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Embeddings between complex Grassmannians

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Abstract

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In this thesis, we characterize the linearity of holomorphic embeddings of the complex Grassmannian $Gr(2, m)$ into the complex Grassmannian $Gr(2, n)$. We study such embeddings by finding all possible total Chern classes of the pullback bundles E of the dual bundles of the universal bundles on $Gr(2, n)$ under these embeddings. We first take a $\mathbb{Z}$-module basis of the cohomology ring of $Gr(2, m)$ which is useful for further works, and express every cohomology classes as a linear combination with respect to this basis. For each holomorphic embedding of $Gr(2, m)$ into $Gr(2, n)$, the total Chern class of E is written uniquely as a linear combination of basis elements with three integral coefficients, the linearity of the embedding is determined completely by these integers. We obtain several conditions on the three integers, including a upper bound of the specific integer, by solving 3-variate Diophantine equations which are constructed from the Chern classes and the Euler class of the normal bundle induced by the embedding, together with a criterion of the numerical non-negativity of Chern classes of holomorphic vector bundles. This upper bound enables us to apply W. Barth and A. Van de Ven's results to E, and we find conditions on m and n for which any holomorphic embedding of $Gr(2, m)$ into $Gr(2, n)$ is linear.

Key words: Complex Grassmannians, Holomorphic embeddings, Schubert cycles, Chern classes
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Contents

Chapter 1

Introduction

Let d, m be positive integers with $d < m$. For an m-dimensional complex vector space V, a complex Grassmannian $Gr(d, V)$ is the space parameterizing all d-dimensional subspaces of V. Since the description of $Gr(d, V)$ is concrete and explicit, and concerns matrices and vector spaces over the complex field $\mathbb{C}$, many practical and computable techniques to study it have been developed. There are algebraic varieties which are generalized from $Gr(d, V)$, for instance, an orthogonal Grassmannian $Gr_q(d, V)$, a symplectic Grassmannian $Gr_\omega(d, V)$ and a flag variety $F(d_1, d_2, \cdots, d_{k-1}, m)$. In addition, after replacing $\mathbb{C}$ by a field k, it is possible to construct another Grassmannian $Gr_k(d, V)$ and its structure depends substantially on the base field k. For this reason, many mathematicians have been interested in complex Grassmannians with their generalizations, and have studied them from various perspectives and purposes.

A complex Grassmannian is a smooth projective variety, and a fundamental and significant object of algebraic geometry. Many features of complex Grassmannians, including homology and the cohomology groups, automorphism groups, holomorphic embedding of them into complex projective spaces and defining ideals, are well-known. In particular, a complex Grassmannian $Gr(d, V)$ admits a cell decomposition. The closure of each cell is called a *Schubert variety* and Schubert varieties play an important role in understanding $Gr(d, V)$. Using Poincaré duality, the homology class of each Schubert variety corresponds to the cohomology class, which is called a *Schubert cycle*, and Schubert cycles on $Gr(d, V)$ can be classified by the d-tuples of non-negative integers satisfying some inequality. The set of all Schubert cycles forms a $\mathbb{Z}$-module basis of the cohomology ring of $Gr(d, V)$, and the multiplications of Schubert cycles are determined by a combinatorial rule, namely Pieri's formula.

There are two canonical holomorphic vector bundles on $Gr(d, V)$, *the universal bundle* $E(d, V)$ and *the universal quotient bundle* $Q(d, V)$, which are defined in natural ways: the fiber of $E(d, V)$ at $x \in Gr(d, V)$ is given by

the d-dimensional subspace L_x of V which corresponds to x,

and the fiber of $Q(d, V)$ at x is given by the quotient space ${}^{V}\!/\!{}_{L_x}$. Every Chern class of $E(d, V)$ and $Q(d, V)$ is a Schubert cycle (up to sign) and the cohomology ring of $Gr(d, V)$ is generated by the set of all Chern classes of $E(d, V)$ as a ring. Furthermore, the tangent bundle of $Gr(d, V)$ is isomorphic to $\check{E}(d, V) \otimes Q(d, V)$ where $\check{E}(d, V)$ is the dual bundle of $E(d, V)$. When $V = \mathbb{C}^m$, we denote $Gr(d, V)$, $E(d, V)$ and $Q(d, V)$ simply by $Gr(d, m)$, $E(d, m)$ and $Q(d, m)$, respectively.

In this thesis, we discuss holomorphic embeddings between complex Grassmannians. For any $d_1 < m$ and $d_2 < n$ with $d_1 \leq d_2$ and $m - d_1 \leq n - d_2$, there is a natural holomorphic embedding of $Gr(d_1, m)$ into $Gr(d_2, n)$:

$\boxed{1}$ Let $f\colon \mathbb{C}^m \hookrightarrow \mathbb{C}^n$ be an injective linear map and let W be a $(d_2 - d_1)$-dimensional subspace of $\mathbb{C}^n$ satisfying $W \cap f(\mathbb{C}^m) = 0$;

$\boxed{2}$ the pair (f, W) induces a holomorphic embedding $\tilde{f}_W\colon Gr(d_1, m) \hookrightarrow Gr(d_2, n)$ which is given by

$$L_{\tilde{f}_W(x)} := L_{f(x)} \oplus W, \qquad x \in Gr(d_1, m).$$

We call such an embedding $\tilde{f}_W$ to be *linear*. Consider the following question:

Question. For $d_1 < m$ and $d_2 < n$ with $d_1 \leq d_2$ and $m - d_1 \leq n - d_2$, what is a sufficient condition for the linearity of holomorphic embeddings of $Gr(d_1, m)$ into $Gr(d_2, n)$? More generally, how can we classify such embeddings?

The most fundamental answer for Question is about the case when $d_1 = d_2\,(=: d)$ and $m = n$. In this case, a holomorphic embedding $\varphi\colon Gr(d, m) \hookrightarrow$

$Gr(d,m)$ is an automorphism of $Gr(d,m)$. To describe a non-linear automorphism of $Gr(d,m)$, fix a basis $\mathcal{B} := \{e_1, \cdots, e_m\}$ of $\mathbb{C}^m$ and let $\{e_1^*, \cdots e_m^*\}$ be the dual basis of $\mathcal{B}$. The choice of $\mathcal{B}$ induces a linear isomorphism $\iota\colon (\mathbb{C}^m)^* \to \mathbb{C}^m$ which is determined by $\iota(e_j^*) = e_j$ for all $1 \leq j \leq m$. Define a map $\phi\colon Gr(d,m) \to Gr(m-d,m)$ by

$$L_{\phi(x)} := \iota\left(L_x^{\perp}\right), \qquad x \in Gr(d,m) \tag{1.0.1}$$

where $L_x^{\perp} \subset (\mathbb{C}^m)^*$ is the annihilator of L_x. We call such a map ϕ a *dual map.* In particular, when $m = 2d$, a dual map $\phi\colon Gr(d,2d) \to Gr(d,2d)$ is an automorphism. In [Cho49], W.-L. Chow classified all automorphisms of $Gr(d,m)$ and showed that every automorphism of $Gr(d,m)$ is linear except when $m = 2d$.

Theorem 1.0.1 ([Cho49, Theorem XI and XV])**.** *The automorphism group of $Gr(d,m)$ is*

$$\mathrm{Aut}(Gr(d,m)) = \begin{cases} \mathbf{P}GL(m,\mathbb{C}), & \textit{if } m \neq 2d \\ \begin{array}{l} \mathbf{P}GL(2d,\mathbb{C}) \sqcup (\phi \circ \mathbf{P}GL(2d,\mathbb{C})) \\ = \mathbf{P}GL(2d,\mathbb{C}) \sqcup (\mathbf{P}GL(2d,\mathbb{C}) \circ \phi) \end{array}, & \textit{if } m = 2d \end{cases}$$

where $\phi\colon Gr(d,2d) \to Gr(d,2d)$ is a dual map.

In [Mok08], N. Mok considered holomorphic embeddings $\varphi\colon Gr(d_1,m) \hookrightarrow Gr(d_2,n)$ with $2 \leq d_1 \leq d_2$ and $2 \leq m-d_1 \leq n-d_2$, and obtained geometric condition on φ for the linearity. For $x \in Gr(d_1,m)$, we regard each tangent vector of $Gr(d_1,m)$ at x as an element of $\check{E}(d_1,m)_x \otimes Q(d_1,m)_x$ where $\mathcal{E}_x$ denotes the fiber of $\mathcal{E}$ at x. We call a tangent vector of $Gr(d_1,m)$ at x to be *decomposable* if it can be written as $v \otimes w$ for some $v \in \check{E}(d_1,m)_x$ and $w \in Q(d_1,m)_x$. N. Mok characterized linear embeddings $\varphi\colon Gr(d_1,m) \hookrightarrow Gr(d_2,n)$ when the differential $d\varphi$ of φ preserves the decomposability of tangent vectors.

Theorem 1.0.2 ([Mok08, Proposition 1, 3 and 4])**.** *Let $\varphi\colon Gr(d_1,m) \hookrightarrow Gr(d_2,n)$ be a holomorphic embedding with $2 \leq d_1 \leq d_2$ and $2 \leq m-d_1 \leq n-d_2$. Assume that $d\varphi$ transforms decomposable tangent vectors into decomposable tangent vectors. Then either φ is linear up to automorphisms* *of $Gr(d_1, m)$ or $Gr(d_2, n)$, or the image of φ lies on some projective space in $Gr(d_2, n)$ (Here, Y is a projective space in $Gr(d_2, n)$ if and only if $i(Y)$ is a projective space in $\mathbb{P}^{\binom{n}{d_2}-1}$ where $i\colon Gr(d_2, n) \hookrightarrow \mathbb{P}^{\binom{n}{d_2}-1}$ is the Plücker embedding).*

Although N. Mok studied holomorphic embeddings between complex Grassmannians by the *pushforward* of vector fields, there have been several approaches to study them by the *pullback* of vector bundles.

Consider more general situations: holomorphic maps from a compact complex manifold Z into the complex Grassmannian $Gr(d, n)$. When we write $Gr(d, n) = Gr(d, \mathbb{C}^n)$ definitely, the space $\Gamma(Gr(d, \mathbb{C}^n), \check{E}(d, \mathbb{C}^n))$ of all holomorphic global sections of $\check{E}(d, \mathbb{C}^n)$ is naturally identified with $(\mathbb{C}^n)^*$.

Any holomorphic map $\psi\colon Z \to Gr(d, n)$ is determined completely by the pullback bundle $\psi^*(\check{E}(d, n))$ because the fiber of $\psi^*(\check{E}(d, n))$ at $z \in Z$ is equal to $(L_{\psi(z)})^* \subset (\mathbb{C}^n)^* = \Gamma(Gr(d, n), \check{E}(d, n))$. Let $\mathcal{E} := \psi^*(\check{E}(d, n))$ and $\pi\colon (\mathbb{C}^n)^* \otimes \mathcal{O}_Z \to \mathcal{E}$ be the pullback of the canonical surjective vector morphism $(\mathbb{C}^n)^* \otimes \mathcal{O}_{Gr(d,n)} \to \check{E}(d, n)$ under the holomorphic map ψ, then the pair $(\mathcal{E}, \pi)$ satisfies that

- $\mathcal{E}$ is a holomorphic vector bundle on Z of rank d;
- $\pi\colon (\mathbb{C}^n)^* \otimes \mathcal{O}_Z \to \mathcal{E}$ is a surjective holomorphic vector morphism.

(1.0.2)

Conversely, assume that a pair $(\mathcal{E}, \pi)$ satisfies (1.0.2). For each $z \in Z$, π induces a surjective linear map $\pi_z\colon (\mathbb{C}^n)^* \to \mathcal{E}_z$ between fibers at z. From these maps, define a holomorphic map $\psi\colon Z \to Gr(d, n)$ by the composition of holomorphic maps

$$Z \quad \overset{\tilde{\pi}}{\to} \quad Gr(n-d, (\mathbb{C}^n)^*) \quad \overset{\perp}{\to} \quad Gr(d, (\mathbb{C}^n)^{**}) = Gr(d, n) \tag{1.0.3}$$

where $L_{\tilde{\pi}(z)} := \ker(\pi_z) \subset (\mathbb{C}^n)^*$ and $L_{\perp(y)} \subset \mathbb{C}^n$ is defined by the annihilator of $L_y \subset (\mathbb{C}^n)^*$. Then we can show that $\pi^*\colon \check{\mathcal{E}} \to \check{\mathcal{F}}$ is an isomorphism and $\pi^*\colon \check{\mathcal{E}} \to \mathbb{C}^n \otimes \mathcal{O}_Z$ is a composition of the map $\check{\mathcal{F}} \to (\mathbb{C}^n)^* \otimes \mathcal{O}_Z$ with $\pi^*\colon \check{\mathcal{E}} \to \check{\mathcal{F}}$. For more details on the correspondence between maps and vector bundles, see [BT82, Section 23] (in the differential category), [GH94, page 207–209] or [Huy05, Remark 4.3.21].

The subsequent results are proved from this aspect.

In [Fed65], S. Feder considered holomorphic embeddings $\varphi\colon \mathbb{P}^m \hookrightarrow \mathbb{P}^n$ and classified them by means of their degrees. Let $\mathcal{O}_{\mathbb{P}^k}(1)$ be the complex line bundle which corresponds to a hyperplane H of $\mathbb{P}^k$, then every complex line bundle on $\mathbb{P}^k$ can be expressed as $\mathcal{O}_{\mathbb{P}^k}(r) := \mathcal{O}_{\mathbb{P}^k}(1)^{\otimes r}$ for some $r \in \mathbb{Z}$ up to isomorphisms. Given a holomorphic map $\psi\colon \mathbb{P}^m \to \mathbb{P}^n$, the pullback bundle of $\mathcal{O}_{\mathbb{P}^n}(1)$ under the map ψ is isomorphic to $\mathcal{O}_{\mathbb{P}^m}(r)$ for a unique integer r, and we call it the *degree of* φ. A holomorphic embedding φ is linear if and only if the degree of φ is 1.

Theorem 1.0.3 ([Fed65, Theorem 1.2, 2.1 and 2.2])**.**

(a) *Let $\varphi\colon \mathbb{P}^m \hookrightarrow \mathbb{P}^n$ be a holomorphic embedding. Then we have*

$$\textit{the degree of } \varphi = \begin{cases} 1, & \textit{if } n < 2m \\ 1 \textit{ or } 2, & \textit{if } n = 2m \end{cases}.$$

(b) *If $n > 2m$, then for any $r > 0$, there is a holomorphic embedding $\mathbb{P}^m \hookrightarrow \mathbb{P}^n$ of degree r.*

In [Tan74], H. Tango considered holomorphic embeddings $\varphi\colon \mathbb{P}^{n-2} \hookrightarrow Gr(2,n)$ with $n \geq 4$ and classified their images. In this case, φ is linear if and only if the image of φ equals $\{x \in Gr(2,n) \mid p \in L_x\}$ for some $p \in \mathbb{C}^n$. To state H. Tango's result, we need to define some subvarieties of $Gr(2,n)$ which are biholomorphic to $\mathbb{P}^{n-2}$. For $x \in Gr(2,n)$, choose a basis $\{v_1, v_2\}$ of $L_x \subset \mathbb{C}^n$, and construct the $2 \times n$ matrix of rank 2 whose i^{th} row is the transpose of v_i for $i = 1$ and 2. The choice of bases of L_x is not unique, but $\{w_1, w_2\}$ is a basis of L_x if and only if the change of basis from $\{v_1, v_2\}$ to $\{w_1, w_2\}$ is an invertible 2×2 matrix. So we express an element in $Gr(2,n)$ as the equivalence class of a $2 \times n$ matrix of rank 2

$$\left[\begin{pmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \end{pmatrix}\right]$$

where the equivalence relation is given by

$$A \sim B \quad \text{if and only if} \quad A = g\,B \text{ for some } g \in GL(2,\mathbb{C}).$$

When $n \geq 4$, define subvarieties $X^0_{n-1,1}$ and $X^1_{n-1,1}$ of $Gr(2,n)$ by

$$\begin{aligned} X^0_{n-1,1} &:= \left\{ \left[\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & x_0 & \cdots & x_{n-3} & x_{n-2} \end{pmatrix} \right] \;\middle|\; [\mathbf{x}] \in \mathbb{P}^{n-2} \right\}; \\ X^1_{n-1,1} &:= \left\{ \left[\begin{pmatrix} x_0 & x_1 & \cdots & x_{n-2} & 0 \\ 0 & x_0 & \cdots & x_{n-3} & x_{n-2} \end{pmatrix} \right] \;\middle|\; [\mathbf{x}] \in \mathbb{P}^{n-2} \right\} \end{aligned} \tag{1.0.4}$$

where $[\mathbf{x}] := [x_0 : x_1 : \cdots : x_{n-2}]$, and define subvarieties $\check{X}^0_{3,1}$ and $\check{X}^1_{3,1}$ of $Gr(2,4)$ by

$$\begin{aligned} \check{X}^0_{3,1} &:= \phi(X^0_{3,1}); \\ \check{X}^1_{3,1} &:= \phi(X^1_{3,1}) \end{aligned} \tag{1.0.5}$$

where $\phi\colon Gr(2,4) \to Gr(2,4)$ is a dual map. For a quadric hypersurface S of $\mathbb{P}^4$, define a subvariety $X_q(S)$ of $Gr(2,5)$ by

$$X_q(S) := \{x \in Gr(2,5) \mid L_x \subset C(S)\} \tag{1.0.6}$$

where $C(S) \subset \mathbb{C}^5$ is the affine cone over S.

Theorem 1.0.4 ([Tan74, Theorem 5.1 and 6.2])**.** *Let $\varphi\colon \mathbb{P}^{n-2} \hookrightarrow Gr(2,n)$ be a holomorphic embedding and X be the image of φ.*

(a) *If $n = 4$, then $X \simeq X^0_{3,1}$, $X^1_{3,1}$, $\check{X}^0_{3,1}$ or $\check{X}^1_{3,1}$.*

(b) *If $n = 5$, then $X \simeq X^0_{4,1}$, $X^1_{4,1}$ or $X_q(S)$ where S is a fixed non-singular quadric hypersurface of $\mathbb{P}^4$.*

(c) *If $n \geq 6$, then $X \simeq X^0_{n-1,1}$ or $X^1_{n-1,1}$.*

(Here, $X \simeq X_0$ if and only if $X = g\,X_0$ for some $g \in \mathbf{P}GL(n,\mathbb{C})$.)

In [SU06], J. C. Sierra and L. Ugaglia classified all the holomorphic embeddings $\varphi\colon \mathbb{P}^m \hookrightarrow Gr(2,n)$ such that the composition of the Plücker embedding $Gr(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$ with them is given by a linear system of quadrics in $\mathbb{P}^m$.

Theorem 1.0.5 ([SU06, Theorem 2.12])**.** *Let $\varphi\colon \mathbb{P}^m \hookrightarrow Gr(2,n)$ be a holomorphic embedding satisfying that the line bundle $\wedge^2\varphi^*(\check{E}(2,n))$ is isomorphic to $\mathcal{O}_{\mathbb{P}^m}(2)$. Let $E := \varphi^*(\check{E}(2,n))$, then one of the following holds:*

(a) $E \simeq \mathcal{O}_{\mathbb{P}^m} \oplus \mathcal{O}_{\mathbb{P}^m}(2)$.

(b) $E \simeq \mathcal{O}_{\mathbb{P}^m}(1) \oplus \mathcal{O}_{\mathbb{P}^m}(1)$.

(c) *$m = 3$ and E is the kernel of a surjective holomorphic bundle morphism $T_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(2)$, with a resolution of the form:*

$$0 \ \to \ \mathcal{O}_{\mathbb{P}^3}(-2) \ \to \ \bigoplus^{4} \mathcal{O}_{\mathbb{P}^3}(-1) \ \to \ \bigoplus^{5} \mathcal{O}_{\mathbb{P}^3} \ \to \ E \ \to \ 0.$$

(c) *$m = 2$ and E has a resolution of the form:*

$$0 \ \to \ \mathcal{O}_{\mathbb{P}^2}(1) \ \to \ E \ \to \ m_P \otimes \mathcal{O}_{\mathbb{P}^2}(1) \ \to \ 0$$

where m_P denotes the ideal sheaf of a point $P \in \mathbb{P}^2$.

