이학박사학위논문

Pricing Barrier Options: A Probabilistic Approach

확률론적으로 접근한 베리어 옵션의 가치평가

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Abstract

Pricing Barrier Options: A Probabilistic Approach

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We investigate the use of theoretical and computational methods from physics in finance, particularly in the areas of contingent claim valuation. We apply these methods to simplify the analysis of complicated barrier options.

The trivariate joint probability density function of Brownian motion and its maximum and minimum can be expressed as an infinite series of normal probability density functions. We show that the infinite series converges uniformly, and use the uniform convergence to prove it satisfies the Fokker-Planck equation. Also, we express the infinite series as a product form using Jacobi's triple product identity. Moreover, we present some error bounds of an approximation of the infinite series by a finite series.

However, practitioners and researchers who have handled financial market data know that asset returns do not behave according to the Gaussian or normal distribution. Indeed, the use of Gaussian models when the asset return distributions are not normal could lead to the underestimation of extreme losses or mispriced derivative products. Consequently, non-Gaussian models are gaining popularity among financial market practitioners. We tried to calculate value of the barrier options when the asset return distributions are heavy-tailed GB2 distribution.

Keywords : barrier option, reflection principle, brownian motion, maximum, minimum, joint probability distribution, error bound, Jacobi's triple product identity, heavy-tailed distribution, GB2

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Contents

Chapter 1

Introduction

Since Black and Scholes' (BS) option pricing model gained an almost immediate acceptance among the professional and academic communities, the trading of derivative securities suddenly increased by a very large amount. Derivative securities are financial securities whose payoffs depends on other underlying securities, and the BS model was the first universally accepted modeling of these financial instruments.

However, in recent years, financial engineeers have created a variety of complex options that are collectively called *exotic options*.

The payoffs on these options are considerably more diverse than the payoffs on standard BS options or on other straightforward generalizations of them.

Most of the mathematical methods involved in the analysis of financial systems have been based, so far, on the simulation of stochastic processes by diffusion equations coupled to stochastic sources, i.e. stochastic equations of Langevin type. More recently, there has been an interest in the analysis of various financial instruments using the path integral formulation.

Use of a path integral formulation has some advantages. First, it is in close relation to the lagrangean description of diffusion processes, second, it opens the way to the use of quantum mechanical methods.

In chapter 1, after a description of the path integral in the Black Scholes model, we turn our attention to the analysis of barrier options. Barrier options are studied here by an artificial quantum mechanical model in which a potential $V(x)$ is added to the Black Scholes lagrangean, as first suggested in ref. [2].

The trivariate joint probability density function of Brownian motion and its maximum and minimum can be expressed as an infinite series of normal probability density functions. In chapter 2, we show that the infinite series converges uniformly, and we express the infinite series as a product form using Jacobi's triple product identity. Moreover, we present some error bounds of an approximation of the infinite series by a finite series.

In chapter 3, we tried to calculate value of the barrier options when the asset return distributions are heavy-tailed GB2 distribution.

1.1 Langevin Evolution

Baaquie, et al [1] have introduced path integral method for the analysis of barrier options. The rest of this chapter has been adapted from Baaquie, et al [1].

In the description of theoretical finance, a security $S(t)$ follows a random walk described by a Ito-Weiner process (or Langevin equation) as

$$\frac{dS(t)}{S(t)} = \phi dt + \sigma R(t) dt, \tag{1.1}$$

where R(t) is a Gaussian white noise with zero mean and uncorrelated values at time t and t' $\langle R(t)R(t')\rangle = \delta(t-t')$. ϕ is the drift term or expected return, while σ is a constant factor multiplying the random source $R(t)$, termed *volatility*.

As a consequence of Ito calculus, differentials of functions of random variables, say $f(S,t)$, do not satisfy Leibnitz's rule, and for a Ito-Weiner process with drift (1.1) one easily obtains for the time derivative of $f(S,t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \phi S \frac{\partial f}{\partial S} + \sigma S \frac{\partial f}{\partial S} R. \tag{1.2}$$

The Black-Scholes model is obtained by removing the randomness of the stochastic process shown above by introducing a random process correlated to (1.2). This operation, termed *hedging*, allows to remove the dependence on the white noise function $R(t)$, by constructing a *portfolio* Π, whose evolution is given by the short-term risk free interest rate r

$$\frac{d\Pi}{dt} = r\Pi. \tag{1.3}$$

A possibility is to choose $\Pi = f - \frac{\partial f}{\partial S} S$. This is a portfolio in which the investor holds an option f and short sells an amount of the underlying security S proportional to $\frac{\partial f}{\partial S}$. A combination of (1.2) and (1.3) yields the Black-Scholes equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} = rf. \tag{1.4}$$

There are some assumptions underlying this result. We have assumed absence of arbitrage, constant spot rate r, continuous balance of the portfolio, no transaction costs and infinite divisibility of the stock.

The quantum mechanical version of this equation is obtained by a change of variable $S = e^x$, with x a real variable. This yields

$$\frac{\partial f}{\partial t} = H_{BS} f \tag{1.5}$$

with an Hamiltonian H_{BS} given by

$$H_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r. \tag{1.6}$$

Notice that one can introduce a quantum mechanical formalism and interpret the option price as a ket $|f\rangle$ in the basis of $|x\rangle$, the underlying security price. Using Dirac notation, we can formally reinterpret $f(x,t) = \langle x|f(t)\rangle$, as a projection of an abstract quantum state $|f(t)\rangle$ on the chosen basis.

In this notation, the evolution of the option price can be formally written as $|f,t\rangle = e^{tH}|f,0\rangle$, for an appropriate Hamiltonian H.

1.2 Options and Barrier Options

1.2.1 Generalities

Let the price at time t of a security be $S(t)$. A specific good can be traded at time t at the price $S(t)$ between a buyer and a seller. The seller (short position) agrees to sell the goods to the buyer (long position) at some time T in the future at a price $F(t,T)$ (the contract price). Notice that contract prices have a 2-time dependence (actual time t and maturity time T). Their difference $\tau = T - t$ is usually called *time to maturity*. Equivalently, the actual price of the contract is determined by the prevailing actual prices and interest rates and by the time to maturity.

Entering into a forward contract requires no money, and the value of the contract for long position holders and short position holders at maturity T will be

$$(-1)^p \left(S(T) - F(t,T)\right) \tag{1.7}$$

where $p = 0$ for long positions and $p = 1$ for short positions. *Futures Contracts* are similar, except that the after the contract is entered, any changes in the market value of the contract are settled by the parties. Hence, the cashflows occur all the way to expiry unlike in the case of the forward where only one cashflow occurs. They are also highly regulated and involve a third party (a *clearing house*). Forward, futures contracts and, as we will see, *op-*

tions go under the name of *derivative products*, since their contract price $F(t,T)$ depend on the value of the underlying security $S(T)$.

In the simplest option, such as a call option, we have seen that the payoff function is defined to be the value of the option at maturity time $(\tau = 0)$. Therefore, the specific path followed by the underlying security is not relevant in order to establish the price at maturity, except for its final value.

Barrier options are, instead, path-dependent. This means that the payoff is dependent on the realized asset path, and certain aspects of the contract are triggered if the asset price, from start to end of the contract, becomes too high or too low.

Barrier options are very popular for various reasons. An investor may have very precise views about the behaviour of a security or he may use them for hedging specific cashflows, to decide to purchase them. In the following, when comparing path dependent options to the simplest options, such as standard calls or puts, we will refer to the latter as to *vanilla* options, using a common financial jargon.

1.2.2 Terminology and Definitions

There are some advantages -and natural limitations- in purchasing a financial instrument such as a barrier option. If the purchaser wants the same payoff typical of a vanilla option, but believes that the upward movement of the underlying will not be likely, then he may decide to buy an *up-and-out* call option. The cost of this contract will be cheaper than the purchase of a corresponding plain vanilla option, but there will be severe limitations on the upward movement of the option.

The physical picture of an up-and-out option is that of a brownian motion of the underlying asset (x) that is immediately killed as soon as the asset hits (from below) the barrier B $(x = B)$, which is specified in the contract.

Similarly, a *down and out* provision renders the option worthless as soon as the asset price hits a barrier B from above. The payoffs in the two cases are given by

$$\begin{aligned} g_{UO}(x,K) &= max(S_T - K)\theta(B - x) \\ g_{DO}(x,K) &= max(S_T - K)\theta(x - B) \end{aligned} \qquad (1.8)$$

for a up-and-out (UO) and a down-and-out (DO) option call respectively. Here, $\theta()$ denotes the standard step function. A terminology used to describe contracts with these features is *knocked out options*. In contracts of this type it is agreed there will not be any payoff if the barrier B is hit.

Similarly, the market offers contract with additional limitations on the allowed variation of the underlying asset. For instance, *double knock out* options have restrictions on the asset variability delimited by two barriers $(B_- < B_+)$ both from above (B_+) and from below (B_-), and give zero payoff if any of the two barriers is hit by the asset from inception time t to expiry time T.

Knock in options are dual, in an obvious sense, to knock out options. *Knock in* options, in fact, are contracts that pay off as long as the barrier B is hit before expiry. If the barrier is hit, then the option is said to have *knocked* *in*, otherwise their payoff is null.

Furtherly categorizing these latter types of options, the position of the barrier respect to the initial value of the underlying allows to distinguish between *up-and-in* options and *down-and-in* options. The payoffs of these contracts are given by

$$\begin{aligned} g_{UI}(x,K) &= max(S_T - K)\theta(x - B) \\ g_{DI}(x,K) &= max(S_T - K)\theta(B - x). \end{aligned} \tag{1.9}$$

For definiteness, in the analysis that follows up, we will focus our attention to knocked out payoffs of the types described in eq. (1.8).

In knocked out options, single or double, killing of the brownian motion is, needless to say, instantaneous, and takes place as soon as the brownian motion of the asset hits any of the barriers.

This aspect of the contract is an unpleasent feature since it introduces a discontinuity in the dynamics, with attached risk management problems both for option buyers and sellers. Such risks, for instance, are those due to erroneous price movements, or to an instantaneous spiky behaviour of an asset, moving upward or downward and penetrating a given barrier, which can lead an investor to the loss of all his investment. In other unpleasent situations, when large positions of options accumulate in the market and are all characterized by the same barrier, trading can drive the asset to the barrier, generating massive losses.

There are various ways by which more conservative and safer contracts can be defined, while maintaining some of the features of knock out options. This is achieved by introducing a finite knock out rate, thereby smoothing out the effect of the barrier. Our goal is to show how it can be implemented in a self-consistent path integral formulation and characterize the pricing of these path dependent options.

1.3 Quantum Methods in Finance

To establish a path integral description of a stochastic process we need a lagrangean and the corresponding action. This can be easily worked out for the BS model, starting from the Hamiltonian given in eq. (1.6). We easily gets

$$L_{BS} = -\frac{1}{2\sigma^2}\left(\frac{dx}{dt} + r - \frac{1}{2}\sigma^2\right)^2 - r \tag{1.10}$$

and the corresponding action, expressed in terms of time to maturity τ

$$S_{BS} = \int_0^\tau L_{BS}(t')dt' \tag{1.11}$$

which can be used to define a corresponding path integral for a fictitious quantum mechanical process in the variable x, the logarithm of the underlying asset

$$\langle x_f|e^{-\tau H_{BS}}|x_i\rangle = \Pi_{t_i<t<t_f}\int_{-\infty}^{+\infty} dx(t)e^{S[x]} \tag{1.12}$$

with the boundary conditions $x(t_i) = x_i$ and $x(t_f) = x_f$. The variable $x = \log(S)$ which identifies the quantum mechanical state of the system will be refered to as to the stock price. The pricing kernel for the stock price is given by the

$$\begin{aligned} p_{BS}(x,x',\tau) &= \int DX_{BS} e^{S_{BS}} \\ &= \langle x | e^{-\tau H_{BS}} | x' \rangle \end{aligned} \tag{1.13}$$

with

$$\int DX_{BS} = \Pi_{t=0}^{\tau} \int_{-\infty}^{\infty} dx(t). \tag{1.14}$$

1.3.1 Generalized Potential

For barrier options it is tempting [2] to introduce a potential $V(x)$ in order to set up a constraint on the stochastic process described by the stock price x.

The corresponding generalized Hamiltonian now reads

$$H_V = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - V(x)\right)\frac{\partial}{\partial x} + V(x). \tag{1.15}$$

It can be shown [2] that H_V obeys the martingale condition, and hence can be used for studying processes in finance.

The non-Hermiticity of H_V is of a particularly simple nature, and it can be shown [2] that for arbitrary V, H_V is equivalent by a similarity transformation to a Hermetian Hamiltonian $H_{\rm Eff}$ given by

$$H_{\rm Eff} = e^{-s} H_V e^s \tag{1.16}$$

where

$$H_{\rm Eff} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\partial V}{\partial x} + \frac{1}{2\sigma^2}V^2 + \frac{1}{2}V + \frac{\sigma^2}{8} \tag{1.17}$$

and

$$s = \frac{1}{2}x - \frac{1}{\sigma^2}\int_0^x dy V(y) \tag{1.18}$$

Note that $H_{\rm Eff}$ is Hermetian and hence its eigenfunctions form a complete basis; from this it follows that the Hamiltonian H_V can also be diagonalized using the eigenfunctions of $H_{\rm Eff}$. In particular

$$H_{\rm Eff}|\phi_n> = E_n|\phi_n> \tag{1.19}$$

$$\Rightarrow H_V|\psi_n> = E_n|\psi_n> \tag{1.20}$$

where

$$|\psi_n> = e^s|\phi_n> \tag{1.21}$$

$$<\tilde{\psi}_n| = e^{-s} <\phi_n| \neq <\psi_n| \tag{1.22}$$

For the Black-Scholes Hamiltonian H_{BS} we have $V(x) = r$ and hence

$$H_{BS} = e^s H_{\rm Eff} e^{-s} \tag{1.23}$$

$$= e^{\alpha x}\Big[-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \gamma\Big]e^{-\alpha x} \tag{1.24}$$

where

$$\gamma = \frac{(r+\sigma^2/2)^2}{2\sigma^2} \quad ; \quad \alpha = \frac{\sigma^2/2 - r}{\sigma^2} \tag{1.25}$$

1.4 Solving the double knock out barrier option

A double barrier option is an option whose value reduces to zero whenever the price of the underlying instrument hits the barriers which we denote by e^a and e^b. Hence, the price of a double knock out barrier European call option expiring at time T and with strike price K at time t_0 provided it has not already been knocked out will be given by

$$e^{-r(T-t_0)}E_t[(e^{x(T)}-K)_+]\mathbf{1}_{a<x(t')<b,t_0<t'<T} \tag{1.26}$$

where $\mathbf{1}$ stands for the indicator function. It is sufficient to solve for the probability distribution of $x(T)$ for those paths which do not go outside the barriers (in other words, the pricing kernel).

