이학박사 학위논문

Analytic valuation of American path-dependent options

(경로에 의존하는 미국형 옵션의 해석적 가치 평가)

2016년 8월

서울대학교 대학원

수리과학부

전준기

Analytic valuation of American path-dependent options

(경로에 의존하는 미국형 옵션의 해석적 가치 평가)

지도교수 강 명 주

이 논문을 이학박사 학위논문으로 제출함

2016년 4월

서울대학교 대학원

수리과학부

전준기

전준기의 이학박사 학위논문을 인준함

2016년 6월

위 원 장 ________________________ (인)

부 위 원 장 ________________________ (인)

위 원 ________________________ (인)

위 원 ________________________ (인)

위 원 ________________________ (인)

Analytic valuation of American path-dependent options

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

Junkee Jeon

Dissertation Director : Professor Myungjoo Kang

Department of Mathematical Sciences
Seoul National University

August 2016

Abstract

American options are type of options that can be exercised anytime during their life. Therefore, the valuation of such options is usually classified as optimal stopping problems or free boundary problems. I derive the analytic pricing formulas and integral equations of American chained options, Russian options with finite time horizon, American floating strike lookback options, and American maximum quanto options. To verify the derived pricing formula and the integral equation satisfied by the free boundary are correct, we numerically solve the derived integral equations using recursive integration method or simple iterative method.

Key words: Option pricing, Path-dependent option, American option, Integral equation, Free boundary problem, Mellin transform
Student Number: 2009-22894

Contents

Chapter 1

Introduction

This thesis is organized as follows. In chapter 2, we consider the pricing of American chained knock-in options. We prove mathematically that the value of knock-in American chained barrier options are expressed in terms of the value of knock-in American options by using reflection principle of Brownian motions. That is, we derive the integral equation satisfied by American chained barrier options. This leads to more accurate valuation of American chained barrier options. Our method is also useful for valuing European chained options as well.

In chapter 3, we deals with the analytic valuation of Russian option with finite time horizon. We derive analytic solution for the inhomogeneous Black-Scholes equation with mixed boundary conditions by using Mellin transform approach. Mixed boundary condition usually arises in the option pricing problem when the underlying asset involves maximum process. We formulate Russian options as a PDE with mixed boundary conditions and obtain the integral equation satisfied by Russian option values by using the analytic formula we derived. We get numerical solutions for the integral equation by applying recursive iteration method, Also, we present the computational speed and accuracy of recursive integration by comparing its numerical results with some of existing methods in the literature.

In chapter 4, we presents our study of American floating strike lookback options written on dividend-paying assets. The valuation of these options can be mathematically formulated as a free boundary inhomogeneous Black-Scholes PDE with a Neumann boundary condition, which we, by using a Mellin transform, convert into a relatively simple ordinary differential equation with Dirichlet boundary conditions. We then use these results to derive an integral equation that can be used to calculate the price of American floating strike lookback options. In addition, we also used Mellin transforms to derive the closed-form of the perpetual case.

In chapter 5, we derive analytic pricing formula for American exchange rate quanto lookback options which apply the value of realized maximum exchange rate until maturity. We first formulate two dimensional inhomogeneous Black-Scholes PDE with mixed boundary conditions whose solution is the value of American maximum exchange rate quanto lookback options. To solve the formulated PDE, we apply the double Mellin transform techniques to get an analytic solution for a general two-dimensional inhomogeneous Black-Scholes PDE with mixed boundary conditions. Using the analytic solution, we finally derive the pricing formula for American maximum exchange rate quanto lookback options and the integral equation satisfied by the free boundary of such options. To verify that our theoretically derived integral equation is indeed correct, we solve the derived equation numerically using 'extended simple iterative method' and compare this solution with a benchmark solution which is obtained by modified binomial tree method. We also plot the free boundary and value of American quanto lookback options with different parameters such as domestic/foreign risk-neutral interest rate, volatility, dividend yield.

Chapter 2

Chained knock-in barrier option

Barrier options are one of the most popular path-dependent derivatives in various markets, particularly in OTC markets and FX markets, since barrier options are cheaper and they are more liquid than vanilla options. Therefore, many people have researched on barrier options so far. Reiner and Ruinstein [61] derived a formula for barrier options. Rich [62] also got the value of barrier option under mathematical framework.

As barrier options have become popular, a variety of new barrier options which consist of more complicated contract emerged. For example, German and Yor [21] derived the price of double barrier options using Laplace transforms. Heynan and Kat [26] studied partial barrier options as well. Especially, we note the paper by Jun and Ku [39] which handle a special type of barrier option with two barrier levels. Different from usual double barrier options, the second barrier level for this option is activated only when the underlying asset hits the first barrier level. Therefore, option is worthless if it does not cross two barrier levels in a specific order. These kind of options were first treated by Pfeffer [60] and Li [55], and extended more generally by Jun and Ku. This types of options have recently become popular in the Japanese over-the-counter equity and foreign exchange derivative markets.

Jun and Ku named such options as chained barrier options and studied on pricing them when two barrier levels are given by exponential functions of time [40]. Furthermore, in [41], they approximated the value of American chained barrier options using the approximation method of Ingersoll [28], which is based on constant exercise policies of barrier options.

2.1 Preliminaries

The usual assumptions for the Black-Scholes option pricing framework are adopted in this work. The stock price S_t is assumed to follow the risk neutral process

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t \tag{2.1.1}$$

where r is the risk-free interest rate, σ and q are the volatility and dividend yields of S, respectively, and W_t is a one-dimensional standard Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0} \equiv \mathbb{F}$ is the natural filtration generated by W_t.

Consider a knock-in American call option where the knock-in trigger clause entitles the holder to receive an American call option with strike price K when underlying asset S passes the threshold level D. Then, in view of optimal stopping problems, the value $CA_{id}(t,S,K,D)$ of the American knock-in call option with expiration T is expressed by

$$CA_{id}(t,S,K,D) = \sup_{t\leq\tau\leq T} E^P[e^{-r(\tau-t)}(S_\tau - K)^+ \mathbf{1}_{\{\min_{0\leq\gamma\leq\tau} S_\gamma \leq D\}} \mid S_t = S] \tag{2.1.2}$$

under the risk-neutral measure P. Similarly, the value of $PA_{iu}(t,S,K,U)$ of the American knock-in put option with expiration T is given by

$$PA_{id}(t,S,K,D) = \sup_{t\leq\tau\leq T} E^P[e^{-r(\tau-t)}(K - S_\tau)^+ \mathbf{1}_{\{\min_{0\leq\gamma\leq\tau} S_\gamma \leq D\}} \mid S_t = S] \tag{2.1.3}$$

Dai and Kwok [10] proved the following theorems.

Theorem 2.1.1 (The value of American knock-in barrier options)

(1) (American down-and-knock-in barrier call options : CA_{id})

Let $CE_{id}(t,S,K,D)$ and $CA_{id}(t,S,K,D)$ denote the pricing function of the European down-and-knock-in barrier call options and American down-and-knock-in barrier call options, respectively, both with down-and-in barrier D, strike price K and expiration T. For $D \leq K \max(1, r/q)$,

$$\begin{aligned} &CA_{id}(t,S,K,D) \\ &= \left(\frac{D}{S}\right)^{\frac{2(r-q)}{\sigma^2}-1}\left[CA(t,\frac{D^2}{S},K) - CE(t,\frac{D^2}{S},K)\right] \\ &\quad + CE_{id}(t,S,K,D) \end{aligned} \tag{2.1.4}$$

where $CE(t, S, K)$ and $CA(t, S, K)$ are the pricing functions of European vanilla call options, American vanilla call options, respectively, both with strike price K and expiration T .
Especially, if $D \leq K$, CA_{id} can be simplified as follows.

$$CA_{id}(t, S, K, D) = \left(\frac{D}{S}\right)^{\frac{2(r-q)}{\sigma^2}-1} CA(t, \frac{D^2}{S}, K) \tag{2.1.5}$$

(2) (American up-and-knock-in barrier put options : PA_{iu})

Let $PE_{id}(t, S, K, U)$ and $PA_{id}(t, S, K, U)$ denote the pricing function of the European up-and-knock-in barrier barrier put option and American up-and-knock-in barrier put option, respectively, both with up-and-in barrier U, strike price K and expiration T. For $U \geq K \min(1, r/q)$,

$$\begin{aligned} & PA_{id}(t, S, K, U) \\ = & \left(\frac{U}{S}\right)^{\frac{2(r-q)}{\sigma^2}-1} \left[PA(t, \frac{U^2}{S}, K) - PE(t, \frac{U^2}{S}, K)\right] \\ & + PE_{id}(t, S, K, U) \end{aligned} \tag{2.1.6}$$

where $PE(t, S, K)$ and $PA(t, S, K)$ are the pricing functions of European vanilla put options, American vanilla put options, respectively, both with strike price K and expiration T.
Especially, if $U \geq K$, PA_{iu} can be simplified as follows.

$$PA_{id}(t, S, K, U) = \left(\frac{U}{S}\right)^{\frac{2(r-q)}{\sigma^2}-1} PA(t, \frac{U^2}{S}, K) \tag{2.1.7}$$

In [10], Dai and Kwok analyzed the value of knock-in put options PA_{iu} according to the value of U and K(for put options). They showed that for $U < K \min(1, r/q)$, there is no analytic formula for PA_{id} as (2.1.6) in Theorem 2.1.1.
Therefore, in this chapter, we assume that $U \geq K \min(1, r/q)$ (for put option) and $D \leq K \max(1, r/q)$ (for call option).

2.2 Analytic Valuation of Chained American Barrier Options

In this section, we derive mathematically that knock-in American chained barrier options are expressed in terms of knock-in American options.

In [41], Jun and Ku obtained the value of American chained knock-in put options using approximation method introduced in [28]. But this method works only for $U \geq K$. We derive the analytic formula for American chained knock-in options using Theorem 2.1.1 and the relation of PA_{id} and PA_{diu}. Our formula works for a wide range, $U \geq K \min(1, r/q)$.

For S_t defined in (2.1.1), we have

$$S_t = S_0 \exp(\mu t + \sigma W_t) \tag{2.2.1}$$

where $\mu = r - q - \frac{\sigma^2}{2}$.

We fix an upper barrier $U(> S_0)$ and a down barrier $D(< S_0)$. For convenience, we also define a function $g(x) := \frac{1}{\sigma} \log(x/S_0)$ for $x > 0$ and let $u := g(U)$, $d := g(D)$ and $L_t := g(S_t)$.

2.2.1 Crossing a single barrier

In this subsection, we consider the pricing formula for knock-in European and American chained barrier options which are activated when the state variable S hits the upper barrier U or the lower barrier D. The following theorem 2.2.1 is the pricing formula for down-and-knock-in European and American chained barrier options, and we state analogous formula for up-and-knock-in chained barrier options in corollary 2.2.2

Remark The chained down(up)-and-knock-in barrier European call options are denoted by $CE_{uid}(CE_{diu})$ and means that the lower(upper) barrier level $D(U)$ activates only when the underlying asset hits the upper(lower) barrier level $U(D)$ first. The same notation is used for the chained put options, PE_{uid}, PE_{diu}.

We proved the relation of chained options and the knock-in options via the following theorem. To the best of our knowledge, there are no papers which states the relation of American chained options and American knock-in options.

Theorem 2.2.1 Let us consider the chained down-and-knock-in European(CE_{uid}) and
American(CA_{uid}) call options expiring at T with strike price $K(\geq D)$, which are activated at time $\tau_u = \min\{t > 0 \mid S_t = U\}$.
For $t_0 < \tau_u$,

$$\begin{aligned} CE_{uid}(t_0, S_{t_0}, K, U, D) &= \left(\frac{U}{S_{t_0}}\right)^{\frac{2\mu}{\sigma^2}} CE_{id}(t_0, \frac{U^2}{S_{t_0}}, K, D) \\ CA_{uid}(t_0, S_{t_0}, K, U, D) &= \left(\frac{U}{S_{t_0}}\right)^{\frac{2\mu}{\sigma^2}} CA_{id}(t_0, \frac{U^2}{S_{t_0}}, K, D) \end{aligned} \tag{2.2.2}$$

Proof First, we consider the chained down-and-knock-in European call option(CE_{uid}).

For $t_0 < \tau_u$, under the risk-neutral measure P,

$$\begin{aligned} &CE_{uid}(t_0, S_{t_0}, K, U, D) \\ &= e^{-r(T-t_0)} E^P[(S_T - K)^+ \mathbf{1}_{\{\min_{\tau_u \leq \gamma \leq T} S_\gamma \leq D,\ \tau_u \leq T,\ S_{\tau_u} = U\}} \mid \mathcal{F}_{t_0}] \\ &= e^{-r(T-t_0)} E^P[(S_0 e^{\sigma L_T} - K)^+ \mathbf{1}_{\{\min_{\tau_u \leq \gamma \leq T} L_\gamma \leq d,\ \tau_u \leq T,\ L_{\tau_u} = u\}} \mid \mathcal{F}_{t_0}] \end{aligned}$$

where the second equation follows from Girsanov's theorem which asserts that $L_t := W_t + \frac{\mu}{\sigma}t$ is a standard Brownian motion under Q measure, defined by

$$\frac{dQ}{dP} = e^{-\frac{\mu}{\sigma}W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \tag{2.2.3}$$

Then,

$$\begin{aligned} &CE_{uid}(t_0, S_{t_0}, K, U, D) \\ =&E^Q[(S_0 e^{\sigma L_T} - K)^+ e^{\frac{\mu}{\sigma}W_T + \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \mathbf{1}_{\{\min_{\tau_u \leq \gamma \leq T} L_\gamma \leq d,\ \tau_u \leq T,\ L_{\tau_u} = u\}} \mid \mathcal{F}_{t_0}] \\ &\times e^{-r(T-t_0)} e^{-\frac{\mu}{\sigma}W_{t_0} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t_0} \end{aligned}$$

For $0 \leq t \leq T$, let us define a process $\tilde{L}_t$ defined by the formula

$$\tilde{L}_t = \begin{cases} L_t & t \leq \tau_u \\ 2u - L_t & t > \tau_u \end{cases} \tag{2.2.4}$$

By reflection principle, $\tilde{L}_t$ is a standard Brownian motion under Q measure. Now note that $2u - d > u$, $\left\{\tilde{L}_\gamma | \max_{\tau_u \leq \gamma \leq T} \tilde{L}_\gamma \geq 2u - d,\ \tau_u \leq T\right\} = \left\{\tilde{L}_\gamma | \max_{0 \leq \gamma \leq T} \tilde{L}_\gamma \geq 2u - d\right\}$. Therefore,

$$\begin{aligned}
&CE_{uid}(t_0, S_{t_0}, K, U, D) \\
=&E^Q[(S_0 e^{\sigma(2u - \tilde{L}_T)} - K)^+ e^{\frac{\mu}{\sigma}(2u - \tilde{L}_T) - \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \mathbf{1}_{\{\max_{\tau_d \leq \gamma \leq T} \tilde{L}_\gamma \geq 2u - d,\ \tau_u \leq T\}} \mid \mathcal{F}_{t_0}] \\
=&E^Q[(S_0 e^{\sigma(2u - \tilde{L}_T)} - K)^+ e^{-\frac{\mu}{\sigma}\tilde{L}_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \mathbf{1}_{\{\max_{0 \leq \gamma \leq T} \tilde{L}_\gamma \geq 2u - d\}} \mid \mathcal{F}_{t_0}] \\
&\times e^{-r(T - t_0)} e^{\frac{2\mu u}{\sigma}} e^{-\frac{\mu}{\sigma}W_{t_0} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t_0}
\end{aligned} \tag{2.2.5}$$

Let us again defined an equivalent probability measure $\tilde{P}$ by setting

$$\frac{d\tilde{P}}{dQ} = e^{-\frac{\mu}{\sigma}\tilde{L}_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \tag{2.2.6}$$

so that the process $\tilde{W}_t := \tilde{L}_t + \frac{\mu}{\sigma}t$, $t \in [0, T]$ follows a standard Brownian motion under $\tilde{P}$.

$$\begin{aligned}
&CE_{uid}(t_0, S_{t_0}, K, U, D) \\
=&E^{\tilde{P}}[(S_0 e^{2\sigma u} e^{\mu T + \sigma(-\tilde{W}_T)} - K)^+ \mathbf{1}_{\{\min_{0 \leq \gamma \leq T} (\mu\gamma + \sigma(-\tilde{W}_\gamma) \leq \sigma(d - 2u)\}} \mid \mathcal{F}_{t_0}] \\
&\times e^{-r(T - t_0)} e^{\frac{2\mu u}{\sigma}} e^{-\frac{\mu}{\sigma}W_{t_0} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t_0} e^{-\frac{\mu}{\sigma}\tilde{L}_{t_0} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t_0} \\
=&E^{\tilde{P}}[(S_0 e^{2\sigma u} e^{\mu T + \sigma(-\tilde{W}_T)} - K)^+ \mathbf{1}_{\{\min_{0 \leq \gamma \leq T} (\mu\gamma + \sigma(-\tilde{W}_\gamma) \leq \sigma(d - 2u)\}} \mid \mathcal{F}_{t_0}] \\
&\times e^{-r(T - t_0)} \left(\frac{U}{S_0}\right)^{\frac{2\mu}{\sigma^2}} \left(\frac{S_0}{S_{t_0}}\right)^{\frac{2\mu}{\sigma^2}}
\end{aligned}$$

The last equality follows from the stock price process (2.2.1) and the definition of u.

Now define new process V_t for $0 \leq t \leq T$ as follows.

$$V_t = S_0 e^{2\sigma u} e^{\mu t + \sigma(-\tilde{W}_t)} \tag{2.2.7}$$

For $M_t := \mu t + \sigma(-\tilde{W}_t)$, by Ito's formula,

$$\begin{aligned} dV_t &= S_0 e^{2\sigma u} e^{\mu t+\sigma(-\tilde{W}_t)} \cdot dM_t + \frac{1}{2} S_0 e^{2\sigma u} e^{\mu t+\sigma(-\tilde{W}_t)} \cdot (dM_t)^2 \\ &= V_t \cdot (\mu \cdot dt - \sigma d\tilde{W}_t) + \frac{1}{2} V_t \cdot \sigma^2 \cdot dt \end{aligned} \tag{2.2.8}$$

Then, V_t satisfies following SDE :

$$dV_t = (r-q)V_t dt + \sigma V_t d(-\tilde{W}_t) \tag{2.2.9}$$

where $-\tilde{W}_t$ is a standard Brownian motion under $\tilde{P}$ measure.

By definition of L_t and $\tilde{L}_t$,

$$\tilde{W}_t = \begin{cases} W_t + \frac{2\mu}{\sigma} t & t \le \tau_u \\ 2u - W_t & t > \tau_u \end{cases} \tag{2.2.10}$$

Since $t_0 < \tau_u$,

$$V_{t_0} = S_0 \left(\frac{U}{S_0}\right)^2 e^{-\mu t_0 - \sigma W_{t_0}} = \frac{U^2}{S_{t_0}} \tag{2.2.11}$$

Therefore,

$$\begin{aligned} CE_{uid}(t_0, S_{t_0}, K, U, D) &= \left(\frac{U}{S_{t_0}}\right)^{\frac{2\mu}{\sigma^2}} e^{-r(T-t_0)} E^{\tilde{P}}[(V_T - K)^+ \mathbf{1}_{\{\min_{0\le\gamma\le T} V_\gamma \le D\}} \mid \mathcal{F}_{t_0}] \\ &= \left(\frac{U}{S_{t_0}}\right)^{\frac{2\mu}{\sigma^2}} CE_{id}(t_0, \frac{U^2}{S_{t_0}}, K, D) \end{aligned} \tag{2.2.12}$$

By (2.2.11), (2.2.12) and standard theory of optimal stopping problem,

$$\begin{aligned} & CA_{uid}(t_0, S_{t_0}, K, U, D) \\ &= \sup_{t_0 \le \tau \le T} e^{-r(\tau-t_0)} E^{\tilde{P}}[(S_\tau - K)^+ \mathbf{1}_{\{\min_{\tau_u\le\gamma\le\tau} S_\gamma \le D,\ \tau_u \le \tau,\ S_{\tau_u}=U\}} \mid \mathcal{F}_{t_0}] \\ &= \sup_{t_0 \le \tau \le T} \left(\frac{U}{S_{t_0}}\right)^{\frac{2\mu}{\sigma^2}} e^{-r(\tau-t_0)} E^{\tilde{P}}[(V_\tau - K)^+ \mathbf{1}_{\{\min_{0\le\gamma\le\tau} V_\gamma \le D\}} \mid \mathcal{F}_{t_0}] \\ &= \left(\frac{U}{S_{t_0}}\right)^{\frac{2\mu}{\sigma^2}} CA_{id}(t_0, \frac{U^2}{S_{t_0}}, K, D) \end{aligned} \tag{2.2.13}$$

Hence, we have proved the desired result.
□

The following corollary can be obtained by using similar techniques in Theorem 2.2.1.

Corollary 2.2.2 Consider the chained up-and-knock in European(PE_{diu}) and American(PA_{diu}) put options expiring at T with strike price $K(< U)$ which are activated at time $\tau_d = \min\{t > 0 \mid S_t = D\}$. Then, for $t_0 < \tau_d$,

$$\begin{aligned} PE_{diu}(t_0, S_{t_0}, K, U, D) &= \left(\frac{D}{S_{t_0}}\right)^{\frac{2\mu}{\sigma^2}} PE_{iu}(t_0, \frac{D^2}{S_{t_0}}, K, D) \\ PA_{diu}(t_0, S_{t_0}, K, U, D) &= \left(\frac{D}{S_{t_0}}\right)^{\frac{2\mu}{\sigma^2}} PA_{iu}(t_0, \frac{D^2}{S_{t_0}}, K, D) \end{aligned} \tag{2.2.14}$$

2.2.2 Crossing two barriers

This subsection addresses a knock-in European and American chained barrier option activated in the event that the underlying asset hits the downstream barrier D followed by reaching the upstream barrier U, or vice versa.

Remark The chained down(up)-and-knock-in barrier European call options crossing two barriers are denoted by $CE_{duid}(CE_{uidu})$ and means that the last lower(upper) barrier level $D(U)$ activates only when the hits the barrier $D(U)$ followed by reaching the barrier $U(D)$. The same notation is used for the chained put options, PE_{duid}, PE_{udiu}.

Theorem 2.2.3 Consider the chained down-and-knock-in European(CE_{duid}) and American(CA_{duid}) call options expiring at T with strike price $K(\geq D)$, activated at time
$\tau_{du} = \min\{t > \tau_d \mid S_t = U,\ \tau_d = \min\{t > 0 \mid S_t = D,\ D < S_0\}\}$.
For $t_0 < \tau_d$,

$$\begin{aligned} CE_{duid}(t_0, S_{t_0}, K, U, D) &= \left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} CE_{id}(t_0, \left(\frac{U}{D}\right)^2 S_{t_0}, K, D) \\ CA_{duid}(t_0, S_{t_0}, K, U, D) &= \left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} CA_{id}(t_0, \left(\frac{U}{D}\right)^2 S_{t_0}, K, D) \end{aligned} \tag{2.2.15}$$

Proof First, we consider the chained down-and-knock-in European call option(CE_{duid}).

For $t_0 < \tau_{du}$, under the risk-neutral measure P,

$$
\begin{aligned}
&CE_{duid}(t_0, S_{t_0}, K, U, D)\\
&= e^{-r(T-t_0)}E^P[(S_T - K)^+ \mathbf{1}_{\{\min_{\tau_{du} \le \gamma \le T} S_\gamma \le D,\ \tau_{du} \le T,\ ,S_{\tau_d}=D,\ S_{\tau_{du}}=U\}} \mid \mathcal{F}_{t_0}]\\
&= e^{-r(T-t_0)}E^P[(S_0 e^{\sigma L_T} - K)^+ \mathbf{1}_{\{\min_{\tau_{du} \le \gamma \le T} L_\gamma \le d,\ \tau_{du} \le T,\ ,L_{\tau_d}=d,\ L_{\tau_{du}}=u\}} \mid \mathcal{F}_{t_0}]\\
&= E^Q[(S_0 e^{\sigma L_T} - K)^+ e^{\frac{\mu}{\sigma}W_T + \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \mathbf{1}_{\{\min_{\tau_{du} \le \gamma \le T} L_\gamma \le d,\ \tau_{du} \le T,\ ,L_{\tau_d}=d,\ L_{\tau_{du}}=u\}} \mid \mathcal{F}_{t_0}]\\
&\times e^{-r(T-t_0)} e^{-\frac{\mu}{\sigma}W_{t_0} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t_0}
\end{aligned}
$$

where the process L_t and measure Q are defined in (2.2.3), respectively.

For $0 \le t \le T$, let us define a process $\hat{L}_t$ defined by the formula

$$
\hat{L}_t = \begin{cases} L_t & t \le \tau_d \\ 2d - L_t & t > \tau_d \end{cases} \tag{2.2.16}
$$

By the virtue of reflection principle, $\hat{L}_t$ is a standard Brownian motion under Q measure.

$$
\begin{aligned}
&CE_{duid}(t_0, S_{t_0}, K, U, D)\\
&= E^Q[(S_0 e^{\sigma L_T} - K)^+ e^{\frac{\mu}{\sigma}W_T + \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \mathbf{1}_{\{\min_{\tau_{du} \le \gamma \le T} L_\gamma \le d,\ \tau_{du} \le T,\ ,L_{\tau_d}=d,\ L_{\tau_{du}}=u\}} \mid \mathcal{F}_{t_0}]\\
&\quad \times e^{-r(T-t_0)} e^{-\frac{\mu}{\sigma}W_{t_0} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t_0}\\
&= E^Q[(S_0 e^{\sigma(2d - \hat{L}_T)} - K)^+ e^{\frac{\mu}{\sigma}(2d - \hat{L}_T) - \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \mathbf{1}_{\{\max_{\tau_{du} \le \gamma \le T} \hat{L}_\gamma \ge d,\ \tau_{du} \le T,\ \hat{L}_{\tau_{du}}=2d-u\}} \mid \mathcal{F}_{t_0}]\\
&\quad \times e^{-r(T-t_0)} e^{-\frac{\mu}{\sigma}W_{t_0} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t_0}
\end{aligned}
$$

Here, we apply the reflection principle again. Let us introduce a process G_t, $t \in [0, T]$, defined by

$$
G_t = \begin{cases} \hat{L}_t & t \le \tau_{du} \\ 2(2d - u) - \hat{L}_t & t > \tau_{du} \end{cases} \tag{2.2.17}
$$

Then,

$$
\begin{aligned}
&CE_{duid}(t_0, S_{t_0}, K, U, D)\\
&= E^Q[(S_0 e^{\sigma(2(u-d)+G_T)} - K)^+ e^{\frac{\mu}{\sigma}G_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \mathbf{1}_{\{\min_{\tau_{du}\le\gamma\le T} G_\gamma \le 3d-2u,\ \tau_{du}\le T,\ G_{\tau_{du}}=2d-u\}} \mid \mathcal{F}_{t_0}]\\
&\times e^{-r(T-t_0)} e^{\frac{2\mu}{\sigma}(u-d)} e^{-\frac{\mu}{\sigma}W_{t_0} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t_0}
\end{aligned}
$$

By the assumption $D < S_0 < U$, $d = \frac{1}{\sigma}\log(\frac{D}{S_0}) < 0 < u = \frac{1}{\sigma}\log(\frac{U}{S_0})$, which in turn yields $0 > 2d - u > 3d - 2u$.