(e) *$m = 2$ and E has a resolution of the form:*

$$0 \ \to \ \bigoplus^{2} \mathcal{O}_{\mathbb{P}^2}(-1) \ \to \ \bigoplus^{4} \mathcal{O}_{\mathbb{P}^2} \ \to \ E \ \to \ 0.$$

For each holomorphic vector bundle $\mathcal{E}$ on $\mathbb{P}^m$ classified in Theorem 1.0.5, the pair $(\mathcal{E}, \Gamma(\mathbb{P}^m, \mathcal{E}))$ induces a holomorphic embedding $\varphi\colon \mathbb{P}^m \hookrightarrow Gr(2, n)$ where $n := \dim(\Gamma(\mathbb{P}^m, \mathcal{E}))$ uniquely up to linear automorphisms of $Gr(2, n)$ (For the construction of φ, see (1.0.3) and its next paragraph). We provide here examples of such embeddings φ.

Example 1.0.6 ([SU06, Example 1.5−1.9 and Remark 3.5])**.** For each item, let $\mathcal{E}$ be the holomorphic vector bundle on $\mathbb{P}^m$ in the same item of Theorem 1.0.5 and $V := \Gamma(\mathbb{P}^m, \mathcal{E})$.

(a) The pair $(\mathcal{E}, V)$ induces a holomorphic embedding $\varphi\colon \mathbb{P}^m \hookrightarrow Gr\left(2, \binom{m+2}{2} + 1\right)$ which is given by the family of ruling lines of a cone over the second Veronese embedding $v_2(\mathbb{P}^m) \subset \mathbb{P}^{\binom{m+2}{2}-1}$ with vertex a point.

(b) The pair $(\mathcal{E}, V)$ induces a holomorphic embedding $\varphi\colon \mathbb{P}^m \hookrightarrow Gr(2, 2m + 2)$ which is given by the family of lines joining the corresponding points on two disjoint $\mathbb{P}^m$'s in $Gr(2, 2m + 2)$. Moreover, the holomorphic embedding $\varphi_0\colon \mathbb{P}^m \hookrightarrow Gr(2, m + 2)$ whose image equals $X^1_{m+1,1}$ given as in

(1.0.4) can be obtained by projecting from $\varphi(\mathbb{P}^m) \subset Gr(2, 2m+2)$ to $Gr(2, m+2)$ and the pullback bundle $\varphi_0^*(\check{E}(2, m+2))$ is also isomorphic to $\mathcal{E}$.

(c) The pair $(\mathcal{E}, V)$ induces a holomorphic embedding $\varphi\colon \mathbb{P}^3 \hookrightarrow Gr(2,5)$ whose image equals the subvariety $X_q(S)$ given as in (1.0.6).

(d) The pair $(\mathcal{E}, V)$ induces a holomorphic embedding $\varphi\colon \mathbb{P}^2 \hookrightarrow Gr(2,5)$ which is a composition of the holomorphic embedding $\mathbb{P}^3 \hookrightarrow Gr(2,5)$ in (c) with a linear embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^3$.

(e) The pair $(\mathcal{E}, V)$ induces a holomorphic embedding $\varphi\colon \mathbb{P}^2 \hookrightarrow Gr(2,4)$ whose image equals the subvariety $\check{X}^1_{3,1}$ given as in (1.0.5).

Similarly, in [Huh11], S. Huh classified all the holomorphic embeddings $\mathbb{P}^m \hookrightarrow Gr(2,n)$ such that the composition of the Plücker embedding $Gr(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$ with them is given by a linear system of cubics in $\mathbb{P}^m$.

Motivated by the previous results, we consider holomorphic embeddings $\varphi\colon Gr(2,m) \hookrightarrow Gr(2,n)$ and obtain the following numerical conditions on m and n for the linearity of φ:

Main Theorem. *Let $\varphi\colon Gr(2,m) \hookrightarrow Gr(2,n)$ be a holomorphic embedding.*

(a) *If $9 \leq m$ and $n \leq \frac{3m-6}{2}$, then φ is linear.*

(b) *If $4 \leq m$ and $n = m+1$, then either φ is linear, or $m = 4$ and φ is a composition of a linear holomorphic embedding of $Gr(2,4)$ into $Gr(2,5)$ with a dual map $\phi\colon Gr(2,4) \to Gr(2,4)$.*

Main Theorem follows from Theorem 4.1.1 and 4.2.1. We do not have enough examples of non-linear embeddings $\varphi\colon Gr(2,m) \hookrightarrow Gr(2,n)$ except when $m \geq 3$ and $n > m(m-1)$ (see Example 3.1.3 (b)). Since $m(m-1)$ is much greater than both $\frac{3m-6}{2}$ and $m+1$, the assumptions in Main Theorem can be improved. To find a sharp condition of m and n for the linearity will be an interesting problem.

Although S. Feder and H. Tango dealt with different cases, they used similar numerical techniques. These methods can be applied to every holomorphic embedding $\varphi\colon Gr(d_1, m) \hookrightarrow Gr(d_2, n)$ if d_1, d_2 are fixed and $d_2(n-d_2) \leq 2d_1(m-d_1)$ as follows:

Step 1. Let E be the pullback bundle of $\check{E}(d_2, n)$ under the embedding φ. If φ is linear, then E is isomorphic to $\check{E}(d_1, m) \oplus (\bigoplus^{d_2-d_1} \mathcal{O}_{Gr(d_1,m)})$, thus the first Chern class $c_1(E)$ of E equals $c_1(\check{E}(d_1, m))$. Conversely, if $c_1(E) = c_1(\check{E}(d_1, m))$, then $d\varphi$ preserves the decomposability of tangent vectors (see Remark 3.2.3), thus either φ is linear up to automorphisms of $Gr(d_1, m)$ or $Gr(d_2, n)$, or φ embeds $Gr(d_1, m)$ into some projective space in $Gr(d_2, n)$ by Theorem 1.0.2. To distinguish the linear case from the others, we need additional conditions.

Step 2. Choose a $\mathbb{Z}$-module basis $\mathcal{B}$ of the cohomology ring of $Gr(d_1, m)$. The total Chern class of E can be written uniquely as a linear combination of elements in $\mathcal{B}$ with coefficients $a, b, \cdots, c$ in $\mathbb{Z}$ (Here, a is determined so that $c_1(E) = a\, c_1(\check{E}(d_1, m))$). Let N be the pullback bundle of the normal bundle of $\varphi(Gr(d_1, m))$ in $Gr(d_2, n)$ under the embedding φ. Using canonical short exact sequences for $E(d_1, m)$ and for $E(d_2, n)$, we construct an equation of the total Chern class of N in terms of the total Chern classes of E, $E(d_1, m)$, their dual bundles and tensor product bundles. Thus each Chern class of N can be written as a linear combination of elements in $\mathcal{B}$ with coefficients in the 3-variate polynomial ring $\mathbb{Z}[a, b, c]$ over $\mathbb{Z}$. The Euler class of $N_{\mathbb{R}}$, which is the real vector bundle corresponding to N, equals the pullback bundle of the Poincaré dual to the homology class of $\varphi(Gr(d_1, m))$ under the embedding φ. So we also express the Euler class of $N_{\mathbb{R}}$ as a linear combination of elements in $\mathcal{B}$ with coefficients in $\mathbb{Z}[a, b, c]$. By definitions of Chern classes, the top Chern class of N equals the Euler class of $N_{\mathbb{R}}$, and the k^{th} Chern class of N equals 0 if $k > \text{rank}(N)$. Since $a, b, \cdots, c$ are integers, these equations are Diophantine equations (If $\text{rank}(N) > \dim(Gr(d_1, m))$ or, equivalently, $d_2(n - d_2) > 2d_1(m - d_1)$, then we cannot obtain any equation).

Step 3. In general, it is hard to solve these kinds of equations. To overcome this difficulty, we need additional conditions on $a, b, \cdots c$, such as inequalities in a, b and c. Applying a criterion of the numerical non-negativity of Chern classes of holomorphic vector bundles to suitable holomorphic vector bundles on $Gr(d_1, m)$, we obtain some inequalities in $a, b, \cdots, c$.

> To obtain other conditions, we have to look for a useful method case by case.

In this way, we can think of the problem on the classification of holomorphic embeddings between complex Grassmannians as the problem on solving the obtained Diophantine equations and inequalities. We prove Main Theorem by applying the above numerical techniques together with further results to our case.

The thesis consists of four chapters.

In Chapter 2, we introduce backgrounds about complex Grassmannians $Gr(d, m)$, such as Schubert cycles, the universal and the universal quotient bundles. While most subjects in this chapter are basic and well-known, there are two remarkable subjects which play significant roles in reach our goal. First, we provide two $\mathbb{Z}$-module bases of the cohomology ring of $Gr(2, m)$. One is the set of all Schubert cycles on $Gr(2, m)$ and the other is the set of all monomials of the form $(c_1(\check{E}(2, m)))^i\,(c_2(\check{E}(2, m)))^j$ satisfying a certain condition on i and j (Proposition 2.1.7). As we already mentioned, the formal basis arises from a cell decomposition of $Gr(2, m)$, thus it is useful to verify geometric features of $Gr(2, m)$. The latter basis, denoted by $\mathcal{C}$, arises from a ring generator $\{c_1(\check{E}(2, m)), c_2(\check{E}(2, m))\}$ of the cohomology ring of $Gr(2, m)$, thus it is useful to express multiplications of cohomology classes until the degree is not greater than $2(m-2)$. For this reason, we use the basis $\mathcal{C}$ to express cohomology classes as linear combinations like $\mathcal{B}$ in Step 2. Second, we provide W. Barth and A. Van de Ven's results, which are about the decomposability of holomorphic vector bundles on complex Grassmannians of rank 2 (Proposition 2.2.3 and 2.2.4). If a holomorphic vector bundle $\mathcal{E}$ on $Gr(2, m)$ of rank 2 satisfies the assumptions of their results, then we can handle $\mathcal{E}$ easily.

In Chapter 3, we consider holomorphic embeddings $\varphi\colon Gr(2, m) \hookrightarrow Gr(2, n)$ and their linearity. As in Step 2, we set the integral coefficients a, b and c to express the total Chern class of E with respect to the basis $\mathcal{C}$, and provide an equivalent condition on the pair (a, b, c) for the linearity of φ (Proposition 3.2.2). When we focus on the coefficients of the powers of $c_1(\check{E}(2, m))$ (resp. $c_2(\check{E}(2, m))$) in the equation of the total Chern class of N in Step 2, we derive a refined equation whose both sides are polynomials in one variable $c_1(\check{E}(2, m))$

(resp. $c_2(\check{E}(2,m))$). If $\operatorname{rank}(N) = 2n - 2m$ is not greater than $m - 2$, then these two refined equations preserve the coefficients of the cohomology classes of degree $2n - 2m$ (Proposition 3.3.4). Solving the refined equations and the equation of the Euler class of $N_{\mathbb{R}}$ in Step 2 together with the inequalities in Step 3, we obtain a lower bound of the coefficient of $(c_1(\check{E}(2,m)))^{2n-2m}$ in the top Chern class of N with respect to $\mathcal{C}$ (Lemma 3.4.7) and an inequality in a, b (Proposition 3.4.9).

In Chapter 4, we prove Main Theorem (a) and (b) separately. For the proof of (a), we first obtain a upper bound of a (Proposition 4.1.2 (b)) from all the previous results. This bound enables us to apply W. Barth and A. Van de Ven's results to E, thus we can solve the refined equation in $c_1(\check{E}(2,m))$ more easily (Theorem 4.1.1). For the proof of (b), we solve the equality of the top Chern class of N and the Euler class of $N_{\mathbb{R}}$ directly (Theorem 4.2.1). It is reasonable because $\operatorname{rank}(N) = 2$ is sufficiently small.

Throughout the thesis, a Grassmannian means a complex Grassmannian, a map means a holomorphic map, and a vector bundle means a holomorphic vector bundle by abuse of terminology.

Chapter 2

Preliminaries

We introduce here basic concepts about Grassmannians $Gr(d, m)$ for further use.

The chapter consists of two sections. In Section 2.1, we provide Schubert varieties, Schubert cycles on $Gr(d, m)$ and Pieri's formula which describes the multiplications of Schubert cycles on $Gr(d, m)$. In general, the set of all Schubert cycles on $Gr(d, m)$ is a $\mathbb{Z}$-module basis of the cohomology ring of $Gr(d, m)$. When $d = 2$, we provide another $\mathbb{Z}$-module basis of the cohomology ring of $Gr(2, m)$, which is motivated by its ring generator, and the relation between these two bases. In Section 2.2, we provide the universal bundle, the universal quotient bundle on $Gr(d, m)$ and their total Chern classes. In addition, we provide W. Barth and A. Van de Ven's results which are about the decomposability of vector bundles on complex Grassmannians of rank 2. For more details on Section 2.1, see [Arr96] and [GH94, Section 1.5], and for more details on Section 2.2, see [Tan74], [BVdV74a] and [BVdV74b].

2.1 Schubert cycles on Grassmannians

For a partial flag $0 \subset A_1 \subsetneq \cdots \subsetneq A_d \subset \mathbb{C}^m$, let $\omega(A_1, \cdots, A_d)$ be the subvariety of $Gr(d, m)$ which is given by

$$\{x \in Gr(d, m) \mid \dim(L_x \cap A_i) \geq i \text{ for all } 1 \leq i \leq d\}.$$

We call such a subvariety $\omega(A_1, \cdots, A_d)$ a *Schubert variety of type* $(a_1, \cdots, a_d)$ where $a_i := m - d + i - \dim(A_i)$ for $1 \leq i \leq d$. The (complex) codimension of $\omega(A_1, \cdots, A_d)$ in $Gr(d, m)$ is $\sum_{i=1}^{d} a_i$. We sometimes denote $(a_1, \cdots, a_d)$ simply by the bold lowercase letter $\boldsymbol{a}$.

Example 2.1.1. There are some familiar Schubert varieties on $Gr(d, m)$, which are sub-Grassmannians of $Gr(d, m)$. Given a type $\star$, let $X_\star := \omega(A_1, \cdots, A_d)$ be a Schubert variety of type $\star$.

(a) $\boldsymbol{a} = (m-d, \cdots, m-d, 0)$: Since $\dim(A_i) = i$ for all $1 \leq i \leq d-1$ and $\dim(A_d) = m$,

$$\begin{aligned} A_i &= \mathrm{span}(\{v_1, \cdots, v_i\}) \quad \text{for all } 1 \leq i \leq d-1; \\ A_d &= \mathbb{C}^m \end{aligned}$$

for some linearly independent vectors $v_1, \cdots, v_{d-1} \in \mathbb{C}^m$. So we have

$$\begin{aligned} X_{\boldsymbol{a}} &= \{x \in Gr(d, m) \mid A_{d-1} \subset L_x\} \\ &\simeq \mathbb{P}\left(\mathbb{C}^m / A_{d-1}\right) \simeq \mathbb{P}^{m-d} \end{aligned}$$

(When $d = 2$, $X_{\boldsymbol{a}} \simeq X^0_{n-1,1}$ where $X^0_{n-1,1}$ is given as in (1.0.4)).

$X_{\boldsymbol{a}}$ is a maximal projective space in $Gr(d, m)$, that is, there is not a projective space in $Gr(d, m)$ containing it properly.

(b) $\boldsymbol{b} = (\underbrace{m-d, \cdots, m-d}_{k}, 0, \cdots, 0)$: Since $\dim(A_i) = i$ for all $1 \leq i \leq k$ and $\dim(A_j) = m - d + j$ for all $k+1 \leq j \leq d$,

$$\begin{aligned} X_{\boldsymbol{b}} &= \{x \in Gr(d, m) \mid A_k \subset L_x\} \\ &\simeq Gr(d-k, \mathbb{C}^m / A_k) \simeq Gr(d-k, m-k). \end{aligned}$$

(c) $\boldsymbol{c} = (k, \cdots, k)$: Since $\dim(A_i) = m - d + i - k$ for all $1 \leq i \leq d$,

$$\begin{aligned} X_{\boldsymbol{c}} &= \{x \in Gr(d, m) \mid L_x \subset A_d\} \\ &= Gr(d, A_d) \simeq Gr(d, m-k). \end{aligned}$$

So any subvariety $Gr(d, H)$ of $Gr(d, m)$ where H is a subspace of $\mathbb{C}^m$ is of this form.

(d) $\boldsymbol{d} = (\underbrace{m-d, \cdots, m-d}_{k}, l, \cdots, l)$: Combining the results of (b) and (c),

$$X_{\boldsymbol{d}} \simeq Gr(d-k, m-k-l),$$

which is contained in a Schubert variety of the form $X_{\boldsymbol{b}}$. The inclusion $X_{\boldsymbol{d}} \subset X_{\boldsymbol{b}}$ corresponds to the inclusion $X_{\boldsymbol{c}} \subset Gr(d, m)$ in (c).

Two Schubert varieties of types $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same homology class if and only if $\boldsymbol{a} = \boldsymbol{b}$. We denote the Poincaré dual to a Schubert variety of type $(a_1, \cdots, a_d)$ by $\omega_{a_1, \cdots, a_d}$ and call it the *Schubert cycle of type* $(a_1, \cdots, a_d)$. Since the codimension of a Schubert variety of type $(a_1, \cdots, a_d)$ is $\sum_{i=1}^d a_i$,

$$\omega_{a_1, \cdots, a_d} \in H^{2\left(\sum_{i=1}^d a_i\right)}(Gr(d, m), \mathbb{Z})$$

and the set of all Schubert cycles describes every cohomology group of $Gr(d, m)$ completely as follows:

$$H^i(Gr(d, m), \mathbb{Z}) = \begin{cases} 0, & \text{if } i \text{ is odd} \\ \operatorname{span}(\mathcal{B}_k), & \text{if } i\,(=2k) \text{ is even} \end{cases}$$

where $\mathcal{B}_k$ is a basis which is given by

$$\left\{ \omega_{a_1, \cdots, a_d} \;\middle|\; m-d \geq a_1 \geq \cdots \geq a_d \geq 0;\ \sum_{i=1}^{d} a_i = k \right\}.$$

In particular, when $d = 2$,

$$\{\omega_{k-i, i} \mid m-2 \geq k-i \geq i \geq 0\} \tag{2.1.1}$$

is a basis of $H^{2k}(Gr(2, m), \mathbb{Z})$. For $k = 2m-4$, $H^{2(2m-4)}(Gr(2, m), \mathbb{Z}) \simeq \mathbb{Z}$ is generated by $\omega_{m-2, m-2} = \omega_{1,1}^{m-2}$. Every $\Gamma \in H^{2(2m-4)}(Gr(2, m), \mathbb{Z})$ is of the form $c_\Gamma\, \omega_{1,1}^{m-2}$ for some integer c_Γ, thus we identify Γ with $c_\Gamma \in \mathbb{Z}$.

By Example 2.1.1 (c), the Poincaré dual to the homology class of the subvariety $Gr(d,H) \subset Gr(d,m)$ where H is a subspace of $\mathbb{C}^m$ is $\omega_{k,\cdots,k}$ for some $0 \leq k \leq m-d$. The next Proposition is about its converse.

Proposition 2.1.2 ([Wal97, Theorem 7 and Corollary 5] or [Bry01, Example 11])**.** *For $m \geq d \geq 2$, let X_0 be a subvariety of $Gr(d,m)$ satisfying that the Poincaré dual to the homology class of X_0 is $\omega_{k,\cdots,k}$ for some $0 \leq k \leq m-d$. Then $X_0 = Gr(d,H)$ for some $(m-k)$-dimensional subspace H of $\mathbb{C}^m$.*

The multiplications of Schubert cycles are commutative and satisfy the following rule, named Pieri's formula.

