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On the Cucker-Smale-Fokker-Planck Model under Random Environment

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On the Cucker-Smale-Fokker-Planck Model under Random Environment

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／# Abstract

In this dissertation, we mainly focus on a kinetic Cucker–Smale–Fokker–Planck (CS-FP) type equation with a degenerate diffusion coefficient. The CS-FP equation is described in a differential equation for a probability distribution function f of the infinitely many Cucker–Smale flocking particles in a random environment. We will present a priori estimates for proving the global existence of classical solutions to the CS-FP equation. The global existence of classical solutions under a given sufficiently smooth initial datum will be obtained by applying sobolev embedding theorem to the a priori estimates and iterating the solutions of uniformly parabolic equations which approximates the CS-FP equation. We also present the Cucker-Smale-Kuramoto model which describes flocking and synchronization coupled phenomena. Sufficient conditions for the asymptotic flocking and synchronization will be derived with the Lyapunov functional approach. We provide the numerical compuations for a special case to suggest the future works on clustering.

Key words: Cucker-Smale Model, Flocking, Cucker-Smale-Fokker-Planck equation, Threshold phenomena, Cucker-Smale-Kuramoto equation, Synchronization
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Contents

List of Figures

Chapter 1

Introduction

The collective behaviour among species on the earth, such as a swarm of insects, a flight of birds, a school of fish, are often observed. Such groups display arranged formations surprisingly in order and they travel without scattering as if there are some governing rules. We call such phenomenon as flocking. Flocking can be more precisely defined as a phenomenon in which self-propelled particles organize into an ordered motion with only limited environmental influences and simple rules. Flocking has drawn attention to many mathematicians. Taking interactions into account, we now have several models that intuitively make sense and fit into the phenomena. Flocking is expected to be further applied to develop unmanned vehicles and sensor networks etc. In this dissertation, the derivation of flocking model will be discussed first and current and future work will be introduced. We mainly take the Cucker and smale [13, 14]'s flocking model and its modified models. The Cucker-Smale model(CS model) is a time-continuous first-order ODE system with position and velocity variables. In this model, each agent is regarded as a point particle, so the volume is neglected. The time derivatives of velocity variables in this dynamical system are expressed as an average value of the other particles' relative velocities multiplied by the communication weight coefficients, which depends on the distance of a pair of particles. The communication weight tends to increase when two particles are located close.

The dynamics in which multiplicative noise is grafted onto the C-S model (2.1.2) is rewritten[2] as follows :

$$\begin{cases} dx_t^i &= v_t^i dt, \qquad t > 0, \quad 1 \le i \le N, \\ dv_t^i &= \dfrac{K}{N} \displaystyle\sum_{j=1}^{N} \psi(|x_t^j - x_t^i|)(v_t^j - v_t^i)dt + \sqrt{2\sigma}(v_t^c - v_t^i)dB_t. \end{cases} \tag{1.0.1}$$

In case that the number of CS particles is large enough, the kinetic mean-field model corresponding to (1.0.1), called the Cucker-Smale-Fokker-Planck model(CS-FP model), for the one-particle distribution function is used to further study its dynamics. The CS-FP model is described as follows. Let $f = f(x, v, t)$ be the one-particle distribution function of the CS ensemble. The evolution of the kinetic density function f is governed by the following Cauchy problem for the kinetic CS-FP equation:

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (L[f]f) = \sigma \Delta_v (|v - v^c|^2 f), \quad x, v \in \mathbb{R}^d, \ t > 0, \\ &L[f](x, v, t) := -K \int_{\mathbb{R}^{2d}} \psi(|x - y|)(v - v_*) f(y, v_*, t) dv_* dy, \end{aligned} \tag{1.0.2}$$

subject to suitable initial datum

$$f(x, v, 0) = f_0(x, v), \quad \int_{\mathbb{R}^{2d}} f_0 = 1, \tag{1.0.3}$$

where K and σ are nonnegative constants that represent the coupling strength and noise strength of noise in a random environment, respectively. v^c is defined as the average velocity, i.e.,

$$v^c(t) := \frac{\int_{\mathbb{R}^{2d}} v f dv dx}{\int_{\mathbb{R}^d} f dv dx}.$$

Note that v^c is a conserved quantity along the dynamics in (1.0.2)–(5.1.3). It will be shown.

The main purpose of this thesis is to classify the large-time dynamics of the CS-FP equation in (1.0.2) depending on the relative ratio between the coupling strength K and the diffusion coefficient σ.

There have been a lot of literatures that dealt with the flocking phenomena: the global existence theories of classical solutions, measure-valued solutions, weak solutions and their flocking estimates in [25, 27, 31, 32], the rigorous mean-field limit from the particle system [8, 25], coupling with fluid equations [3, 4, 9, 10], and the macroscopic C-S model and its asymptotic justification [21, 22, 30].

In this thesis, we present the analysis of the CS-FP model and other flocking dynamics in the following order: First, we introduce the a priori asymptotic dynamics of (1.0.2) under an assumption of the positivity and boundedness of the communication weight ψ. To measure the degree of velocity alignment, we use the velocity variance of the kinetic density function f, so that the velocity variance decay implies the formation of velocity alignment. In the course of the proof for Theorem 2.2.1, we derive an identity representing the competition between the velocity alignment forcing and the nonuniform diffusion. Thus, we have two dichotomies of the large-time behavior of the velocity variance.

Second, we present the global existence of classical solutions to the kinetic mean-field equation in (1.0.2). In Theorem 4.1.1, we show that the CS-FP equation admits a global smooth solution for an H^k_α ($k > d+2,\ \alpha > \frac{d+2}{2}$)-initial datum with finite mass and energy. The smallness of the initial datum is not needed in the a priori estimate. Of course, we cannot expect a uniform bound for the H^k_α-norm of f because we have the formation of the velocity alignment for $K \gg \sigma$, which reflects the unlimited growth of the H^k_α-norm of f.

The thesis after the introduction is organized as follows. In Chapter 2, we briefly discuss the flocking models. In Chapter 3, the Cucker-Smale model with white noise is introduced. In Chapter 4, we present the global H^k_α solvability of the CS-FP equation in (1.0.2)–(5.1.3) in a suitable admissible function space using the energy method. In Chapter 5, we present the Cucker-Smale model coupled with phase interaction. Numerical experiments are included to supplement the future works. Finally, Chapter 6 is devoted to a summary of our main results.

Notations for the CS-FP Equation: For a measurable function $u = u(x,v)$ in the phase space $\mathbb{R}^{2d}$, we set

$$\|u\|_{L^1_2} := \|(1+|v|^2)u\|_{L^1(\mathbb{R}^{2d})}, \quad \|u\|^2_{L^2_\alpha} := \int_{\mathbb{R}^{2d}} (1+|v|^2)^\alpha |f|^2 dv dx, \quad \alpha \geq 0,$$
$$\|u\|^2_{H^k_\alpha} := \|u\|^2_{L^2_\alpha} + \sum_{1 \leq i+j \leq k} \|\partial^i_x \partial^j_v u\|^2_{L^2_\alpha}, \quad k \in \mathbb{N} \cup \{0\}.$$

For $\alpha = 0$, we denote $\|u\|^2_{H^k} := \|u\|^2_{H^k_0}$.

Notations for the CSK Equation: For $\mathbf{x} = (x_1, \cdots, x_N)$, $\mathbf{v} = (v_1, \cdots, v_N)$, $\theta = (\theta_1, \cdots, \theta_N)$ and $\omega = (\omega_1, \cdots, \omega_N)$, we set

$$\begin{aligned} \|\mathbf{x}\| &:= \Big(\textstyle\sum_{i=1}^N |x_i|^2 \Big)^{\frac{1}{2}}, \qquad & D(\mathbf{x}) &:= \max_{1 \leq i,j \leq N} \|x_i - x_j\|, \\ D(\mathbf{v}) &:= \max_{1 \leq i,j \leq N} \|v_i - v_j\|, \qquad & D(\theta) &:= \max_{1 \leq i,j \leq N} \|\theta_i(t) - \theta_j(t)\|, \\ D(\omega) &:= \max_{1 \leq i,j \leq N} \|\omega_i - \omega_j\|. \end{aligned}$$

Chapter 2

Preliminaries

In this chapter, we present Cucker-Smale flocking models from microscopic to macroscopic scales [15] and relavent results from previous literatures. We also consider how to transform the CS model with random communication into a stochastic model with multiplicative noise.

2.1 The Cucker-Smale Model

2.1.1 The Vicsek Model

Vicsek et al presented a simple phase transition model in [43]. The model is motivated from the effort to understand self-ordered behaviour of biological systems such as clustering, migration, and various pattern formations. The basic rule of the model is that at each time step the velocity of each particle driven with a constant absolute velocity is updated according to the average direction of the neighboring particles' moves with some random perturbation added. In this context, its neighborhood particles within radius r are involed in the interaction.

The numerical simulations of the Vicsek model is conducted in [43] under these conditions: i) Initially, N particles are randomly distributed in the cell(a

square shaped cell of length L with periodic boundary).
ii) All the particles have the same absolute velocity value.
iii) The initial moving direction θ of each particle is randomly distributed, and, at each time step, the moving direction and position are updated in the following manner:

$$\begin{cases} x_i(t+1) &= x_i(t) + v_i(t)\Delta t, \\ \theta_i(t+1) &= \langle\theta(t)\rangle_r + \Delta\theta, \end{cases} \tag{2.1.1}$$

where $\langle\theta(t)\rangle_r$ represents the average direction of the velocities of particles within a neighborhood of radius r and $\Delta\theta$ is chosen from a continuous uniform distribution with a finite support $[-\frac{\eta}{2}, \frac{\eta}{2}]$. By changing three free parameters such as a density $\rho = \frac{N}{L^2}$, noise η, and a velocity size $|v|$, Vicsek et al presents orderedness for particles.

2.1.2 The Cucker-Smale Model

Felipe Cucker and Steve Smale suggested a Newton type microscopic model for an interactive multiple number of particle system which present a flocking phenomenon in their works [14]. Motivated by the work of Vicsek et al in [43] , Cucker and Smale worked on sufficient conditions for an asymptotic flocking in terms of interaction coefficients and initial configuration, and they showed a rigorous flocking estimates. The Cucker-Smale model(C-S model) is described as the system of ODEs for $N-$particles:

$$\begin{cases} \dfrac{dx_i}{dt} &= v_i, \quad t > 0, i = 1, ..., N, \\ \dfrac{dv_i}{dt} &= \frac{K}{N}\sum_{j=1}^N \psi(|x_j - x_i|)(v_j - v_i), \end{cases} \tag{2.1.2}$$

where K and $\psi(s) = \dfrac{1}{(1+s^2)^{\frac{\beta}{2}}}, \quad \beta \geq 0$ represent a nonnegative coupling strength and a communication weight reflecting the intensity of communication, respectively. We recall the definition of flocking:

Definition 2.1.1. [2, 27] *Let $\mathcal{B} = \{(x_i, v_i)\}_{i=1}^N$ be a solution to the deterministic system* (2.1.2)*. Then, the system $\mathcal{B}$ exhibits global (or mono-cluster) flocking if and only if it satisfies the following two conditions.*

1. *The spatial diameter of $\mathcal{B}$ is uniformly bounded, i.e.,*

$$\sup_{0 \le t < \infty} \max_{1 \le i,j \le N} \|x_j(t) - x_i(t)\| < \infty. \tag{2.1.3}$$

2. *The velocity diameter of $\mathcal{B}$ tends to zero asymptotically, i.e.,*

$$\lim_{t \to \infty} \max_{1 \le i,j \le N} \|v_j(t) - v_i(t)\| = 0, \tag{2.1.4}$$

where $|\cdot|$ is the standard ℓ_2-norm in $\mathbb{R}^d$.

Remark 2.1.1. *when the conditions in* (2.1.3) *and* (2.1.4) *hold for a stochastic interacting system almost surely, the system is said to be presenting the strong stochastic flocking.*

For an all-to-all case, the communication weight is $\psi \equiv 1$, i.e., $\beta = 0$. The C-S model has been extensively studied in many literatures [13, 14, 25, 27, 2, 17, 24] etc, and in [...] it is verified that the threshold between a conditional and unconditional flocking is $\beta = 1$. Under ψ is a long range interaction, i.e., $\beta \le 1$, flocking unconditionally occurs no matter how scarce the given initial configuration is.

2.1.3 The Kinetic Cucker-Smale Model

As N grows sufficiently large, i.e., the CS system (2.1.2) is rewritten as a partial differential equation for $f = f(x, v, t)$, a one particle distribution function, described as:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F(f)f) = 0, \quad x, v \in \mathbb{R}^d, \ t > 0, \\ F(f)(x, v, t) = -K \int_{\mathbb{R}^{2d}} \psi(|x - y|)(v - v_*) f(y, v_*, t) dv_* dy. \end{cases} \tag{2.1.5}$$

Let us denote $f^N = f^N(x_1, ..., x_N, v_1, ..., v_N, t)$ by the $N-$particle probability density function. The density function f^N does not change its value by interchanging any two space-phace arguments, i.e., for any j and k

$$f^N(..., x_j, ..., x_k, ..., v_j, ..., v_k, ..., t) = f^N(..., x_k, ..., x_j, ..., v_k, ..., v_j, ..., t)$$

holds. In [15], the time evolution of f^N is written in a form of Liouville equation as follows:

$$\partial_t f^N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f^N + \frac{1}{N}\sum_{i=1}^N \nabla_{v_i} \cdot \left(\sum_{j=1}^N \psi(|x_i - x_j|)(v_j - v_i) f^N \right) = 0.$$

We set the marginal distribution $\hat{f}^N = \hat{f}^N(x_1, v_1, t)$ as

$$\hat{f}^N(x_1, v_1, t) = \int_{\mathbb{R}^{2d(N-1)}} f^N(x_1, x_-, v_1, v_-, t) dx_- dv_-,$$

with $(x_-, v_-) := (x_2, ..., x_N, v_2, ..., v_N)$. We then obtain the kinetic C-S equation by integrating the above Liouville type equation with respect to (x_-, v_-) and taking the mean-field limit $N \to \infty$.

The communication between particles can also be affected by their surroundings. The roles of the environment can be subtle to be accurately measured. Using the stochastic noise, not only we take the environmental effect into account, but also we present its indeterministic feature. Let x_i and v_i be the position and velocity, respectively, of the i-th particle in $\mathbb{R}^d$. Recall the CS flocking model:

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \quad t > 0, \quad 1 \le i \le N, \\ \frac{dv_i}{dt} &= \frac{K}{N}\sum_{j=1}^N \bar{\psi}(|x_j - x_i|)(v_j - v_i). \end{aligned}$$

After Cucker and Smale's seminal works in [13, 14], further extensions of the CS model in (2.1.2) have been attempted in [8, 11, 12, 23, 24, 26]. Let us now consider the case that the Gaussian white noise is involed in $\bar{\psi}$, so that it is rewritten as

$$\bar{\psi}(|x_j - x_i|) = \underbrace{\psi(|x_j - x_i|)}_{\text{deterministic}} + \underbrace{\frac{\sqrt{2\sigma}}{K}\eta_t}_{\text{random noise}} \quad , \tag{2.1.6}$$

with the d-dimensional Gaussian white noise $\eta_t = (\eta_t^1, ..., \eta_t^d)$ satisfying

$$\langle d\eta_t^i \rangle = 0, \quad \langle d\eta_t^i, d\eta_s^j \rangle = \delta_{ij} d(t \wedge s), \quad 1 \le i, j \le N, \ t, s > 0.$$

Combined with the random communication part in (2.1.6), the C-S model in (2.1.2) turns into the stochastic CS model with multiplicative noise [2]:

$$\begin{aligned} dx_i &= v_i dt, \qquad t > 0, \quad 1 \le i \le N, \\ dv_i &= \frac{K}{N} \sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i) dt + \sqrt{2\sigma}(v_c - v_i) dB_t, \end{aligned} \tag{2.1.7}$$

where the average velocity v_c is defined as

$$v_c := \frac{1}{N} \sum_{j=1}^{N} v_j.$$

Note that system in (2.1.7) conserves the total momentum. Hence, if asymptotic flocking occurs in the sense of Definition 2.1.1, then the flocking velocity is given by the initial average velocity v_0^c, as in the original CS model [25, 27]. It is well known [2, 17, 24] that noise can stabilize and destabilize deterministic dynamical systems depending on its type. For example, additive noise destabilizes flocking states in the C-S model, whereas multiplicative noise can stabilize flocking states (see [2, 12, 24]). For the reader's interest, we quote the stabilization result of multiplicative noise in C-S flocking without proof.

Remark 2.1.2. *If the conditions in* (2.1.3) *and* (2.1.4) *hold a.s. for a stochastic interacting system, then we say that strong stochastic flocking occurs.*

2.2 The Cucker-Smale-Fokker-Planck Equation

In this subsection, we study the estimates for the conservation laws and the exponential flocking estimate in (1.0.2). Recall that the kinetic CS-FP equation is written as follows:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (L[f]f) = \sigma \Delta_v (|v - v^c|^2 f), \quad x, v \in \mathbb{R}^d, \ t > 0, \\ L[f](x, v, t) = -K \int_{\mathbb{R}^{2d}} \psi(|x - y|)(v - v_*) f(y, v_*, t) dv_* dy, \end{cases}$$

subject to an initial datum

$$f(x,v,0)=f_0(x,v),\qquad \int_{\mathbb{R}^{2d}} f_0(x,v)dxdv=1.$$

Lemma 2.2.1. (Conservation laws) *Let $f=f(x,v,t)$ be a smooth solution to the CS-FP equation* (1.0.2)–(5.1.3) *which vanishes at infinity and has the finite first moments, i.e.,*

$$\int_{\mathbb{R}^{2d}}(1+|v|)f_0(x,v)dvdx<+\infty,\quad t\geq 0.$$

Then, the total mass and total momentum are conserved:

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} fdvdx=0,\qquad \frac{d}{dt}\int_{\mathbb{R}^{2d}} vfdvdx=0\quad t>0.$$

Proof. The equation in (1.0.2) can be written in a divergent form:

$$\partial_t f+\nabla_x\cdot(vf)+\nabla_v\cdot\Big[L[f]f-\sigma\nabla_v(|v-v^c|^2f)\Big]=0. \tag{2.2.8}$$

In order to obtain the conservation of mass, we integrate (2.2.8) with respect to (x,v). For the conservation of momentum, we first multiply the CS-FP equation(1.0.2) by v, so that we find the local balanced law for vf:

$$\begin{aligned}\partial_t(vf)+\nabla_x\cdot(v\otimes vf)+\nabla_v\cdot\Big[v\otimes(L[f]f)-\sigma v\otimes\nabla_v(|v-v^c|^2f)\Big]\\ =L(f)f-\sigma\nabla_v(|v-v^c|^2f).\end{aligned} \tag{2.2.9}$$

By integrating the above the relation (2.2.9) with respect to (x,v) and assigning the definition of $L(f)$, i.e.,

$$\int_{\mathbb{R}^{2d}} L(f)fdvdx=-K\int_{\mathbb{R}^{4d}}\psi(|x-y|)(v-v_*)f(y,v_*,t)f(x,v,t)dvdv_*dydx=0$$

to the relation, we derive the conservation of total momentum. □

We define a functional $\mathcal{F}(f)$ so that the velocity variance of the kinetic density function f is measured:

$$\mathcal{F}(f(t)):=\int_{\mathbb{R}^{2d}}|v-v_c(t)|^2fdvdx=\int_{\mathbb{R}^{2d}}|v-v_c(0)|^2fdvdx,$$

where $v_c(t) = v_c(0),\ t > 0$ due to the conservation of momentum. Note that the zero convergence of $\mathcal{F}(f(t))$ as $t \to \infty$ is interpreted as the formation of velocity alignment in probability. The connection between the zero convergence of $\mathcal{F}(f(t))$ and the probabilistic velocity alignment is easily shown by the Chebyshev inequality as the following: let us define $f(t)$ as a probability density function in (x, v). For any $\varepsilon > 0$, we have

$$\begin{aligned}\mathcal{F}(f(t)) &= \int_{\mathbb{R}^{2d}} |v - v_c(t)|^2 f dv dx \\ &\geq \int_{|v-v_c(0)|>\varepsilon} |v - v_c(0)|^2 f dv dx \\ &\geq \varepsilon^2 \int_{|v-v_c(0)|>\varepsilon} f dv dx = \varepsilon^2 \mathbb{P}[|v - v_c(0)| > \varepsilon].\end{aligned}$$

This yields the following inequality:

$$\lim_{t\to\infty} \mathbb{P}[|v - v_c(0)| > \varepsilon] \leq \frac{1}{\varepsilon^2} \lim_{t\to\infty} \mathcal{L}(f(t)) = 0.$$

The first main result in this thesis is the asymptotic threshold phenomenon of $\mathcal{F}$ depending on the relative strengths between K and σ.