Written as a path integral, the formula is

$$e^{-r(T-t_0)}\int \mathcal{D}x\Theta(x(t)-a)\Theta(b-x(t))e^{S_{BS}(x(t))}(e^{x(T)}-K)_+ \tag{1.27}$$

where S_{BS} is the Black-Scholes action

$$S_{BS}=-\frac{1}{2\sigma^2}\int dt(\dot{x}+r-\frac{\sigma^2}{2})^2 \tag{1.28}$$

While the step functions look complicated in the path integral, they can be seen to be having the effect of an infinite potential barrier since they effectively prohibit the path from entering the forbidden region outside the barriers. Hence, the problem might be better solved using the Hamiltonian and this is indeed the case.

In the Schrödinger formulation, the above problem is to find the pricing kernel for a system with the Hamiltonian

$$\hat{H} = \hat{H}_{BS} + V(x) \tag{1.29}$$

where the Black-Scholes Hamiltonian is given by

$$\hat{H}_{BS} = -\frac{\sigma^2}{2}\frac{\partial}{\partial x^2} + (\frac{\sigma^2}{2} - r)\frac{\partial}{\partial x} \tag{1.30}$$

and the potential $V(x)$ is given by

$$V(x) = \begin{cases} \infty & x < a \\ 0 & a < x < b \\ \infty & x > b \end{cases} \tag{1.31}$$

This is very similar to the well known problem of a particle in an infinite potential well except that the Hamiltonian has an extra term involving $\frac{\partial}{\partial x}$ which makes it non-Hermitian.

This problem can be solved by transforming the underlying wave functions. By making the transformation $\langle x \mid \phi \rangle = e^{-\alpha(x-a)} \langle x \mid \psi \rangle$ and $\langle \phi \mid x \rangle = e^{\alpha(x-a)} \langle \psi \mid x \rangle$, where $|\phi\rangle$ are the vectors in the new (Hilbert) space, $|\psi\rangle$ and $\langle\tilde{\psi}|$ are the original vectors and their duals respectively and $\alpha = \frac{\sigma^2/2 - r}{\sigma^2}$. In this new space, the Black-Scholes Hamiltonian takes the simple Hermitian form $-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}$.

The problem is now identical to that of a quantum mechanical particle of mass $\frac{1}{\sigma^2}$ (in units where $\hbar = 1$) in an infinite potential well. As is well known in this case, the allowed momenta are $p_n = \frac{n\pi}{b-a}$. The eigenfunctions are hence given by

$$\langle x \mid \psi_n \rangle = e^{\alpha(x-a)} \langle x \mid \phi_n \rangle = \sqrt{\frac{2}{b-a}} i e^{\alpha(x-a)} \sin p_n (x-a) \tag{1.32}$$

$$\langle \tilde{\psi}_n \mid x \rangle = e^{-\alpha(x-a)} \langle \phi_n \mid x \rangle = -\sqrt{\frac{2}{b-a}} i e^{-\alpha(x-a)} \sin p_n (x-a) \tag{1.33}$$

where $\langle x \mid \phi_n \rangle$ are the eigenfunctions of the quantum mechanical particle in an infinite potential well.

The eigenfunctions are orthonormal and form a complete basis since

$$\begin{aligned}
&\sum_{n=1}^{\infty} \langle x \mid \psi_n \rangle \langle \tilde{\psi}_n \mid x' \rangle = \frac{2}{b-a} e^{\alpha(x-x')} \sum_{n=1}^{\infty} \sin p_n (x-a) \sin p_n (x'-a) \\
&= \frac{1}{2(b-a)} e^{\alpha(x-x')} \sum_{n=-\infty}^{\infty} \left(\exp \frac{in\pi}{b-a}(x-x') - \exp \frac{in\pi}{b-a}(x+x'-2a) \right) \\
&= \frac{\pi}{b-a} e^{\alpha(x-x')} \left(\delta\left(\frac{\pi(x-x')}{b-a} \right) - \delta\left(\frac{\pi(x+x'-2a)}{b-a} \right) \right) \\
&= \delta(x-x')
\end{aligned} \tag{1.34}$$

since $a < x < b$ and $a < x' < b$.

The pricing kernel is hence given by

$$
\begin{aligned}
\langle x|e^{-\tau\hat{H}}|x'\rangle =& \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\langle x|\psi_n\rangle\langle\tilde{\psi}_n|e^{-\tau\hat{H}}|\psi_{n'}\rangle\langle\tilde{\psi}_{n'}|x'\rangle \\
=& \sum_{n=1}^{\infty}\langle x \mid \psi_n\rangle\langle\tilde{\psi}_n \mid x'\rangle e^{-\tau E_n} \\
=& \frac{1}{2(b-a)}\exp\left(-\frac{\tau\sigma^2\beta}{2}+\alpha(x-x')\right) \\
& \sum_{n=-\infty}^{\infty}\exp\left(-\frac{\tau\sigma^2 p_n^2}{2}\right)(e^{ip_n(x-x')}-e^{ip_n(x+x'-2a)}) \\
=& \frac{1}{2(b-a)}\exp\left(-\frac{\tau\sigma^2\beta}{2}+\alpha(x-x')\right)\sum_{n=-\infty}^{\infty}\int dy\delta(y-n)\exp\left(-\frac{y^2\pi^2\tau\sigma^2}{2(b-a)^2}\right) \\
& \left(\exp\frac{iy\pi(x-x')}{b-a}-\exp\frac{iy\pi(x+x'-2a)}{b-a}\right) \\
=& \sqrt{\frac{1}{2\pi\tau\sigma^2}}\exp\left(-\frac{\tau\sigma^2\beta}{2}+\alpha(x-x')\right) \\
& \sum_{n=-\infty}^{\infty}\left(\exp-\frac{(x-x'+2n(b-a))^2}{2\tau\sigma^2}-\exp-\frac{(x+x'-2a-2n(b-a))^2}{2\tau\sigma^2}\right)
\end{aligned}
\tag{1.35}
$$

where

$$\beta = \frac{(\sigma^2/2+r)^2}{\sigma^4} \tag{1.36}$$

and the identity

$$\delta(y-n) = \sum_{n=-\infty}^{\infty} e^{2\pi i n y} \tag{1.37}$$

has been used.

Hence, we see that the pricing kernel (apart from the drift terms) is given by an infinite sum of Gaussians. To check its reasonableness, we check the value in the limits $b \to \infty$ and $a \to -\infty$. In the former case, only the $n = 0$

term contributs and in the latter, only the $n = 0$ and $n = 1$ terms contribute. It is easy to see that, in both cases, the result reduces to the solution for the single knockout barrier pricing kernel. When both limits are simultaneously active, only the first term in the $n = 0$ term exists and it is easily seen that gives rise to the well known Black-Scholes pricing kernel.

We can now evaluate the price of a double barrier European call option using the pricing kernel from (1.35). The result is seen to be

$$\begin{aligned} f = \sum_{n=-\infty}^{\infty} \Big(& e^{-2n\alpha(b-a)} \left(e^{2n(b-a)} SN(d_{n1}) - Ke^{-r\tau} N(d_{n2}) \right) \\ & - S^{2\alpha} e^{-2\alpha(n(b-a)-a)} \left(e^{2n(b-a)} \frac{e^{2a}}{S} N(d_{n3}) - Ke^{-r\tau} N(d_{n4}) \right) \Big) \end{aligned} \tag{1.38}$$

where

$$d_{n1} = \frac{\ln(\frac{S}{K}) + 2n(b-a) + \tau\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}} \tag{1.39}$$

$$d_{n2} = \frac{\ln(\frac{S}{K}) + 2n(b-a) + \tau\left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}} = d_{n1} - \sigma\sqrt{\tau} \tag{1.40}$$

$$d_{n3} = \frac{\ln(\frac{e^{2a}}{SK}) + 2n(b-a) + \tau\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}} \tag{1.41}$$

$$d_{n4} = \frac{\ln(\frac{e^{2a}}{SK}) + 2n(b-a) + \tau\left(r - \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}} = d_{n3} - \sigma\sqrt{\tau} \tag{1.42}$$

Chapter 2

On the Trivariate Joint Distribution of Brownian Motion and its Maximum and Minimum

2.1 Introduction

Consider a standard Brownian motion $\{W_t | t \geq 0\}$ with $W_0 = 0$. Denote its maximum and minimum, respectively, by

$$l_t = \min_{0 \leq s \leq t} W_s \quad \text{and} \quad u_t = \max_{0 \leq s \leq t} W_s. \tag{2.1}$$

It is known that the trivariate joint distribution of (W_t, l_t, u_t) is expressed as

$$\begin{aligned} &P\left(a \leq l_t \leq u_t \leq b, W_t \in dx\right) \\ &= \frac{1}{\sqrt{2\pi t}} \sum_{k=-\infty}^{\infty} \left[\exp\left(-\frac{1}{2t}\left\{x - 2k(b-a)\right\}^2\right)\right. \\ &\qquad \left. - \exp\left(-\frac{1}{2t}\left\{x - 2b - 2k(b-a)\right\}^2\right)\right] dx, \end{aligned} \tag{2.2}$$

where $a \leq 0 \leq b$. This equation and its variants are found in the literature of probability such as Bachelier (1901), Lévy (1948, p. 213), Darling and Siegert (1953), Cox and Miller (1965, p. 222), Freedman (1970, pp. 26-7), Feller (1970, p. 341), Csáki (1978), Shorack and Wellner (1986, pp. 33-36), Teunen and Goovaerts (1994), Revuz and Yor (1998, p. 111), Borodin and Salminen (2002, p. 174), etc. The purposes of this chapter are to show uniform convergence of the infinite series of Equation (2.2), to show that it is a solution to the Fokker-Planck equation, to present some approximations of the trivariate joint probability density function, and to analyze their error bounds.

2.2 Uniform Convergence

For a fixed $t \in (0,\infty)$, define two sequences of functions $\{q_k(x;t) | k = \cdots, -1, 0, 1, \cdots\}$ and $\{r_k(x;t) | k = \cdots, -1, 0, 1, \cdots\}$, respectively, by

$$q_k(x;t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\{x - 2k(b-a)\}^2\right), \quad (a \le x \le b) \tag{2.3}$$

and

$$r_k(x;t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\{x - 2b - 2k(b-a)\}^2\right), \quad (a \le x \le b). \tag{2.4}$$

Clearly, $q_k(x;t)$ and $r_k(x;t)$ are positive for each k. Equation (2.2) becomes

$$P(a \le l_t \le u_t \le b, W_t \in dx) = \sum_{k=-\infty}^{\infty} \{q_k(x;t) - r_k(x;t)\}\, dx. \tag{2.5}$$

Equation (2.3) implies

$$\frac{q_{k+1}(x;t)}{q_k(x;t)} = \exp\left(\frac{2(b-a)}{t}\{x - b + a - 2k(b-a)\}\right), \tag{2.6}$$

which can be written as

$$q_{k+1}(x;t) = \beta(x)\alpha^k q_k(x;t), \quad (k = \cdots, -1, 0, 1, \cdots), \tag{2.7}$$

where

$$\alpha = \exp\left(-\frac{4(b-a)^2}{t}\right) \quad \text{and} \quad \beta(x) = \exp\left(\frac{2}{t}(b-a)(x-b+a)\right). \tag{2.8}$$

It can be driven from Equation (2.7) that

$$q_k(x;t) = \beta^k(x)\alpha^{\binom{k}{2}} q_0(x;t), \quad (k = \cdots, -1, 0, 1, \cdots), \tag{2.9}$$

where

$$q_0(x;t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}x^2\right) \quad \text{and} \quad \binom{k}{2} = \frac{k(k-1)}{2}. \tag{2.10}$$

Equations (2.7) and (2.8) imply

$$\lim_{k\to\infty} \frac{q_{k+1}(x:t)}{q_k(x;t)} = \lim_{k\to\infty} \beta(x)\alpha^k = 0. \tag{2.11}$$

The ratio test indicates that $\sum_{k=1}^{\infty} q_k(x;t)$ converges for any $x \in [a,b]$. For each $k \geq 1$, $q_k(x;t)$ is increasing on $[a,b]$, and then, $0 < q_k(x;t) \leq q_k(b;t)$. Since $\sum_{k=1}^{\infty} q_k(b;t)$ is convergent, the series $\sum_{k=1}^{\infty} q_k(x;t)$ converges uniformly on the compact set $[a,b]$. For details of this uniform convergence, readers may refer to Rudin (1976, p.148). Equation (2.7) implies that

$$\lim_{k\to-\infty} \frac{q_k(x;t)}{q_{k+1}(x:t)} = \lim_{k\to-\infty} \frac{1}{\beta(x)\alpha^k} = 0. \tag{2.12}$$

The ratio test implies that $\sum_{k=-\infty}^{-1} q_k(x;t)$ is convergent for any $x \in [a,b]$. For each $k \leq -1$, $q_k(x;t)$ is decreasing on $[a,b]$, and then, $0 < q_k(x;t) \leq q_k(a;t)$. Since $\sum_{k=-\infty}^{-1} q_k(a;t)$ is convergent, $\sum_{k=-\infty}^{-1} q_k(x;t)$ converges uniformly on the compact set $[a,b]$. Therefore, the series $\sum_{k=-\infty}^{\infty} q_k(x;t)$ converges uniformly on $[a,b]$. We know from Equation (2.9) that

$$\sum_{k=-\infty}^{\infty} q_k(x;t) = q_0(x;t) \sum_{k=-\infty}^{\infty} \beta^k(x)\alpha^{\binom{k}{2}}, \tag{2.13}$$

which can be represented by Jacobi's triple product identity (see, e.g., Zwillinger [2003, p. 48]) as follows.

$$\sum_{k=-\infty}^{\infty} q_k(x;t) = q_0(x;t) \prod_{j=1}^{\infty} \left\{ \left(1-\alpha^j\right) \left(1+\frac{1}{\beta(x)}\alpha^j\right) \left(1+\beta(x)\alpha^{j-1}\right) \right\} \tag{2.14}$$

Equation (2.4) implies

$$\frac{r_{k+1}(x:t)}{r_k(x;t)} = \exp\left(\frac{2(b-a)}{t}\{x-3b+a-2k(b-a)\}\right), \tag{2.15}$$

which can be written as

$$r_{k+1}(x:t) = \gamma(x)\alpha^k r_k(x;t), \quad (k=\cdots,-1,0,1,\cdots), \tag{2.16}$$

where

$$\gamma(x) = \exp\left(\frac{2}{t}(b-a)(x-3b+a)\right) = \beta(x)\exp\left(-\frac{4}{t}b(b-a)\right). \tag{2.17}$$

It can be driven from Equation (2.16) that

$$r_k(x;t) = \gamma^k(x)\alpha^{\binom{k}{2}} r_0(x;t), \quad (k=\cdots,-1,0,1,\cdots), \tag{2.18}$$

where

$$r_0(x;t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}[x-2b]^2 \right). \tag{2.19}$$

Applying the same method as before, we can show that the series $\sum_{k=-\infty}^{\infty} r_k(x;t)$ converges uniformly on $[a,b]$, and that its sum is

$$\sum_{k=-\infty}^{\infty} r_k(x;t) = r_0(x;t) \sum_{k=-\infty}^{\infty} \gamma^k(x) \alpha^{\binom{k}{2}}, \tag{2.20}$$

which can be expressed as

$$\sum_{k=-\infty}^{\infty} r_k(x;t) = r_0(x;t) \prod_{j=1}^{\infty} \left\{ \left(1-\alpha^j\right) \left(1+\frac{1}{\gamma(x)}\alpha^j\right) \left(1+\gamma(x)\alpha^{j-1}\right) \right\}. \tag{2.21}$$

We now summarize uniform convergence of $\sum_{k=-\infty}^{\infty} \{q_k(x;t) - r_k(x;t)\}$ as follows.

