathcal{C}^\psi$. Take $K' := K^\psi \cap \psi(\mathcal{D} \cup \mathcal{G})$. Then, K' is compact because it is closed subset of a compact set K^ψ.

Now show that $\psi^{-1}(K') \cap \mathcal{D}$ is compact. Let us consider an open cover $\cup_{\alpha \in A}\{U_\alpha\}$ of $\psi^{-1}(K') \cap \mathcal{D}$. Then, we construct a collection of open sets $\cup_{\alpha \in A}\{V_\alpha\}$ as follows:

- $Z_\alpha := U_\alpha \cap (\mathcal{C} \backslash \mathcal{G})$ (it is an open set in $\mathcal{C}$ because $\mathcal{G}$ is closed in $\mathcal{C}$);
- $V_\alpha := Z_\alpha \cup W_\alpha$ for all $\alpha \in A$ where W_α is an open neighborhood of $g(U_\alpha \cap \mathcal{D})$ satisfying that $W_\alpha \cap \mathcal{D} \cap \mathcal{G} = g(U_\alpha \cap \mathcal{D})$;

Note that W_α always exists because $\mathcal{G}$ is a smooth part of $\partial\mathcal{C}$. In addition, since $\psi^{-1}(\psi(V_\alpha)) = V_\alpha$, by (G5), $\psi(V_\alpha)$ is open in $\mathcal{C}^\psi$. Thus, $\cup_{\alpha \in A}\{\psi(V_\alpha)\}$ is an open cover of K'. Therefore, since K' are compact, there exists a finite set $B \subset A$ such that $\cup_{\alpha \in B}\{\psi(V_\alpha)\}$ is a finite open cover of K'. Then, it follows that $\cup_{\alpha \in B}\{V_\alpha\}$ is a finite open cover of $\psi^{-1}(K') \cap \mathcal{D}$. By the definition of V_α and W_α, we obtain that $\cup_{\alpha \in B}\{U_\alpha\}$ is a finite open cover of $\psi^{-1}(K') \cap \mathcal{D}$. Therefore, $\psi^{-1}(K') \cap \mathcal{D}$ is compact. Similarly, we can show that $\psi^{-1}(K') \cap \mathcal{G}$ is also compact.

Since $\mathcal{D}_K := \psi^{-1}(K') \cap \mathcal{D}$ and $\mathcal{G}_K := \psi^{-1}(K') \cap \mathcal{G}$ are compact, we can take an $\epsilon > 0$ such that

$$\mathcal{O}_\mathcal{D}^K \cap \mathcal{O}_\mathcal{D}^K = \varnothing$$

where $\mathcal{O}_\mathcal{D}^K := \{x \in K : d_{\mathcal{D}_K}(x) < \epsilon\}$ and $\mathcal{O}_\mathcal{G}^K := \{x \in K : d_{\mathcal{G}_K}(x) < \epsilon\}$. Since $\mathcal{O}_\mathcal{D}^K$ and $\mathcal{O}_\mathcal{G}^K$ are open in K, $K^\mathcal{D} := K \backslash \mathcal{O}_\mathcal{D}^K$ and $K^\mathcal{G} := K \backslash \mathcal{O}_\mathcal{G}^K$ are closed. In addition $K^\mathcal{D} \cup K^\mathcal{G} = K$. Since $\psi|_{K^\mathcal{D}}$ is a C^1 embedding, $\psi(K^\mathcal{D})$ is a closed subset of the compact set K^ψ, so that $\psi(K^\mathcal{D})$ is compact. Similarly, we have that $\psi(K^\mathcal{G})$ is a compact. Since $\psi|_{K^\mathcal{D}}$ and $\psi|_{K^\mathcal{G}}$ are C^1 embeddings and their images are compact, we obtain that $K^\mathcal{G}$ and $K^\mathcal{D}$ are compact. Therefore, we have that $K = K^\mathcal{D} \cup K^\mathcal{G}$ is compact. □

It follows from (6.2.7) that, for $x(t)$, there exists t_1 such that $x(t)$ and $r(t)$ are included in the compact set $\mathcal{R}_\gamma$ all $t \geq t_1$. Since $\mathcal{D} \cap \mathcal{G} = \varnothing$ and $\mathcal{R}_\gamma$ is compact, we can take $\epsilon^* > 0$ such that $\mathcal{O}_\mathcal{D}(\epsilon^*) \cap \mathcal{O}_\mathcal{G}(\epsilon^*) = \varnothing$ where

- $\mathcal{O}_\mathcal{D}(\epsilon) := \{x \in \mathcal{R}_\gamma : d_\mathcal{D}(x) < \epsilon\}$;

- $\mathcal{O}_{\mathcal{D}}(\epsilon) := \{x \in \mathcal{R}_\gamma : d_{\mathcal{G}}(x) < \epsilon\}$.

Then, it follows that $\mathcal{R}_\gamma = (\mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{D}}(\epsilon)) \cup (\mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{G}}(\epsilon))$ if $\epsilon \le \epsilon^*$. Let $\delta := \min(\epsilon, \epsilon^*)$. Then, by Lemma 4.3.2, it holds that $\psi|_{\mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{D}}(\delta)}$ is injective and there exists $L_1 > 0$ such that

$$|x_1 - x_2| \le L_1 |\psi(x_1) - \psi(x_2)|$$

for $(x_1, x_2) \in \mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{D}}(\delta) \times \mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{D}}(\delta)$. Likewise, $\psi|_{\mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{G}}(\delta)}$ is injective and there exists $L_2 > 0$ such that

$$|x_1 - x_2| \le L_2 |\psi(x_1) - \psi(x_2)|$$

for $(x_1, x_2) \in \mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{G}}(\delta) \times \mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{G}}(\delta)$. Let $L := \max(L_1, L_2)$. By (6.2.7), there exists $T > t_1 > 0$ such that $|\psi(x(t)) - \zeta_{\mathrm{r}}(t)| < \min(\frac{\delta}{L}, \gamma)$ for all $t > T$. Moreover, it is satisfied that $r(t) \in \mathcal{R}_\gamma \backslash (\mathcal{O}_{\mathcal{D}}(\epsilon) \cup \mathcal{O}_{\mathcal{G}}(\epsilon)) = (\mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{D}}(\epsilon)) \cap (\mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{G}}(\epsilon)) \subset (\mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{D}}(\delta)) \cap (\mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{G}}(\delta))$ when $\inf_{\theta \in \mathcal{D} \cup \mathcal{G}} |r(t) - \theta| \ge \epsilon$. Since $x(t) \in \mathcal{R}_\gamma = (\mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{D}}(\delta)) \cup (\mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{G}}(\delta))$, the one of the following conditions holds:

- $(x(t), r(t)) \in \mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{D}}(\delta) \times \mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{D}}(\delta)$ and $|x(t) - r(t)| \le L_1 |\psi(x(t)) - \psi(r(t))| < \delta$.

- $(x(t), r(t)) \in \mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{G}}(\delta) \times \mathcal{R}_\gamma \backslash \mathcal{O}_{\mathcal{G}}(\delta)$ and $|x(t) - r(t)| \le L_2 |\psi(x(t)) - \psi(r(t))| < \delta$.

Therefore, it holds that $|x(t) - r(t)| < \delta \le \epsilon$ for $t \in \{t > T : \inf_{\theta \in \mathcal{D} \cup \mathcal{G}} |r(t) - \theta| \ge \epsilon\}$.

□

Remark 6.2.1. Sufficient conditions for the asymptotic tracking controller of the glued system are well-known. For example, the condition relying on Lyapunov function is that there exist C^1 function $V : \mathcal{C}^\psi \times \mathcal{C}^\psi \times \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}_{\ge 0}$, functions

$\alpha_1, \alpha_2 \in \mathcal{K}$, and scalar $c < 0$ such that

$$\alpha_1(|\zeta - \zeta_{\mathrm{r}}|) \leq V(z) \leq \alpha_2(|\zeta - \zeta_{\mathrm{r}}|)$$

$$\mathrm{d}V(z) \begin{bmatrix} f^{\psi}(\zeta_{\mathrm{r}}) + \omega^{\psi}(\zeta_{\mathrm{r}}, u_{\mathrm{r}}(t)) \\ f^{\psi}(\zeta) + \omega^{\psi}(\zeta, u_{\mathrm{c}}^{\psi}(t, \zeta_{\mathrm{r}}, \zeta, \eta)) \\ f_{\mathrm{c}}^{\psi}(t, \zeta_{\mathrm{r}}, \zeta, \eta) \end{bmatrix} \leq cV(z)$$

for all $z := (\zeta_{\mathrm{r}}, \zeta, \eta) \in \mathcal{C}^{\psi} \times \mathcal{C}^{\psi} \times \mathbb{R}^{n_{\mathrm{c}}}$ and for all $t \geq 0$. □

Example 6.2.1. Consider $\mathcal{H}_{\omega}$ given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u =: Ax + Bu \quad && \text{when } x \in \{x \in \mathbb{R}^2 : |x_1| \leq \pi\} =: \mathcal{C}, \\ x^+ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x =: Jx && \text{when } x \in \{x \in \mathcal{C} : x_1 = \pi\} =: \mathcal{D}, \end{aligned}$$

where $x = (x_1, x_2)$. Note that the system satisfies Assumption 4.3.1 and Assumption 6.1.1. The reference to be considered is a state trajectory starting from $r_0 = (0, 5)$ under $u = u_{\mathrm{r}}(t)$ where $u_{\mathrm{r}}(t)$ is piecewise continuous and $\int_0^t u_{\mathrm{r}}(s)\mathrm{d}s \leq 1$ for all $t \geq 0$. It implies that $r(t) =: (r_1(t), r_2(t)) \in [-\pi, \pi] \times [4, 6]$ for all $t \geq 0$. Therefore, Assumption 6.1.2 holds.