Lemma 2.1.3 (Pieri's formula)**.** *In $Gr(d,m)$, for $m-d \geq a_1 \geq a_2 \geq \cdots \geq a_d \geq 0$ and $m-d \geq h \geq 0$,*

$$\omega_{a_1,a_2,\cdots,a_d}\,\omega_{h,0,\cdots,0} = \sum_{(b_1,b_2,\cdots,b_d)\in I} \omega_{b_1,b_2,\cdots,b_d} \tag{2.1.2}$$

where I is the set of all pairs $(b_1,b_2,\cdots,b_d) \in \mathbb{Z}^d$ satisfying

$$m-d \geq b_1 \geq a_1 \geq b_2 \geq a_2 \geq \cdots \geq b_d \geq a_d \geq 0;$$
$$\left(\sum_{i=1}^{d} a_i\right) + h = \sum_{i=1}^{d} b_i.$$

For the proof of Lemma 2.1.3, see [GH94, page 203]. To multiply two general Schubert cycles on $Gr(d,m)$ by using Lemma 2.1.3, we need to express this multiplication as a composition of finite multiplications of the form (2.1.2). In general, it is not easy. But when $d=2$, we have a refined Pieri's formula which enables us to multiply any two general Schubert cycles easily. Before describing this formula, we adopt the following convention:

Convention 2.1.4. In $Gr(2,m)$, let $\omega_{k,l} = 0$ unless $m-2 \geq k \geq l \geq 0$.

From now on, we always assume Convention 2.1.4 when $d=2$.

Corollary 2.1.5 (Refined Pieri's formula)**.** *Schubert cycles on $Gr(2,m)$ satisfy the following relations:*

(a) ([Tan74, Lemma 4.2 (i)]) $\omega_{i,j}\,\omega_{1,1} = \omega_{i+1,j+1}$.

(b) (Restate of Lemma 2.1.3) $\omega_{i,0}\,\omega_{j,0} = \omega_{i+j,0} + \omega_{i+j-1,1} + \cdots + \omega_{i+1,j-1} + \omega_{i,j}$.

Using Corollary 2.1.5 and the commutativity of multiplications, we can multiply Schubert cycles on $Gr(2,m)$ easily. For example,

$$\begin{aligned}\omega_{8,5}\,\omega_{7,3} &= (\omega_{3,0}\,\omega_{1,1}^5)\,(\omega_{4,0}\,\omega_{1,1}^3) = \omega_{4,0}\,\omega_{3,0}\,\omega_{1,1}^8\\ &= (\omega_{7,0} + \omega_{6,1} + \omega_{5,2} + \omega_{4,3})\,\omega_{1,1}^8\\ &= \omega_{15,8} + \omega_{14,9} + \omega_{13,10} + \omega_{12,11}\end{aligned}$$

(Some terms can be omitted if $m < 17$).

Furthermore, using Corollary 2.1.5, we obtain the result on the multiplications of two Schubert cycles of complementary degrees.

Corollary 2.1.6 ([Tan74, Lemma 4.2 (ii)])**.** *In $Gr(2,m)$, let i, j, k and l be integers with $m-2 \geq i \geq j \geq 0$, $m-2 \geq k \geq l \geq 0$ and $i+j+k+l = 2m-4$. Then we have*

$$\omega_{i,j}\,\omega_{k,l} = \begin{cases} 1, & \text{if } i+l = m-2 = j+k \\ 0, & \text{otherwise} \end{cases}.$$

By Corollary 2.1.6, for each $0 \leq p \leq 2m-4$, there is a bijection

$$\begin{aligned}\tau_p\colon \{\omega_{i,j} \mid m-2 \geq i \geq j \geq 0;\ i+j = p\}\\ \to \{\omega_{k,l} \mid m-2 \geq k \geq l \geq 0;\ k+l = 2m-4-p\}\end{aligned}$$

which is defined by the property: the multiplication of $\omega_{i,j}$ and $\tau_p(\omega_{i,j})$ is equal to 1. We call the image $\tau_p(\omega_{i,j}) = \omega_{m-2-j,m-2-i}$ the *dual Schubert cycle of* $\omega_{i,j}$.

Note that the cohomology ring of $Gr(2,m)$

$$H^\bullet(Gr(2,m),\mathbb{Z}) = \bigoplus_{k=0}^{2m-4} H^{2k}(Gr(2,m),\mathbb{Z})$$

is generated by $\omega_{1,0}$ and $\omega_{1,1}$ as a ring. Motivated by this fact, we find a new basis of $H^{2k}(Gr(2,m),\mathbb{Z})$ whose elements are expressed by $\omega_{1,0}$ and $\omega_{1,1}$.

Proposition 2.1.7. (a) *For $0 \le k \le 2m-4$, the set of Schubert cycles*

$$\left\{\omega_{1,0}^{k-2i}\,\omega_{1,1}^{i} \mid m-2 \ge k-i \ge i \ge 0\right\} \tag{2.1.3}$$

forms a basis of $H^{2k}(Gr(2,m),\mathbb{Z})$. In particular, when $0 \le k \le m-2$, the set of Schubert cycles

$$\left\{\omega_{1,0}^{k-2i}\,\omega_{1,1}^{i} \;\middle|\; 0 \le i \le \left\lfloor \frac{k}{2} \right\rfloor\right\} \tag{2.1.4}$$

forms a basis of $H^{2k}(Gr(2,m),\mathbb{Z})$ where $\lfloor \bullet \rfloor$ is the maximal integer which does not exceed $\bullet$.

(b) *For $0 \le k \le m-2$, let $\Gamma \in H^{2k}(Gr(2,m),\mathbb{Z})$ be a cohomology class. Then the coefficient of $\omega_{k,0}$ in Γ with respect to the basis* (2.1.1) *coincides with that of $\omega_{1,0}^{k}$ in Γ with respect to the basis* (2.1.4)*.*

Proof. (a) Using Corollary 2.1.5, for each $i \in \mathbb{Z}$ with $m-2 \ge k-i \ge i \ge 0$,

$$\begin{aligned} \omega_{1,0}^{k-2i}\,\omega_{1,1}^{i} &= \left(\omega_{k-2i,0} + \sum_{j=1}^{h} a_{i,j}\,\omega_{k-2i-j,j}\right)\omega_{1,1}^{i} \\ &= \omega_{k-i,i} + \sum_{j=1}^{h} a_{i,j}\,\omega_{k-i-j,i+j} \end{aligned} \tag{2.1.5}$$

for some non-negative integers $a_{i,j}$ and $h := \left\lfloor \frac{k-2i}{2} \right\rfloor$. Since $m-2 \ge k-i \ge i \ge 0$, the leading term $\omega_{k-i,i}$ of (2.1.5) is not a zero.

Note that $m-2 \ge k-i \ge i \ge 0$ if and only if $i_0 \le i \le \left\lfloor \frac{k}{2} \right\rfloor$ where $i_0 := \max\{0, k-m+2\}$, so we have

$$(2.1.1) = \left\{\omega_{k-i,i} \mid i_0 \le i \le \left\lfloor \frac{k}{2} \right\rfloor\right\}. \tag{2.1.6}$$

Let $\mathcal{A}_i := \operatorname{span}\left\{\omega_{1,0}^{k-2j}\,\omega_{1,1}^{j} \mid i \le j \le \left\lfloor \frac{k}{2} \right\rfloor\right\}$ for $i_0 \le i \le \left\lfloor \frac{k}{2} \right\rfloor$ and $\mathcal{A}_{\lfloor \frac{k}{2} \rfloor + 1} := 0$

(zero $\mathbb{Z}$-module). Then we have by (2.1.5),

$$\mathcal{A}_{i+1} \subset \mathcal{A}_i \setminus \{\omega_{1,0}^{k-2i}\,\omega_{1,1}^{i}\}; \qquad \omega_{k-i,i} \in \omega_{1,0}^{k-2i}\,\omega_{1,1}^{i} + \mathcal{A}_{i+1} \subset \mathcal{A}_i \tag{2.1.7}$$

for all $i_0 \leq i \leq \lfloor \frac{k}{2} \rfloor$. So the basis (2.1.6) is contained in $\mathcal{A}_{i_0}$, which is the $\mathbb{Z}$-submodule generated by (2.1.3). Furthermore, the basis (2.1.6) and the set (2.1.3) have the same number of elements. Hence, (2.1.3) is also a basis of $H^{2k}(Gr(2,m),\mathbb{Z})$.

(b) Since $k \leq m-2$, $i_0 = 0$. By (2.1.7), we have $\omega_{k,0} \in \omega_{1,0}^{k} + \mathcal{A}_1$, $\omega_{1,0}^{k} \neq \mathcal{A}_1$ and $\omega_{k-i,i} \in \mathcal{A}_i$ for all $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$. Hence, the coefficient of $\omega_{k,0}$ in Γ with respect to the basis (2.1.1) is equal to that of $\omega_{1,0}^{k}$ in Γ with respect to (2.1.4). □

Remark 2.1.8. By Proposition 2.1.7 (a), the set of Schubert cycles

$$\left\{\omega_{1,0}^{k-2i}\,\omega_{1,1}^{i} \;\middle|\; 0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \right\} = \left\{\omega_{1,0}^{k},\; \omega_{1,0}^{k-2}\,\omega_{1,1}, \cdots, \omega_{1,0}^{k-2\lfloor k/2 \rfloor}\,\omega_{1,1}^{\lfloor k/2 \rfloor}\right\}$$

is linearly independent if $0 \leq k \leq m-2$, but it is linearly dependent if $m-1 \leq k \leq 2m-4$. For this reason, we assume that $(\mathrm{rank}(N) =)\, 2n-2m \leq m-2$ in Proposition 3.3.4 where N is the vector bundle on $Gr(2,m)$ which is given as in the introductory part of Section 3.3.

Corollary 2.1.9. *For $0 \leq k \leq 2m-4$, let $\mathcal{C}_k$ be the basis of $H^{2k}(Gr(2,m),\mathbb{Z})$ which is given as in* (2.1.3)*. Let $\mathcal{Q}_{1,0}$ and $\mathcal{Q}_{1,1}$ be the quotient $\mathbb{Z}$-modules*

$$\begin{aligned} \mathcal{Q}_{1,0} &:= H^{\bullet}(Gr(2,m),\mathbb{Z})\big/\mathcal{M}_{1,0}\,; \\ \mathcal{Q}_{1,1} &:= H^{\bullet}(Gr(2,m),\mathbb{Z})\big/\mathcal{M}_{1,1} \end{aligned} \tag{2.1.8}$$

where $\mathcal{M}_{1,0}$ is the $\mathbb{Z}$-submodule of $H^{\bullet}(Gr(2,m),\mathbb{Z})$ which is generated by the basis

$$\left(\bigsqcup_{k=0}^{2m-4} \mathcal{C}_k\right) \setminus \left\{\omega_{1,0}^{k} \mid 0 \leq k \leq m-2\right\},$$

and $\mathcal{M}_{1,1}$ is the $\mathbb{Z}$-submodule of $H^{\bullet}(Gr(2,m),\mathbb{Z})$ which is generated by the *basis*

$$\left(\bigsqcup_{k=0}^{2m-4} \mathcal{C}_k\right) \setminus \left\{\omega_{1,1}^k \;\middle|\; 0 \leq k \leq \left\lfloor \frac{m-2}{2} \right\rfloor\right\}.$$

Then we have

$$\mathcal{Q}_{1,0} \simeq \mathbb{Z}[\omega_{1,0}]\Big/(\omega_{1,0}^{m-1})\ ; \qquad \mathcal{Q}_{1,1} \simeq \mathbb{Z}[\omega_{1,1}]\Big/(\omega_{1,1}^{\lfloor (m-2)/2 \rfloor + 1})$$

as both a $\mathbb{Z}$*-module and a ring (Here,* (x^k) *is an ideal in* $\mathbb{Z}[x]$ *generated by* x^k*).*

Proof. Since $\mathcal{M}_{1,0}$ is an ideal in $H^\bullet(Gr(2,m),\mathbb{Z})$, the quotient $\mathbb{Z}$-module $\mathcal{Q}_{1,0}$ has a canonical ring structure. By Proposition 2.1.7 (a), $\bigsqcup_{k=0}^{2m-4} \mathcal{C}_k$ is a $\mathbb{Z}$-module basis of $H^\bullet(Gr(2,m),\mathbb{Z})$, and $\{\omega_{1,0}^k \mid 0 \leq k \leq m-2\} \subset \bigsqcup_{k=0}^{m-2} \mathcal{C}_k$. Hence, $\mathcal{Q}_{1,0}$ is isomorphic to

$$\operatorname{span}\left(\{\omega_{1,0}^k \mid 0 \leq k \leq m-2\}\right) \simeq \mathbb{Z}[\omega_{1,0}]\Big/(\omega_{1,0}^{m-1}) \tag{2.1.9}$$

as a $\mathbb{Z}$-module. Furthermore, $\mathcal{Q}_{1,0}$ is isomorphic to the right hand side of (2.1.9) as a ring. The proof for $\mathcal{Q}_{1,1}$ is similar. ☐

2.2 Vector bundles on Grassmannians

Let $E(d,m)$ be the *universal bundle on* $Gr(d,m)$ whose total space is

$$\{(x,v) \in Gr(d,m) \times \mathbb{C}^m \mid v \in L_x\}.$$

Denote the universal bundle on $Gr(d,V)$ by $E(d,V)$. We have the following canonical short exact sequence:

$$0 \;\to\; E(d,m) \;\to\; \bigoplus^m \mathcal{O}_{Gr(d,m)} \;\to\; Q(d,m) \;\to\; 0 \tag{2.2.1}$$

where $Q(d,m) := \left(\bigoplus^m \mathcal{O}_{Gr(d,m)}\right)\Big/E(d,m)$, which is called the *universal quotient bundle on* $Gr(d,m)$.

Recall that automorphisms of $Gr(d,m)$ are classified in Theorem 1.0.1. The following Lemma is about the relations between $E(d,m)$, $Q(d,m)$ and these automorphisms.

Lemma 2.2.1. *Let φ be an automorphism of $Gr(d,m)$.*

(a) *If $\varphi \in \mathbf{P}GL(m,\mathbb{C})$, then $\varphi^*(E(d,m)) \simeq E(d,m)$.*

(b) *If $m = 2d$ and $\varphi \in \mathbf{P}GL(2d,\mathbb{C}) \circ \phi$, then $\varphi^*(\check{E}(d,2d)) \simeq Q(d,2d)$ where $\check{E}(d,2d)$ is the dual bundle of $E(d,2d)$.*

The proof of Lemma 2.2.1 is clear by definitions of $E(d,m)$ and $Q(d,m)$.

Proposition 2.2.2 ([Tan74, Lemma 1.3 and 1.4])**.** *In $Gr(d,m)$, the total Chern classes of $E(d,m)$ and $Q(d,m)$ are as follows:*

(a) $c(E(d,m)) = 1 + \sum_{k=1}^{d} (-1)^k \, \omega_{\underbrace{1,\cdots,1}_{k},0,\cdots,0}$.

(b) $c(Q(d,m)) = 1 + \sum_{k=1}^{m-d} \omega_{k,0,\cdots,0}$.

Next, we consider vector bundles on complex Grassmannians of rank 2. In [BVdV74a] and [BVdV74b], W. Barth and A. Van de Ven found criteria of the decomposability of such vector bundles.

Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^k$ of rank 2. For a projective line ℓ in $\mathbb{P}^k$, the restriction $\mathcal{E}\big|_{\ell}$ is decomposable by Grothendieck theorem ([OSS11, Theorem 2.1.1]), that is,

$$\mathcal{E}\big|_{\ell} = \mathcal{O}_{\ell}(a_1) \oplus \mathcal{O}_{\ell}(a_2)$$

for some integers a_1 and a_2 unique up to permutations. For such a_1 and a_2, define $b(\mathcal{E}\big|_{\ell})$ by the integer $\left\lfloor \frac{|a_1 - a_2|}{2} \right\rfloor$ and using this, let

$$B(\mathcal{E}) := \max \left\{ b(\mathcal{E}\big|_{\ell}) \mid \mathbb{P}^1 \simeq \ell \subset \mathbb{P}^k \right\}. \tag{2.2.2}$$

The following proposition tells us a sufficient condition for the decomposability of vector bundles on $\mathbb{P}^k$ of rank 2.

Proposition 2.2.3 ([BVdV74a, Theorem 5.1])**.** *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^k$ of rank 2 satisfying $B(\mathcal{E}) < \frac{k-2}{4}$. Then $\mathcal{E}$ is decomposable.*

Also, there is a sufficient condition for which vector bundles on $Gr(d,m)$ of rank 2 is either decomposable or isomorphic to some special form.

Proposition 2.2.4 ([BVdV74a, Theorem 4.1])**.** *Let $\mathcal{E}$ be a vector bundle on $Gr(d,m)$ of rank 2 with $m-d \geq 2$ satisfying that the restrictions $\mathcal{E}\big|_Y$ are decomposable for all Schubert varieties $Y \subset Gr(d,m)$ of type $(m-d,\cdots,m-d,0)$. Then either $\mathcal{E}$ is decomposable, or $d=2$ and $\mathcal{E} \simeq E(2,m) \otimes L$ for some line bundle L on $Gr(2,m)$.*

By Example 2.1.1 (a), every Schubert variety of type $(m-d,\cdots,m-d,0)$ is biholomorphic to a maximal projective space $\mathbb{P}^{m-d}$, thus the conclusion of Proposition 2.2.3 relates to the assumption of Proposition 2.2.4. Proposition 2.2.3 and 2.2.4 play an important role in proving Theorem 4.1.1.

Chapter 3

Embeddings of $Gr(2, m)$ into $Gr(2, n)$

For $n \geq m > d$, let $\varphi\colon Gr(d, m) \hookrightarrow Gr(d, n)$ be an embedding. As we already mentioned in Chapter 1, we characterize such an embedding φ by means of the pullback bundle E of the dual bundle of the universal bundle on $Gr(d, n)$ under the embedding φ.

Let $d = 2$ and $m \geq 4$, and let $\mathcal{C}_k$ be the set of cohomology classes which is given as in (2.1.3) for each $0 \leq k \leq 2m - 4$. Then $\mathcal{C} := \bigcup_{k=0}^{2m-4} \mathcal{C}_k$ is a $\mathbb{Z}$-module basis of the cohomology ring of $Gr(2, m)$ by Proposition 2.1.7 (a). The total Chern class of E can be written uniquely as a linear combination of $\omega_{0,0}$, $\omega_{1,0}$, $\omega_{1,0}^2$ and $\omega_{1,1}$ with integral coefficients $1, a, b$ and c, respectively. To determine possible pairs (a, b, c), we consider the pullback bundle N of the normal bundle of $\varphi(Gr(d, m))$ in $Gr(d, n)$ under the embedding φ, and use the descriptions of Chern classes in terms of an Euler class (Note 3.3.1).

The chapter consists of four sections. We assume that $d = 2$ and $m \geq 4$ except Section 3.1. In Section 3.1, we define the linearity and the twisted linearity of embeddings $\varphi\colon Gr(d, m) \hookrightarrow Gr(d, n)$ and present relations between the (twisted) linearity of φ and the image of φ (Proposition 3.1.4 and Corollary 3.1.5). In Section 3.2, we set the integral coefficients a, b and c, and present an equivalent condition on the pair (a, b, c) for the (twisted) linearity of φ (Proposition 3.2.2). In Section 3.3, we construct an equation of the Euler class of $N_{\mathbb{R}}$ (Proposition 3.3.2) and an equation of the total Chern class of N

(Lemma 3.3.3), by independent ways. From these two equations, we express the k^{th} Chern classes of N ($k \geq \text{rank}(N)$) and the Euler class of $N_{\mathbb{R}}$ as linear combinations of elements in the basis $\mathcal{C}$ with coefficients in $\mathbb{Z}[a, b, c]$. Using Note 3.3.1, we obtain several Diophantine equations in variables a, b and c. When we assume that $\text{rank}(N) = 2n - 2m \leq m - 2$ and focus on the coefficients of $\omega_{1,0}^k$ and $\omega_{1,1}^l$, we derive two refined equations from the equation of the total Chern class of N (Proposition 3.3.4). In Section 3.4, we obtain inequalities in variables a, b and c (Proposition 3.4.4 and 3.4.9) by various ways: a criterion of the numerical non-negativity of Chern classes of holomorphic vector bundles (Proposition 3.4.2), two refined equations in one variable, the result on the degrees of Schubert cycles (Lemma 3.4.6) and so on.