Theorem 2.2.1. *Suppose that the communication weight function ψ is positive and bounded below and above, i.e., there exists positive constants ψ_M and ψ_m satisfying*

$$\psi_m \leq \psi(s) \leq \psi_M, \quad s \geq 0.$$

In addition, let $f = f(x, v, t)$ be a classical solution to (1.0.2) *which quickly vanishes at infinity and satisfies the finite second moments*

$$\int_{\mathbb{R}^{2d}} (1 + |v|^2) f(x, v, t) dv dx < \infty, \quad t \geq 0.$$

i) For $K > \dfrac{d\sigma}{\psi_m \|f_0\|_{L^1}}$, there is a positive constant $K_m := 2\left(K\psi_m\|f_0\|_{L^1} - d\sigma\right)$ which satisfies

$$\mathcal{F}(f(t)) \leq \mathcal{F}(f_0) e^{-K_m t}, \quad t \geq 0.$$

ii) If $K < \dfrac{d\sigma}{\psi_M \|f_0\|_{L^1}}$, there is a positive constant $K_M := 2\left(d\sigma - K\psi_M\|f_0\|_{L^1}\right)$ which satisfies

$$\mathcal{F}(f(t)) \geq \mathcal{F}(f_0) e^{K_M t}, \quad t \geq 0.$$

Proof. First, let us multiply (1.0.2) by $|v-v^c|^2$ in order to obtain

$$\begin{aligned}&\partial_t(|v-v^c|^2 f)+\nabla_x\cdot(v|v-v^c|^2 f)\\&\quad+\nabla_v\cdot\Big(|v-v^c|^2L(f)f-\sigma|v-v^c|^2\nabla_v(|v-v^c|^2 f)+2\sigma(v-v^c)|v-v^c|^2 f\Big)\\&\quad=2(v-v^c)\cdot(L(f)f)+2d\sigma|v-v^c|^2 f.\end{aligned}\tag{2.2.10}$$

We then integrate the relation in (2.2.10) to obtain

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}}|v-v^c|^2 f dvdx=2\int_{\mathbb{R}^{2d}}(v-v^c)\cdot(L(f)f)dvdx+2d\sigma\int_{\mathbb{R}^{2d}}|v-v^c|^2 f dvdx.\tag{2.2.11}$$

Note that, by exchanging v and v_*, the first term on the right side of (2.2.11) is rewritten as

$$\begin{aligned}&\int_{\mathbb{R}^{2d}}(v-v^c)\cdot(L(f)f)dvdx\\&\quad=-K\int_{\mathbb{R}^{4d}}\psi(|x-y|)(v-v^c)\cdot(v-v_*)f(y,v_*,t)f(x,v,t)dv_*dvdydx\\&\quad=K\int_{\mathbb{R}^{4d}}\psi(|x-y|)(v_*-v^c)\cdot(v-v_*)f(y,v_*,t)f(x,v,t)dv_*dvdydx\\&\quad=-\frac{K}{2}\int_{\mathbb{R}^{4d}}\psi(|x-y|)|v-v_*|^2 f(y,v_*,t)f(x,v,t)dv_*dvdydx.\end{aligned}\tag{2.2.12}$$

We combine (2.2.11) and (2.2.12) to obtain a dissipation estimate for $\mathcal{F}(f(t))$:

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^{2d}}|v-v^c|^2 f dvdx\\&\quad=-K\int_{\mathbb{R}^{4d}}\psi(|x-y|)|v-v_*|^2 f(y,v_*,t)f(x,v,t)dv_*dvdydx\\&\quad+2d\sigma\int_{\mathbb{R}^{2d}}|v-v_c|^2 f dvdx.\end{aligned}\tag{2.2.13}$$

In (2.2.13), we now take the lower bound for ψ to derive the corresponding

Gronwall's inequality for $\mathcal{F}$:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^{2d}}|v-v_c|^2 f dv dx &\le -2(K\psi_m\|f_0\|_{L^1}-d\sigma)\int_{\mathbb{R}^{2d}}|v-v_c|^2 f dv dx\\ &= -K_m\int_{\mathbb{R}^{2d}}|v-v_c|^2 f dv dx,\end{aligned}\tag{2.2.14}$$

where $K_m := 2\left(K\psi_m\|f_0\|_{L^1}-d\sigma\right)$. Meanwhile, we take the upper bound for ψ to derive

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^{2d}}|v-v_c|^2 f dv dx &\ge -2(K\psi_M\|f_0\|_{L^1}-d\sigma)\int_{\mathbb{R}^{2d}}|v-v_c|^2 f dv dx\\ &= K_M\int_{\mathbb{R}^{2d}}|v-v_c|^2 f dv dx,\end{aligned}\tag{2.2.15}$$

where $K_M := 2\left(d\sigma-K\psi_M\|f_0\|_{L^1}\right)$. (2.2.14) and (2.2.15) lead to the desired flocking estimate. □

Remark 2.2.1. *1. The results in Theorem 2.2.1 suggest the possible existence of a critical coupling strength K_c from the diffusing phase to the flocking phase as we increase the coupling strength. In fact, for the all-to-all coupling case $\psi = 1$, such a critical coupling strength K_c is exactly given by the value $\frac{d\sigma}{\|f_0\|_{L^1}}$, and we have the following threshold phenomenon:*

$$\lim_{t\to\infty}\mathcal{F}(f(t)) = \begin{cases}\infty & K<K_c,\ \textit{subcritical regime},\\ \mathcal{F}(f_0) & K=K_c,\ \textit{critical regime},\\ 0 & K>K_c,\ \textit{supercritical regime}.\end{cases}$$

This phenomenon is reminiscent of the existence of a critical coupling strength from the incoherent state (disordered phase) to the partially ordered state in Kuramoto synchronization [1].

2. Note that the asymptotic formation of flocking states containing the factor $\delta(\cdot - v^c(0))$ is mainly due to the nonuniform diffusion coefficients $\sigma|v|^2$ in (1.0.2). *For the additive white noise case, the formation of flocking is not* *possible. This can be easily seen from the following kinetic CS-FP equation from [6]:*

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (L[f]f) = \sigma \Delta_v f. \tag{2.2.16}$$

For the all-to-all coupling case $\psi = 1$ *and suitable normalization conditions:*

$$\int_{\mathbb{R}^{2d}} f dv dx = 1, \quad \int_{\mathbb{R}^{2d}} v f dv dx = 0,$$

the linear alignment forcing term $L[f]$ *becomes linearly damped:*

$$L[f](x, v, t) = -Kv.$$

Thus, the equation in (2.2.16) *becomes a linear Vlasov–Fokker–Planck equation:*

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (Kvf + \sigma \nabla_v f).$$

Then, it is easy to check that the above equation has a space-homogeneous equilibrium f_∞*:*

$$f_\infty(v) = e^{-\frac{K}{2\sigma}|v|^2}, \quad v \in \mathbb{R}^d;$$

thus, there is no emergent velocity alignment for any positive K*.*

Lemma 2.2.2. *Let* $f = f(x, v, t)$ *be a classical solution to* (1.0.2) *that quickly decays to zero at infinity and satisfies the finite second moments:*

$$\int_{\mathbb{R}^{2d}} v f dv dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^{2d}} (1 + |v|^2) f dv dx < \infty, \quad t \geq 0.$$

Then, we have

$$\begin{aligned} &(i) \ \|\nabla_v \cdot L[f]\|_{L^\infty} \leq dK \|\psi\|_{L^\infty} \|f_0\|_{L^1}, \\ &(ii) \ \||v| f\|_{L^1} \leq e^{-\frac{K_m t}{2}} \sqrt{\|f_0\|_{L^1} \||v|^2 f_0\|_{L^1}}. \end{aligned}$$

Here K_m *is a constant appearing in Theorem 2.2.1.*

Proof. (i) For the first estimate, we use the definition of the linear operator $L[f]$ in (1.0.2) and use Lemma 2.2.1 to obtain

$$\begin{aligned} |\nabla_v \cdot L[f]| &= \left| dK \int_{\mathbb{R}^{2d}} \psi(|x - y|) f(y, v_*, t) dv_* dy \right| \\ &\leq dK \|\psi\|_{L^\infty} \|f(t)\|_{L^1} = dK \|\psi\|_{L^\infty} \|f_0\|_{L^1}. \end{aligned}$$

(ii) We use the Cauchy–Schwarz inequality to obtain

$$\||v|f\|_{L^1} \leq \sqrt{\|f\|_{L^1}\||v|^2 f\|_{L^1}} \leq e^{-\frac{K_m t}{2}} \sqrt{\|f_0\|_{L^1}\||v|^2 f_0\|_{L^1}}.$$

□

Chapter 3

The Cucker-Smale Model with White Noise

In this chapter, we summarize the results of the CS model with additive noise and multiplicative noise which appeared in [24] and [2], respectively.

3.1 The Additive Noise Case

The C-S flocking with additive noise is discussed in [24]. The dynamics of N particles are described in terms of position and velocity, $(x_i(t), v_i(t)) \in \mathbb{R}^d \times \mathbb{R}^d$, as follows:

$$\begin{cases} dx_i = v_i dt, \\ dv_i = \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} \psi(|x_i - x_j|)(v_j - v_i)dt + \sqrt{D}dW_i(t), \end{cases} \tag{3.1.1}$$

subject to deterministic initial data $(x_i(0), v_i(0)),\ i = 1, ..., N$. Here $dW_i(t)$ is the d-dimensional Brownian motion satisfying $\langle dW^a(t)\rangle = 0,\ \langle dW_i^a(t)dW_j^b(s)\rangle = \delta(a-b)\delta(i-j)d(t \wedge s),\ \langle\cdot\rangle$: ensemble average. In this system, K is interpreted as a repulsive coupling constant if $K < 0$ and a attractive coupling constant if $K > 0$. The concept of asumptotic flocking is expressed as

Definition 3.1.1. *[24] The system* (3.1.1) *exhibits a (time-asymptotic)* ***flocking*** *if and only if* $\{(x_i, v_i)\}_{i=1}^N$ *satisfy the two conditions:*

i) For any $1 \le i, j \le N$, *the expectation differences of the pairwise velocity asymptotically vanish, i.e.,*

$$\lim_{t\to\infty} |\langle v_i(t)\rangle - \langle v_j(t)\rangle| = 0,$$

ii) For any $1 \le i, j \le N$, *the average diameter of a group is uniformly bounded in* t, *i.e.,*

$$\sup_{0\le t<\infty} |\langle x_i(t)\rangle - \langle x_j(t)\rangle| < \infty.$$

By setting the following notations as

$$x_c := \frac{1}{N}\sum_{i=1}^N x_i,\ v_c := \frac{1}{N}\sum_{i=1}^N v_i,\ \hat{x}_i = x_i - x_c,\ \hat{v}_i = v_i - v_c,$$

we will analyze the system both macroscopically and microscopically.

We notice that the macroscopic quantities(ensemble averages) satisfy

$$\begin{cases} dx_c = v_c dt \\ dv_c = \dfrac{\sqrt{D}}{N} \sum_{i=1}^N dW_i(t). \end{cases}$$

The main macroscopic analysis is the following:

Proposition 3.1.1. *[24] Let* (x_c, v_c) *satisfy the above system of equations and* $a \in 1, ..., d$. *Then we have*

i) $\langle v_c(t)\rangle = v_c(0), \quad var[v_c^a(t)] = \dfrac{Dt}{N},$

ii) $\langle x_c(t)\rangle = x_c(0) + tv_c(0), \quad var[x_c^a(t)] = \dfrac{Dt^2}{2N}.$

Remark 3.1.1. *From the proposition, we find the average macroscopic velocity converges to* $v_c(0)$ *as the number of agents increases.*

$$v_c(t) - v_c(0) \to 0, \quad a.s.$$

The microscopic quantities(fluctuations), on the other hand, satisfy

$$\begin{cases} d\hat{x}_i = \hat{v}_i dt, \\ d\hat{v}_i = \dfrac{K}{N}\displaystyle\sum_{j=1}^{N}\psi(|\hat{x}_i - \hat{x}_j|)(\hat{v}_j - \hat{v}_i)dt + \sqrt{D}\left(1 - \dfrac{1}{N}\right)dW_i - \dfrac{\sqrt{D}}{N}\displaystyle\sum_{i\neq j} dW_j, \end{cases}$$

with the initial data $(\hat{x}_i(0), \hat{v}_i(0))$ and their zero sum constraint

$$\sum \hat{x}_i(0) = \sum \hat{v}_i(0) = 0, \quad t \geq 0.$$

The momentum is conserved. The authors in [24] provide the following proposition:

Proposition 3.1.2. *[24] Let $(\hat{x}_i, \hat{v}_i)$ satisfy the above microscopic system of equations when $\psi \equiv 1$. Let $\hat{v}_i^a$ and $\hat{x}_i^a$ be a-th components of each vector. Then for $1 \leq i, j \leq N$ and $1 \leq a \leq d$, the followings hold: for $t \geq 0$,*

i) $\langle \hat{v}_i^a(t)\rangle = e^{-Kt}\hat{v}_i^a(0)$,

ii) $var(\hat{v}_i^a(t)) = \dfrac{D}{2K}\left(1 - \dfrac{1}{N}\right)(1 - e^{-2Kt})$,

iii) $|\langle \hat{v}_i^a(t)\rangle - \langle \hat{v}_j^a(t)\rangle| = e^{-Kt}|\langle \hat{v}_i^a(0)\rangle - \langle \hat{v}_j^a(0)\rangle|$,

iv) $\langle |\hat{v}_i^a(t) - \hat{v}_j^a(t)|^2\rangle = e^{-2Kt}|\hat{v}_i^a(0) - \hat{v}_j^a(0)|^2 + \dfrac{D}{K}(1 - e^{-2Kt})$.

From this proposition, we find the variance of the velocity perturbation does not disappear as time goes to ∞. In the next theorem, the flocking estimate in all-to-all interaction case ($\psi \equiv 1$) is covered.

Theorem 3.1.1. *[24] Assume $(\hat{x}_i, \hat{v}_i)$ be a solution to the above microscopic system of equations when $\psi \equiv 1$. Then we obtain,*

i) $\lim_{t\to\infty} |\langle \hat{v}_i(t)\rangle - \langle \hat{v}_j(t)\rangle| = 0, \quad \sup_{0\leq t<\infty} |\langle \hat{x}_i(t)\rangle - \langle \hat{x}_j(t)\rangle| < \infty$,

ii) $\lim_{t\to\infty} \mathbb{P}(|\hat{v}_i(t) - \hat{v}_j(t)|^2 > \varepsilon) \leq \dfrac{D}{K\varepsilon}$, *for any* $\varepsilon > 0$.

The authors in [24] try to extend this result to radially symmetric communication weight case when ψ has a nonnegative lower bound condition. Using energy estimates, the following theorem is derived.

Theorem 3.1.2. *[24] Let(x_i, v_i) be a solution to the microscopic system with a nonnegative communication weight $\psi(r)$, which is nonincreasing in $r \in \mathbb{R}$ and uniformly bounded below by $\psi_* \geq 0$ for any $t > 0$. Let us define $\mathcal{X}$ and $\mathcal{V}$ as $\mathcal{X}(t) := \sum_{i=1}^N \|x_i(t)\|^2$, $\mathcal{V}(t) := \sum_{i=1}^N \|v_i(t)\|^2$. Then $(\mathcal{X}, \mathcal{V})$ satisfies*

$$\begin{aligned} d\mathcal{X} &\leq 2\sqrt{\mathcal{X}}\sqrt{\mathcal{V}}dt, \quad t > 0, \\ d\mathcal{V} &\leq -2N\psi(2\mathcal{X})\mathcal{V}dt + dD\left(1 - \frac{1}{N}\right)dt \\ &+ \sqrt{D}\sum_{i=1}^N v_i \cdot \left[\left(1 - \frac{1}{N}\right)dW_i - \frac{\sqrt{D}}{N}\sum_{j \neq i} dW_j\right]. \end{aligned}$$

Moreover,

$$\langle \mathcal{V}(t) \rangle \leq \langle \mathcal{V}(0) \rangle e^{-2N\psi_* t} + \frac{dD}{2N\psi_*}\left(1 - \frac{1}{N}\right)(1 - e^{-2N\psi_* t}).$$

Remark 3.1.2. *Unless $D = 0$, the asymptotic flocking is not guaranteed. However, this theorem implies the uniform boundedness for the variance of fluctuations*

$$\lim_{t \to \infty} Var(v_i^a(t)) \leq \frac{dD}{2N\psi_*},$$

where v_i^a is the a-th component of v_i.