Finding a suitable gluing function for the system is based on the insight of making the set $\mathcal{C}$ like a rolled paper in $\mathbb{R}^3$. For this purpose, we take $\psi : \mathcal{C} \to \mathbb{R}^3$, $(x_1, x_2) \mapsto (\cos x_1, \sin x_1, x_2)$. It is easily checked that ψ is of class C^1 and satisfies (G1–3) and (G5). In addition, we have that

$$\operatorname{rank}\big(\mathrm{d}\psi(x)\big) = \operatorname{rank}\left(\begin{bmatrix} -\sin x_1 & 0 \\ \cos x_1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2 \text{ for all } x \in \mathcal{C},$$

which implies that (G4) also holds. Therefore, ψ is a gluing function of $\mathcal{H}$. Moreover, since

$$\mathrm{d}\psi(x)Ax = (0, -x_2, 0) = \mathrm{d}\psi(Jx)AJx \qquad \text{for all } x \in \mathcal{D},$$
$$\mathrm{d}\psi(x)B = (0, 0, 1) = \mathrm{d}\psi(Jx)B \qquad \text{for all } x \in \mathcal{D},$$

the condition (6.2.1) holds and we obtain a continuous glued system (6.2.2) as

$$\begin{aligned}\dot{\zeta}_1 &= -\zeta_2\zeta_3,\\ \dot{\zeta}_2 &= \zeta_1\zeta_3,\\ \dot{\zeta}_3 &= u,\end{aligned}$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathcal{C}^\psi$. Note that a glued reference $\zeta_\mathrm{r}(t) = \psi(r(t))$ is a solution to the above system when $u = u_\mathrm{r}(t)$.

For the system and the reference, we propose a static tracking controller satisfying (C5) as

$$u_\mathrm{c}^\psi(t, \zeta_\mathrm{r}, \zeta) = u_\mathrm{r}(t) - \zeta_2\zeta_{\mathrm{r}1} + \zeta_1\zeta_{\mathrm{r}2} - k(\zeta_3 - \zeta_{\mathrm{r}3})$$

with $\zeta(0) \in \mathcal{V}^\psi := \{\zeta \in \mathcal{C}^\psi : 2(1 - \zeta_1\zeta_{\mathrm{r}1} - \zeta_2\zeta_{\mathrm{r}2}) + (\zeta_3 - \zeta_{\mathrm{r}3})^2 < 4\}$ where $k > 0$. Take $z = (z_1, z_2, z_3) := (1 - \zeta_1\zeta_{\mathrm{r}1} - \zeta_2\zeta_{\mathrm{r}2}, \zeta_2\zeta_{\mathrm{r}1} - \zeta_1\zeta_{\mathrm{r}2}, \zeta_3 - \zeta_{\mathrm{r}3})$ and $V(z) := z^\top z$. In fact, $z_1 = 1 - \cos(x_1 - r_1)$ and $z_2 = \sin(x_1 - r_1)$. Therefore, it follows that $z_1^2 + z_2^2 = 2z_1$ and $V(z) = 2z_1 + z_3^2$. Then, from the dynamics of z

$$\begin{aligned}\dot{z}_1 &= z_2z_3,\\ \dot{z}_2 &= (1 - z_1)z_3,\\ \dot{z}_3 &= -z_2 - kz_3,\end{aligned}$$

we have that $\dot{V} = -2kz_3^2$. Since V is non-increasing as t increases, we obtain that $z_3(t) = |\zeta_3(t) - \zeta_{\mathrm{r}3}(t)| \leq \sqrt{V(z(0))} < 2$. Moreover, since $|\zeta_{\mathrm{r}3}(t) - 5| \leq 1$, it follows that $\zeta(t) \in \{\zeta \in \mathcal{C}^\psi : |\zeta_3 - 5| \leq 3\} =: \mathcal{Z}^\psi$ and (C6) holds. For $\mathcal{Z} := \psi^{-1}(\mathcal{Z}^\psi) = [-\pi, \pi] \times [2, 8]$, (C7) trivially holds. Therefore, Assumption 6.2.1 is satisfied. In addition, by LaSalle's theorem, $z_2, z_3 \to 0$ and $z_2^2 = z_1(2 - z_1) \to 0$ as $t \to \infty$. Since $V(z(t)) \leq V(z(0)) < 4 = V|_{z=(2,0,0)}$, we obtain that $z_1 \to 0$.

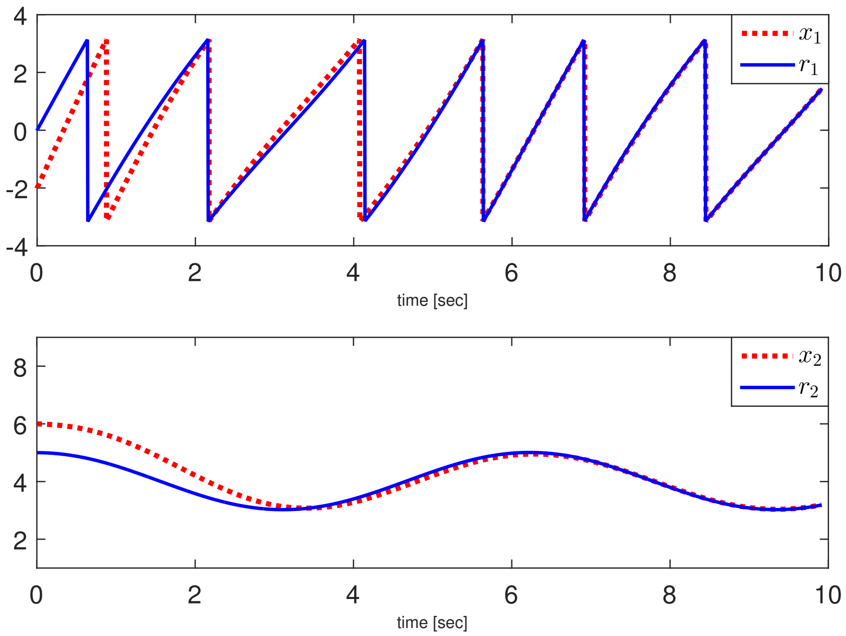

Figure 6.1: State and reference trajectories of $\mathcal{H}_\omega$ in Examples 6.2.1 under (6.2.8) when $x_0 = (-2, 6)$, $r_0 = (0, 5)$, $u_{\mathrm{r}}(t) = -\sin t$, and $k = 1$.

Therefore, it is concluded that $|\zeta(t) - \zeta_{\mathrm{r}}(t)| \to 0$ as $t \to \infty$. Figure 6.1 shows a simulation result under

$$\begin{aligned} u = u_{\mathrm{c}}(t, r, x) &= u_{\mathrm{c}}^{\psi}(t, \psi(r), \psi(x)) \\ &= u_{\mathrm{r}}(t) - \sin(x_1 - r_1) - k(x_2 - r_2) \end{aligned} \tag{6.2.8}$$

with the initial condition x_0 belonging to $\mathcal{V} = \psi^{-1}(\mathcal{V}^{\psi}) = \{(x_1, x_2) \in \mathcal{C} : -2\cos(x_1 - r_1(0)) + (x_2 - r_2(0))^2 < 2\}$. □

6.3 Using Discontinuous Feedback to Counteract Dynamics Jumps

In fact, finding gluing functions of $\mathcal{H}_\omega$ satisfying (6.2.1) may not be a trivial task. If (6.2.1) does not hold, the system in the glued domain may have discontinuous vector fields under a continuous feedback control (even though state jumps disappear), and may be regarded as a (state-triggered) switched system.

Nevertheless, there is a possibility to counteract the discontinuity by feedback control in some cases. This possibility is exploited in this section.