3.1 Linear embeddings

In Chapter 1, we defined a *linear* embedding $\tilde{f}_W \colon Gr(d_1, m) \hookrightarrow Gr(d_2, n)$, which is induced by an injective linear map $f \colon \mathbb{C}^m \hookrightarrow \mathbb{C}^n$ and a $(d_2 - d_1)$-dimensional subspace W of $\mathbb{C}^n$ satisfying $f(\mathbb{C}^m) \cap W = 0$. When $d_1 = d_2 \,(=: d)$, we do not have to consider an extra summand W of $L_{\tilde{f}_W(x)}$ ($x \in Gr(d, m)$), so the definition of the linearity is simpler. Furthermore, when either $m = 2d$ or $n = 2d$, there is a non-linear, but natural embedding because of the existence of a dual map ϕ, which is a non-linear automorphism of $\text{Aut}(Gr(d, 2d))$.

Definition 3.1.1. Let $\varphi \colon Gr(d, m) \hookrightarrow Gr(d, n)$ be an embedding.

(a) An embedding φ is *linear* if φ is induced by an injective linear map $f \colon \mathbb{C}^m \hookrightarrow \mathbb{C}^n$, that is,

$$L_{\varphi(x)} = \text{the } d\text{-dimensional subspace } f(L_x) \text{ of } \mathbb{C}^n$$

for all $x \in Gr(d, m)$.

(b) When $m = 2d$ (resp. $n = 2d$), an embedding φ is *twisted linear* if φ is of the form $\varphi_0 \circ \phi$ (resp. $\phi \circ \varphi_0$) where $\varphi_0 \colon Gr(d, m) \hookrightarrow Gr(d, n)$ is a linear embedding and $\phi \colon Gr(d, 2d) \to Gr(d, 2d)$ is a dual map.

Remark 3.1.2. Assume that $m = 2d = n$ and $\varphi\colon Gr(d,2d) \hookrightarrow Gr(d,2d)$ is a twisted linear embedding of the form $\phi \circ \varphi_0$ where $\varphi_0\colon Gr(d,2d) \hookrightarrow Gr(d,2d)$ is a linear embedding. In this case, ϕ and φ_0 are automorphisms of $Gr(d,2d)$, and since φ_0 is linear, $\varphi_0 \in \mathbf{P}GL(2d,\mathbb{C})$. By Theorem 1.0.1, $\mathbf{P}GL(2d,\mathbb{C}) \circ \phi = \phi \circ \mathbf{P}GL(2d,\mathbb{C})$, thus we have

$$\varphi = \phi \circ \varphi_0 = \varphi_1 \circ \phi$$

for some $\varphi_1 \in \mathbf{P}GL(2d,\mathbb{C})$.

Of course, every embedding of $Gr(d,m)$ into $Gr(d,n)$ is not always linear. When $d = 1$, S. Feder showed the existence of a non-linear embedding $\mathbb{P}^m \hookrightarrow \mathbb{P}^n$ for $n > 2m$ by Theorem 1.0.3 (b). The following example provides some non-linear embeddings for $d \geq 2$.

Example 3.1.3 (Non-linear embeddings)**.** Let $\varphi\colon Gr(d,m) \hookrightarrow Gr(d,n)$ be an embedding.

(a) If either $m = 2d$ or $n = 2d$, then every twisted linear embedding is not linear.

(b) Consider the Plücker embedding $i\colon Gr(d,m) \hookrightarrow \mathbb{P}(\wedge^d \mathbb{C}^m) = \mathbb{P}^N$ where $N := \binom{m}{d} - 1$. There is a maximal projective space $Y \simeq \mathbb{P}^{n-d}$ of $Gr(d,n)$, which is a Schubert variety of type $(n-d, \cdots, n-d, 0)$ (Example 2.1.1 (a)), and let $j\colon Y \hookrightarrow Gr(d,n)$ be an inclusion. Apply S. Feder's result to this situation. If $n - d > 2N$, then there is a non-linear embedding $\psi\colon \mathbb{P}^N \hookrightarrow \mathbb{P}^{n-d} \simeq Y$ by Theorem 1.0.3 (b). The composition of maps $\varphi := j \circ \psi \circ i$ is an embedding of $Gr(d,m)$ into $Gr(d,n)$, but it is not linear.

(c) Consider an embedding $\varphi\colon Gr(2,3) \to Gr(2,4)$. Let $\psi := \varphi \circ \phi$ be the composition of maps where $\phi\colon \mathbb{P}^2 \to Gr(2,3)$ is a dual map, then ψ is an embedding of $\mathbb{P}^2$ into $Gr(2,4)$. By Theorem 1.0.4, there are the following 4 types of $X := \psi(\mathbb{P}^2) = \varphi(Gr(2,3))$:

$$X^0_{3,1}; \quad X^1_{3,1}; \quad \check{X}^0_{3,1}; \quad \check{X}^1_{3,1}$$

which are given as in (1.0.4) and (1.0.5), up to linear automorphisms of $Gr(2,4)$. In particular, $X^0_{3,1}$ is a Schubert variety of type $(2,0)$ and $\check{X}^0_{3,1}$ is a Schubert variety of type $(1,1)$. If $X = \check{X}^0_{3,1}$ up to linear automorphisms, then φ is linear, and if $X = X^0_{3,1}$ up to linear automorphisms, then φ is twisted linear. On the other hand, if $X = X^1_{3,1}$ or $\check{X}^1_{3,1}$ up to linear automorphisms, then φ is neither linear nor twisted linear.

There is a relation between the (twisted) linearity of an embedding φ of $Gr(d,m)$ into $Gr(d,n)$ and the image of φ.

Proposition 3.1.4. *For $n \geq m \geq d$, let $\varphi\colon Gr(d,m) \hookrightarrow Gr(d,n)$ be an embedding. Then the image of φ is equal to $Gr(d,H_\varphi)$ for some m-dimensional subspace H_φ of $\mathbb{C}^n$ if and only if one of the following conditions holds:*

- *φ is linear;*
- *$m = 2d$ and φ is twisted linear.*

Proof. During this proof, we denote the image of φ by X.

If φ is linear, then φ is induced by an injective linear map $f\colon \mathbb{C}^m \hookrightarrow \mathbb{C}^n$, thus $X = Gr(d, f(\mathbb{C}^m))$. Moreover, if φ is twisted linear with $m = 2d$, then $\varphi = \varphi_0 \circ \phi$ for some linear embedding $\varphi_0\colon Gr(d,2d) \hookrightarrow Gr(d,n)$ (For the case when $n = m$, see Remark 3.1.2). So we have

$$X = \varphi_0(\phi(Gr(d,2d))) = \varphi_0(Gr(d,2d)),$$

thus the image of $\varphi = \varphi_0 \circ \phi$ is $Gr(d,H)$ for some $(2d)$-dimensional subspace H of $\mathbb{C}^n$.

Conversely, assume that $X = Gr(d,H_\varphi)$ where H_φ is an m-dimensional subspace of $\mathbb{C}^m$. Fix a biholomorphism $\psi_\varphi\colon Gr(d,H_\varphi) \to Gr(d,m)$ which is induced by a linear isomorphism $H_\varphi \to \mathbb{C}^m$. Let $\varphi_1\colon Gr(d,m) \to Gr(d,H_\varphi)$ be a biholomorphism which is obtained by restricting the codomain of φ to its image $Gr(d,H_\varphi)$. Then $\psi_\varphi \circ \varphi_1$ is an automorphism of $Gr(d,m)$ as follows:

$$\begin{tikzcd}
 & Gr(d,m) & \\
Gr(d,m) \arrow[ur, "\psi_\varphi \circ \varphi_1"] \arrow[r, "\simeq", "\varphi_1"'] \arrow[ur, phantom, shift right=4ex, "\circlearrowleft"] & Gr(d,H_\varphi) \arrow[u, "\psi_\varphi"', "\wr"] \arrow[r, "\subset"] & Gr(d,n)
\end{tikzcd}$$

If $m \neq 2d$, then $\psi_\varphi \circ \varphi_1 \in \mathbf{P}GL(m,\mathbb{C})$ by Theorem 1.0.1, thus φ is linear. If $m = 2d$, then $\psi_\varphi \circ \varphi_1 \in \mathbf{P}GL(2d,\mathbb{C}) \sqcup (\mathbf{P}GL(2d,\mathbb{C}) \circ \phi)$ by Theorem 1.0.1, thus φ is either linear or twisted linear. $\square$

Proposition 3.1.4 covers all the cases when $\varphi\colon Gr(d,m) \hookrightarrow Gr(d,n)$ is either linear or twisted linear except when $n = 2d > m$ and φ is twisted linear. The following corollary is an analogous result for this exceptional case.

Corollary 3.1.5. *For $2d > m$, let $\varphi\colon Gr(d,m) \hookrightarrow Gr(d,2d)$ be an embedding. Then the image of φ is equal to $\{x \in Gr(d,2d) \mid V_\varphi \subset L_x\}$ for some $(2d-m)$-dimensional subspace V_φ of $\mathbb{C}^{2d}$ if and only if φ is twisted linear.*

Proof. Since $2d > m$, φ is twisted linear if and only if $\varphi = \phi \circ \varphi_0$ for some linear embedding $\varphi_0\colon Gr(d,m) \hookrightarrow Gr(d,2d)$ or, equivalently, the image of $\phi \circ \varphi$ is equal to $Gr(d,H)$ for some m-dimensional subspace H of $\mathbb{C}^{2d}$ by Proposition 3.1.4. Since $\phi \circ \phi = id$, to complete the proof, it suffices to show that given a $(2d-m)$-dimensional subspace V_φ of $\mathbb{C}^{2d}$,

$$\phi(\{x \in Gr(d,2d) \mid V_\varphi \subset L_x\}) = Gr(d,H)$$

for some m-dimensional subspace H of $\mathbb{C}^{2d}$. By definition (1.0.1) of a dual map ϕ,

$$\begin{aligned}
\phi\left(\{x \in Gr(d,2d) \mid V_\varphi \subset L_x\}\right) &= \{\phi(x) \in Gr(d,2d) \mid V_\varphi \subset L_x\} \\
&= \{\phi(x) \in Gr(d,2d) \mid L_{\phi(x)} = \iota(L_x^\perp) \subset \iota(V_\varphi^\perp)\} \\
&= \{y \in Gr(d,2d) \mid L_y \subset \iota(V_\varphi^\perp)\} \quad (y := \phi(x)) \\
&= Gr(d, \iota(V_\varphi^\perp)).
\end{aligned}$$

Here, the dimension of $\iota(V_\varphi^\perp)$ is $2d - (2d-m) = m$ as desired. $\square$

3.2 Equivalent conditions for the linearity

Let $\varphi\colon Gr(2,m) \hookrightarrow Gr(2,n)$ be an embedding and $E := \varphi^*(\check{E}(2,n))$ where $\check{E}(2,n)$ is the dual bundle of $E(2,n)$. To distinguish Schubert cycles on two Grassmannians $Gr(2,m)$ and $Gr(2,n)$, denote Schubert cycles on $Gr(2,n)$

(resp. $Gr(2,m)$) by $\tilde{\omega}_{i,j}$ (resp. $\omega_{k,l}$) (Of course, all properties in Chapter 2 hold for $Gr(2,n)$ and $\tilde{\omega}_{i,j}$). By Proposition 2.2.2 (a),

$$c(\check{E}(2,n)) = 1 + \tilde{\omega}_{1,0} + \tilde{\omega}_{1,1}$$

and

$$\begin{aligned} c_1(E) &= \varphi^*(\tilde{\omega}_{1,0}) = a\,\omega_{1,0}, \\ c_2(E) &= \varphi^*(\tilde{\omega}_{1,1}) = b\,\omega_{1,0}^2 + c\,\omega_{1,1} = b\,\omega_{2,0} + (b+c)\,\omega_{1,1} \end{aligned} \tag{3.2.1}$$

for some $a,b,c \in \mathbb{Z}$. By Proposition 2.1.7 (a), $\{\omega_{0,0}, \omega_{1,0}, \omega_{1,0}^2, \omega_{1,1}\}$ is a basis of the $\mathbb{Z}$-module $\bigoplus_{k=0}^{2} H^{2k}(Gr(2,m),\mathbb{Z})$ if and only if $m \geq 4$. So the coefficients a, b and c given as in (3.2.1) are determined uniquely for each φ. For this reason, we always assume that $m \geq 4$.

Lemma 3.2.1. *For $n \geq m \geq 4$, let $\varphi\colon Gr(2,m) \hookrightarrow Gr(2,n)$ be an embedding. Then the Poincaré dual to the homology class of $X := \varphi(Gr(2,m))$ is*

$$\sum_{i=0}^{n-m} (X \cdot \tilde{\omega}_{n-2-i,2m-n-2+i})\,\tilde{\omega}_{2n-2m-i,i}$$

where $X \cdot \tilde{\omega}_{n-2-i,2m-n-2+i}$ is the intersection number in $Gr(2,n)$.

Proof. Since the codimension of $X \simeq Gr(2,m)$ in $Gr(2,n)$ is $2n-2m$, the Poincaré dual to the homology class of X is

$$\sum_{i=0}^{n-m} d_i\,\tilde{\omega}_{2n-2m-i,i}$$

for some integers d_i. By Corollary 2.1.6,

$$X \cdot \tilde{\omega}_{n-2-j,2m-n-2+j} = \sum_{i=0}^{n-m} d_i\,(\tilde{\omega}_{2n-2m-i,i} \cdot \tilde{\omega}_{n-2-j,2m-n-2+j}) = d_j$$

as desired. $\square$

Now we ready to prove the following proposition on the equivalent conditions for the (twisted) linearity of an embedding.

Proposition 3.2.2. *For $n \geq m \geq 4$ (resp. $n \geq m = 4$), let $\varphi\colon Gr(2,m) \hookrightarrow Gr(2,n)$ be an embedding and a,b,c be the the integers given as in* (3.2.1)*. The following are equivalent:*

(a) *φ is linear (resp. φ is twisted linear);*

(b) *$E \simeq \check{E}(2,m)$ (resp. $E \simeq Q(2,4)$);*

(c) *$(a,b,c) = (1,0,1)$ (resp. $(a,b,c) = (1,1,-1)$).*

Proof. During this proof, we denote the image of φ by X.

- (a) $\Rightarrow$ (b) : Assume that φ is linear, then $X = Gr(2, H_\varphi)$ for some m-dimensional subspace H_φ of $\mathbb{C}^n$ by Proposition 3.1.4. The total space of $E(2,n)|_X$ is

$$\begin{aligned} &\{(x,v) \in X \times \mathbb{C}^n \mid v \in L_x \subset \mathbb{C}^n\} \\ =&\{(x,v) \in Gr(2,H_\varphi) \times H_\varphi \mid v \in L_x \subset H_\varphi\} \end{aligned}$$

  which is exactly equal to the total space of $E(2,H_\varphi)$. So $E(2,n)|_X = E(2,H_\varphi)$ and we have

$$E = \varphi^*(\check{E}(2,n)) = \varphi^*(\check{E}(2,n)|_X) = \varphi^*(\check{E}(2,H_\varphi)) \simeq \check{E}(2,m).$$

  If φ is twisted linear, then $\varphi \circ \phi$ is linear. By the previous result, $(\varphi \circ \phi)^*(E) \simeq \check{E}(2,m)$. So $\varphi^*(E) \simeq \phi^*(\check{E}(2,m))$, which is isomorphic to $Q(2,m)$ by Lemma 2.2.1 (b).

- (b) $\Rightarrow$ (c) : By Proposition 2.2.2, we have

$$\begin{aligned} c(\check{E}(2,m)) &= 1 + \omega_{1,0} + \omega_{1,1} \\ c(Q(2,m)) &= 1 + \omega_{1,0} + \omega_{2,0} = 1 + \omega_{1,0} + \omega_{1,0}^2 - \omega_{1,1}. \end{aligned}$$

  So if $E \simeq \check{E}(2,m)$ (resp. $E \simeq Q(2,4)$), then $(a,b,c) = (1,0,1)$ (resp. $(a,b,c) = (1,1,-1)$).

- (c) $\Rightarrow$ (a) : Assume that $(a,b,c) = (1,0,1)$. Then $\varphi^*(\tilde{\omega}_{1,0}) = \omega_{1,0}$ and $\varphi^*(\tilde{\omega}_{1,1}) = \omega_{1,1}$. Note that $H^\bullet(Gr(2,n),\mathbb{Z})$ (resp. $H^\bullet(Gr(2,m),\mathbb{Z})$) is generated by $\tilde{\omega}_{1,0}$ and $\tilde{\omega}_{1,1}$ (resp. $\omega_{1,0}$ and $\omega_{1,1}$) as a ring. Moreover, the multiplicative structures of $H^\bullet(Gr(2,n),\mathbb{Z})$ and $H^\bullet(Gr(2,m),\mathbb{Z})$ are exactly same when we adopt Convention 2.1.4. So we have

$$\varphi^*(\tilde{\omega}_{i,j}) = \omega_{i,j} \tag{3.2.2}$$

for all $n-2 \geq i \geq j \geq 0$, thus the Poincaré dual to the homology class of X is

$$\sum_{i=0}^{n-m} (X \cdot \tilde{\omega}_{n-2-i,2m-n-2+i})\, \tilde{\omega}_{2n-2m-i,i} \tag{3.2.3}$$

by Lemma 3.2.1, and so we have

$$\begin{aligned}
(3.2.3) =& \sum_{i=0}^{n-m} (\varphi^*(\tilde{\omega}_{n-2-i,2m-n-2+i}))\, \tilde{\omega}_{2n-2m-i,i} \\
=& \sum_{i=0}^{n-m} (\omega_{n-2-i,2m-n-2+i})\, \tilde{\omega}_{2n-2m-i,i} \quad (\because \text{ Equation (3.2.2)}) \\
=& (\omega_{m-2,m-2})\, \tilde{\omega}_{n-m,n-m} \quad (\because\ \omega_{n-2-i,2m-n-2+i} = 0 \\
& \qquad\qquad \text{unless } m-2 \geq n-2-i \geq 2m-n-2+i \geq 0) \\
=& \tilde{\omega}_{n-m,n-m}.
\end{aligned}$$

By Proposition 2.1.2, any subvariety of $Gr(2,n)$ which corresponds to $\tilde{\omega}_{n-m,n-m}$ is of the form $Gr(2,H)$ where H is an m-dimensional subspace of $\mathbb{C}^n$. Applying Proposition 3.1.4 and the implication of (a)$\Rightarrow$(c), we obtain the desired result.

Assume that $(a,b,c) = (1,1,-1)$. To prove the implication of (c)$\Rightarrow$(a) for this case, it suffices to show that $\varphi \circ \phi$ is linear. The total Chern class of $(\varphi \circ \phi)^*(E) = \phi^*(\varphi^*(E))$ equals

$$\begin{aligned}
c((\varphi \circ \phi)^*(E)) =& c(\phi^*(\varphi^*(E))) = \phi^*(c(\varphi^*(E))) \\
=& \phi^*(1 + \omega_{1,0} + \omega_{1,0}^2 - \omega_{1,1}) \\
=& 1 + \omega_{1,0} + \omega_{1,0}^2 - (\omega_{1,0}^2 - \omega_{1,1}) \\
=& 1 + \omega_{1,0} + \omega_{1,1},
\end{aligned}$$

thus $\varphi \circ \phi$ is linear by the implication of (c) $\Rightarrow$ (a) for the case of linearity. Hence, φ is twisted linear.