3.2 The Multiplicative Noise Case

The C-S flocking with multiplicative noise is discussed in [2]. The dynamics of N particles are described in terms of position and velocity, $(x_i(t), v_i(t))$, as follows:

$$\begin{cases} dx_i = v_i dt, \quad t > 0, \\ dv_i = \dfrac{K}{N}\displaystyle\sum_{j=1}^N \psi(|x_i - x_j|)(v_j - v_i)dt + D(v_i - v_e)dW(t), \end{cases} \tag{3.2.2}$$

with a $d-$dimensional constant state v_e. Here $dW(t)$ is the one-dimensional Brownian motion satisfying $\langle dW(t)\rangle = 0$, $\langle dW(t)dW(s)\rangle = \delta(t-s)$, $\langle\cdot\rangle$: ensemble average. In this system, K is interpreted as a repulsive coupling constant if $K < 0$ and a attractive coupling constant if $K > 0$. The definition of the strong stochastic flocking is expressed as

Definition 3.2.1. *[24] The system* (3.2.2) *exhibits an* ***asymptotic strong stochastic flocking*** *if and only if* $\{x_i, v_i\}, i = 1, ..., N$ *satisfy the two conditions:*

1. *For any* $1 \le i, j \le N$, *the pairwise velocity differences asymptotically vanish, i.e.,*
$$\lim_{t\to\infty} |v_i(t) - v_j(t)| = 0, a.s.$$
2. *For any* $1 \le i, j \le N$, *the diameter of a group is uniformly bounded in* t, *i.e.,*
$$\sup_{0\le t<\infty} |x_i(t) - x_j(t)| < \infty, a.s.$$

We notice that the macroscopic quantities(ensemble averages) satisfy

$$\begin{cases} dx_c = v_c dt \\ dv_c = D(v_c - v_e)dW(t). \end{cases}$$

The main macroscopic analysis is as follows:

Proposition 3.2.1. *[2] Let* (x_c, v_c) *satisfy the above system of equations. Then there exists* $T > 0$ *such that*

i) $\langle v_c(t) - v_e\rangle = 0, \quad \langle x_c(t) - x_c(0)\rangle = v_c(0)t,$

ii) $|v_c(t) - v_e| = |v_c(0) - v_e| \exp\{-(D^2/2)t + DW(t)\},$

iii) $|x_c(t) - x_c(0) - tv_e| \le C|v_c(0) - v_e|$, *where* $C = C(T, D)$.

From the proposition, we figure out the ensemble average of velocity goes to zero in time, and the distance between $x_c(t)$ and $x_c(0) + tv_e$ does not blow up at any $t > 0$.

The microscopic quantities(fluctuations), on the other hand, satisfy

$$\begin{cases} d\hat{x}_i = \hat{v}_i dt, \quad t > 0, \\ d\hat{v}_i = \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} \psi(|\hat{x}_i - \hat{x}_j|)(\hat{v}_j - \hat{v}_i)dt + D\hat{v}_i dW(t), \end{cases}$$

with zero sum constraint

$$\sum \hat{x}_i = \sum \hat{v}_i = 0, \quad t \geq 0.$$

The authors of [2] also provide the corresponding proposition as follows:

Proposition 3.2.2. *[2] Let* (x_c, v_c) *satisfy the above microscopic system of equations. Let* v_i^a *and* x_i^a *be a-th components of each vector. Then for* $1 \leq i, j \leq N$ *and* $1 \leq a \leq d$*, the followings hold:*

i) $\langle \hat{v}_i^a(t) \rangle = e^{-Kt}\hat{v}_i^a(0), \quad Var(\hat{v}_i^a(t)) = e^{-2Kt}(e^{D^2 t} - 1)(\hat{v}_i^a(0))^2,$

ii) $|\langle \hat{v}_i^a(t) \rangle - \langle \hat{v}_j^a(t) \rangle| \leq e^{-Kt}|\langle \hat{v}_i^a(0) \rangle - \langle \hat{v}_j^a(0) \rangle|,$
$\langle |\hat{v}_i^a(t) - \hat{v}_j^a(t)|^2 \rangle \leq e^{-(2K-D^2)t}|\hat{v}_i^a(0) - \hat{v}_j^a(0)|^2.$

From this proposition, we find the velocity perturbation exponentially diminishes in time provided that $K > \frac{D^2}{2}$.

Theorem 3.2.1. *[2] Suppose the coupling constant* K *and noise coefficient* D *satisfy* $K > \dfrac{D^2}{2}$*, and let* $(\hat{x}_i, \hat{v}_i)$ *be a solution to the above microscopic system with bounded initial data. Then the asymptotic strong stochastic flocking occurs. To be more specific,*

i) $\lim_{t\to\infty} |\hat{v}_i(t) - \hat{v}_j(t)| = 0, \quad \sup_{0\leq t<\infty} |\hat{x}_i(t) - \hat{x}_j(t)| < \infty,$ *and*

ii) $\lim_{t\to\infty} \mathbb{P}(|\hat{v}_i(t) - \hat{v}_j(t)|^2 > \varepsilon) = 0,$ *for any* $\varepsilon > 0$*,*

hold.

In the next theorem, under radially symmetric communication weight ψ with nonnegative lower bound condition, the asymptotic strong stohastic flocking of (x_i, v_i) is derived.

Theorem 3.2.2. *[2] Let*$(\hat{x}_i, \hat{v}_i)$ *be a solution of the microscopic system with a nonnegative communication weight* $\psi(r)$, *which is nonincreasing in* $r \in \mathbb{R}$ *and bounded below by* $\bar{\psi}_* \geq 0$. *Then the asymptotic strong stochastic flocking occurs:*

i) $\lim_{t\to\infty} |v_i(t) - v_j(t)| = 0$, *for any* $1 \leq i, j \leq N$,

ii) $\sup_{0\leq t<\infty} |x_i(t) - x_j(t)| < \infty$, *for any* $1 \leq i, j \leq N$.

Chapter 4

Wellposedness of the Cucker-Smale-Fokker-Planck Equation

In this chapter, we aim to prove the global existence of classical solution to the CS-FP equation. We first present a priori $H^k_\alpha-$estimates and then show the local existence of the solution to extend the argument to the global sense.

Furthermore, the estimates of a solution to the Cucker-Smale-Mckean-Vlasov equation will be also discussed. The contents of this chapter are based on a joint work with Ha, S-Y, Noh, S-E, and Xiao, Q-H [20].

4.1 Estimates of Classical Solutions

In this section, the global existence of classical solutions to the Cauchy problem of the CS-FP equation with $v^c = 0$ will be mainly verified. Consider the Cauchy problem as follows:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v(L[f]f) = \sigma \Delta_v (|v|^2 f), \quad x, v \in \mathbb{R}^d, \ t > 0, \\ L[f](x,v,t) = -K \displaystyle\int_{\mathbb{R}^{2d}} \psi(|x-y|)(v-v_*) f(y,v_*,t) dv_* dy, \\ f(x,v,0) = f_0(x,v). \end{cases} \tag{4.1.1}$$

We begin this section by recalling the definition of a classical solution to (4.1.1).

Definition 4.1.1. *Let $f = f(x, v, t)$ be a classical solution in $\mathbb{R}^d \times \mathbb{R}^d \times I$, for an interval $I \subset (0, \infty)$, of (4.1.1) with a nonnegative datum f_0 if and only if the following conditions hold.*

1. *f is continuous in $\mathbb{R}^d \times \mathbb{R}^d \times I$ and continuously differentiable once with respect to (x, t) and continuously differentiable twice with respect to v.*
2. *For all $x, v \in \mathbb{R}^d$ and $t \in I$,*
$$(y, v_*) \to \psi(|x-y|)(v-v_*)f(y, v_*, t) \in L^1(\mathbb{R}^d \times \mathbb{R}^d).$$
3. *f satisfies the equation in (4.1.1) in a pointwise sense and $f|_{t=0} = f_0$.*

Let us now define a function space in which we will look for a classical solution. For $T > 0$, we set

$$\mathcal{X}_{k,\alpha}(T) := \left\{ f \in \mathcal{C}(0, T; (H^k_\alpha \cap L^1_2)(\mathbb{R}^{2d})) \ : \ \sup_{t \in [0,T)} (||f(t)||_{H^k_\alpha} + ||f(t)||_{L^1_2}) < \infty \right\}.$$

The global existence of classical solutions is one of the main results in this thesis. This is described as follows:

Theorem 4.1.1. *Let $T \in (0, \infty)$ be a given constant. We assume that the initial datum f_0 satisfies*

$$f_0 \in H^k_\alpha \cap L^1_2(\mathbb{R}^{2d}), \quad \text{for some positive constants } k > 2 + d \text{ and } \alpha > \frac{d+2}{2}.$$

Then, there exists a unique global classical solution to the Cauchy problem in (4.1.1) in the function space $\mathcal{X}_{k,\alpha}(T)$.

Remark 4.1.1. *The standard Sobolev imbedding theorem implies that*

$$||f(t)||_{H^k} < \infty, \ k > 2 + d \quad \Longrightarrow \quad ||f(t)||_{\mathcal{C}^2} < \infty$$

holds. The unique solution in Theorem 4.1.1 is a classical solution we look for.

In the next two subsections, we exhibit a priori estimates and the local existence to prove Theorem 4.1.1.

4.1.1 A priori Estimates

In this subsection, we study the a priori $H^k_\alpha-$ estimates for the solution to the Cauchy problem in (4.1.1). For simplicity, we introduce simplified notation for the $L^2_{x,v}$-norm as follows:

$$\|g\| := \|g\|_{L^2}, \quad \|g\|_\alpha := \|g\|_{L^2_\alpha}.$$

Lemma 4.1.1. (Zeroth-order estimate) *Let a constant $T \in (0, \infty]$ be given. We denote $f \in \mathcal{X}_{k,\alpha}(T)$ as a classical solution to (4.1.1) in $[0,T)$. There is a constant $C_0 = C_0(d, K, \sigma, \alpha, \|\psi\|_{L^\infty}, \|f_0\|_{L^1})$ satisfying*

$$\|f\|_\alpha^2 + \sigma \int_0^t \||v|\nabla_v f(s)\|_\alpha^2 ds \le C_0 e^{C_0 t}\|f_0\|_\alpha^2, \quad t \in [0,T).$$

Proof. In the first chapter, we defined $\|f\|_\alpha$ as

$$\|f\|_\alpha^2 = \int_{\mathbb{R}^{2d}} (1+|v|^2)^\alpha |f|^2 dv dx.$$

We recall the CS-FP equation (4.1.1) in $[0,T)$ as follows:

$$\begin{aligned}
&\partial_t f + v\cdot\nabla_x f + \nabla_v \cdot (L[f]f) = \sigma \Delta_v (|v-v^c|^2 f), \quad x, v \in \mathbb{R}^d,\ 0<t<T,\\
&L[f](x,v,t) = -K\int_{\mathbb{R}^{2d}} \psi(|x-y|)(v-v_*)f(y,v_*,t)dv_* dy,
\end{aligned}$$

subject to an initial datum

$$f(x,v,0) = f_0(x,v), \quad \int_{\mathbb{R}^{2d}} f_0 = 1.$$

We now present the estimate in two cases.

Case 1 ($\alpha = 0$): Multiplying (4.1.1) by $2f$, we obtain

$$\begin{aligned}
&\partial_t\left(f^2\right) + \nabla_x\cdot\left(vf^2\right) + \nabla_v\cdot\left[f^2 L[f] - 2\sigma f\nabla_v(|v|^2 f)\right] + 2\sigma|v|^2|\nabla_v f|^2\\
&= f^2\nabla_v\cdot L[f] - 4\sigma f v\cdot\nabla_v f
\end{aligned} \tag{4.1.2}$$

from (4.1.1). Applying the Cauchy–Schwarz inequality and the result in Lemma 2.2.2(i) to the integration of (4.1.2) with respect to (x, v), an inequality is derived as follows:

$$\begin{aligned}\frac{d}{dt}\|f\|^2 + 2\sigma\||v|\nabla_v f\|^2 &\leq \int_{\mathbb{R}^{2d}} f^2|\nabla_v \cdot L[f]|dvdx + 4\sigma\int_{\mathbb{R}^{2d}} |fv\cdot\nabla_v f|dvdx \\ &\leq dK\|f_0\|_{L^1}\|\psi\|_{L^\infty}\|f\|^2 + \sigma\||v|\nabla_v f\|^2 + 4\sigma\|f\|^2.\end{aligned}$$

Furthermore, the Gronwall's inequality for $\|f\|^2$ yields

$$\frac{d}{dt}\|f\|^2 + \sigma\||v|\nabla_v f\|^2 \leq \left((dK\|f_0\|_{L^1}\|\psi\|_{L^\infty} + 4\sigma)t\right)\|f\|^2, \quad t \in (0, T),$$

which implies

$$\|f(t)\|^2 \leq \|f_0\|^2 e^{(dK\|f_0\|_{L^1}\|\psi\|_{L^\infty}+4\sigma)t}. \tag{4.1.3}$$

Case 2 ($\alpha > 0$): Let us now consider the weighted estimate. We, in this case, multiply (4.1.2) by $(1+|v|^2)^\alpha$, so that the following is acquired:

$$\begin{aligned}&\partial_t\left[(1+|v|^2)^\alpha f^2\right] + 2\sigma|v|^2(1+|v|^2)^\alpha|\nabla_v f|^2 + 8\sigma\alpha|v|^2(1+|v|^2)^{\alpha-1}f^2 \\ &\quad +\nabla_x\cdot\left[v(1+|v|^2)^\alpha f^2\right] + \nabla_v\cdot\left\{(1+|v|^2)^\alpha\left[Kf^2L[f] - 2\sigma f\nabla_v(|v|^2 f)\right]\right\} \\ &\quad = (1+|v|^2)^\alpha f^2\nabla_v\cdot L[f] + 2\alpha(1+|v|^2)^{\alpha-1}f^2 v\cdot L[f] \\ &\quad -4\sigma(1+|v|^2)^{\alpha-1}[1+(1+\alpha)|v|^2]fv\cdot\nabla_v f.\end{aligned}$$

Integrating the above equality with respect to (x, v) and using the simplified notation for $\|f\|_\alpha$, we derive the following inequality:

$$\begin{aligned}&\frac{d}{dt}\|f\|_\alpha^2 + 2\sigma\||v|\nabla_v f\|_\alpha^2 + 8\sigma\alpha\||v|f\|_{\alpha-1}^2 \\ &\qquad \leq K\|f\|_\alpha^2\|\nabla_v\cdot L[f]\|_{L^\infty} + 2K\alpha\int_{\mathbb{R}^{2d}}(1+|v|^2)^{\alpha-1}f^2|v\cdot L[f]|dvdx \\ &\qquad + 4\sigma(1+\alpha)\|f\|_\alpha\||v|\nabla_v f\|_\alpha \\ &\qquad =: \mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{13}.\end{aligned} \tag{4.1.4}$$

(Estimates of $\mathcal{J}_{1i},\ i = 1, 2, 3$): From the result in lemma 2.2.2 (i) and the Cauchy-Schwartz inequality, we derive

$$\mathcal{J}_{11} + \mathcal{J}_{13} \leq K\|\psi\|_{L^\infty}\|f_0\|_{L^1}\|f\|_\alpha^2 + \sigma\||v|\nabla_v f\|_\alpha^2 + 4\sigma(1+\alpha)^2\|f\|_\alpha^2. \quad (4.1.5)$$

In order to deal with the second term $\mathcal{J}_{12}$, we combine the definition of $L[f]$, Lemma 2.2.2 and (4.1.3), so that we obtain the following estimate:

$$\begin{aligned}
\mathcal{J}_{12} &= 2K\alpha \int_{\mathbb{R}^{2d}} (1+|v|^2)^{\alpha-1} f^2 v \cdot \left[\int_{\mathbb{R}^{2d}} \psi(|x-y|)(v_* - v) f(y, v_*, t) dv_* dy\right] dvdx \\
&\leq 2K\alpha \int_{\mathbb{R}^{2d}} (1+|v|^2)^{\alpha-1} f^2 |v| \left[\int_{\mathbb{R}^{2d}} \psi(|x-y|)|v_*| f(y, v_*, t) dv_* dy\right] dvdx \\
&\leq 2K\alpha\|\psi\|_{L^\infty}\|f\|_\alpha^{\frac{2\alpha-1}{\alpha}}\|f\|^{\frac{1}{\alpha}}\||v|f\|_{L^1} \\
&\leq 2K\alpha\|\psi\|_{L^\infty}\sqrt{\|f_0\|_{L^1}}\|f\|_\alpha^{\frac{2\alpha-1}{\alpha}}\left[\sqrt{\||v|^2 f_0\|_{L^1}^\alpha} e^{-\frac{K_m t}{2}}\|f\|\right]^{\frac{1}{\alpha}} \\
&\leq 2K\alpha\|\psi\|_{L^\infty}\sqrt{\|f_0\|_{L^1}}\left(\|f\|_\alpha^2 + \||v|^2 f_0\|_{L^1}^\alpha \|f_0\|^2 e^{(dK\|f_0\|_{L^1}\|\psi\|_{L^\infty} + 4\sigma - \alpha K_m)t}\right), \qquad (4.1.6)
\end{aligned}$$

where we used (4.1.3) and the generalized Hölder's inequality in the last line. To complete the estimate, we gather (4.1.5) and (**??**) in (4.1.4) to acquire

$$\begin{aligned}
&\frac{d}{dt}\|f\|_\alpha^2 + \sigma\||v|\nabla_v f\|_\alpha^2 \\
&\quad \leq \left[4\sigma(1+\alpha)^2 + K\|\psi\|_{L^\infty}(\|f_0\|_{L^1} + 2\alpha\sqrt{\|f_0\|_{L^1}})\right]\|f\|_\alpha^2 \\
&\quad + 2K\alpha\|\psi\|_{L^\infty}\sqrt{\|f_0\|_{L^1}}\||v|^2 f_0\|_{L^1}^\alpha\|f_0\|^2 e^{(dK\|f_0\|_{L^1}\|\psi\|_{L^\infty} + 4\sigma - \frac{\alpha K_m}{2})t}.
\end{aligned}$$

Using the Gronwall's inequality and choosing proper C_0 lead to the desired result. Here C_0 may depend on $K, \sigma, d, \alpha, \|\psi\|_{L^\infty}$, and $\|f_0\|_{L^1}$. □

Lemma 4.1.2. (First-order estimate) *Let a constant $T \in (0, \infty]$ be given. Denote $f \in \mathcal{X}_{k,\alpha}(T)$ as a classical solution to (4.1.1)-*(5.1.3) *in the time interval $[0, T)$. Then, there is a constant $C_1 = C_1(K, \sigma, d, \alpha, \|\psi\|_{L^\infty}, \|\psi'\|_{L^\infty}, \|f_0\|_{L^1}, \||v|^2 f_0\|_{L^1})$ satisfying*

$$\|\nabla_x f\|^2 + \|\nabla_v f\|^2 + \sigma\int_0^t \left(\||v|\nabla_v\nabla_x f\|_\alpha^2 + \||v|\nabla_v^2 f\|^2\right)(s) ds \leq C_1 e^{C_1 t}\|f_0\|_{H^1},$$

for $t \in (0, T)$.