Consider the plant of which the flow map is modeled by an input affine form,

$$f(x) + \omega(x,u) =: a(x) + b(x)u$$

where a and b are locally Lipschitz. Then, the condition (6.2.1) holds when the conditions

$$\mathrm{d}\psi(x)a(x) = \mathrm{d}\psi(g(x))a(g(x)) \quad \text{for all } x \in \mathcal{D}, \tag{6.3.1}$$

$$\mathrm{d}\psi(x)b(x) = \mathrm{d}\psi(g(x))b(g(x)) \quad \text{for all } x \in \mathcal{D}, \tag{6.3.2}$$

hold. In general, the gluing function satisfying both (6.3.1) and (6.3.2) is difficult to find. Furthermore, it may not exist. However, the condition (6.2.1) may be relaxed through a feedback. Suppose that there exist C^1 functions $\gamma(x) : \mathcal{C} \to \mathbb{R}^{p\times p}$ and $\kappa(x) : \mathcal{C} \to \mathbb{R}^p$ such that, for $x \in \mathcal{C}$, $\gamma(x)$ is invertible and

$$\mathrm{d}\psi(x)a(x) + \mathrm{d}\psi(x)b(x)\gamma(x)\kappa(x) = \mathrm{d}\psi(g(x))a(g(x)) + \mathrm{d}\psi(x)b(x)\gamma(x)\kappa(g(x)) \tag{6.3.3}$$

$$\mathrm{d}\psi(x)b(x)\gamma(x) = \mathrm{d}\psi(g(x))b(g(x))\gamma(g(x)). \tag{6.3.4}$$

We call (6.3.3) and (6.3.4) *relaxed vector field matching condition* and *relaxed input matching condition*, respectively. Under these conditions, we obtain that

$$\begin{aligned}\mathrm{d}\psi(x)f(x,u) &= \mathrm{d}\psi(x)a(x) + \mathrm{d}\psi(x)b(x)u \\ &= \mathrm{d}\psi(x)a(x) + \mathrm{d}\psi(x)b(x)\gamma(x)\kappa(x) + \mathrm{d}\psi(x)b(x)\gamma(x)(\gamma(x)^{-1}u - \kappa(x)).\end{aligned}$$

Via ψ^{-1} in (4.3.4), the glued system is described as

$$f^{\psi}(\zeta,u) =: a^{\psi}_{\kappa\gamma}(\zeta) + b^{\psi}_{\gamma}(\zeta)(\gamma^{-\psi}(\zeta)u - \kappa^{\psi}(\zeta)), \tag{6.3.5}$$

where

$$\begin{aligned}
a^{\psi}_{\kappa\gamma}(\zeta) &:= \mathrm{d}\psi(\psi^{-1}(\zeta))a(\psi^{-1}(\zeta)) + b^{\psi}_{\gamma}(\zeta)\kappa^{\psi}(\zeta),\\
b^{\psi}_{\gamma}(\zeta) &:= \mathrm{d}\psi(\psi^{-1}(\zeta))b(\psi^{-1}(\zeta))\gamma^{\psi}(\zeta),\\
\kappa^{\psi}(\zeta) &:= \kappa(\psi^{-1}(\zeta)),\\
\gamma^{\psi}(\zeta) &:= \gamma(\psi^{-1}(\zeta)),\\
\gamma^{-\psi}(\zeta) &:= \gamma(\psi^{-1}(\zeta))^{-1}.
\end{aligned}$$

By (G3), (6.3.3)–(6.3.4), and Lemma 4.3.2, $a^{\psi}_{\kappa\gamma}$ and b^{ψ}_{γ} are continuous. On the other hand, $\gamma^{-\psi}(\zeta)$ and $\kappa^{\psi}(\zeta)$ may have discontinuities for $\zeta \in \psi(\mathcal{D})$ by the definition (4.3.4), which cause discontinuities in (6.3.5). However, such discontinuities can be canceled by the discontinuous feedback

$$u = \gamma^{\psi}(\zeta)v + \kappa^{\psi}(\zeta).$$

After these operations, we obtain the (feedback) continuous form

$$\dot{\zeta} = a^{\psi}_{\kappa\gamma}(\zeta) + b^{\psi}_{\gamma}(\zeta)v \quad \text{for } \zeta \in \mathcal{C}^{\psi}. \tag{6.3.6}$$

Note that $\zeta_{\mathrm{r}}(t)$ is a solution to (6.3.6) when $v = \gamma^{-\psi}(\zeta_{\mathrm{r}}(t))(u_{\mathrm{r}}(t) - \kappa^{\psi}(\zeta_{\mathrm{r}}(t))) =: v_{\mathrm{r}}(t)$. Since $a^{\psi}_{\kappa\gamma}$ and b^{ψ}_{γ} are continuous, we may proceed as in the previous section to find a tracking controller $(f^{\psi}_{\mathrm{c}}, v^{\psi}_{\mathrm{c}})$ for (6.3.6) with the resulting closed-loop system

$$\begin{aligned}
\dot{\zeta}_{\mathrm{r}} &= a^{\psi}_{\kappa\gamma}(\zeta_{\mathrm{r}}) + b^{\psi}_{\gamma}(\zeta_{\mathrm{r}})v_{\mathrm{r}}(t),\\
\dot{\zeta} &= a^{\psi}_{\kappa\gamma}(\zeta) + b^{\psi}_{\gamma}(\zeta)v^{\psi}_{\mathrm{c}}(t, \zeta_{\mathrm{r}}, \zeta, \eta),\\
\dot{\eta} &= f^{\psi}_{\mathrm{c}}(t, \zeta_{\mathrm{r}}, \zeta, \eta).
\end{aligned}$$

Then, a tracking controller $(f_{\mathrm{c}}, u_{\mathrm{c}})$ for $\mathcal{H}_{\omega}$ is obtained as follows:

$$\begin{aligned}
f_{\mathrm{c}}(t, x, \eta) &:= f^{\psi}_{\mathrm{c}}(t, \psi(r(t)), \psi(x), \eta),\\
u_{\mathrm{c}}(t, x, \eta) &:= \gamma(x)\left(v^{\psi}_{\mathrm{c}}(t, \psi(r(t)), \psi(x), \eta) + \kappa(x)\right).
\end{aligned} \tag{6.3.7}$$

Remark 6.3.1. For (6.3.7), we can also apply Theorem 6.2.1 by taking $u_{\mathrm{c}}^{\psi}(t,\zeta_{\mathrm{r}},\zeta,\eta) := \gamma^{\psi}(\zeta)(v_{\mathrm{c}}^{\psi}(t,\zeta_{\mathrm{r}},\zeta,\eta)+\kappa^{\psi}(\zeta))$. In this case, u_{c}^{ψ} may not be locally Lipschitz because of $\kappa^{\psi}(\zeta)$ and $\gamma^{\psi}(\zeta)$. However, the condition on u_{c}^{ψ} in (C3) can be replaced by the condition on v_{c}^{ψ} that $v_{\mathrm{c}}^{\psi} : \mathbb{R}_{\geq 0} \times \mathcal{C}^{\psi} \times \mathcal{C}^{\psi} \times \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{p}$ are locally Lipschitz in $(\zeta_{\mathrm{r}},\zeta,\eta)$ and piecewise continuous in t. □

Example 6.3.1. Consider a second-order hybrid system with the state $x = (x_1, x_2)$

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x + \begin{bmatrix} 0 \\ b \end{bmatrix} u \\
&=: Ax + Bu \qquad \text{when } x \in \mathcal{C} = \{x \in \mathbb{R}^2 : (x_1 \geq 0) \wedge (|x| > 0)\}, \\
x^{+} &= -x \qquad \text{when } x \in \mathcal{D} = \{x \in \mathcal{C} : (x_1 = 0) \wedge (x_2 < 0)\},
\end{aligned} \tag{6.3.8}$$

where $a_{12} > 0$ and $b \neq 0$. Notice that the system satisfies Assumption 4.3.1 and Assumption 6.1.1. Suppose that the reference $r(t)$ satisfies Assumption 6.1.2.

We first consider the following system in $\mathbb{R}^3$ whose behavior is the same as (6.3.8), for which a gluing function is sought for. Indeed, with $\bar{x} = (x, p)$, the system is defined as

$$\begin{aligned}
\dot{\bar{x}} &= \bar{f}(\bar{x}) + \bar{b}(\bar{x})u = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \qquad \text{when } \bar{x} \in \bar{\mathcal{C}} := \mathcal{C} \times \{-1, 1\}, \\
\bar{x}^{+} &= \bar{g}(\bar{x}) := -\bar{x} \qquad \text{when } \bar{x} \in \bar{\mathcal{D}} := \mathcal{D} \times \{-1, 1\}.
\end{aligned} \tag{6.3.9}$$

Note that the first 2-elements of the state trajectory of (6.3.9) and the state trajectory of (6.3.8) coincide. We say that the system is a twins system[1] for (6.3.8).

We take the reference $\bar{r}(t) := (r(t), p_{\mathrm{r}}(t))$, which is a state trajectory of (6.3.9) when $\bar{r}_0 := (r_0, 1)$ and $u = u_{\mathrm{r}}(t)$. This system and reference also satisfy Assump-

[1] In [Pek14], a projection-based modeling having a similar role to the glued system of the twins system is proposed for unilaterally constrained Hamiltonian systems.