$\square$

Remark 3.2.3. Assume that $n < 2m-2$ and $a = 1$. Using Theorem 1.0.2, we can prove the implication of (c) $\Rightarrow$ (a) in Proposition 3.2.2 for this case, without computing the Poincaré dual to the homology class of X. Since $a = 1$, φ maps each projective line in $Gr(2,m)$ to a projective line in $Gr(2,n)$, thus for any $x \in Gr(2,m)$, the differential $d\varphi$ preserves the decomposability of tangent vectors of $Gr(2,m)$ at x. So by Theorem 1.0.2, either φ is linear up to automorphisms of $Gr(2,m)$ or $Gr(2,n)$, or the image of φ lies on some projective space in $Gr(2,n)$. But the latter case is impossible because the dimension of $\varphi(Gr(2,m))$ is greater than the dimension of each maximal projective space in $Gr(2,n)$, that is, $2m-4 > n-2$. By Theorem 1.0.1, there is a non-linear automorphism of $Gr(2,m)$ only for the case when $m = 4$. Hence, if $m \geq 5$, then φ is linear, and if $m = 4$, then φ is either linear or twisted linear.

3.3 Equations in a, b and c

Let a, b and c be the integers which are given as in (3.2.1), and N be the pullback bundle of the normal bundle of $X := \varphi(Gr(2,m))$ in $Gr(2,n)$ under the embedding φ. In this section, we construct an equation of the total Chern class $c(N)$ of N (Lemma 3.3.3) and under the assumption when $n \leq \frac{3m-2}{2}$, we deduce the refined equations in one variable (Proposition 3.3.4) from the equation of $c(N)$. Using the definition of the Chern classes of N in terms of the Euler class of $N_{\mathbb{R}}$ (Note 3.3.1), we obtain several Diophantine equations in variables a, b and c.

3.3.1 Euler class of $N_{\mathbb{R}}$

There are two methods to construct the Chern classes of a complex vector bundle $\mathcal{E}$: one is via Chern-Weil theory and the other is via the Euler class $E(\mathcal{E}_{\mathbb{R}})$ of $\mathcal{E}_{\mathbb{R}}$. For more details on the first and the second methods to construct

Chern classes, see [BT82, Section 20] and [MS74, Section 14], respectively. For more details on Euler classes, see [BT82, Section 11] or [MS74, Section 9].

By the second method (or deriving from the first method), we have the following result on the k^{th} Chern classes $c_k(\mathcal{E})$ of $\mathcal{E}$ for $k \geq \operatorname{rank}(\mathcal{E})$:

Note 3.3.1 ([BT82, (20.10.4) and (20.10.6)] or [MS74, page 158]). Let Z be a real manifold and $\mathcal{E}$ be a complex vector bundle on Z. Then we have

$$c_k(\mathcal{E}) = \begin{cases} e(\mathcal{E}_{\mathbb{R}}), & \text{if } k = \operatorname{rank}(\mathcal{E}) \\ 0, & \text{if } k > \operatorname{rank}(\mathcal{E}) \end{cases}$$

where $\mathcal{E}_{\mathbb{R}}$ is the real vector bundle on Z which corresponds to $\mathcal{E}$.

As in Note 3.3.1, let $N_{\mathbb{R}}$ be the real vector bundle which corresponds to N. Then we can compute the Euler class of $N_{\mathbb{R}}$ as follows:

Proposition 3.3.2. *For $n \geq m \geq 4$, the Euler class of $N_{\mathbb{R}}$ is*

$$e(N_{\mathbb{R}}) = \sum_{i=0}^{n-m} (X \cdot \tilde{\omega}_{n-2-i,2m-n-2+i})\, \varphi^*(\tilde{\omega}_{2n-2m-i,i}). \tag{3.3.1}$$

In particular, when $n = m+1$,

$$\begin{aligned} e(N_{\mathbb{R}}) &= (X \cdot \tilde{\omega}_{m-1,m-3})\, \varphi^*(\tilde{\omega}_{2,0}) + (X \cdot \tilde{\omega}_{m-2,m-2})\, \varphi^*(\tilde{\omega}_{1,1}) \\ &= \{(X \cdot \tilde{\omega}_{m-1,m-3})\,(a^2 - b) + (X \cdot \tilde{\omega}_{m-2,m-2})\, b\}\, \omega_{1,0}^2 \\ &\quad + (X \cdot \tilde{\omega}_{m-2,m-2} - X \cdot \tilde{\omega}_{m-1,m-3})\, c\, \omega_{1,1}. \end{aligned}$$

Proof. By [Fed65, Theorem 1.3],

$$e(N_{\mathbb{R}}) = \varphi^*(\varphi_*(1))$$

where 1 is the cohomology class in $H^{\bullet}(Gr(2,m), \mathbb{Z})$ which corresponds to $Gr(2,m)$ itself. Since $\varphi_*(1)$ is the cohomology class in $H^{\bullet}(Gr(2,n), \mathbb{Z})$ which corresponds to X,

$$\begin{aligned} e(N_{\mathbb{R}}) &= \varphi^*\left(\sum_{i=0}^{n-m}(X\cdot\tilde{\omega}_{n-2-i,2m-n-2+i})\,\tilde{\omega}_{2n-2m-i,i}\right) \\ &= \sum_{i=0}^{n-m}(X\cdot\tilde{\omega}_{n-2-i,2m-n-2+i})\,\varphi^*(\tilde{\omega}_{2n-2m-i,i}) \end{aligned}$$

by Lemma 3.2.1. $\square$

The cohomology ring $H^\bullet(Gr(2,n),\mathbb{Z})$ of $Gr(2,n)$ is generated by $\{\tilde{\omega}_{1,0},\,\tilde{\omega}_{1,1}\}$ as a ring, thus any $\tilde{\Gamma}\in H^{2k}(Gr(2,n),\mathbb{Z})$ can be written as $f_{\tilde{\Gamma}}(\tilde{\omega}_{1,0},\,\tilde{\omega}_{1,1})$ for some polynomial $f_{\tilde{\Gamma}}\in\mathbb{Z}[x,y]$. Since $\varphi^*\colon H^\bullet(Gr(2,n),\mathbb{Z})\to H^\bullet(Gr(2,m),\mathbb{Z})$ is a ring homomorphism, we have $\varphi^*(\tilde{\Gamma}) = f_{\tilde{\Gamma}}(a\,\omega_{1,0},\, b\,\omega_{1,0}^2 + c\,\omega_{1,1})$, which is expressed as $g_{\tilde{\Gamma},\varphi}(\omega_{1,0},\,\omega_{1,1})$ for some $g_{\tilde{\Gamma},\varphi}\in\mathbb{Z}[x,y]$.

Apply this method to the Euler class $e(N_{\mathbb{R}})$. In each summand of (3.3.1), $X\cdot\tilde{\omega}_{n-2-i,2m-n-2+i}\in\mathbb{Z}$ and $\varphi^*(\tilde{\omega}_{2n-2m-i,i})\in H^{2n-2m}(Gr(2,m),\mathbb{Z})$, so we can express $e(N_{\mathbb{R}})$ as

$$e(N_{\mathbb{R}}) = \sum_{i=0}^{n-m} A_i\,\omega_{1,0}^{2n-2m-2i}\,\omega_{1,1}^{i} \tag{3.3.2}$$

for some polynomials $A_i\in\mathbb{Z}[a,b,c]$.

3.3.2 Total Chern class of N

Lemma 3.3.3. *Let $n\geq m\geq 4$. Then the total Chern class $c(N)$ of N satisfies the following equation:*

$$\begin{aligned} &c(N)(1+(4b-a^2)\,\omega_{1,0}^2+4c\,\omega_{1,1})(1+\omega_{1,0}+\omega_{1,1})^m \\ &\quad= (1+a\,\omega_{1,0}+b\,\omega_{1,0}^2+c\,\omega_{1,1})^n(1-\omega_{1,0}^2+4\,\omega_{1,1}) \end{aligned} \tag{3.3.3}$$

which is satisfied in $H^\bullet(Gr(2,m),\mathbb{Z})$. Moreover, the first and the second Chern classes of N are

$$\begin{aligned} c_1(N) &= (an-m)\,\omega_{1,0};\\ c_2(N) &= \left\{\binom{n}{2}a^2 - amn + m^2 - \binom{m}{2} + a^2 - 1 + b(n-4)\right\}\omega_{1,0}^2 \\ &\qquad\qquad + \{c(n-4) - m + 4\}\,\omega_{1,1}. \end{aligned}$$

Proof. Taking the tensor product of (2.2.1) with $\check{E}(2,m)$, we obtain a short exact sequence

$$0 \to E(2,m)\otimes\check{E}(2,m) \to \bigoplus^{m}\check{E}(2,m) \to Q(2,m)\otimes\check{E}(2,m) \to 0, \quad (3.3.4)$$

and after replacing m with n, we also obtain a short exact sequence

$$0 \to E(2,n)\otimes\check{E}(2,n) \to \bigoplus^{n}\check{E}(2,n) \to Q(2,n)\otimes\check{E}(2,n) \to 0. \quad (3.3.5)$$

Since $T_{Gr(2,n)} \simeq Q(2,n)\otimes\check{E}(2,n)$ and $T_{Gr(2,m)} \simeq Q(2,m)\otimes\check{E}(2,m)$,

$$\begin{aligned} c(\varphi^*(T_{Gr(2,n)})) &= \frac{\varphi^*(c(\check{E}(2,n)))^n}{\varphi^*(c(E(2,n)\otimes\check{E}(2,n)))} = \frac{c(E)^n}{c(\check{E}\otimes E)};\\ c(T_{Gr(2,m)}) &= \frac{c(\check{E}(2,m))^m}{c(E(2,m)\otimes\check{E}(2,m))} \end{aligned}$$

by (3.3.4) and (3.3.5). So we have the following equation:

$$c(N) = \frac{c(\varphi^*(T_{Gr(2,n)}))}{c(T_{Gr(2,m)})} = \frac{c(E)^n/c(\check{E}\otimes E)}{c(\check{E}(2,m))^m/c(E(2,m)\otimes\check{E}(2,m))},$$

that is,

$$c(N)c(\check{E}\otimes E)c(\check{E}(2,m))^m = c(E)^n c(E(2,m)\otimes\check{E}(2,m)). \quad (3.3.6)$$

Note that

$$\begin{aligned} c(E) &= 1 + a\,\omega_{1,0} + b\,\omega_{1,0}^2 + c\,\omega_{1,1}; \\ c(\check{E} \otimes E) &= 1 + (4b - a^2)\,\omega_{1,0}^2 + 4c\,\omega_{1,1}; \\ c(\check{E}(2,m)) &= 1 + \omega_{1,0} + \omega_{1,1}; \\ c(E(2,m) \otimes \check{E}(2,m)) &= 1 - \omega_{1,0}^2 + 4\,\omega_{1,1}. \end{aligned} \tag{3.3.7}$$

Putting (3.3.7) into (3.3.6), we obtain Equation (3.3.3).

Comparing the cohomology classes of degree 2 in both sides of (3.3.3), we have

$$c_1(N) + m\,\omega_{1,0} = an\,\omega_{1,0},$$

so we obtain

$$c_1(N) = (an - m)\,\omega_{1,0}. \tag{3.3.8}$$

Comparing the cohomology classes of degree 4 in both sides of (3.3.3), we have

$$\begin{aligned} c_2(N) + c_1(N) \cdot m\,\omega_{1,0} + (4b - a^2)\,\omega_{1,0}^2 + 4c\,\omega_{1,1} + \binom{m}{2}\,\omega_{1,0}^2 + m\,\omega_{1,1} \\ = \binom{n}{2} a^2\,\omega_{1,0}^2 + bn\,\omega_{1,0}^2 + cn\,\omega_{1,1} - \omega_{1,0}^2 + 4\,\omega_{1,1}. \end{aligned} \tag{3.3.9}$$

Putting (3.3.8) into (3.3.9), we obtain

$$\begin{aligned} c_2(N) = \left\{ \binom{n}{2} a^2 - amn + m^2 - \binom{m}{2} + a^2 - 1 + b(n-4) \right\} \omega_{1,0}^2 \\ + \{c(n-4) - m + 4\}\,\omega_{1,1} \end{aligned}$$

as desired. $\square$

By replacing $c(N)$ by Γ, regard (3.3.3) as an equation with a variable Γ. Write a solution Γ of (3.3.3) as $\Gamma = \sum_{k=0}^{2m-4} \Gamma_k$ where $\Gamma_k \in H^{2k}(Gr(2,m), \mathbb{Z})$ for all $0 \le k \le 2m-4$. In the proof of Lemma 3.3.3, we compute Γ_k for $k = 1$ and 2 by the following steps:

$\boxed{1}$ Obtain an equation $(\star_k)$, which is satisfied in $H^{2k}(Gr(2,m), \mathbb{Z})$, after comparing the cohomology classes of degree $2k$ in both sides of (3.3.3);

$\boxed{2}$ compute Γ_k by putting Γ_i into $(\star_k)$ for all $0 \le i < k$.

Repeat these operations from $k = 3$ to $2m-4$. After that, we can express Γ_k $(= c_k(N))$ for $0 \le k \le 2m-4$ as follows:

$$\begin{aligned}
\Gamma_0 =& 1 = B_{0,0}\,\omega_{1,0}^0 = B_{0,0}\,\omega_{1,1}^0 \\
\Gamma_1 =& B_{1,0}\,\omega_{1,0} \\
\Gamma_2 =& B_{2,0}\,\omega_{1,0}^2 + B_{2,1}\,\omega_{1,1} \\
&\vdots \\
\Gamma_k =& B_{k,0}\,\omega_{1,0}^k + B_{k,1}\,\omega_{1,0}^{k-2}\,\omega_{1,1} + \cdots + B_{k,h_k}\,\omega_{1,0}^{k-2h_k}\,\omega_{1,1}^{h_k} \\
&\vdots \\
\Gamma_{2m-4} =& B_{2m-4,0}\,\omega_{1,0}^{2m-4} + B_{2m-4,1}\,\omega_{1,0}^{2m-6}\,\omega_{1,1} + \cdots + B_{2m-4,m-2}\,\omega_{1,1}^{m-2}
\end{aligned} \tag{3.3.10}$$

where $h_k := \left\lfloor \frac{k}{2} \right\rfloor$ and $B_{k,i} \in \mathbb{Z}[a,b,c]$ for all $0 \le k \le 2m-4$, $0 \le i \le h_k$. In addition, $\Gamma_k = 0$ for all $k > 2m-4$.

Since the rank of N is $2n-2m$, we have by Note 3.3.1,

$$c_k(N) = \begin{cases} e(N_{\mathbb{R}}), & \text{if } k = 2n-2m \\ 0, & \text{if } 2n-2m < k \le 2m-4 \end{cases}. \tag{3.3.11}$$

If $2n-2m > 2m-4$, then we cannot obtain any further information from (3.3.11). On the other hand, if $2n-2m \le 2m-4$, then we obtain several Diophantine equations in a, b and c by applying (3.3.2) and (3.3.10) to (3.3.11). For this reason, the condition $2n-2m \le 2m-4$ (or equivalently, $n \le 2m-2$) is essential in order to find all possible pairs (a,b,c) of integers from these Diophantine equations.

If $m-2 < 2n-2m \le 2m-4$, then the choices of coefficients A_i in (3.3.2) is not unique and the choice of coefficients $B_{k,i}$ in (3.3.10) is not unique for all $2n-2m \le k \le 2m-4$ by Proposition 2.1.7 (a) (or Remark 2.1.8). To apply (3.3.2) and (3.3.10) to (3.3.11), we need to write $e(N_{\mathbb{R}})$ and Γ_{2n-2m} with respect to the same basis of $H^{2(2n-2m)}(Gr(2,m),\mathbb{Z})$, and write Γ_k with respect to a basis of $H^{2k}(Gr(2,m),\mathbb{Z})$ for each $2n-2m < k \le 2m-4$ again.

If $2n-2m \le m-2$, then we do not have to care the uniqueness of coefficients in $e(N_{\mathbb{R}})$ and Γ_k $(2n-2m \le k \le m-2)$, thus we have the following Diophantine equations directly:

$$\begin{array}{rlrlcrl} A_0 = & B_{2n-2m,0}; & A_1 = & B_{2n-2m,1}; & \cdots & A_{n-m} = & B_{2n-2m,n-m}; \\ 0 = & B_{k,0}; & 0 = & B_{k,1}; & \cdots & 0 = & B_{k,h_h} \end{array} \tag{3.3.12}$$

for all $2n-2m < k \leq m-2$. However, if $n-m$ is big, then it is difficult to find all Diophantine equations in (3.3.12). So we need simpler equations than (3.3.3).

3.3.3 Refined equations in $\omega_{1,0}$ and $\omega_{1,1}$

Using Equation (3.3.3) and the basis (2.1.4) of a cohomology group together with Corollary 2.1.9, we obtain two refined equations in one variable as follows:

Proposition 3.3.4. *Let $m \leq n \leq \frac{3m-2}{2}$. For $0 \leq k \leq 2n-2m$, write the k^{th} Chern class $c_k(N)$ of N as*

$$c_k(N) = \begin{cases} \alpha_0 = 1 = \beta_0, & \text{if } k = 0 \\ \alpha_1\,\omega_{1,0} = \beta_1\,\omega_{1,0}, & \text{if } k = 1 \\ \alpha_k\,\omega_{1,0}^k + \cdots + \beta_k\,\omega_{1,1}^{k/2}, & \text{if } k \geq 2 \text{ is even} \\ \alpha_k\,\omega_{1,0}^k + \cdots + \beta_k\,\omega_{1,0}\omega_{1,1}^{(k-1)/2}, & \text{if } k \geq 3 \text{ is odd} \end{cases} \tag{3.3.13}$$

with respect to the basis (2.1.4)*. Then we have two equations*

$$\begin{aligned} \left(\sum_{k=0}^{2n-2m} \alpha_k\,\omega_{1,0}^k\right) & (1 + (4b - a^2)\,\omega_{1,0}^2)(1+\omega_{1,0})^{m-1} \\ & = (1 + a\,\omega_{1,0} + b\,\omega_{1,0}^2)^n (1 - \omega_{1,0}) \end{aligned} \tag{3.3.14}$$

which is satisfied in $\mathbb{Z}[\omega_{1,0}]\Big/(\omega_{1,0}^{m-1})$, and

$$\left(\sum_{k=0}^{n-m} \beta_{2k}\,\omega_{1,1}^k\right)(1 + 4c\,\omega_{1,1})(1+\omega_{1,1})^m = (1 + c\,\omega_{1,1})^n(1 + 4\,\omega_{1,1}) \tag{3.3.15}$$

which is satisfied in $\mathbb{Z}[\omega_{1,1}]\Big/(\omega_{1,1}^{\lfloor (m-2)/2 \rfloor + 1})$.

Proof. By Proposition 2.1.7 (a), we can express each $c_k(N)$ where $0 \leq k \leq 2n - 2m$ uniquely as in (3.3.13) because $2n - 2m \leq m - 2$.

By Corollary 2.1.9, there are ring isomorphisms $\rho_0 \colon \mathcal{Q}_{1,0} \to \mathbb{Z}[\omega_{1,0}] \big/ (\omega_{1,0}^{m-1})$ and $\rho_1 \colon \mathcal{Q}_{1,1} \to \mathbb{Z}[\omega_{1,1}] \big/ (\omega_{1,1}^{\lfloor (m-2)/2 \rfloor + 1})$ where $\mathcal{Q}_{1,0}$ and $\mathcal{Q}_{1,1}$ are the quotient rings given as in (2.1.8). Consider the images of both sides of (3.3.3) under the composition of the isomorphism ρ_0 with the canonical projection $H^\bullet(Gr(2, m), \mathbb{Z}) \to \mathcal{Q}_{1,0}$. To find their images, it suffices to consider the terms involving in $\omega_{1,0}$, we have

$$\left(\sum_{k=0}^{2n-2m} \alpha_k \, \omega_{1,0}^k \right) (1 + (4b - a^2)\, \omega_{1,0}^2)(1 + \omega_{1,0})^m = (1 + a\, \omega_{1,0} + b\, \omega_{1,0}^2)^n (1 - \omega_{1,0}^2),$$

and after dividing both sides by $1 + \omega_{1,0}$, we obtain Equation (3.3.14). Similarly, we can obtain Equation (3.3.15) by considering the image of both sides of (3.3.3) under the composition of ρ_1 with the canonical projection $H^\bullet(Gr(2, m), \mathbb{Z}) \to \mathcal{Q}_{1,1}$. □

Remark 3.3.5. Assume that $n \leq \frac{3m-2}{2}$. After comparing the notations in (3.3.10) and 3.3.13, $\alpha_k = A_{k,0}$ and $\beta_k = B_{k, \lfloor k/2 \rfloor}$ for all $0 \leq k \leq 2n - 2m$. In particular, $\beta_{2i} = B_{2i,i}$ for all $0 \leq i \leq n - m$.