[Theorem 1] For integers M and N satisfying $M+N \geq 0$, let

$$S_{-M,N}(x,t) = \sum_{k=-M}^{N} \{q_k(x;t) - r_k(x;t)\},$$

where $a \leq x \leq b$ and $t > 0$. For a fixed t, as $M \to \infty$ and $N \to \infty$, $S_{-M,N}(x,t)$ converges uniformly to $S_{-\infty,\infty}(x,t)$ on the set $\{a \leq x \leq b\}$, which is equal to

$$\sum_{k=-\infty}^{\infty} \{q_k(x;t) - r_k(x;t)\} = \sum_{k=-\infty}^{\infty} \alpha^{\binom{k}{2}} \left\{q_0(x;t)\beta^k(x) - r_0(x;t)\gamma^k(x)\right\}.$$

The limit can be also expressed as

$$q_0(x;t)\prod_{j=1}^{\infty}\left\{\left(1-\alpha^j\right)\left(1+\frac{1}{\beta(x)}\alpha^j\right)\left(1+\beta(x)\alpha^{j-1}\right)\right\}$$
$$-r_0(x;t)\prod_{j=1}^{\infty}\left\{\left(1-\alpha^j\right)\left(1+\frac{1}{\gamma(x)}\alpha^j\right)\left(1+\gamma(x)\alpha^{j-1}\right)\right\}. \qquad \square$$

Consider the Fokker-Planck equation

$$\frac{\partial f(x,t)}{\partial t}=\frac{1}{2}\frac{\partial^2 f(x,t)}{\partial x^2}. \tag{2.22}$$

It is known (see, e.g., Cox and Miller [1965, p. 222]) that $S_{-\infty,\infty}(x,t) = \sum_{k=-\infty}^{\infty}\{q_k(x;t)-r_k(x;t)\}$ satisfies the Fokker-Planck equation (2.22). To prove it minutely, we need to show that the orders of infinite summation and differential operators of $\sum_{k=-\infty}^{\infty}\{q_k(x;t)-r_k(x;t)\}$ can be exchangeable, i.e., the infinite series is differentiable term by term. However, as far as the authors know, it has not been proven before. It can be shown as in Appendix that, for each integer k, $q_k(x;t)$ and $r_k(x;t)$ satisfy the Fokker-Planck equation (2.22), i.e.,

$$\frac{\partial q_k(x;t)}{\partial t}=\frac{1}{2}\frac{\partial^2 q_k(x;t)}{\partial x^2} \quad \text{and} \quad \frac{\partial r_k(x;t)}{\partial t}=\frac{1}{2}\frac{\partial^2 r_k(x;t)}{\partial x^2}. \tag{2.23}$$

Thus, $q_k(x;t)-r_k(x;t)$ is also a solution to the Fokker-Planck equation (2.22), and so is the linear superposition $S_{-M,N}(x,t)$ for any integers M and

N. As shown in Appendix using uniform convergence, we know that

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{\sum_{k=-\infty}^{\infty} q_k(x;t)\right\} = \frac{\partial}{\partial t}\left\{\sum_{k=-\infty}^{\infty} q_k(x;t)\right\}, \tag{2.24}$$

and

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{\sum_{k=-\infty}^{\infty} r_k(x;t)\right\} = \frac{\partial}{\partial t}\left\{\sum_{k=-\infty}^{\infty} r_k(x;t)\right\}. \tag{2.25}$$

Equations (2.24) and (2.25) imply

$$\frac{1}{2}\frac{\partial^2 S_{-\infty,\infty}(x,t)}{\partial x^2} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sum_{k=-\infty}^{\infty}\{q_k(x;t)-r_k(x;t)\}\right]$$
$$= \frac{\partial}{\partial t}\left[\sum_{k=-\infty}^{\infty}\{q_k(x;t)-r_k(x;t)\}\right] = \frac{\partial S_{-\infty,\infty}(x,t)}{\partial t}. \tag{2.26}$$

Also, it can be shown as in Appendix that, for any $t \in (0,\infty)$,

$$S_{-\infty,\infty}(a,t) = 0 \text{ and } S_{-\infty,\infty}(b,t) = 0. \tag{2.27}$$

[Theorem 2] The infinite series $S_{-\infty,\infty}(x,t) = \sum_{k=-\infty}^{\infty}\{q_k(x;t)-r_k(x;t)\}$ satisfies the following Fokker-Planck equation on $a < x < b$ and $t > 0$

$$\frac{\partial S_{-\infty,\infty}(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 S_{-\infty,\infty}(x,t)}{\partial x^2},$$

and the boundary conditions are $S_{-\infty,\infty}(a,t) = 0$ and $S_{-\infty,\infty}(b,t) = 0$ for any $t > 0$. □

2.3 Approximation and Error Bound

It can be shown that

$$\frac{r_k(x;t)}{q_k(x;t)} = \exp\left(\frac{2b}{t}\{x-b-2k(b-a)\}\right). \tag{2.28}$$

Equation (2.28) can be written as

$$r_k(x;t) = \eta(x)\delta^k q_k(x;t), \quad (k = \cdots, -1, 0, 1, \cdots), \tag{2.29}$$

where

$$\delta = \exp\left(-\frac{4b(b-a)}{t}\right) \quad \text{and} \quad \eta(x) = \exp\left(\frac{2b}{t}(x-b)\right). \tag{2.30}$$

Equation (2.29) implies that, for each $k(= \cdots, -1, 0, 1, \cdots)$,

$$q_k(x;t) - r_k(x;t) = \left\{1-\eta(x)\delta^k\right\} q_k(x;t) = \left\{\frac{1}{\eta(x)}\delta^{-k} - 1\right\} r_k(x;t) \tag{2.31}$$

Equations (2.7) and (2.29) imply that, for each $k(= \cdots, -1, 0, 1, \cdots)$,

$$r_k(x;t) - q_{k+1}(x;t) = \left\{\eta(x)\delta^k - \beta(x)\alpha^k\right\} q_k(x;t). \tag{2.32}$$

It is clear that, for any $x \in [a,b]$,

$$0 < \alpha < \delta < 1, \quad 0 < \beta(x) < \eta(x) \leq 1, \quad 0 < \gamma(x) < 1. \tag{2.33}$$

We know from Equations (2.31)-(2.33) that, for any $x \in [a,b]$,

$$\cdots \geq q_k(x:t) \geq r_k(x:t) \geq q_{k+1}(x;t) \geq r_{k+1}(x;t) \geq \cdots \to 0, \;\; (k = 1,2,\cdots), \tag{2.34}$$

and

$$\cdots \geq r_{k+1}(x;t) \geq q_{k+1}(x;t) \geq r_k(x:t) \geq q_k(x:t) \geq \cdots \to 0, \;\; (k = -1,-2,\cdots). \tag{2.35}$$

For any integers M and N satisfying $M+N \geq 0$, let

$$\varepsilon_{-M,N}(x,t) = S_{-\infty,\infty}(x,t) - S_{-M,N}(x,t). \tag{2.36}$$

The function $\varepsilon_{-M,N}(x,t)$ is the remainder of orders $(-M,N)$ or the error term for approximation of $S_{-\infty,\infty}(x,t)$ by $S_{-M,N}(x,t)$. It is clear that

$$S_{-M,N}(x,t) = S_{1,N}(x,t) + S_{0,0}(x,t) + S_{-M,-1}(x,t). \tag{2.37}$$

Equations (2.9), (2.18), and (2.31) imply that

$$S_{0,0}(x,t) = \{1-\eta(x)\}\, q_0(x;t) = \left\{\frac{1}{\eta(x)}-1\right\} r_0(x;t), \quad (2.38)$$

$$\begin{aligned} S_{1,N}(x,t) &= \sum_{k=1}^{N} \{1-\eta(x)\delta^k\}\, q_k(x;t) \\ &= q_0(x;t) \sum_{k=1}^{N} \{1-\eta(x)\delta^k\}\, \beta^k(x)\alpha^{\binom{k}{2}}, \quad (2.39) \\ S_{-M,-1}(x,t) &= \sum_{k=-M}^{-1} \left\{\frac{1}{\eta(x)\delta^k}-1\right\} r_k(x;t) \\ &= r_0(x;t) \sum_{k=-M}^{-1} \left\{\frac{1}{\eta(x)\delta^k}-1\right\} \gamma^k(x)\alpha^{\binom{k}{2}}. \quad (2.40) \end{aligned}$$

When we calculate $S_{1,N}(x,t)$ and $S_{-M,-1}(x,t)$, we would rather use Horner's method for computational efficiency.

For positive integers m and n, let

$$R_{N,0}(x,t) = q_{N+1}(x;t), \quad (2.41)$$

$$R_{N,n}(x,t) = q_{N+1}(x;t) - \sum_{k=N+1}^{N+n} \{r_k(x;t) - q_{k+1}(x;t)\}, \quad (2.42)$$

$$R_{-M,0}(x,t) = r_{-M-1}(x;t), \quad (2.43)$$

and

$$R_{-M,-m}(x,t) = r_{-M-1}(x;t) - \sum_{k=-M-m}^{-M-1} \{q_k(x;t) - r_{k-1}(x;t)\}. \quad (2.44)$$

Equations (2.34) and (2.35) imply that, for any $x \in [a,b]$ and $t > 0$,

$$R_{N,0}(x,t) \geq R_{N,1}(x,t) \geq R_{N,2}(x,t) \geq \cdots \geq R_{N,\infty}(x,t) > 0 \tag{2.45}$$

and

$$R_{-M,0}(x,t) \geq R_{-M,-1}(x,t) \geq R_{-M,-2}(x,t) \geq \cdots \geq R_{-M,-\infty}(x,t) > 0. \tag{2.46}$$

We know from the definition of $q_k(x;t)$ and $\eta(x)$ that

$$\frac{dq_k(x;t)}{dx} = \frac{1}{t}\{x - 2k(b-a)\} q_k(x;t), \tag{2.47}$$

$$\frac{d^2 q_k(x;t)}{dx^2} = \frac{1}{t}\left[1 - \frac{1}{t}\{x - 2k(b-a)\}^2\right] q_k(x;t), \tag{2.48}$$

$$\frac{d\eta(x)}{dx} = \frac{2b}{t}\eta(x). \tag{2.49}$$

Equations (2.31), (2.47), and (2.49) imply that

$$\frac{\partial}{\partial x}\{q_k(x;t) - r_k(x;t)\} = -\eta'(x)\delta^k q_k(x;t) + \left\{1 - \eta(x)\delta^k\right\} \frac{\partial q_k(x;t)}{\partial x}$$
$$= -\frac{2b}{t}\delta^k \eta(x) q_k(x;t) - \frac{1}{t}\left\{1 - \eta(x)\delta^k\right\}\{x - 2k(b-a)\} q_k(x;t) \tag{2.50}$$

Let

$$d_k(x) = 2b\eta(x)\delta^k + \left\{1 - \eta(x)\delta^k\right\}\{x - 2k(b-a)\}, \tag{2.51}$$

which is equal to

$$d_k(x) = \{-x + 2(k+1)b - 2ka\}\eta(x)\delta^k + \{x - 2k(b-a)\}. \tag{2.52}$$

Equation (2.50) can be written as

$$\frac{\partial}{\partial x}\{q_k(x;t) - r_k(x;t)\} = -\frac{q_k(x;t)}{t} d_k(x). \tag{2.53}$$

If $k > K_+ = \max\left\{\frac{3b}{2(b-a)}, \frac{t\ln 2}{4b(b-a)}\right\}$ and $x \in [a,b]$, then

$$\begin{aligned}
& d_k(x) \le \{-x + 2(k+1)b - 2ka\}\,\eta(b)\delta^k + \{x - 2k(b-a)\} \\
& = \{-x + 2(k+1)b - 2ka\}\,\delta^k + \{x - 2k(b-a)\} \\
& < \{-x + 2(k+1)b - 2ka\}\,\frac{1}{2} + \{x - 2k(b-a)\} \\
& \le \frac{3}{2}b - kb + ka < 0,
\end{aligned} \tag{2.54}$$

where the first inequality holds because $\eta(x)$ is increasing on $[a,b]$, the first equality does by $\eta(b) = 1$, the second inequality does because $\delta^k < 1/2$, and the last inequality does because $k > 3b/\{2(b-a)\}$. Equations (2.53) and (2.54) show that $q_k(x;t) - r_k(x;t)$ is increasing on $[a,b]$. Also, for any $k > 0$, $q_k(x;t)$ is increasing on $[a,b]$ for $k > K_+$. Thus, the following proposition holds;

$$N > K_+, n \ge 0, \text{ and } a \le x \le b \;\; \Rightarrow R_{N,n}(a,t) \le R_{N,n}(x,t) \le R_{N,n}(b,t). \tag{2.55}$$

If $k < K_- = \min\left\{\frac{a-4b}{2(b-a)}, -\frac{t\ln 2}{4b(b-a)} - \frac{1}{2}\right\}$ and $x \in [a,b]$, then

$$\begin{aligned}
& -d_k(x) \ge \{x - 2k(b-a) - 2b\}\,\eta(a)\delta^k - \{x - 2k(b-a)\} \\
& > \{x - 2k(b-a) - 2b\}\,\eta(a) \cdot \frac{2}{\eta(a)} - \{x - 2k(b-a)\} \\
& \ge a - 4b - 2k(b-a) > 0,
\end{aligned} \tag{2.56}$$

where the first inequality holds because $\eta(x)$ is increasing on $[a,b]$, the second inequality does because $\delta^k > 2/\eta(a)$, and the last inequality does because $k < (a-4b)/\{2(b-a)\}$. Equations (2.53) and (2.56) show that $r_k(x;t) - q_k(x;t)$ is decreasing on $[a,b]$ for $k < K_-$. Also, for any $k < 0$, $r_k(x;t)$ is decreasing on $[a,b]$. Thus, the following proposition holds;

$$M > -K_-,\ m \geq 0, \text{ and } a \leq x \leq b \ \Rightarrow R_{-M,-m}(b,t) \leq R_{-M,-m}(x,t) \leq R_{-M,-m}(a,t). \tag{2.57}$$

We now summarize properties of the error bounds of $S_{-M,N}(x,t)$ as follows.