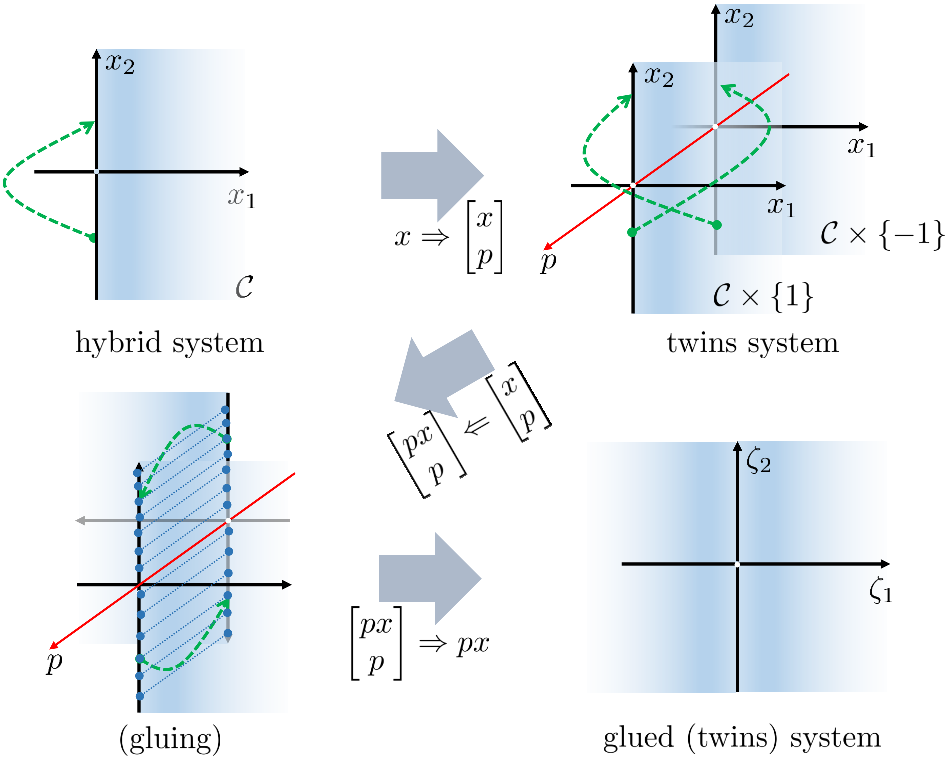

Figure 6.2: Process of the proposed gluing method for (6.3.8), where green dash-lines are instances of state jumps and blue dot-lines mean links between points which should be glued.

tion 4.3.1 and Assumptions6.1.1–6.1.2. For this system, we can take a gluing function $\psi(\bar{x}) = px$. Moreover, the condition (6.3.1) holds because, for $\bar{x} \in \bar{\mathcal{D}}$,

$$\mathrm{d}\psi(\bar{x})\bar{f}(\bar{x}) = \begin{bmatrix} pI_2 & x \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \bar{x} = Apx$$

and

$$\mathrm{d}\psi(\bar{g}(\bar{x}))\bar{f}(\bar{g}(\bar{x})) = \begin{bmatrix} -pI_2 & -x \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} (-\bar{x}) = Apx.$$

However, the condition (6.3.2) does not hold, because, for $\bar{x} \in \bar{\mathcal{D}}$, $\mathrm{d}\psi(\bar{x})\bar{b}(\bar{x}) = pB$ and $\mathrm{d}\psi(\bar{g}(\bar{x}))\bar{b}(\bar{g}(\bar{x})) = -pB$ are not the same.

Take $\gamma(\bar{x}) := p$ (and $\kappa(\bar{x}) := 0$). Note that $\gamma(\bar{x})$ is continuously differentiable and non-zero for $\bar{x} \in \bar{\mathcal{C}}$. Then, since

$$\mathrm{d}\psi(\bar{x})\bar{b}(\bar{x})\gamma(\bar{x}) = \begin{bmatrix} pI_2 & x \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} p = B$$

and

$$\mathrm{d}\psi(\bar{g}(\bar{x}))\bar{b}(\bar{g}(\bar{x}))\gamma(g(\bar{x})) = \begin{bmatrix} -pI_2 & -x \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} (-p) = B$$

for $\bar{x} \in \bar{\mathcal{D}}$, the relaxed input matching condition (6.3.4) is satisfied.

We obtain that $\bar{a}^{\psi}_{\gamma}(\zeta) = A\zeta$ due to $px = \zeta$. In addition, since the inverse gluing function in (4.3.4) is obtained as

$$\psi^{-1}(\zeta) := \mathrm{sgn}(\zeta) \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ 1 \end{bmatrix} \quad \text{for } \zeta = (\zeta_1, \zeta_2) \in \bar{\mathcal{C}}^{\psi} = \mathbb{R}^2 \backslash 0_2,$$

we have that $\gamma^{\psi}(\zeta) = \mathrm{sgn}(\zeta)$ and $\bar{b}^{\psi}_{\gamma}(\zeta) = B$, where $\mathrm{sgn}(\zeta)$ is $\zeta_1/|\zeta_1|$ if $\zeta_1 \neq 0$ and $\zeta_2/|\zeta_2|$ otherwise. Moreover, since $\left(\gamma^{\psi}(\zeta)\right)^2 = 1$, we obtain that $\gamma^{-\psi}(\zeta) =$

$\gamma^\psi(\zeta) = \mathrm{sgn}(\zeta)$. Then, the vector field (6.3.5) on the glued domain is derived as

$$\dot{\zeta} = A\zeta + B\mathrm{sgn}(\zeta)u \quad \text{for } \zeta \in \bar{\mathcal{C}}^\psi,$$

which can be seen as a (state-triggered) switched system, because $\mathrm{sgn}(\zeta)$ is piecewise continuous on $\bar{\mathcal{C}}^\psi$. However, since $\mathrm{sgn}(\zeta)$ is invertible on $\bar{\mathcal{C}}^\psi$, its discontinuity can be canceled by input. Since (A, B) is controllable, we can make a state tracking control law for (6.3.6) as

$$v_{\mathrm{c}}^\psi(t, \zeta_{\mathrm{r}}, \zeta) = K_{\mathrm{c}}(\zeta - \zeta_{\mathrm{r}}) + v_{\mathrm{r}}(t) \quad \text{for } |\zeta(0) - \zeta_{\mathrm{r}}(0)| < \epsilon$$

where $A + BK_{\mathrm{c}}$ is Hurwitz and ϵ is designed as a function of r_0 and $A + BK_{\mathrm{c}}$ to guarantee that $\zeta(t) \neq 0_2$ for all $t \geq 0$. Then, through (6.3.7), we can find the local tracking control law for (6.3.9)

$$\begin{aligned} u &= u_{\mathrm{c}}(t, \bar{x}) \\ &= \gamma(\bar{x}) v_{\mathrm{c}}^\psi(t, \psi(\bar{r}(t)), \psi(\bar{x})) \\ &= p v_{\mathrm{c}}^\psi(t, p_{\mathrm{r}}(t) r(t), px) \\ &= p\big(K_{\mathrm{c}}(px - p_{\mathrm{r}}(t) r(t)) + p_{\mathrm{r}}(t) u_{\mathrm{r}}(t)\big) \\ &= K_{\mathrm{c}}(x - p_{\mathrm{r}}(t) r(t) p) + p_{\mathrm{r}}(t) u_{\mathrm{r}}(t) p \end{aligned} \tag{6.3.10}$$

with the constraint that $|p(0)x(0) - p_{\mathrm{r}}(0)r(0)| = |x_0 - r_0| < \epsilon$. For the simulation, we take

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, r_0 = (0, 6),\ u_{\mathrm{r}}(t) = \begin{cases} -3 & \text{if } \mathrm{mod}(t, 10) \in [0, 4), \\ -2 & \text{otherwise,} \end{cases} \tag{6.3.11}$$

and illustrate the result in Figure 6.3. □

Remark 6.3.2. In Example 6.3.1, the system (6.3.8) can represent the bouncing ball system when