Each of (3.3.14) and (3.3.15) is satisfied in a quotient ring of the form $\mathbb{Z}[x] \big/ (x^k)$, which is isomorphic to the $\mathbb{Z}$-submodule of all polynomials in $\mathbb{Z}[x]$ of degree $< k$ as a $\mathbb{Z}$-module. When we express each element in $\mathbb{Z}[x] \big/ (x^k)$ as a polynomial of degree $< k$, we use (3.3.14) and (3.3.15) as follows:

- We can compare the coefficients of $\omega_{1,0}^k$ in both sides of (3.3.14) for $0 \leq k \leq m - 2$;
- we can compare the coefficients of $\omega_{1,1}^k$ in both sides of (3.3.15) for $0 \leq 2k \leq m - 2$.

In this way, we obtain the leftest and the rightest Diophantine equations among (3.3.12).

3.4 Inequalities in a, b and c

In this section, we derive some inequalities in a, b, c and α_k (Proposition 3.4.4, 3.4.5 and 3.4.9) by using numerical non-negativity of Chern classes and solving two refined equations (3.3.14) and (3.3.15) in one variable.

3.4.1 Numerical non-negativity of Chern classes

Definition 3.4.1. Let Y be a non-singular variety. A cohomology class $\Gamma \in H^{2k}(Y, \mathbb{Z})$ is *numerically non-negative* if the intersection numbers $\Gamma \cdot Z$ are non-negative for all subvarieties Z of Y of dimension k.

In our case when $Y = Gr(2,m)$, a cohomology class $\Gamma \in H^{2k}(Gr(2,m), \mathbb{Z})$ is numerically non-negative means that when we write Γ as the linear combination with respect to the basis (2.1.1), every coefficient in Γ is non-negative. The following proposition tells us a sufficient condition for the numerical non-negativity of all Chern classes of vector bundles $\mathcal{E}$.

Proposition 3.4.2 ([Tan74, Proposition 2.1 (i)])**.** *Let Z be a non-singular variety and let $\mathcal{E}$ be a vector bundle of arbitrary rank on Z which is generated by global sections. Then each Chern class $c_i(\mathcal{E})$ of $\mathcal{E}$ is numerically non-negative for all $i = 1, 2, \cdots \dim(Z)$.*

We find vector bundles on $Gr(2,m)$ satisfying the assumption of Proposition 3.4.2 and obtain inequalities in a, b and c.

Lemma 3.4.3. *Each of vector bundles $E = \varphi^*(\check{E}(2,n))$, $\varphi^*(Q(2,n))$ and N is generated by global sections, and its Chern classes are all numerically non-negative.*

Proof. Note that a vector bundle $\mathcal{E}$ on Z is generated by global sections if and only if there is a surjective bundle morphism from a trivial bundle on Z (of any rank) to $\mathcal{E}$. By the short exact sequence after replacing m in (2.2.1) with n and its dual, E and $\varphi^*(Q(2,n))$ are generated by global sections. Moreover, by the short exact sequence (3.3.5), $T_{Gr(2,n)} \simeq Q(2,n) \otimes \check{E}(2,n)$ is generated by global sections and from this, we can conclude that $N = \varphi^*(T_{Gr(2,n)}) \big/ T_{Gr(2,m)}$ is generated by global sections. Hence, for $\mathcal{E} = E$, $\varphi^*(Q(2,n))$ and N, $c_0(\mathcal{E}) =$

$1 \geq 0$ and $c_1(\mathcal{E}), \cdots, c_{2m-4}(\mathcal{E})$ are all numerically non-negative by Proposition 3.4.2. $\square$

Proposition 3.4.4. *Let $n \geq m \geq 4$.*

(a) $a \geq 1,\ b \geq 0$ *and* $b + c \geq 0$ *with* $a^2 \geq b$.

(b) *If* $m \geq 5$, *then* $a^2 \geq 2b$.

(c) *If* $m \geq 6$, *then* $a^2 > 2b$.

(d) *For* $n \leq \frac{3m-2}{2}$, *let* $\alpha_k\ (0 \leq k \leq 2n-2m)$ *be the integers which are given as in* (3.3.13). *Then* $\alpha_k \geq 0$ *for all* $0 \leq k \leq 2n-2m$.

Proof. (a) By Lemma 3.4.3, each Chern class of E and N is numerically non-negative. By (3.2.1),

$$c(E) = 1 + a\,\omega_{1,0} + b\,\omega_{2,0} + (b+c)\,\omega_{1,1},$$

so a, b and $b + c$ are non-negative. By Lemma 3.3.3,

$$c_1(N) = (an - m)\,\omega_{1,0},$$

so $a \geq 1$.

By Lemma 3.4.3 again,

$$\begin{aligned} c_2(\varphi^*(Q(2,n))) &= \varphi^*(\tilde{\omega}_{2,0}) \quad (\because \text{ Proposition 2.2.2 (b)}) \\ &= \varphi^*(\tilde{\omega}_{1,0}^2 - \tilde{\omega}_{1,1}) \quad (\because \text{ Corollary 2.1.5}) \\ &= (a\,\omega_{1,0})^2 - (b\,\omega_{1,0}^2 + c\,\omega_{1,1}) \\ &= (a^2 - b)\,\omega_{1,0}^2 - c\,\omega_{1,1} \\ &= (a^2 - b)\,\omega_{2,0} + (a^2 - b - c)\,\omega_{1,1} \end{aligned}$$

is numerically non-negative. Since $m \geq 4$, $\omega_{2,0}$ is not zero, so we have $a^2 \geq b$.

(b) By Lemma 3.4.3,

$$\begin{aligned} c_3(\varphi^*(Q(2,n))) &= \varphi^*(\tilde{\omega}_{3,0}) \quad (\because \text{ Proposition 2.2.2 (b)}) \\ &= \varphi^*(\tilde{\omega}_{1,0}^3 - 2\,\tilde{\omega}_{1,0}\,\tilde{\omega}_{1,1}) \quad (\because \text{ Corollary 2.1.5}) \\ &= (a\,\omega_{1,0})^3 - 2a\,\omega_{1,0}\,(b\,\omega_{1,0}^2 + c\,\omega_{1,1}) \\ &= a(a^2-2b)\,\omega_{1,0}^3 - 2ac\,\omega_{1,0}\,\omega_{1,1} \\ &= a(a^2-2b)\,\omega_{3,0} + 2a(a^2-2b-c)\,\omega_{2,1} \end{aligned}$$

is numerically non-negative. Since $m \geq 5$, $\omega_{3,0}$ is not zero, so we have $a(a^2 - 2b) \geq 0$. By (a), $a \geq 1$, thus $a^2 \geq 2b$.

(c) By Lemma 3.4.3,

$$\begin{aligned} c_4(\varphi^*(Q(2,n))) &= \varphi^*(\tilde{\omega}_{4,0}) \quad (\because \text{ Proposition 2.2.2 (b)}) \\ &= \varphi^*(\tilde{\omega}_{1,0}^4 - 3\,\tilde{\omega}_{1,0}^2\,\tilde{\omega}_{1,1} + \tilde{\omega}_{1,1}^2) \quad (\because \text{ Corollary 2.1.5}) \\ &= (a\,\omega_{1,0})^4 - 3(a\,\omega_{1,0})^2(b\,\omega_{1,0}^2 + c\,\omega_{1,1}) + (b\,\omega_{1,0}^2 + c\,\omega_{1,1})^2 \\ &= (a^4 - 3a^2b + b^2)\,\omega_{1,0}^4 + (-3a^2c + 2bc)\,\omega_{1,0}^2\,\omega_{1,1} + c^2\,\omega_{1,1}^2 \\ &= (a^4 - 3a^2b + b^2)\,\omega_{4,0} + \alpha\,\omega_{3,1} + \beta\,\omega_{2,2} \quad (\because \text{ Proposition 2.1.7}(b)) \end{aligned}$$

is numerically non-negative ($\alpha, \beta \in \mathbb{Z}[a,b,c]$). Since $m \geq 6$, $\omega_{4,0}$ is not zero, so we have $a^4 - 3a^2b + b^2 \geq 0$, that is, either $a^2 \geq \left(\frac{3+\sqrt{5}}{2}\right) b$ or $a^2 \leq \left(\frac{3-\sqrt{5}}{2}\right) b$. But $a^2 \leq \left(\frac{3-\sqrt{5}}{2}\right) b < b$ is impossible by (a). Hence, we have

$$a^2 \geq \left(\frac{3+\sqrt{5}}{2}\right) b > 2b.$$

(d) By Proposition 2.1.7 (b), the coefficient of $\omega_{k,0}$ in $c_k(N)$ with respect to the basis (2.1.1) is equal to α_k. By Lemma 3.4.3, each $c_k(N)$ is numerically non-negative, thus $\alpha_k \geq 0$ for all $0 \leq k \leq 2n - 2m$. □

3.4.2 Refined equation in $\omega_{1,1}$

Solving the refined equation (3.3.15) in one variable $\omega_{1,1}$, each β_k with $1 \leq k \leq n - m$ is a polynomial in only one variable c. Comparing the coefficients of $\omega_{1,1}^k$ in both sides of (3.3.15) for a suitable power k, we obtain a numerical condition on c.

Proposition 3.4.5. *For $m \le n \le \frac{3m-6}{2}$, we have $c \ge 1$.*

Proof. We complete the proof by showing that the following two cases are impossible:

Case 1. $c \le -1$; Case 2. $c = 0$.

Case 1. Suppose that $c \le -1$. By (3.3.15), we have

$$\begin{aligned}&\left(\sum_{k=0}^{n-m} \beta_{2k}\,\omega_{1,1}^k\right)(1+4c\,\omega_{1,1})\\ &= (1+c\,\omega_{1,1})^n(1+\omega_{1,1})^{-m}(1+4\,\omega_{1,1}),\end{aligned} \tag{3.4.1}$$

which is satisfied in $\mathbb{Z}[\omega_{1,1}]\big/(\omega_{1,1}^{\lfloor (m-2)/2\rfloor+1})$. Since $2(n-m+2) \le m-2$, we can compare the coefficient of $\omega_{1,1}^{n-m+2}$ in both sides of (3.4.1). Thus we have

$$\begin{aligned} 0 =& \sum_{k=0}^{n-m+2} \binom{n}{k}\binom{m+(n-m+2-k)-1}{n-m+2-k} c^k\,(-1)^{n-m+2-k}\\ &+ 4\sum_{k=0}^{n-m+1} \binom{n}{k}\binom{m+(n-m+1-k)-1}{n-m+1-k} c^k\,(-1)^{n-m+1-k}\\ =& \sum_{k=0}^{n-m+2} \binom{n}{k}\binom{n+1-k}{m-1}(-c)^k\,(-1)^{n-m+2}\\ &+ 4\sum_{k=0}^{n-m+1} \binom{n}{k}\binom{n-k}{m-1}(-c)^k\,(-1)^{n-m+1}.\end{aligned} \tag{3.4.2}$$

After dividing both sides of (3.4.2) by $(-1)^{n-m+1}$,

$$4\sum_{k=0}^{n-m+1} \binom{n}{k}\binom{n-k}{m-1}(-c)^k = \sum_{k=0}^{n-m+2} \binom{n}{k}\binom{n+1-k}{m-1}(-c)^k. \tag{3.4.3}$$

Since

$$\begin{aligned}\binom{n}{k}\binom{n-k}{m-1} &= \frac{n!}{k!(n-k)!}\cdot\frac{(n-k)!}{(m-1)!(n-m+1-k)!}\\ &= \frac{n!}{(m-1)!(n-m+1)!}\cdot\frac{(n-m+1)!}{k!(n-m+1-k)!}\\ &= \binom{n}{m-1}\binom{n-m+1}{k},\end{aligned}$$

the left hand side of (3.4.3) is equal to

$$\begin{aligned}(\mathrm{LHS}) &= 4\sum_{k=0}^{n-m+1}\binom{n}{m-1}\binom{n-m+1}{k}(-c)^k\\ &= 4\binom{n}{m-1}(1-c)^{n-m+1}.\end{aligned}\tag{3.4.4}$$

Similarly, since

$$\begin{aligned}\binom{n}{k}\binom{n+1-k}{m-1} &= \frac{n!}{k!(n-k)!}\cdot\frac{(n+1-k)!}{(m-1)!(n-m+2-k)!}\\ &= \frac{(n+1)!}{(m-1)!(n-m+2)!}\cdot\frac{(n-m+2)!}{k!(n-m+2-k)!}\cdot\frac{n+1-k}{n+1}\\ &= \binom{n+1}{m-1}\binom{n-m+2}{k}\cdot\frac{n+1-k}{n+1}\\ &\geq \binom{n+1}{m-1}\binom{n-m+2}{k}\cdot\frac{m-1}{n+1}\\ &= \binom{n}{m-2}\binom{n-m+2}{k}\end{aligned}$$

for $0 \leq k \leq n-m+2$, the right hand side of (3.4.3) satisfies

$$(\mathrm{RHS}) \;\geq\; \binom{n}{m-2}(1-c)^{n-m+2}.\tag{3.4.5}$$

Applying (3.4.4) and (3.4.5) to (3.4.3), since $1-c \geq 0$, we have

$$(1-c)\binom{n}{m-2} \;\leq\; 4\binom{n}{m-1}.$$

So $(1-c)\,(m-1) \;\leq\; 4\,(n-m+2)$, that is,

$$4n \geq (5-c)\,m + c - 9.$$

But since $4n \leq 6m-12$, we have

$$(c+1)\,m \geq c+3. \tag{3.4.6}$$

If $c=-1$, then (3.4.6) is impossible clearly. If $c \leq -2$, then $c+1 \leq -1$, so we have

$$m \;\leq\; \frac{c+3}{c+1} \;=\; 1 + \frac{2}{c+1} \;<\; 1$$

by (3.4.6), which implies a contradiction.

Case 2. Suppose that $c=0$. Putting $c=0$ into (3.3.15), we have

$$\sum_{k=0}^{n-m} \beta_{2k}\,\omega_{1,1}^{k} = (1+\omega_{1,1})^{-m}(1+4\,\omega_{1,1}), \tag{3.4.7}$$

which is satisfied in $\mathbb{Z}[\omega_{1,1}]\Big/(\omega_{1,1}^{\lfloor (m-2)/2\rfloor+1})$. Comparing the coefficient of $\omega_{1,1}^{n-m+i}$ in both sides of (3.4.7) for $i=1$ and 2, we have

$$\begin{aligned}
0 &= \binom{m+(n-m+i)-1}{n-m+i}(-1)^{n-m+i} \\
&\qquad + 4\binom{m+(n-m+i-1)-1}{n-m+i-1}(-1)^{n-m+i-1} \\
&= (-1)^{n-m+i}\left\{\binom{n+i-1}{m-1} - 4\binom{n+i-2}{m-1}\right\} \\
&= (-1)^{n-m+i}\binom{n+i-1}{m-1}\left(1 - 4\cdot\frac{n-m+i}{n+i-1}\right).
\end{aligned}$$

So we obtain
$$3n = 4m - 3i - 1$$
for $i = 1$ and 2, which implies a contradiction.

Hence, we conclude that $c \geq 1$. □

3.4.3 Refined equation in $\omega_{1,0}$

The refined equation (3.3.14) in one variable $\omega_{1,0}$ is harder than the refined equation (3.3.15) in $\omega_{1,1}$ to solve. To overcome this difficulty, using Note 3.3.1, the previous inequalities together with Lemma 3.4.6, we obtain a lower bound of α_{2n-2m} (Lemma 3.4.7). This bound has a key role to solve (3.3.14).

Lemma 3.4.6 ([Ful98, Example 14.7.11] (or [HP52, page 364]))**.** *In $Gr(2,m)$,*
$$\omega_{i,j}\,\omega_{1,0}^{2m-4-i-j} = \frac{(2m-4-i-j)!(i-j+1)!}{(m-2-i)!(m-1-j)!}$$
for $m-2 \geq i \geq j \geq 0$.

Lemma 3.4.7. *Let $m \leq n \leq \frac{3m-6}{2}$ and α_{2n-2m} be the integer given as in* (3.3.13)*. Then*
$$\alpha_{2n-2m} \geq \frac{(2m-4)!}{(m-2)!(m-1)!} \cdot b^{n-2}.$$

Proof. By Note 3.3.1, $c_{2n-2m}(N) = e(N_{\mathbb{R}})$. So α_{2n-2m} is equal to the coefficient of $\omega_{1,0}^{2n-2m}$ in
$$e(N_{\mathbb{R}}) = \sum_{i=0}^{n-m} d_i\,\varphi^*(\tilde{\omega}_{2n-2m-i,i})$$
where $d_i := X \cdot \tilde{\omega}_{n-2-i,2m-n-2+i}$, by Proposition 3.3.2. Here, $d_i \geq 0$ for all $0 \leq i \leq n-m$ because each d_i is the intersection number of two subvarieties of $Gr(2,n)$.

Let Γ_i be the coefficient of $\omega_{1,0}^{2n-2m-2i}$ in $\varphi^*(\tilde{\omega}_{2n-2m-2i,0})$ with respect to the basis (2.1.4). Then the coefficient of $\omega_{1,0}^{2n-2m}$ in
$$\begin{aligned}\varphi^*(\tilde{\omega}_{2n-2m-i,i}) &= \varphi^*(\tilde{\omega}_{2n-2m-2i,0})\,\varphi^*(\tilde{\omega}_{1,1}^{i}) \quad (\because \text{ Corollary 2.1.5})\\ &= \varphi^*(\tilde{\omega}_{2n-2m-2i,0})\,(b\,\omega_{1,0}^2 + c\,\omega_{1,1})^i\end{aligned}$$

with respect to (2.1.4) is equal to $\Gamma_i\, b^i$. By Proposition 2.1.7 (b), Γ_i is equal to the coefficient of $\omega_{2n-2m-2i,0}$ in $\varphi^*(\tilde{\omega}_{2n-2m-2i,0})$ with respect to (2.1.1), and by Lemma 3.4.3 it is non-negative for all $0 \le i \le n-m$. So we have

$$\begin{aligned}
\alpha_{2n-2m} &= \sum_{i=0}^{n-m} d_i\, \Gamma_i\, b^i \\
&\ge d_{n-m}\, \Gamma_{n-m}\, b^{n-m} \quad (\because\ b \ge 0 \text{ by Proposition 3.4.4 (a)}) \\
&= d_{n-m}\, b^{n-m} \quad (\because\ \text{Since } \varphi^*(\tilde{\omega}_{0,0}) = \omega_{0,0},\ \Gamma_{n-m} = 1).
\end{aligned} \tag{3.4.8}$$

Applying

$$\begin{aligned}
d_{n-m} &= \varphi^*(\tilde{\omega}_{1,1}^{m-2}) = (b\,\omega_{1,0}^2 + c\,\omega_{1,1})^{m-2} \\
&= \sum_{i=0}^{m-2} \binom{m-2}{i} b^{m-2-i}\, c^i\, \omega_{1,0}^{2m-4-2i}\, \omega_{1,1}^i \\
&\ge b^{m-2}\, \omega_{1,0}^{2m-4} \quad (\because\ b,\ c,\ \omega_{1,0}^{2m-4-2i}\, \omega_{1,1}^i \ge 0 \\
&\qquad \text{by Proposition 3.4.4 (a), 3.4.5 and Lemma 3.4.6)} \\
&= \frac{(2m-4)!}{(m-2)!(m-1)!} \cdot b^{m-2} \quad (\because\ \text{Lemma 3.4.6})
\end{aligned}$$

to (3.4.8), we obtain the desired inequality. $\square$

Now, we ready to prove an inequality in a and b better than that of Proposition 3.4.4 (c). Before proving this, we use the following notation.