Also, one can easily show that $G_\gamma > 3d - 2u$ in $(0, \tau_{du})$ since $L_\gamma > d$ for $0 < t < \tau_d$ and $L_\gamma < u$ for $\tau_d < \tau_{du}$.

Thus, $\{G_\gamma| \min\limits_{\tau_{du}\le\gamma\le T} G_\gamma \le 3d - 2u,\ \tau_{du} \le T\} = \{G_\gamma| \min\limits_{0\le\gamma\le T} G_\gamma \le 3d - 2u\}$.

Hence,

$$
\begin{aligned}
&CE_{duid}(t_0, S_{t_0}, K, U, D)\\
=&E^Q[(S_0 e^{\sigma(2(u-d)+G_T)} - K)^+ e^{\frac{\mu}{\sigma}G_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \mathbf{1}_{\{\min_{0\le\gamma\le T} G_\gamma \le 3d-2u\}} \mid \mathcal{F}_{t_0}]\\
&\times e^{-r(T-t_0)} e^{\frac{2\mu}{\sigma}(u-d)} e^{-\frac{\mu}{\sigma}W_{t_0} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t_0}
\end{aligned}
$$

Let us again an equivalent probability measure P^* by setting

$$
\frac{dP^*}{dQ} = e^{\frac{\mu}{\sigma}G_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \tag{2.2.18}
$$

so that the process $H_t = G_t - \frac{\mu}{\sigma}t,\ t \in [0, T]$ is a Brownian motion under P^* measure.

$$
\begin{aligned}
&CE_{duid}(t_0, S_{t_0}, K, U, D)\\
=&E^{P^*}[(S_0 e^{\sigma(2(u-d)+H_T+\frac{\mu}{\sigma}T)} - K)^+ \mathbf{1}_{\{\min_{0\le\gamma\le T}(H_\gamma+\frac{\mu}{\sigma}\gamma)\le 3d-2u\}} \mid \mathcal{F}_{t_0}]\\
&\times e^{-r(T-t_0)} e^{\frac{2\mu}{\sigma}(u-d)}
\end{aligned}
$$

For $t \in [0, T]$, define a process Z_t as

$$
Z_t = S_0 e^{2\sigma(u-d)} e^{\mu t + \sigma H_t} \tag{2.2.19}
$$

By using Ito's formula and proceeding as in (2.2.8), we can derive the following SDE satisfied by Z_t :

$$
dZ_t = (r - q)Z_t dt + \sigma Z_t dH_t \tag{2.2.20}
$$

where H_t is a standard Brownian motion under P^* measure and

$$\begin{aligned} &CE_{duid}(t_0, S_{t_0}, K, U, D) \\ &= \left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} e^{-r(T-t_0)} E^{P^*}[(Z_T - K)^+ \mathbf{1}_{\{\min_{0\le\gamma\le T} Z_\gamma \le D\}} \mid \mathcal{F}_{t_0}] \end{aligned} \tag{2.2.21}$$

For $t_0 < \tau_d$,

$$Z_{t_0} = \left(\frac{U}{D}\right)^2 S_0 e^{\mu t_0 + \sigma W_{t_0}} = \left(\frac{U}{D}\right)^2 S_{t_0} \tag{2.2.22}$$

Similar to (2.2.12) and (2.2.13),

$$\begin{aligned} CE_{duid}(t_0, S_{t_0}, K, U, D) &= \left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} CE_{id}(t_0, \left(\frac{U}{D}\right)^2 S_{t_0}, K, D) \\ CA_{duid}(t_0, S_{t_0}, K, U, D) &= \left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} CA_{id}(t_0, \left(\frac{U}{D}\right)^2 S_{t_0}, K, D) \end{aligned} \tag{2.2.23}$$

□

The following corollary can be obtained by using similar techniques in theorem 2.2.3.

Corollary 2.2.4 Consider the chained up-and-knock in European(PE_{udiu}) and American(PA_{udiu}) put options expiring at T with strike price $K(< U)$ activated at time
$\tau_{ud} = \min\{t > \tau_u \mid S_t = D,\ \tau_u = \min\{t > 0 \mid S_t = U,\ U > S_0\}\}$. Then, for $t_0 < \tau_u$,

$$\begin{aligned} PE_{udiu}(t_0, S_{t_0}, K, U, D) &= \left(\frac{D}{U}\right)^{\frac{2\mu}{\sigma^2}} PE_{iu}(t_0, \frac{D^2}{U^2} S_{t_0}, K, D) \\ PA_{udiu}(t_0, S_{t_0}, K, U, D) &= \left(\frac{D}{U}\right)^{\frac{2\mu}{\sigma^2}} PA_{iu}(t_0, \frac{D^2}{U^2} S_{t_0}, K, D) \end{aligned} \tag{2.2.24}$$

Remark 2.2.5
By using European barrier formulas in [25, Huang p.152-153] and relations in Theorem 2.2.1 and Theorem 2.2.3, we can obtain the results of [39].

2.3 Numerical results

In this section, we compute the value of American knock-in chained barrier options by making use of recursive integration method [27].

For $U \geq K\min(1, r/q)$, by using theorem 2.1.1 and corollary 2.2.2, the value of American chained up-and-in put option $PA_{diu}(t, S_t, K, U, D)$ is expressed by

$$
\begin{aligned}
&PA_{diu}(t, S_t, K, U, D) \\
&= \left(\frac{U}{D}\right)^{\frac{2(r-q)}{\sigma^2}-1} \left[PA(t, \frac{U^2 S_t}{D^2}, K) - PE(t, \frac{U^2 S_t}{D^2}, K)\right] \\
&+ \left(\frac{D}{S}\right)^{\frac{2(r-q)}{\sigma^2}-1} PE_{id}(t, \frac{D^2}{S_t}, K, U)
\end{aligned}
\tag{2.3.1}
$$

By the integral equation representation of the value of American options in [43], the American put $PA(t, s, K)$ in (2.3.1) can be decomposed as follows.

$$
\begin{aligned}
&PA(t, S, K) \\
=&PE(t, S, K) + \int_t^T \left[rKe^{-r(\xi-t)}\mathcal{N}(-d(\xi - t, \frac{S}{S_p^*(\xi)})) + \sigma\sqrt{\xi - t}) - qSe^{-q(\xi-t)}\right. \\
&\left.\times\mathcal{N}(-d(\xi - t, \frac{S}{S_p^*(\xi)}))\right] d\xi \\
&= Ke^{-r(T-t)}\mathcal{N}(-d(T - t, \frac{S}{K}) - \sigma\sqrt{T - t}) - Se^{-q(T-t)}\mathcal{N}(-d(T - t, \frac{S}{K})) \\
&+ \int_t^T \left[rKe^{-r(\xi-t)}\mathcal{N}(-d(\xi - t, \frac{S}{S_p^*(\xi)})) + \sigma\sqrt{\xi - t}) - qSe^{-q(\xi-t)}\mathcal{N}(-d(\xi - t, \frac{S}{S_p^*(\xi)}))\right] d\xi
\end{aligned}
\tag{2.3.2}
$$

where $d(t, x) = \dfrac{\log x + (r - q + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$ and $S_p^*(t)$ is the optimal stopping time of American put option and is the solution of the integral equation

$$
\begin{aligned}
&K - S_p^*(t) = \\
&Ke^{-r(T-t)}\mathcal{N}(-d(T - t, \frac{S_p^*(t)}{K}) - \sigma\sqrt{T - t}) - Se^{-q(T-t)}\mathcal{N}(-d(T - t, \frac{S_p^*(t)}{K})) \\
&+ \int_t^T \left[rKe^{-r(\xi-t)}\mathcal{N}(-d(\xi - t, \frac{S_p^*(t)}{S_p^*(\xi)}) + \sigma\sqrt{\xi - t}) - qSe^{-q(\xi-t)}\mathcal{N}(-d(\xi - t, \frac{S_p^*(t)}{S_p^*(\xi)})\right] d\xi
\end{aligned}
\tag{2.3.3}
$$

Hence,

$$
\begin{aligned}
&PA_{diu}(t, S_t, K, U, D) \\
&= \left(\frac{U}{D}\right)^{\frac{2(r-q)}{\sigma^2}-1} \left(\int_t^T \left[rKe^{-r(\xi-t)}\mathcal{N}(-d(\xi-t, \frac{U^2 S_t}{D^2 S_p^*(\xi)}) + \sigma\sqrt{\xi-t}) - q\frac{U^2}{D^2}S_t e^{-q(\xi-t)}\right.\right. \\
&\left.\left.\times\mathcal{N}(-d(\xi-t, \frac{U^2 S_t}{D^2 S_p^*(\xi)})\right] d\xi\right) + \left(\frac{D}{S_t}\right)^{\frac{2(r-q)}{\sigma^2}-1} PE_{iu}(t, \frac{D^2}{S_t}, K, U).
\end{aligned}
\tag{2.3.4}
$$

Also in [25], the price formula for European up-and-knock-in put options are given as follows.

(1) If $U \geq K$,

$$
\begin{aligned}
&PE_{iu}(t, S_t, K, U) \\
&= \left(\frac{U}{S_t}\right)^{\frac{2(r-q)}{\sigma^2}-1} \left[Ke^{-r(T-t)}\mathcal{N}\left(-d(T-t, \frac{U^2}{S_t K}) + \sigma\sqrt{T-t}\right) - \frac{U^2}{S_t}e^{-q(T-t)}\right. \\
&\left.\mathcal{N}\left(-d(T-t, \frac{U^2}{S_t K})\right)\right]
\end{aligned}
\tag{2.3.5}
$$

(2) Otherwise,

$$
\begin{aligned}
&PE_{iu}(t, S_t, K, U) \\
&= Ke^{-r(T-t)}\mathcal{N}\left(-d(T-t, \frac{S_t}{K}) + \sigma\sqrt{T-t}\right) - S_t e^{-q(T-t)}\mathcal{N}\left(-d(T-t, \frac{S_t}{K})\right) \\
&- Ke^{-r(T-t)}\mathcal{N}\left(-d(T-t, \frac{S_t}{U}) + \sigma\sqrt{T-t}\right) + S_t e^{-q(T-t)}\mathcal{N}\left(-d(T-t, \frac{S_t}{U})\right) \\
&+ \left(\frac{U}{S_t}\right)^{\frac{2(r-q)}{\sigma^2}-1} \left[Ke^{-r(T-t)}\mathcal{N}\left(-d(T-t, \frac{U}{S_t}) + \sigma\sqrt{T-t}\right) - \frac{U^2}{S_t}e^{-q(T-t)}\right. \\
&\left.\mathcal{N}\left(-d(T-t, \frac{U}{S_t})\right)\right]
\end{aligned}
\tag{2.3.6}
$$

Table 2.1: Comparison of American chained put option PA_{diu} with varying S and K, option parameter: $T = 0.5,\ U = 105,\ D = 95,\ r = 0.05,\ q = 0,\ \sigma = 0.3$

S	K	Jun& Ku (100)	Recursive Integration Method	Benchmark (Monte Carlo)
96	95	1.5138	1.5169	1.5188
	97.5	1.9503	1.9531	1.9699
	100	2.4670	2.4730	2.4400
	102.5	3.0732	3.0830	3.0531
	105	3.7747	3.7888	3.7686
98	95	1.2575	1.2572	1.2600
	97.5	1.6317	1.6328	1.6418
	100	2.0812	2.0846	2.0604
	102.5	2.6131	2.6197	2.6260
	105	3.2334	3.2441	3.2050
100	95	1.0399	1.0389	1.0522
	97.5	1.3609	1.3609	1.3698
	100	1.7503	1.7519	1.7632
	102.5	2.2150	2.2190	2.2426
	105	2.7616	2.7688	2.7339
102	95	0.8576	0.8561	0.8714
	97.5	1.1319	1.1310	1.1463
	100	1.4676	1.4679	1.4624
	102.5	1.8720	1.8740	1.8939
	105	2.3515	2.3560	2.3535
104	95	0.7053	0.7036	0.7164
	97.5	0.9389	0.9375	0.9482
	100	1.2273	1.2266	1.2218
	102.5	1.5575	1.5782	1.5634
	105	1.9965	1.9990	2.0296
	RSME	1.7256e-02	1.9428e-02	
	CPU time	1.3254e-01	6.7480e-02	

Table 2.2: Comparison of American chained put option PA_{diu} with varying U and D, option parameter: $T = 0.5,\ S = 100,\ r = 0.05,\ q = 0,\ \sigma = 0.3$

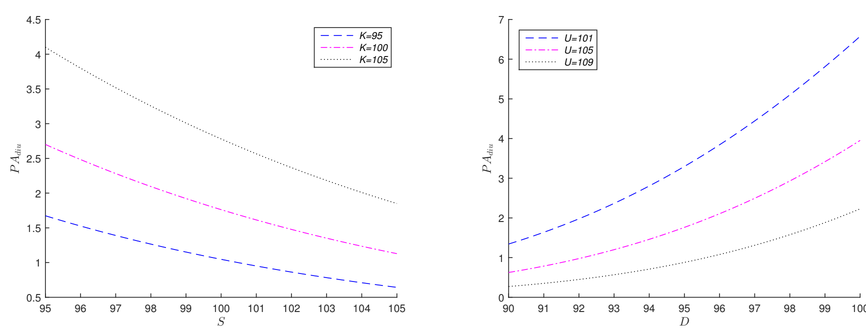
U	D	Jun& Ku (100)	Recursive Integration Method	Benchmark (Monte Carlo)
101	91	1.6270	1.6279	1.6456
	93	2.3508	2.3559	2.3232
	95	3.2754	3.2868	3.2221
	97	4.4148	4.4336	4.3007
	99	5.7743	5.7997	5.5791
103	91	1.1373	1.1363	1.1600
	93	1.6890	1.6903	1.6780
	95	2.4152	2.4207	2.4029
	97	3.3358	3.3476	3.2997
	99	4.4636	4.4826	4.4202
103	91	0.7808	0.7791	0.8034
	93	1.1922	1.1914	1.1694
	95	1.7503	1.7519	1.7632
	97	2.4782	2.4842	2.4800
	99	3.3946	3.4068	3.4294
105	91	0.5267	0.5251	0.5506
	93	0.8272	0.8256	0.8389
	95	1.2471	1.2464	1.2648
	97	1.8108	1.8127	1.8144
	99	2.5400	2.5465	2.5248
107	91	0.3494	0.3481	0.3802
	93	0.5643	0.5626	0.5927
	95	0.8739	0.8723	0.9034
	97	1.3016	1.3012	1.3057
	99	1.8706	1.8728	1.8706
	RSME	5.1155e-02	5.8472e-02	
	CPU time	1.3052e-01	9.0884e-02	

From (2.3.3), (2.3.4), (2.3.5) and (2.3.6), we can obtain analytic formula of American chained knock-in put options.
To sum up, detailed methods for getting analytic formula of American chained options are as follows.

Algorithm : Numerical Method for American chained option PA_{diu}.
Step 1: By using recursive integration method, calculate the free boundary $S_p^*(t)$ of American put option in (2.3.3).
Step 2: For numerical solution $S_p^*(t)$ in **Step 1**, calculate value of American put option PA in (2.3.2).
Step 3: Using approximate solution PA obtained in **Step 2** and formula (2.3.5), (2.3.6), calculate American chained put option PA_{diu}, numerically.

We compare our results with those of Jun and Ku [41] in Table 2.1 and with those of Monte Carlo simulation in Table 2.2 For the computation, the data were collected from [41]. Table 2.1 compares option value by fixing two barrier levels as U=105 and D=95 while varying S and K. Table 2.2 represents the behavior of option values according to various barrier levels U and D, while holding S and K to 100.

As you can see from RMSE and CPU times in Table 2.1 and Table 2.2, the valuation with analytic formula using recursive iteration method is an efficient method to use in real computations.

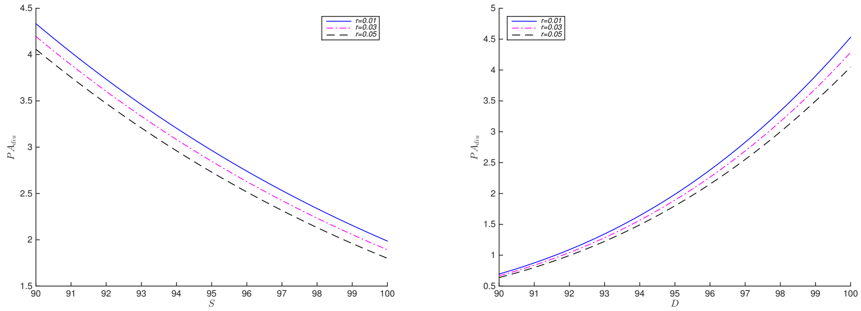


(a) Value change of PA_{diu} with respect to S for $D = 95$

(b) Value change of PA_{diu} with respect to D for $S = 100$

Figure 2.1: Values of PA_{diu} for $T = 0.5,\ r = 0.05,\ q = 0,\ \sigma = 0.3$

Figure 2.1-(a) shows how the value of knock-in American chained barrier put options(PA_{diu}) with two fixed barrier levels changes as the value of underlying asset varies. In the figure, the option price decreases as stock



(a) Value change of PA_{diu} with respect to S for $D = 95$

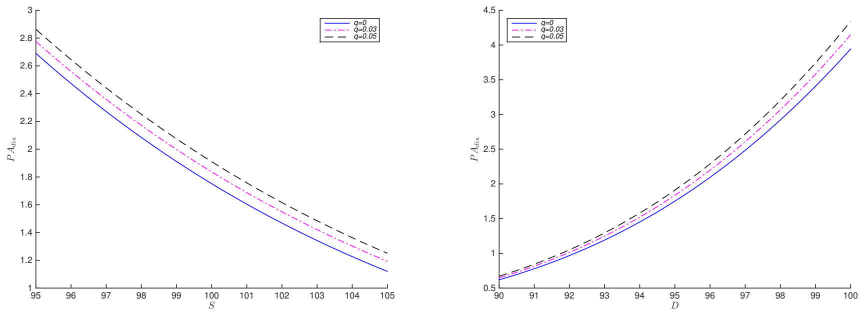
(b) Value change of PA_{diu} with respect to D for $S = 100$

Figure 2.2: Values of PA_{diu} for $K = 100,\ T = 0.5,\ q = 0.02,\ \sigma = 0.3$

price increases, while it increases as the strike price increases. Figure 2.1-(b) represents a graph of option prices according to two barrier levels U and D, while holding S and K to 100. It is worth notable that the option price increases as D increases, and it decreases as U increases. These are due to the fact that larger D implies high possibility of activating options and larger U implies small chance of knock-in. In Figure 2.2, the behavior of PA_{diu} versus the value of underlying asset S, lower barrier D with different interest rate are shown. Figure 2.2-(a) shows the option value decreases as the underlying asset or interest rate increases. Figure 2.2-(b) represents that the option value increases as lower barrier D moves upward. In Figure 2.3, we plot the option value with different dividends q. The option value becomes large with larger dividends. In Figure 2.4, we plot the case when $K \min(1, r/q) \le U \le K$, which was not dealt within [41].

2.4 Summary

We have presented the analytic price formulas for knock-in American chained-barrier options under the Black-Scholes pricing framework. The chained-barrier option is a type of barrier options where monitoring of other barrier activates at a time when the underlying asset first hits a specific barrier line. In this paper we show how to price knock-in chained-barrier options utilizing a knock-in American barrier option formula, using the reflection princi-

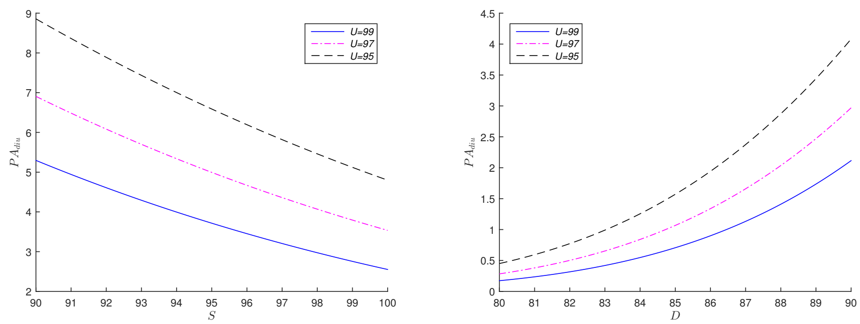


(a) Value change of PA_{diu} with respect to S for $D = 95$

(b) Value change of PA_{diu} with respect to D for $S = 100$

Figure 2.3: Values of PA_{diu} for $K = 100,\ T = 0.5,\ r = 0.05,\ \sigma = 0.3$

ple. This formula enables fast and accurate valuation of knock-in American chained-barrier options. Also, we provide some numerical solutions and plots of the value of knock-in American chained-barrier options



(a) Value change of PA_{diu} with respect to S for $D = 91$

(b) Value change of PA_{diu} with respect to D for $S = 100$

Figure 2.4: Values of PA_{diu} for $K = 100,\ T = 0.5,\ r = 0.05,\ q = 0.06,\ \sigma = 0.3$

Chapter 3

Russian option with finite time horizon

A Russian option is a kind of path-dependent American option which entitles the holder to either buy or sell the underlying asset at the best price at which it is traded during the life of the option. Because American option holders can exercise their rights at any instant before maturity, the valuation of such options is usually classified as optimal stopping problems or free boundary problems. In addition, the value of these options contains an additional early exercise premium compared to European type options. A considerable amount of research has been conducted on options which combine features of American options with those of path-dependent options. For example, Dai and Kwok studied American floating lookback options [12], and Lai and Lim researched American fixed strike options [52].

The Russian option was introduced by Shepp and Shiryaev in [63] and can be considered a type of perpetual American fixed strike lookback option. Ekström analyzed the regularity of the free boundary of Russian options with a finite time horizon and derived partial differential equations (PDEs) satisfied by these options [15]. Peskir drew out integral equations satisfied by Russian options with finite maturity using a stochastical local time-space formula [59]. An integral equation was first used to solve option pricing problems in the valuation of American options. Kim, [43], was the first to derive the integral equations satisfied by the value of American options, which are known to have no closed form solutions. In general, it is not possible to solve such integral equations analytically; instead, numerical methods have to be found that would allow the solutions to be approximated. To date, there have been various numerical approaches. Huang et al. used recursive integration methods [27], and Ju [38] utilized the multipiece exponential function method to solve such integral equations numerically. The interested reader can refer to [34], [51], which contain numerous approaches for solving American option problems from basic finite difference methods to methods using binomial trees. In this chapter, we use the recursive iteration method proposed by Huang et al. [27] to obtain the numerical solution of the integral equation satisfied by Russian options, which we derive in subsequent sections.

A Russian option with a finite time horizon can be formulated into a parabolic PDE with mixed boundary conditions. Kimura [45], instead of solving the PDE directly, expressed the solution using a Laplace transform. In this chapter, we derive integral equations satisfied by Russian options with a finite time horizon by solving the parabolic PDE directly using Mellin transform techniques. The Mellin transform is a type of integral transform and can be considered as a two-sided Laplace transform. Especially, it converts a Black-Scholes type PDE into a simple ordinary differential equation (ODE). Therefore, the use of the inverse Mellin transform enables the analytical representation of the value function of Russian options to be easily obtained. For this reason, the Mellin transform is widely used in option pricing. To list some examples, Panini first introduced option pricing using the Mellin transform [56],[57], whereas Yoon and Kim obtained the closed solution for vulnerable options using double Mellin transforms [69]. Jeon et al. [30] drew a pricing formula for the path-dependent option with two underlying assets and Jeon et al. derived integral equations satisfied by American floating strike lookback options [29].

3.1 Model Formulation: Free Boundary Problem

The usual assumptions for the Black-Scholes option pricing framework are adopted in this work. Let $(S_t)_{t\geq 0}$ denote the price of an underlying asset of a Russian option under a risk-neutral probability measure $\mathbb{P}$. The stochastic dynamics of S_t is described by

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t, \quad S_0 = s \tag{3.1.1}$$

where $r > 0$ is the risk-free interest rate, $q \geq 0$ is the continuous dividend rate, and $\sigma > 0$ is the constant volatility of S_t. W_t is a one-dimensional standard Brownian motion process on a filtered probability space $(\Omega, \mathcal{F}_{t\geq 0}, \mathbb{P})$, where $\mathcal{F}_{t\geq 0} \equiv \mathbb{F}$ is the natural filtration generated by $(W_t)_{t\geq 0}$. For the pricing process $(S_t)_{t\geq 0}$, we define the maximum process as

$$M_t = (\max_{0\leq\gamma\leq t} S_\gamma) \vee m \tag{3.1.2}$$

where $m \geq s > 0$ are given and fixed.

Consider a Russian option with a given finite time horizon $T > 0$. In the absence of arbitrage opportunities, the value $R(t, s, m)$ is a solution of the *optimal stopping problem*

$$R(t, s, m) = \sup_{0\leq\tau_t\leq T-t} \mathbb{E}\left[e^{-r\tau_t} M_{\tau_t} \mid S_0 = s, M_0 = m\right] \tag{3.1.3}$$

where τ_t is the stopping time of the filtration $\mathbb{F}$ and the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$. The random variable $\tau_t \in [0, T-t]$ is considered an optimal stopping time if it is able to provide the supremum value of the right hand side of (3.1.3).