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u = -\rho + v,$$

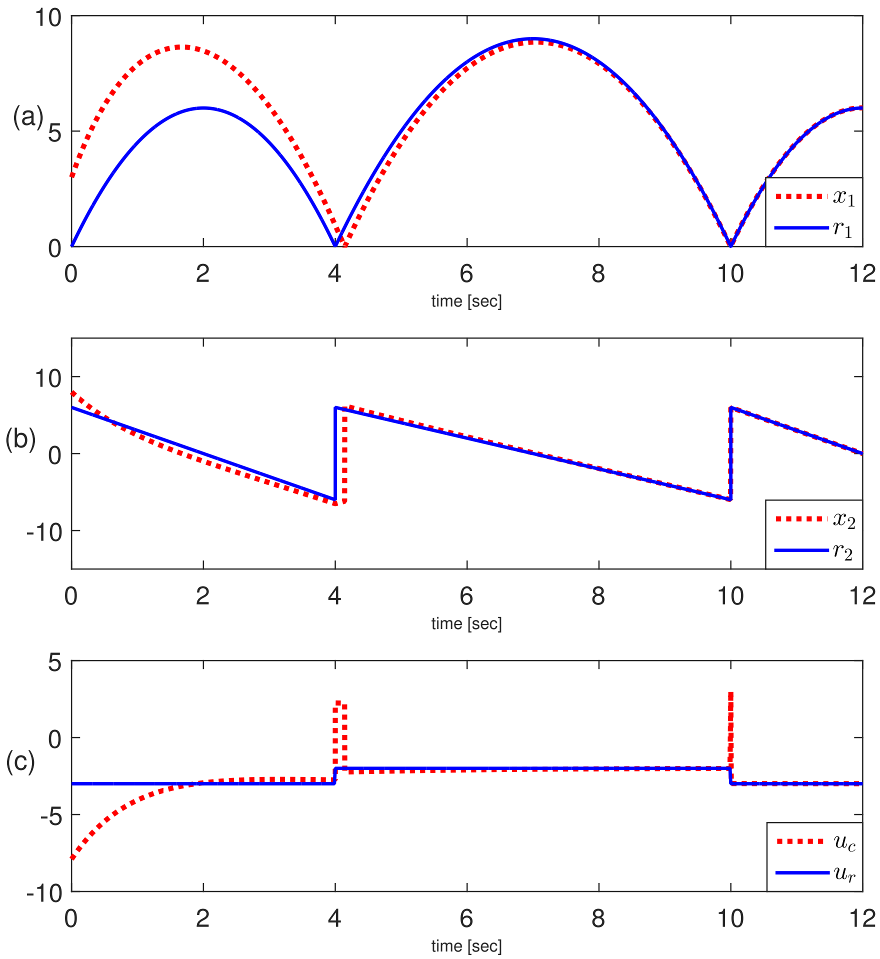

Figure 6.3: (a) and (b) depict reference and plant state trajectories and (c) shows reference input $u_{\mathrm{r}}(t)$ and control input $u_{\mathrm{c}}(t)$ for (6.3.11) where $x_0 = (3, 8)$ and the control law is (6.3.10) with $K_{\mathrm{c}} = [-0.6, \ -1.55]$.

where v is the control acceleration and ρ is the gravity constant. In this case, the controller form is similar to the one proposed in [FTZ13]. □

Example 6.3.2. Consider $\mathcal{H}_\omega$ with an input given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} x_2^2+1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u =: a(x) + bu \quad && \text{when } x \in \{x \in \mathbb{R}^2 : |x_1| \le \frac{\pi}{2}\} =: \mathcal{C}, \\ x^+ &= -x =: g(x) && \text{when } x \in \{x \in \mathcal{C} : x_1 = \frac{\pi}{2}\} =: \mathcal{D}, \end{aligned} \tag{6.3.12}$$

where $x = (x_1, x_2)$, $\mathcal{U} = \{u \in \mathbb{R} : |u| \le M\}$, and $M > 0$ is sufficiently large. Note that Assumption 4.3.1 and Assumption 6.1.1 hold. Suppose that a reference $r(t) =: (r_1(t), r_2(t))$ satisfies Assumption 6.1.2 and $r_2(t)$ does not converge to zero.

Intuitively, we wanted to glue the set $\mathcal{D}$ to $\mathcal{G}$ like a Mobius strip in $\mathbb{R}^3$; that is, it is like the rolled paper but the point $(\pi/2, x_2)$ is glued to $(-\pi/2, -x_2)$ not to $(-\pi/2, x_2)$. However, this work may be rather complicated. Instead, we first consider the following twins system for (6.3.12) to find a simple gluing function. Indeed, with $\bar{x} = (x, p)$, the system is defined as

$$\begin{aligned} \dot{\bar{x}} &= \bar{a}(\bar{x}) + \bar{b}u := \begin{bmatrix} x_2^2+1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \quad && \text{when } \bar{x} \in \bar{\mathcal{C}} := \mathcal{C} \times \{-1, 1\}, \\ \bar{x}^+ &= \bar{g}(\bar{x}) := -\bar{x} && \text{when } \bar{x} \in \bar{\mathcal{D}} := \mathcal{D} \times \{-1, 1\}, \end{aligned} \tag{6.3.13}$$

in which, the initial condition is $\bar{x}_0 = (x_0, 1)$. Note that the first 2-elements of the state trajectory to (6.3.13) and the state trajectory to (6.3.12) coincide.

Note that the reference $\bar{r}(t) := (r(t), p_r(t))$ is a state trajectory to (6.3.13) when the initial condition is $\bar{r}_0 := (r_0, 1)$ and $u = u_r(t)$. For (6.3.13), we can take a gluing function $\psi(\bar{x}) = (\cos(x_1 - \frac{\pi}{2}p), \sin(x_1 - \frac{\pi}{2}p), px_2)$. Then, the vector field

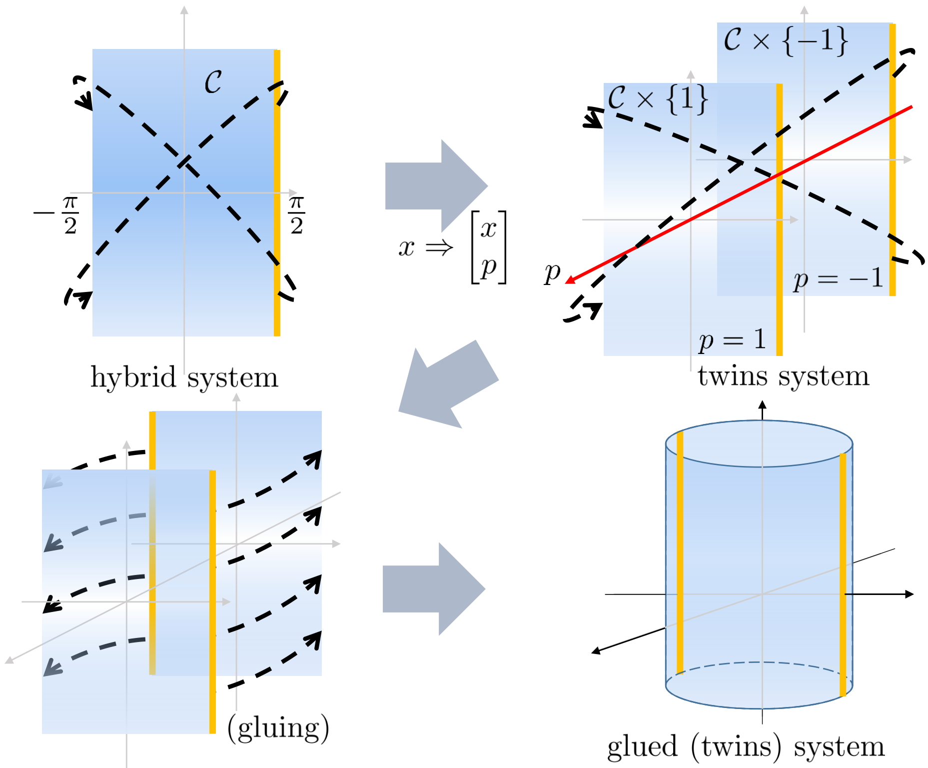

Figure 6.4: Process of the proposed gluing method for (6.3.12).

matching condition (6.3.1) holds because, for all $\bar{x} = (x_1, x_2, p) \in \bar{\mathcal{D}}$,

$$\begin{aligned} \mathrm{d}\psi(\bar{x})\bar{a}(\bar{x}) &= \begin{bmatrix} -\sin(x_1 - \frac{\pi}{2}p) & 0 & * \\ \cos(x_1 - \frac{\pi}{2}p) & 0 & * \\ 0 & p & * \end{bmatrix} \begin{bmatrix} x_2^2 + 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\sin(-x_1 + \frac{\pi}{2}p) & 0 & * \\ \cos(-x_1 + \frac{\pi}{2}p) & 0 & * \\ 0 & -p & * \end{bmatrix} \begin{bmatrix} (-x_2)^2 + 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \mathrm{d}\psi(\bar{g}(\bar{x}))\bar{a}(\bar{g}(\bar{x})). \end{aligned} \tag{6.3.14}$$

However, the input matching condition (6.3.2) does not hold, since

$$\mathrm{d}\psi(\bar{x})\bar{b} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix} \text{ and } \mathrm{d}\psi(\bar{g}(\bar{x}))\bar{b} = \begin{bmatrix} 0 \\ 0 \\ -p \end{bmatrix} \text{ for all } \bar{x} \in \bar{\mathcal{D}}.$$

So, let us take $\gamma(\bar{x}) := p$. Note that $\gamma(\bar{x})$ is of class C^1 and non zero for $\bar{x} \in \bar{\mathcal{C}}$. Since it holds that

$$\mathrm{d}\psi(\bar{x})\bar{b}\gamma(\bar{x}) = \begin{bmatrix} 0 \\ 0 \\ p^2 \end{bmatrix} = \mathrm{d}\psi(\bar{g}(\bar{x}))\bar{b}\gamma(g(\bar{x})) \tag{6.3.15}$$

for all $\bar{x} \in \bar{\mathcal{D}}$, (6.3.4) is satisfied. Then, because $\zeta_3^2 = x_2^2$, by (6.3.14) and (6.3.15), we obtain that

$$\bar{a}^{\psi}(\zeta) = \begin{bmatrix} -\zeta_2(\zeta_3^2 + 1) \\ \zeta_1(\zeta_3^2 + 1) \\ 0 \end{bmatrix} \text{ and } \bar{b}_{\gamma}^{\psi}(\zeta) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where $\psi(\bar{x}) =: \zeta = (\zeta_1, \zeta_2, \zeta_3)$. Fortunately, we can find a tracking controller for $(\bar{a}^{\psi}, \bar{b}_{\gamma}^{\psi})$ as

$$v_c^{\psi}(t, \zeta_r, \zeta) = v_r(t) - (\zeta_2\zeta_{r1} - \zeta_1\zeta_{r2})(\zeta_3 + \zeta_{r3}) - k\zeta_{e3}$$

where $\zeta_e = (\zeta_{e1}, \zeta_{e2}, \zeta_{e3}) := \zeta - \zeta_r$ and $k > 0$. Let us take $z = (z_1, z_2, z_3) :=$

$(1-\zeta_1\zeta_{r1}-\zeta_2\zeta_{r2}, \zeta_2\zeta_{r1}-\zeta_1\zeta_{r2}, \zeta_3-\zeta_{r3})$ and $V(z) := z^\top z$. Then, it follows from Barbalat's lemma that $|\zeta_e(t)| \to 0$ as $t \to \infty$ and, when $|\psi(\bar{x}_0) - \psi(\bar{r}_0)| < 2$, the controller for (6.3.13) is given by

$$u = u_c(t, \bar{r}, \bar{x}) = \gamma(\bar{x}) v_c^{\psi}(t, \psi(\bar{r}), \psi(\bar{x})). \tag{6.3.16}$$

A simulation result is illustrated in Figure 6.5. □

6.4 Output Tracking Controller for Normal Form

In this section, we consider a output tracking control problem for a class of hybrid systems. To deal with the problem, we define a hybrid system with input and output.