Proof. **Case 1** (estimate of $\nabla_x f$): First, we take the spatial gradient ∇_x to (4.1.1) to obtain the following partial differential equation:

$$\partial_t \nabla_x f + \nabla_x \cdot (v \nabla_x f) + \nabla_v \cdot (L(f) \nabla_x f) + \nabla_v \cdot ((\nabla_x L(f)) f) = \sigma \Delta_v (|v|^2 \nabla_x f). \tag{4.1.7}$$

Using the dot product, we multiply the above relation in (4.1.7) by $2\nabla_x f$ and acquire

$$\begin{aligned}
&\partial_t \left(|\nabla_x f|^2\right) + 2\sigma |v|^2 |\nabla_v \nabla_x f|^2 + \nabla_x \cdot \left(v |\nabla_x f|^2\right) \\
&+ \nabla_v \cdot \left(|\nabla_x f|^2 L[f] - 2\sigma \nabla_v \left(|v|^2 \nabla_x f\right) \cdot \nabla_x f\right) \\
&- \nabla_v \cdot \left\{ 2K \left[f \nabla_x f \cdot \int_{\mathbb{R}^{2d}} v \nabla_x \psi(|x-y|) f(y, v_*, t) dv_* \right] dy \right\} \\
&= |\nabla_x f|^2 \nabla_v \cdot L[f] - 4\sigma \nabla_x f \cdot (v \cdot \nabla_v) \nabla_x f \\
&- 2K \nabla_v \nabla_x f \cdot \left[f \int_{\mathbb{R}^{2d}} v \nabla_x \psi(|x-y|) f(y, v_*, t) dv_* dy \right] \\
&- 2K \nabla_x f \cdot \int_{\mathbb{R}^{2d}} \left(\nabla_v f \cdot v_*\right) \nabla_x \psi(|x-y|) f(y, v_*, t) dv_* dy.
\end{aligned} \tag{4.1.8}$$

We integrate (4.1.8) over $(x, v) \in \mathbb{R}^{2d}$ and apply Lemma 2.2.2 to derive

$$\begin{aligned}
&\frac{d}{dt} \|\nabla_x f\|^2 + 2\sigma \||v| \nabla_v \nabla_x f\|^2 \\
&\qquad \leq \|\nabla_v \cdot L[f]\|_{L^\infty} \|\nabla_x f\|^2 + 4\sigma \|\nabla_x f\| \||v| \nabla_v \nabla_x f\| \\
&\qquad + \ 2K \|\psi'\|_{L^\infty} \|f_0\|_{L^1} \int_{\mathbb{R}^{2d}} |v| f |\nabla_v \nabla_x f| dv dx \\
&\qquad + \ 2K \|\psi'\|_{L^\infty} \||v| f\|_{L^1} \int_{\mathbb{R}^{2d}} |\nabla_v f \cdot \nabla_x f| dv dx \\
&\qquad =: \mathcal{J}_{21} + \mathcal{J}_{22} + \mathcal{J}_{23} + \mathcal{J}_{24}.
\end{aligned} \tag{4.1.9}$$

Using the result in Lemma 2.2.2 (i) and the Cauchy–Schwarz inequality, we verify that

$$\begin{aligned}
&\mathcal{J}_{21} \leq K \|\psi\|_{L^\infty} \|f_0\|_{L^1} \|\nabla_x f\|^2, \\
&\mathcal{J}_{22} \leq \frac{\sigma}{2} \||v| \nabla_v \nabla_x f\|^2 + 8\sigma \|\nabla_x f\|^2.
\end{aligned} \tag{4.1.10}$$

For $\mathcal{J}_{23}$, Young's inequality with $p = q = 2$ is applied to obtain

$$\mathcal{J}_{23} \leq \frac{\sigma}{2}\||v|\nabla_v\nabla_x f\|^2 + \frac{2K^2\|\psi'\|_{L^\infty}^2\|f_0\|_{L^1}^2}{\sigma}\|f\|^2. \tag{4.1.11}$$

For $\mathcal{J}_{24}$, Lemma 2.2.2 (ii) is used in order to find

$$\mathcal{J}_{24} \leq K\|\psi'\|_{L^\infty}\sqrt{\|f_0\|_{L^1}\||v|^2 f_0\|_{L^1}}e^{-\frac{K_m t}{2}}(\|\nabla_x f\|^2 + \|\nabla_v f\|^2). \tag{4.1.12}$$

We put (4.1.10), (4.1.11), and (4.1.12) altogether into (4.1.9) and acquire

$$\begin{aligned}\frac{d}{dt}\|\nabla_x f\|^2 + \sigma\||v|\nabla_v\nabla_x f\|^2 \leq& \left(\|\nabla_x f\|^2 + \|\nabla_v f\|\right)\\ &\times\left\{K\|\psi\|_{L^\infty}\|f_0\|_{L^1} + 8\sigma + K\|\psi'\|_{L^\infty}\sqrt{\|f_0\|_{L^1}\||v|^2 f_0\|_{L^1}}e^{-\frac{K_m t}{2}}\right\}\\ &+ \frac{2K^2\|\psi'\|_{L^\infty}^2\|f_0\|_{L^1}^2}{\sigma}\|f\|^2.\end{aligned} \tag{4.1.13}$$

Case 2 (estimate of $\nabla_v f$): Analogous to the **Case 1**, the partial differential equation for $\partial_{v_i} f$ is derived as

$$\partial_t\left(\partial_{v_i} f\right) + \partial_{x_i} f + \nabla_x \cdot \left(v\partial_{v_i} f\right) + \nabla_v \cdot \left(\partial_{v_i}\left(L[f]f\right)\right) = \sigma\Delta_v\left(\partial_{v_i}\left(|v|^2 f\right)\right). \tag{4.1.14}$$

After we multiply (4.1.14) by $2\partial_{v_i} f$, integrate over $\mathbb{R}^{2d}$, and sum up from $i = 1$ to $i = d$, we obtain

$$\begin{aligned}\partial_t\|\nabla_v f\|^2 &+ \sum_{i=1}^{d}\int_{\mathbb{R}^{2d}} 2(\partial_{v_i} f)(\partial_{x_i} f)dvdx\\ &= -\sum_{i=1}^{d}\int_{\mathbb{R}^{2d}} 2(\partial_{v_i} f)\nabla_v(\partial_{v_i}(L[f]f))dvdx\\ &+ \sum_{i=1}^{d}\int_{\mathbb{R}^{2d}} 2\sigma(\partial_{v_i} f)\Delta_v\left(\partial_{v_i}(|v|^2 f)\right)dvdx =: -\mathcal{J}_{31} + \mathcal{J}_{32}.\end{aligned} \tag{4.1.15}$$

(Estimate of $\mathcal{J}_{31}$): We can notice the integrand of $\mathcal{J}_{31}$ can be simplified as

$$\begin{aligned}
&2(\partial_{v_i} f)\Big(L[f]\cdot\nabla_v(\partial_{v_i} f)+(\partial_{v_i} f)\nabla_v\cdot L[f]+\partial_{v_i}L[f]\cdot\nabla_v f\Big)\\
&\quad= L[f]\cdot\nabla_v(|\partial_{v_i} f|^2)+2|\partial_{v_i} f|^2\nabla_v\cdot L[f]+2(\partial_{v_i} f)(\partial_{v_i}L[f])\cdot\nabla_v f\\
&\quad=\nabla_v\cdot\big(|\partial_{v_i} f|^2 L[f]\big)+|\partial_{v_i} f|^2\nabla_v\cdot L[f]+2(\partial_{v_i} f)(\partial_{v_i}L[f])\cdot\nabla_v f.
\end{aligned}\tag{4.1.16}$$

Using (4.1.16) and Lemma 2.2.2, we find

$$\begin{aligned}
|\mathcal{I}_{31}| &\le \|\nabla_v f\|^2\|\nabla_v\cdot L[f]\|_{L^\infty}+2\|\nabla_v\cdot L[f]\|_{L^\infty}\|\nabla_v f\|^2\\
&\le 3dK\|\psi\|_{L^\infty}\|f_0\|_{L^1}\|\nabla_v f\|^2.
\end{aligned}\tag{4.1.17}$$

(Estimate of $\mathcal{J}_{32}$): We use integration by parts technique over and over to acquire

$$\begin{aligned}
\mathcal{I}_{32} &= -2\sigma\sum_{i=1}^{d}\int_{\mathbb{R}^{2d}}\nabla_v(\partial_{v_i} f)\cdot\nabla_v\left(\partial_{v_i}(|v|^2 f)\right)dvdx\\
&= -2\sigma\sum_{i=1}^{d}\int_{\mathbb{R}^{2d}}\nabla_v(\partial_{v_i} f)\cdot\big(\nabla_v(2v_i)f+2v_i\nabla_v f+2v\partial_{v_i} f+|v|^2\nabla_v\partial_{v_i} f\big)\,dvdx\\
&= -2\sigma\sum_{i=1}^{d}\int_{\mathbb{R}^{2d}}2(\partial_{v_i}^2 f)f+\nabla_v(\partial_{v_i} f)\cdot\big(2v_i\nabla_v f+2v\partial_{v_i} f+|v|^2\nabla_v\partial_{v_i} f\big)\,dvdx\\
&= -2\sigma\sum_{i=1}^{d}\int_{\mathbb{R}^{2d}}-2(\partial_{v_i} f)^2+\nabla_v(\partial_{v_i} f)\cdot\big(2v_i\nabla_v f+2v\partial_{v_i} f+|v|^2\nabla_v\partial_{v_i} f\big)\,dvdx\\
&\le 4\sigma\|\nabla_v f\|^2+8\sigma\|\nabla_v f\|\||v|\nabla_v^2 f\|-2\sigma\||v|\nabla_v^2 f\|^2.
\end{aligned}\tag{4.1.18}$$

Replacing (4.1.15) by (4.1.17) and (4.1.18), we find

$$\begin{aligned}
&\frac{d}{dt}\|\nabla_v f\|^2+2\sigma\||v|\nabla_v^2 f\|^2\\
&\quad\le\|\nabla_v f\|^2+\|\nabla_x f\|^2+(3dK\|\psi\|_{L^\infty}\|f_0\|_{L^1}+4\sigma)\|\nabla_v f\|^2+8\sigma\|\nabla_v f\|\||v|\nabla_v^2 f\|\\
&\quad\le\sigma\||v|\nabla_v^2 f\|^2+(1+3dK\|\psi\|_{L^\infty}\|f_0\|_{L^1}+20\sigma)\|\nabla_v f\|^2+\|\nabla_x f\|^2.
\end{aligned}$$

This implies

$$\begin{aligned} &\frac{d}{dt}\|\nabla_v f\|^2 + \sigma\||v|\nabla_v^2 f\|^2 \\ &\qquad \leq (1 + 3dK\|\psi\|_{L^\infty}\|f_0\|_{L^1} + 20\sigma)\|\nabla_v f\|^2 + \|\nabla_x f\|^2. \end{aligned} \tag{4.1.19}$$

Using Lemma 4.1.1 and a suitable constant C_1, we get the desired estimate.

□

With analogous estimates to Lemma 4.1.1 or Lemma 4.1.2, higher-order estimates can also be acquired. Putting all the estimates together, we obtain a priori H^k_α−estimates as follows:

Proposition 4.1.1. (A priori H^k_α estimates) *Let a constant* $T \in (0, \infty]$ *be given. Let* $f \in \mathcal{X}_{k,\alpha}(T)$ *be a classical solution to (4.1.1) in* $[0, T)$. *Then, for an integer* k *in the range of* 0 *to* N, *there is constants* $\{C_k\}_{k=0}^N$ *such that* $C_k = C_k(K, \sigma, d, \alpha, \|\psi\|_{L^\infty}, \dots, \|\psi^{(k)}\|_{L^\infty}, \|f_0\|_{L^1}, \||v|^2 f_0\|_{L^1})$ *satisfying*

$$\|f\|^2_{H^k_\alpha} \leq C_k e^{C_k t}\|f_0\|^2_{H^k_\alpha}, \quad t \in [0, T).$$

4.2 A Local Existence Result

In this section, we approach the local existence of (4.1.1) by approximating through known results. The local and global existence theory for Vlasov–Fokker–Planck-type equations with constant diffusion coefficients is extensively studied in many literatures [7, 16, 38, 42]. We begin this section by introducing the idea of the local existence theory and compares our problem with the standard Vlasov–Fokker–Planck equation with constant diffusion coefficients. Notice a degenerate and variable diffusion coefficient term $\sigma|v|^2$ in (4.1.1). This is a difficulty why the direct application of the standard existence theory [18] for parabolic equations does not work here.

We build approximate solutions $\{f^n\}_{n=0}^\infty$ using a successive iteration scheme as follows. At $n = 0$, we initially put

$$f^0(x, v, t) := f_0(x, v), \quad x, v \in \mathbb{R}^d, \ t > 0. \tag{4.2.20}$$

Next to the initial step, let us suppose the approximate solutions $\{f^{N-1}\}_{N\geq 1}$ be created. The N-iterated function f^N is then set to be the unique solution of the linear parabolic equation

$$\begin{cases}\partial_t f^N + v\cdot\nabla_x f^N + \nabla_v\cdot(L[f^{N-1}]f^N) = \sigma\Delta_v(|v|^2 f^N),\\ f^N(x,v,0) = f_0(x,v).\end{cases} \tag{4.2.21}$$

Granted that the unique solvability of the linear Vlasov–Fokker–Planck equation in (4.2.21) is widely verified, for example, in [7], we consider the well-definedness of $\{f^N\}$ with (4.2.20) and (4.2.21) to be true without proof. To sum up, the existence of a local solution in $\mathcal{X}_k(T)$ is described as the following:

Proposition 4.2.1. (Local existence of a classical solution) *Let f_0 be a non-negative and sufficiently regular initial datum satisfying the following condition:*

$$f_0 \in (H^k_\alpha \cap L^1_2)(\mathbb{R}^{2d}), \quad \alpha > 2+d.$$

Then there are a positive constant C and a sufficiently small constant $T_* > 0$, which the Cauchy problem in (4.1.1) admits a unique solution $f \in \mathcal{X}_{k,\alpha}(T_*)$, such that

$$\begin{aligned}&(i)\ f(x,v,t)\geq 0, \quad f(\cdot,\cdot,t)\in L^1_2(\mathbb{R}^{2d}), \quad x,v\in\mathbb{R}^d,\ 0\leq t< T_*,\\ &(ii)\ \sup_{0\leq t\leq T_*}\|f(t)\|_{H^k_\alpha} \leq C\|f_0\|_{H^k_\alpha}.\end{aligned}$$

Proof. We provide a brief sketch of the proof in the following.

Step 1 Let us denote a sequence of the approximate solutions as $\{f^N\}_{N=0}^\infty$, which is created by using the iteration scheme in (4.2.20)–(4.2.21). Let an integer $N\geq 0$ and a small constant $\epsilon > 0$ be given. For each N and ϵ, we include a viscosity term on the right side of the equation (4.2.21) and construct

$$\begin{aligned}&\partial_t \tilde{f}^N + v\cdot\nabla_x \tilde{f}^N + \nabla_v\cdot(L[f^{N-1}]\tilde{f}^N) = \sigma\Delta_v(|v|^2\tilde{f}^N) + \epsilon(\triangle_x \tilde{f}^N + \triangle_v \tilde{f}^N),\\ &\tilde{f}^N(x,v,0) = f_0(x,v).\end{aligned}$$

From the maximum principle of the uniformly parabolic equation, the nonnegativity of the solution to the above equation can be shown. The limit $\epsilon \to 0$ in this equation implies that the solution f^N to (4.2.21) is nonnegative, i.e., $f^N(x,v,t) \geq 0$ for $x, v \in \mathbb{R}^{2d},\ t > 0$. Furthermore, $H^k(\mathbb{R}^{2d}) \hookrightarrow C^{1,2,1}_{x,v,t}$ for $k > d+2$ holds and the corresponding strong pointwise limit function of f^N also preserves the nonnegativity.

Step 2 We can claim that there exist positive constants t_1 and t_2 with $0 < t_2 \leq t_1$ such that

$$\begin{aligned}
&\|f^N\|_{L^1} = \|f_0\|_{L^1}, \quad \sup_{0\leq t\leq t_1} \||v|^2 f^N(t)\|_{L^1} \leq 2\||v|^2 f_0\|_{L^1}, \quad N \geq 0,\\
&\sup_{0\leq t\leq t_1} \|f^N(t)\|_{H^k_\alpha} \leq 2\|f_0\|_{H^k_\alpha}, \qquad \text{and}\\
&\sup_{0\leq t\leq t_2} \|(f^{N+1} - f^N)(t)\|_{H^k_\alpha} \leq \frac{1}{2} \sup_{0\leq t\leq t_2} \|(f^N - f^{N-1})(t)\|_{H^k_\alpha}.
\end{aligned} \tag{4.2.22}$$

This is similar to Lemma 2.2.1, 2.2.2 and a priori estimates in Proposition 4.1.1.

Proof of claim (4.2.22)*:* The approximate solution f^n to (4.2.21) satisfy

$$\|f^N\|_{L^1} = \|f_0\|_{L^1},$$

and

$$\begin{aligned}
\||v|^2 f^{N+1}\|_{L^1} \leq{}& e^{(2d\sigma + 3K\|\psi\|_{L^\infty}\|f_0\|_{L^1})t} \||v|^2 f_0\|_{L^1}\\
&+ K\|\psi\|_{L^\infty}\|f_0\|_{L^1} \int_0^t e^{(2d\sigma+3K\|\psi\|_{L^\infty}\|f_0\|_{L^1})(t-s)} \||v|^2 f^N(s)\|_{L^1} ds.
\end{aligned}$$

Inductively, we derive the following inequality:

$$\begin{aligned}
&\||v|^2 f^{N+1}\|_{L^1}\\
&\leq e^{(2d\sigma+3K\|\psi\|_{L^\infty}\|f_0\|_{L^1})t} \||v|^2 f_0\|_{L^1} + \sum_{i=1}^\infty \frac{(K\|\psi\|_{L^\infty}\|f_0\|_{L^1})^i}{i!} \||v|^2 f_0\|_{L^1}\\
&\leq \left(e^{\left(2d\sigma+3K\|\psi\|_{L^\infty}\|f_0\|_{L^1}\right)t} + e^{K\|\psi\|_{L^\infty}\|f_0\|_{L^1} t} - 1 \right) \||v|^2 f_0\|_{L^1}\\
&\leq e^{(2d\sigma+4K\|\psi\|_{L^\infty}\|f_0\|_{L^1})t} \||v|^2 f_0\|_{L^1}.
\end{aligned}$$

Refer to the proof of Proposition 4.1.1 for noticing that $L[f^{N-1}]$ of (4.2.21) is estimated by $\|f^{N-1}\|_{L^1}$ with $\||v|^2 f^{N-1}\|_{L^1}$ which can be all uniformly estimated as shown. Repeatedly applying analogous computations to the proof of proposition 4.1.1, we choose a positive constant t_0 satisfying

$$\|f^N(t)\|_{H^k}^2 \le \tilde{C}_k e^{\tilde{C}_k t}\|f_0\|_{H^k}^2$$

for any t with $t < t_0$. In this inequality, the value of $\tilde{C}_k$ is depending on $K, \sigma, d, \alpha, \|\psi\|_{L^\infty}, \dots, \|\psi^{(k)}\|_{L^\infty}, \|f_0\|_{L^1},$ and $\||v|^2 f_0\|_{L^1}$. A small number t_1 is then decided such that

$$\||v|^2 f^{N+1}\|_{L^1} \le 2\||v|^2 f_0\|_{L^1}, \qquad \|f^N(t)\|_{H^k_\alpha} \le 2\|f_0\|_{H^k_\alpha}$$

for any $t \in [0, t_1]$.