Solving the optimal stopping problem (3.1.3) is equivalent to finding the points (t, S_t, M_t) for which early exercise before maturity would be optimal.

Let

$$\mathcal{D} = \{(t, s, m) \in [0, T-t] \times (0, m] \times \mathbb{R}_+\} \tag{3.1.4}$$

Then, domain $\mathcal{D}$ of the pricing model can be divided into two regions: the stopping region $\mathcal{S} = \{(t, s, m) \in \mathcal{D} \mid \ 0 < s < s^*(t, m)\}$, and the continuation region $\mathcal{S}^C = \{(t, s, m) \in \mathcal{D} \mid \ s^*(t, m) \ < s \leq m\}$. Here, $s^*(t, m)$ is termed the *free boundary* or *early exercise boundary* and is given by

$$s^*(t, m) = \inf\left\{s \in [0, m] \mid (t, s, m) \in \mathcal{S}^C\right\}$$

The linear complementarity formulation which governs $R(t, s, m)$ is given by

$$\begin{aligned} &\mathcal{L}R(t, s, m) \leq 0, \quad R(t, s, m) \geq m \\ &[\mathcal{L}R(t, s, m)] \cdot (R(t, s, m) - m) = 0, \quad m \geq s > 0, \quad 0 \leq t \leq T \end{aligned} \tag{3.1.5}$$

with auxiliary conditions

$$\begin{aligned} &\frac{\partial R}{\partial m}(t, s, s) = 0 \\ &R(T, s, m) = m. \end{aligned} \tag{3.1.6}$$

The operator $\mathcal{L}$ is defined by

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + (r-q)s\frac{\partial}{\partial s} - r. \tag{3.1.7}$$

Furthermore, by the standard argument using Feynman-Kac formula, the optimal stopping problem (3.1.3) can be reduced to a *free boundary problem*. For the free boundary $s^*(t,m)$, this problem is equivalent to solving the following PDE:

$$\mathcal{L}R(t,s,m) = 0, \quad s^*(t,m) < s \leq m$$

with boundary condition (3.1.6). Those who are interested in the theories regarding to the optimal stopping problem and the free boundary problem can refer to [58].

At the free boundary $s = s^*(t,m)$, arbitrage arguments show that the option price $R(t,s,m)$ satisfies the *smooth pasting condition*.

$$\begin{aligned} &\lim_{s \downarrow s^*(t,m)} \frac{\partial R}{\partial s} = 0 \\ &\lim_{s \downarrow s^*(t,m)} R(t,s,m) = m \end{aligned} \tag{3.1.8}$$

By changing the variables $x := s/m$ and $\bar{R}(t,x) = R(t,s,m)/m$, the dimension of the above formulation can be reduced by one.

Then, we can rewrite the linear complementary form (3.1.5) as

$$\begin{aligned} &\mathcal{L}\bar{R}(t,x) \leq 0, \quad \bar{R}(t,x) \geq 1 \\ &[\mathcal{L}\bar{R}(t,x)] \cdot (\bar{R}(t,x) - 1) = 0, \quad 0 < x \leq 1, \quad 0 \leq t \leq T. \end{aligned} \tag{3.1.9}$$

Further,

$$\mathcal{L}\bar{R}(t,x) = \frac{\partial \bar{R}}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \bar{R}}{\partial x^2} + (r-q)x\frac{\partial \bar{R}}{\partial x} - r\bar{R} = 0, \quad x^*(t) < x \leq 1, \tag{3.1.10}$$

where $x^*(t) = \bar{s}(t,m)/m$ ($\in [0,1]$), with auxiliary conditions:

$$\begin{aligned}
&\bar{R}(T,x) = 1, \\
&\lim_{x \downarrow x^*} \bar{R}(t,x) = 1, \\
&\lim_{x \downarrow x^*} \frac{\partial \bar{R}}{\partial x} = 0, \\
&\lim_{x \uparrow 1} \left(\bar{R} - \frac{\partial \bar{R}}{\partial x} \right) = 0.
\end{aligned} \tag{3.1.11}$$

In terms of the value function $\bar{R}(t,x)$, the stopping region $\bar{\mathcal{S}}$ is defined by

$$\begin{aligned}
\bar{\mathcal{S}} &:= \{(t,x) \mid \ 0 \le t < T, 0 < x < x^*(t)\} \\
&= \left\{(t,x) \mid \bar{R}(t,x) = 1,\ 0 \le t < T, 0 < x \le 1 \right\}
\end{aligned}$$

and the continuation region $\bar{\mathcal{S}}^C$ is given by

$$\begin{aligned}
\bar{\mathcal{S}}^C &:= \{(t,x) \mid \ 0 \le t < T, x^*(t) < x \le 1\} \\
&= \left\{(t,x) \mid \bar{R}(t,x) > 1,\ 0 \le t < T, 0 < x \le 1 \right\}.
\end{aligned}$$

Therefore, the value $\bar{R}(t,x)$ is the solution of the following inhomogeneous Black-Scholes equation:

$$\mathcal{L}\bar{R}(t,x) = \frac{\partial \bar{R}}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \bar{R}}{\partial x^2} + (r-q)x\frac{\partial \bar{R}}{\partial x} - r\bar{R} = \zeta(t,x), \tag{3.1.12}$$

where

$$\zeta(t,x) = \begin{cases} -r & \text{for} \quad 0 \le x < x^*(t) \\ 0 & \text{for} \quad x^*(t) < x < 1 \end{cases} \tag{3.1.13}$$

with a *mixed boundary condition* in (3.1.11) and domain
$\{(t,x) \mid 0 \le t < T,\ 0 < x \le 1\}$

3.2 Inhomogeneous Black-Scholes Partial Differential Equation: Mixed Boundary Problem

We formulated the inhomogeneous Black-Scholes PDE with mixed boundary conditions regarding the value function of Russian options with a finite time horizon in section 3.1. In this section, we solve the inhomogeneous mixed boundary Black-Scholes PDEs by extending the idea of Buchen [4], which was used to solve the homogeneous case. Our new methodology makes possible for valuing options with complicated mixed boundary conditions.

For the PDE operator $\mathcal{L}$ defined in (3.1.7), consider the following *mixed boundary condition* PDE problem:

$$\mathcal{L}V(t,x) = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}x^2\frac{\partial^2 V}{\partial x^2} + (r-q)x\frac{\partial V}{\partial x} - rV = f(t,x) \tag{3.2.1}$$

with boundary condition

$$\begin{aligned} &V(T,x) = h(x) \\ &\frac{\partial V}{\partial x}(t,1) = V(t,1) \end{aligned} \tag{3.2.2}$$

with domain $\{(t,x) \mid 0 \le t < T,\ 0 < x \le 1\}$ and we assume that $h(x), f(t,x)$ are smooth functions and that Mellin transforms of $h, f, x\frac{dh}{dx}, x\frac{df}{dx}$ exist in the proper domain.

For arbitrary smooth function $U(t,x)$, define the differential operator $\mathcal{H}[\cdot]$ as follows:

$$\mathcal{H}[U(t,x)] := x\frac{\partial U}{\partial x}(t,x) - U(t,x). \tag{3.2.3}$$

From (3.2.1),

$$\begin{aligned} \mathcal{H}[\mathcal{L}V(t,x)] &= \mathcal{H}\left[\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}x^2\frac{\partial^2 V}{\partial x^2} + (r-q)x\frac{\partial V}{\partial x} - rV\right] \\ &= \frac{\partial \mathcal{H}[V]}{\partial t} + \frac{\sigma^2}{2}x^2\frac{\partial^2 \mathcal{H}[V]}{\partial x^2} + (r-q)x\frac{\partial \mathcal{H}[V]}{\partial x} - r\mathcal{H}[V] \\ &= \mathcal{H}[f(t,x)] \end{aligned}$$

and

$$\begin{aligned} &\mathcal{H}[V(T,x)] = \mathcal{H}[h(x)] \\ &\left(\frac{\partial V}{\partial x}(t,1) - V(t,1)\right) = \mathcal{H}[V(t,1)] = 0. \end{aligned}$$

Let

$$P(t,x) := \mathcal{H}[V(t,x)],\ \phi(x) := \mathcal{H}[h(x)],\ \chi(t,x) := \mathcal{H}[f(t,x)]$$

Then, PDE problem (3.2.1) with boundary condition (3.2.2) is converted to

$$\begin{aligned} &\mathcal{L}P(t,x) = \chi(t,x) \\ &P(T,x) = \phi(x) \\ &P(t,1) = 0 \end{aligned} \tag{3.2.4}$$

with domain $\{(t,x) \mid 0 \le t < T,\ 0 < x \le 1\}$.

The following theorem enables us to obtain the solution of PDE (3.2.4)

Theorem 3.2.1 (Inhomogeneous Black-Scholes PDE problem with Dirichlet boundary condition)
The solution $P(t,x)$ of PDE (3.2.4) is given by

$$P(t,x) = \bar{P}(t,x) - \left(\frac{1}{x}\right)^{(1-k_2)} \bar{P}(t,\frac{1}{x}), \tag{3.2.5}$$

where $\bar{P}(t,x)$ is the solution of PDE $\mathcal{L}\bar{P}(t,x) = \chi(t,x)\mathbf{1}_{\{x<1\}}$ with $\bar{P}(T,x) = \phi(x)\mathbf{1}_{\{x<1\}}$ $(k_2 = \frac{2(r-q)}{\sigma^2})$.

Proof To solve PDE (3.2.4), we consider an unrestricted domain PDE problem:

$$\begin{aligned} &\mathcal{L}\bar{P}(t,x) = \chi(t,x)\mathbf{1}_{\{x<1\}} \\ &\bar{P}(T,x) = \phi(x)\mathbf{1}_{\{x<1\}} \end{aligned} \tag{3.2.6}$$

with domain $\{(t,x) \mid 0 \le t < T, 0 < x < \infty\}$.

Denote $\hat{P}(t,w)$ as the Mellin transform of $\bar{P}(t,x)$. Then, by the inverse Mellin transform,

$$\bar{P}(t,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{P}(t,w)x^{-w}dw. \tag{3.2.7}$$

Then, PDE (3.2.6) is changed by the following ODE:

$$\begin{aligned}&\frac{d\hat{P}}{dt}(t,w)+\frac{1}{2}\sigma^2Q(w)\hat{P}(t,w)=\hat{\chi}(t,w)\\&Q(w)=w^2+w(1-k_2)-k_1,\end{aligned} \tag{3.2.8}$$

where $\hat{\chi}$ is the Mellin transform $\chi(t,x)\mathbf{1}_{\{x<1\}}$ and $k_1=\frac{2r}{\sigma^2}$, $k_2=\frac{2(r-q)}{\sigma^2}$.

The inhomogeneous ODE (3.2.8) yields

$$\hat{P}(t,w)=e^{\frac{1}{2}\sigma^2Q(w)(T-t)}\hat{\phi}(w)-\int_t^T e^{\frac{1}{2}\sigma^2Q(w)(\eta-t)}\hat{\chi}(\eta,w)d\eta. \tag{3.2.9}$$

and

$$\begin{aligned}\bar{P}(t,x)=&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\frac{1}{2}\sigma^2Q(w)(T-t)}\hat{\phi}(w)x^{-w}dw\\&-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T e^{\frac{1}{2}\sigma^2Q(w)(\eta-t)}\hat{\chi}(\eta,w)x^{-w}d\eta dw\end{aligned} \tag{3.2.10}$$

where $\hat{\phi}$ is the Mellin transform of $\phi(x)\mathbf{1}_{\{x<1\}}$.

In addition, to calculate (3.2.10), let us consider

$$\mathcal{B}(t,x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\frac{1}{2}\sigma^2Q(w)t}x^{-w}dw. \tag{3.2.11}$$

Because $e^{\frac{1}{2}\sigma^2Q(w)(T-t)}$, $\hat{\phi}(w)$, and $\hat{\chi}(\eta,w)$ are the Mellin transforms of $\mathcal{B}(t,x)$, $\phi(x)\mathbf{1}_{\{x<1\}}$, and $\chi(\eta,x)\mathbf{1}_{\{x<1\}}$, respectively, by using the Mellin convolution in Proposition A.1.1 in Appendix A.1, we obtain

$$\begin{aligned}\bar{P}(t,x)=&\int_0^\infty\phi(u)\mathbf{1}_{\{u<1\}}\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}du\\&-\int_t^T\int_0^\infty\chi(\eta,u)\mathbf{1}_{\{u<1\}}\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta,\end{aligned} \tag{3.2.12}$$

where

$$\mathcal{B}(t,x)=e^{-\frac{\sigma^2}{2}\{(\frac{1-k_2}{2})^2+k_1\}t}\frac{x^{\frac{1-k_2}{2}}}{\sigma\sqrt{2\pi t}}\exp\left\{-\frac{1}{2}\frac{(\log x)^2}{\sigma^2t}\right\}. \tag{3.2.13}$$

Because $Q(w) = \frac{1}{2}\sigma^2 \left[\left(w + \frac{1-k_2}{2} \right)^2 - (\frac{1-k_2}{2})^2 - k_1 \right]$, we obtain

$$Q(w) = Q(k_2 - 1 - w)$$

and by the properties of the Mellin transform and direct computation,

$$\mathcal{B}(t,x) = x^{(1-k_2)}\mathcal{B}(t, \frac{1}{x}). \tag{3.2.14}$$

From (3.2.14), it leads to

$$\begin{aligned} x^{(1-k_2)}\bar{P}(t, \frac{1}{x}) = & \int_0^\infty \phi(\frac{1}{u})u^{(1-k_2)}\mathbf{1}_{\{u>1\}}\mathcal{B}(T-t, \frac{1}{u})\frac{1}{u}du \\ & - \int_t^T \int_0^\infty \chi(\eta, u)u^{(1-k_2)}\mathbf{1}_{\{u>1\}}\mathcal{B}(\eta - t, \frac{1}{u})\frac{1}{u}dud\eta. \end{aligned} \tag{3.2.15}$$

Denote $\bar{P}^*(t,x) := x^{(1-k_2)}\bar{P}(t, \frac{1}{x})$.

Then by (3.2.15), $\bar{P}^*(t,x)$ is the solution of the following PDE:

$$\begin{aligned} &\mathcal{L}\bar{P}^*(t,x) = \chi(t,x)x^{(1-k_2)}\mathbf{1}_{\{x>1\}} \\ &\bar{P}^*(T,x) = \phi(\frac{1}{x})x^{(1-k_2)}\mathbf{1}_{\{x>1\}}. \end{aligned}$$

Let $P(t,x) = \bar{P}(t,x) - \bar{P}^*(t,x)$, then $P(t,x)$ satisfies the following equations:

$$\begin{aligned} &\mathcal{L}P(t,x) = \chi(t,x)\mathbf{1}_{\{x<1\}} - \chi(t, \frac{1}{x})x^{(1-k_2)}\mathbf{1}_{\{x>1\}}, \\ &P(T,x) = \phi(x)\mathbf{1}_{\{x<1\}} - \phi(\frac{1}{x})x^{(1-k_2)}\mathbf{1}_{\{x>1\}}, \end{aligned}$$

and $P(t,1) = \bar{P}(t,1) - \bar{P}(t,1) = 0$.

Hence, the solution of PDE (3.2.4) is given by

$$P(t,x) = \bar{P}(t,x) - x^{(1-k_2)}\bar{P}(t, \frac{1}{x})$$

□

Theorem 3.2.2 (Inhomogeneous Black-Scholes PDE problem with mixed boundary condition)
The solution $V(t,x)$ of PDE (3.2.1) with boundary condition (3.2.2) satisfies the following PDE:

$$\begin{aligned}\mathcal{L}V(t,x) =& f(t,x)\mathbf{1}_{\{x<1\}} + f(t,\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}}\\ &+ (k_2+1)x\left[\int_1^x f(t,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2+1)}dy\right]\mathbf{1}_{\{x>1\}}\end{aligned}$$

$$\begin{aligned}V(T,x) =& h(x)\mathbf{1}_{\{x<1\}} + h(\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}}\\ &+ (k_2+1)x\left[\int_1^x h(\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2+1)}dy\right]\mathbf{1}_{\{x>1\}}.\end{aligned}$$

Proof For $P(t,x)$ defined in Theorem 3.2.1 and $\mathcal{H}[\cdot]$ defined in (3.2.3), define

$$V(t,x) = \mathcal{H}^{-1}[P(t,x)]$$

(i.e., $\mathcal{H}[V(t,x)] = P(t,x)$).

Then,

$$P(t,1) = \mathcal{H}[V(t,1)] = \frac{\partial V}{\partial x}(t,1) - V(t,1) = 0 \tag{3.2.16}$$

and

$$V(t,x) = \mathcal{H}^{-1}[P(t,x)] = \mathcal{H}^{-1}[\bar{P}(t,x)] - \mathcal{H}^{-1}[\bar{P}^*(t,x)],$$

where $\bar{P}(t,x)$ and $\bar{P}^* = x^{(1-k_2)}\bar{P}(t,\frac{1}{x})$ are defined in Theorem 3.2.1.

By the properties of the Mellin transform in Appendix A.1 and the definition of $\mathcal{H}$, $\mathcal{M}_x(\mathcal{H}^{-1}[U(x)];w) = -(w+1)\mathcal{M}_x(U(x);w)$ for any function $U(x)$.

Therefore, for $\bar{V}(t,x) := \mathcal{H}^{-1}[\bar{P}(t,x)]$ and $\bar{V}^*(t,x) := \mathcal{H}^{-1}[\bar{P}^*(t,x)] =$

$\mathcal{H}^{-1}[x^{(1-k_2)}\bar{P}(t,\frac{1}{x})]$

$$\begin{aligned}\hat{V}(t,w) &= -(w+1)\hat{P}(t,w)\\ \hat{V}^*(t,w) &= -(w+1)\hat{P}^*(t,w),\end{aligned} \tag{3.2.17}$$

where $\hat{P}$,$\hat{P}^*$, $\hat{V}$, and $\hat{V}^*$ are Mellin transforms of $\bar{P}$,$\bar{P}^*$, $\bar{V}$, and $\bar{V}^*$, respectively.

By Theorem 3.2.1, $\bar{P}(t,x)$ is the solution of the following PDE:

$$\begin{aligned}&\mathcal{L}\bar{P}(t,x) = \chi(t,x)\mathbf{1}_{\{x<1\}}\\ &\bar{P}(T,x) = \phi(x)\mathbf{1}_{\{x<1\}}\end{aligned}$$

and

$$\frac{d\hat{P}}{dt}(t,w) + \frac{1}{2}\sigma^2 Q(w)\hat{P}(t,w) = \hat{\chi}(t,w),$$

where $\hat{P}$ and $\hat{\chi}$ are the Mellin transforms of $\bar{P}(t,x)$ and $\chi(t,x)\mathbf{1}_{\{x<1\}}$, respectively.

Since $\hat{P}(t,w) = -(w+1)\hat{V}(t,w)$ and $\hat{\chi}(t,w) = -(w+1)\hat{f}(t,w)$,

$$\frac{d\hat{V}}{dt}(t,w) + \frac{1}{2}\sigma^2 Q(w)\hat{V}(t,w) = \hat{f}(t,w),$$

where $\hat{f}$ is the Mellin transform of $f(t,x)\mathbf{1}_{\{x<1\}}$. Hence,

$$\mathcal{L}\bar{V}(t,x) = f(t,x)\mathbf{1}_{\{x<1\}}. \tag{3.2.18}$$

Similarly, $\bar{P}^*(t,x)$ satisfies the PDE:

$$\begin{aligned}&\mathcal{L}\bar{P}^*(t,x) = \chi(t,\frac{1}{x})x^{(1-k_2)}\mathbf{1}_{\{x>1\}}\\ &\bar{P}^*(T,x) = \phi(\frac{1}{x})x^{(1-k_2)}\mathbf{1}_{\{x>1\}}\end{aligned}$$

and

$$\frac{d\hat{P}^*}{dt}(t,w) + \frac{1}{2}\sigma^2 Q(w)\hat{P}^*(t,w) = \hat{\chi}(t, k_2 - 1 - w).$$

Since $\hat{\chi}(t, k_2 - 1 - w) = -(k_2 - 1 - w + 1)\hat{f}(t, k_2 - 1 - w)$,

$$\frac{d\hat{V}^*}{dt}(t,w) + \frac{1}{2}\sigma^2 Q(w)\hat{V}^*(t,w) = -\hat{f}(t, k_2 - 1 - w) + \frac{k_2+1}{w+1}\hat{f}(t, k_2 - 1 - w)$$

By the inverse Mellin transform,

$$\begin{aligned}\mathcal{L}\bar{V}^*(t,x) = &- f(t, \frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)} \mathbf{1}_{\{x>1\}} \\ &- (k_2+1)x \int_1^x \left(\frac{1}{y}\right)^{(k_2+1)} f(t, \frac{1}{y}) dy \cdot \mathbf{1}_{\{x>1\}}.\end{aligned}$$

Therefore, since $V(t,x) = \bar{V}(t,x) - \bar{V}^*(t,x)$,

$$\begin{aligned}\mathcal{L}V(t,x) = &f(t,x)\mathbf{1}_{\{x<1\}} + f(t, \frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)} \mathbf{1}_{\{x>1\}} \\ &+ (k_2+1)x \left[\int_1^x f(t, \frac{1}{y})\left(\frac{1}{y}\right)^{(k_2+1)} dy\right] \mathbf{1}_{\{x>1\}}.\end{aligned} \tag{3.2.19}$$

The same procedure enables us to obtain

$$\begin{aligned}V(T,x) = &h(x)\mathbf{1}_{\{x<1\}} + h(\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)} \mathbf{1}_{\{x>1\}} \\ &+ (k_2+1)x \left[\int_1^x h(\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2+1)} dy\right] \mathbf{1}_{\{x>1\}}.\end{aligned} \tag{3.2.20}$$

By (3.2.18) and (3.2.19), we have proved the desired result.
□

3.3 Integral Equation of Russian Options with Finite Time Horizon: Premium Decomposition

The value of an American type option with an early exercise policy can be decomposed into two parts: the value of the European type option and the early exercise premium. In this section, we first decompose Russian options in the same way using the solution of the mixed boundary problem derived in section 3.2, and then we derive the integral equation satisfied by the free boundary of Russian options.

By Theorem 3.2.2, the value function $\bar{R}(t,x)$ is the solution of the following PDE:

$$\begin{aligned}\mathcal{L}\bar{R}(t,x) =& \zeta(t,x)\mathbf{1}_{\{x<1\}} + \zeta(t,\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} \\ &+ (k_2+1)x\left[\int_1^x \zeta(t,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2+1)} dy\right]\mathbf{1}_{\{x>1\}}\end{aligned} \tag{3.3.1}$$

$$\begin{aligned}\bar{R}(T,x) =& \mathbf{1}_{\{x<1\}} + \left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} \\ &+ (k_2+1)x\left[\int_1^x \left(\frac{1}{y}\right)^{(k_2+1)} dy\right]\mathbf{1}_{\{x>1\}}\end{aligned} \tag{3.3.2}$$

where $\zeta(t,x) = -r\mathbf{1}_{\{x<x^*(t)\}}$ is defined in (4.2.13).

Define $\bar{R}(t,x) = \bar{R}_E(t,x) + \bar{R}_e(t,x)$, where $\bar{R}_E(t,x)$ and $\bar{R}_e(t,x)$ satisfy the following PDEs, respectively.

$$\begin{aligned}\mathcal{L}\bar{R}_E(t,x) =& 0 \\ \bar{R}_E(T,x) =& \mathbf{1}_{\{x<1\}} + \left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} \\ &+ (k_2+1)x\left[\int_1^x \left(\frac{1}{y}\right)^{(k_2+1)} dy\right]\mathbf{1}_{\{x>1\}}\end{aligned} \tag{3.3.3}$$

and

$$
\begin{aligned}
\mathcal{L}\bar{R}_e(t,x) =& \zeta(t,x)\mathbf{1}_{\{x<1\}} + \zeta(t,\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} \\
&+ (k_2+1)x\left[\int_1^x \zeta(t,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2+1)}dy\right]\mathbf{1}_{\{x>1\}} \\
\bar{R}_e(T,x) =& 0.
\end{aligned}
$$

Here, it should be noted that $\bar{R}_E(t,x)$ is the price of a European Russian option with a finite time horizon and $\bar{R}_e$ is the early exercise premium of a Russian option with a finite time horizon.

Then, $\bar{R}_E(t,x)$ and $\bar{R}_e(t,x)$ are given by

$$
\begin{aligned}
\bar{R}_E(t,x) = \int_0^\infty & \left(\mathbf{1}_{\{u<1\}} + \mathbf{1}_{\{u>1\}}\left(\frac{1}{u}\right)^{(k_2-1)} + (k_2+1)u\left[\int_1^u \left(\frac{1}{y}\right)^{(k_2+1)}dy\right]\right. \\
&\left.\mathbf{1}_{\{u>1\}}\right)\mathcal{B}(T-t,\frac{x}{u})\frac{1}{u}du
\end{aligned}
\tag{3.3.4}
$$

$$
\begin{aligned}
\bar{R}_e(t,x) = -\int_t^T\int_0^\infty & \left(\zeta(\eta,u)\mathbf{1}_{\{u<1\}} + \zeta(\eta,\frac{1}{u})\mathbf{1}_{\{u>1\}}\left(\frac{1}{u}\right)^{(k_2-1)} + (k_2+1)u\right. \\
&\left.\times\left[\int_1^u \zeta(\eta,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2+1)}dy\right]\mathbf{1}_{\{u>1\}}\right)\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta.
\end{aligned}
\tag{3.3.5}
$$

The analytical representations of the solutions $\bar{R}_E(t,x)$ and $\bar{R}_e(t,x)$ admit different forms, depending on whether $r \neq q$ or $r = q$.