Definition 6.4.1. A hybrid system $\mathcal{H} = (\mathcal{C}, f, \mathcal{D}, g)$ with input and output is denoted by $\mathcal{H}_\omega^h := (\mathcal{C}, f, \mathcal{D}, g, \omega, h)$ such that

$$\mathcal{H}_\omega^h \begin{cases} \dot{x} &= f(x) + \omega(x)u \qquad \text{when } x \in \mathcal{C} \\ x^+ &= g(x) \qquad \text{when } x \in \mathcal{D} \\ y &= h(x), \end{cases}$$

where $h : \mathcal{C} \to \mathbb{R}$ is an output map and $\omega : \mathcal{C} \to T\mathcal{C}$ is an input map. □

To simplify the presentation, we only consider the case of the single input and single output system. In addition, we assume that the flow is an input affine. Under the basic Assumption 4.3.1, we impose the flow, input, and output map.

Assumption 6.4.1. The flow map f, input map ω, and output map h are smooth. Furthermore, $\omega : \mathcal{C} \to T\mathcal{C}$ is not outward-pointing and not inward-pointing on $\mathcal{D} \cup \mathcal{G}$. □

The main idea of this section is to find a normal form via an auxiliary output. The introduction of the auxiliary output to consider the hybrid dynamical system as the continuous-time dynamical system are proposed in Section 5.3 and [MT16].

Assumption 6.4.2. For $\mathcal{H}_\omega^h$, there exist a smooth map $\phi : \mathbb{R} \to \mathbb{R}$ and a positive integer ρ, $1 \le \rho \le n$, such that with $h^* = \phi \circ h$

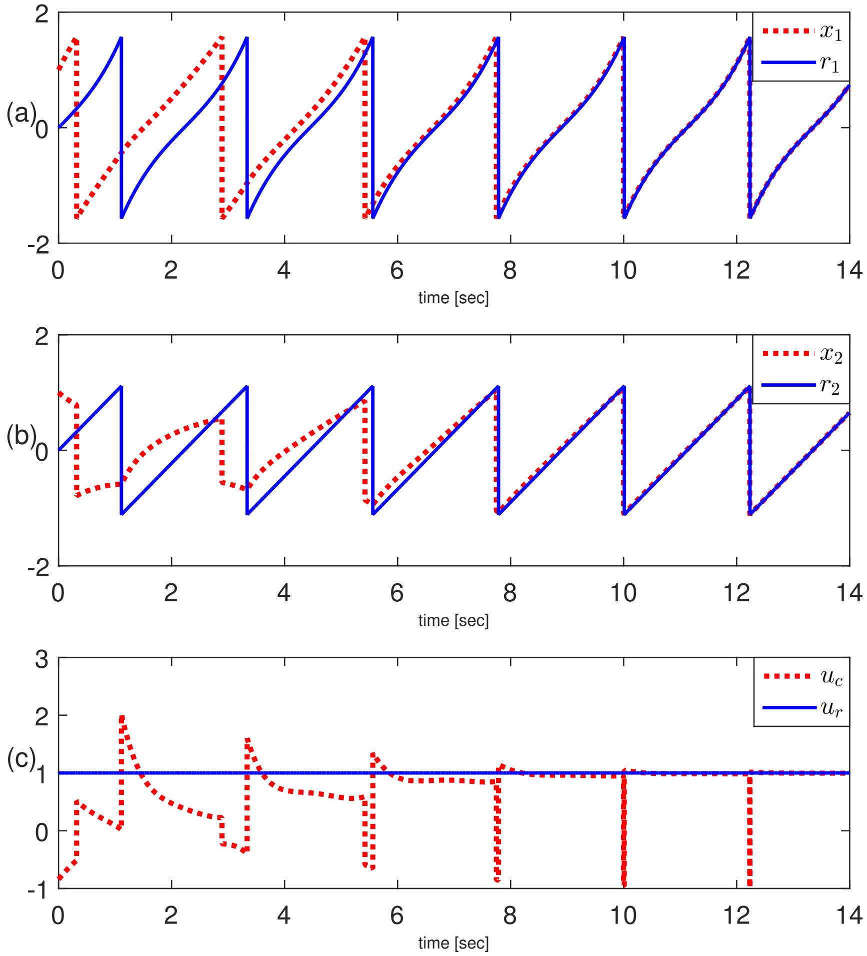

Figure 6.5: (a) and (b) depict the first 2-elements of the state and reference trajectories and (c) shows the control and reference inputs of (6.3.13) under (6.3.16) when $x_0 = (1, 1)$, $r_0 = (0, 0)$, $u_r(t) \equiv 1$, and $k = 1$.

- $L_\omega L_f^{i-1} h^*(x) = 0$ for all $i = 1, \ldots \rho - 1$ and for all $\forall x \in \mathcal{C}$;
- $L_\omega L_f^{\rho-1} h^*(x) \neq 0$ for all $x \in \mathcal{C}$;
- $L_f^i h^*(x) = L_f^i h^*(x_g)|_{x_g = g(x)}$ for all $i = 0, \ldots, \rho$ and for all $x \in \mathcal{D}$.

□

Note that $y^* = \phi(y)$ can be regarded as an auxiliary output. The first and second conditions come from the definition of the relative degree ρ of the system. The third condition guarantees that the auxiliary output trajectory $y^*(t) = h^*(x(t))$ is differentiable up to ρ-th order with respect to t. Let $\zeta = T_2(x) := (h(x), \ldots, L_f^{\rho-1} h^*(x))$. By the definition, this map is C^1 and immersion.

It is well-known that, if $\rho = n$, then for every $x \in \mathcal{C}$, a neighborhood N of x exists such that the map $T(x) = T_2(x)$, restricted to N, is a diffeomorphism on N. Moreover, when $\rho < n$, for every $x \in \mathcal{C}$, a neighborhood N of x in $\mathcal{C}$ and continuously differentiable functions $\varphi_1(x), \ldots, \varphi_{n-\rho}(x)$ exists such that

$$\frac{\partial \varphi_i}{\partial x} \omega(x) = 0 \text{ for } 1 \leq i \leq n - \rho, \ \forall x \in N$$

and the map $T(x) = (T_1(x), T_2(x))$, restricted to N, is a diffeomorphism on N where $T_1(x) := (\varphi_1(x), \ldots, \varphi_{n-\rho}(x))$.

Assumption 6.4.3. If $\rho = n$, then $T_2(x)$ is injective on $\mathcal{C} \backslash \mathcal{D}$. If $\rho < n$, then $T(x)$ is injective on $\mathcal{C} \backslash \mathcal{D}$. □

Suppose that Assumption 4.3.1 and Assumptions 6.4.1–6.4.3 hold. Similar to the change of variables, via $z = (\eta, \zeta) := T(x)$, the system $\mathcal{H}_\omega^h = (\mathcal{C}, f, \mathcal{D}, g, \omega, h)$

is changed into

$$\begin{aligned}
&\dot{\eta} = f_0(\eta,\zeta) \\
&\dot{\zeta} = A_c\zeta + B_c[L_f^{\rho}h^*(x) + L_\omega L_f^{\rho-1}h^*(x)u]
\end{aligned} \qquad (\eta,\zeta) \in T(\mathcal{C})$$

$$\begin{aligned}
\eta^+ &= \begin{bmatrix} \varphi_1(g(x)) \\ \vdots \\ \varphi_{n-\rho}(g(x)) \end{bmatrix} \\
\zeta^+ &= \zeta
\end{aligned} \qquad (\eta,\zeta) \in T(\mathcal{D})$$

$$y^* = \varphi(h(x)) = C_c\zeta$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}_{\rho\times\rho}, B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\rho\times 1}, C_c \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{1\times\rho}.$$

For the system, the state feedback control

$$u = \frac{1}{L_\omega L_f^{\rho-1}h^*(x)}[-L_f^{\rho}h^*(x) + v] \tag{6.4.1}$$

converts the external (continuous-time) dynamics into a chain of ρ integrator, $y^{(\rho)} = v$, and makes the remaining internal hybrid dynamics unobservable from the auxiliary output.