Notation 3.4.8. Let R be a ring $\mathbb{Z}$ or $\mathbb{R}$. Identifying $R[x]/(x^k)$ with the R-module which is generated by a basis $\{1, x, x^2, \cdots, x^{k-1}\}$, express an element in $R[x]/(x^k)$ uniquely as a linear combination of $1, x, x^2, \cdots x^{k-1}$ with integral coefficients. Denote by $f(x) \preceq g(x)$ in $R[x]/(x^k)$ if the coefficient of x^i in $g(x) - f(x)$ is non-negative for all $0 \le i < k$.

Proposition 3.4.9. *For $m \le n \le \frac{3m-6}{2}$, let a and b be the integers which are given as in* (3.2.1)*. Then $a^2 > 4b$.*

Proof. Suppose that $a^2 \le 4b$. By Proposition 3.4.4 (a), $a \ge 1$, so we have

$b \geq 1$. Then by (3.3.14), we have

$$\begin{aligned}\sum_{k=0}^{2n-2m} \alpha_k\, \omega_{1,0}^k &\preceq (1 + a\,\omega_{1,0} + b\,\omega_{1,0}^2)^n \\ &\preceq (1 + \sqrt{b}\,\omega_{1,0})^{2n}\end{aligned} \tag{3.4.9}$$

in $\mathbb{Z}[\omega_{1,0}]\Big/(\omega_{1,0}^{m-1})$. Comparing the coefficient of $\omega_{1,0}^{2n-2m}$ in both sides of (3.4.9),

$$\alpha_{2n-2m} \;\leq\; \binom{2n}{2n-2m} b^{n-m} \;=\; \binom{2n}{2m} b^{n-m}. \tag{3.4.10}$$

By Lemma 3.4.7 and (3.4.10),

$$\frac{(2m-4)!}{(m-2)!(m-1)!} \cdot b^{n-2} \;\leq\; \binom{2n}{2m} b^{n-m},$$

so we have

$$\begin{aligned} b^{m-2} &\leq \binom{2n}{2m} \cdot \frac{(m-2)!(m-1)!}{(2m-4)!} \\ &\leq \binom{3m-6}{2m} \cdot \frac{(m-2)!(m-1)!}{(2m-4)!} \qquad (\because\; n \leq 2m;\quad 2n \leq 3m-6) \\ &= \frac{(3m-6)(3m-7)\cdots(2m+1)}{(2m-4)(2m-5)\cdots(m+3)} \cdot \frac{(m-2)(m-3)(m-4)}{(m+2)(m+1)m} \cdot (m-5) \\ &\leq 2^{m-6} \cdot (m-5). \end{aligned} \tag{3.4.11}$$

Hence, we conclude that $b \leq 2$.

Since $m \leq \frac{3m-6}{2}$, we have $m \geq 6$. By Proposition 3.4.4 (c), $2b < a^2 \leq 4b$, so the only possible pair (a,b) is $(2,1)$. Putting $(a,b) = (2,1)$ into (3.3.14), we have

$$\sum_{k=0}^{2n-2m} \alpha_k\, \omega_{1,0}^k = (1+\omega_{1,0})^{2n-m+1}(1-\omega_{1,0}) \tag{3.4.12}$$

which is satisfied in $\mathbb{Z}[\omega_{1,0}]\Big/(\omega_{1,0}^{m-1})$. Since $2n-2m+1 \leq m-2$, we can compare the coefficient of $\omega_{1,0}^{2n-2m+1}$ in both sides of (3.4.12). Then we have

$$\begin{aligned} 0 &= \binom{2n-m+1}{2n-2m+1} - \binom{2n-m+1}{2n-2m} \\ &= \binom{2n-m+1}{2n-2m+1}\left(1 - \frac{2n-2m+1}{m+1}\right). \end{aligned}$$

So $\frac{2n-2m+1}{m+1} = 1$, that is, $3m = 2n$ $(\leq 3m-6)$ which implies a contradiction. Hence, we conclude that $a^2 > 4b$. $\square$

Chapter 4

Characterization of linear embeddings

Recall Main Theorem in Chapter 1, which characterizes linear embeddings of $Gr(2, m)$ into $Gr(2, n)$.

Main Theorem. *Let $\varphi\colon Gr(2, m) \hookrightarrow Gr(2, n)$ be a holomorphic embedding.*

(a) *If $9 \le m$ and $n \le \frac{3m-6}{2}$, then φ is linear.*

(b) *If $4 \le m$ and $n = m + 1$, then either φ is linear, or $m = 4$ and φ is a composition of a linear embedding of $Gr(2, 4)$ into $Gr(2, 5)$ with a dual map $\phi\colon Gr(2, 4) \to Gr(2, 4)$.*

Combining Main Theorem and Proposition 3.1.4, we have the following corollary:

Corollary 4.0.1. *Let $\varphi\colon Gr(2, m) \hookrightarrow Gr(2, n)$ be a holomorphic embedding. Either if $9 \le m$ and $n \le \frac{3m-6}{2}$, or if $4 \le m$ and $n = m + 1$, then the image of φ equals $Gr(2, H)$ for some m-dimensional subspace H of $\mathbb{C}^n$.*

Denote each assumption on Main Theorem as follows:

- General case : $9 \le m$ and $n \le \frac{3m-6}{2}$;
- Special case : $4 \le m$ and $n = m + 1$.

The chapter consists of two sections. In Section 4.1, we prove Main Theorem for general case (Theorem 4.1.1) by applying W. Barth and A. Van de Ven's results (Proposition 2.2.3 and 2.2.4) to $E = \varphi^*(\check{E}(2,n))$. The upper bound of a (Proposition 4.1.2 (b)) makes it possible. In Section 4.2, we prove Main Theorem for special case (Theorem 4.2.1) by comparing $c_2(N)$ with $e(N_{\mathbb{R}})$ directly.

4.1 General case

In this section, we prove Main Theorem for the case when $9 \le m$ and $n \le \frac{3m-6}{2}$.

Theorem 4.1.1. *If $9 \le m$ and $n \le \frac{3m-6}{2}$, then any embedding $\varphi\colon Gr(2,m) \hookrightarrow Gr(2,n)$ is linear.*

Before proving Theorem 4.1.1, we first prove the following inequalities in a and b:

Proposition 4.1.2. *Under the same assumption with Theorem 4.1.1, we have the following inequalities:*

(a) $\sqrt{a^2-4b} < \frac{m-4}{3}$.

(b) $a < \frac{m-4}{2}$.

Proof. (a) First, by Proposition 3.4.9, the expression $\sqrt{a^2-4b}$ is well defined. By (3.3.14), we have

$$\left(\sum_{k=0}^{2n-2m} \alpha_k\, \omega_{1,0}^k\right)(1+\omega_{1,0})^{m-1}(1+(4b-a^2)\,\omega_{1,0}^2) \\ = (1+a\,\omega_{1,0}+b\,\omega_{1,0}^2)^n(1-\omega_{1,0})$$

which is satisfied in $\mathbb{Z}[\omega_{1,0}]\big/(\omega_{1,0}^{m-1})$. For convenience, let

$$\sum_{k=0}^{m-2} A_k\, \omega_{1,0}^k := \left(\sum_{i=0}^{2n-2m} \alpha_i\, \omega_{1,0}^i\right)(1+\omega_{1,0})^{m-1};$$
$$\sum_{k=0}^{m-2} B_k\, \omega_{1,0}^k := (1+a\,\omega_{1,0}+b\,\omega_{1,0}^2)^n$$

which are satisfied in $\mathbb{Z}[\omega_{1,0}]\Big/(\omega_{1,0}^{m-1})$. Then we have

$$A_{k+2} + (4b - a^2)A_k = B_{k+2} - B_{k+1}$$

for all $0 \leq k \leq m-4$. In particular, when $k = 2n - 2m + 2$ ($\leq m - 4$),

$$A_{2n-2m+4} + (4b - a^2)\, A_{2n-2m+2} = B_{2n-2m+4} - B_{2n-2m+3}. \tag{4.1.1}$$

Suppose that $\sqrt{a^2 - 4b} \geq \frac{m-4}{3}$. We derive a contradiction by comparing the signs of both sides of (4.1.1).

- Since $A_k = \sum_{i=0}^{2n-2m} \alpha_i \binom{m-1}{k-i}$ for all $k \geq 2n - 2m$, we have

$$\begin{aligned}
(\text{LHS}) &= A_{2n-2m+4} + (4b - a^2)\, A_{2n-2m+2} \\
&= \sum_{i=0}^{2n-2m} \alpha_i \left\{ \binom{m-1}{2n-2m+4-i} - (a^2 - 4b)\binom{m-1}{2n-2m+2-i} \right\} \\
&= \sum_{i=0}^{2n-2m} \alpha_i \binom{m-1}{2n-2m+2-i} \cdot C_i
\end{aligned}$$

  where $C_i := \frac{(-2n+3m-3+i)(-2n+3m-4+i)}{(2n-2m+4-i)(2n-2m+3-i)} - (a^2 - 4b)$. Since

$$\begin{aligned}
C_i &\leq \frac{(m-3)(m-4)}{4 \cdot 3} - (a^2 - 4b) \\
&\leq \frac{(m-4)(-m+7)}{36} < 0 \qquad (\because\ m \geq 9)
\end{aligned}$$

  for all $0 \leq i \leq 2n - 2m$ and since $\alpha_i \geq 0$ by Proposition 3.4.4 (d), the left hand side of (4.1.1) is negative.

- Since $B_k = \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{n}{i}\binom{n-i}{k-2i} a^{k-2i}\, b^i$, we have

$$\begin{aligned}
\text{(RHS)} =& B_{2n-2m+4} - B_{2n-2m+3} \\
\geq& \sum_{i=0}^{n-m+1} \binom{n}{i} a^{2n-2m+3-2i}\, b^i \cdot \\
& \qquad \left\{ \binom{n-i}{2n-2m+4-2i} a - \binom{n-i}{2n-2m+3-2i} \right\} \\
=& \sum_{i=0}^{n-m+1} \binom{n}{i} a^{2n-2m+3-2i}\, b^i \binom{n-i}{2n-2m+4-2i} \cdot D_i
\end{aligned}$$

where $D_i := a - \frac{2n-2m+4-2i}{-n+2m-3+i}$. Since

$$\begin{aligned}
D_i &\geq \sqrt{a^2-4b} - \frac{2n-2m+4}{-n+2m-3} \\
&\geq \frac{m-4}{3} - \frac{(3m-6)-2m+4}{-\frac{3m-6}{2}+2m-3} \qquad \left(\because\ n \leq \frac{3m-6}{2}\right) \\
&= \frac{m-4}{3} - \frac{2m-4}{m} \\
&= \frac{(m-5)^2-13}{3m} > 0 \qquad (\because\ m \geq 9)
\end{aligned}$$

for all $0 \leq i \leq n-m+1$ and since $a, b \geq 0$ by Proposition 3.4.4 (a), the right hand side of (4.1.1) is positive.

As a result, the equality (4.1.1) does not hold, thus this implies a contradiction. Hence, $\sqrt{a^2-4b} < \frac{m-4}{3}$.

(b) Suppose that $a \geq \frac{m-4}{2}$. Then by (a),

$$a^2 - 4b < \frac{(m-4)^2}{9} \leq \left(\frac{2a}{3}\right)^2. \tag{4.1.2}$$

Regard (3.3.14) as an equation over $\mathbb{R}$. By (3.3.14) and (4.1.2), we have

$$\begin{aligned} \sum_{k=0}^{2n-2m} \alpha_k\, \omega_{1,0}^k &\preceq (1 + a\,\omega_{1,0} + b\,\omega_{1,0}^2)^n \left\{ \sum_{i=0}^{2m-4} \left(\sqrt{a^2 - 4b}\,\omega_{1,0} \right)^i \right\} \\ &\preceq \left(1 + \frac{a}{2}\,\omega_{1,0}\right)^{2n} \left\{ \sum_{i=0}^{2m-4} \left(\frac{2a}{3}\,\omega_{1,0} \right)^i \right\} \end{aligned} \tag{4.1.3}$$

which is satisfied in $\mathbb{R}[\omega_{1,0}] \big/ (\omega_{1,0}^{m-1})$. Comparing the coefficients of $\omega_{1,0}^{2n-2m}$ in both sides of (4.1.3),

$$\begin{aligned} \alpha_{2n-2m} &\leq \sum_{i=0}^{2n-2m} \binom{2n}{i} \left(\frac{a}{2}\right)^i \left(\frac{2a}{3}\right)^{2n-2m-i} \\ &= \left\{ \sum_{i=0}^{2n-2m} \binom{2n}{i} \left(\frac{1}{2}\right)^i \left(\frac{2}{3}\right)^{2n-i} \right\} \cdot \left(\frac{3}{2}\right)^{2m} a^{2n-2m} \\ &\leq \left\{ \frac{1}{2} \sum_{i=0}^{2n} \binom{2n}{i} \left(\frac{1}{2}\right)^i \left(\frac{2}{3}\right)^{2n-i} \right\} \cdot \left(\frac{3}{2}\right)^{2m} a^{2n-2m} \quad (\because\ 2n - 2m \leq n) \\ &= \frac{81}{32} \left(\frac{1}{2} + \frac{2}{3}\right)^{2n} \cdot \left(\frac{3}{2}\right)^{2m-4} a^{2n-2m} \\ &\leq \frac{81}{32} \left(\frac{7}{6}\right)^{3m-6} \cdot \left(\frac{3}{2}\right)^{2m-4} a^{2n-2m} \quad (\because\ 2n \leq 3m - 6) \\ &= \frac{81}{32} \left(\frac{7^3}{96}\right)^{m-2} a^{2n-2m}. \end{aligned}$$

By Lemma 3.4.7, we have

$$\frac{(2m-4)!}{(m-2)!(m-1)!} \cdot b^{n-2} \ \leq\ \frac{81}{32} \left(\frac{7^3}{96}\right)^{m-2} a^{2n-2m}. \tag{4.1.4}$$

By (4.1.2), $b > \frac{5}{36}a^2$, so the left hand side of (4.1.4) satisfies

$$\begin{aligned}
\frac{(2m-4)!}{(m-2)!(m-1)!}\cdot b^{n-2} &= \frac{(2m-4)(2m-5)\cdots m}{(m-2)(m-3)\cdots 2}\cdot b^{n-2} \\
&\geq 2^{m-3}\cdot\left(\frac{5}{36}\right)^{n-2} a^{2n-4} \\
&\geq 2^{m-3}\cdot\left(\frac{5}{36}\right)^{(3m-10)/2} a^{2n-4} \quad \left(\because\ n \leq \frac{3m-6}{2}\right) \\
&\geq \frac{1}{2}\left(\frac{36}{5}\right)^{2}\cdot 2^{m-2}\cdot\left(\frac{5}{36}\right)^{(3m-6)/2} a^{2n-4} \\
&> 24\left(\frac{5\sqrt{5}}{3\cdot 6^2}\right)^{m-2} a^{2n-4},
\end{aligned}$$

thus,

$$24\left(\frac{5\sqrt{5}a^2}{3\cdot 6^2}\right)^{m-2} < \frac{81}{32}\left(\frac{7^3}{96}\right)^{m-2}.$$

So $\frac{5\sqrt{5}a^2}{3\cdot 6^2} \leq \frac{7^3}{96}$, that is, $a^2 \leq \frac{9\cdot 7^3}{40\sqrt{5}} \simeq 34.5137$, thus $a \leq 5$.

Since $4b < a^2 < \frac{36}{5}b$, the only possible pairs (a,b) are

$$(3,2);\quad (4,3);\quad (5,4);\quad (5,5);\quad (5,6).$$

However they are all impossible by Lemma 4.1.4 which is provided later. Hence, $a < \frac{m-4}{2}$ as desired. $\square$

Lemma 4.1.3. *If $m \geq 7$, then 12 divides $ab\,(a^2-b+3)$.*

Proof. By [Tan74, Lemma 4.10], if $k \geq 5$ and $\mathcal{E}$ is a vector bundle on $\mathbb{P}^k$ of rank 2 with

$$c(\mathcal{E}) = 1 + \alpha\, H + \beta\, H^2$$

where H is a hyperplane of $\mathbb{P}^k$, then $\alpha\beta\,(\alpha^2-\beta+3)$ is divisible by 12. In our case, $Gr(2,m)$ contains a Schubert variety $Y \simeq \mathbb{P}^{m-2}$ of type $(m-2,0)$ and the total Chern class of the restriction of E to Y is

$$c(E\big|_Y) = 1 + a\, H + b\, H^2.$$

Since $m-2 \geq 5$, $ab\,(a^2-b+3)$ is divisible by 12. $\square$

Lemma 4.1.4. *The following pairs (a,b) of integers are impossible:*

(a) $(3,2),(4,3)$ *and* $(5,4)$ *for* $m \leq n \leq \frac{3m-4}{2}$*;*

(b) $(5,5)$ *for* $7 \leq m \leq n$*;*

(c) $(5,6)$ *for* $m \leq n \leq \frac{3m-2}{2}$.

Proof. (a) In this case, $a=b+1$ with $b=2,3$ or 4, so $1+a\,\omega_{1,0}+b\,\omega_{1,0}^2 = (1+\omega_{1,0})(1+b\,\omega_{1,0})$ and $a^2-4b=(b-1)^2$. By (3.3.14), we have

$$\begin{aligned}&\left(\sum_{k=0}^{2n-2m}\alpha_k\,\omega_{1,0}^k\right)(1-(b-1)^2\,\omega_{1,0}^2)\\ &= (1+\omega_{1,0})^{n-m+1}(1+b\,\omega_{1,0})^n(1-\omega_{1,0})\end{aligned} \tag{4.1.5}$$

which is satisfied in $\mathbb{Z}[\omega_{1,0}]\big/(\omega_{1,0}^{m-1})$. Comparing the coefficient of $\omega_{1,0}^{2n-2m+2}$ in both sides of (4.1.5),

$$\begin{aligned}&-(b-1)^2\,\alpha_{2n-2m}\\ =&\sum_{k=0}^{n-m+1}\binom{n-m+1}{k}\left\{\binom{n}{2n-2m+2-k}b^{2n-2m+2-k}\right.\\ &\qquad\left.-\binom{n}{2n-2m+1-k}b^{2n-2m+1-k}\right\}\\ =&\sum_{k=0}^{n-m+1}\binom{n-m+1}{k}\binom{n}{2n-2m+2-k}b^{2n-2m+1-k}\cdot A_k\end{aligned} \tag{4.1.6}$$

where $A_k := b-\frac{2n-2m+2-k}{-n+2m-1+k}$. By Proposition 3.4.4 (d), the left hand side of

(4.1.6) is non-positive. On the other hand, since

$$\begin{aligned} A_k &\geq b - \frac{2n-2m+2}{-n+2m-1} \\ &\geq b - \frac{(3m-4)-2m+2}{-\left(\frac{3m-4}{2}\right)+2m-1} \qquad \left(\because\ n \leq \frac{3m-4}{2}\right) \\ &= b - \frac{2m-4}{m+2} \\ &> 0 \qquad (\because\ b = 2,\, 3 \text{ or } 4) \end{aligned}$$

for $0 \leq k \leq n-m+1$, the right hand side of (4.1.6) is positive which implies a contradiction.

(b) Since $ab\,(a^2-b+3) = 25\cdot(25-5+3) = 25\cdot 23$ is not divisible by 12, this case is impossible by Lemma 4.1.3.