3.3.1 Case of $r \neq q$

From (3.3.4) and (3.3.5),

$$
\begin{aligned}
\bar{R}_E(t,x) =& \int_0^\infty \left(\mathbf{1}_{\{u<1\}} + \mathbf{1}_{\{u>1\}}\left(\frac{1}{u}\right)^{(k_2-1)} - (1+\frac{1}{k_2})\left(u^{-k_2+1}-u\right)\mathbf{1}_{\{u>1\}}\right) \\
&\mathcal{B}(T-t,\frac{x}{u})\frac{1}{u}du \\
\bar{R}_e(t,x) =& r\int_t^T\int_0^\infty \left(\mathbf{1}_{\{u<x^*(\eta)\}} + \mathbf{1}_{\{u>\frac{1}{x^*(\eta)}\}}\left(\frac{1}{u}\right)^{(k_2-1)} - (1+\frac{1}{k_2})\left[u^{(1-k_2)}\right.\right. \\
&\left.\left.-x^*(\eta)^{k_2}u\right]\mathbf{1}_{\{u>\frac{1}{x^*(\eta)}\}}\right)\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta.
\end{aligned}
$$

Lemma 3.3.1 For $\mathcal{B}(t,x)$ defined in (3.2.13) with $k_1 = \frac{2r}{\sigma^2}$, $k_2 = \frac{2(r-q)}{\sigma^2}$,

$$\int_0^b u^{-\alpha}\mathcal{B}(T-t,\frac{x}{u})\frac{1}{u}du$$
$$= x^{-\alpha}e^{-\frac{1}{2}\sigma^2(T-t)\{(\frac{1-k_2}{2})^2+k_1-(\frac{1-k_2}{2}+\alpha)^2\}}\mathcal{N}\left(\frac{-\log\frac{x}{b}+\sigma^2(T-t)(\frac{1-k_2}{2}+\alpha)}{\sigma\sqrt{T-t}}\right)$$
$$\int_b^\infty u^{-\alpha}\mathcal{B}(T-t,\frac{x}{u})\frac{1}{u}du$$
$$= x^{-\alpha}e^{-\frac{1}{2}\sigma^2(T-t)\{(\frac{1-k_2}{2})^2+k_1-(\frac{1-k_2}{2}+\alpha)^2\}}\mathcal{N}\left(\frac{\log\frac{x}{b}-\sigma^2(T-t)(\frac{1-k_2}{2}+\alpha)}{\sigma\sqrt{T-t}}\right)$$

By Lemma 3.3.1,

$$\begin{aligned}\bar{R}_E(t,x) =& e^{-r(T-t)}\mathcal{N}\left(-d_-(T-t,x)\right) - \frac{1}{k_2}e^{-r(T-t)}\left(\frac{1}{x}\right)^{(k_2-1)}\mathcal{N}\left(-d_-(T-t,\frac{1}{x})\right)\\ &+ (1+\frac{1}{k_2})xe^{-q(T-t)}\mathcal{N}\left(d_+(T-t,x)\right)\\ \bar{R}_e(t,x) =& \int_t^T re^{-r(\eta-t)}\mathcal{N}\left(-d_-(\eta-t,\frac{x}{x^*(\eta)})\right)d\eta\\ &-\frac{1}{k_2}\left(\frac{1}{x}\right)^{(k_2-1)}\int_t^T re^{-r(\eta-t)}\mathcal{N}\left(-d_-(\eta-t,\frac{1}{x^*(\eta)x})\right)d\eta\\ &+(1+\frac{1}{k_2})x\int_t^T rx^*(\eta)^{k_2}e^{-q(\eta-t)}\mathcal{N}\left(d_+(\eta-t,x^*(\eta)x)\right)d\eta\end{aligned}$$

where

$$d_\pm(T-t,x) = \frac{\log x + (r-q\pm\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \qquad (3.3.6)$$

By the *smooth pasting condition* of (3.1.11), at the free boundary $x = x^*(t)$,

$$
\begin{aligned}
1 =& e^{-r(T-t)}\mathcal{N}\left(-d_-(T-t, x^*(t))\right) - \frac{1}{k_2}e^{-r(T-t)}\left(\frac{1}{x^*(t)}\right)^{(k_2-1)} \\
&\times \mathcal{N}\left(-d_-(T-t, \frac{1}{x^*(t)})\right) + (1+\frac{1}{k_2})x^*(t)e^{-q(T-t)}\mathcal{N}\left(d_+(T-t, x^*(t))\right) \\
&+ \int_t^T re^{-r(\eta-t)}\mathcal{N}\left(-d_-(\eta-t, \frac{x^*(t)}{x^*(\eta)})\right)d\eta \\
&- \frac{1}{k_2}\left(\frac{1}{x^*(t)}\right)^{(k_2-1)}\int_t^T re^{-r(\eta-t)}\mathcal{N}\left(-d_-(\eta-t, \frac{1}{x^*(\eta)x^*(t)})\right)d\eta \\
&+(1+\frac{1}{k_2})x^*(t)\int_t^T rx^*(\eta)^{k_2}e^{-q(\eta-t)}\mathcal{N}\left(d_+(\eta-t, x^*(\eta)x^*(t))\right)d\eta.
\end{aligned}
\tag{3.3.7}
$$

3.3.2 Case of $r = q$

Since $k_2 = 0$, by (3.3.4) and (3.3.5),

$$
\begin{aligned}
\bar{R}_E(t,x) =& \int_0^\infty \left(\mathbf{1}_{\{u<1\}} + u\cdot\mathbf{1}_{\{u>1\}} + u\log u\mathbf{1}_{\{u>1\}}\right)\mathcal{B}(T-t, \frac{x}{u})\frac{1}{u}du \\
\bar{R}_e(t,x) =& r\int_t^T\int_0^\infty \Big(\mathbf{1}_{\{u<x^*(\eta)\}} + u\cdot\mathbf{1}_{\{u>\frac{1}{x^*(\eta)}\}} + [u\log u - \log x^*(\eta)u] \\
&\mathbf{1}_{\{u>\frac{1}{x^*(\eta)}\}}\Big)\mathcal{B}(\eta-t, \frac{x}{u})\frac{1}{u}dud\eta.
\end{aligned}
$$

Lemma 3.3.2 For $\mathcal{B}(t,x)$ defined in (3.2.13) with $k_1 = \frac{2r}{\sigma^2}$ and $k_2 = 0$ (i.e., $r = q$),

$$
\begin{aligned}
&\int_b^\infty u\log u\,\mathcal{B}(T-t, \frac{x}{u})\frac{1}{u}du \\
&= x\,e^{-r(T-t)}\left(\log x + \frac{\sigma^2(T-t)}{2}\right)\mathcal{N}\left(\frac{\log\frac{x}{b} + \sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\
&+ x\,e^{-r(T-t)}\sigma\sqrt{T-t}\cdot n\left(\frac{\log\frac{x}{b} + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}
$$

where n is the probability density function of the standard normal distribution.

Proof By the definition of $\mathcal{B}(t,x)$,

$$\mathcal{B}(t,x) = e^{-\frac{\sigma^2}{2}\{\frac{1}{4}+k_1\}t}\frac{x^{\frac{1}{2}}}{\sigma\sqrt{2\pi t}}\exp\left\{-\frac{1}{2}\frac{(\log x)^2}{\sigma^2 t}\right\}$$

Then, by using Lemma 3.3.1,

$$\begin{aligned}
&\int_b^\infty u\log u\,\mathcal{B}(T-t,\frac{x}{u})\frac{1}{u}du \\
=&\int_b^\infty u(\log x-\log\frac{x}{u})\,\mathcal{B}(T-t,\frac{x}{u})\frac{1}{u}du \\
=&\log x\int_b^\infty u\,\mathcal{B}(T-t,\frac{x}{u})\frac{1}{u}d\eta - xe^{-\frac{\sigma^2}{2}\{\frac{1}{4}+k_1\}t}\int_b^\infty \frac{u}{x}\log\frac{x}{u}\,\frac{\left(\frac{x}{u}\right)^{\frac{1}{2}}}{\sigma\sqrt{2\pi(T-t)}} \\
&\times\exp\left\{-\frac{1}{2}\frac{(\log\frac{x}{u})^2}{\sigma^2(T-t)}\right\}\frac{1}{u}du \\
&= x\log x\;e^{-r(T-t)}\mathcal{N}\left(\frac{\log\dfrac{x}{b}+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) - x\;e^{-r(T-t)}\frac{1}{\sigma\sqrt{2\pi(T-t)}} \\
&\times\int_{-\infty}^{\log\frac{x}{b}} q\cdot\exp\left\{-\frac{1}{2}\left(\frac{q+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)^2\right\}dq \\
&= x\log x\;e^{-r(T-t)}\mathcal{N}\left(\frac{\log\dfrac{x}{b}+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) \\
&+x\;\frac{\sigma^2(T-t)}{2}\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\cdot\int_{-\infty}^{\log\frac{x}{b}}\exp\left\{-\frac{1}{2}\left(\frac{q+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)^2\right\}dq \\
&-x\;\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{\log\frac{x}{b}}\left(q+\frac{\sigma^2}{2}(T-t)\right)\cdot\exp\left\{-\frac{1}{2}\left(\frac{q+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)^2\right\}dq \\
=&x\;e^{-r(T-t)}\left(\log x+\frac{\sigma^2(T-t)}{2}\right)\mathcal{N}\left(\frac{\log\dfrac{x}{b}+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) \\
&+x\;e^{-r(T-t)}\sigma\sqrt{T-t}\cdot n\left(\frac{\log\dfrac{x}{b}+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

□

By Lemma 3.3.1 and Lemma 3.3.2, we can obtain

$$
\begin{aligned}
&\bar{R}_E(t,x)\\
&= e^{-r(T-t)}\mathcal{N}\left(d^{\,0}(T-t,\frac{1}{x})\right) + xe^{-r(T-t)}\left(\log x + 1 + \frac{\sigma^2}{2}(T-t)\right)\\
&\times \mathcal{N}\left(d^{\,0}(T-t,x)\right) + xe^{-r(T-t)}\sigma\sqrt{T-t}\cdot n\left(d^{\,0}(T-t,x)\right)\\
&\bar{R}_e(t,x)\\
&= \int_t^T re^{-r(\eta-t)}\mathcal{N}\left(d^{\,0}(\eta-t,\frac{x^*(\eta)}{x})\right)d\eta\\
&+ \int_t^T re^{-r(\eta-t)}\cdot x\,\sigma\sqrt{\eta-t}\cdot n\left(d^{\,0}(\eta-t,x^*(\eta)x)\right)d\eta\\
&+ \int_t^T rxe^{-r(\eta-t)}\left(\log x + \log x^*(\eta) + 1 + \frac{\sigma^2}{2}(\eta-t)\right)\mathcal{N}\left(d^{\,0}(\eta-t,x^*(\eta)x)\right)d\eta,
\end{aligned}
$$

where

$$
d^{\,0}(T-t,x) = \frac{\log x + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}. \tag{3.3.8}
$$

Then, at $x = x^*(t)$ and the *smooth pasting condition* (3.1.11),

$$
\begin{aligned}
1 &= e^{-r(T-t)}\mathcal{N}\left(d^{\,0}(T-t,\frac{1}{x^*(t)})\right)\\
&+ x^*(t)e^{-r(T-t)}\left(\log x^*(t) + 1 + \frac{\sigma^2}{2}(T-t)\right)\cdot\mathcal{N}\left(d^{\,0}(T-t,x^*(t))\right)\\
&+ x^*(t)\sigma\sqrt{T-t}\cdot e^{-r(T-t)}n\left(d^{\,0}(T-t,\frac{1}{x^*(t)})\right)\\
&+ \int_t^T re^{-r(\eta-t)}\cdot x^*(t)\,\sigma\sqrt{\eta-t}\cdot n\left(d^{\,0}(\eta-t,x^*(\eta)x^*(t))\right)d\eta\\
&+ \int_t^T rx^*(t)e^{-r(\eta-t)}\left(\log x^*(t) + \log x^*(\eta) + 1 + \frac{\sigma^2}{2}(\eta-t)\right)\\
&\times \mathcal{N}\left(d^{\,0}(\eta-t,x^*(\eta)x^*(t))\right)d\eta + \int_t^T re^{-r(\eta-t)}\mathcal{N}\left(d^{\,0}(\eta-t,\frac{x^*(\eta)}{x^*(t)})\right)d\eta.
\end{aligned} \tag{3.3.9}
$$

From Case 3.3.1, Case 3.3.2, and $x = \frac{s}{m}$, $x^*(t) = \frac{s^*(t,m)}{m}$, we obtain the integral equation of a Russian option with a finite time horizon.

Theorem 3.3.1 [Premium decomposition of Russian option with finite time horizon]
The price $R(t,s,m)$ of a Russian option with a finite time horizon defined in (3.1.3) is expressed by

$$R(t,s,m) := R_E(t,s,m) + R_e(t,s,m),$$

and $R_E(t,s,m)$ and $R_e(t,s,m)$ is given by

(1) $r \neq q$

$$\begin{aligned}
&R_E(t,s,m)\\
=&e^{-r(T-t)}\mathcal{N}\left(-d_-(T-t,\frac{s}{m})\right) - \frac{1}{k_2}e^{-r(T-t)}\left(\frac{m}{s}\right)^{(k_2-1)}\mathcal{N}\left(-d_-(T-t,\frac{m}{s})\right)\\
&+(1+\frac{1}{k_2})\frac{s}{m}e^{-q(T-t)}\mathcal{N}\left(d_+(T-t,\frac{s}{m})\right)\\
&R_e(t,s,m)\\
=&\int_t^T re^{-r(\eta-t)}\mathcal{N}\left(-d_-(\eta-t,\frac{s}{s^*(\eta,m)})\right)d\eta\\
&-\frac{1}{k_2}\left(\frac{m}{s}\right)^{(k_2-1)}\int_t^T re^{-r(\eta-t)}\mathcal{N}\left(-d_-(\eta-t,\frac{m^2}{s^*(\eta,m)s})\right)d\eta\\
&+(1+\frac{1}{k_2})\frac{s}{m}\int_t^T r\left(\frac{s^*(\eta,m)}{m}\right)^{k_2}e^{-q(\eta-t)}\mathcal{N}\left(d_+(\eta-t,\frac{s^*(\eta,m)s}{m^2})\right)d\eta
\end{aligned}$$

and the free boundary $s^*(t)$ satisfies the following integral equation:

$$\begin{aligned}
1 =&e^{-r(T-t)}\mathcal{N}\left(-d_-(T-t,\frac{s^*(t,m)}{m})\right)\\
&-\frac{1}{k_2}e^{-r(T-t)}\left(\frac{m}{s^*(t,m)}\right)^{(k_2-1)}\mathcal{N}\left(d_-(T-t,\frac{s^*(t,m)}{m})\right)\\
&+\int_t^T re^{-r(\eta-t)}\mathcal{N}\left(-d_-(\eta-t,\frac{s^*(t,m)}{s^*(\eta,m)})\right)d\eta\\
&-\frac{1}{k_2}\left(\frac{m}{s^*(t,m)}\right)^{(k_2-1)}\int_t^T re^{-r(\eta-t)}\mathcal{N}\left(-d_-(\eta-t,\frac{m^2}{s^*(\eta,m)s^*(t)})\right)d\eta\\
&+(1+\frac{1}{k_2})\frac{s^*(t,m)}{m}\int_t^T r\left(\frac{s^*(\eta,m)}{m}\right)^{k_2}e^{-q(\eta-t)}\mathcal{N}\left(d_+(\eta-t,\frac{s^*(\eta,m)s^*(t,m)}{m^2})\right)d\eta.\\
&+(1+\frac{1}{k_2})\frac{s}{m}e^{-q(T-t)}\mathcal{N}\left(d_+(T-t,\frac{s^*(t,m)}{m})\right)
\end{aligned}$$

(2) $r = q$

$$
\begin{aligned}
&R_E(t,s,m)\\
&= e^{-r(T-t)}\mathcal{N}\left(d^{\,0}(T-t,\frac{m}{s})\right) + \frac{se^{-r(T-t)}}{m}\left(\log\frac{s}{m} + 1 + \frac{\sigma^2}{2}(T-t)\right)\\
&\times\mathcal{N}\left(d^{\,0}(T-t,\frac{s}{m})\right) + \frac{se^{-r(T-t)}}{m}\sigma\sqrt{T-t}\cdot n\left(d^{\,0}(T-t,\frac{s}{m})\right)
\end{aligned}
$$

$$
\begin{aligned}
&R_e(t,s,m)\\
&= \int_t^T re^{-r(\eta-t)}\mathcal{N}\left(d^{\,0}(\eta-t,\frac{s^*(\eta,m)}{s})\right)d\eta\\
&+ \int_t^T re^{-r(\eta-t)}\cdot\frac{s}{m}\sigma\sqrt{\eta-t}\cdot n\left(d^{\,0}(\eta-t,\frac{s^*(\eta,m)s}{m^2})\right)d\eta\\
&+ \int_t^T r\frac{s}{m}e^{-r(\eta-t)}\left(\log\frac{s}{m} + \log\frac{s^*(\eta,m)}{m} + 1 + \frac{\sigma^2}{2}(\eta-t)\right)\\
&\times\mathcal{N}\left(d^{\,0}(\eta-t,\frac{s^*(\eta,m)s}{m^2})\right)d\eta
\end{aligned}
$$

and the free boundary $s^*(t)$ is the solution of the integral equation:

$$
\begin{aligned}
1 &= e^{-r(T-t)}\mathcal{N}\left(d^{\,0}(T-t,\frac{m}{s^*(t,m)})\right)\\
&+ \frac{s^*(t,m)}{m}e^{-r(T-t)}\left(\log\frac{s^*(t,m)}{m} + 1 + \frac{\sigma^2}{2}(T-t)\right)\cdot\mathcal{N}\left(d^{\,0}(T-t,\frac{s^*(t,m)}{m})\right)\\
&+ \frac{s^*(t,m)}{m}e^{-r(T-t)}\sigma\sqrt{T-t}\cdot n\left(d^{\,0}(T-t,\frac{s^*(t,m)}{m})\right)\\
&+ \int_t^T re^{-r(\eta-t)}\cdot\frac{s^*(t,m)}{m}\sigma\sqrt{\eta-t}\cdot n\left(d^{\,0}(\eta-t,\frac{s^*(\eta,m)s^*(t,m)}{m^2})\right)d\eta\\
&+ \int_t^T r\frac{s^*(t,m)}{m}e^{-r(\eta-t)}\left(\log\frac{s^*(t,m)}{m} + \log\frac{s^*(\eta,m)}{m} + 1 + \frac{\sigma^2}{2}(\eta-t)\right)\\
&\times\mathcal{N}\left(d^{\,0}(\eta-t,\frac{s^*(\eta,m)s^*(t,m)}{m^2})\right)d\eta,\\
&+ \int_t^T re^{-r(\eta-t)}\mathcal{N}\left(d^{\,0}(\eta-t,\frac{s^*(\eta,m)}{s^*(t,m)})\right)d\eta
\end{aligned}
$$

where $k_2 = \frac{2(r-q)}{\sigma^2}$ and $d_\pm(T-t,x)$, $d^0_\pm(T-t,x)$ are defined in (3.3.6) and (3.3.8), respectively.

Theorem 3.3.2 For $r, q > 0$ and the free boundary $x^*(t)$ defined in (3.1.11), we have

$$\lim_{t \to T} x^*(t) = 1.$$

Proof Since $x^*(t) < 1$ for all t, it is clear that $x^*(T-) \leq 1$. Let $x^*(T-) < 1$, then there exist x such that $x \in (x^*(T-), 1)$ and $(T-, x)$ belong to continuation region $\bar{S}^C$ defined in section 3.1. In addition, $\bar{R}(T-, x) = 1$.

For $(T-, x)$,

$$\frac{\partial \bar{R}}{\partial t}(T-, x) = -\left[\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \bar{R}}{\partial x^2}(T-, x) + (r-q)x\frac{\partial \bar{R}}{\partial x}(T-, x) - r\bar{R}(T-, x)\right] = r > 0.$$

Therefore, for some point (t, x) in the continuation region $\bar{S}^C$, $\bar{R}(t, x) < 1$. However, this is a contradiction for $(t, x) \in \bar{S}^C$. Hence, $x^*(T-) = 1$.
□

3.4 Valuing Russian Options: Perpetual Case

In this section, we derive the closed form solution for the value of perpetual Russian options using the Mellin transform representation of the value of Russian options with a finite time horizon, which was derived in section 3.3.

Note that at any given time t, an infinite amount of time remains until maturity, and therefore the free boundary for a perpetual Russian option is constant. Let the free boundary of a perpetual Russian option be s^*_∞.

Theorem 3.4.1 For $q > 0$, the price $R_\infty(s, m)$ of a perpetual Russian option is given by

$$R_\infty(s, m) = \frac{m\beta_1}{\beta_1 - \beta_2}\left(\frac{s^*_\infty}{s}\right)^{\beta_2} - \frac{m\beta_2}{\beta_1 - \beta_2}\left(\frac{s^*_\infty}{s}\right)^{\beta_1}, \qquad s > s^*_\infty$$

and s^*_∞ is the free boundary of the perpetual Russian option

$$s^*_\infty = m\left(\frac{\beta_2(1+\beta_1)}{\beta_1(1+\beta_2)}\right)^{\frac{1}{\beta_2 - \beta_1}}$$

where β_1 and β_2 are the two real roots of the quadratic equation

$$w^2 + (1-k_2)w - k_1 = 0, \;\; k_1 = \frac{2r}{\sigma^2}, \; k_2 = \frac{2(r-q)}{\sigma^2}.$$

Proof From (3.3.3), $\bar{R}_e(t,x)$ is the solution of the PDE

$$\begin{aligned}\mathcal{L}\bar{R}_e(t,x) =& \zeta(t,x)\mathbf{1}_{\{x<1\}} + \zeta(t,\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} \\ &+ (k_2+1)x\left[\int_1^x \zeta(t,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2+1)} dy\right]\mathbf{1}_{\{x>1\}} \\ \bar{R}_e(T,x) =& 0.\end{aligned}$$

Define $\bar{R}_e^1(t,x), \bar{R}_e^2(t,x)$, and $\bar{R}_e^3(t,x)$ such that $\bar{R}_e(t,x) = \bar{R}_e^1(t,x) + \bar{R}_e^2(t,x) + \bar{R}_e^3(t,x)$ and satisfies the following PDEs, respectively.

$$\mathcal{L}\bar{R}_e^1(t,x) = \zeta(t,x)\mathbf{1}_{\{x<1\}}, \quad \bar{R}_e^1(T,x) = 0$$

$$\mathcal{L}\bar{R}_e^2(t,x) = \zeta(t,\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}}, \quad \bar{R}_e^2(T,x) = 0$$

and

$$\mathcal{L}\bar{R}_e^3(t,x) = (k_2+1)x\left[\int_1^x \zeta(t,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2+1)} dy\right]\mathbf{1}_{\{x>1\}}, \quad \bar{R}_e^3(T,x) = 0.$$

Since $\zeta(t,x) = -r\mathbf{1}_{\{x<x^*(t)\}}$, the Mellin transform of $\zeta(t,x)$ is $\hat{\zeta}(t,w) = -r\dfrac{x^*(t)^w}{w}$.

Then, for $Re(w) > 0$, by the inverse Mellin transform,

$$\bar{R}_e^1(t,x) = \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} x^{-w}\int_t^T r\frac{x^*(t)^w}{w}e^{\frac{1}{2}\sigma^2 Q(w)(\eta-t)}d\eta dw \qquad (3.4.1)$$

$$\frac{d\bar{R}_e^1}{dx}(t,x) = -\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} x^{-w-1}\int_t^T rx^*(t)^w e^{\frac{1}{2}\sigma^2 Q(w)(\eta-t)}d\eta dw.$$

Let β_1 and β_2 be solutions of $Q(w) = w^2 + w(1-k_2) - k_1$ defined in (3.2.8). If $\beta_1 < \beta_2$, then it is easy to show that for $r > 0$, $\beta_1 < -1$ and $\beta_2 > 0$. Hence, for $\beta_1 < Re(w) < \beta_2$, $Re(Q(w)) < 0$.

Let $0 < Re(c_1) < \beta_2$, then

$$\frac{d\bar{R}_e^1}{dx}(x) = \frac{k_1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{1}{x_\infty^*}\left(\frac{x_\infty^*}{x}\right)^{w+1}\frac{1}{(w-\beta_1)(w-\beta_2)}dw$$

as $T \to \infty$.