[Theorem 3] For $M > -K_-$, $N > K_+$, $m \geq 0$, $n \geq 0$, $x \in [a,b]$, and $t > 0$, the following inequalities hold.

$$R_{N,n}(a,t) \leq R_{N,n}(x,t) \leq R_{N,n}(b,t)$$

$$R_{-M,-m}(b,t) \leq R_{-M,-m}(x,t) \leq R_{-M,-m}(a,t)$$

$$R_{N,n}(a,t) - R_{-M,-m}(a,t) \leq R_{N,n}(x,t) - R_{-M,-m}(x,t) \leq R_{N,n}(b,t) - R_{-M,-m}(b,t)$$

$$R_{N,\infty}(a,t) - R_{-M,-\infty}(a,t) \leq \varepsilon_{-M,N}(x,t) \leq R_{N,\infty}(b,t) - R_{-M,-\infty}(b,t)$$

□

Equations (2.45) and (2.46) and Theorem 3 imply that, for $M > -K_-$,

$N > K_+$, $m \geq 0$, $n \geq 0$, and $x \in [a,b]$,

$$-R_{-M,-m}(a,t) \leq -R_{-M,\infty}(a,t) \leq \varepsilon_{-M,N}(x,t) \leq R_{N,\infty}(b,t) \leq R_{N,n}(b,t) \tag{2.58}$$

and

$$|\varepsilon_{-M,N}(x,t)| \leq \max\{R_{-M,-m}(a,t), R_{N,n}(b,t)\}. \tag{2.59}$$

Jacobi's triple product identity representation in Theorem 1 implies that

$$S_{-\infty,\infty}(x,t) = J_L(x,t) + O\left(\alpha^L\right), \tag{2.60}$$

where $J_L(x,t)$ is defined by

$$q_0(x;t)\prod_{j=1}^{L}\left\{\left(1-\alpha^j\right)\left(1+\frac{1}{\beta(x)}\alpha^j\right)\left(1+\beta(x)\alpha^{j-1}\right)\right\}$$
$$-r_0(x;t)\prod_{j=1}^{L}\left\{\left(1-\alpha^j\right)\left(1+\frac{1}{\gamma(x)}\alpha^j\right)\left(1+\gamma(x)\alpha^{j-1}\right)\right\}. \tag{2.61}$$

Equation (2.60) means that the function $J_L(x,t)(x)$ is an approximation of $S_{-\infty,\infty}(x,t)$ with the remainder $O\left(\alpha^L\right)$.

(Example 1) We know that, for $N \geq 0$,

$$R_{N,0}(b,t) = q_{N+1}(b;t) = \beta^{N+1}(b)\alpha^{\binom{N+1}{2}}q_0(b;t) \leq \alpha^{\binom{N+1}{2}}\frac{1}{\sqrt{2\pi t}} \leq \alpha^{N^2/2}\frac{1}{\sqrt{2\pi t}}.$$

Hence, the following proposition holds.

$$N > \frac{1}{b-a}\sqrt{-\frac{t}{2}\ln\left(\varepsilon\sqrt{2\pi t}\right)} \;\Rightarrow\; R_{N,0}(b,t) < \varepsilon$$

Also, we know that, for $M \geq 1$,

$$R_{-M,0}(a,t) = r_{-M-1}(a;t) = \gamma^{-M-1}(a)\alpha^{\binom{-M-1}{2}} r_0(a;t)$$
$$\leq \left(\frac{\alpha^{3/2}}{\gamma(a)}\right)^{M+1} \alpha^{\binom{M+2}{2}-3(M+1)/2} r_0(a;t) \leq \alpha^{(M-1)^2/2}\frac{1}{\sqrt{2\pi t}},$$

where the first inequality holds because

$$\frac{\alpha^{3/2}}{\gamma^{M+1}(a)} = \exp\left(\frac{2}{t}(b-a)a\right) < 1.$$

Hence, the following proposition holds.

$$M-1 > \frac{1}{b-a}\sqrt{-\frac{t}{2}\ln\left(\varepsilon\sqrt{2\pi t}\right)} \;\Rightarrow\; R_{-M,0}(a,t) < \varepsilon$$

Thus, Equation (2.59) implies the following proposition.

$$\max\{M-1,N\} > \frac{1}{b-a}\sqrt{-\frac{t}{2}\ln\left(\varepsilon\sqrt{2\pi t}\right)} \;\Rightarrow\; |\varepsilon_{-M,N}(x,t)| < \varepsilon$$

As an example, let $a=-1,\ b=2,\ t=2,\ x=0.5$ and $\varepsilon = 10^{-15}$, then

$$\frac{1}{b-a}\sqrt{-\frac{t}{2}\ln\left(\varepsilon\sqrt{2\pi t}\right)} = 1.9228.$$

Thus, we may choose $M=3$ and $N=2$. The asymptotic values are as follows.

$(-M,N)$	$S_{-M,N}(0.5,2)$	$(-M,N)$	$S_{-M,N}(0.5,2)$
(0,0)	0.071034228403985	(-1,0)	0.054355942725271
(-1,1)	0.054397289152109	(-2,1)	0.054397288013575
(-2,2)	0.054397288013575	(-3,2)	0.054397288013575
(-3,3)	0.054397288013575	(-4,3)	0.054397288013575

We know from the above table that a pair of orders $(-M,N) = (-2,1)$ is good enough to obtain a finite series approximate value with absolute error less than 10^{-15}.

Let $L = \ln\alpha/\ln\varepsilon$. Then, $L = 1.9188$. The asymptotic values $J_l(0.5,2)$ of Jacob's triple product identity representation are as followes.

l	$J_l(0.5,2)$	l	$J_l(0.5,2)$
0	0.071034228403985	1	0.054397289405489
2	0.054397288013575	3	0.054397288013575
4	0.054397288013575	5	0.054397288013575

□

We know from the above table that an order $L = 2$ is good enough to obtain an approximate value of Jacob's triple product identity representation with absolute error less than 10^{-15}.

2.4 Conclusion

In this chapter, it is shown that the infinite series of the trivariate joint probability density function of Brownian motion and its maximum and minimum converges uniformly, and that it satisfies the Fokker-Planck equation. Also, the joint density function is represented through Jacobi's triple product identity. Moreover, some properties of error bounds to approximate the infinite series by a finite series are presented.

Chapter 3

Fitting the Risk-Neutral Density Function: The Generalized Beta Approach

3.1 Introduction

In this chapter we introduce a generalized distribution for describing security returns. McDonald, et al [23] have introduced generalized beta distribution for security returns. Many part of this introduction of this chapter has been adapted from McDonald, et al and Rebonato, et al [24]. The distribution has the feature of being extremely flexible, and it includes a large number of well-known distributions, such as the log-normal, log-t, and log-Cauchy distributions, as special or limiting cases. Distributions with large, even infinite higher moments can be specified by the choice of parameters. This flexibility allows a direct representation of different degrees of fat tails in the distribution. The generalized distribution also has a natural relation to much of the literature on mixed distributions since a wide range of mixed distributions can be described as special cases of this distribution.

There are two common approaches to the study of the distribution of security returns in the finance literature. The first begins by describing the process that gives rise to the returns, and the second begins by seeking to represent in a usable form a distribution function that empirically fits the observed return distribution. Much of the literature that relies on mixed distributions takes the first approach as its starting point and in doing so emphasizes the market process and the relation between various market variables, such as price variability and trading volume. A number of these papers lead to well-defined distributions. Others, which examine the trading process in greater detail, such as those of Epps and Epps (1976) [25], Oldfield, Rogalski, and Jarrow (1977) [26], and Tauchen and Pitts (1983) [27], lead to distributions that cannot be represented in explicit form or are difficult to specify and use in application.

The second approach serves as the starting point for a line of research that has its roots in the work of Fama (1963, 1965) [28] [29] and Mandelbrot (1963) [30]. This work begins with the empirical observation that stock returns are more peaked and have thicker tails than the log-normal and then finds a distribution function that fits this observation. One such set of distributions is characterized by a set of symmetric-stable distributions with characteristic exponents between one and two. For the details of these distributions, refer to Appendix B. These distributions are chosen both because of their fit to the observed distributions and because they have the attractive property of closure under multiplication. That is, the product of security returns will retain the same distributional form as for individual returns. There appears to have been little if any work to link this set of distributions to the actual mechanism of security trading. In this respect, these distributions remain only an empirical description of the fitted distributions.

The generalized distribution we present in this chapter has the advantage of being easily interpreted as a mixed distribution and has an easily expressible density function that makes it amenable to both empirical and theoretical work in which the density must be expressed explicitly.

A new approach is proposed in this chapter, by means of which an equity price or an interest or FX rate is modeled in such a way that its terminal distribution is assumed to have a particular four-parameter functional form that encompasses the log-normal distribution as a special case. For each expiry, the best combination of parameters that gives rise to an optimal (in a sense to be described) match to market call and put prices can be found using a very efficient and rapid procedure. The approach can prove useful in the marking-to-model of out-of-the-money options and in the creation of the smooth strike/expiry smile volatility surface needed as input for all process-driven pricing models. Further desirable features of the method stem from the fact that closed-form solutions are presented, not only for call and put prices consistent with this distribution but also for the cumulative distribution arising from the chosen density. Thanks to these analytic solutions, the search procedure needed to calibrate the model to market prices can be rendered extremely fast.

The advantages of the approach presented in this chapter should be important:

• To begin with, since the distribution function is directly modeled, the resulting density is ensured by construction to assume a well-behaved and “plausible” shape. Since, as noted above, very small changes in input prices can correspond to very different distributions, it conversely follows that an approach starting directly from the distribution can fit a great variety of market prices with little loss of precision.

• Wildly fluctuating local volatilities (a common by-product of spot-based tree implementations) are no longer encountered.

• We express the closed-form solutions for calls, puts, and their derivatives in terms of the integrals of a family of functions to which the log-normal distribution belongs as a special case; in addition, the functional form of these closed-form solutions is such that they have an easily recognizable Black-like appearance, making their use easy and straightforward for practitioners accustomed to pricing using the market-standard Black formula. The existence of a smiley volatility quote simply stems from the familiarity of the market participants with the Black conceptual and computational framework.

Alternative techniques have been proposed in order to fit the market implied volatility surface: the mixture of two log-normals, the Edgeworth expansion, or even a spline-fitting to the smile curve. We believe that the approach we recommend in this chapter displays noticeable advantages over these techniques. Spline-fitting is notoriously unstable; being based on a series of polynomials, asymptotically it produces answers that bear no similarity to the function to be fitted (an implied volatility surface); it does not allow closed-form pricing formulas. Most importantly, spline-fitting to the volatility surface generates an implied density via the second derivative of the call prices with respect to the strike. Since splines are not linked in any fundamental way to the underlying density, there is in general no guarantee that double differentiation will give rise to an admissible density. The approach described in this chapter guarantees that this will not happen because the density itself is the starting point, rather than the by-product of a double differentiation.

As for the other approaches mentioned above, the Generalized Beta 2 (GB2 in the following) method can be implemented so as to be significantly more parsimonious: after the first moment is matched by enforcing the correct pricing of the forward rate/price, and the equivalent volatility determined from the market data, there remain only two free parameters per maturity. Nonetheless, in all the tests we have run, the fit has always proved to be excellent, plausibly indicating that the deviations from the log-normal density are well captured by just two moments above the second. Furthermore, the similarity of the pricing equations with the familiar Black formula is only encountered with the GB2 approach and should constitute a powerful incentive for its adoption by the financial community.

Stephen J. Taylor, et al [32] have introduced theoretical risk-neutral densities. The rest of this introduction of this chapter has been adapted from Stephen J. Taylor, et al [32]. Breeden and Litzenberger (1978) [31] show that a unique risk-neutral density g for a subsequent asset price S_T can be inferred from European call prices $C(K)$ when contracts are priced for all strikes K and there are no arbitrage opportunities. The riskneutral density (RND) is then

$$g(K) = e^{rT} \frac{\partial^2 C}{\partial K^2} \tag{3.1}$$

and

$$C(K) = e^{-rT} \int_K^\infty (x-K) g(x) dx \tag{3.2}$$

with r the risk-free rate and T the time remaining until all options expire. The forward price F, for time T, is the risk-neutral expectation of S_T ; it is also a futures price, assuming non-stochastic interest rates and dividend payments. These relationships between the RND and derivative prices are the basis for empirical derivations of implied RNDs, despite the impossibility of obtaining option data for a continuum of strikes.