(c) Since $a^2-4b=1$, we have by (3.3.14),

$$\begin{aligned} \sum_{k=0}^{2n-2m} \alpha_k\,\omega_{1,0}^k &\preceq (1+5\,\omega_{1,0}+6\,\omega_{1,0}^2)^n \\ &\preceq \left(1+\frac{5}{2}\,\omega_{1,0}\right)^{2n} \end{aligned} \tag{4.1.7}$$

which is satisfied in $\mathbb{Z}[\omega_{1,0}]\big/(\omega_{1,0}^{m-1})$. Comparing the coefficient of $\omega_{1,0}^{2n-2m}$ in both sides of (4.1.7),

$$\begin{aligned} \alpha_{2n-2m} &\leq \binom{2n}{2n-2m}\left(\frac{5}{2}\right)^{2n-2m} \\ &= \binom{2n}{2m}\left(\frac{5}{2}\right)^{2n-4}\left(\frac{2}{5}\right)^{2m-4} \end{aligned} \tag{4.1.8}$$

Applying Lemma 3.4.7 to (4.1.8),

$$\begin{aligned}\left(\frac{24}{25}\right)^{n-2} \leq & \binom{2n}{2m} \cdot \frac{(m-2)!(m-1)!}{(2m-4)!}\left(\frac{2}{5}\right)^{2m-4} \\ \leq & 2^{m-6} \cdot (m-5)\left(\frac{2}{5}\right)^{2m-4} \\ & (\because \text{ The same argument with (3.4.11)}) \\ = & \frac{m-5}{16}\left(\frac{8}{25}\right)^{m-2}.\end{aligned} \tag{4.1.9}$$

But since $n \leq \frac{3m-6}{2}$,

$$\left(\frac{24}{25}\right)^{n-2} \geq \left(\frac{25}{24}\right)^{2}\left\{\left(\frac{24}{25}\right)^{3/2}\right\}^{m-2},$$

thus by (4.1.9), we have $\frac{48\sqrt{6}}{125} = \left(\frac{24}{25}\right)^{3/2} \leq \frac{8}{25}$ which implies a contradiction. □

Proof of Theorem 4.1.1. Let $Y \simeq \mathbb{P}^{m-2}$ be a Schubert variety of $Gr(2,m)$ of type $(m-2,0)$. Then the total Chern class of the restriction of $E := \varphi^*(\check{E}(2,n))$ to Y is

$$c(E\big|_Y) = 1 + aH + bH^2$$

where H is a hyperplane of Y. For any projective line ℓ in Y,

$$(E\big|_Y)\big|_\ell = E\big|_\ell \simeq \mathcal{O}_\ell(a_1) \oplus \mathcal{O}_\ell(a_2)$$

for some integers a_1, a_2 with $a_1 + a_2 = a$. Since $(E\big|_Y)\big|_\ell$ is generated by global sections by Lemma 3.4.3, a_1 and a_2 are non-negative. So we have

$$B(E\big|_Y) \leq \frac{a-0}{2} < \frac{m-4}{4}$$

by Proposition 4.1.2 (b) (For the definition of $B(E\big|_Y)$, see (2.2.2)). Hence, $E\big|_Y$ is decomposable by Proposition 2.2.3. Since $Y \simeq \mathbb{P}^{m-2}$ is arbitrary, E is either decomposable or isomorphic to $E(2,m) \otimes L$ for some line bundle L on $Gr(2,m)$ by Proposition 2.2.4.

Case 1. $E \simeq E(2,m) \otimes L$: Let $c(L) = 1 + r\,\omega_{1,0}$ with $r \in \mathbb{Z}$. Then

$$c(E(2,m) \otimes L) = 1 + (2r+1)\,\omega_{1,0} + r(r+1)\,\omega_{1,0}^2 + \omega_{1,1},$$

that is,

$$a = 2r+1; \qquad b = r(r+1); \qquad c = 1 \tag{4.1.10}$$

with $r \geq 0$ by Proposition 3.4.4 (a). After putting (4.1.10) into (3.3.14), we have

$$\left(\sum_{k=0}^{2n-2m} \alpha_k\,\omega_{1,0}^k\right)(1+\omega_{1,0})^m = (1 + r\,\omega_{1,0})^n(1+(r+1)\,\omega_{1,0})^n \tag{4.1.11}$$

which is satisfied in $\mathbb{Z}[\omega_{1,0}]\big/(\omega_{1,0}^{m-1})$. Comparing the coefficients of $\omega_{1,0}^{2n-2m}$ in both sides of (4.1.11),

$$\alpha_{2n-2m} \;\leq\; \binom{2n}{2n-2m}(r+1)^{2n-2m} \;=\; \binom{2n}{2m}(r+1)^{2n-2m}. \tag{4.1.12}$$

By Lemma 3.4.7, we have

$$\begin{aligned}(r(r+1))^{n-2} &\leq \binom{2n}{2m} \cdot \frac{(m-2)!(m-1)!}{(2m-4)!} \cdot (r+1)^{2n-2m} \\ &\leq 2^{m-6} \cdot (m-5)(r+1)^{2n-2m} \\ &\quad (\because \text{ The same argument with (3.4.11)}).\end{aligned} \tag{4.1.13}$$

After dividing both sides of (4.1.13) by $(r+1)^{2n-2m}$,

$$r^{n-2}(r+1)^{-n+2m-2} \leq \frac{m-5}{16} \cdot 2^{m-2}. \tag{4.1.14}$$

Since $-n+2m-2 \geq -\left(\frac{3m-6}{2}\right) + 2m - 2 = \frac{m+2}{2} > 0$, we have

$$r^{2m-4} \leq r^{n-2}(r+1)^{-n+2m-2}. \tag{4.1.15}$$

Combining (4.1.14) and (4.1.15), $r^2 \leq 2$, thus, $r = 0$ or 1.

If $r = 1$, then $(a,b) = (3,2)$ by (4.1.10), which is impossible by Lemma

4.1.4 (a). Hence, $r = 0$.

Case 2. $E \simeq L_1 \oplus L_2$: Let $c(L_1) = 1 + r_1\,\omega_{1,0}$ and $c(L_2) = 1 + r_2\,\omega_{1,0}$ with $r_1, r_2 \in \mathbb{Z}$. Then

$$c(L_1 \oplus L_2) = 1 + (r_1 + r_2)\,\omega_{1,0} + r_1 r_2\,\omega_{1,0}^2,$$

that is,

$$a = r_1 + r_2; \qquad b = r_1 r_2; \qquad c = 0.$$

However by Proposition 3.4.5, $c = 0$ cannot be happened.

As a result, $E \simeq E(2,m) \otimes L$ where L is the trivial line bundle on $Gr(2,m)$ and thus $(a,b,c) = (1,0,1)$. Hence, $\varphi\colon Gr(2,m) \hookrightarrow Gr(2,n)$ is linear by Proposition 3.2.2. □

4.2 Special case

In this section, we prove Main Theorem for the case when $4 \leq m$ and $n = m+1$.

Theorem 4.2.1. *Let $\varphi\colon Gr(2,m) \hookrightarrow Gr(2,m+1)$ be an embedding.*

(a) *If $m = 4$, then any embedding φ is either linear or twisted linear.*

(b) *If $m \geq 5$, then any embedding φ is linear.*

Since $9 \leq m = n - 1$ satisfies the conditions $9 \leq m$ and $n \leq \frac{3m-6}{2}$, the result of Theorem 4.2.1 for that case is already verified and we do not have to prove it again. However, we prove Theorem 4.2.1 for whole cases without using any results in Section 4.1.

When m is too small, we cannot apply some results in Section 3.3 and 3.4. But since $\operatorname{rank}(N) = 2$ is small, we can compute the top Chern class of N and the Euler class of $N_{\mathbb{R}}$ by hand, and construct explicit Diophantine equations in a, b and c.

Proposition 4.2.2. *Let $m \geq 4$.*

(a) *The following two equations hold:*

$$\binom{m+1}{2}(a-1)^2 + a^2 - 1 + b\,(m-3) = (a^2 - b)\,d_0 + b\,d_1 \tag{4.2.1}$$

and

$$c\,(m-3) - m + 4 = c\,(d_1 - d_0) \tag{4.2.2}$$

where $d_0 := \varphi^*(\tilde{\omega}_{m-1,m-3})$ and $d_1 := \varphi^*(\tilde{\omega}_{m-2,m-2})$.

(b) *If $m \geq 5$, then c divides $m-4$. Moreover, $2b + 2c - a^2 > 0$ and $c \geq 1$.*

Proof. (a) By Note 3.3.1, $c_2(N) = e(N_{\mathbb{R}})$, and we compute $e(N_{\mathbb{R}})$ and $c_2(N)$ in Proposition 3.3.2 and Lemma 3.3.3, respectively. Comparing the coefficient of $\omega_{1,0}^2$ (resp. $\omega_{1,1}$) in $c_2(N)$ with that in $e(N_{\mathbb{R}})$, we obtain the desired equation (4.2.1) (resp. (4.2.2)).

(b) If $m \geq 5$, then c divides $m-4$ by (4.2.2) and in particular, $c \neq 0$. After dividing both sides of (4.2.2) by c,

$$\begin{aligned}
& m - 3 - \frac{m-4}{c} \\
= \;& d_1 - d_0 \\
= \;& \{(2b - a^2)\,\omega_{1,0}^2 + 2c\,\omega_{1,1}\}(b\,\omega_{1,0}^2 + c\,\omega_{1,1})^{m-3} \\
= \;& \{(2b - a^2)\,\omega_{2,0} + (2b + 2c - a^2)\,\omega_{1,1}\}\,(b\,\omega_{2,0} + (b+c)\,\omega_{1,1})^{m-3} \\
= \;& \sum_{i=0}^{m-3} \binom{m-3}{i} b^i\,(b+c)^{m-3-i} \cdot \\
& \quad \left\{(2b - a^2)\,\omega_{2,0}^{i+1}\,\omega_{1,1}^{m-3-i} + (2b + 2c - a^2)\,\omega_{2,0}^{i}\,\omega_{1,1}^{m-2-i}\right\}.
\end{aligned} \tag{4.2.3}$$

Since $m - 3 - \frac{m-4}{c} \geq m - 3 - \frac{m-4}{1} = 1$, the right hand side of (4.2.3) is positive. Furthermore, since b, $b + c$, $a^2 - 2b \geq 0$ by Proposition 3.4.4 (a), (b), and since $\omega_{2,0}^{i+1}\,\omega_{1,1}^{m-3-i}$, $\omega_{2,0}^{i}\,\omega_{1,1}^{m-2-i} \geq 0$, we have

$$2b + 2c - a^2 > 0.$$

Hence, $c > \frac{1}{2}\,(a^2 - 2b) \geq 0$ by Proposition 3.4.4 (b) again. $\square$

Proof of Theorem 4.2.1. (a) Note that since $\omega_{1,0}^4 = 2$ and $\omega_{1,0}^2\,\omega_{1,1} = 1 = \omega_{1,1}^2$ in $H^8(Gr(2,4),\mathbb{Z}) \simeq \mathbb{Z}$,

$$\begin{aligned} d_0 &= b\,(a^2 - b) + (b+c)\,(a^2 - b - c); \\ d_1 &= b^2 + (b+c)^2. \end{aligned} \tag{4.2.4}$$

By (4.2.1) and (4.2.2), we have

$$10\,(a-1)^2 + a^2 - 1 + b = (a^2 - b)\,d_0 + b\,d_1 \tag{4.2.5}$$

and

$$c = c\,(d_1 - d_0). \tag{4.2.6}$$

Case 1. $c \neq 0$: By (4.2.4) and (4.2.6),

$$\begin{aligned} 1 &= d_1 - d_0 \\ &= b\,(2b - a^2) + (b+c)\,(2b + 2c - a^2). \end{aligned} \tag{4.2.7}$$

Applying (4.2.7) to (4.2.5), we have

$$\begin{aligned} 11a^2 - 20a + 9 + b &= a^2\,d_0 + b\,(d_1 - d_0) \\ &= a^2\,d_0 + b. \end{aligned} \tag{4.2.8}$$

So a divides 9 and a^2 divides $-20a + 9$, thus, $a = 1$. Furthermore, in this case,

$$0 = d_0 = -b^2 + 2b + c - (b+c)^2$$

by putting $a = 1$ into the first equation in (4.2.4), that is,

$$b\,(b-1) = (b+c)\,\{1 - (b+c)\}. \tag{4.2.9}$$

Since the right hand side of (4.2.9) is less than or equal to $\frac{1}{4}$, $b = 0$ or 1. The only possible pairs (b,c) satisfying (4.2.7) and (4.2.9) are $(0,1)$ and $(1,-1)$. Hence, $(a,b,c) = (1,0,1)$ or $(1,1,-1)$.

Case 2. $c = 0$: Applying (4.2.4) to (4.2.5), we have

$$10\,(a-1)^2 + a^2 - 1 + b = 2b\,\left\{(a^2 - b)^2 + b^2\right\}. \tag{4.2.10}$$

Suppose that $b > 0$. After dividing both sides of (4.2.10) by b,

$$2b^2 \left\{ \left(\frac{a^2}{b} - 1 \right)^2 + 1 \right\} \; < \; 11 \cdot \frac{a^2}{b} + 1,$$

that is,

$$2b^2\, t^2 - (4b^2 + 11)\, t + 4b^2 - 1 < 0 \tag{4.2.11}$$

where $t := \frac{a^2}{b}$. The discriminant of (4.2.11) is

$$\begin{aligned} D &= (4b^2 + 11)^2 - 4 \cdot 2b^2 \cdot (4b^2 - 1) \\ &= -16b^4 + 96b^2 + 121 \\ &= -16(b^2 - 3)^2 + 265. \end{aligned}$$

If $b \geq 3$, then $D < 0$, so (4.2.11) is impossible. For $b = 1$ or 2, we can show directly that (4.2.10) does not have any integral solution a. Hence, $b = 0\,(= c)$, so $a = 1$ by (4.2.10).

By Proposition 3.2.2, if $(a, b, c) = (1, 0, 1)$, then φ is linear and if $(a, b, c) = (1, 1, -1)$, then φ is twisted linear. To complete the proof, it suffices to show that the pair $(a, b, c) = (1, 0, 0)$ is impossible. In this case, by Lemma 3.3.3, we have

$$c(N) = 1 + \omega_{1,0} = c(E). \tag{4.2.12}$$

After dividing both sides of (3.3.3) by (4.2.12), we have

$$(1 - \omega_{1,0}^2)(1 + \omega_{1,0} + \omega_{1,1})^4 = (1 + \omega_{1,0})^4(1 - \omega_{1,0}^2 + 4\,\omega_{1,1}). \tag{4.2.13}$$

Comparing the cohomology classes of degree 6 in (4.2.13),

$$\binom{4}{3} \omega_{1,0}^3 + \binom{4}{2} \cdot 2\,\omega_{1,0}\,\omega_{1,1} - \binom{4}{1} \omega_{1,0}^3 = \binom{4}{3} \omega_{1,0}^3 + \binom{4}{1} \omega_{1,0}\,(-\omega_{1,0}^2 + 4\,\omega_{1,1}),$$

and from this, we have

$$(4 \cdot 2 + 12 - 4 \cdot 2)\,\omega_{2,1} = \{4 \cdot 2 + 4 \cdot (-2 + 4)\}\;\omega_{2,1}$$

because $\omega_{1,0}^3 = 2\,\omega_{2,1}$. Hence, this implies a contradiction.

(b) By Proposition 3.4.4 (b), $a^2 - b \geq b$, so we have

$$\begin{aligned}
& \binom{m+1}{2}(a-1)^2 + a^2 - 1 + b\,(m-3) \\
\geq\ & b\,(d_0 + d_1) \qquad (\because\ \text{Equation (4.2.1)}) \\
=\ & a^2\, b\,\omega_{1,0}^2\,(b\,\omega_{1,0}^2 + c\,\omega_{1,1})^{m-3} \\
\geq\ & a^2\, b^{m-2}\,\omega_{1,0}^{2m-4} \qquad\qquad (4.2.14)\\
& (\because\ b,\ c,\ \omega_{1,0}^{2m-4-2i}\,\omega_{1,1}^{i} \geq 0 \quad \text{by} \quad \text{Proposition 3.4.4 (a),} \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{4.2.2 (b) and Lemma 3.4.6)} \\
=\ & \frac{(2m-4)!}{(m-2)!(m-1)!} \cdot a^2\, b^{m-2} \qquad (\because\ \text{Lemma 3.4.6}).
\end{aligned}$$

The left hand side of (4.2.14) is less than

$$a^2 \left\{ \binom{m+1}{2} + m - 2 \right\}$$

because $a^2 \geq b$ by Proposition 3.4.4 (a). After dividing both sides of (4.2.14) by a^2,

$$\binom{m+1}{2} + m - 2 \geq \frac{(2m-4)!}{(m-2)!(m-1)!} \cdot b^{m-2},$$

thus we have

$$b = \begin{cases} 0 \text{ or } 1, & \text{if } m = 5 \text{ or } 6 \\ 0, & \text{if } m \geq 7 \end{cases}.$$

Assume that $b = 1$ with $m = 5$ or 6. By Proposition 3.4.4 (b) and Proposition 4.2.2 (b),

$$2 \;=\; 2b \;\leq\; a^2 \;<\; 2b + 2c \;=\; 2 + 2c$$

and $c\,(\geq 1)$ divides $m - 4$. The only possible pair (m, a, c) satisfying these properties is $(6, 2, 2)$. Apply $(m, a, b, c) = (6, 2, 1, 2)$ to (4.2.14), then

$$27 \;=\; \binom{7}{2} + 6 \;\geq\; \frac{8!}{4! \cdot 5!} \cdot 4 \;=\; 56$$

which implies a contradiction.

Assume that $b = 0$ with $m \geq 5$. Then by (4.2.2),

$$\begin{aligned} c\,(m-3) - m + 4 &= c\,(-a^2\,\omega_{1,0}^2 + 2c\,\omega_{1,1})\,(c\,\omega_{1,1})^{m-3} \\ &= (2c - a^2)\,c^{m-2} \\ &\geq c^{m-2} \quad (\because\ \text{Proposition 4.2.2 (b)}), \end{aligned}$$

thus $c = 1$. By Proposition 4.2.2 (b), $2 - a^2 > 0$, so $a = 1$.

Hence, $(a, b, c) = (1, 0, 1)$ for $m \geq 5$, thus any embedding $\varphi\colon Gr(2, m) \hookrightarrow Gr(2, m+1)$ is linear by Proposition 3.2.2. $\square$

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국문초록

이 논문에서는 복소 그라스만 다양체 $Gr(2,m)$ 에서 복소 그라스만 다양체 $Gr(2,n)$ 으로 가는 복소해석적 매장의 선형성에 대해 규명하고자 한다. 복소 그라스만 다양체 $Gr(2,n)$ 위의 보편벡터다발의 쌍대벡터다발을, 주어진 매장으로 끌어당긴 벡터다발을 E 라고 하면, 가능한 모든 E 의 전 천 특성류들을 조사함으로써 이러한 매장들에 대한 성질을 파악할 수 있다. 먼저, 앞으로의 논리를 전개하는데 유용하게 사용될 $Gr(2,m)$ 의 코호몰로지환의 $\mathbb{Z}$-가군 기저를 하나 잡고, 모든 코호몰로지류들은 이 기저에 대한 선형결합으로 표현한다. 복소 그라스만 다양체 $Gr(2,m)$ 에서 $Gr(2,n)$ 으로 가는 각각의 복소해석적 매장에 대해, 벡터다발 E 의 전 천 특성류를 이 기저로 표현하면, 세 개의 정수계수를 가지는 선형결합으로 유일하게 쓸 수 있는데, 매장의 선형성은 이러한 정수들에 의해 완전히 결정된다. 주어진 매장으로부터 유도된 법벡터다발의 천 특성류와 오일러 특성류로부터, 3-변수 디오판토스 방정식들을 얻을 수 있고, 해석적벡터다발의 천 특성류가 음이 아닐 판정 기준과 함께 이 방정식들을 풀면, 특정 정수의 상계를 비롯한, 세 개의 정수들이 만족해야 하는 몇 가지 조건들을 얻을 수 있다. 이 정수 상계로부터 W. Barth 와 A. Van de Ven 이 증명한 결과들을 벡터다발 E 에 적용할 수 있고, 복소 그라스만 다양체 $Gr(2,m)$ 에서 $Gr(2,n)$ 으로 가는 복소해석적 매장이 항상 선형이 되기 위한 m 과 n 의 조건들을 구할 수 있다.

주요어휘 : 복소 그라스만 다양체, 복소해석적 매장, 슈베르트 싸이클, 천 특성류
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