By the residue theorem,

$$\begin{aligned}\frac{d\bar{R}_e^1}{dx}(x_\infty^*) &= \frac{k_1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{1}{x_\infty^*}\frac{1}{(w-\beta_1)(w-\beta_2)}dw \\ &= -\frac{k_1}{x_\infty^*}\frac{1}{\beta_2-\beta_1}.\end{aligned} \tag{3.4.2}$$

Similarly, for $Re(w) < k_2 - 1$,

$$\bar{R}_e^2(t,x) = \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}x^{-w}\int_t^T r\frac{x^*(t)^{k_2-1-w}}{k_2-1-w}e^{\frac{1}{2}\sigma^2Q(w)(\eta-t)}d\eta dw. \tag{3.4.3}$$

$$\frac{d\bar{R}_e^2}{dx}(t,x) = -\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}x^{-w-1}\int_t^T r\frac{wx^*(t)^{k_2-1-w}}{k_2-1-w}e^{\frac{1}{2}\sigma^2Q(w)(\eta-t)}d\eta dw.$$

For $\beta_1 < Re(c_2) < k_2 - 1$, letting $T \to \infty$,

$$\frac{d\bar{R}_e^2}{dx}(x) = \frac{k_1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}x^{-w-1}\frac{wx^*(t)^{k_2-1-w}}{(k_2-1-w)(w-\beta_1)(w-\beta_2)}dw.$$

Hence,

$$\begin{aligned}\frac{d\bar{R}_e^2}{dx}(x_\infty^*) &= \frac{k_1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\frac{w{x_\infty^*}^{k_2-2-2w}}{(k_2-1-w)(w-\beta_1)(w-\beta_2)}dw \\ &= k_1\frac{\beta_1{x_\infty^*}^{k_2-2-2\beta_1}}{(\beta_1-\beta_2)(k_2-1-\beta_1)}.\end{aligned} \tag{3.4.4}$$

Finally, for $Re(w) < \min\{k_2 - 1, -1\}$,

$$\bar{R}_e^3(t,x) = -\frac{1}{2\pi i}\int_{c_3 - i\infty}^{c_3 + i\infty} x^{-w} \int_t^T r(k_2+1)\frac{x^*(t)^{k_2-1-w}}{(w+1)(k_2-1-w)} \\ e^{\frac{1}{2}\sigma^2 Q(w)(\eta - t)} d\eta dw \tag{3.4.5}$$

$$\frac{d\bar{R}_e^3}{dx}(t,x) = \frac{1}{2\pi i}\int_{c_3 - i\infty}^{c_3 + i\infty} x^{-w-1} \int_t^T r(k_2+1)\frac{wx^*(t)^{k_2-1-w}}{(w+1)(k_2-1-w)} \\ \times e^{\frac{1}{2}\sigma^2 Q(w)(\eta - t)} d\eta dw.$$

For $\beta_1 < Re(c_3) < -1$, letting $T \to \infty$,

$$\frac{d\bar{R}_e^3}{dx}(x) = -k_1(k_2+1)\frac{1}{2\pi i}\int_{c_3 - i\infty}^{c_3 + i\infty} \frac{x^{-w-1} w x^*(t)^{k_2-1-w}}{(w+1)(k_2-1-w)(w-\beta_1)(w-\beta_2)} dw.$$

Then,

$$\begin{aligned} \frac{d\bar{R}_e^3}{dx}(x_\infty^*) &= -\frac{1}{2\pi i}\int_{c_3 - i\infty}^{c_3 + i\infty} \frac{k_1(k_2+1) w x^*(t)^{k_2-2-2w}}{(w+1)(k_2-1-w)(w-\beta_1)(w-\beta_2)} dw \\ &= -k_1(k_2+1)\frac{\beta_1 {x_\infty^*}^{(k_2-2-2\beta_1)}}{(1+\beta_1)(k_2-1-\beta_1)(\beta_1-\beta_2)}. \end{aligned} \tag{3.4.6}$$

From section 3.3, it is easy to verify that

$$\frac{\partial \bar{R}_E}{\partial x}(t,x) \to 0, \quad as \;\; T \to \infty \tag{3.4.7}$$

by the *smooth pasting condition*,

$$\lim_{x \to x_\infty^*} \frac{\partial \bar{R}}{\partial x}(t,x) = 0. \tag{3.4.8}$$

Therefore, by (3.4.2)∼(3.4.8) and $\beta_1 + \beta_2 = k_2 - 1$, $\beta_1 \cdot \beta_2 = -k_1$,

$$x_\infty^* = \left(\frac{\beta_2(1+\beta_1)}{\beta_1(1+\beta_2)}\right)^{\frac{1}{\beta_2 - \beta_1}}. \tag{3.4.9}$$

Since $x_\infty^* = \frac{s_\infty^*}{m}$, the free boundary of a perpetual Russian option is given by

$$s_\infty^* = m\left(\frac{\beta_2(1+\beta_1)}{\beta_1(1+\beta_2)}\right)^{\frac{1}{\beta_2-\beta_1}}. \tag{3.4.10}$$

For the same reason, from (3.4.1), (3.4.3), and (3.4.5),

$$\begin{aligned}
\bar{R}_e^1(x) &= -\frac{k_1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} x^{-w}\frac{{x_\infty^*}^w}{w}\frac{1}{(w-\beta_1)(w-\beta_2)}dw. \\
&= k_1 x^{-\beta_2}\frac{{x_\infty^*}^{\beta_2}}{\beta_2(\beta_2-\beta_1)} \\
\bar{R}_e^2(x) &= -\frac{k_1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} x^{-w}\frac{{x_\infty^*}^{(k_1-1-w)}}{(k_2-1-w)(w-\beta_1)(w-\beta_2)}dw. \\
&= -k_1 x^{-\beta_1}\frac{{x_\infty^*}^{(k_2-1-\beta_1)}}{(k_2-1-\beta_1)(\beta_1-\beta_2)} \\
\bar{R}_e^3(x) &= \frac{k_1(1+k_2)}{2\pi i}\int_{c_3-i\infty}^{c_3+i\infty} x^{-w}\frac{{x_\infty^*}^{(k_1-1-w)}}{(w+1)(k_2-1-w)(w-\beta_1)(w-\beta_2)}dw. \\
&= (1+k_2)k_1 x^{-\beta_1}\frac{{x_\infty^*}^{(k_2-1-\beta_1)}}{(1+\beta_1)(k_2-1-\beta_1)(\beta_1-\beta_2)}.
\end{aligned} \tag{3.4.11}$$

Moreover,

$$\lim_{T\to\infty}\bar{R}_E(t,x) = 0; \tag{3.4.12}$$

By using (3.4.9),(3.4.11), and (3.4.12), we obtain

$$\bar{R}(x) = \frac{\beta_1}{\beta_1-\beta_2}\left(\frac{x_\infty^*}{x}\right)^{\beta_2} - \frac{\beta_2}{\beta_1-\beta_2}\left(\frac{x_\infty^*}{x}\right)^{\beta_1}.$$

Therefore, for $s > s_\infty^*$, the value of a perpetual Russian option is given by

$$R_\infty(s,m) = \frac{m\beta_1}{\beta_1-\beta_2}\left(\frac{s_\infty^*}{s}\right)^{\beta_2} - \frac{m\beta_2}{\beta_1-\beta_2}\left(\frac{s_\infty^*}{s}\right)^{\beta_1}. \tag{3.4.13}$$

□

3.5 Numerical Results

In this section we approximate the value of Russian options by numerically solving the integral equation satisfied by the free boundary, which was derived in section 3.4. There is a variety of numerical algorithms for solving the integral equation which appears in American type option problems. Here, we use the recursive integration method, which was first introduced by Huang et al. [27]. We first explain how to apply the recursive integration method for solving our integral equation. In subsequent subsections, we analyze the results qualitatively and quantitatively. For quantitative analysis, we use binary tree model with number of timesteps $n = 10000$ as a benchmark. Laplace-Carlson transform method is also introduced to compare results quantitatively. The results of each method is then compared with the benchmark.

3.5.1 Recursive Integration Method for Russian Option with Finite Time Horizon

We first convert the integral equations (3.3.7),(3.3.8) for the free boundary $x^*(t)$, into more convenient form. By letting $\tau = T - t$, $\xi = \eta - t$ and $y^*(t) := x^*(T-t)$, we have

(1) $r \neq q$

$$
\begin{aligned}
1 =& e^{-r\tau}\mathcal{N}\left(-d_-(\tau, y^*(\tau))\right) - \frac{1}{k_2}e^{-r\tau}\left(\frac{1}{y^*(\tau)}\right)^{(k_2-1)}\mathcal{N}\left(-d_-(\tau, \frac{1}{y^*(\tau)})\right) \\
&+(1+\frac{1}{k_2})y^*(\tau)e^{-q\tau}\mathcal{N}\left(d_+(\tau, y^*(\tau))\right) \\
&+\int_0^\tau re^{-r\xi}\mathcal{N}\left(-d_-(\xi, \frac{y^*(\tau)}{y^*(\tau-\xi)})\right)d\xi \\
&-\frac{1}{k_2}\left(\frac{1}{y^*(\tau)}\right)^{(k_2-1)}\int_0^\tau re^{-r\xi}\mathcal{N}\left(-d_-(\xi, \frac{1}{y^*(\tau-\xi)y^*(\tau)})\right)d\xi \\
&+(1+\frac{1}{k_2})y^*(\tau)\int_0^\tau ry^*(\tau-\xi)^{k_2}e^{-q\xi}\mathcal{N}\left(d_+(\xi, y^*(\tau-\xi)y^*(\tau))\right)d\xi.
\end{aligned}
\tag{3.5.1}
$$

(2) $r = q$

$$
\begin{aligned}
1 &= e^{-r\tau}\mathcal{N}\left(d^{\,0}(\tau, \frac{1}{y^*(\tau)})\right) \\
&+ y^*(t)e^{-r\tau}\left(\log y^*(\tau) + 1 + \frac{\sigma^2}{2}\tau\right)\cdot\mathcal{N}\left(d^{\,0}(\tau, y^*(\tau))\right) \\
&+ y^*(\tau)e^{-r\tau}\sigma\sqrt{\tau}\cdot n\left(d^{\,0}(\tau, y^*(\tau))\right) \\
&+ \int_0^\tau re^{-r\xi}\cdot y^*(\tau)\,\sigma\sqrt{\xi}\cdot n\left(d^{\,0}(\xi, y^*(\tau-\xi)y^*(\tau))\right)d\xi \\
&+ \int_0^\tau ry^*(\tau)e^{-r\xi}\left(\log y^*(\tau) + \log y^*(\tau-\xi) + 1 + \frac{\sigma^2}{2}\xi\right) \\
&\times\mathcal{N}\left(d^{\,0}(\xi, y^*(\tau-\xi)y^*(\tau))\right)d\xi + \int_0^\tau re^{-r\xi}\mathcal{N}\left(d^{\,0}(\xi, \frac{y^*(\tau-\xi)}{y^*(\tau)})\right)d\xi,
\end{aligned}
\tag{3.5.2}
$$

with $k_2 = \frac{2(r-q)}{\sigma^2}$ and $d_\pm(t,x) = \dfrac{\log x + (r - q \pm \frac{\sigma^2}{2})}{\sigma\sqrt{t}}$, $d^0(t,x) = \dfrac{\log x + \frac{\sigma^2}{2}t}{\sigma\sqrt{t}}$.
For $i = 1, 2,$ define $F_i(\tau, y^*(\tau))$ and $G_i(\tau, \xi, y^*(\tau), y^*(\tau-\xi))$ as when $r \neq q$,

$$
\begin{aligned}
&F_1(\tau, y^*(\tau)) := e^{-r\tau}\mathcal{N}\left(-d_-(\tau, y^*(\tau))\right) - \frac{1}{k_2}e^{-r\tau}\left(\frac{1}{y^*(\tau)}\right)^{(k_2-1)} \\
&\times\mathcal{N}\left(-d_-(\tau, \frac{1}{y^*(\tau)})\right) + (1 + \frac{1}{k_2})y^*(\tau)e^{-q\tau}\mathcal{N}\left(d_+(\tau, y^*(\tau))\right) \\
&G_1(\tau, \xi, y^*(\tau), y^*(\tau-\xi)) := re^{-r\xi}\mathcal{N}\left(-d_-(\xi, \frac{y^*(\tau)}{y^*(\tau-\xi)})\right) \\
&-\frac{1}{k_2}\left(\frac{1}{y^*(\tau)}\right)^{(k_2-1)} re^{-r\xi}\mathcal{N}\left(-d_-(\xi, \frac{1}{y^*(\tau-\xi)y^*(\tau)})\right) \\
&+ r(1 + \frac{1}{k_2})y^*(\tau)y^*(\tau-\xi)^{k_2}e^{-q\xi}\mathcal{N}\left(d_+(\xi, y^*(\tau-\xi)y^*(\tau))\right)
\end{aligned}
\tag{3.5.3}
$$

and when $r = q$,

$$
\begin{aligned}
&F_2(\tau, y^*(\tau)) := e^{-r\tau}\mathcal{N}\left(d^{\,0}(\tau, \frac{1}{y^*(\tau)})\right) + y^*(t)e^{-r\tau}\left(\log y^*(\tau) + 1 + \frac{\sigma^2}{2}\tau\right) \\
&\times\mathcal{N}\left(d^{\,0}(\tau, y^*(\tau))\right) + y^*(\tau)e^{-r\tau}\sigma\sqrt{\tau}\cdot n\left(d^{\,0}(\tau, y^*(\tau))\right) \\
&G_2(\tau, \xi, y^*(\tau), y^*(\tau-\xi)) := re^{-r\xi}\cdot y^*(\tau)\,\sigma\sqrt{\xi}\cdot n\left(d^{\,0}(\xi, y^*(\tau-\xi)y^*(\tau))\right) \\
&+ re^{-r\xi}\mathcal{N}\left(d^{\,0}(\xi, \frac{y^*(\tau-\xi)}{y^*(\tau)})\right) \\
&+ ry^*(\tau)e^{-r\xi}\left(\log y^*(\tau) + \log y^*(\tau-\xi) + 1 + \frac{\sigma^2}{2}\xi\right)\mathcal{N}\left(d^{\,0}(\xi, y^*(\tau-\xi)y^*(\tau))\right).
\end{aligned}
\tag{3.5.4}
$$

Then, integral equations (3.5.3),(3.5.4) are expressed by

$$1 = F_i(\tau, y^*(\tau)) + \int_0^\tau G_i(\tau, \xi, y^*(\tau), y^*(\tau - \xi))d\xi \tag{3.5.5}$$

To solve the integral equation (3.5.5), we apply the recursive iteration method, which is frequently used for solving integral equation. The procedures are as follows.

We divide the interval $[0, \tau]$ into n subintervals with end points $\tau_j,\ j = 0, 1, 2, ..., n$, where $\tau_0 = 0$,$\tau_n = \tau$ and $\Delta\tau = \frac{\tau}{n}$. Let y_j^* denote the numerical approximation to $y^*(\tau_j),\ j = 0, 1, ..., n$.

For $\tau = \tau_1$, by the trapezoidal rule, integral equation (3.5.5) is approximated by

$$1 = F_i(\tau_1, y_1^*) + \frac{\Delta\tau}{2}\left[G_i(\tau_1, \tau_0, y_1^*, y_1^*) + G_i(\tau_1, \tau_1, y_1^*, y_0^*)\right]. \tag{3.5.6}$$

Since y_0^* is known to be 1 by theorem 3.3.2, the only unknown in (3.5.6) is y_1^*. We can solve nonlinear equation (3.5.6) by utilizing the numerical root-finding method such as the bisection method. Similarly, for $\tau = \tau_2$, we have

$$\begin{aligned}1 =& F_i(\tau_2, y_2^*) + \frac{\Delta\tau}{2}\left[G_i(\tau_2, \tau_0, y_2^*, y_2^*) + 2G_i(\tau_2, \tau_1, y_2^*, y_1^*)\right. \\ &\left. + G_i(\tau_2, \tau_2, y_2^*, y_0^*)\right]\end{aligned} \tag{3.5.7}$$

Since y_1^* is known from previous step, equation (3.5.7) can be solved for y_2^* by the same procedure. Hence, for $y_k^*,\quad k = 2, 3, ...n$, recursively, y_k^* is the solution of the following integral equation,

$$\begin{aligned}1 =& F_i(\tau_k, y_k^*) + \frac{\Delta\tau}{2}\left[G_i(\tau_k, \tau_0, y_k^*, y_k^*) + 2\sum_{j=1}^{k-1} G_i(\tau_k, \tau_{k-j}, y_k^*, y_j^*)\right. \\ &\left. + G_i(\tau_k, \tau_k, y_k^*, y_0^*)\right]\end{aligned} \tag{3.5.8}$$

Now from the values of $\{y_i^*\}_{i=1}^n$, $\bar{R}(t,x)$ can be approximated by

$$\begin{aligned}\bar{R}(\tau,x) \approx R_n(\tau,x) :=& F_i(\tau,x) + \frac{\Delta\tau}{2}\left[G_i(\tau_n,\tau_0,x,y_n^*)\right.\\ &\left.+2\sum_{j=1}^{n-1} G_i(\tau_n,\tau_{n-j},x,y_j^*) + G_i(\tau_n,\tau_n,x,y_0^*)\right].\end{aligned} \tag{3.5.9}$$

For sufficiently large number of subintervals n, the approximated free boundary y_n^* converges to $y^*(\tau)$ and therefore R_n converges to $\bar{R}$ as well.

Furthermore, we can accelerate the convergence of recursive integration method.by applying the Geske and Johnson formula [22] using a three-point Richardson extrapolation scheme

$$\bar{R} \approx \frac{9R_3 - 8R_2 + R_1}{2}. \tag{3.5.10}$$

where $R_i,\ i = 1,2,3$, is the price of an i-times exercisable Russian option. Since the value of $R(t,s,m)$ is $m\bar{R}(t,\frac{s}{m})$, this enables us to obtain the price of a Russian option with a finite time horizon numerically.

To sum up, we implement the recursive integration method according to the following procedure.

Algorithm : Recursive Integration Method for Russian option.

Step 0: Set $(n+1)$ to be the number of time nodes dividing the interval $[0,\tau]$ into n equal subintervals.

Step 1: Approximate optimal stopping boundary $y^*(\tau)$.

Step 1-1: For $y_0^* = 1$, obtain y_1^* in equation (3.5.6) using numerical root-finding method(e.g, bisection method).

Step 1-2: Calculate y_i^* $(i = 2,3,...,n)$, by solving nonlinear equation (3.5.8), recursively.

Step 2: Approximate to value of $\bar{R}(t,x)$ Russian option with finite time horizon.

Step 2-1: For $\{y_i^*\}_{i=1}^n$ in **Step 1-2**, calculate n-times exercisable Russian option $\bar{R}_n(t,x)$ in (3.5.10)

Step 2-2: Calculate $\bar{R}(t,x)$ using the Geske and Johnson extrapolation scheme in (3.5.10).

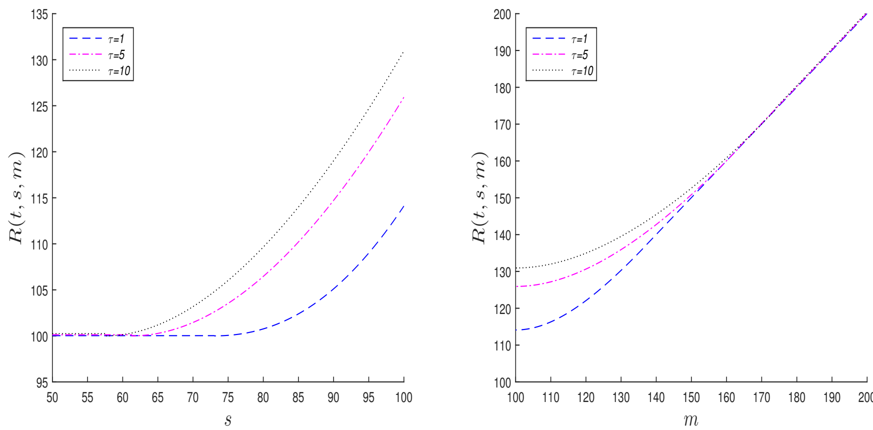
3.5.2 Results : Qualitative analysis

The price of Russian options are plotted in Fig 3.1(a) and Fig 3.1(b) according to their maturity. Fig 3.1(a) exhibits the relation between stock prices and option prices when m held constant. Fig 3.1(b) shows the relation between option prices and the maximum value of stock prices when s are held constant. It is easily seen that option values increase as the time to maturity is extended. Furthermore, in Fig 3.1, the value function is constant while initial stock prices are less than or equal to the free boundary, after which the value increases. Especially, in Fig 3.1(b), the value increases linearly beyond the free boundary. The option price as a function of the interest rate and value of the free boundary, respectively, is plotted in Fig 3.2. The decrease in the option price as the interest rate increases is obvious, and the exercise region shrinks. Fig 3.3 shows the relation between the dividend rate and the free boundary. As in the case of interest rates, the option price and exercise region decrease as the dividend rate increases. Fig 3.4 shows a plot of the option price and free boundary versus volatility. As expected, our results show that the value of Russian options and the exercise region of the free boundary both increase as the volatility increases.

3.5.3 Results : Comparison with Other Methods

In this subsection, we compare our numerical results for the value of Russian options with those of two other methods; binomial tree model(BTM) and Laplace-Carlson transform(LCT) method. We first give a brief introduction of these two frequently used methods.

BTM is one of the most popular pricing method for vanilla options. It is first proposed by Cox, Ross, and Rubinstein in 1979 [8]. Although the model is very simple and easy to implement, it enables us to solve a variety of option pricing problems numerically. The original BTM model assumes that time is discretized by $t_0 = 0$, $t_1 = \Delta t, \ldots, t_n = n\Delta t = T$ with $\Delta t = \frac{T}{n}$. For each time, the underlying asset has only two possible moves. It can move up



(a) Values of $R(t,s,m)$ with respect to s for $m = 100$.

(b) Values of $R(t,s,m)$ with respect to m for $s = 100$.

Figure 3.1: Values of $R(t,s,m)$ $\quad (r = 0.03,\ q = 0.03,\ \sigma = 0.2)$

by a factor of $u \geq 1$ with a probability p or move down by a factor of $d \leq 1$ with a probability $1-p$. Therefore the price of underlying asset will either be uS_0 or dS_0 at next period if the current price of underlying asset is S_0. The factors u,d are usually calculated as $u = e^{\sigma\sqrt{\Delta t}}, d = e^{-\sigma\sqrt{\Delta t}}$ by using the assumption that S follows log-normal distribution of variance $\sigma^2\Delta t$. The risk-neutral probability p is given by $p = \dfrac{e^{(r-q)\Delta t} - d}{u-d}$. Note that at time t_j there are $j+1$ possible values for the underlying asset S_j according to the number of ups $k = 0, \ldots, j$. We denote such possible stock price at time t_j with k ups as $S_j^k = u^k d^{j-k} S_0$. Given that the payoff at maturity V_n^k is known, we can obtain the option price at current time by recursively discounting the option price at succesive time using $V_j^k = e^{-r\Delta t}[pV_{j+1}^{k+1} + (1-p)V_{j+1}^k]$.

For Russian options, careful consideration should be given to apply BTM for valuation. Since Russian options are path-dependent options, the payoff at maturity is determined not from the price of underlying asset but from the path underlying asset has taken. Therefore we need to compute the payoff in each path. In our computation, we use Forward Shooting Grid(FSG) algorithm, which is an application of BTM for path-dependent options. Those who are interested in the details of FSG can refer to [1]. One can also doubt the convergence of BTM for path-dependent options. However, the results

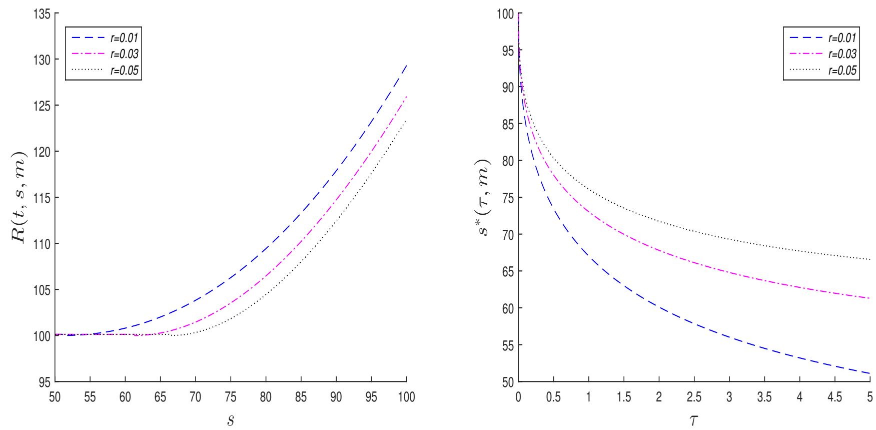


Figure 3.2: Values of $R(t,s,m)$ and free boundary $s^*(t,m)$ $\quad(\tau = 5$, $q = 0.03$, $\sigma = 0.3$, and $m = 100)$

of Jiang and Dai [35] assures that BTM converges for these options as well.

LCT method([45],[46],[47],[48]) is also frequently used for valuing American path-dependent options. It utilizes Laplace-Carlson method, a variant of traditional Laplace transform, to convert PDE satisfied by Russian options into ODE. The method gives formula for transformed option values and the original values are obtained by inverting them using Gaver-Stehfest algorithm. The algorithm produces a double sequence G_m^n such that the diagonal component $G_n = G_n^n$ converges to the Russian option value as $n \to \infty$. The method accelerate its convergence by using n-point Richardson extrapolation scheme, $\overline{G}_n = \sum_{k=1}^{n} \frac{(-1)^{(n-k)}k^n}{k!(n-k)!} G_k$. As suggested and computed in [47], we choose $n = 4$ for fast and efficient computations. For recursive iteration method, the results of $n = 3$ are not significantly different from those of $n = 4$. Therefore we suggest to use $n = 3$ for computational efficiency.

For each numerical experiments we choose BTM using FSG with $n = 10,000$ as a benchmark result and consider it as a exact value of Russian options since the convergence of BTM model is guaranteed. As we have different form of integral equation whether $r = q$ or not, we first consider two cases, $r = q = 0.05$ and $r = 0.05, q = 0.03$. Also, we investigate the option value when there are no dividends, i.e. $r = 0.05, q = 0$. Each of re-

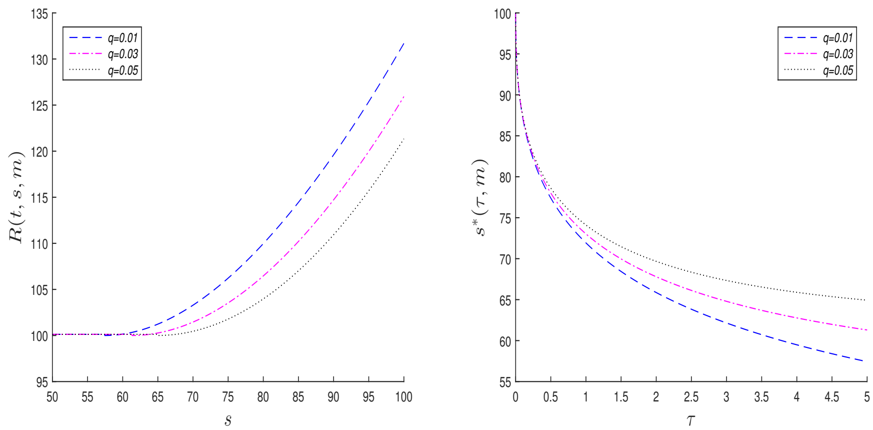


Figure 3.3: Values of $R(t, s, m)$ and free boundary $s^*(t, m)$ $(\tau = 5, r = 0.03, \sigma = 0.3$, and $m = 100)$

sults are summarized in Table 3.1, 3.2, 3.3. Column 5,6 show the numerical results using binomial tree with $N = 150, 500$. Column 7 reports the results of LCT with $n = 4$, and column 8 shows the results of recursive iteration method with 3-point Richardson extrapolation. The accuracy of a method is measured by its root of the mean squared error(RSME), as shown in the second-to-last row. The CPU time in seconds is also shown in the last column.

Table 3.1, 3.2, 3.3 indicates that the recursive iteration method is better method in many ways. First of all, RMSE of recursive iteration method is much smaller than that of other methods in all cases. Second, the results of recursive iteration method are stable. You can see that some results of LCT when $s/m = 0.8$ seem to be running away from the Benchmark. Recursive iteration method does not exhibit such a behavior. Although recursive iteration method takes longer time than LCT, it just take less than 0.3s on average, and it is much faster than widely-used BTM. These reasons illustrate that the recursive iteration method can be a better tool for valuing Russian options.