Parametric families of RNDs are estimated in this chapter. a parameter vector θ is estimated by minimizing the average squared difference between observed market prices and theoretical option prices, namely

$$\frac{1}{N}\sum_{i=1}^{N}(C_{market}(K_i) - C(K_i|\theta))^2, \tag{3.3}$$

with

$$C(K_i|\theta) = e^{-rT}\int_{K_i}^{\infty}(x - K_i)g(x|\theta)dx, 1 \leq i \leq N. \tag{3.4}$$

In these equations, N is the number of prices obtained from option quotes or trades during a particular day and $g(x|\theta)$ is a parametric density function that produces the theoretical option pricing formula $C(K|\theta)$ given by equation(3.2). We choose specific parametric densities for the RNDs because they enable us to obtain closed-form real-world densities.

3.2 The relation of the GB2 to log-normal distribution

The GB2 includes the generalized gamma (GG) as a limiting case:

$$GG(x;a,\beta,p) = \lim_{q\to\infty} g_{GB2}(x;a,\beta q^{\frac{1}{a}},p,q) \tag{3.5}$$

Further limits applied to the GG lead to the log-normal density(LN) as a special limiting case of the GB2:

$$LN(x;\mu,\sigma) = \lim_{a\to 0} GG[x;a,\beta = (\sigma^2 a^2)^{\frac{1}{a}}, p = (a\mu+1)/\beta^a] \tag{3.6}$$
$$= \lim_{a\to 0}\lim_{q\to\infty} g_{GB2}(x;a,b = (\sigma^2 a^2 q)^{\frac{1}{a}}, p = \frac{a\mu+1}{\sigma^2 a^2}, q)$$

3.3 Risk-neutral condition of generalized beta distribution of the second kind

For general distribution $f_g^Q(S_T)$, we choose generalized beta distribution of the second kind (GB2). Refer to McDonald(1987) [23] for details of GB2 distribution. The GB2 probability density function is defined as follows

$$g_{GB2}(x;a,b,p,q) = \frac{|a|x^{ap-1}}{b^{ap}B(p,q)[1+(x/b)^a]^{p+q}} = f_g^Q(x). \tag{3.7}$$

The density is risk-neutral if

$$\langle x\rangle = \int_0^\infty x\cdot g_{GB2}(x;a,b,p,q)dx \tag{3.8}$$
$$= \frac{bB(p+\frac{1}{a}, q-\frac{1}{a})}{B(p,q)} = S_0 e^{rT}$$

On the other hand, if we set $x = S_0 e^y$, then we can describe risk-neutral condition as follows

$$\begin{aligned} \langle y \rangle &= \int_{-\infty}^{\infty} y \cdot g_{GB2}(S_0 e^y; a, b, p, q) S_0 e^y dy \\ &= \int_0^{\infty} \ln \frac{x}{S_0} \cdot g_{GB2}(x; a, b, p, q) dx \\ &= \ln \frac{b}{S_0} + \frac{1}{a}\left(\frac{\Gamma'(p)}{\Gamma(p)} - \frac{\Gamma'(q)}{\Gamma(q)}\right) = rT \end{aligned} \tag{3.9}$$

If we take logarithm at equation(3.8), then we can see that

$$\ln S_0 + rT = \ln b + \ln \Gamma(P + \frac{1}{a}) - \ln \Gamma(P) + \ln \Gamma(q + \frac{1}{a}) - \ln \Gamma(q) \tag{3.10}$$

And If take the 1st order Taylor expansion of logarithm of gamma function in equation(3.10), then we can find that equation(3.10) have the same result as equation(3.9)

$$rT \simeq \ln \frac{b}{S_0} + \frac{1}{a}\left(\frac{d \ln \Gamma(P)}{dp} - \frac{d \ln \Gamma(q)}{dq}\right) \tag{3.11}$$

3.4 GB2 Option Pricing

The theoretical option pricing formula depends on the cumulative distribution function of GB2 density, denoted G_{GB2}. And G_{GB2} is a function of cumulative distribution function of the beta distribution, denoted I_z and called the incomplete beta function:

$$G_{GB2}(x;a,b,p,q) = G_{GB2}((x/b)^a;1,1,p,q) = I_z(p,q) \tag{3.12}$$

with $z(x,a,b) = (x/b)^a/(1+(x/b)^a)$. If the density is risk-neutral, so that the constraint in equation(3.8) applies, then European call option prices are given by

$$\begin{aligned} C(K;\theta) &= e^{-rT}\int_K^{\infty}(x-K)g_{GB2}(x;a,b,p,q)dx \\ &= S_0\left[1-G_{GB2}(K;a,b,p+\frac{1}{a},q-\frac{1}{a})\right] - Ke^{-rT}\left[1-G_{GB2}(K;a,b,p,q)\right] \\ &= S_0\left[1-I_z(p+\frac{1}{a},q-\frac{1}{a})\right] - Ke^{-rT}\left[1-I_z(p,q)\right]. \end{aligned} \tag{3.13}$$

And, put option prices are given by

$$\begin{aligned} P(K;\theta) &= e^{-rT}\int_0^{K}(K-x)g_{GB2}(x;a,b,p,q)dx \\ &= Ke^{-rT}G_{GB2}(K;a,b,p,q) - S_0G_{GB2}(K;a,b,p+\frac{1}{a},q-\frac{1}{a}) \\ &= Ke^{-rT}I_z(p,q) - S_0I_z(p+\frac{1}{a},q-\frac{1}{a}). \end{aligned} \tag{3.14}$$

We can see that Equation(3.13) and (3.14) satisfy Put-Call Parity condition.

$$S_0 + P(K;\theta) = Ke^{-rT} + C(K;\theta) \tag{3.15}$$

3.5 GB2 Single Barrier Option Pricing

Using the Reflection Principle of Geometric Brownian Motion(GBM), we can find the payoff function of Up-and-Out call option, as follows;

$$\begin{aligned} C_T^{UO} =& [S_T - K]1(S_T > K) - [S_T - K]1(S_T > S_U) \\ & - \alpha[Y_T - K]1(Y_T > K) + \alpha[Y_T - K]1(Y_T > S_U) \end{aligned} \tag{3.16}$$

where

$$\alpha = \left[\frac{S_U}{S_0}\right]^{\frac{2r}{\sigma^2}-1} \tag{3.17}$$

Therefore, an Up-and-Out call option for a general distribution can be priced as follows;

$$\begin{aligned} C_0^{UO} =& e^{-rT} \int [S_T - K] I(S_T > K) f_g^Q(S_T) dS_T \\ & - e^{-rT} \int [S_T - K] I(S_T > S_U) f_g^Q(S_T) dS_T \\ & - e^{-rT} \int \alpha [Y_T - K] I(Y_T > K) f_g^Q(Y_T) dY_T \\ & + e^{-rT} \int \alpha [Y_T - K] I(Y_T > S_U) f_g^Q(Y_T) dY_T, \end{aligned} \tag{3.18}$$

Where $f_g^Q(S_T)$ is a probability density function(PDF) under risk-neutral measure Q.

That is to say, path dependent barrier options can be calculated with payoff condition of equation(3.16).

$$C^{UO} = e^{-rT}\int_{K}^{B_+} [x-K]\, g_{GB2}(x;a,b,p,q)dx \qquad (3.19)$$
$$- e^{-rT}\int_{K}^{B_+} \alpha(x)\left[\frac{B_+^2}{x} - K\right] g_{GB2}(x;a,b,p,q)dx$$

where $\alpha(x) = \left[\frac{B_+}{x}\right]^{\frac{2r}{\sigma^2}-1} = \left[\frac{B_+}{x}\right]^{\lambda}$

$$\begin{aligned}
C^{UO} =& S_0\left[I_{Z_2}(p+\frac{1}{a},q-\frac{1}{a}) - I_{Z_1}(p+\frac{1}{a},q-\frac{1}{a})\right] \qquad (3.20)\\
& - Ke^{-rT}\left[I_{Z_2}(p,q) - I_{Z_1}(p,q)\right]\\
& + \frac{B_+^{\lambda+2}B(p-\frac{\lambda+1}{a},q+\frac{\lambda+1}{a})e^{-rT}}{b^{\lambda+1}B(p,q)}\\
& \times\left[I_{Z_3}(p-\frac{\lambda+1}{a},q+\frac{\lambda+1}{a}) - I_{Z_2}(p-\frac{\lambda+1}{a},q+\frac{\lambda+1}{a})\right]\\
& - \frac{KB_+^{\lambda}B_+(p-\frac{\lambda}{a},q+\frac{\lambda}{a})e^{-rT}}{b^{\lambda}B(p,q)}\left[I_{Z_3}(p-\frac{\lambda}{a},q+\frac{\lambda}{a}) - I_{Z_2}(p-\frac{\lambda}{a},q+\frac{\lambda}{a})\right]\\
=& S_0\left[I_{Z_2}(p+\frac{1}{a},q-\frac{1}{a}) - I_{Z_1}(p+\frac{1}{a},q-\frac{1}{a})\right]\\
& - Ke^{-rT}\left[I_{Z_2}(p,q) - I_{Z_1}(p,q)\right]\\
& + B_+e^{-rT}E\left[\left(\frac{B_+}{S_T}\right)^{\lambda+1}\right]\left[I_{Z_3}(p-\frac{\lambda+1}{a},q+\frac{\lambda+1}{a}) - I_{Z_2}(p-\frac{\lambda+1}{a},q+\frac{\lambda+1}{a})\right]\\
& - Ke^{-rT}E\left[\left(\frac{B_+}{S_T}\right)^{\lambda}\right]\left[I_{Z_3}(p-\frac{\lambda}{a},q+\frac{\lambda}{a}) - I_{Z_2}(p-\frac{\lambda}{a},q+\frac{\lambda}{a})\right]
\end{aligned}$$

where $Z_1 = \frac{\left(\frac{K}{b}\right)^a}{1+\left(\frac{K}{b}\right)^a}, Z_2 = \frac{\left(\frac{B_+}{b}\right)^a}{1+\left(\frac{B_+}{b}\right)^a}, Z_3 = \frac{\left(\frac{B_+^2}{bK}\right)^a}{1+\left(\frac{B_+^2}{bK}\right)^a}$ and $I_Z(p,q)$ is a incomplete beta function.

3.6 GB2 double barrier option

3.6.1 Log-normal distribution Case

Up and Out Down and Out Call Option can be priced according to equation(1.38)

$$
\begin{aligned}
C_{\text{LN}}^{\text{UODO}} e^{rT} &= \int_K^{B_+} [S_T - K] LN^Q(S_T) dS_T \\
&+ \sum_{n=1}^{\infty} (-1)^n \int_K^{B_+} \prod_1^n \alpha_i [Y_n - K] LN^Q(Y_n) dY_n \\
&+ \sum_{n=1}^{\infty} (-1)^n \int_K^{B_+} \prod_1^n \bar{\alpha}_i [\bar{Y}_n - K] LN^Q(\bar{Y}_n) d\bar{Y}_n
\end{aligned}
\tag{3.21}
$$

where

$$
\begin{aligned}
Y_{2m+1} &= \frac{B_+^2}{S_0} \left(\frac{B_+}{B_-}\right)^{2m} \\
Y_{2m} &= S_0 \left(\frac{B_-}{B_+}\right)^{2m} \\
\alpha_{2m+1} &= \left(\frac{Y_{2m+1}}{B_+}\right)^{\frac{2r}{\sigma^2}-1} \\
\alpha_{2m} &= \left(\frac{Y_{2m}}{B_-}\right)^{\frac{2r}{\sigma^2}-1}
\end{aligned}
\tag{3.22}
$$

$$C_{\text{LN}}^{\text{UODO}} \tag{3.23}$$
$$= \sum_{-\infty}^{\infty} \left[\left(\frac{B_-}{B_+} \right)^n \right]^\lambda \left\{ \left(\frac{B_-}{B_+} \right)^{2n} S_0[N(d_{n1}) - N(d_{n3})] - Ke^{-rT}[N(d_{n2}) - N(d_{n4})] \right\}$$
$$+ \sum_{-\infty}^{\infty} \left[\left(\frac{B_+}{B_-} \right)^n \frac{B_+}{S_0} \right]^\lambda \left\{ \left(\frac{B_+}{B_-} \right)^{2n} \frac{B_+^2}{S_0}[N(d_{n5}) - N(d_{n7})] - Ke^{-rT}[N(d_{n6}) - N(d_{n8})] \right\}$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2} \right) T}{\sigma\sqrt{T}}, \;\; d_2 = d_1 - \sigma\sqrt{T}, \\ d_3 &= \frac{\ln \frac{S_0}{B_+} + \left(r + \frac{\sigma^2}{2} \right) T}{\sigma\sqrt{T}}, \;\; d_4 = d_3 - \sigma\sqrt{T}, \\ d_5 &= \frac{\ln \frac{B_+^2}{S_0 K} + \left(r + \frac{\sigma^2}{2} \right) T}{\sigma\sqrt{T}}, \;\; d_6 = d_5 - \sigma\sqrt{T}, \\ d_7 &= \frac{\ln \frac{B_+}{S_0} + \left(r + \frac{\sigma^2}{2} \right) T}{\sigma\sqrt{T}}, \;\; d_8 = d_7 - \sigma\sqrt{T} \end{aligned} \tag{3.24}$$

for $i = 1, 2, 3, 4$

$$d_{ni} = d_i + \frac{2n \ln \frac{B_-}{B_+}}{\sigma\sqrt{T}} \tag{3.25}$$

for $i = 5, 6, 7, 8$

$$d_{ni} = d_i - \frac{2n \ln \frac{B_-}{B_+}}{\sigma\sqrt{T}}. \tag{3.26}$$

3.6.2 Generalized Beta distribution Case

With the help of the payoff function of double barrier option, we can integrate and find the closed form solution of double barrier option. For the case of GB2 distribution, the formula can be calculated as following.