Let us consider a output reference $y_r(t)$ which is bounded and differentiable up to ρ-order. Then, $y_r^*(t) := \phi(y_r(t))$ is also differentiable up to ρ-order and it is an output to the system

$$\dot{\zeta}_r = A_c\zeta_r + B_c y_r^{*(\rho)}(t) \tag{6.4.2}$$

$$y^* = C_c\zeta_r \tag{6.4.3}$$

Under the feedback (6.4.1), we can construct an output tracking dynamic controller as

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c v + L(C_c x_c - y^*) \\ v &= K(x_c - \zeta_r) + y_r^{*(\rho)}(t) \end{aligned} \tag{6.4.4}$$

where L and K are designed to make that $A_c + B_c K$ and $A_c + LC_c$ are Hurwitz. Then the closed-loop system $\mathcal{H}_\omega^h$ with (6.4.2) is

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \zeta) \\ \dot{\zeta} &= A_c \zeta + B_c(K(x_c - \zeta_r) + y_r^{*(\rho)}(t)) && (\eta, \zeta) \in T(\mathcal{C}) \\ \dot{\zeta}_r &= A_c \zeta_r + B_c y_r^{*(\rho)}(t) \\ \dot{x}_c &= A_c x_c + B_c(K(x_c - \zeta_r) + y_r^{*(\rho)}(t)) + LC_c(x_c - \zeta) \\ \eta^+ &= \begin{bmatrix} \varphi_1(g(x)) \\ \vdots \\ \varphi_{n-\rho}(g(x)) \end{bmatrix} \\ \zeta^+ &= \zeta && (\eta, \zeta) \in T(\mathcal{D}). \\ \zeta_r^+ &= \zeta_r \\ x_c^+ &= x^+ \\ y^* &= \phi(h(x)) = C_c \zeta \end{aligned}$$

Let $e = \begin{bmatrix} e_r^\top & e_o^\top \end{bmatrix}^\top := \begin{bmatrix} \zeta_r^\top - \zeta^\top & x_c^\top - \zeta^\top \end{bmatrix}^\top$. Since ζ_r, ζ, x_c do not jump whenever the discrete events of the system occur, the dynamics of the error e is obtained as the continuous-time dynamics

$$\begin{aligned} \dot{e}_r &= A_c e_r + B_c K(e_r - e_o) \\ \dot{e}_o &= A_c e_o + LC_c e_o. \end{aligned}$$

From the above dynamics, we have that $e \to 0$ as time goes to infinity. Consequently, it follows that $|y_r^*(t) - \phi(y(t))| \to 0$.

However, if $\rho < n$, then there is the internal dynamics of the system

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \zeta) && (\eta, \zeta) \in T(\mathcal{C})\\ \eta^+ &= \begin{bmatrix} \varphi_1(g(x)) \\ \vdots \\ \varphi_{n-\rho}(g(x)) \end{bmatrix} = \begin{bmatrix} \varphi_1(g(T^{-1}(z))) \\ \vdots \\ \varphi_{n-\rho}(g(T^{-1}(z))) \end{bmatrix} && (\eta, \zeta) \in T(\mathcal{D}).\end{aligned}$$

where $T^{-1}(z) := (T|_{\mathcal{C}\setminus\mathcal{D}})^{-1}(z)$. To prevent the diverge of the internal state, we need to impose the additional assumption on this dynamics.

Assumption 6.4.4. For any C^ρ bounded input function $u_\eta(t) \in \mathbb{R}^\rho$ remaining on $T_2(\mathcal{C})$, the execution of the hybrid dynamics

$$\begin{aligned}\dot{\eta} &= f_0(\eta, u_\eta(t)) && (\eta, u_\eta(t)) \in T(\mathcal{C})\\ \eta^+ &= \begin{bmatrix} \varphi_1(g(T^{-1}([\eta^\top \;\; u_\eta(t)^\top]^\top))) \\ \vdots \\ \varphi_{n-\rho}(g(T^{-1}([\eta^\top \;\; u_\eta(t)^\top]^\top))) \end{bmatrix} && (\eta, u_\eta(t)) \in T(\mathcal{D}).\end{aligned}$$

is bounded. □

Under this assumption, the internal state η does not blow up and the closed-loop system is well-defined for all $t \geq 0$. Therefore, we have that $|y_r^*(t) - \phi(y(t))| \to 0$ as $t \to \infty$.

Note that, as $t \to \infty$, $|y_r^*(t) - \phi(y(t))| = |\phi(y_r)(t) - \phi(y(t))| \to 0$ does not always imply $|y_r(t) - y(t)| \to 0$ because ϕ may not be a homeomorphism. If ϕ restricted to $h(\mathcal{C})$ is a homeomorphic to its image $\phi(h(\mathcal{C}))$, we obtain that $|y_r(t) - y(t)| \to 0$ as time goes to infinity. However, it is not general because of the third condition of Assumption 6.4.2. When $i = 0$, the condition means that

$$\phi(h(x)) = \phi(h(g(x))) \text{ for all } x \in \mathcal{D}.$$

From the above condition, at least to guarantee the injectivity of ϕ on $\mathcal{C}$, it is necessary that

$$h(x) = h(g(x)) \text{ for all } x \in \mathcal{D},$$

which means that the output trajectory is continuous with respect to t.

Example 6.4.1. Let us consider a simple hybrid system $\mathcal{H}_\omega^h$

$$\begin{aligned}
\dot{x} &= Fx + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + Wu =: f(x) + \omega(x)u \quad && \text{when } x \in \{x \in \mathbb{R}^3 : |x_3| \leq 1\} =: \mathcal{C} \\
x^+ &= Gx =: g(x) && \text{when } x \in \{x \in \mathcal{C} : x_3 = 1\} =: \mathcal{D}. \\
y &= Hx =: h(x)
\end{aligned}$$

where

$$F := \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, W := \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}, H = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

It is easy to check that Assumption 4.3.1 and Assumption 6.4.1 hold. Let $\phi(y) := y$. Then, it follows that, for all $x \in \mathcal{C}$,

$$L_\omega h^*(x) = HW = 0$$

which implies the first condition of Assumption 6.4.2 with $\rho = 2$. Furthermore, it holds that, for all $x \in \mathcal{C}$,

$$L_\omega L_f h^*(x) = HFW = 1$$

which implies the second condition of Assumption 6.4.2. Finally, since, for $x \in \mathcal{D}$,

$$\begin{aligned} h^*(x) &= Hx = x_1 + x_2 + 1 \\ L_f h^*(x) &= H(Fx - \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}) = HFx = x_2 - 1 \\ L_f^2 h^*(x) &= HF(Fx - \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}) = -x_1 - x_2 - 1 \end{aligned}$$

and

$$\begin{aligned} h^*(x_g)|_{x_g=g(x)} &= HGx = x_1 + x_2 + 1 \\ L_f h^*(x_g)|_{x_g=g(x)} &= H(FGx - \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}) = HFGx = x_2 - 1 \\ L_f^2 h^*(x_g)|_{x_g=g(x)} &= HF(FGx - \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}) = HF^2Gx = -x_1 - x_2 - 1 \end{aligned}$$

the third condition of Assumption 6.4.2 is also satisfied. Therefore, Assumption 6.4.2 hold.

Take $\eta = \varphi(x) := x_3$. Then it follows that

$$\frac{\partial \varphi}{\partial x}\omega(x) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 0 \text{ for all } x \in \mathcal{C}.$$

Moreover, since $T(x) := \begin{bmatrix} \varphi(x) & Hx & FHx \end{bmatrix}^\top = \begin{bmatrix} x_3 & x_1 + x_2 + x_3 & x_2 - x_3 \end{bmatrix}^\top$ is injective on $\mathcal{C}$, Assumption 6.4.3 holds. By the change of variables via $z =$

$[\eta\ \zeta_1\ \zeta_2]^\top := T(x)$, the system is changed into

$$\begin{aligned}
\dot{\eta} &= 1 && \text{when } \eta \in [-1,1] \\
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= -\zeta_1 + u \\
\eta^+ &= -\eta && \text{when } \eta \in \{1\}. \\
\zeta_1^+ &= \zeta_1 \\
\zeta_2^+ &= \zeta_2
\end{aligned}$$

In this case, the flow set $T(\mathcal{C}) = \{(\eta,\zeta_1,\zeta_2) \in \mathbb{R}^3 : \eta \in [-1,1]\}$ and jump set $T(\mathcal{D}) = \{(\eta,\zeta_1,\zeta_2) \in \mathbb{R}^3 : \eta = 1\}$ only depend on η. The internal dynamics is obtained as

$$\begin{aligned}
\dot{\eta} &= 1 && \text{when } \eta \in [-1,1] \\
\eta^+ &= -\eta && \text{when } \eta \in \{1\}
\end{aligned}$$

which trivially satisfies Assumption 6.4.4.