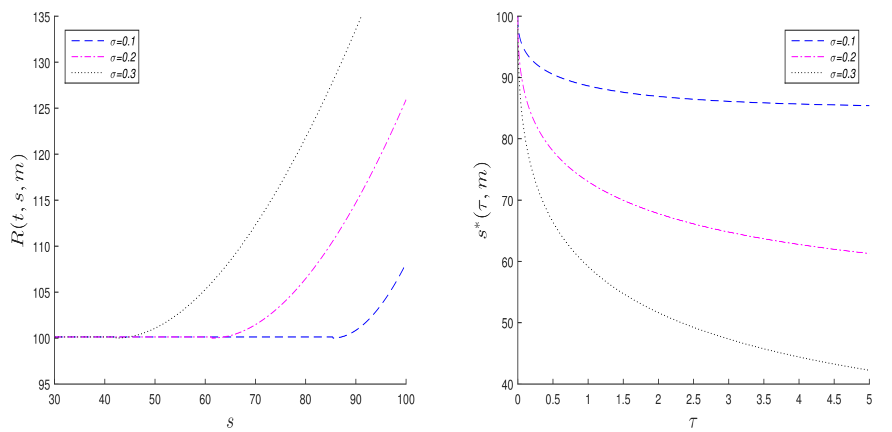


Figure 3.4: Values of $R(t, s, m)$ and free boundary $s^*(t, m)$ $\quad(\tau = 5, r = 0.03, q = 0.03, \text{ and } m = 100)$

Table 3.1: Russian option values $R(t, s, m)/m$ with dividends $(r = 0.05,\ q = 0.05)$

σ	T(yr)	s/m	Benchmark	Binomial (150)	Binomial (500)	LCT	Recursive Iteration Method
0.2	0.0833	1	1.0428	1.0408	1.0418	1.0427	1.0432
		0.9	1.0000	1.0000	1.0001	0.9998	1.0000
		0.8	1.0000	1.0000	1.0000	1.0000	1.0000
	0.3333	1	1.0797	1.0755	1.0776	1.0789	1.0804
		0.9	1.0106	1.0094	1.0100	1.0084	1.0106
		0.8	1.0000	1.0000	1.0000	1.0000	1.0000
	0.5833	1	1.1004	1.0947	1.0978	1.0989	1.1011
		0.9	1.0221	1.0199	1.0210	1.0194	1.0221
		0.8	1.0000	1.0000	1.0000	1.0000	1.0003
0.3	0.0833	1	1.0667	1.0634	1.0650	1.0667	1.0671
		0.9	1.0061	1.0056	1.0058	1.0053	1.0061
		0.8	1.0000	1.0000	1.0000	1.0000	1.0000
	0.3333	1	1.1287	1.1220	1.1254	1.1280	1.1298
		0.9	1.0428	1.0397	1.0412	1.0412	1.0430
		0.8	1.0055	1.0047	1.0051	0.9412	1.0057
	0.5833	1	1.1661	1.1569	1.1616	1.1646	1.1675
		0.9	1.0711	1.0661	1.6860	1.0688	1.0715
		0.8	1.0179	1.0159	1.0169	1.0129	1.0180
0.4	0.0833	1	1.0908	1.0865	1.0887	1.0911	1.0915
		0.9	1.0185	1.0172	1.0179	1.0181	1.0187
		0.8	1.0005	1.0004	1.0004	1.0094	1.0007
	0.3333	1	1.1795	1.1702	1.1749	1.1791	1.1811
		0.9	1.0826	1.0773	1.0799	1.0815	1.0833
		0.8	1.0259	1.0236	1.0248	1.0235	1.0259
	0.5833	1	1.2351	1.2222	1.2287	1.2339	1.2372
		0.9	1.1285	1.1202	1.1243	1.1266	1.1296
		0.8	1.0554	1.0508	1.0532	1.0522	1.0555
		RSME		4.854e-03	2.399e-03	1.259e-02	7.232e-04
		CPU time		3.454e-01	1.288e+01	5.000e-03	2.794e-01

Table 3.2: Russian option values $R(t,s,m)/m$ with dividends ($r = 0.05$, $q = 0.03$)

σ	T(yr)	s/m	Benchmark	Binomial (150)	Binomial (500)	LCT	Recursive Iteration Method
0.2	0.0833	1	1.0437	1.0416	1.0427	1.0436	1.0440
		0.9	1.0001	1.0001	1.0001	1.0001	1.0013
		0.8	1.0000	1.0000	1.0000	1.0000	1.0000
	0.3333	1	1.0832	1.0789	1.0811	1.0823	1.0839
		0.9	1.0121	1.0108	1.0115	1.0098	1.0121
		0.8	1.0000	1.0000	1.0000	1.0000	1.0000
	0.5833	1	1.1065	1.1007	1.1037	1.1049	1.1073
		0.9	1.0255	1.0232	1.0244	1.0227	1.0255
		0.8	1.0000	1.0000	1.0000	0.9997	1.0005
0.3	0.0833	1	1.0675	1.0643	1.0658	1.0676	1.0680
		0.9	1.0064	1.0058	1.0061	1.0056	1.0064
		0.8	1.0000	1.0000	1.0000	1.0000	1.0000
	0.3333	1	1.1324	1.1257	1.1291	1.1317	1.1335
		0.9	1.0452	1.0421	1.0436	1.0437	1.0454
		0.8	1.0062	1.0053	1.0058	0.9560	1.0064
	0.5833	1	1.1727	1.1635	1.1681	1.1712	1.1742
		0.9	1.0761	1.0709	1.0735	1.0737	1.0765
		0.8	1.0203	1.0181	1.0192	1.0160	1.0203
0.4	0.0833	1	1.0917	1.0874	1.0896	1.0920	1.0924
		0.9	1.0190	1.0176	1.0183	1.0185	1.0191
		0.8	1.0005	1.0004	1.0005	1.0099	1.0007
	0.3333	1	1.1834	1.1740	1.1788	1.1830	1.1850
		0.9	1.0855	1.0802	1.0828	1.0844	1.0863
		0.8	1.0275	1.0251	1.0263	1.0250	1.0275
	0.5833	1	1.2421	1.2292	1.2357	1.2409	1.2444
		0.9	1.1342	1.1258	1.1357	1.1323	1.1354
		0.8	1.0592	1.0544	1.0568	1.0559	1.0592
		RSME		4.902e-03	2.432e-03	9.961e-03	8.010e-04
		CPU time		3.253e-01	1.254e+01	5.000e-03	8.778e-02

Table 3.3: Russian option values $R(t, s, m)/m$ with no dividends ($r = 0.05$, $q = 0$)

σ	T(yr)	s/m	Benchmark	Binomial (150)	Binomial (500)	LCT	Recursive Iteration Method
0.2	0.0833	1	1.0450	1.0429	1.0440	1.0499	1.0453
		0.9	1.0002	1.0001	1.0001	1.0010	1.0014
		0.8	1.0000	1.0000	1.0000	1.0000	1.0000
	0.3333	1	1.0887	1.0844	1.0865	1.0878	1.0894
		0.9	1.0146	1.0132	1.0139	1.0122	1.0145
		0.8	1.0000	1.0000	1.0000	1.0000	1.0000
	0.5833	1	1.1162	1.1105	1.1134	1.1145	1.1172
		0.9	1.0314	1.0288	1.0301	1.0283	1.0313
		0.8	1.0005	1.0003	1.0004	0.9989	1.0084
0.3	0.0833	1	1.0688	1.0657	1.0673	1.0689	1.0693
		0.9	1.0068	1.0062	1.0065	1.0060	1.0068
		0.8	1.0000	1.0000	1.0000	0.9999	1.0000
	0.3333	1	1.1381	1.1314	1.1348	1.1374	1.1393
		0.9	1.0491	1.0458	1.0474	1.0475	1.0493
		0.8	1.0075	1.0065	1.0070	1.0092	1.0076
	0.5833	1	1.1831	1.1738	1.1785	1.1815	1.1845
		0.9	1.0839	1.0785	1.0812	1.0815	1.0845
		0.8	1.0242	1.0218	1.0230	1.0198	1.0242
0.4	0.0833	1	1.0931	1.0888	1.0910	1.0933	1.0938
		0.9	1.0196	1.0182	1.0190	1.0192	1.0198
		0.8	1.0006	1.0005	1.0005	1.0099	1.0008
	0.3333	1	1.1894	1.1800	1.1847	1.1890	1.1909
		0.9	1.0901	1.0846	1.0873	1.0890	1.0909
		0.8	1.0299	1.0274	1.0286	1.0274	1.0299
	0.5833	1	1.2531	1.2401	1.2467	1.2519	1.2554
		0.9	1.1432	1.1345	1.1389	1.1413	1.1445
		0.8	1.0654	1.0600	1.0625	1.0617	1.0652
		RSME		2.484e-05	6.150e-06	7.173e-06	2.968e-06
		CPU time		3.413e-01	1.303e+01	5.000e-03	1.4544e-01

3.6 Summary

In conclusion, we described a general method for valuing options with finite maturity, which usually can be mathematically formulated as the free boundary inhomogeneous Black-Scholes PDE with mixed boundary conditions after a suitable change of variables. The main idea of our approach was to convert the given PDE into the relatively simple ODE using Mellin transforms. After solving the ODE with some analytical manipulations, we inverted the ODE solutions to obtain the general solutions for the Black-Scholes equation with a mixed boundary condition using inverse Mellin transforms.

As an illustration of our method, we yielded the integral equation satisfied by the value function of Russian options by applying our anlaytic formula for the inhomogeneous Black-Scholes PDE with mixed boundary conditions. We then valued perpetual (infinite time horizon) Russian options using Mellin transforms as well as basic complex analysis theories.

Furthermore, we numerically solved the derived integral equations using recursive integration methods, and presented the varying option price and free boundary according to the chosen parameters. Our numerical results confirm that the integral equation we derived is correct. Also, we compared our recursive integration method to some existing American path-dependent option valuation techniques such as binary tree model and Laplace-Carlson transform method. By comparing RMSE and computational time of the results, we concluded that our method is accurate and efficient forcomputing Russian option values with finite time horizon.

In conclusion, our Mellin transform based pricing techniques are distinguished from traditional methods in that our approaches not only give us a value of Russian options but also give a analytical representation for the solution of general inhomogeneous Black-Scholes equation with mixed boundary conditions. To the best of our knowledge, there have been no approaches which give an analytic representation of the solution of inhomogeneous Black-Scholes equation with mixed boundary conditions. Since we analytically present the general solution of Black-Scholes equation with mixed boundary conditions, our methodology can be applied to solve free boundary problems for a variety of option pricing problems.

Chapter 4

American floating strike lookback option

Lookback options are path-dependent options with payoffs depending on the maximum or the minimum of the underlying asset price during the lifetime of the option. A popular form of lookback options in the insurance field is equity-indexed annuities (EIA), although other kinds of lookback options are also traded worldwide in the exchange market (refer to [20],[54] for further details on the subject and related topics.) Various researchers have published results relating to the pricing of European lookback options; for example, Goldman et al. [2e] and Conze and Viswanathan [7] derived the exact formula for the value function of European lookback options and Dai et al. [11] presented a formula for quanto lookback options regarding two underlying assets.

As American option holders can exercise their options at any instant before expiry, the early exercise policy should be considered when valuing American options. This is the reason why problems involving American options are usually referred to as optimal stopping problems or free boundary problems. Regarding American option theories, Kwok [58] gave an elaborate description, whereas Peskir and Shiryaev [58] established a number of theories related to optimal stopping problems. Furthermore, Kim [43] derived an integral equation satisfied by American options, because the closed-form solution of the American option did not yet exist at the time.

American lookback options can be thought of as a combination of American options and lookback options. Therefore, they have the properties of both of these types of options. Especially, valuing them requires a solution for the free boundary problems, an approach which is similar to the valuation of other American options. In addition, the presence of a lookback state variable results in a Neumann boundary condition. American lookback options can be divided into two categories: American fixed lookback options and American floating strike lookback options. Both of these types of options solve the same partial differential equation (PDE), but their payoff functions are different. There is also a relationship between American fixed strike lookback options and Russian options, where the latter could be considered a kind of perpetual version of the former. Anyone interested in Russian options can refer to [15],[59],[63]. The distinctive property of American floating strike lookback options is the homogeneity of their value functions. Such homogeneity makes it possible to reduce the dimension of the problem by one; thus, the structure of American floating strike lookback options is relatively simpler than that of the usual American fixed strike options. We focus on American floating strike options in this chapter.

American floating strike lookback options have been studied previously. For example, Yu et al. studied the exercise boundary of American floating strike lookback options [70], and Dai and Kwok characterized the optimal stopping region of American lookback options [12],[13]. Lai and Lim [53] proposed a way to calculate the value of American floating strike lookback options by using a numerical approach known as the Bernoulli walk approach. Kimura performed a premium decomposition for American floating lookback options employing Laplace transforms [48]. Finally, we remark that Dai succeeded in obtaining a closed-form solution of American options [9].

In this work, our approach was to mainly use the Mellin transform, which is a type of integral transform that can be considered a two-sided Laplace transform. Especially, a Mellin transform is widely used in solving option problems because it can be used to convert a Black-Scholes PDE into a simple ordinary differential equation (ODE). Remarkable results have been achieved by following the approach based on the Mellin transform; for example, Panini and Srivastav priced European, American, as well as perpetual American options using Mellin transforms [56],[57], respectively. Frontczak [18] defined a modified Mellin transform and used it to derive an integral equation satisfied by an American call option. Yoon and Kim obtained a closed form of vulnerable options using double Mellin transforms [69]. Yoon also obtained a solution for European options with a stochastic interest model [68]. Jeon and Yoon [31] and Buchen [4] analyzed the pricing of lookback type options with Mellin transform techniques. In addition, Jeon et al.[30] derived a semi-closed form solution of vulnerable lookback options by utilizing double Mellin transforms.

4.1 Model formulation

Let S_t denote the underlying asset of the floating strike lookback option under a risk-neutral probability measure $\mathbb{P}$.

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t \quad (r > q)$$

where $r(> 0)$ is the riskless interest rate, σ and $q(> 0)$ are the volatility and dividend yield of X, respectively, and W_t is a one-dimensional standard Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0} \equiv \mathbb{F}$ is the natural filtration generated by $\mathbb{F}$.

For the process $(S_t)_{t\geq 0}$, define the minimum process as

$$m_t = \min_{0\leq\gamma\leq t} S_\gamma, \quad t \geq 0$$

Consider an *American floating strike lookback call* option with a given finite time horizon $T > 0$. The payoff at maturity is given by $(S_T - m_T)$. In the absence of arbitrage opportunities, the value $C(t, S_t, m_t)$ is a solution of an *optimal stopping problem* (see [67])

$$C(t, s, m) = \sup_{\tau\in[t,T]} \mathbb{E}\left[e^{-r(\tau-t)}(S_\tau - m_\tau) \mid S_t = s, m_t = m\right] \tag{4.1.1}$$

where τ is the stopping time of the filtration $\mathbb{F}$ and the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$.

It is known that the *optimal stopping problem* (4.1.1) can be reduced to a *free boundary problem.*
Define the differential operator $\mathcal{L}$ by

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + (r - q)s\frac{\partial}{\partial s} - r$$

Then, the free boundary problem can be written in a linear complementary form (see [51] and [66]) as

$$\begin{aligned} &\mathcal{L}\,C(t, s, m) \leq 0, \;\; C(t, s, m) \geq S - m, \\ &(\mathcal{L}\,C(t, s, m))\,(C(t, s, m) - (s - m)) = 0, \;\; s > m > 0, \;\; 0 \leq t < T, \end{aligned} \tag{4.1.2}$$

together with auxiliary conditions

$$\begin{aligned} &C(T,s,m) = s - m \\ &\lim_{x \downarrow m} \frac{\partial C}{\partial m} = 0 \end{aligned} \tag{4.1.3}$$

The free boundary of problem (4.1.1) is given by the critical stock price $s^*(t,m)$ (This is termed the *early exercise boundary*). Arbitrage arguments show that the option price $C(t,s,m)$ must also satisfy the "smooth pasting conditions" at $s^*(t,m)$.

$$\begin{aligned} &\lim_{s \uparrow s^*} C(t,s,m) = s^*(t,m) - m \\ &\lim_{s \uparrow s^*} \frac{\partial C}{\partial s} = 1 \end{aligned} \tag{4.1.4}$$

With the change of variables

$$x := \frac{m}{s} \tag{4.1.5}$$

and

$$\bar{C}(t,x) = \frac{C(t,s,m)}{s}. \tag{4.1.6}$$

Then, we can rewrite the linear complementary form (4.1.2) as

$$\begin{aligned} &\bar{\mathcal{L}}\,\bar{C}(t,x) \le 0, \;\; \bar{C}(t,x) \ge 1 - x, \\ &\left(\bar{\mathcal{L}}\,\bar{C}(t,x)\right)\left(\bar{C}(t,x) - (1-x)\right) = 0, \quad 0 < x < 1, \; 0 \le t < T, \end{aligned} \tag{4.1.7}$$

with auxiliary conditions:

$$\begin{aligned} &\bar{C}(T,x) = \phi(x) := 1 - x \\ &\lim_{x \downarrow 1} \frac{\partial \bar{C}}{\partial x} = 0 \\ &\lim_{x \uparrow x^*} \bar{C}(t,x) = 1 - x^*(t) \\ &\lim_{x \uparrow x^*} \frac{\partial \bar{C}}{\partial x} = -1 \end{aligned} \tag{4.1.8}$$

where the critical value is $x^* = \frac{m}{s^*}$ and the operator $\bar{\mathcal{L}}$ is given by

$$\bar{\mathcal{L}} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + (q-r)x\frac{\partial}{\partial x} - q.$$

Hence, solving the optimal stopping problem (4.1.1) is equivalent to finding the points $(t, x^*(t))$. Let

$$\mathcal{D} = \{(t,x) \mid 0 \le t \le T,\, 0 < x \le 1\}$$

and let $\mathcal{S}$ and $\mathcal{S}^{\mathcal{C}}$ denote the *stopping region* and *continuation region*, respectively. In terms of the value function $\bar{C}(t,x)$, the stopping region $\mathcal{S}$ is defined by

$$\begin{aligned}\mathcal{S} &= \{(t,x) \in \mathcal{D} \mid \bar{C}(t,x) = 1 - x\} \\ &= \{(t,x) \mid 0 < x < x^*(t),\, 0 \le t \le T\}\end{aligned}$$

The continuation region $\mathcal{S}^C$ is given by

$$\begin{aligned}\mathcal{S}^C &= \{(t,x) \in \mathcal{D} \mid \bar{C}(t,x) > 1 - x\} \\ &= \{(t,x) \mid x^*(t) < x < 1,\, 0 \le t \le T\}\end{aligned}$$

Hence, the value function $\bar{C}(t,x)$ satisfies the following inhomogeneous Black-Scholes PDE :

$$\bar{\mathcal{L}}\bar{C}(t,x) = \frac{\partial \bar{C}}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \bar{C}}{\partial x^2} + (q-r)x\frac{\partial \bar{C}}{\partial x} - q\bar{C} = f(t,x) \qquad (4.1.9)$$

where

$$f = f(t,x) = \begin{cases} rx - q & \text{for} \quad 0 \le x \le x^*(t) \\ 0 & \text{for} \quad x^*(t) < x < 1 \end{cases} \qquad (4.1.10)$$

with Neumann boundary conditions in (4.1.8).

In the following section, we exhibit the closed-form representation of the solution of the general inhomogeneous Black-Scholes PDE with Dirichlet and Neumann boundary conditions. The application of such a closed-form formula to (4.1.9) leads to a representation of the value function $\bar{C}(t,x)$.

4.2 Inhomogeneous Black-Scholes equation with Neumann boundary condition

Although Buchen already analyzed real option problems with Neumann boundary conditions using a Mellin transform [4], he used a homogeneous Black-Scholes equation. In this section, we extend his idea to the inhomogeneous case, with the aim of converting an inhomogeneous Black-Scholes PDE with Neumann boundary conditions into an inhomogeneous Black-Scholes PDE with Dirichlet boundary conditions. We subsequently use these results to solve the reduced equation with the aid of Mellin transform techniques (the definition and properties of the Mellin transform are summarized in the appendix of this paper.)

Define the PDE operator $\mathcal{L}_{BS}$ as

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + (q-r)x\frac{\partial}{\partial x} - q.$$

Consider the following *Neumann boundary condition* PDE problem :

$$\begin{aligned} &\mathcal{L}_{BS}\, P(t,x) = h(t,x), \quad P(T,x) = g(x) \\ &\frac{\partial P}{\partial x}(t,1) = 0 \end{aligned} \tag{4.2.1}$$

on domain $\{(t,x) \mid 0 \le t < T,\, 0 \le x < 1\}$.

We assume that $h(t,x)$,$g(x)$ are smooth functions and Mellin transforms of $h, g, x\dfrac{dh}{dx}, x\dfrac{dg}{dx}$ are well-defined in proper domains, respectively.

Let $V(t,x) = x\dfrac{\partial P}{\partial x}(t,x)$, $\psi(t,x) = x\dfrac{\partial h}{\partial x}(t,x)$ and $\zeta(x) = x\dfrac{dg}{dx}(x)$, then PDE (4.2.1) is converted to

$$\begin{aligned} &\mathcal{L}_{BS}\, V(t,x) = \psi(t,x) \\ &V(t,1) = 0 \\ &V(T,x) = \zeta(x) \end{aligned} \tag{4.2.2}$$

on domain $\{(t,x) \mid 0 \le t < T,\, 0 \le x < 1\}$.

To solve PDE (4.2.2), we consider an unrestricted inhomogeneous PDE:

$$\begin{aligned}
&\mathcal{L}_{BS}\, Q(t,x) = \psi(t,x)\mathbf{1}_{\{x<1\}} \\
&Q(T,x) = \zeta(x)\mathbf{1}_{\{x<1\}}
\end{aligned} \tag{4.2.3}$$

on domain $\{(t,x) \mid 0 \le t < T,\, 0 \le x < \infty\}$.

Then, we can define $\hat{Q}(t,w)$ the Mellin transform of $Q(t,x)$.

$$\hat{Q}(t,w) = \int_0^\infty Q(t,x)x^{w-1}dw$$

From PDE (4.2.3), $\hat{Q}(t,w)$ satisfies the following ODE

$$\frac{d\hat{Q}}{dt} + \left(\frac{1}{2}\sigma^2 w(w+1) - (q-r)w - q\right)\hat{Q} = \hat{\psi}(t,w) \tag{4.2.4}$$

where $\hat{Q}$ and $\hat{\psi}$ are the Mellin transforms of $Q(t,x)$ and $\psi(t,x)\mathbf{1}_{\{x<1\}}$, respectively.

Let $A(w) = w^2 + w(1-k_2) - k_1$ where $k_1 = \frac{2q}{\sigma^2},\ k_2 = \frac{2(q-r)}{\sigma^2}$.

Then,

$$\hat{Q}(t,w) = e^{\frac{1}{2}\sigma^2 A(w)(T-t)}\hat{\zeta}(w) - \int_t^T e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}\hat{\psi}(\eta,w)d\eta \tag{4.2.5}$$

where $\hat{\zeta}(w)$ is the Mellin transform of $\zeta(x)\mathbf{1}_{\{x<1\}}$.

By the inverse Mellin transform,

$$\begin{aligned}
Q(t,x) =& \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}\sigma^2 A(w)(T-t)}\hat{\zeta}(w)x^{-w}dw \\
&- \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}\hat{\psi}(\eta,w)x^{-w}d\eta dw
\end{aligned} \tag{4.2.6}$$

To compute (4.2.6), let

$$\mathcal{B}(t,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}\sigma^2 A(w)t}x^{-w}dw \tag{4.2.7}$$

Then,

$$\mathcal{B}(t,x) = e^{-\frac{\sigma^2}{2}\{(\frac{1-k_2}{2})^2+k_1\}t}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{\sigma^2}{2}(w+\frac{1-k_2}{2})^2 t}x^{-w}dw.$$

According to the property of the Mellin transform in the Appendix A.1

$$\mathcal{B}(t,x) = e^{-\frac{\sigma^2}{2}\{(\frac{1-k_2}{2})^2+k_1\}t}\frac{x^{\frac{1-k_2}{2}}}{\sigma\sqrt{2\pi t}}\exp\left\{-\frac{1}{2}\frac{(\log x)^2}{\sigma^2 t}\right\}. \tag{4.2.8}$$

Because $e^{\frac{\sigma^2}{2}A(w)(T-t)}$, $\hat{\zeta}(w)$, and $\hat{\psi}(\eta,w)$ are the Mellin transforms of $\mathcal{B}(T-t,x)$,$\zeta(x)\mathbf{1}_{\{x<1\}}$, and $\psi(\eta,x)\mathbf{1}_{\{x<1\}}$, respectively, according to the Mellin convolution property in the Appendix A.1,

$$\begin{aligned} Q(t,x) = &\int_0^\infty \zeta(u)\mathbf{1}_{\{u<1\}}\mathcal{B}(T-t,\frac{x}{u})\frac{1}{u}du \\ &-\int_t^T\int_0^\infty \psi(\eta,u)\mathbf{1}_{\{u<1\}}\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta. \end{aligned} \tag{4.2.9}$$

The following lemma is essential in that it provides a tool for the domain extension of the PDE (4.2.2).

Lemma 4.2.1 For $\mathcal{B}(t,x)$ defined in (5.2.7),

$$\mathcal{B}(t,x) = x^{(1-k_2)}\mathcal{B}(t,\frac{1}{x}) \tag{4.2.10}$$

Proof See the proof of Theorem 3.2.1 in Section 3.2.
□

Using Lemma 4.2.1, we can prove the following theorem which extends the solution of the PDE (4.2.2) to $[0,T)\times[0,\infty)$

Theorem 4.2.1 (Inhomogeneous Black-Scholes equation with Dirichlet condition)
In the domain $\{(t,x) \mid 0 \le t < T, 0 \le x < \infty\}$, $V(t,x)$ the solution of PDE (4.2.2) is given by

$$V(t,x) = Q(t,x) - x^{(1-k_2)}Q(t,\frac{1}{x}) \tag{4.2.11}$$

where $Q(t,x)$ defined in (4.2.3) is expressed by

$$\begin{aligned} Q(t,x) = &\int_0^\infty \zeta(u)\mathbf{1}_{\{u<1\}}\mathcal{B}(T-t,\frac{x}{u})\frac{1}{u}du \\ &- \int_t^T \int_0^\infty \psi(\eta,u)\mathbf{1}_{\{u<1\}}\mathcal{B}(\tau-t,\frac{x}{u})\frac{1}{u}dud\eta \end{aligned}$$

Proof See the proof of Theorem 3.2.1 in Section 3.2.
□

Note that we extended the domain of the PDE (4.2.2) to non-negative real numbers using Theorem 4.2.1. Therefore, we can apply Mellin transforms to the extended PDE, which would enable us to obtain the representation for the solution of an inhomogeneous Black-Scholes PDE with Neumann boundary conditions.