$$
\begin{aligned}
& e^{rT} C^{\mathrm{UODO}} \qquad (3.27)\\
= \quad & \sum_{n=-\infty}^{\infty} \int_{K\left(\frac{B_+}{B_-}\right)^{2n}}^{B_+\left(\frac{B_+}{B_-}\right)^{2n}} \left(\frac{B_-}{B_+}\right)^{n\lambda} \left[\left(\frac{B_-}{B_+}\right)^{2n} x - K\right] g_{GB2}(x;a,b,p,q)dx \\
& - \sum_{n=-\infty}^{\infty} \int_{\frac{B_+^2}{K}\left(\frac{B_+}{B_-}\right)^{2n}}^{B_+\left(\frac{B_+}{B_-}\right)^{2n}} \left[\left(\frac{B_-}{B_+}\right)^{n} \frac{B_+}{x}\right]^{\lambda} \left[\left(\frac{B_+}{B_-}\right)^{2n} \frac{B_+^2}{x} - K\right] g_{GB2}(x;a,b,p,q)dx \\
= \quad & \sum_{n=-\infty}^{\infty} \frac{bB(p+\frac{1}{a}, q-\frac{1}{a})}{B(p,q)} \left(\frac{B_-}{B_+}\right)^{n(\lambda+2)} \left[I_{Z_{n2}}(p+\frac{1}{a}, q-\frac{1}{a}) - I_{Z_{n1}}(p+\frac{1}{a}, q-\frac{1}{a})\right] \\
& - \sum_{n=-\infty}^{\infty} K \left(\frac{B_-}{B_+}\right)^{n\lambda} \left[I_{Z_{n2}}(p,q) - I_{Z_{n1}}(p,q)\right] \\
& - \sum_{n=-\infty}^{\infty} \frac{B_+^{\lambda+2} B(p-\frac{\lambda+1}{a}, q+\frac{\lambda+1}{a})}{b^{\lambda+1} B(p,q)} \left(\frac{B_+}{B_-}\right)^{n(\lambda+2)} \\
& \times \left[I_{Z_{n2}}(p-\frac{\lambda+1}{a}, q+\frac{\lambda+1}{a}) - I_{Z_{n3}}(p-\frac{\lambda+1}{a}, q+\frac{\lambda+1}{a})\right] \\
& + \sum_{n=-\infty}^{\infty} \frac{KB(p-\frac{\lambda}{a}, q+\frac{\lambda}{a})}{b^{\lambda} B(p,q)} \left[\left(\frac{B_+}{B_-}\right)^{n} B_+\right]^{\lambda} \left[I_{Z_{n2}}(p-\frac{\lambda}{a}, q+\frac{\lambda}{a}) - I_{Z_{n3}}(p-\frac{\lambda}{a}, q+\frac{\lambda}{a})\right]
\end{aligned}
$$

$$= \sum_{n=-\infty}^{\infty} S_0 e^{rT} \left(\frac{B_-}{B_+}\right)^{n(\lambda+2)} \left[I_{Z_{n2}}(p+\frac{1}{a}, q-\frac{1}{a}) - I_{Z_{n1}}(p+\frac{1}{a}, q-\frac{1}{a})\right] \tag{3.28}$$

$$- \sum_{n=-\infty}^{\infty} K \left(\frac{B_-}{B_+}\right)^{n\lambda} [I_{Z_{n2}}(p,q) - I_{Z_{n1}}(p,q)]$$

$$- \sum_{n=-\infty}^{\infty} B_+ \left\langle \left(\frac{B_+}{x}\right)^{\lambda+1} \right\rangle \left(\frac{B_+}{B_-}\right)^{n(\lambda+2)}$$

$$\times \left[I_{Z_{n2}}(p-\frac{\lambda+1}{a}, q+\frac{\lambda+1}{a}) - I_{Z_{n3}}(p-\frac{\lambda+1}{a}, q+\frac{\lambda+1}{a})\right]$$

$$+ \sum_{n=-\infty}^{\infty} K \left\langle \left[\left(\frac{B_+}{B_-}\right)^n \frac{B_+}{x}\right]^{\lambda} \right\rangle \left[I_{Z_{n2}}(p-\frac{\lambda}{a}, q+\frac{\lambda}{a}) - I_{Z_{n3}}(p-\frac{\lambda}{a}, q+\frac{\lambda}{a})\right]$$

where

$$Z_{n1} = \frac{\left[\frac{K}{b}\left(\frac{B_+}{B_-}\right)^{2n}\right]^a}{1+\left[\frac{K}{b}\left(\frac{B_+}{B_-}\right)^{2n}\right]^a}, \tag{3.29}$$

$$Z_{n2} = \frac{\left[\frac{B_+}{b}\left(\frac{B_+}{B_-}\right)^{2n}\right]^a}{1+\left[\frac{B_+}{b}\left(\frac{B_+}{B_-}\right)^{2n}\right]^a},$$

$$Z_{n3} = \frac{\left[\frac{B_+^2}{bK}\left(\frac{B_+}{B_-}\right)^{2n}\right]^a}{1+\left[\frac{B_+^2}{bK}\left(\frac{B_+}{B_-}\right)^{2n}\right]^a},$$

$$\lambda = \frac{2r}{\sigma^2} - 1$$

and where $I_Z(p,q)$ is a incomplete beta function. The equation (3.28) holds because

$$\langle x^n \rangle = \frac{b^n B(p+\frac{n}{a}, q-\frac{n}{a})}{B(p,q)} \tag{3.30}$$
$$S_0 e^{rT} = \frac{bB(p+\frac{1}{a}, q-\frac{1}{a})}{B(p,q)}\ .$$

With these equations, we can plot various option graphs.

Figure. 1 shows various call option prices including up-and-out down-and-out GB2, and Black-scholes option prices.

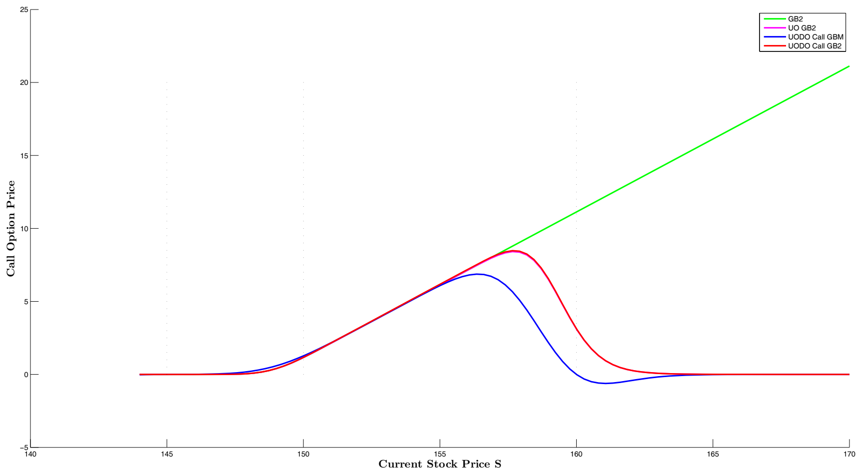

Figure. 1: Call Option Prices (strike price: 150, interest rate: 0.04, up barrier: 160, down barrier: 145)

We made the 3D plots with the axis of current stock price, time to maturity and option price. Figure.2 and 3 compares Black-sholes and GB2 Call option prices.

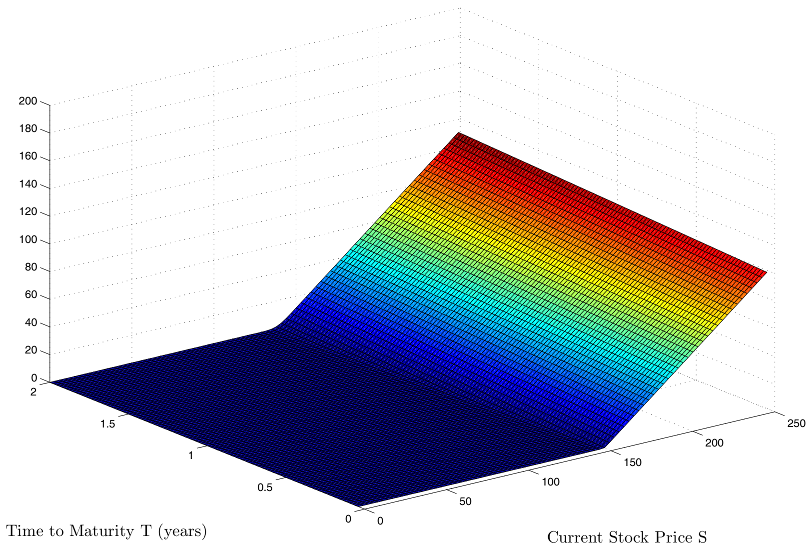

Figure. 2: Black-scholes Call Option (strike price: 150, interest rate: 0.04)

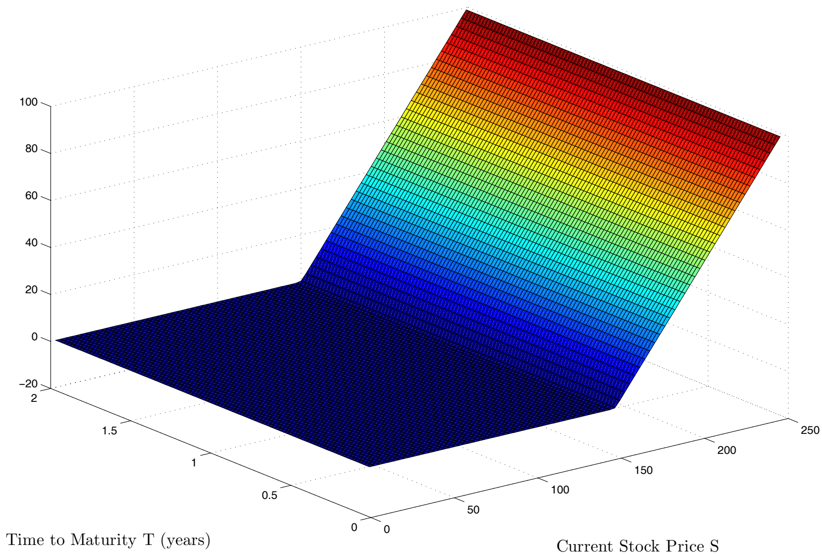

Figure. 3: GB2 Call Option (strike price: 150, interest rate: 0.04)

Figure.4 and 5 compares Black-sholes and GB2 Up-and-Out Down-and-Out call option prices.

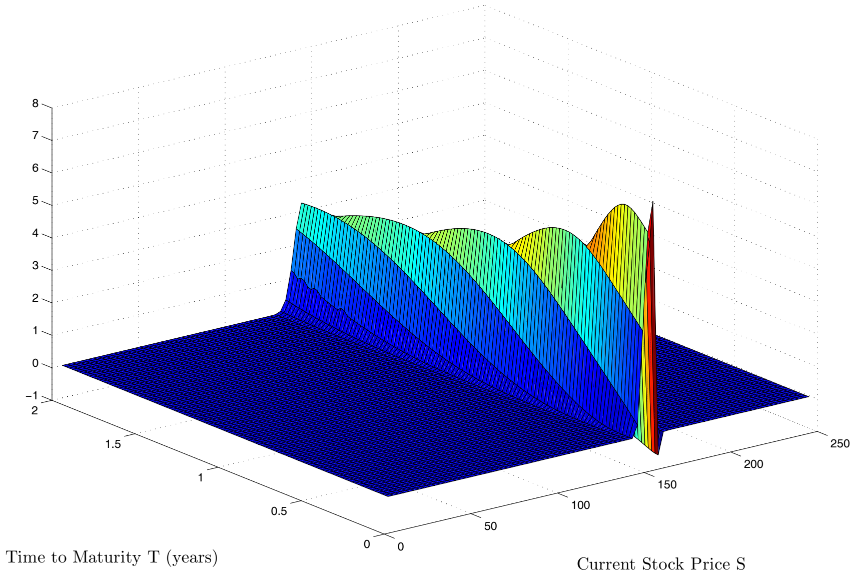

Figure. 4: Up-and-Out Down-and-Out Black-Scholes Call Option (strike price: 150, interest rate: 0.04, up barrier: 160, down barrier: 145)

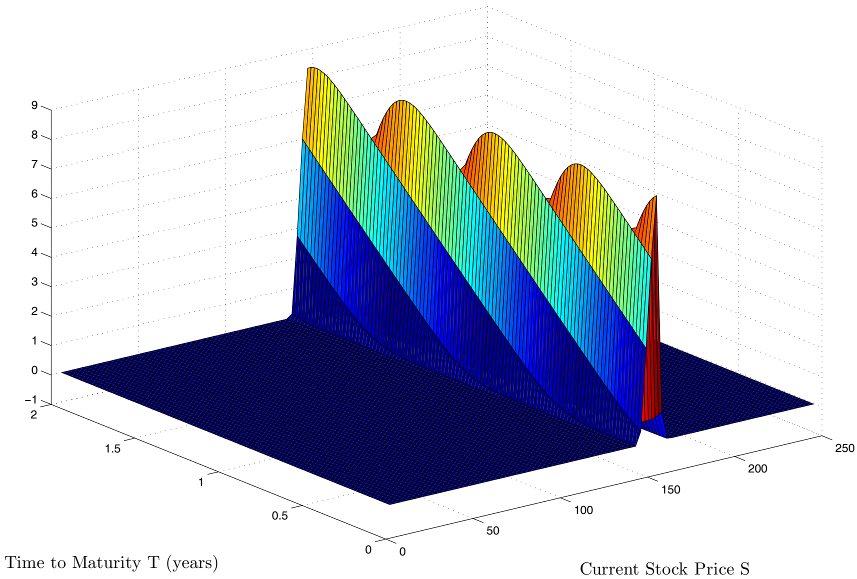

Figure. 5: Up-and-Out Down-and-Out GB2 Call Option(strike price: 150, interest rate: 0.04, up barrier: 160, down barrier: 145)

Chapter 4

Conclusions

4.1 Conclusion

It is shown that the infinite series of the trivariate joint probability density function of Brownian motion and its maximum and minimum converges uniformly, and that it satisfies the Fokker-Planck equation. Also, the joint density function is represented through Jacobi's triple product identity. Moreover, some properties of error bounds to approximate the infinite series by a finite series are presented.

Using the method developed in log-normal distribution(reflection principle), we made the pay-off conditions of the barrier options. With the pay-off conditions and risk-neutral condition of generalized beta distribution of second kind, we calculated closed form solution of single and double barrier option prices.