On the other hand, $(f^{N+1} - f^N)$ satisfies the following equation

$$\begin{cases} \partial_t(f^{N+1} - f^N) + v \cdot \nabla_x(f^{N+1} - f^N) + \nabla_v \cdot (L[f^{N-1}](f^{N+1} - f^N)) \\ +\nabla_v \cdot (L[f^N - f^{N-1}]f^N) = \sigma\Delta_v(|v|^2(f^{N+1} - f^N)), \\ (f^{N+1} - f^N)(x, v, 0) = 0. \end{cases} \tag{4.2.23}$$

We take a similar idea from the proof in Proposition 4.2.1 and derive this estimate:

$$\begin{aligned} \frac{d}{dt}\|(f^{N+1} - f^N)\|_{H^k_\alpha}^2 + \sigma\||v|\nabla_v(f^{N+1} - f^N)\|_{H^k_\alpha}^2 \\ \le \bar{C}_1\|(f^{N+1} - f^N)\|_{H^k_\alpha}^2 \\ +\bar{C}_2\|f^N\|_{H^k_\alpha}\|(f^{N+1} - f^N)\|_{H^k_\alpha}\|(1 + |v|)(f^N - f^{N-1})\|_{L^1}, \end{aligned}$$

where constants $\bar{C}_1$ and $\bar{C}_2$ are chosen depending on $K, \sigma, d, \alpha, \|\psi\|_{L^\infty}, \dots,$ $\|\psi^{(k)}\|_{L^\infty}, \|f_0\|_{L^1},$ and $\||v|^2 f_0\|_{L^1}$.
We note that for $\alpha > \frac{d+2}{2}$,

$$\begin{aligned} &\|(1 + |v|)(f^N - f^{N-1})\|_{L^1} \\ &\le \left[\int_{\mathbb{R}^d}(1 + |v|^2)^{1-\alpha}dv \int_{\mathbb{R}^{2d}}(1 + |v|^2)^\alpha dv|f^N - f^{N-1}|^2 dvdx\right]^{\frac{1}{2}} \\ &\le C(\alpha)\|f^N - f^{N-1}\|_{L^2_\alpha} \end{aligned} \tag{4.2.24}$$

holds.
From this result, we find

$$2\frac{d}{dt}\|(f^{N+1} - f^N)\|_{H^k_\alpha} \le \bar{C}_1\|(f^{N+1} - f^N)\|_{H^k_\alpha} + C(\alpha)\bar{C}_2\|f^N\|_{H^k_\alpha}\|(f^N - f^{N-1})\|_{L^2_\alpha}.$$

Taking $(f^{N+1} - f^N)(x, v, 0) = 0$ and $\|f^N(t)\|^2_{H^k} \le e^{\tilde{C}_k t}\|f_0\|^2_{H^k}$ into account, this inequality implies

$$\begin{aligned}
&\sup_{0\le t\le t_2} \|(f^{N+1} - f^N)(t)\|_{H^k_\alpha} \\
&\qquad \le \frac{C(\alpha)\bar{C}_2}{2} \sup_{0\le t\le t_2} e^{\frac{(\bar{C}_1+\tilde{C}_k)t}{2}} \int_0^t e^{-\frac{\bar{C}_1 s}{2}} ds \sup_{0\le t\le t_2} \|(f^N - f^{N-1})(t)\|_{H^k_\alpha} \\
&\qquad \le \frac{C(\alpha)\bar{C}_2}{2} \sup_{0\le t\le t_2} te^{\frac{(\bar{C}_1+\tilde{C}_k)t}{2}} \sup_{0\le t\le t_2} \|(f^N - f^{N-1})(t)\|_{H^k_\alpha} \\
&\qquad \le \frac{1}{2} \sup_{0\le t\le t_2} \|(f^N - f^{N-1})(t)\|_{H^k_\alpha}.
\end{aligned}$$

Choosing $0 < t_2 \le t_1$ sufficiently small, we can make

$$\frac{C(\alpha)\bar{C}_2}{2} t_2 e^{\frac{(\bar{C}_1+\tilde{C}_k)t_2}{2}} \le \frac{1}{2},$$

and this means the sequence $\{f^N\}_{N=1}^\infty$ is Cauchy in H^k_α for $t \in [0, t_2]$. Combined with the nonnegativity of $\{f^N\}_{N=1}^\infty$, this result suggests that the corresponding limit function $f \in \mathcal{X}_{k,\alpha}(t_2)$ satisfies $f \ge 0$ since

$$H^k(\mathbb{R}^{2d}) \hookrightarrow C^{1,2,1}_{x,v,t} \text{ for } k > d + 2.$$

Furthermore, from Lemma 2.2.1 and Lemma 2.2.2, it follows that we obtain

$$\|f(t)\|_{L^1} = \|f_0\|_{L^1} \quad \text{and} \quad \||v|^2 f(t)\|_{L^1} \le e^{-Kt}\sqrt{\|f_0\|_{L^1}\||v|^2 f_0\|_{L^1}}, \text{ for } t \ge 0.$$

Thus, $f(x, v, t) \in L^1_2(\mathbb{R}^{2d})$ for $t > 0$ and $f(x, v, t) \in \mathcal{X}_{k,\alpha}(T_*)$ for $T_* = t_2$ hold. □

4.2.1 Extention of Local Existence

So far, we make preparations to give the proof of our main theorem. Let T_∞ be the maximal lifespan of a regular solution as follows:

$$\mathcal{T} := \{T \in (0, \infty] : \text{Cauchy problem in (1.0.2) admits a unique global solution } f \in \mathcal{X}_{k,\alpha}(T)\},$$

$$T_\infty := \sup \mathcal{T}.$$

From Proposition 4.1.1, it brings that $T_0 \in \mathcal{T}$, i.e., the set $\mathcal{T}$ is nonempty; thus, $T_\infty \geq T_0$ holds.
We will now prove $T_\infty = \infty$.
Suppose not. If $T_\infty < \infty$ is satisfied, according to proposition 4.1.1, we have

$$\|f(T_\infty)\|_{H^k_\alpha} < \infty.$$

Therefore, the Cauchy problem in (1.0.2) with the initial datum $f(T_\infty)$ can be solved. In addition, we can apply the local existence result in Proposition 4.2.1 to see if there is a constant $\delta > 0$ such that the Cauchy problem in (1.0.2) is solvable even in the time interval $[0, T_\infty + \delta)$. This contradicts the maximality assumption in T_∞. Thus, both $T_\infty = \infty$ and the desired global existence are obtained. This completes the proof of our main theorem.

Theorem 4.2.1. *Let $T \in (0, \infty)$ be a positive constant and assume that the initial datum f_0 satisfies*

$$f_0 \in H^k_\alpha \cap L^1_2(\mathbb{R}^{2d}), \quad \textit{for some positive constants } k > 2 + d,\ \alpha > \frac{d+2}{2}.$$

Then, there exists a unique global classical solution to the Cauchy problem in (4.1.1) *in the function space $\mathcal{X}_{k,\alpha}(T)$.*

Chapter 5

The Cucker-Smale-Kuramoto Model

In this chapter, we consider an ensemble of Cucker-Smale particles combined with a periodic internal state. The Cucker-Smale-Kuramoto model(CSK model) and The CSK model with a Hebbian coupling will be dealt with. The latter model can be viewed as a special case for CSK model. Our model is basically created by combining the Cucker-Smale model and Kuramoto model together, so that velocities, positions, and phases are affected by each other. Before starting the first section, recall that the Kuramoto model is described as follows: Let $\{\theta_i(t)\}_{i=1}^N$ be $N-$ Kuramoto phase oscillators. The dynamics are governed by

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_j \sin(\theta_j - \theta_i),$$

where K and Ω_i are a coupling coefficient and a natural frequency of i-th oscillator, respectively. The contents of this chapter are based on the joint work with Ha, S-Y, Noh, S-E, Park J-Y [19].

5.1 The Cucker-Smale-Kuramoto Models

Let x_i, v_i and θ_i be the position, velocity and phase of the i-th particle, respectively. In this setting, the coupled Cucker-Smale-Kuramoto model reads as follows.

$$\begin{cases} \dfrac{dx_i}{dt} & = v_i, \ i = 1, ..., N, \ t > 0, \\ \dfrac{dv_i}{dt} & = \dfrac{K_1}{N} \displaystyle\sum_j \psi_1(|x_i - x_j|, \theta_i - \theta_j)(v_j - v_i), \\ \dfrac{d\theta_i}{dt} & = \Omega_i + \dfrac{K_2}{N} \sum_j \psi_2(|x_j - x_i|) \sin(\theta_j - \theta_i), \end{cases} \tag{5.1.1}$$

where K_1 and K_2 are coupling strengths and $\psi_l, \ l = 1, 2$ are defined as

$$\begin{aligned} \psi_1(|x_i - x_j|, \theta_i - \theta_j) &:= \frac{\cos(\theta_i - \theta_j)}{(1 + |x_i - x_j|^2)^{\beta_1}}, \text{ and} \\ \psi_2(|x_i - x_j|) &:= \frac{1}{(1 + |x_i - x_j|^2)^{\beta_2}}, \end{aligned}$$

for some real number β_1, and β_2. Without loss of generality, we discuss all the results in case that the average position and average velocity are zero, i.e.,

$$\sum_i \Omega_i = 0, \ \sum_i x_i = 0, \text{ and } \sum_i v_i = 0.$$

We next introduce the other model. The CSK model with Hebbian Coupling is a special case for the CSK model. We similarly assume that the velocity interaction weight between C-S particles and the phase interaction weight are given by the ansatz $\psi_v(\theta_j - \theta_i, x_j - x_i)$ and $\psi_\theta(\|x_j - x_i\|)$, but the coupling strength is not necessarily a constant. The coupling strength k_{ij} in phase dynamics is the Hebbian like adaptive law. The equation is expressed as

$$\begin{cases} \dfrac{d}{dt}x_i &= v_i, \quad t > 0, \quad 1 \le i, j \le N, \\ \dfrac{d}{dt}v_i &= \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} \psi_v(\theta_j - \theta_i, x_j - x_i)(v_j - v_i), \\ \dfrac{d}{dt}\theta_i &= \Omega_i + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} k_{ij}\psi_\theta(\|x_j - x_i\|)\sin(\theta_j - \theta_i), \\ \dfrac{d}{dt}k_{ij} &= \varepsilon(\alpha\cos(\theta_j - \theta_i) - k_{ij}), \end{cases} \tag{5.1.2}$$

subject to initial data:

$$x_i(0) = x_i^0, \quad v^i(0) = v_i^0, \quad \theta_i(0) = \theta_i^0 \quad \text{and} \quad k^{ij}(0) = k_{ij}^0. \tag{5.1.3}$$

Here, ε and α are called learning rate and learning enhancement factor, respectively. We consider that $\psi_v(\theta, x)$ and $\psi_\theta(r)$ are nonnegative functions which are not increasing in the variables $\|x\|$ and r, respectively.

Before closing this section, we recall the definition of synchronization.

Definition 5.1.1. *Let $\theta(t) := (\theta_1(t), \cdots, \theta_N(t))$ be a dynamic solution to a system. Then we have the following solution concepts for synchronization:*

1. *The phase configuration $\theta(t)$ exhibits asymptotic complete synchronization (ACS) if and only if the following two conditions hold:*

$$\sup_{t\ge 0}\max_{1\le i,j\le N}|\theta_i(t) - \theta_i(t)| < \infty, \qquad \lim_{t\to\infty}\max_{1\le i,j\le N}|\theta_i(t) - \theta_j(t)| = 0.$$

2. *The phase configuration $\theta(t)$ exhibits asymptotic complete-frequency synchronization (ACFS) if and only if the following two conditions hold:*

$$\sup_{t\ge 0}\max_{1\le i,j\le N}|\theta_i(t) - \theta_i(t)| < \infty, \qquad \lim_{t\to\infty}\max_{1\le i,j\le N}|\dot\theta_i(t) - \dot\theta_j(t)| = 0.$$

3. *The phase configuration $\theta(t)$ exhibits asymptotic practical synchronization (APS) if and only if the following condition holds:*

$$\lim_{k_m\to\infty}\limsup_{t\to\infty}\max_{1\le i,j\le N}|\theta_i(t) - \theta_j(t)| = 0,$$

5.2 Frameworks

In this section, we briefly introduce the assumptions for the following sections and the main results.

The first set of assumptions which will be brought up whenever we deal with (5.1.1) is given as

($\mathcal{A}1$) $0 < D(\theta^0) < D^\infty < \frac{\pi}{2}$,

($\mathcal{A}2$) There exists x_* such that

$$\frac{K_1 \cos D^\infty}{2} \int_{2\|x^0\|}^{2x_*} \psi(s)ds = \|v^{i0}\|,\ \psi(s) := \frac{1}{(1+s^2)^{\beta_1}},$$

($\mathcal{A}3$) $K_2 \gg \dfrac{D(\Omega)}{D(\theta^0)\psi(x_*)}$.

The following is the main theorem on the CSK model. The proof will be provided later.

Theorem 5.2.1. *(Flocking and Synchronization) Assume ($\mathcal{A}1$)-($\mathcal{A}3$). Then we have*

(i) $\lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0, \quad \sup_{0\le t<\infty} \|x_i(t) - x_j(t)\| < \infty, \quad 1 \le i, j \le N,$

(ii) $\lim_{0\le t<\infty} \|\dot{\theta}_i(t) - \dot{\theta}_j(t)\| = 0, \quad \lim_{0\le t<\infty} \|\theta_i(t) - \theta_j(t)\| \le \dfrac{D^\infty D(\Omega)}{2K_2 \sin D^\infty \psi_2(2x_*)}.$

Another set of assumptions is regarding the CSK model with Hebbian Coupling. In this thesis, we particularly deal with the system under the settings:

$$\psi_v(\theta, x) := \frac{\cos(\theta)}{(1+\|x\|^2)^{\beta_1}},\ \text{and}\ \psi_\theta(\|x\|) := \frac{1}{(1+\|x\|^2)^{\beta_2}},$$

for some $\beta_1, \beta_2 > 0$.

($\mathcal{B}1$) $\dfrac{\alpha}{\varepsilon} > k_m := \min_{i,j} K_\theta^{ij}(0)$ holds. We denote the number $D^\infty \in [0, \frac{\pi}{2})$ by $\cos D^\infty = \dfrac{\varepsilon K_\theta^m}{\alpha}$,

($\mathcal{B}2$) There exists x_* such that

$$\frac{K \cos D^\infty}{2}\Psi(2x_*) = \|v^0\|, \text{ where } \Psi(l) := \int_{2\|x^0\|}^{l} \psi(s)ds = \int_{2\|x^0\|}^{l} \frac{1}{(1+s^2)^{\beta_1}}ds,$$

($\mathcal{B}3$) $D(\Omega) < \dfrac{2k_m D(\Theta(0))\psi_\theta(2x^*)\sin D^\infty}{D^\infty}$,

where k_m is the minimum value of k_{ij} .

Remark 5.2.1. *Note that $\beta_1 < \frac{1}{2}$ guarantees the unconditional existence of x_* in ($\mathcal{B}2$) due to the non-integrability of*

$$\psi(s) = \frac{1}{(1+s^2)^{\beta_1}}$$

over $\mathbb{R}_+$. In case of $\beta_1 \geq \frac{1}{2}$, such x_* conditionally exists if we set K to be sufficiently large.

Theorem 5.2.2. *(Main Flocking and Synchronization) Suppose that the parameters ε, α and initial position and velocity (x^0, v^0), phase configuration and coupling strength satisfy the framework ($\mathcal{B}1$) − ($\mathcal{B}3$). Then for any solution (x, v, θ) to (5.1.2) we have*

(i) $\lim_{t\to\infty} D(v(t)) = 0, \ \sup_{0\le t<\infty} D(x(t)) < \infty,$

(ii) $\lim_{0\le t<\infty} \max_{i,j} \|\dot\theta_i(t) - \dot\theta_j(t)\| = 0,$

$$\lim_{0\le t<\infty} \max_{i,j} \|\theta_i(t) - \theta_j(t)\| \le \frac{D^\infty D(\Omega)}{2K \sin D^\infty \psi_\theta(2x_*)},$$

for all $1 \le i, j \le N$.

5.3 Estimates in the CSK model

In this section, we presents flocking estimates in the CSK model (5.1.1). We begin with the following proposition.

Proposition 5.3.1. *Assume ($\mathcal{A}1$)-($\mathcal{A}3$). Then a solution $\{(x_i, v_i, \theta_i)\}_{i=1}^N$ to* (5.1.1) *satisfy*

$$
\begin{aligned}
\left|\frac{d}{dt}\|x\|\right| &\leq \|v\|, \quad \frac{d}{dt}\|v\| \leq -K_1 \cos(D^\infty)\psi(2\|x\|)\|v\|,\\
\frac{d}{dt}D(\theta) &\leq D(\Omega) - \frac{2K_2 \sin D^\infty}{D^\infty}\psi_2(2\|x\|)D(\theta), \ t > 0.
\end{aligned}
$$

Proof. Applying the Cauchy-Schwartz inequality, we obtain

$$
\left|\frac{d}{dt}\|x\|^2\right| = \left|\sum_{i=1}^N \frac{d}{dt}\|x_i\|_{L^2}^2\right| = \left|\sum_{i=1}^N 2 < x_i, v_i >\right| \leq 2\|x\|\|v\|,
$$

and

$$
\begin{aligned}
\frac{d}{dt}\|v\|^2 &= \sum_{i=1}^N \frac{d}{dt}\|v_i\|_{L^2}^2\\
&= \frac{2K_1}{N}\sum_{j=1}^N\sum_{i=1}^N \langle v_i, \psi_1(|x_i - x_j|, \theta_i - \theta_j)(v_j - v_i)\rangle\\
&\leq -2K_1 \cos(D^\infty)\psi(2\|x\|)\|v\|^2.
\end{aligned}
$$

In order to show the third inequality, we consider at most countable time intervals $(t_{k-1}, t_k) \quad k = 1, 2, 3, ...,$ where $0 = t_0 < t_1 < t_2 < t_3 < ...$ such that $D(\theta(t)) = \theta_M(t) - \theta_m(t)$ holds for some fixed indices M and m during $t \in (t_{k-1}, t_k)$.

$$
\begin{aligned}
&\frac{d}{dt}D(\theta) = \Omega_M - \Omega_m\\
&\quad + \frac{K}{N}\sum_j \{\psi_2(|x_j - x_*|)\sin(\theta_j - \theta_M) - \psi_2(|x_j - x_*|)\sin(\theta_j - \theta_m)\}
\end{aligned}
$$

$$\leq D(\Omega) - \frac{2K_2 \sin D^\infty}{D^\infty}\psi_2(2\|x\|)D(\theta).$$

□

We develop the argument from the proposition 5.3.1, so that $\|v\|$ and $\|\omega\|$ ($\omega := \dot{\theta}$) are shown to be decaying in time.