Theorem 4.2.2 (Inhomogeneous Black-Scholes equation with Neumann boundary conditions)
$P(t,x)$, the solution of PDE (4.2.2), satisfies the following PDE:

$$\begin{aligned} \mathcal{L}_{BS}P(t,x) =& h(t,x)\mathbf{1}_{\{x<1\}} + h(t,\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} \\ &+ (k_2-1)\left[\int_1^x h(t,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}} \end{aligned}$$

$$\begin{aligned} P(T,x) =& g(x)\mathbf{1}_{\{x<1\}} + g(\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} \\ &+ (k_2-1)\left[\int_1^x g(\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}} \end{aligned}$$

with the domain $\{(t,x) \mid 0 \le t < T, 0 < x < \infty\}$.

Proof By Theorem 4.2.1, on domain $\{(t,x)|0 \le t < T, 0 < x < \infty\}$, $Q(t,x)$ satisfies the following PDE:

$$\begin{aligned} &\mathcal{L}_{BS}V(t,x) = \psi(t,x)\mathbf{1}_{\{x<1\}} - \psi(t,\frac{1}{x})x^{(1-k_2)}\mathbf{1}_{\{x>1\}} \\ &V(T,x) = \zeta(x)\mathbf{1}_{\{x<1\}} - \zeta(\frac{1}{x})x^{(1-k_2)}\mathbf{1}_{\{x>1\}} \end{aligned} \tag{4.2.12}$$

Define

$$H(t,x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{V}(t,w)\frac{1}{w}x^{-w}dw$$

i.e.,$H(t,x)$ is the inverse Mellin transformation of $\dfrac{-\hat{V}(t,w)}{w}$.

Clearly, $-w\hat{H}(t,w) = \hat{V}(t,w)$. According to the property of the Mellin transform,

$$x\frac{\partial}{\partial x}H(t,x) = V(t,x).$$

Because $V(t,1) = 0$,

$$1 \cdot \frac{\partial H}{\partial x}(t,1) = 0. \tag{4.2.13}$$

Similarly, by (4.2.4),

$$\frac{d\hat{V}}{dt} + \left(\frac{1}{2}\sigma^2 w(w+1) - (q-r)w - q\right)\hat{V} = \hat{\psi}(t,w) - \hat{\psi}(t,k_2-1-w)$$

where $\hat{V}(t,w)$ is the Mellin transform of $V(t,x)$.

From $-w\hat{H}(t,w) = \hat{V}(t,w)$,

$$\begin{aligned} &-w\left[\frac{d\hat{H}}{dt} + \left(\frac{1}{2}\sigma^2 w(w+1) - (q-r)w - q\right)\hat{H}\right] \\ &= -w\hat{h}(t,w) + (k_2-1-w)\hat{h}(t,k_2-1-w) \end{aligned}$$

where $\hat{h}$ is Mellin transform of $h(t,x)\mathbf{1}_{\{x<1\}}$.

Hence,

$$\begin{aligned} &\frac{d\hat{H}}{dt} + \left(\frac{1}{2}\sigma^2 w(w+1) - (q-r)w - q\right)\hat{H} \\ &= \hat{h}(t,w) + \hat{h}(t,k_2-1-w) - \frac{k_2-1}{w}\hat{h}(t,k_2-1-w). \end{aligned} \tag{4.2.14}$$

By the inverse Mellin transform of both sides of (4.2.14),

$$\begin{aligned}\mathcal{L}_{BS}\, H(t,x) =& h(t,x)\mathbf{1}_{\{x<1\}} + h(t,\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} \\ &+ (k_2-1)\left[\int_1^x h(t,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}.\end{aligned} \tag{4.2.15}$$

By the same procedure,

$$\begin{aligned}H(T,x) =& g(x)\mathbf{1}_{\{x<1\}} + g(\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} \\ &+ (k_2-1)\left[\int_1^x g(\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}.\end{aligned} \tag{4.2.16}$$

By (4.2.13), (4.2.15), and (4.2.16), $P(t,x)$, which is the solution of PDE (4.2.1), satisfies

$$\begin{aligned}\mathcal{L}_{BS}P(t,x) =& h(t,x)\mathbf{1}_{\{x<1\}} + h(t,\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} \\ &+ (k_2-1)\left[\int_1^x h(t,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}\end{aligned}$$

$$\begin{aligned}P(T,x) =& g(x)\mathbf{1}_{\{x<1\}} + g(\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} \\ &+ (k_2-1)\left[\int_1^x g(\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}\end{aligned}$$

with the domain $\{(t,x) \mid 0 \le t < T, 0 < x < \infty\}$.

□

From Theorem 4.2.2, we define the absorbing Neumann boundary operator $\mathcal{T}$ as follows:

Definition 4.2.1 Let $U(t,x)$ be any function of t and x. Then, the image of the absorbing *Neumann boundary operator* of U with respect to $x=1$ is defined to be the function

$$\begin{aligned}\mathcal{T}[U(t,x)] =& U(t,x)\mathbf{1}_{\{x<1\}} + U(t,\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}} \\ &+ (k_2-1)\left[\int_1^x U(t,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}.\end{aligned} \tag{4.2.17}$$

4.3 Integral equation representation of American floating strike lookback option

The value of American type options is usually decomposed into the European option value and the early exercise value terms. Numerous approaches and considerable effort have been devoted to obtaining the value of American options. For example, I.Kim demonstrated the inclusion of early exercise premium terms in the integral equation [43], Lai and Lim derived the integral equation representation of American floating strike options using reflection Brownian motion [53], and Kimura performed a premium decomposition using Laplace transforms [54].

In this section, we derive an integral equation satisfied by American floating lookback options using our approach involving a PDE based on a Mellin transform, as described in section 4.2.

By Theorem 4.2.2, the solution $\bar{C}(t,x)$ of the PDE (4.1.9) satisfies the following PDE when absorbing the Neumann boundary condition:

$$\bar{\mathcal{L}}\bar{C}(t,x) = \frac{\partial\bar{C}}{\partial t} + \frac{1}{2}\sigma^2x^2\frac{\partial^2\bar{C}}{\partial x^2} + (q-r)x\frac{\partial\bar{C}}{\partial x} - q\bar{C} = \mathcal{T}[f(t,x)] \tag{4.3.1}$$

where $\mathcal{T}$ is the absorbing Neumann boundary operator in Definition 4.2.1 and

$$f = f(t,x) = \begin{cases} rx-q & \text{for} \quad 0 \le x \le x^*(t) \\ 0 & \text{for} \quad x^*(t) < x < 1 \end{cases}$$

with auxiliary condition

$$
\begin{aligned}
&\bar{C}(T,x) = \mathcal{T}[\phi(x)] \\
&\lim_{x \uparrow x^*} \bar{C}(t,x) = 1 - x^*(t) \\
&\lim_{x \uparrow x^*} \frac{\partial \bar{C}}{\partial x} = -1
\end{aligned} \tag{4.3.2}
$$

in domain $\{(t,x) \mid 0 \leq t < T, 0 < x < \infty\}$.

Let

$$
\bar{C}(t,x) = \bar{C}_E(t,x) + \bar{C}_P(t,x)
$$

where $\bar{C}_E(t,x)$ and $\bar{C}_P(t,x)$ satisfy following PDEs :

$$
\begin{aligned}
\tilde{\mathcal{L}}\bar{C}_E(t,x) &= 0 \\
\bar{C}_E(T,x) &= \mathcal{T}[\phi(x)]
\end{aligned} \tag{4.3.3}
$$

and

$$
\tilde{\mathcal{L}}\bar{C}_P(t,x) = \mathcal{T}[f(t,x)], \quad \bar{C}_P(T,x) = 0 \tag{4.3.4}
$$

Define $C_E(t,s,m) = s\tilde{C}_E(t,\frac{m}{s})$ and $C_P(t,s.m) = s\tilde{C}_P(t,\frac{m}{s})$.

Clearly, $C(t,s,m) = C_E(t,s,m) + C_P(t,s,m)$ and $C_E(t,s,m)$ is the value of the European lookback call option with terminal payoff $(S_t - m_t)$. We rewrite the value of $C_E(t,s,m)$, which was obtained by Kimura and can be found in [54], to obtain,

$$
\begin{aligned}
&C_E(t,s,m) \\
=& se^{-q(T-t)}\mathcal{N}(d_1^+(\frac{s}{m},T-t)) - me^{-r(T-t)}\mathcal{N}(d_1^-(\frac{s}{m},T-t)) \\
&+\frac{\sigma^2 s}{2(r-q)}\left\{e^{-r(T-t)}(\frac{m}{s})^{\frac{2(r-q)}{\sigma^2}}\mathcal{N}(d_1^-(\frac{m}{s},T-t)) - e^{-q(T-t)}\mathcal{N}(-d_1^+(\frac{s}{m},T-t))\right\}
\end{aligned} \tag{4.3.5}
$$

where N is the standard cumulative normal distribution and

$$
d^{\pm}(\xi, T-t) := \frac{1}{\sigma\sqrt{T-t}}\left\{\log\xi + (r-q\pm\frac{1}{2}\sigma^2)(T-t)\right\}. \tag{4.3.6}
$$

Therefore,

$$
\begin{aligned}
&\bar{C}_E(t,x) \\
=&e^{-q(T-t)}\mathcal{N}(d^+(\frac{1}{x},T-t)) - xe^{-r(T-t)}\mathcal{N}(d^-(\frac{1}{x},T-t)) \\
+&\frac{\sigma^2}{2(r-q)}\Big\{e^{-r(T-t)}x^{\frac{2(r-q)}{\sigma^2}}\mathcal{N}(d^-(x,T-t)) \\
&-e^{-q(T-t)}\mathcal{N}(-d^+(\frac{1}{x},T-t))\Big\}.
\end{aligned}
\tag{4.3.7}
$$

The following lemma is useful for the derivation of the integral equation representation of American lookback options.

Lemma 4.3.1

$$
\begin{aligned}
\int_0^A u^{-\alpha}\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}d\eta =&x^{-\alpha}e^{-\frac{1}{2}\sigma^2(\eta-t)\{(\frac{1-k_2}{2})^2+k_1-(\frac{1-k_2}{2}+\alpha)^2\}} \\
&\times\mathcal{N}\left(\frac{-\log\frac{x}{A}+\sigma^2(\eta-t)(\frac{1-k_2}{2}+\alpha)}{\sigma\sqrt{\eta-t}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\int_A^\infty u^{-\alpha}\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}d\eta =&x^{-\alpha}e^{-\frac{1}{2}\sigma^2(\eta-t)\{(\frac{1-k_2}{2})^2+k_1-(\frac{1-k_2}{2}+\alpha)^2\}} \\
&\times\mathcal{N}\left(\frac{\log\frac{x}{A}-\sigma^2(\eta-t)(\frac{1-k_2}{2}+\alpha)}{\sigma\sqrt{\eta-t}}\right)
\end{aligned}
$$

where $k_1 = \frac{2q}{\sigma^2}$, $k_2 = \frac{2(q-r)}{\sigma^2}$.

Now, we state the theorem regarding the price of an American lookback option as follows.

Theorem 4.3.1 (Premium decomposition of American floating strike lookback option)
The price of an *American floating strike lookback call option*, as defined in (4.1.1), is

$$C(t,s,m) = C_E(t,s,m) + C_P(t,s,m),$$

where

$$\begin{aligned}C_E(t,s,m) =& se^{-q(T-t)}\mathcal{N}(d_1^+(\frac{s}{m})) - me^{-r(T-t)}\mathcal{N}(d_1^-(\frac{s}{m})) \\ &-\frac{1}{k_2}\left\{e^{-r(T-t)}(\frac{s}{m})^{k_2}\mathcal{N}(d_1^-(\frac{m}{s})) - e^{-q(T-t)}\mathcal{N}(-d_1^+(\frac{s}{m}))\right\}\end{aligned}$$

and

$$\begin{aligned}&C_P(t,s,m)\\ &= -r\cdot m\int_t^T e^{-r(\eta-t)}\mathcal{N}\left(d^-(\frac{s}{s^*(\eta,m)},\eta-t)\right)d\eta \\ &+ q\cdot s\int_t^T \cdot e^{-q(\eta-t)}\mathcal{N}\left(d^+(\frac{s}{s^*(\eta,m)},\eta-t)\right)d\eta \\ &- \frac{r\cdot s}{k_2}\int_t^T \left(\frac{m}{s}\right)^{k_2} e^{-r(\eta-t)}\mathcal{N}\left(d^-(\frac{m^2}{s\cdot s^*(\eta,m)},\eta-t)\right)d\eta \\ &- r(1-\frac{1}{k_2})\int_t^T e^{-q(\eta-t)}\left(\frac{m^2}{s^*(\eta,m)}\right)^{k_2}\mathcal{N}\left(-d^+(\frac{m^2}{s\cdot s^*(\eta,m)},\eta-t)\right)d\eta \\ &+ q\cdot s\int_t^T e^{-q(\eta-t)}\left(\frac{m^2}{s^*(\eta,m)}\right)^{k_2-1}\mathcal{N}\left(-d^+(\frac{m^2}{s\cdot s^*(\eta,m)},\eta-t)\right)d\eta.\end{aligned}$$

For the *free boundary of American floating lookback option* $s^*(t,m)$, we define $x^*(t) = \dfrac{m}{s^*(t,m)}$. Then x^* satisfies the following integral equation.

$$\begin{aligned}&1-x^*(t)\\ &= \bar{C}_E(t,x^*(t)) - r\int_t^T x^*(t)e^{-r(\eta-t)}\mathcal{N}\left(d^-(\frac{x^*(\eta)}{x^*(t)},\eta-t)\right)d\eta \\ &+ q\int_t^T e^{-q(\eta-t)}\mathcal{N}\left(d^+(\frac{x^*(\eta)}{x^*(t)},\eta-t)\right)d\eta \\ &- \frac{r}{k_2}\int_t^T x^*(t)^{-k_2}e^{-r(\eta-t)}\mathcal{N}\left(d^-(x^*(\eta)x^*(t),\eta-t)\right)d\eta \\ &- r(1-\frac{1}{k_2})\int_t^T e^{-q(\eta-t)}x^*(\eta)^{k_2}\mathcal{N}\left(-d^+(\frac{1}{x^*(\eta)x^*(t)},\eta-t)\right)d\eta \\ &+ q\int_t^T e^{-q(\eta-t)}x^*(\eta)^{(k_2-1)}\mathcal{N}\left(-d^+(\frac{1}{x^*(\eta)x^*(t)},\eta-t)\right)d\eta.\end{aligned}$$

where $k_2 = \frac{2(q-r)}{\sigma^2}$, $d^{\pm}$ is defined in (4.3.6) and $\bar{C}_E(t,x)$ is defined in (4.3.7)

Proof By (4.3.1),

$$f(t,x) = (rx-q)\mathbf{1}_{\{x<x^*(t)\}}$$

In PDE (4.3.4), $\bar{C}_P(t,x)$ is expressed by

$$\bar{C}_P(t,x) = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1(t,x) &:= -\int_t^T \int_0^\infty f(\eta,u)\mathbf{1}_{\{u<1\}}\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta \\
I_2(t,x) &:= -\int_t^T \int_0^\infty f(\eta,\frac{1}{u})\mathbf{1}_{\{u>1\}}\left(\frac{1}{u}\right)^{(k_2-1)}\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta \\
I_3(t,x) &:= -(k_2-1)\int_t^T \int_0^\infty \left[\int_1^u f(\eta,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{u>1\}}\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta
\end{aligned}$$

By using Lemma 4.3.1,

$$\begin{aligned}
&I_1(t,x) \\
&= -\int_t^T \int_0^\infty f(\eta,u)\mathbf{1}_{\{u<1\}}\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta \\
&= -\int_t^T \int_0^{x^*(\eta)} (ru-q)\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta \\
&= -r\int_t^T \int_0^{x^*(\eta)} u\cdot\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta + q\int_t^T \int_0^{x^*(\eta)} 1\cdot\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta \\
&= -r\int_t^T xe^{-r(\eta-t)}\mathcal{N}\left(d^-(\frac{x^*(\eta)}{x},\eta-t)\right)d\eta + q\int_t^T e^{-q(\eta-t)}\mathcal{N}\left(d^+(\frac{x^*(\eta)}{x},\eta-t)\right)d\eta.
\end{aligned}$$

From $I_2(t,x) = \left(\frac{1}{x}\right)^{(k_2-1)} I_1(t,\frac{1}{x})$,

$$\begin{aligned}
I_2(t,x) = &-r\int_t^T \left(\frac{1}{x}\right)^{k_2} e^{-r(\eta-t)}\mathcal{N}\left(d^-(x^*(\eta)x,\eta-t)\right)d\eta \\
&+q\int_t^T e^{-q(\eta-t)}\left(\frac{1}{x}\right)^{(k_2-1)}\mathcal{N}\left(d^+(x^*(\eta)x,\eta-t)\right)d\eta.
\end{aligned}$$

Because $f(t,\frac{1}{y}) = (\frac{r}{y} - q)\mathbf{1}_{\{y>\frac{1}{x^*(\eta)}\}}$ and

$$\begin{aligned}
&\int_0^\infty \left[\int_1^u f(\eta,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{u>1\}}\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}du \\
&=\int_0^\infty \left[\int_{\frac{1}{x^*(\eta)}}^u f(\eta,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{u>\frac{1}{x^*(\eta)}\}}\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}du \\
&=\int_{\frac{1}{x^*(\eta)}}^\infty \left[\int_{\frac{1}{x^*(\eta)}}^u \left(\frac{r}{y}-q\right)\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}du \\
&=\int_{\frac{1}{x^*(\eta)}}^\infty \left[-r\frac{u^{-k_2}}{k_2}+q\frac{u^{-(k_2-1)}}{k_2-1}+r\frac{x^*(\eta)^{k_2}}{k_2}-q\frac{x^*(\eta)^{(k_2-1)}}{k_2-1}\right]\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}du.
\end{aligned}$$

Hence,

$$\begin{aligned}
&I_3 = -(k_2-1)\int_t^T\int_0^\infty \left[\int_1^u f(\eta,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{u>1\}}\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta \\
&=\int_t^T\int_{\frac{1}{x^*(\eta)}}^\infty \left[r\frac{k_2-1}{k_2}u^{-k_2}-qu^{-(k_2-1)}-r\frac{k_2-1}{k_2}x^*(\eta)^{k_2}+qx^*(\eta)^{(k_2-1)}\right] \\
&\times\mathcal{B}(\eta-t,\frac{x}{u})\frac{1}{u}dud\eta \\
&=r(1-\frac{1}{k_2})\int_t^T x^{-k_2}e^{-r(\eta-t)}\mathcal{N}\left(d^-(x^*(\eta)x,\eta-t)\right)d\eta \\
&-q\int_t^T x^{-(k_2-1)}e^{-q(\eta-t)}\mathcal{N}\left(d^+(x^*(\eta)x,\eta-t)\right)d\eta \\
&-r(1-\frac{1}{k_2})\int_t^T e^{-q(\eta-t)}x^*(\eta)^{k_2}\mathcal{N}\left(-d^+(\frac{1}{x^*(\eta)x},\eta-t)\right)d\eta \\
&+q\int_t^T e^{-q(\eta-t)}x^*(\eta)^{(k_2-1)}\mathcal{N}\left(-d^+(\frac{1}{x^*(\eta)x},\eta-t)\right)d\eta.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\bar{C}_P(t,x)\\
&= -r\int_t^T xe^{-r(\eta-t)}\mathcal{N}\left(d^-(\frac{x^*(\eta)}{x},\eta-t)\right)d\eta + q\int_t^T e^{-q(\eta-t)}\mathcal{N}\left(d^+(\frac{x^*(\eta)}{x},\eta-t)\right)d\eta\\
&-\frac{r}{k_2}\int_t^T x^{-k_2}e^{-r(\eta-t)}\mathcal{N}\left(d^-(x^*(\eta)x,\eta-t)\right)d\eta\\
&-r(1-\frac{1}{k_2})\int_t^T e^{-q(\eta-t)}x^*(\eta)^{k_2}\mathcal{N}\left(-d^+(\frac{1}{x^*(\eta)x},\eta-t)\right)d\eta\\
&+q\int_t^T e^{-q(\eta-t)}x^*(\eta)^{(k_2-1)}\mathcal{N}\left(-d^+(\frac{1}{x^*(\eta)x},\eta-t)\right)d\eta.
\end{aligned}
\tag{4.3.8}
$$

By (4.3.2), the critical value of $x^*(t)$ satisfies the following integral equation.

$$
\begin{aligned}
&1-x^*(t)\\
&=\bar{C}_E(t,x^*(t))-r\int_t^T x^*(t)e^{-r(\eta-t)}\mathcal{N}\left(d^-(\frac{x^*(\eta)}{x^*(t)},\eta-t)\right)d\eta\\
&+q\int_t^T e^{-q(\eta-t)}\mathcal{N}\left(d^+(\frac{x^*(\eta)}{x^*(t)},\eta-t)\right)d\eta\\
&-\frac{r}{k_2}\int_t^T x^*(t)^{-k_2}e^{-r(\eta-t)}\mathcal{N}\left(d^-(x^*(\eta)x^*(t),\eta-t)\right)d\eta\\
&-r(1-\frac{1}{k_2})\int_t^T e^{-q(\eta-t)}x^*(\eta)^{k_2}\mathcal{N}\left(-d^+(\frac{1}{x^*(\eta)x^*(t)},\eta-t)\right)d\eta\\
&+q\int_t^T e^{-q(\eta-t)}x^*(\eta)^{(k_2-1)}\mathcal{N}\left(-d^+(\frac{1}{x^*(\eta)x^*(t)},\eta-t)\right)d\eta.
\end{aligned}
$$

From $x^*(t)=\frac{m}{s^*(t,m)}$, we obtain the desired result.
□

Remark 4.3.1 In fact, $f(t,x)$ is not differentiable at $x=x^*(t)$. But, by Appendix C.4 in [17], there exist sequences $\{f_n(t,x)\}\in C^\infty\left((0,T)\times(\frac{1}{n},1-\frac{1}{n})\right)$ such that $f_n\to f$ a.e. and $f_n\to f$ in $L^1\left((0,T)\times(0,1)\right)$ as $n\to\infty$. Therefore, by applying Theorem 4.2.2 to a smooth sequence of functions $\{f_n(t,x)\}$ and letting $n\to\infty$, we obtain the same result for $f(t,x)$ as well.

4.4 Perpetual American floating strike lookback option

In this section, we derive the closed-form expressions of a perpetual American lookback call option using a Mellin transform and elementary complex analysis.

Theorem 4.4.1 (Free boundary of perpetual American floating strike lookback call option)
If $T \to \infty$, we denote $s^*_\infty(m)$ as the free boundary of the perpetual American lookback call option. Then, $x^*_\infty = \frac{m}{s^*_\infty}$ satisfies

$$x^*_\infty{}^{(\lambda_2-\lambda_1)} = \frac{\lambda_1}{\lambda_2}\frac{(1+\lambda_2)x^*_\infty - \lambda_2}{(1+\lambda_1)x^*_\infty - \lambda_1},$$

where λ_1, λ_2 are the two roots of the equation $\frac{\sigma^2}{2}\lambda^2 + (\frac{\sigma^2}{2} - (q-r)) - q = 0$.

Proof In PDE (4.3.4), let $\bar{C}_P(t,x) := \bar{C}^1_P(t,x) + \bar{C}^2_P(t,x) + \bar{C}^3_P(t,x)$, where

$$\tilde{\mathcal{L}}\bar{C}^1_P(t,x) = f(t,x)\mathbf{1}_{\{x<1\}}, \quad \bar{C}^1_P(T,x) = 0 \tag{4.4.1}$$

$$\tilde{\mathcal{L}}\bar{C}^2_P(t,x) = f(t,\frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}\mathbf{1}_{\{x>1\}}, \quad \bar{C}^2_P(T,x) = 0, \tag{4.4.2}$$

and

$$\tilde{\mathcal{L}}\bar{C}^3_P(t,x) = (k_2-1)\left[\int_1^x f(t,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}},\ \bar{C}^3_P(T,x) = 0. \tag{4.4.3}$$

From PDE (4.4.1),

$$\frac{d\hat{C}_1}{dt} + \left(\frac{1}{2}\sigma^2 w(w+1) - (q-r)w - q\right)\hat{C}_1 = \hat{f}(t,w),$$

where $\hat{C}_1(t,w)$ is the Mellin transform of $\bar{C}_P^1(t,x)$ and

$$\begin{aligned}\hat{f}(t,w) &= \int_0^\infty f(t,x)x^{w-1}dx \\ &= \int_0^{x^*(t)} (rx-q)x^{w-1}dx \\ &= \frac{rx^*(t)^{w+1}}{w+1} - \frac{qx^*(t)^w}{w}\end{aligned}$$

and by the inverse Mellin transform

$$\begin{aligned}\bar{C}_P^1(t,x) =& \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^T \left[\frac{rx^*(\eta)^{w+1}}{w+1} - \frac{qx^*(\eta)^w}{w}\right] \\ &\times e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}d\eta dw\end{aligned} \tag{4.4.4}$$

and

$$\begin{aligned}\frac{\partial \bar{C}_P^1}{\partial x} =& \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w-1}\int_t^T \left[\frac{rw}{w+1}x^*(\eta)^{w+1} - qx^*(\eta)^w\right] \\ &\times e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}d\eta dw,\end{aligned}$$

where $Re(w) > 0$ and

$$\begin{aligned}A(w) &= w^2 + w(1-k_2) - k_1 \\ &= (w-\lambda_1)(w-\lambda_2), \quad \lambda_1 < \lambda_2\end{aligned}$$

Then, it is easy to verify that if $q > 0$,

$$\lambda_1 < -1, \lambda_2 > 0.$$

Letting $T \to \infty$,

$$\begin{aligned}\frac{\partial \bar{C}_P^1}{\partial x} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w-1}\int_t^\infty \left[\frac{rw}{w+1}{x_\infty^*}^{w+1} - q{x_\infty^*}^{w}\right] e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}d\eta dw \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2}{\sigma^2}\frac{1}{A(w)}\left[q\left(\frac{x_\infty^*}{x}\right)^{(w+1)}\frac{1}{x_\infty^*} - \frac{rw}{w+1}\left(\frac{x_\infty^*}{x}\right)^{(w+1)}\right] dw.\end{aligned} \tag{4.4.5}$$

Note that at any time t, there is infinite time to maturity, and therefore the free boundary of the perpetual American lookback call is constant, i.e., $S_\infty^*(t) = S_\infty^*$ for all t. Hence, $x_\infty^* = \dfrac{m}{s_\infty^*}$ is constant, too.