Chapter A

Appendix A - Proof of Equations

A.1 Proof of Equation in Chapter 2

[Proof of Equation (2.23)]

For a real number m, let

$$f_m(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(x-m)^2\right). \tag{A.1.1}$$

We can show that

$$\frac{\partial f_m}{\partial t} = -\frac{t-(x-m)^2}{2t^2} f_m(x,t), \tag{A.1.2}$$

$$\frac{\partial^2 f_m}{\partial t^2} = \frac{(x-m)^4 - 6t(x-m)^2 + 3t^2}{4t^4} f_m(x,t), \tag{A.1.3}$$

$$\frac{\partial f_m}{\partial x} = \frac{m-x}{t} f_m(x,t), \tag{A.1.4}$$

$$\frac{\partial^2 f_m}{\partial x^2} = -\frac{t-(x-m)^2}{t^2} f_m(x,t), \tag{A.1.5}$$

$$\frac{\partial^3 f_m}{\partial x^3} = \frac{(x-m)\left\{3t-(x-m)^2\right\}}{t^3} f_m(x,t). \tag{A.1.6}$$

Equations (A.1.2) and (A.1.5) imply that $f(x,t)$ satisfies the following Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}. \tag{A.1.7}$$

Thus, $q_k(x;t)$ and $r_k(x;t)$ satisfy the Fokker-Planck equation.

[Proof of Equations (2.24) and (2.25)]

Let t be a positive constant. If $k > \frac{b+\sqrt{3t}}{2(b-a)}$, then Equation (A.1.6) implies that $\frac{\partial^3 q_k(x;t)}{\partial x^3}$ is positive on $[a,b]$, and that $\frac{\partial^2 q_k(x;t)}{\partial x^2}$ is increasing on $[a,b]$. Thus, we know that, for any $x \in [a,b]$,

$$0 < \frac{\partial^2 q_k(x;t)}{\partial x^2} < \frac{\partial^2 q_k(b;t)}{\partial x^2} = -\frac{t - \{b - 2k(b-a)\}^2}{t^2} q_k(b;t), \quad \text{(A.1.8)}$$

where the first inequality and the equality hold by Equation (A.1.5). Equation (2.11) implies

$$\lim_{k \to \infty} \frac{-\frac{t-\{b-2(k+1)(b-a)\}^2}{t^2} q_{k+1}(b;t)}{-\frac{t-\{b-2k(b-a)\}^2}{t^2} q_k(b;t)} = 0. \quad \text{(A.1.9)}$$

The ratio test in Equation (A.1.9) implies that $\sum_{k=1}^{\infty} \frac{\partial^2 q_k(b;t)}{\partial x^2}$ converges. Thus, Equation (A.1.8) implies that $\sum_{k=1}^{\infty} \frac{\partial^2 q_k(x;t)}{\partial x^2}$ converges uniformly on the compact set $[a,b]$. Similarly, we can show that $\sum_{k=-\infty}^{-1} \frac{\partial^2 q_k(x;t)}{\partial x^2}$ converges uniformly on the compact set $[a,b]$, and so does $\sum_{k=-\infty}^{\infty} \frac{\partial^2 q_k(x;t)}{\partial x^2}$. Therefore,

$$\frac{\partial^2}{\partial x^2} \left\{ \sum_{k=-\infty}^{\infty} q_k(x;t) \right\} = \sum_{k=-\infty}^{\infty} \frac{\partial^2 q_k(x;t)}{\partial x^2}. \quad \text{(A.1.10)}$$

It can be shown using the same method that $\sum_{k=-\infty}^{\infty} \frac{\partial^2 r_k(x;t)}{\partial x^2}$ converges uniformly on the compact set $[a,b]$, and that

$$\frac{\partial^2}{\partial x^2} \left\{ \sum_{k=-\infty}^{\infty} r_k(x;t) \right\} = \sum_{k=-\infty}^{\infty} \frac{\partial^2 r_k(x;t)}{\partial x^2}. \quad \text{(A.1.11)}$$

Let $x \in (a,b)$ be fixed. For $m > 0$, Equation (A.1.3) implies that the equation $\frac{\partial^2 f_m(x,t)}{\partial t^2} = 0$ holds at $t = t_{1,m} = \left(1 + \sqrt{\frac{2}{3}}\right)(x-m)^2$ and $t = t_{2,m} = \left(1 - \sqrt{\frac{2}{3}}\right)(x-m)^2$. Therefore, $\left|\frac{\partial f_m(x,t)}{\partial t}\right|$ has its supremum at one of the points $\{0, t_1, t_2, \infty\}$. Equations (A.1.1) and (A.1.2) imply that

$$\lim_{t\to 0}\left|\frac{\partial f_m}{\partial t}\right| = \lim_{t\to 0}\frac{\left|t-(x-m)^2\right|}{2t^2} f(x,t) = 0, \tag{A.1.12}$$

$$\lim_{t\to \infty}\left|\frac{\partial f_m}{\partial t}\right| = \lim_{t\to \infty}\frac{\left|t-(x-m)^2\right|}{2t^2} f(x,t) = 0, \tag{A.1.13}$$

$$\begin{aligned}\left|\frac{\partial f_m(x,t_{1,m})}{\partial t}\right| &= \left|\frac{t_{1,m}-(x-m)^2}{2t_{1,m}^2} f(x,t_{1,m})\right| \\ &= \frac{\sqrt{2/3}}{2\sqrt{2\pi}\left(1+\sqrt{2/3}\right)^{5/2}}\exp\left(\frac{-1}{2\left(1+\sqrt{2/3}\right)}\right)\frac{1}{|x-m|^3}\end{aligned} \tag{A.1.14}$$

and

$$\begin{aligned}\left|\frac{\partial f_m(x,t_{2,m})}{\partial t}\right| &= \left|\frac{t_{2,m}-(x-m)^2}{2t_{2,m}^2} f(x,t_{2,m})\right| \\ &= \frac{\sqrt{2/3}}{2\sqrt{2\pi}\left(1-\sqrt{2/3}\right)^{5/2}}\exp\left(\frac{-1}{2\left(1-\sqrt{2/3}\right)}\right)\frac{1}{|x-m|^3}\end{aligned} \tag{A.1.15}$$

Equations (A.1.12)-(A.1.15) imply that there exists a constant $c \in (0,\infty)$ satisfying

$$\max_{0<t<\infty}\left|\frac{\partial f_m(x,t)}{\partial t}\right| \le c\frac{1}{|x-m|^3}. \tag{A.1.16}$$

The integral test indicates

$$\sum_{m=1}^{\infty}\frac{1}{|x-m|^3} < \infty \tag{A.1.17}$$

Equations (A.1.16) and (A.1.17) imply that $\sum_{m=1}^{\infty}\frac{\partial f_m(x,t)}{\partial t}$ converges uniformly on the set $\{0 < t < \infty\}$. For this uniform convergence property, readers may refer to Rudin (1976, p. 152). Similarly, it can be proved that $\sum_{m=-\infty}^{-1} \frac{\partial f_m(x,t)}{\partial t}$ converges uniformly on the set $\{0 < t < \infty\}$. So does $\sum_{m=-\infty}^{\infty} \frac{\partial f_m(x,t)}{\partial t}$. This uniform convergence implies that

$$\sum_{k=-\infty}^{\infty} \frac{\partial f_k(x,t)}{\partial t} = \frac{\partial}{\partial t}\left\{\sum_{k=-\infty}^{\infty} f_k(x,t)\right\}. \tag{A.1.18}$$

Equation (A.1.18) implies that

$$\sum_{k=-\infty}^{\infty} \frac{\partial q_k(x;t)}{\partial t} = \frac{\partial}{\partial t}\left\{\sum_{k=-\infty}^{\infty} q_k(x;t)\right\} \tag{A.1.19}$$

and

$$\sum_{k=-\infty}^{\infty} \frac{\partial r_k(x;t)}{\partial t} = \frac{\partial}{\partial t}\left\{\sum_{k=-\infty}^{\infty} r_k(x;t)\right\}. \tag{A.1.20}$$

Equations (2.23), (A.1.10), and (A.1.19) implies Equation (2.24), and Equations (2.23), (A.1.11), and (A.1.20) does Equation (2.25).

[Proof of Equation (2.27)]

Equations (2.8) and (2.17) imply that

$$\beta(b)\gamma(b) = \exp\left(\frac{2}{t}[b-a][2a-2b]\right) = \alpha. \tag{A.1.21}$$

Thus,

$$\frac{1}{\beta(b)}\alpha^j + \beta(b)\alpha^{j-1} = \gamma(b)\alpha^{j-1} + \frac{1}{\gamma(b)}\alpha^j. \tag{A.1.22}$$

It is clear from Equations (2.10) and (2.19) that

$$q_0\,(b;t) = r_0\,(b;t)\,. \tag{A.1.23}$$

Equations (A.1.22) and (A.1.23) and Theorem 1 yield

$$\begin{aligned} & q_0(b;t)\prod_{j=1}^{\infty}\left\{\left(1-\alpha^j\right)\left(1+\frac{1}{\beta(b)}\alpha^j\right)\left(1+\beta(b)\alpha^{j-1}\right)\right\} \\ & = r_0(b;t)\prod_{j=1}^{\infty}\left\{\left(1-\alpha^j\right)\left(1+\frac{1}{\gamma(b)}\alpha^j\right)\left(1+\gamma(b)\alpha^{j-1}\right)\right\}, \end{aligned} \tag{A.1.24}$$

which implies $S_{-\infty,\infty}(b,t) = 0$.

Equations (2.8) and (2.17) imply that

$$\beta(a)\gamma(a) = \exp\left(\frac{2}{t}[b-a][4a-4b]\right) = \alpha^2. \tag{A.1.25}$$

Equations (2.8), (2.10), and (2.19) imply that

$$r_0\,(a;t)\,\beta(a) = q_0\,(a;t)\,\alpha. \tag{A.1.26}$$

It can be shown that

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} r_k(a;t) = r_0(a;t)\sum_{k=-\infty}^{\infty}\gamma^k(a)\alpha^{\binom{k}{2}} = r_0(a;t)\sum_{k=-\infty}^{\infty}\gamma^{k-1}(a)\alpha^{\binom{k-1}{2}} \\ & = r_0(a;t)\sum_{k=-\infty}^{\infty}\beta^{1-k}(a)\alpha^{2(k-1)}\alpha^{\binom{k-1}{2}} = q_0(a;t)\sum_{k=-\infty}^{\infty}\beta^{-k}(a)\alpha^{\binom{-k}{2}} \\ & = q_0(a;t)\sum_{k=-\infty}^{\infty}\beta^k(a)\alpha^{\binom{k}{2}} = \sum_{k=-\infty}^{\infty} q_k(a;t), \end{aligned} \tag{A.1.27}$$

where the first equality holds by Equation (2.18), the third does by Equation (A.1.25), and the fourth does by Equation (A.1.26). Equation (A.1.27) implies $S_{-\infty,\infty}(a,t) = 0$.

Chapter B

Appendix B - Lévy Diffusion and Fractional Fokker–Planck Equation

B.1 Generalized Langevin equation

This Appendix B has been adapted from A.V. Tour, et al [33].

We start with the Langevin–like equation for a stochastic quantity $X(t)$:

$$\frac{dX(t)}{dt} = Y(t) \tag{B.1.1}$$

In the classical theory of a Brownian motion, $X(t)$ is the location of Brownian particle under the influence of stochastic pulses $Y(t)$. The statistical properties of this stochastic forcing will be specified below. We first need to derive an equation for the distribution function

$$p(x,t) = \langle \delta[x - X(t)] \rangle \tag{B.1.2}$$

where the brackets $\langle ... \rangle$ denote statistical averaging over stochastic force realisations. Due to the fact that the Dirac function is the Fourier transform of the unity, we have:

$$\delta[x - X(t)] = \int_{-\infty}^{\infty} \frac{dk}{2\pi} exp\{-ik[x - X(t)]\} \tag{B.1.3}$$

When averaged, Eq.B.1.3 yields merely that the probability is the inverse Fourier transform of the characteristic function $Z_X(k,t)$:

$$Z_X(k,t) = \langle exp(ikX(t)]) \rangle \tag{B.1.4}$$

$$p(x,t) = F^{-1}[Z_X(k,t)] \tag{B.1.5}$$

where F and F^{-1} denote respectively the Fourier–transform and its inverse:

$$F[f] = \hat{f}(k) = \int_{-\infty}^{\infty} dx \;\; exp(ikx) f(x) \tag{B.1.6}$$
$$F^{-1}[\hat{f}] = f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \;\; exp(-ikx) \hat{f}(k)$$

On the other hand, Eq.B.1.1 can be integrated into:

$$X(t) = X(0) + \int_0^t d\tau Y(\tau) \tag{B.1.7}$$

Since we can assume [1] without loss of generality that $X(0) = 0$, we obtain the following equation:

$$\frac{\partial p}{\partial t} = F^{-1}[\frac{\partial}{\partial t} \left\langle exp\left[ik \int_0^t d\tau Y(\tau)\right] \right\rangle] \tag{B.1.8}$$

Now, to make a further step, it is necessary to specify the statistical properties of the stochastic source. We consider the particular example [56] when the source is represented as a sum of independent stochastic "pulses" acting at equally spaced times t_j:

[1] Indeed, we are considering only the 'forward' Fokker-Planck equation.

$$Y(t) = \sum_{j=0}^{\infty} Y_{j,\Delta} \Delta \delta(t - t_j) \quad . \tag{B.1.9}$$

where $t_0 = 0, t_{j+1} - t_j = \Delta$ $(j = 0, 1, 2, ...)$ and the pulses $Y_{j,\Delta}$ are independent stochastic variables having stable Lévy distribution $P\{Y_{j,\Delta}\}$ for all j and which has the following characteristic function [34]

$$Z_{Y_{j,\Delta}}(k) = \langle exp(ikY_{j,\Delta}) \rangle = exp\Delta \left\{ i\gamma k - D|k|^{\alpha} \left[1 - i\beta \frac{k}{|k|} \omega(k, \alpha) \right] \right\} \tag{B.1.10}$$

where α, β, γ, D are real constants $(0 < \alpha \leq 2, -1 \leq \beta \leq 1, D \geq 0)$ and $\omega(k, \alpha)$ is defined as:

$$\alpha \neq 1 : \ \ \omega(k, \alpha) = tan \frac{\pi \alpha}{2}; \quad \alpha = 1 : \ \ \omega(k, \alpha) = \frac{\pi}{2} log|k| \tag{B.1.11}$$

α and β classify the type of the stable distributions up to translations and dilatations: with given α and β, γ and D can vary without changing the type of a stable distribution. The parameter α characterizes the asymptotic behaviour of the stable distribution:

$$p(x) \sim x^{-1-\alpha}, x \rightarrow \infty \tag{B.1.12}$$

hence, corresponds to the critical order of moments for their divergence:

$$\mu \geq \alpha : \langle x^{\mu} \rangle = \infty, \tag{B.1.13}$$

For (additive) walks α is also related to the fractal dimension of the trail [41], whereas for the generator of the (multiplicative) universal multifractals it measures their multifractality [50]. The parameter β characterizes the degree of asymmetry of distribution function. Indeed, if $\beta = 0$, then negative and positive values of $Y_{j,\Delta}$ occur with equal probabilities, while if $\beta = 1$ or $\beta = -1$ (maximally asymmetric distributions) then, for $0 < \alpha < 1$ and $\gamma = 0$ $P\{Y_{j,\Delta}\}$ vanishes outside from $[0, +\infty]$ or respectively from $[-\infty, 0]$ [2]. We already mentioned that maximal asymmetry is required for generators of universal multifractals; let us add that in this case the Laplace transform is more convenient than the Fourier transform. The nonzero value of β implies the existence of a primary direction of the stochastic pulses (that is, the direction to plus or minus infinity), and thus the existence of a drift for particles in this direction. For more details concerning the properties of stable laws see, e.g. [57]. The meaning of γ and D will be discussed and clarified below.