Proposition 5.3.2. *Suppose that ($\mathcal{A}1$)-($\mathcal{A}3$). For a solution* $(x, v, \theta) = \{(x_i, v_i, \theta_i)\}_{i=1}^N$ *to* (5.1.1), *we have*

$$\begin{aligned}
\|v(t)\| &\leq \|v^0\| exp(-K_1 \cos(D^\infty)\psi(2x_*)t), \quad t > 0, \\
\|\omega(t)\|^2 &\leq \|\omega^0\|^2 exp\left(\left\{-2K_2\psi_2(2x_*)\cos D^\infty + \varepsilon K_2\left(-\psi_2'\left(\frac{1}{2\beta_2+1}\right)\right)\right\}t\right) \\
&+ \frac{K_2}{\varepsilon}\left(-\psi_2'\left(\frac{1}{2\beta_2+1}\right)\right)\|v^0\|^2 exp(-2K_2\cos D^\infty\psi(2x_*)t).
\end{aligned}$$

Proof. First inequality is easily obtained from Proposition 5.3.1 by applying Gronwall's inequality. In order to derive the second one, we calculate the time derivative of $\|\omega\|^2$ as follows.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\|\omega_i\|^2\right) &= \frac{K_2}{N}\sum_j \left\{\psi_2'(\|x_j - x_i\|)\left(\frac{x_j - x_i}{\|x_j - x_i\|}\cdot(v_j - v_i)\right)\sin(\theta_j - \theta_i)\omega_i\right. \\
&+ \left.\psi_2(\|x_j - x_i\|)\cos(\theta_j - \theta_i)(\omega_j - \omega_i)\omega_i\right\},
\end{aligned}$$

for $i = 1, 2, ..., N$. Summing these equations from $i = 1$ up to $i = N$, we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\sum_i \|\omega_i\|^2\right) &= \frac{K_2}{N}\sum_{i,j}\left\{\psi_2'(\|x_j - x_i\|)\left(\frac{x_j - x_i}{\|x_j - x_i\|}\cdot(v_j - v_i)\right)\right. \\
&\times \left.\sin(\theta_j - \theta_i)\omega_i + \psi_2(\|x_j - x_i\|)\cos(\theta_j - \theta_i)(\omega_j - \omega_i)\omega_i\right\} \\
&= \frac{K_2}{2N}\sum_{i,j}\left\{-\psi_2'(\|x_j - x_i\|)\left(\frac{x_j - x_i}{\|x_j - x_i\|}\cdot(v_j - v_i)\right)\right. \\
&\times \sin(\theta_j - \theta_i)\,(\omega_j - \omega_i) - \psi_2(\|x_j - x_i\|)\cos(\theta_j - \theta_i) \\
&\times \left.\|\omega_j - \omega_i\|^2\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{K_2}{2N}\left(-\psi_2'\left(\frac{1}{2\beta_2+1}\right)\right)\sum_{i,j}\left[\frac{\|v_j-v_i\|^2}{2\epsilon}+\frac{\epsilon\|\omega_j-\omega_i\|^2}{2}\right] \\
&- \frac{K_2}{2N}\psi_2(2x_*)\cos D^\infty \|\omega_j-\omega_i\|^2.
\end{aligned}
$$

By exchanging indices $i \leftrightarrow j$, the second equality is deduced. Young's inequality and minimum and maximum values of $\psi_2, \psi_2', and\ \cos(\theta)$ yield the last line. Using the result $\sum_j v_j = \sum_j \omega_j = 0$, from the above relation, we have the following inequality:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\sum_i \|\omega_i\|^2\right) &\leq \frac{K_2}{2\varepsilon}\left[-\psi_2'\left(\frac{1}{2\beta_2+1}\right)\right]\|v\|^2 \\
&- \frac{K}{2}\left[2\psi_2(2x_*)\cos D^\infty - \varepsilon\left(-\psi_2'(\frac{1}{2\beta_2+1})\right)\right]\|\omega\|^2.
\end{aligned}
$$

Applying the Gronwall's inequality, we obtain

$$
\begin{aligned}
\|\omega(t)\|^2 &\leq \|\omega^0\|^2 exp\left(\left\{-2K_2\psi_2(2x_*)\cos D^\infty + \varepsilon K_2\left(-\psi_2'\left(\frac{1}{2\beta_2+1}\right)\right)\right\}t\right) \\
&+ \frac{K_2}{\varepsilon}\left(-\psi_2'\left(\frac{1}{2\beta_2+1}\right)\right)\|v^0\|^2 exp(-2K_2\cos D^\infty \psi(2x_*)t).
\end{aligned}
$$

□

We present a dissipation estimate for our models, so that the flocking estimate is covered. Let us define a Lyapunov type functional as follows:

$$
\mathcal{E}(\|x\|,\|v\|) := \|v\| + \frac{K\cos(D^\infty)}{2}\Psi(2\|x\|).
$$

Lemma 5.3.1. *Assume* ($\mathcal{A}1$)*-*($\mathcal{A}3$) *hold, and let* $\{(x_i,v_i)\}_{i=1}^N$ *be the global smoooth solution to the system* (5.1.1)*. Then we have*

(i) $\mathcal{E}(\|x(t)\|,\|v(t)\|) \leq \mathcal{E}(\|x^0\|,\|v^0\|), \quad t>0,$

(ii) $\|v(t)\| + \frac{K\cos(D^\infty)}{2}\int_{2\|x^0\|}^{2\|x(t)\|}\psi(s)ds \leq \|v^0\|.$

Proof. Let Ψ be a potential function of ψ, i.e., $\Psi(\|x\|) := \int_{2\|x^0\|}^{\|x\|} \psi(s)ds$. Using the second inequality in proposition 5.3.1, we have

$$\frac{d}{dt}\Big[\|v\| + \frac{K\cos(D^\infty)}{2}\Psi(2\|x\|)\Big] \le -K\cos(D^\infty)\psi(2\|x\|)\|v\| + \frac{K}{2}\cos(D^\infty)\psi(2\|x\|)2\|v\| \le 0.$$

Integration and $(\mathcal{A}2)$ yield

$$\|v(t)\| + \frac{K\cos(D^\infty)}{2}\int_{2\|x^0\|}^{2\|x(t)\|}\psi(s)ds \le \|v^0\|.$$

□

Proposition 5.3.3. *Let us define*

$$\mathcal{T} := \{t \in \mathbb{R}_+ \quad | \quad D(\theta(t)) < D^\infty\} \;\; and \;\; T^\infty := \sup_{t\in\mathbb{R}_+} \mathcal{T}.$$

Then $T^\infty = \infty$ holds.

Proof. First we claim that $x_* \ge \|x(t)\|$ for $t > 0$. From the Lemma 5.3.1 and $(\mathcal{A}2)$, we have

$$\|v(t)\| + \frac{K\cos(D^\infty)}{2}\int_{2\|x^0\|}^{2\|x(t)\|}\psi(s)ds \le \|v^0\| = \frac{K\cos D^\infty}{2}\int_{2\|x^0\|}^{2x_*}\psi(s)ds.$$

Since $\psi \ge 0$ and $\|v\| \ge 0$ hold, the above implies $\|x(t)\| \le x_*$ for $t > 0$. In addition, $\psi(2\|x\|) \ge \psi(2x_*)$ is derived because ψ is decreasing.

Suppose $T^\infty < \infty$. Combining the claim and proposition 5.3.1, we obtain

$$\frac{d}{dt}D(\theta) \le D(\Omega) - \frac{2K\sin D^\infty}{D^\infty}\psi_2(2x_*)D(\theta),$$

and applying Gronwall's inequality to the above, we have

$$D(\theta) \le D(\theta^0)e^{-\frac{2K\psi_2(2x_*)\sin D^\infty}{D^\infty}t} + D(\Omega)\big(1 - e^{-\frac{2K\psi_2(2x_*)\sin D^\infty}{D^\infty}t}\big).$$

By the construction of T^∞, the following is supposed to hold:

$$\lim_{t\to T^\infty} D(\theta(t)) = D^\infty.$$

However, $(\mathcal{A}3)$ and the previous inequality of $D(\theta)$ imply $D(\theta(T^\infty)) < D^\infty$, which is contradiction. □

Theorem 5.3.1. *(Flocking and Synchronization) Assume ($\mathcal{A}1$)-($\mathcal{A}3$). Then we have*

(i) $\lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0, \ \sup_{0\le t<\infty} \|x_i(t) - x_j(t)\| < \infty,$

(ii) $\lim_{0\le t<\infty} D(\dot{\theta}(t)) = 0, \ \lim_{0\le t<\infty} D(\theta(t)) \le \dfrac{D^\infty D(\Omega)}{2K \sin D^\infty \psi_2(2x_*)},$

for all $1 \le i, j \le N$.

Proof. (i) From Proposition 5.3.2, we have

$$\|v_i(t) - v_j(t)\| \le \sqrt{2}\|v(t)\| \le \sqrt{2}\|v^0\| exp(-K\cos(D^\infty)\psi(2x_*)t).$$

(ii) It is easily obtained by combining the result in (i) with the following inequality:

$$\|x_(t) - x_j(t)\| \le \|x_i(0) - x_j(0)\| + \int_0^t \|v_i(s) - v_j(s)\| ds.$$

(iii) From Proposition 5.3.2, for any integers i, j,

$$\begin{aligned} \|\dot{\theta}_i(t) - \dot{\theta}_j(t)\| &\le \sqrt{2}\|\dot{\theta}(t)\| \\ &\le \sqrt{2}\left\{ \|\omega^0\|^2 e^{-\left\{2K\psi_2(2x_*)\cos D^\infty + \varepsilon K \psi_2'\left(\frac{1}{2\beta_2+1}\right)\right\}t} \right. \\ &+ \left. \frac{K}{\varepsilon}\left(-\psi_2'\left(\frac{1}{2\beta_2+1}\right)\right)\|v^0\|^2 exp(-2K\cos D^\infty \psi(2x_*)t) \right\}^{\frac{1}{2}}. \end{aligned}$$

(iv) Applying the Gronwall's inequality to the third result in Proposition 5.3.1, we derive the inequality. □

5.4 Estimates in the CSK model with Hebbian Coupling

In this section, we study the emergent phenomena of the coupled system (5.1.2) using the Lyapunov functional approach initiated in [25].

Lemma 5.4.1. *Assume* $\{k^{ij}_{\theta,0}\}_{1\le i,j\le N}$ *in* (5.1.2) *is given as a symmetric matrix and (*$\mathcal{B}1$*) holds. Then* $k^{ij}_\theta(t)$ *is symmetric for* $t \ge 0$. *If* $\epsilon > 0,\quad \alpha > 0$, $\{k^{ij}_\theta(0)\}_{1\le i,j\le N}$ *are nonnegative and* $D(\theta(t)) < D^\infty < \frac{\pi}{2}$, *then* $k^{ij}_\theta(t)$ *is nonnegative and bounded below and above by the initial minimum and initial maximum, respectively, for* $t > 0$ *and* $1 \le i,j \le N$.

Proof. Using the fact that $\cos\theta$ is an even function, we derive the following equality:

$$\frac{d}{dt}(k_{ij} - k_{ji}) = -\varepsilon(k_{ij} - k_{ji}).$$

Solving the above first order linear ODE, we obtain

$$(k_{ij} - k_{ji})(t) = (k_{ij} - k_{ji})(0)e^{-\varepsilon t} = (k^0_{ij} - k^0_{ji})e^{-\epsilon t} = 0.$$

Thus, symmetricity of the initial condition for $\{K^{ij}_{\theta,0}\}_{1\le i,j\le N}$ implies the symmetricity of $\{K^{ij}_\theta(t)\}_{1\le i,j\le N}$ for any $t \ge 0$.
By the previous lemma,

$$\dot{k}_{ij} \quad = \quad \varepsilon\alpha\cos(\theta_j - \theta_i) - \varepsilon k_{ij} \ge \varepsilon\alpha\cos(D^\infty) - \varepsilon k_{ij}$$

Using Gronwall's inequality, we obtain

$$k_{ij}(t) \ge e^{-\varepsilon t}k_{ij}(0) + \frac{\alpha\cos(D^\infty)}{\varepsilon}(1 - e^{-\varepsilon t}).$$

Let us define k_m and k_m as $k_m := \min_{i,j} k_{ij}(0)$ and $k_m := \max_{i,j} k_{ij}(0)$. Under the assumption ($\mathcal{B}1$), the previous inequality yields

$$k_{ij}(t) \ge e^{-\varepsilon t}k_m + k_m(1 - e^{-\varepsilon t}) = k_m, \quad t > 0.$$

$k_{ij}(t)$ is also bounded above because

$$\dot{k}_{ij} \quad = \quad \varepsilon\alpha\cos(\theta_j - \theta_i) - \varepsilon k_{ij} \le \varepsilon\alpha - \varepsilon k_{ij}$$

implies

$$k_{ij}(t) \le e^{-\varepsilon t}(k_{ij}(0) - \alpha) + \alpha \le k_m.$$

□

Remark 5.4.1. *If $\varepsilon\alpha < 0$ holds, $k_{ij}(t)$ is nonnegative for $0 < t < T_0$, for some positive real number T_0, and $1 \leq i, j \leq N$. In this section, we will assume $\{k^0_{ij}\}_{1\leq i,j\leq N}$ be given nonnegative and symmetric, and ε and α be given positive.*

Proposition 5.4.1. *Suppose that the following conditions hold.*

1. *the natural frequencies and initial data satisfy*

$$\sum_{i=1}^{N} \Omega_i = 0,\ \sum_{i=1}^{N} \theta_i(0) = 0 and \sum_{i=1}^{N} v_i(0) = 0$$

   in (5.1.2).

2. *the communication weight $\psi_v(\theta, x)$ is an even function in both θ and x.*

3. *The coupling strength matrix $\{k_{ij}\}$ is symmetric. i.e., $k_{ij} = k_{ji}$ holds for $1 \leq i, j \leq N$.*

Then $\displaystyle\sum_{i=1}^{N} \theta_i(t) = 0$ and $\displaystyle\sum_{i=1}^{N} v_i(t) = 0$ hold, for $t \geq 0$.

Proof. We calculate the time derivative of $\displaystyle\sum_{i=1}^{N} \theta_i(t)$ and $\displaystyle\sum_{i=1}^{N} v_i(t)$ as follows:

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{N} v_i(t) &= \frac{K}{N}\sum_{i,j=1}^{N} \psi_v(\theta_j - \theta_i, x_j - x_i)(v_j - v_i) \\
&= \frac{K}{N}\sum_{i,j=1}^{N} \psi_v(\theta_i - \theta_j, x_i - x_j)(v_i - v_j) \\
&= 0,
\end{aligned}$$

the above second equality is obtained by interchanging i and j. Because ψ_v is even, the time derivative of the velocity sum is zero. Likewise,

$$\frac{d}{dt}\sum_{i=1}^{N} \theta_i(t) = \sum_{i=1}^{N} \omega_i + \frac{1}{N}\sum_{i,j=1}^{N} K^{ij}_{\theta}\psi_\theta(\|x_j - x_i\|)\sin(\theta_j - \theta_i)$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{i,j=1}^{N} K_\theta^{ji}\psi_\theta(\|x_j - x_i\|)\sin(\theta_i - \theta_j)\\
&= 0.
\end{aligned}
$$

Thus,

$$\sum_{i=1}^{N}\theta_i(t) = \sum_{i=1}^{N}\theta_i(0) = 0$$

and

$$\sum_{i=1}^{N}v_i(t) = \sum_{i=1}^{N}v_i(0) = 0.$$

□

For the rest of the part, we set the initial conditions as

$$\sum_{i=1}^{N}\omega_i = 0,\ \sum_{i=1}^{N}\theta_i(0) = 0 \text{ and } \sum_{i=1}^{N}v_i(0) = 0.$$

Proposition 5.4.2. *Let* $(x, v, \theta) = \{(x_i, v_i, \theta_i)\}_{i=1}^N$ *be the global smoooth solution to the system* (5.1.2)*. Assume (*$\mathcal{B}1$*)-(*$\mathcal{B}3$*) hold. Then the followings hold:*

(i) $\left|\frac{d}{dt}\|x\|\right| \le \|v\|,\ \frac{d}{dt}\|v\| \le -K\cos(D^\infty)\psi(2\|x\|)\|v\|,$

(ii) $\frac{d}{dt}D(\theta) \le D(\Omega) - \frac{2k_m \sin D^\infty}{D^\infty}\psi_2(2\|x\|)D(\theta).$

Proof. Applying the Cauchy-Schwartz inequality, we obtain

$$\left|\frac{d}{dt}\|x\|^2\right| = \left|\sum_{i=1}^{N}\frac{d}{dt}|x_i|^2\right| = \left|\sum_{i=1}^{N}2\langle x_i, v_i\rangle\right| \le 2\|x\|\|v\|,$$

and

$$\frac{d}{dt}\|v\|^2 = \sum_{i=1}^{N}\frac{d}{dt}\|v_i\|^2$$

$$
\begin{aligned}
&= \frac{2K}{N}\sum_{j=1}^{N}\sum_{i=1}^{N}\langle v_i, \psi_v(x_i - x_j, \theta_i - \theta_j)(v_j - v_i)\rangle \\
&\leq -2K\cos(D^\infty)\psi(2\|x\|)\|v\|^2.
\end{aligned}
$$

To show the third inequality, we consider at most countable time intervals (t_{k-1}, t_k) $k = 1, 2, 3, ...$, where $0 = t_0 < t_1 < t_2 < t_3 < ...$ such that $t_n \to \infty$ as $n \to \infty$ and $D(\theta(t)) = \theta_M(t) - \theta_m(t)$ holds for some indices M and m which are fixed during $t \in (t_{k-1}, t_k)$. Then, we have

$$
\begin{aligned}
\frac{d}{dt}D(\theta) &= \Omega_M - \Omega_m \\
&\quad + \frac{1}{N}\sum_j \left\{ k_{Mj}\psi_\theta(\|x_j - x_M\|)\sin(\theta_j - \theta_M) - k_{mj}\psi_\theta(\|x_j - x_m\|)\sin(\theta_j - \theta_m)\right\} \\
&\leq D(\Omega) - \frac{2k_m \sin D^\infty}{D^\infty}\psi_2(2\|x\|)D(\theta)
\end{aligned}
$$

holds. □

We define a Lyapunov type functional as follows:

$$
\mathcal{E}(\|x\|, \|v\|) := \|v\| + \frac{K\cos(D^\infty)}{2}\Psi(2\|x\|).
$$

Lemma 5.4.2. *Let* $(x, v, \theta) = \{(x_i, v_i, \theta_i)\}_{i=1}^N$ *be the global smoooth solution to the system* (5.1.2). *Suppose that the conditions (*$\mathcal{B}1$*)-(*$\mathcal{B}3$*) hold. Then we, have*

(i) $\mathcal{E}(\|x(t)\|, \|v(t)\|) \leq \mathcal{E}(\|x^0\|, \|v^0\|), \quad t \geq 0,$

(ii) $\|v(t)\| + \frac{K\cos(D^\infty)}{2}\Psi(2\|x(t)\|) \leq \|v^0\|.$

Proof. Using the second inequality in proposition 5.4.2, we have

$$
\frac{d}{dt}\Big[\|v\| + \frac{K\cos(D^\infty)}{2}\Psi(2\|x\|)\Big]
$$

$$\begin{aligned}&\le -K\cos(D^\infty)\psi(2\|x\|)\|v\| + \frac{K}{2}\cos(D^\infty)\psi(2\|x\|)2\|v\|\\&\le 0.\end{aligned}$$

Integration and ($\mathcal{B}2$) yield

$$\|v(t)\| + \frac{K\cos(D^\infty)}{2}\int_{2\|x^0\|}^{2\|x(t)\|}\psi(s)ds \le \|v^0\|.$$

□

Proposition 5.4.3. *Let us define*

$$\mathcal{T} := \left\{t \in \mathbb{R}_+ \cup \{0\} \quad | \quad D(\theta(t)) < D^\infty\right\}$$

and $T^\infty := \sup\mathcal{T}$*. Then* $T^\infty = \infty$*.*

Proof. To begin with, we claim $x_* \ge \|x(t)\|$ is true for any $t > 0$. Using the Lemma 5.4.2 and ($\mathcal{B}2$), we obtain

$$\|v(t)\| + \frac{K\cos(D^\infty)}{2}\int_{2\|x^0\|}^{2\|x(t)\|}\psi(s)ds \le \|v^0\| = \frac{K\cos D^\infty}{2}\int_{2\|x^0\|}^{2x_*}\psi(s)ds.$$

Since $\psi(s)$ and $\|v\|$ are nonnegative, the second term in the left side does not exceed $\|v^0\|$, which equals $\int_{2\|x^0\|}^{2x_*}\psi(s)ds$. Therefore, $\|x(t)\| \le x_*$ is true for $t > 0$. In addition, $\psi(2\|x\|) \ge \psi(2x_*)$ holds for ψ being nonincreasing.