Let us consider an output reference $y_r(t)$, which is bounded and C^2 function. In addition, take $\zeta_r := (y_r, \dot{y}_r)$. Then, we can construct an output tracking controller in (6.4.4) as

$$\begin{aligned}
\dot{x}_c &= A_c x_c + B_c v + L\left(C_c x_c - y\right) := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v + L\left(\begin{bmatrix} 1 & 0 \end{bmatrix} x_c - y\right) \\
v &= K(x_c - \zeta_r) + \ddot{y}_r(t),
\end{aligned}$$

where $A_c + LC_c$ and $A_c + B_c K$ are Hurwitz. Note that, since T is injective on $\mathbb{R}^3$, we have that $|y(t) - y_r(t)|$ converges to zero as time goes to infinity. □

Chapter 7

Conclusions

In this chapter, we summarizes the whole contents of the dissertation that have been addressed so far, and provides some future works. We have dealt with three kinds of problems for hybrid dynamical system as listed below.

- **Gluing boundaries and smoothing vector field**
  We considered a hybrid dynamical system whose domain is a smooth manifold with boundary and discrete-time dynamics happens on the boundary. In this situation, we proposed the idea eliminating this discrete-time dynamics by gluing the boundaries. Through this idea, the hybrid dynamical system may become to a continuous-time dynamical system. The problems were how to glue the boundaries of the domain and how to specify the glued domain and the continuous-time dynamics on the glued domain. At first, we glued the boundaries using the quotient map. In addition, by the Boundary Flowout Theorem, we obtained the glued domain and smooth vector field. However, the process was so complex and the glued domain was abstract, we introduced a notion of gluing function which is intuitive because it just glues the domain on Euclidean space in a topological sense.

- **State estimation problem for hybrid dynamical system**
  We considered the state estimation problem of hybrid dynamical systems with state-triggered jumps using a gluing function. Via the gluing function, we might change the hybrid dynamical system into some continuous-time dynamical system without any state jumps and design the state observer

from conventional observer design methods for the continuous-time dynamical systems such as Luenberger observers, high-gain observer, and so on. From an estimate of these observer, we constructed a state estimate of the hybrid dynamical system. Most previous observer design approaches for the hybrid dynamical systems require knowledge of the state jump time instants, but the proposed observer design technique does not.

- **Tracking control problem for hybrid dynamical system**
  We also considered the state tracking control problem of hybrid dynamical systems with state-triggered jumps using a gluing function. By similar way to the estimation problem, we might obtain some continuous-time dynamical system without any state jumps from a given hybrid dynamical system. Then, via conventional tracking controller design techniques for the continuous-time dynamical systems, we constructed a tracking controller for the hybrid dynamical system. Many previous tracking controller of the hybrid dynamical systems should make the state jump whenever the reference jumps occur while the proposed tracking controller need not do.

Some further issues for future research related to the topics of this dissertation are listed as follows.

- Under some condition, the gluing function always exists but not easily obtainable. Therefore, the problems of determining gluing functions systematically needs further research.

- In the estimation problem, we embed the glued domain in some Euclidean space and construct observer in the Euclidean space. Therefore, a state estimate may not be in the glued domain where the inverse gluing function is well-defined. To solve this problem, we employ a projection map but it is restrictive. The state estimation approach not using the projection map is the topic of on-going research.

- In fact, we use the gluing function as a transformation to consider the hybrid dynamical system as a continuous-time dynamical system, so we may apply this approach in the opposite direction. i.e., the concept of "detecting" needs to be considered because it may change a complex continuous-time dynamical system into a simple hybrid dynamical system.

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국문초록

State Estimation and Tracking Control for Hybrid Systems by Gluing the Domains

상태변수 영역 접합을 통한 하이브리드 시스템의 상태변수 추정 및 추종 제어

본 논문은 하이브리드 동적 시스템 (hybrid dynamical system) 에 대한 상태변수 추정 (state estimation) 과 추종 제어기 (tracking controller) 설계 문제를 다루고 있다. 하이브리드 동적 시스템은 미분 방정식 (differential equation) 으로 모델링되는 연속 시간 동역학 (continuous-time dynamics) 과 차분 방정식 (difference equation) 으로 모델링되는 이산 시간 동역학 (discrete-time dynamics) 이 혼합된 시스템이다. 일반적으로 하이브리드 동적 시스템은 연속 시간 동역학과 이산 시간 동역학이 상호 의존적이기 때문에, 그 특성이 복잡하여 다루기 힘든 시스템으로 알려져 있다.

본 논문에서 다루는 상태변수 추정 문제란 주어진 시스템의 모델링 정보와 실시간 출력 정보를 활용하여 대상 시스템의 상태변수를 추정하는 것이며, 추종 제어기 설계 문제란 주어진 시스템 모델링 정보와 실시간 상태변수 정보를 가지고 상태변수를 주어진 레퍼런스 궤적 (reference trajectory) 으로 추종하게 만드는 제어 입력을 설계하는 것이다. 연속 시간 동역학을 갖는 연속 시간 동적 시스템 (continuous-time dynamical system) 과 이산 시간 동역학을 갖는 이산 시간 동적 시스템 (discrete-time dynamical system) 에 대해서는 상태 변수 추정과 추종 제어기 설계에 대한 많은 결과가 알려져 있지만, 하이브리드 동적 시스템에 대해서는 그 결과가 미미하며, 존재하는 결과들은 일반적인 하이브리드 동적 시스템을 다루기보단, 스위치드 시스템 (swithced system) 이나 생체 호르몬 모델, 파워트레인 시스템 (powertrain system) 등 특정 하이브리드 동적 시스템에을 다루고 있다.

본 논문에서는 먼저 하이브리드 동적 시스템을 쉽게 다루기 위한 글루잉 (gluing) 이란 기술 소개한다. 상태변수가 일정한 값에 도달했을 때만 이산 시간 동역학이 발생하는 시스템 (hybrid system with state-triggered jumps) 에 대하여, 이 기술은 시스템의 상태변수 영역 접합을 통해 하이브리드 동적 시스템이 내재하고 있는 이산 시간 동역학을 제거한다. 따라서, 하이브리드 동적 시스템을 연속 시간 동적 시스템으로 리모델링할 수 있게 한다. 그리고, 특정 시스템에 대해서 그 리모델링

된 연속 시간 동역학을 상태변수에 대한 매끄러운 함수(smooth function)로 만드는 기술(smoothing)을 소개한다. 이 기술들은 미분 다양체(differential manifold)와 미분 위상(differential topology) 등과 같은 수학적 이론에 기반하기 때문에, 본 학위 논문에서는 미분 다양체와 벡터장(vector field)으로 모델링되는 하이브리드 동적 시스템에 대하여 이 기술들의 사용을 다룬다.

본 논문에서는 특히 글루잉 기술에 집중한다. 글루잉을 통해 얻어지는 리모델링된 연속 시간 동적 시스템은 추상적인 미분 다양체와 벡터장으로 주어지기 때문에, 그것을 유클리드 공간(Euclidean space)에 매장함(embedding)으로써 시스템을 구체화한다. 그리고 시스템 변환으로써 의미를 가질 수 있는 글루잉을 보장하는 함수의 필수적인 조건을 제시한다. 특히, 글루잉을 통해 매장된 시스템은 연속 시간 동적 시스템이기 때문에, 연속 시간 동적 시스템들에 대해서 개발된 다양한 기존의 관측기와 추종 제어기 설계 기술을 이용하여, 하이브리드 동적 시스템의 관측기와 추종 제어기를 설계할 수 있다.

기존에 존재하는 대다수의 하이브리드 동적 시스템의 상태변수 관측기의 경우, 하이브리드 동적 특성을 가지며, 관측기의 이산 동역학과 추정 대상 시스템의 이산 시간 동역학이 동시에 일어나야 한다는 한계가 있다. 따라서 기존의 추정기의 경우, 추정 시스템의 이산 시간 동역학 발생 시간 정보를 요구하거나 그 시간을 측정 혹은 추정하는 구조를 가지게 되는데, 이는 그 시간 정보를 추정하는 추가적인 추정기를 필요로 하며 측정 시간 지연에 취약하다. 반면, 글루잉을 통해 설계된 상태변수 추정기는 이산 시간 동역학 발생 시간 정보를 추정하거나 요구하지 않기 때문에 추가적인 추정기가 불필요하고 측정 시간 지연에도 강인하다. 또한, 글루잉 기법을 통해 설계된 추종 제어기는, 시스템 상태변수와 레퍼런스의 불연속점이 발생하는 시간들의 불일치로 인해 발생하는 추종 오차에 취약한 기존의 제어기와 달리, 이 오차에 대해 강인한 특성을 가진다.

마지막으로, 충돌이 있는 기계 시스템이나 AC/DC 변환기의 외란 발생기와 같은 실제 예제들과 이론적 예제들을 통해 글루잉 기법의 적용과 상태변수 관측기 및 추종 제어기 설계를 연습하고, 시뮬레이션을 통해 상태변수 추정과 레퍼런스 추종 결과를 확인한다.

주요어: 하이브리드 동적 시스템, 미분 다양체, 글루잉, 스무딩, 상태변수 추정, 비선형 관측기 설계, 추적 제어
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