Now, using Eq.B.1.9 and the independence condition of the stochastic pulses $Y_{j,\Delta}$ we get:

$$\left\langle exp\left[ik\int_0^t d\tau y(\tau)\right]\right\rangle = \left\langle exp\left[ik\sum_{j=0}^{n} Y_{j,\Delta}\right]\right\rangle \tag{B.1.14}$$

$$= \prod_{j=0}^{n} \langle exp\langle ikY_{j,\Delta}\rangle\rangle = \langle exp(ikY_{j,\Delta})\rangle^n$$

where n is a number of pulses corresponding to the present time $t = n\Delta$.

[2]For $\alpha > 1$, $P\{Y_{j,\Delta}\}$ decays faster than an exponential on the corresponding half axis.

Therefore , with the help of the equation of the characteristic function of the pulses (Eq.B.1.10), we obtain the characteristic function $Z_X(k,t)$ (Eq.B.1.4) of the stable process:

$$Z_X(k,t) = \left\langle exp\left[ik\int_0^t d\tau Y(\tau)\right]\right\rangle \tag{B.1.15}$$
$$= exp\left\{t\left[i\gamma k - D|k|^\alpha\left(1 - i\beta\frac{k}{|k|}\omega(k,\alpha)\right)\right]\right\}$$

The fact that this process has stationary independent increments [58] (i.e. pulses $Y_{j,\Delta}$) gives the possibility to get directly Eq.B.1.15 without using any discretisation of $Y(t)$ as previously done (Eq.B.1.9).

Now inserting this expression of $Z_X(k,t)$ into Eq.B.1.8, one obtains:

$$\frac{\partial p}{\partial t} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}[i\gamma k - D|k|^\alpha + i\beta D\omega(k,\alpha)k|k|^{\alpha-1}]Z_X(k,t)exp(-ikx) \tag{B.1.16}$$

For the sake of the simplicity of notations, we will consider in the following only the case $\alpha \neq 1,\ or\ \beta = 0$. Therefore, Eq.B.1.11 reduces to:

$$\omega(k,\alpha) \equiv \omega(\alpha) = tan\frac{\pi\alpha}{2} \tag{B.1.17}$$

B.2 An expression of the Fractional Fokker–Planck Equation

One can see that in Eq.B.1.16 the following type of integrals appears $F^{-1}(|k|^{\alpha}Z_X]$, which in fact correspond to fractional differentiations. Indeed, one may use Laplacian power for the Riesz's definition of a fractional differentiation since for any function $f(x)$:

$$-\Delta f(x) = F^{-1}(|k|^2\hat{f}(k)) \tag{B.2.1}$$

yields a rather straightforward extension:

$$(-\Delta)^{\alpha/2} f(x) = F^{-1}(|k|^{\alpha}\hat{f}(k)) \tag{B.2.2}$$

Then, Eq.B.1.16 yields:

$$\frac{\partial p}{\partial t} + \gamma \frac{\partial p}{\partial x} = -D\left[(-\Delta)^{\alpha/2} p + \beta\omega(\alpha)\frac{\partial}{\partial x}(-\Delta)^{(\alpha-1)/2} p\right] \tag{B.2.3}$$

which for symmetric laws $\beta = 0$ is a straightforward generalization of the classical Fokker–Planck equation, by:

$$\Delta \to -(-\Delta)^{\alpha/2} \tag{B.2.4}$$

This also points out that the scale parameter D of the Lévy distribution corresponds to the diffusion coefficient of the Fractional Fokker Planck equation. On the other hand, the second term in the left hand side of Eq.B.2.3

has an obvious physical meaning. Independently on the value of α, it describes the convection of particles by the (constant) velocity γ. For $\alpha > 1$, γ corresponds furthermore to the mean value of the source $\langle Y(t)\rangle$, whereas it is no more the case for $\alpha \leq 1$ since the latter is no longer finite. In the latter case, the diffusion term has a a derivation order smaller or equal to the convection term. This confirms that the case $\alpha = 1$ is indeed critical between two rather distinct regimes and it is more involved than other cases. Besides, it is worthwhile to note the role of the term (on the r.h.s.) related to asymmetry $(\beta \neq 0)$. On the one hand, this term can be interpreted as an additional contribution to the convection due to existence of the preferred direction of the pulses related to $(\beta \neq 0)$. On the other hand, such a flow is not proportional to p (as the convective flow does) but rather to $(-\Delta)^{(\alpha-1)/2}p$, which is rather typical for the diffusion flow. In some sense, due to this term the division of flows into convective and diffusion ones (as done in the standard Fokker–Planck equation) becomes rather questionable and presumably no longer relevant for the Fractional Fokker–Planck equation. One may note that a somewhat similar weakening of this distinction occurs also in the classical Fokker-Plank for nonlinear systems [59]. On the other hand, it is easy to check that the Fractional Fokker–Planck equation is Galilean invariant, as is should be: the velocity of the moving framework just add to γ.

B.3 The non uniqueness of the expression of the Fractional Fokker-Planck Equation

One cannot expect to obtain a unique expression for the Fractional the Fokker-Planck equation, since there is not a unique generalization of the differentiation to a fractional order. Indeed, there exist various definitions of the fractional differentiation (see, e.g. [60] and references therein) which are not equivalent. This will be illustrated by two examples in the next section. The first one is related to the fact that there are 'signed' (fractional) differentiation and respectively 'unsigned' (fractional) differentiations, i.e. differentiations which are not invariant and respectively invariant with the mirror symmetry $x \to -x$. In the case of standard differentiation, the question of signs is fixed: 'signed' and 'unsigned' differentiations correspond merely to odd and respectively even orders of differentiation (hence the unique expression of the classical Fokker–Planck equation, which is of second order). This is no longer the case for fractional differentiations.

The second example corresponds to the fact that fractional differentiations are in fact defined by integration, and therefore can depend on the bounds of integration.

Nevertheless, we are convinced that the expression corresponding to Eq.B.2.3 is at the same time the simplest one to derive and the one whose physical significance is the most straightforward. On the other hand, let us emphasize that the existence of distinct expressions for the Fractional Fokker-Planck equation does not question the uniqueness of its solution. Indeed, these distinct expressions are equivalent because their solution should correspond to the unique probability density function corresponding to a given Langevin–like equation (Eq.B.1.1).

The non uniqueness could be rather understood in the following way: corresponding to the distinct fractional differentiations (and their corresponding fractional integrations), there should be distinct ways of solving the Fractional Fokker-Planck equation in order to obtain its unique solution.

B.4 Two alternative expressions of the Fractional Fokker-Planck Equation

Contrary to the unsigned fractional power of a Laplacian Eq.B.2.2, let us consider for instance the following 'signed' fractional differentiation:

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x) = F^{-1}[(-ik)^\alpha \hat{f}(k)]. \tag{B.4.1}$$

With the help of (i) the identity ($\theta(k)$ being the Heaviside function):

$$|k|^\alpha = k^\alpha[\theta(k) + (-1)^\alpha \theta(-k)] \tag{B.4.2}$$

and of (ii) the inverse Fourier transform of the Heaviside function:

$$F^{-1}[\theta(k)] = \frac{1}{2}\delta(x) + \frac{1}{2\pi i x} \tag{B.4.3}$$

as well as of (iii) the property that a Fourier transform of a product corresponds to the convolution of the Fourier transforms, one derives from Eq.B.1.16 an another form of the Fractional Fokker–Planck equation.

$$\frac{\partial p}{\partial t}+\gamma\frac{\partial p}{\partial x}=-D\left(cos\frac{\pi\alpha}{2}+\beta sin\frac{\pi\alpha}{2}tan\frac{\pi\alpha}{2}\right)\frac{\partial^\alpha p}{\partial x^\alpha}$$
$$-D(1-\beta)sin\frac{\pi\alpha}{2}\frac{\partial^\alpha}{\partial x^\alpha}\int_{-\infty}^{\infty}\frac{dx'}{\pi}\frac{p(x',t)}{x-x'} \tag{B.4.4}$$

Indeed, with the help of the following determinations[3] $(-i)^\alpha = e^{-i\frac{\alpha\pi}{2}}, (-1)^\alpha = e^{-i\alpha\pi}$, Eq.B.4.2 yields:

$$|k|^\alpha = (-ik)^\alpha[\theta(k)e^{i\frac{\alpha\pi}{2}}+\theta(-k)e^{-i\frac{\alpha\pi}{2}}] \tag{B.4.5}$$

and with the help of Eqs.B.4.1,B.4.3,B.4.5, it is rather straightforward to derive Eq.B.4.4.

However, Eq.B.4.4 is already rather involved in the case $\beta = 0$, whereas this case is obvious for the equivalent Eq.B.2.3:

$$\frac{\partial p}{\partial t}=-\gamma\frac{\partial p}{\partial x}-Dcos\frac{\pi\alpha}{2}\frac{\partial^\alpha p}{\partial x^\alpha}-Dsin\frac{\pi\alpha}{2}\frac{\partial^\alpha}{\partial x^\alpha}\int_{-\infty}^{\infty}\frac{dx'}{\pi}\frac{p(x',t)}{x-x'} \tag{B.4.6}$$

the last term of the r.h.s. of Eq.B.4.6 is rather complex, whereas indispensable. Indeed, there is a need of signed second term to counterbalance the first signed term of Eq. B.4.6, in order that the r.h.s. of Eq.B.4.6 will correspond to an unsigned differentiation (the fractional power of the Laplacian in Eq.B.2.3). Both terms correspond to the signed fractional differentiation of order α but whereas it is applied to p in the former term, it is applied to

[3] One may note that the existence of other determinations confirms the non uniqueness of the fractional derivative defined in eq.B.4.2. Furthermore, taking another determination will merely modify some prefactors in r.h.s. of Eq.B.4.4

an integration of a zero order of p in the latter term. This zero order integration corresponds to the effective interaction of particles having a scaling law inversely proportional to the distance between them. An analogy with the interaction between dislocation lines [61] can be mentioned. It is plausible that the collective effect corresponding to this the effective interaction of particles could be responsible of the large jumps which are so important in Lévy motions.

An other expression of the Fractional Fokker–Planck equation can be also obtained with the help of the Riemann–Liouville derivatives. The μ-th order Riemann–Liouville derivatives on the real axis are defined as

$$
\begin{aligned}
(\mathbf{D}^{\mu}_{+} f)(x) &= \frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_{-\infty}^{x} dx' \frac{f(x')}{(x-x')^{\mu}} \\
(\mathbf{D}^{\mu}_{-} f)(x) &= -\frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_{x}^{\infty} dx' \frac{f(x')}{(t-x')^{\mu}}
\end{aligned}
\tag{B.4.7}
$$

where $\mathbf{D}^{\mu}_{-}, \mathbf{D}^{\mu}_{+}$ are respectively the left-side and the right-side derivatives of fractional order μ $(0 < \mu < 1)$ and Γ is the Euler's gamma-function. An other expression of the Fractional Fokker–Planck equation can be:

$$
\frac{\partial p}{\partial t} + \gamma \frac{\partial p}{\partial x} = -D \mathbf{D}^{\alpha/2}_{+} \mathbf{D}^{\alpha/2}_{-} p - D \beta \omega(\alpha) \frac{\partial}{\partial x} \mathbf{D}^{(\alpha-1)/2}_{+} \mathbf{D}^{(\alpha-1)/2}_{-} p \tag{B.4.8}
$$

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초 록

통계물리학적인 방법론을 경제시스템에 적용하여 그에 따라 나타나는 금융상품의 가치평가에 대한 연구를 진행하였다. 원자산이 추계 확률 과정(stochastic process)을 따를 때, 그에 따라 금융 상품의 가치는 일반적으로 어느 주어진 시점에서의 확률분포의 기대값으로 구해지게 되어 원자산의 값이 시간에 따라 변해온 경로와는 무관(path independent)하게 된다. 그러나 원자산의 값이 정해진 상한이나 하한을 넘는지 여부를 관찰하는 경로 의존적인(path dependent) 금융상품의 경우, 그 가치를 평가하기가 어려운데, 물리학에서의 반사원리(reflection principle)와 경로적분(path integral)을 활용하여 복잡한 금융상품의 가치평가를 수행하였다. 우선, 원자산의 수익률이 정규분포를 따르고 원자산의 상한과 하한이 동시에 존재하는 금융상품(double barrier option)의 가치가 여러 기대값들의 무한 합으로 구해질 때, 그것이 수렴함을 보였고, 수치적으로 몇 개 항의 합만으로도 정확성을 유지할 수 있다는 것을 보였다. 또한, 원자산의 수익률이 정규분포를 따르지 않고 두터운 꼬리(heavy-tailed distribution)를 가지는 모형 하에서의 경로 의존적인 금융상품의 가치평가도 수행을 하여 해석적인 해를 얻었다.

주요어 : 베리어 옵션, 반사원리, 브라운 운동, 최대값, 최소값, 결합확률분포, 오차범위, 자코비 삼중곱, 꼬리가 두꺼운 분포, GB2

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