Let us suppose $T^\infty < \infty$. Combining the previous claim and proposition 5.4.2, we obtain

$$\frac{d}{dt}D(\theta) \le D(\Omega) - \frac{2k_m\sin D^\infty}{D^\infty}\psi(2x_*)D(\theta).$$

By applying Gronwall's inequality and ($\mathcal{B}3$) to this, we get

$$D(\theta) \le D(\theta(0))e^{-\frac{2k_m\psi(2x_*)\sin D^\infty}{D^\infty}t} + D(\Omega)\left(1 - e^{-\frac{2k_m\psi(2x_*)\sin D^\infty}{D^\infty}t}\right) \le D(\theta(0)).$$

By the construction of T^∞, the following is supposed to hold:

$$\lim_{t\to T^\infty} D(\theta(t)) = D^\infty.$$

However, the previous inequality of $D(\theta)$ implies $D(\theta(T^\infty)) < D^\infty$. Since $D(\theta(t))$ is continous, there is $\delta > 0$ such that $[T^\infty, T^\infty + \delta) \subset \mathcal{T}$. This contradicts to the assumption. □

Corollary 5.4.1. *Let $(x(t), \theta(t))$ be a solution to* (5.1.2)*. Then for x^* in* $(\mathcal{B}2)$*, we have*

$$\|x(t)\| \le x^*, \quad t \ge 0.$$

Proof. From (5.4.2), we know

$$\frac{K\cos(D^\infty)}{2}\Psi(2\|x(t)\|) \le \|v^0\| = \frac{K\cos(D^\infty)}{2}\int_{2\|x^0\|}^{2x^*}\psi(s)ds$$

holds. If $\|x(t)\| > x^*$, for some $t > 0$, the following is true:

$$\frac{K\cos(D^\infty)}{2}\int_{2\|x^0\|}^{2x^*}\psi(s)ds \le \frac{K\cos(D^\infty)}{2}\int_{2\|x^0\|}^{2\|x(t)\|}\psi(s)ds.$$

□

From the above results, we can show $\|v\|$ and $\|\dot\theta\|$ decaying in time.

Proposition 5.4.4. *Assume* $(\mathcal{B}1)$*-*$(\mathcal{B}3)$*. Then we have*

$$\begin{aligned}
\|v(t)\| &\le \|v^0\| exp(-K\cos(D^\infty)\psi(2x_*)t)\\
\|\dot\theta(t)\|^2 &\le \|\dot\theta^0\|^2 exp\Big(\big\{-2K\psi_\theta(2x_*)\cos D^\infty + \varepsilon K\big(-\psi'_\theta\left(\frac{1}{2\beta_2+1}\right)\big)\big\}t\Big)\\
&+ \frac{K}{\varepsilon}\Big(-\psi'_\theta\left(\frac{1}{2\beta_2+1}\right)\Big)\|v^0\|^2 exp(-2K\cos D^\infty\psi(2x_*)t)
\end{aligned}$$

Proof. First inequality is easily obtained from Proposition 5.4.2 by applying Gronwall's inequality. In order to derive the second one, we calculate the time derivative of $\|\dot\theta\|^2$ as follows. For $i = 1, 2, ..., N$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\dot\theta_i\|^2 &= \frac{1}{N}\sum_j\Big\{\Big[\dot{k}_{ij}\psi_\theta(\|x_j-x_i\|) + k_{ij}\psi'_\theta(\|x_j-x_i\|)\Big(\frac{x_j-x_i}{\|x_j-x_i\|}\\
&\cdot(v_j-v_i)\Big)\Big]\sin(\theta_j-\theta_i)\dot\theta_i + k_{ij}\psi_\theta(\|x_j-x_i\|)\cos(\theta_j-\theta_i)(\dot\theta_j-\dot\theta_i)\dot\theta_i\Big\}\\
&= \frac{1}{N}\sum_j\Big\{\Big[\varepsilon(\alpha\cos(\theta_j-\theta_i) - k_\theta^{ij})\psi_\theta(\|x_j-x_i\|) + k_{ij}\psi'_\theta(\|x_j-x_i\|)\\
&\times\Big(\frac{x_j-x_i}{\|x_j-x_i\|}\cdot(v_j-v_i)\Big)\Big]\sin(\theta_j-\theta_i)\dot\theta_i + k_{ij}\psi_\theta(\|x_j-x_i\|)\cos(\theta_j-\theta_i)
\end{aligned}$$

$$\times(\dot{\theta}_j - \dot{\theta}_i)\dot{\theta}_i\Big\}.$$

Summing these equations from $i = 1$ up to $i = N$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\sum_i \|\dot{\theta}_i\|^2\Big) \\
&\quad = \frac{1}{N}\sum_{i,j}\Big\{ \big[\varepsilon(\alpha\cos(\theta_j - \theta_i) - k_\theta^{ij})\psi_\theta(\|x_j - x_i\|) + k_{ij} \\
&\quad \times \psi_\theta'(\|x_j - x_i\|)\Big(\frac{x_j - x_i}{\|x_j - x_i\|}\cdot(v_j - v_i)\Big)\Big]\sin(\theta_j - \theta_i)\dot{\theta}_i + k_{ij}\psi_\theta(\|x_j - x_i\|) \\
&\quad \times\cos(\theta_j - \theta_i)(\dot{\theta}_j - \dot{\theta}_i)\dot{\theta}_i\Big\} \\
&\quad = \frac{1}{N}\sum_{i,j}\Big\{\Big[\varepsilon(\alpha\cos(\theta_j - \theta_i) - k_\theta^{ij})\psi_\theta(\|x_j - x_i\|) + k_{ij}\psi_\theta'(\|x_j - x_i\|) \\
&\quad \times\Big(\frac{x_j - x_i}{\|x_j - x_i\|}\cdot(v_j - v_i)\Big)\Big]\sin(\theta_j - \theta_i)(-\dot{\theta}_j) + k_{ij}\psi_\theta(\|x_j - x_i\|) \\
&\quad \times\cos(\theta_j - \theta_i)(\dot{\theta}_j - \dot{\theta}_i)(-\dot{\theta}_j)\Big\} \\
&\quad = \frac{1}{2N}\sum_{i,j}\Big\{\Big[-\varepsilon(\alpha\cos(\theta_j - \theta_i) - k_\theta^{ij})\psi_\theta(\|x_j - x_i\|) - k_{ij}\psi_\theta'(\|x_j - x_i\|) \\
&\quad \times\Big(\frac{x_j - x_i}{\|x_j - x_i\|}\cdot(v_j - v_i)\Big)\Big]\sin(\theta_j - \theta_i)(\dot{\theta}_j - \dot{\theta}_i) - k_{ij}\psi_\theta(\|x_j - x_i\|) \\
&\quad \times\cos(\theta_j - \theta_i)\|\dot{\theta}_j - \dot{\theta}_i\|^2\Big\} \\
&\quad \le \frac{1}{2N}\sum_{i,j}\Big\{\Big[-\varepsilon(\alpha\cos(\theta_j - \theta_i) - k_\theta^{ij})\psi_\theta(\|x_j - x_i\|) - k_{ij}\psi_\theta'(\|x_j - x_i\|) \\
&\quad \times\Big(\frac{x_j - x_i}{\|x_j - x_i\|}\cdot(v_j - v_i)\Big)\Big]\sin(\theta_j - \theta_i)(\dot{\theta}_j - \dot{\theta}_i) - k_{ij}\psi_\theta(\|x_j - x_i\|) \\
&\quad \times\cos(\theta_j - \theta_i)\|\dot{\theta}_j - \dot{\theta}_i\|^2\Big\}.
\end{aligned}$$

By exchanging indices $i \leftrightarrow j$, the second equality is deduced. Young's inequality and minimum and maximum values of $\psi_\theta, \psi_\theta'$, and $\cos(\theta)$ yield the last line. Using the result $\sum_j v_j = \sum_j \dot{\theta}_j = 0$, from the above relation, we have the following inequality:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big(\sum_i \|\dot{\theta}_i\|^2\Big) \;&\le\; \frac{K}{2\varepsilon}\Big(-\psi_\theta'\left(\frac{1}{2\beta_2+1}\right)\|v\|^2 - \frac{K}{2}\Big[2\psi_\theta(2x_*)\cos D^\infty \\
&- \;\varepsilon\Big(-\psi_\theta'\left(\frac{1}{2\beta_2+1}\right)\Big)\Big]\|\dot{\theta}\|^2.
\end{aligned}
$$

Applying the Gronwall's inequality, we obtain

$$
\begin{aligned}
\|\dot{\theta}(t)\|^2 \;&\le\; \|\dot{\theta}^0\|^2 exp\Big(\left\{-2K\psi_\theta(2x_*)\cos D^\infty + \varepsilon K\left(-\psi_\theta'\left(\frac{1}{2\beta_2+1}\right)\right)\right\}t\Big) \\
&+ \;\frac{K}{\varepsilon}\Big(-\psi_\theta'\left(\frac{1}{2\beta_2+1}\right)\Big)\|v^0\|^2 exp(-2K\cos D^\infty\psi(2x_*)t).
\end{aligned}
$$

□

Lemma 5.4.3. *Let* (x, v, θ) *be a solution to* (5.1.2) *such that* $D(\theta(t)) < D^\infty < \frac{\pi}{2}$*. Then, we have*

(i) $\displaystyle \frac{d}{dt}\|\theta\| \le \|\Omega\| - \frac{\sin D^\infty}{D^\infty}\min_{i,j} k_{ij}(t)\psi_\theta(x^\infty)\|\theta\|, \quad t > 0,$

(ii) $\displaystyle \frac{d}{dt}\|\theta\| \ge -\|\Omega\| - \max_{i,j} k_{ij}(t)\psi_\theta(0)\|\theta\|.$

Proof. By exchanging the indices i and j, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\sum_{i=1}^N \|\theta_i\|^2 \;&=\; \sum_{i=1}^N \dot{\theta}_i\theta_i = \sum_{i=1}^N \omega_i\theta_i + \frac{1}{N}\sum_{i,j=1}^N k_{ij}\psi_\theta(\|x_j - x_i\|)\sin(\theta_j - \theta_i)\theta_i \\
&=\; \sum_{i=1}^N \omega_i\theta_i - \frac{1}{N}\sum_{i,j=1}^N k_{ij}\psi_\theta(\|x_j - x_i\|)\sin(\theta_j - \theta_i)\theta_j \\
&=\; \sum_{i=1}^N \omega_i\theta_i - \frac{1}{2N}\sum_{i,j=1}^N k_{ij}\psi_\theta(\|x_j - x_i\|)\sin(\theta_j - \theta_i)(\theta_j - \theta_i).
\end{aligned}
$$

The second equality in the above equation holds if the symmetricity of $\{k_\theta^{ij}(0)\}_{1\le i,j\le N}$ is satisfied. For any $\tilde{\theta} \in [0, D^\infty)$, $\dfrac{\sin(\tilde{\theta})}{\tilde{\theta}} > \dfrac{\sin D^\infty}{D^\infty}$ holds.

Thus,

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N}\|\theta_i\|^2 \quad \leq \quad \|\Omega\|\|\theta\| - \frac{\sin D^\infty}{D^\infty}\min_{1\leq i,j\leq N} k_\theta^{ij}(t)\psi_\theta(\|x_j - x_i\|)\|\Theta\|^2.$$

□

Theorem 5.4.1. *(Main Flocking and Synchronization) Suppose the parameters* ε, α *and initial position and velocity* (x^0, v^0) *and initial phase configuration and coupling strength satisfy the framework* $(\mathcal{B}1) - (\mathcal{B}3)$. *Then for any solution* (x, v, θ) *to* (5.1.2) *we have*

(i) $\lim_{t\to\infty} D(v(t)) = 0, \ \sup_{0\leq t<\infty} D(x(t)) < \infty,$

(ii) $\lim_{0\leq t<\infty}\max_{i,j}\|\dot{\theta}_i(t) - \dot{\theta}_j(t)\| = 0,$

$$\lim_{0\leq t<\infty}\max_{i,j}\|\theta_i(t) - \theta_j(t)\| \leq \frac{D^\infty D(\Omega)}{2K\sin D^\infty\psi_2(2x_*)},$$

for all $1 \leq i, j \leq N$.

Proof. First, we can verify that $D(v(t)) = \max_{i,j}|v_i(t) - v_j(t)| \leq \max_{i,j}|v_i(t)| + |v_j(t)| \leq 2\|V(t)\|$ holds. Applying the proposition (5.3.2) to this inequality, we obtain

$$D(v(t)) \leq 2\|v^0\|e^{-K\cos D^\infty\psi(2x_*)t}.$$

Letting $t \to \infty$, (i) is shown. The second result can be verified with the help of the first result. For any i, j, $x_i(t) - x_j(t)$ is expressed as $x_i(t) - x_j(t) = x_i(0) - x_j(0) + \int_0^t v_i(s) - v_j(s)ds$. By the triangular inequality, $|x_i(t) - x_j(t)| \leq |x_i(0) - x_j(0)| + \int_0^t |v_i(s) - v_j(s)|ds$ holds. Hence,

$$\begin{aligned} D(x(t)) \quad &\leq \quad D(x(0)) + \int_0^t 2D(v(s))ds \\ &\leq \quad D(x(0)) + 2\|v^0\|\frac{1 - e^{-K\cos D^\infty\psi(2x*)t}}{K\cos D^\infty\psi(2x_*)} \end{aligned}$$

is true, and this implies (ii). □

5.5 Numerical Simulations

In this section, we provide several numerical simulations to system (5.1.2) in order to supplement our results on the CSK model with Hebbian coupling. We set

$$\begin{aligned} \psi_v(\theta_j - \theta_i, x_j - x_i) &= \frac{\cos(\theta_j - \theta_i)}{(1 + \|x_j - x_i\|^2)^{\beta_1}}, \\ \psi_\theta(\|x_j - x_i\|) &= \frac{1}{(1 + \|x_j - x_i\|^2)^{\beta_2}}, \quad \beta_1,\ \beta_2 \geq 0. \end{aligned} \tag{5.5.1}$$

Under the ansatz (5.5.1), the system (5.1.2) with $\varepsilon = 1$ becomes

$$\begin{cases} \dfrac{dx_i}{dt} &= v_i, \quad t > 0, \quad 1 \leq i, j \leq N, \\ \dfrac{dv_i}{dt} &= \dfrac{K}{N} \displaystyle\sum_{j=1}^{N} \dfrac{\cos(\theta_j - \theta_i)}{(1 + \|x_j - x_i\|^2)^{\beta_1}} (v_j - v_i), \\ \dfrac{d\theta_i}{dt} &= \Omega_i + \frac{1}{N} \displaystyle\sum_{j=1}^{N} \dfrac{k_{ij}}{(1 + \|x_j - x_i\|^2)^{\beta_2}} \sin(\theta_j - \theta_i), \\ \dfrac{dk_{ij}}{dt} &= \varepsilon(\alpha \cos(\theta_j - \theta_i) - k^{ij}), \end{cases} \tag{5.5.2}$$

For different choices of β_1 and β_2, we will investigate the dynamic features of system (5.5.2). For simulations, we used the fourth-order Runge-Kutta method and $N = 100$. In all simulations, we consider a planar case $d = 2$ and initial data is randomly drawn.

5.5.1 Natural Frequency

Initial position x_{i0} and velocity v_{i0} are chosen randomly from the box $[-1, 1] \times [-1, 1]$ to satisfy zero sum conditions. To begin with, we check if the time evolution of the system varies depending on the natural frequency Ω_i.

Case 1: In this case, the initial natural frequencies are chosen 1 for the first half of the group and -1 for the other half. In other words,

$$\Omega_i = \begin{cases} 1, & i = 1, ..., 50, \\ -1, & i = 51, ..., 100. \end{cases}$$

This is the case when $D(\theta(0)) > \frac{\pi}{2}$ holds. We set the other parameters as

$$K = 5,\ \beta_1 = 0.1,\ \beta_2 = 0.1,\ \alpha = 5,\ \epsilon = 1,$$

and conduct numerical analysis and display some features. As Figures 5.1-4 show, phases of the particles cluster as two distinct groups.

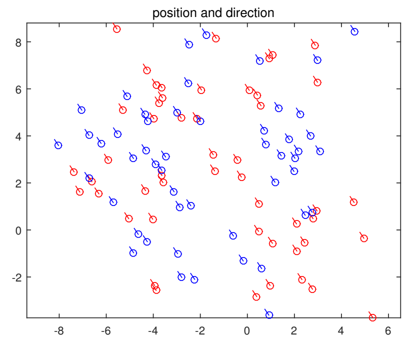

Figure 5.1: Position and direction for $\Omega = \pm 1$

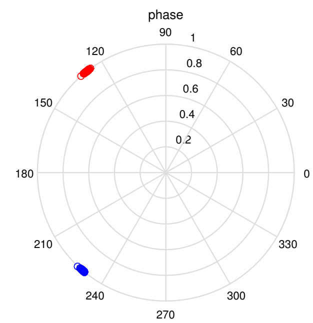

Figure 5.2: Phase for $\Omega = \pm 1$

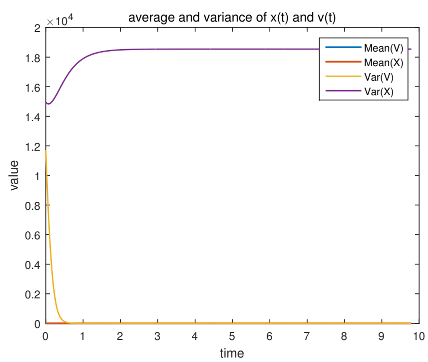

Figure 5.3: Average and Variance of $x(t)$ and $v(t)$ for $\Omega = \pm 1$

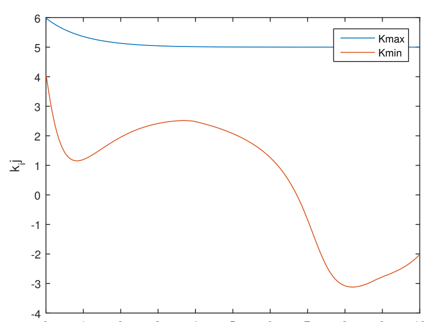

Figure 5.4: k_{max} and k_{min} for $\Omega = \pm 1$

Case 2:

$$\Omega_i \in U\left(-\frac{1}{2}, \frac{1}{2}\right),\ i = 1, ..., 100.$$

This is the case when the natural frequencies are chosen from the uniform distribution and the sufficient conditions hold. As time passes, both synchronization and flocking occur. The results are visualized as follows:

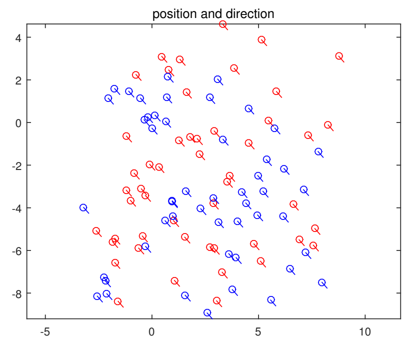

Figure 5.5: Position and direction for $\Omega \in U(-1/2, 1/2)$

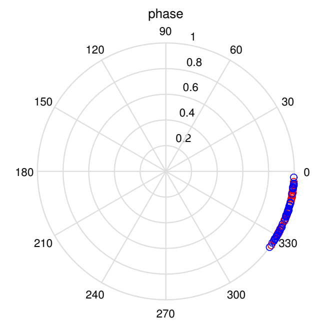

Figure 5.6: Phase for $\Omega \in U(-1/2, 1/2)$

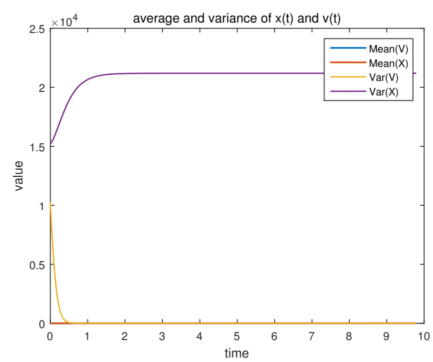

Figure 5.7: Average and Variance of $x(t)$ and $v(t)$ for $\Omega \in U(-1/2, 1/2)$

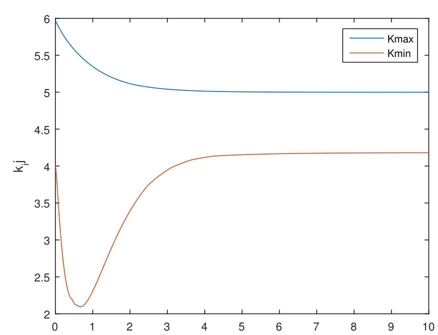

Figure 5.8: k_{max} and k_{min} for $\Omega \in U(-1/2, 1/2)$

Case 3:

$$\Omega_i = 0, \ i = 1, ..., 100.$$

If the natural frequencies are identically zero, the system quickly reaches the flocking and synchronization state. This lead $k_ij(t)$ to approach some constant number.

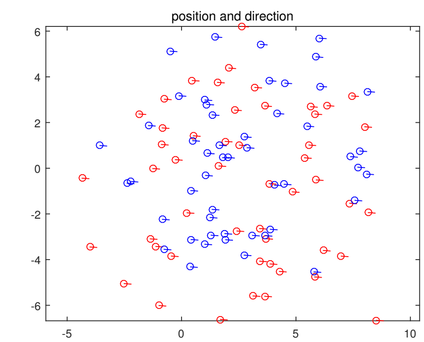

Figure 5.9: Position and direction for $\Omega = 0$

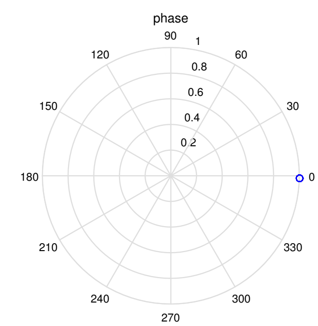

Figure 5.10: Phase for $\Omega = 0$

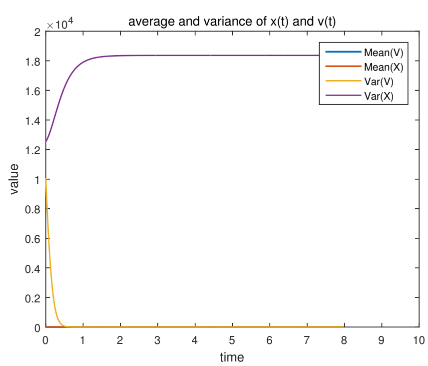

Figure 5.11: Average and Variance of $x(t)$ and $v(t)$ for $\Omega = 0$

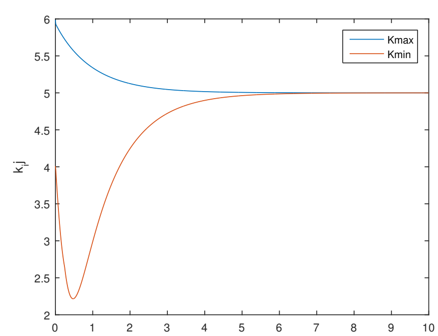

Figure 5.12: k_{max} and k_{min} for $\Omega = 0$

5.5.2 Intensity of Interaction

Case 1: The natural frequencies for oscillators are given as

$$\Omega_i = \begin{cases} 1, & i = 1, ..., 50, \\ -1, & i = 51, ..., 100. \end{cases}$$

In this part, we first raise β_1 to $\beta_1 = 0.5$, make the other parameters remain the same($K = 5$, $\beta_2 = 0.1$, $\alpha = 5$, $\epsilon = 1$) and conduct the numerical analysis. After this, the value of β_2 is changed into $\beta_2 = 0.5$ with the other parameters fixed as $K = 5$, $\beta_1 = 0.1$, $\alpha = 5$, $\epsilon = 1$ and the analysis is repeated. This is the case when a long-range interaction is replaced by a short-range interaction. The results regarding the change of β_1 are visualized as follows:

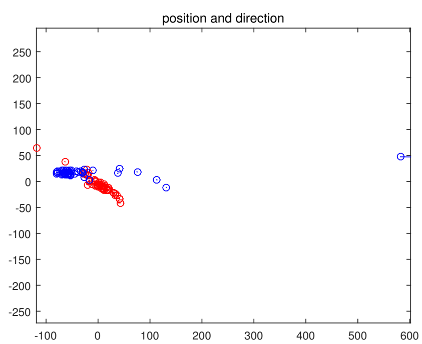

Figure 5.13: Position and direction for $\Omega = \pm 1$

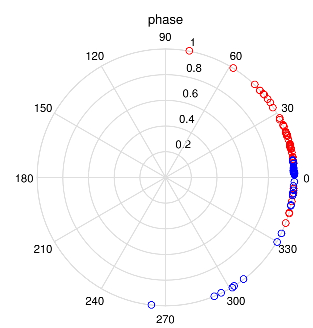

Figure 5.14: Phase for $\Omega = \pm 1$

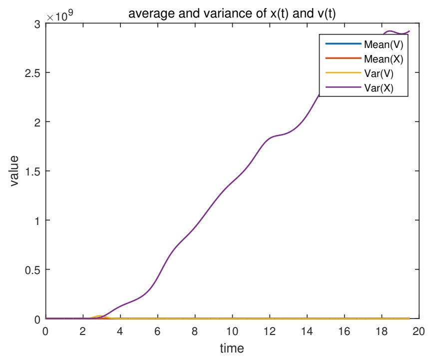

Figure 5.15: Average and Variance of $x(t)$ and $v(t)$ for $\Omega = \pm 1$

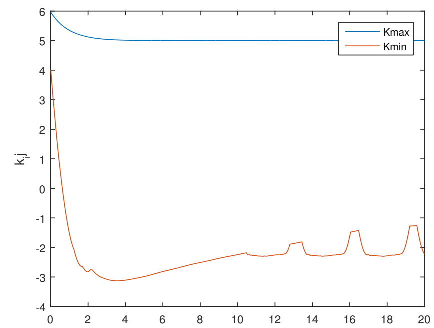

Figure 5.16: k_{max} and k_{min} for $\Omega = \pm 1$

The results regarding the change of β_2 are visualized as follows:

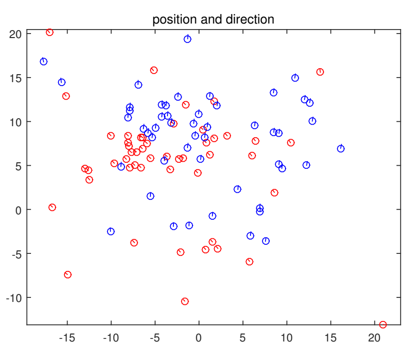

Figure 5.17: Position and direction for $\Omega = \pm 1$

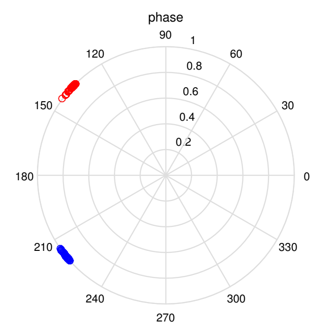

Figure 5.18: Phase for $\Omega = \pm 1$

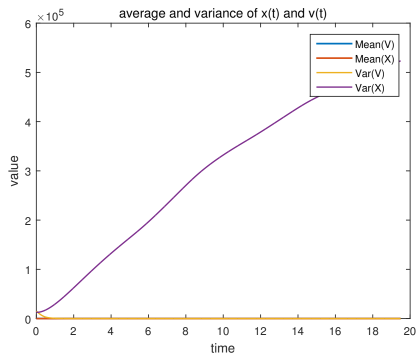

Figure 5.19: Average and Variance of $x(t)$ and $v(t)$ for $\Omega = \pm 1$

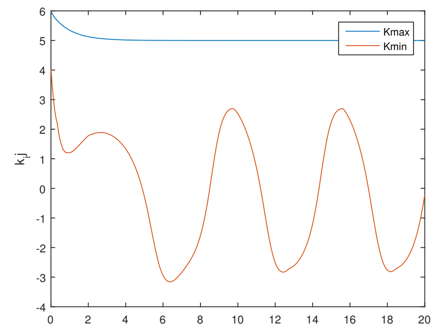

Figure 5.20: k_{max} and k_{min} for $\Omega = \pm 1$

Case 2: In the case 2, the natural frequency for oscillators is picked from a uniform distribution.

$$\Omega_i \in U\left(-\frac{1}{2}, \frac{1}{2}\right)$$

In this part, we change β_1 and β_2 one by one and repeat the same procedure as Case 1. With more repeated numerical experiments, these results contributes to find out critical conditions for clustering, unflocking-synchronization, and flocking-desynchronization. The results regarding the change of β_1 are visualized as follows:

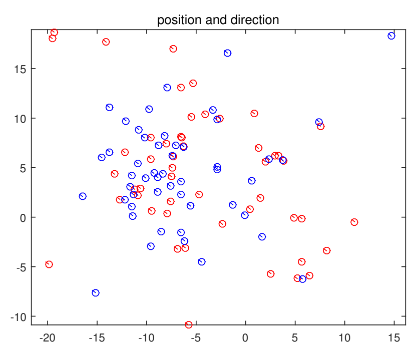

Figure 5.21: Position and direction for $\Omega \in U(-1/2, 1/2)$

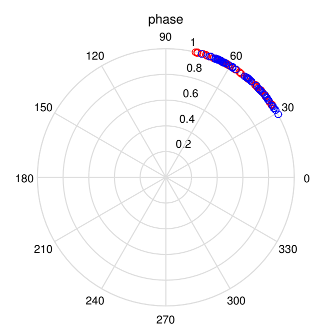

Figure 5.22: Phase for $\Omega \in U(-1/2, 1/2)$

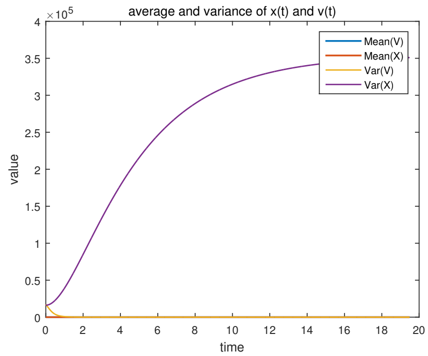

Figure 5.23: Average and Variance of $x(t)$ and $v(t)$ for $\Omega \in U(-1/2, 1/2)$

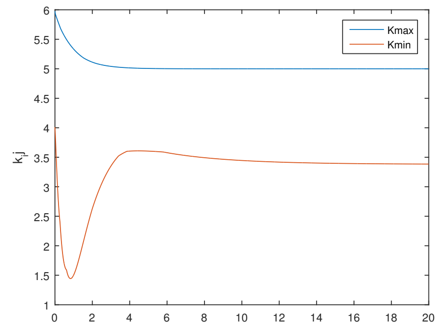

Figure 5.24: k_{max} and k_{min} for $\Omega \in U(-1/2, 1/2)$

The results regarding the change of β_2 are visualized as follows:

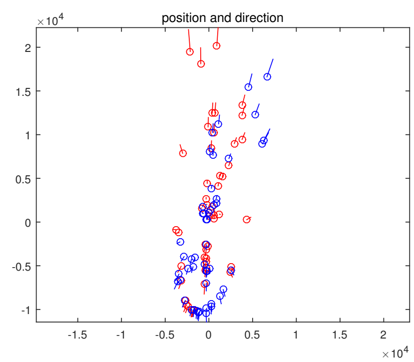

Figure 5.25: Position and direction for $\Omega \in U(-1/2, 1/2)$

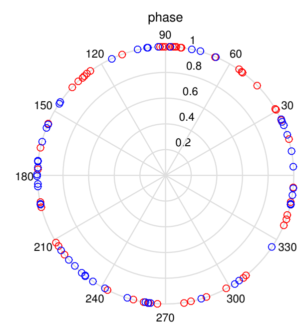

Figure 5.26: Phase for $\Omega \in U(-1/2, 1/2)$

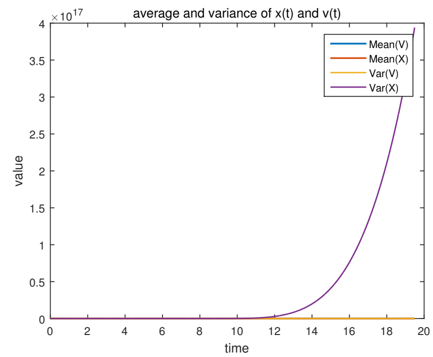

Figure 5.27: Average and Variance of $x(t)$ and $v(t)$ for $\Omega \in U(-1/2, 1/2)$

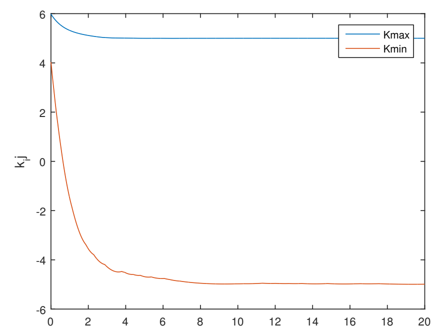

Figure 5.28: k_{max} and k_{min} for $\Omega \in U(-1/2, 1/2)$

Chapter 6

Conclusion

We begin the thesis with reviewing previous flocking models appearing in other literatures and extended the stochastic Cucker-Smale model to a mean-field kinetic model that leads us to derive the Cucker-Smale-Fokker-Planck equation (CS-FP equation). For the main part, we prove the wellposedness of the CS-FP equation by applying the Sobolev embedding theorem to the energy estimates for a weak solution in an admissible set. In addition, we deal with the Cucker-Smale-Kuramoto model(CSK model) and verify sufficient conditions for occuring both flocking and synchronization with the Lyapunov functional approach. As a special case for the CSK model, the CSK model with the Hebbian coupling is introduced, and its numerical simulations are covered. For the Hebbian coupling case, the numerical results suggest clustering and other nontrivial time evolution depending on the range of interaction and the initial configuration.

For the future work in the CS-FP equation, the threshhold phenomena of the energy functional depending of K_c need further be studied for a general communication weight function $\psi(s)$. For the future work in the CSK model, the sufficient conditions need be eased and subdivided so that conditions for the clustering, flocking-desynchronization and unflocking-synchronization are completed.

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국문초록

이 논문에서는 퇴화 확산 계수가 관여된 쿠커-스메일-포커-플랑크형 방정식을 다룬다. 쿠커-스메일-포커-플랑크 방정식은 확률 환경 하에서 무한의 쿠커-스메일 플로킹 입자의 위치, 속도에 관한 분포함수의 편미분 방정식으로 표현된다. 본 연구에서는 분포함수의 고계 편미분 도함수에 대한 에너지 평가식을 유도하여 쏘볼레프 몰입정리를 적용하고, 균등 포물형 방정식으로 해를 근사함으로 초기 조건이 충분히 매끄럽게 주어졌을때 쿠커-스메일-포커-플랑크 방정식의 해가 국소적으로 존재함을 증명하고, 이를 대역해로 확장한다.

다음으로 플로킹과 동기화를 연결지어 세운 쿠커-스메일-쿠라모토 모델을 소개한다. 본 연구에서는 쿠커-스메일-쿠라모토 모델에서 플로킹과 동기화가 발생하는 충분조건을 리아프노프 범함수를 이용하여 확증하고, 특별한 초기 조건에서의 모델의 양상을 수치적 계산 결과와 함께 제시한다.

주요어휘: 쿠커-스메일 모델, 플로킹, 쿠커-스메일-포커-플랑크 방정식, 스레쉬홀드 현상,쿠커-스메일-쿠라모토 방정식, 동기화 현상
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