In addition, it is necessary that $Re(A(w)) < 0$ to ensure that (4.4.5) holds as $T \to \infty$. Hence, $0 < Re(w) < \lambda_2$.

Then,

$$\begin{aligned}\frac{\partial \bar{C}_P^1}{\partial x}(x_\infty^*) =& \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{k_1}{(w-\lambda_1)(w-\lambda_2)}\frac{1}{x_\infty^*} \\ & - (k_1-k_2)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{w}{(w-\lambda_1)(w-\lambda_2)(w+1)} d\eta dw.\end{aligned}$$

Because $0 < Re(w) < \lambda_2$, by application of the residue theorem,

$$\frac{\partial \bar{C}_P^1}{\partial x}(x_\infty^*) = -\frac{k_1}{x_\infty^*}\frac{1}{(\lambda_2-\lambda_1)} + (k_1-k_2)\frac{\lambda_2}{(\lambda_2-\lambda_1)(1+\lambda_2)}. \tag{4.4.6}$$

In case of $\bar{C}_P^2$, the Mellin transform of $f(t, \frac{1}{x})\left(\frac{1}{x}\right)^{(k_2-1)}$ is $\hat{f}(t, k_2-1-w)$ and

$$\hat{f}(t, k_2-1-w) = \frac{rx^*(t)^{(k_2-w)}}{k_2-w} - \frac{qx^*(t)^{(k_2-1-w)}}{k_2-1-w}$$

with $Re(w) < k_2 - 1$.

Similarly, as $T \to \infty$,

$$\begin{aligned}\bar{C}_P^2(t,x) =& -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^T \left[\frac{rx^*(\eta)^{(k_2-w)}}{k_2-w} - \frac{qx^*(\eta)^{(k_2-1-w)}}{k_2-1-w}\right] \\ & \times e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw\end{aligned} \tag{4.4.7}$$

and

$$\begin{aligned}\frac{\partial \bar{C}_P^2}{\partial x} =& \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w-1}\int_t^\infty \left[\frac{rw}{k_2-w}x_\infty^{*\,(k_2-w)} - \frac{qw}{(k_2-1-w)}x_\infty^{*\,(k_2-1-w)}\right] \\ & \times e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw \\ =& \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2}{\sigma^2}\frac{x^{-(w+1)}}{A(w)}\left[q\frac{w}{k_2-w-1}x_\infty^{*\,(k_2-1-w)} - r\frac{w}{k_2-w}x_\infty^{*\,(k_2-w)}\right] dw\end{aligned}$$

with $\lambda_1 < Re(w) < k_2 - 1$.

By application of the residue theorem,

$$\begin{aligned}
\frac{\partial \bar{C}_P^2}{\partial x}(x_\infty^*) &= k_1 \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{w {x_\infty^*}^{(k_2-2-2w)}}{(w-\lambda_1)(w-\lambda_2)(k_2-1-w)} \\
&\quad - (k_1-k_2)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{w {x_\infty^*}^{(k_2-1-2w)}}{(w-\lambda_1)(w-\lambda_2)(k_2-w)} d\eta dw \\
&= k_1 \frac{\lambda_1 {x_\infty^*}^{(k_2-2-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-1-\lambda_1)} - (k_1-k_2)\frac{\lambda_1 {x_\infty^*}^{(k_2-1-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-\lambda_1)}
\end{aligned} \tag{4.4.8}$$

In case of $\bar{C}_P^3$, the Mellin transform of $(k_2-1)\left[\int_1^x f(t,\frac{1}{y})\left(\frac{1}{y}\right)^{(k_2-1)}\frac{1}{y}dy\right]\mathbf{1}_{\{x>1\}}$ is $-\frac{(k_2-1)}{w}\hat{f}(t,k_2-1-w)$ and

$$-\frac{(k_2-1)}{w}\hat{f}(t,k_2-1-w) = r\frac{(1-k_2)x^*(t)^{(k_2-w)}}{w(k_2-w)} - q\frac{(1-k_2)x^*(t)^{(k_2-1-w)}}{w(k_2-1-w)}$$

with $Re(w) < 0$.

Similarly,

$$\begin{aligned}
\bar{C}_P^3(t,x) = &- \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^T \left[\frac{r(1-k_2)x^*(\eta)^{(k_2-w)}}{w(k_2-w)} - \frac{q(1-k_2)x^*(\eta)^{(k_2-1-w)}}{w(k_2-1-w)}\right] \\
&\times e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw
\end{aligned} \tag{4.4.9}$$

and as $T \to \infty$,

$$\begin{aligned}
\frac{\partial \bar{C}_P^3}{\partial x} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w-1}\int_t^\infty \left[r\frac{(1-k_2)}{k_2-w}{x_\infty^*}^{(k_2-w)} - q\frac{(1-k_2)}{(k_2-1-w)}{x_\infty^*}^{(k_2-1-w)}\right] \\
&\quad \times e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2}{\sigma^2}\frac{1}{A(w)} x^{-(w+1)}\left[q\frac{(1-k_2)}{(k_2-1-w)}{x_\infty^*}^{(k_2-1-w)} - \frac{(1-k_2)}{k_2-w}{x_\infty^*}^{(k_2-w)}\right] dw
\end{aligned}$$

with $\lambda_1 < Re(w) < \min\{0, k_2-1\}$.

By application of the residue theorem,

$$\begin{aligned}
&\frac{\partial \bar{C}_P^3}{\partial x}(x_\infty^*) \\
&= \frac{k_1(1-k_2)}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{{x_\infty^*}^{(k_2-2-2w)}}{(w-\lambda_1)(w-\lambda_2)(k_2-1-w)} \\
&- \frac{(k_1-k_2)(1-k_2)}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{{x_\infty^*}^{(k_2-1-2w)}}{(w-\lambda_1)(w-\lambda_2)(k_2-w)}dw \\
&= k_1(1-k_2)\frac{{x_\infty^*}^{(k_2-2-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-1-\lambda_1)} - (1-k_2)(k_1-k_2)\frac{{x_\infty^*}^{(k_2-1-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-\lambda_1)}.
\end{aligned} \tag{4.4.10}$$

By (4.3.5) and $x = \frac{m}{s}$,

$$\begin{aligned}
&\bar{C}_E(t,x) \\
=&e^{-q(T-t)}\mathcal{N}(d_1^+(\frac{1}{x},T-t)) - xe^{-r(T-t)}\mathcal{N}(d_1^-(\frac{1}{x},T-t)) \\
+&\frac{\sigma^2}{2(r-q)}\left\{e^{-r(T-t)}x^{\frac{2(r-q)}{\sigma^2}}\mathcal{N}(d_1^-(x,T-t)) - e^{-q(T-t)}\mathcal{N}(-d_1^+(\frac{1}{x},T-t))\right\}
\end{aligned} \tag{4.4.11}$$

By performing a computation, it is not hard to show that

$$\frac{\partial \bar{C}_E}{\partial x} \to 0 \quad as \quad T \to \infty. \tag{4.4.12}$$

By applying "smooth pasting conditions",

$$\lim_{x \to x^*(t)} \frac{\partial \bar{C}}{\partial x} = -1 \tag{4.4.13}$$

and (4.4.6),(4.4.8),(4.4.10),and (4.4.12),

$$\begin{aligned}
-1 = &-\frac{k_1}{x_\infty^*}\frac{1}{(\lambda_2-\lambda_1)} + (k_1-k_2)\frac{\lambda_2}{(\lambda_2-\lambda_1)(1+\lambda_2)} + k_1\frac{\lambda_1 {x_\infty^*}^{(k_2-2-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-1-\lambda_1)} \\
&- (k_1-k_2)\frac{\lambda_1 {x_\infty^*}^{(k_2-1-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-\lambda_1)} + k_1(1-k_2)\frac{{x_\infty^*}^{(k_2-2-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-1-\lambda_1)} \\
&- (1-k_2)(k_1-k_2)\frac{{x_\infty^*}^{(k_2-1-2\lambda_1)}}{(\lambda_1-\lambda_2)(k_2-\lambda_1)}.
\end{aligned}$$

Using $\lambda_1 + \lambda_2 = k_2 - 1$ and $\lambda_1\lambda_2 = -k_1$, we obtain

$$x_\infty^{*\,(\lambda_2-\lambda_1)} = \frac{\lambda_1}{\lambda_2}\frac{(1+\lambda_2)x_\infty^* - \lambda_2}{(1+\lambda_1)x_\infty^* - \lambda_1}, \tag{4.4.14}$$

which is the critical value of an American floating strike lookback call option derived by Dai [9].
□

We can also prove the following theorem using Theorem 4.4.1

Theorem 4.4.2 (Price of perpetual American floating strike call option)
The closed-form price formula of the perpetual American lookback call option is given by

$$C(s,m) = s\Lambda_1\left(\frac{s}{m}\right)^{\lambda_2} + s\Lambda_2\left(\frac{s}{m}\right)^{\lambda_1}, \quad s > s_\infty^*$$

where

$$\Lambda_1 = \frac{(1+\lambda_1)x_\infty^* - \lambda_1}{(\lambda_2-\lambda_1)x_\infty^{*\,-\lambda_2}}, \quad \Lambda_2 = \frac{(1+\lambda_2)x_\infty^* - \lambda_2}{(\lambda_1-\lambda_2)x_\infty^{*\,-\lambda_1}}$$

and λ_1, λ_2, and x_∞^* are defined in Theorem 5.4.1.

Proof By (4.4.4),

$$\bar{C}_P^1(t,x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^T\left[\frac{rx^*(\eta)^{w+1}}{w+1} - \frac{qx^*(\eta)^w}{w}\right]e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}d\eta dw$$

with $0 < Re(w) < \lambda_2$.

Letting $T \to \infty$, then

$$\begin{aligned}\bar{C}_P^1(x) &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^\infty\left[\frac{rx_\infty^{*\,w+1}}{w+1} - \frac{qx_\infty^{*\,w}}{w}\right]e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)}d\eta dw\\ &= \frac{1}{2\pi i}\frac{2r}{\sigma^2}\int_{c-i\infty}^{c+i\infty}\frac{x_\infty^*}{(w+1)(w-\lambda_1)(w-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^w dw\\ &\quad - \frac{1}{2\pi i}\frac{2q}{\sigma^2}\int_{c-i\infty}^{c+i\infty}\frac{1}{w(w-\lambda_1)(w-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^w dw.\end{aligned}$$

By application of the residue theorem,

$$\bar{C}_P^1(x) = \frac{(1+\lambda_1)x_\infty^* - \lambda_1}{(\lambda_2 - \lambda_1)} \left(\frac{x_\infty^*}{x}\right)^{\lambda_2}. \tag{4.4.15}$$

By (4.4.7),

$$\begin{aligned}\bar{C}_P^2(t,x) = &-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^T \left[\frac{rx^*(\eta)^{(k_2-w)}}{k_2-w} - \frac{qx^*(\eta)^{(k_2-1-w)}}{k_2-1-w}\right] \\ &\times e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw\end{aligned}$$

with $\lambda - 1 < Re(w) < k_2 - 1$.

Letting $T \to \infty$, then

$$\begin{aligned}\bar{C}_P^2(x) &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^\infty \left[\frac{rx_\infty^{*\,(k_2-w)}}{k_2-w} - \frac{qx_\infty^{*\,(k_2-1-w)}}{k_2-1-w}\right] e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw \\ &= \frac{1}{2\pi i}\frac{2r}{\sigma^2}\int_{c-i\infty}^{c+i\infty} \frac{x_\infty^{*\,(k_2-2w)}}{(k_2-w)(w-\lambda_1)(w-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^w dw \\ &\quad - \frac{1}{2\pi i}\frac{2q}{\sigma^2}\int_{c-i\infty}^{c+i\infty} \frac{x_\infty^{*\,(k_2-1-2w)}}{(k_2-1-w)(w-\lambda_1)(w-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^w dw.\end{aligned}$$

By application of the residue theorem,

$$\bar{C}_P^2(x) = \frac{-(1+\lambda_1)x_\infty^* + \lambda_1}{(\lambda_1 - \lambda_2)} \left(\frac{x_\infty^*}{x}\right)^{\lambda_1} x_\infty^{*\,(\lambda_2-\lambda_1)}. \tag{4.4.16}$$

By (4.4.9),

$$\begin{aligned}\bar{C}_P^3(t,x) = &-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^T \left[\frac{r(1-k_2)x^*(\eta)^{(k_2-w)}}{w(k_2-w)} - \frac{q(1-k_2)x^*(\eta)^{(k_2-1-w)}}{w(k_2-1-w)}\right] \\ &\times e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw\end{aligned}$$

with $Re(w) < \min\{0, k_2 - 1\}$.

Letting $T \to \infty$, then

$$\begin{aligned}
&\bar{C}_P^3(x) \\
&= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-w}\int_t^{\infty}\left[\frac{r(1-k_2){x_\infty^*}^{(k_2-w)}}{w(k_2-w)} - \frac{q(1-k_2){x_\infty^*}^{(k_2-1-w)}}{w(k_2-1-w)}\right] \\
&\times e^{\frac{1}{2}\sigma^2 A(w)(\eta-t)} d\eta dw \\
&= \frac{1}{2\pi i}\frac{2r}{\sigma^2}\int_{c-i\infty}^{c+i\infty}\frac{(1-k_2){x_\infty^*}^{(k_2-2w)}}{w(k_2-w)(w-\lambda_1)(w-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^w dw \\
&- \frac{1}{2\pi i}\frac{2q}{\sigma^2}\int_{c-i\infty}^{c+i\infty}\frac{(1-k_2){x_\infty^*}^{(k_2-1-2w)}}{w(k_2-1-w)(w-\lambda_1)(w-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^w dw.
\end{aligned}$$

By application of the residue theorem,

$$\bar{C}_P^3(x) = \frac{(\lambda_1+\lambda_2)\{(1+\lambda_1)x_\infty^* - \lambda_1\}}{\lambda_1(\lambda_1-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^{\lambda_1}{x_\infty^*}^{(\lambda_2-\lambda_1)}. \tag{4.4.17}$$

By Theorem 4.4.1 and (4.4.15),(4.4.16),and (4.4.17)

$$\bar{C}_P(x) = \frac{(1+\lambda_1)x_\infty^* - \lambda_1}{(\lambda_2-\lambda_1)}\left(\frac{x_\infty^*}{x}\right)^{\lambda_2} + \frac{(1+\lambda_2)x_\infty^* - \lambda_2}{(\lambda_1-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^{\lambda_1}.$$

From (4.3.5),

$$\bar{C}_E(t,x) \to 0 \quad as \quad T \to \infty; \tag{4.4.18}$$

therefore,

$$\bar{C}(x) = \bar{C}_P(x) = \frac{(1+\lambda_1)x_\infty^* - \lambda_1}{(\lambda_2-\lambda_1)}\left(\frac{x_\infty^*}{x}\right)^{\lambda_2} + \frac{(1+\lambda_2)x_\infty^* - \lambda_2}{(\lambda_1-\lambda_2)}\left(\frac{x_\infty^*}{x}\right)^{\lambda_1}. \tag{4.4.19}$$

Because $x = \frac{m}{s}$, we obtained the desired result.

4.5 Summary

This paper describes our examination of the valuation of American floating strike lookback options written on dividend-paying assets. Usually, the valuation of American options is equivalent to solving a free boundary problem of the inhomogeneous Black-Scholes PDE. However, in the case of American floating strike lookback options, the Neumann boundary condition should be considered due to the existence of the lookback state variable. Therefore, our examination started with an analysis of the inhomogeneous Black-Scholes PDE with a Neumann boundary condition. The analysis consisted of two steps.

We first converted the PDE with a Neumann boundary condition into the same PDE with a Dirichlet boundary condition by changing the variables such that they were particularly apt to accommodate the properties of the Mellin transform.

The second step consisted of extending the domain of the given PDE. Whereas the Mellin transform is defined on the domain, the given PDE involving the value function is defined only on the domain, and we used a scaling and reflection method to introduce the absorbing Neumann boundary operator. We then used this operator to extend the PDE onto the domain. Using these two steps, we derived the integral equation satisfied by American floating strike lookback options, after which we also derived the closed form of the value of perpetual American options.

In conclusion, we proposed an approach based on the Mellin transform for solving inhomogeneous Black-Scholes PDEs with a Neumann boundary condition. As a result of the general applicability of our methodology, we expect this approach to be useful for solving a variety of option problems.

Chapter 5

American maximum exchange rate quanto lookback option

Quanto options stand for "quantity-adjusting options", in which the options' payoff is both affected by domestic and foreign currency. There are various kinds of quanto options depending on the purpose and they are one of the most popular options. Kwok and Wong [50] studied the payoff for various kinds of quanto options. Especially we note Dai et al. [11] research on American maximum exchange quanto lookback options which take account of maximum exchange rate.

In this chapter, we draw analytic pricing formula for exchange rate quanto lookback options where exchange rate is applied to the maximum value until maturity. In particular, we studied American maximum exchange rate quanto lookback options of which option holders are free to exercise options prior to their expiration date. Such options are not easy to deal with, since they are multi asset options and they have characteristics of American options as well as lookback options. Although Dai et al.[11] proved the various properties for the free boundary of American exchange rate quanto lookback options, they didn't obtain analytic pricing for such options. To the best of our knowledge, there has been no research which draw an analytic formula American exchange rate quanto lookback options.

A lookback option is a path-dependent option, which means that payoff of the lookback option depends on the maximum or minimum of an underlying asset. Lookback options are traded on the market with a variety of form, such as dynamic fund protection(Gerber and Shiu [19]), turbo warrant options(Wong and Chan [67]). Therefore, there have been lots of researches for pricing lookback options. Analytic formula for the European lookback option is derived by Goldman et al.[23], Conze and Viswanathan [7] and pricing formula for European quanto lookback options with two underlying assets was derived by M.Dai and Y.Kwok [11].

An American option can be mathematically seen as an optimal stopping problem in that option holders can exercise options at any time up to the option's expiration. Kim [43] derived the integral equation satisfied by the American option at the very first. Since then, researchers have started to research on various American options. Especially, Russian options have been intensively studied since they are typical example of American lookback style options. Shepp and Shiryaev [63] investigated pricing formula for perpetual Russian options, and after that Ekstr$\ddot{o}$m [15] and Peskir [58] extended formula to Russian options with finite time horizon. Dai and Kwok [13] analyzed the optimal stopping boundary for American floating strike lookback options, and Lai and Lim [52] drew the integral equations satisfied by American fixed strike lookback options.

There are lots of options involving two underlying assets. For example, vulnerable options(Johnson and Stulz [36]) consider the default risk of option holders and writers. Also, maximum options (Johnson [37]) take payoff function equal to the maximum payoff of two underlying assets, and spread options (Carmona and Durrleman [5]) have the payoff given by a function of the difference of two underlying assets. Different from a single asset, in multi asset pricing, additional difficulty and complexity of analysis arises due to the correlation between underlying assets. Broadie and Detemple [3] explored the property of free boundary in American options with multi underlying assets.

In this chapter, the main tool for deriving formula is the double Mellin transform technique. The Mellin transform is a type of integral transformation and can be seen as a two-sided Laplace transform. In particular, it converts a Black-Scholes PDE into a simple ODE. Therefore it is suitable for option pricing and has been widely used. To list some examples, Panini and Srivastav [56] derived the integral equation satisfied by the value of American options. Jeon et al. [32] handled valuing Russian options with finite maturity using the Mellin transform. Also, there has been researches regarding the double Mellin transform, which is a two-dimensional version of the usual Mellin transform. Yoon and Kim [69] priced vulnerable options using the double Mellin transform, and Jeon et al. [33] derived the analytic solution for the general two-dimensional Black-Scholes equation with time-dependent coefficients using the double Mellin transform.

Many researches have focused on numerically solving the integral equation for American options. For single asset problems, Huang et al. [27] proposed the recursive integration method to numerically solve the integral equation derived by Kim [43]. Ju [38] used multipiece exponential method, which involves the approximation of free boundary by exponential functions. Also, Kim et al.[44] suggested an iterative algorithm called 'a simple iterative method' to solve the integral equation, and the algorithm is very effective in the view of computational efficiency, accuracy, and convergence speed.

For multi-asset problems, however, the free boundary depends on not only time but also other state variables. Therefore, solving the integral equation involving such free boundary becomes much more complicated, and actually there are few known algorithms for solving such integral equations. Chiarella [6] suggested such an algorithm for American spread call options written on two underlying assets.

The Black-Scholes equation with mixed boundary conditions frequently arises in option pricing problems involving the maximum or minimum process for underlying asset. For a single asset problem, Jeon et al. [32] drew out a general solution for one-dimensional Black-Scholes PDE. This chapter extends the result of Jeon et al. [32] to two-dimensional option pricing problems.

5.1 Model Formulation : Free boundary problem

In this chapter, the usual assumptions of the Black-Scholes environment are adopted. Let F_t denote the exchange rate at time t, which means that F_t represents the domestic price at time t of one unit of foreign currency. Let S_t be the foreign currency price at time t. Let r_d and r_f be the constant domestic and foreign riskless interest rates, respectively. Then, the stochastic dynamics of S_t and F_t are described by

$$\begin{aligned} dS_t &= \delta_s^d S_t dt + \sigma_s S_t dW_s^d \\ dF_t &= (r_d - r_f) F_t dt + \sigma_f F_t dW_f^d \end{aligned} \tag{5.1.1}$$

where σ_s, σ_f are the volatility of S_t and F_t, respectively. W_s^d, W_f^d are standard Brownian motion under the domestic risk neutral measure Q^d and $dW_s^d dW_f^d = \rho dt$. By using **Quanto Prewashing Technique** introduced by Dravid et al.[14] ,

$$\delta_s^d = r_f - q - \rho\sigma_f\sigma_s$$

where q is the dividend yield of the foreign asset S_t in foreign world.

For the exchange rate process, define the maximum process of F_t as

$$M_t = \max_{0 \leq \gamma \leq t} F_\gamma, \quad t \geq 0$$

Consider the American maximum exchange-rate quanto lookback call option, whose terminal payoff function in the domestic currency world is given by

$$M_T \cdot (S_T - K)^+$$

where K is the strike price in foreign currency and T is the expiration date of call options and $(x)^+ := \max(x, 0)$.

In the absence of arbitrage opportunities, the value $C(t, s, f, m)$ of an optimal stopping problem given by

$$C(t, s, f, m) = \sup_{t \leq \tau \leq T} \mathbb{E}\left[e^{-r_d(\tau - t)} M_\tau (S_\tau - K)^+ \mid S_t = s, F_t = f, M_t = m\right]$$

where τ is the stopping time of the filtration $\mathbb{F}$ generated by (W_s^d, W_f^d) and $\mathbb{E}$ is conditional expectation calculated under the domestic risk neutral measure Q^d.

By standard technique of optimal stopping problem (also known as the variational inequalities),

$$\begin{aligned}
&\min\{\mathcal{L}\, C,\ C - m \cdot (s - K)^+\} = 0 \\
&\frac{\partial C}{\partial m}(t, s, m, m) = 0, \ \ C(T, s, m, f) = m \cdot (s - K)^+ \\
&s > 0,\ 0 < f < m,\ 0 \leq t \leq T,
\end{aligned} \tag{5.1.2}$$

The partial differential equation operator $\mathcal{L}$ is defined by

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{\sigma_s^2}{2} s^2 \frac{\partial^2}{\partial s^2} + \frac{\sigma_f^2}{2} f^2 \frac{\partial^2}{\partial f^2} + \rho \sigma_s \sigma_f s f \frac{\partial^2}{\partial s \partial f} + \delta_s^d s \frac{\partial}{\partial s} + (r_d - r_f) f \frac{\partial}{\partial f} - r_d \mathcal{I}$$

where $\mathcal{I}$ is identity operator.

Let

$$\mathcal{R} = \{(t, s, f, m) \in [0, T] \times \mathbb{R}_+ \times [0, m] \times \mathbb{R}_+\}$$

be the whole region. and let $\mathcal{S}$ and $\mathcal{S}^C$ denote the stopping region and continuation region, respectively. Then, in term of the value function $C(t, s, f, m)$, the stopping region $\mathcal{S}$ is defined by

$$\mathcal{S} = \{(t, s, f, m) \mid C(t, s, f, m) = m(s - K)^+\}$$

The continuation region $\mathcal{S}^C$ is the complement of $\mathcal{S}$ in $\mathcal{R}$,

$$\mathcal{S}^C = \{(t, s, f, m) \mid C(t, s, f, m) > m(s - K)^+\}$$

The boundary that separates $\mathcal{S}$ form $\mathcal{S}^C$ is referred to as the *free boundary*, is given by

$$s^*(t, f, m) = \sup\{s \in \mathbb{R}_+ \mid (t, s, f, m) \in \mathcal{S}^C\}, \quad t \in [0, T]$$

According to Dai et al. [11], similar to American type call options, the stopping region and the exercise region of C correspond to $s > s^*(t, f, m)$ and $s \leq s^*(t, f, m)$, respectively. In terms of free boundary $s^*(t, f, m)$, the continuation region $\mathcal{S}^C$ can be expressed by

$$\mathcal{S}^C = \{(t, s, f, m) \mid 0 < s < s^*(t, f, m)\}$$

Also, at the free boundary $s = s^*(t, f, m)$,

$$\begin{aligned} &C(t, s^*(t, f, m), f, m) = m(s^*(t, f, m) - K) \\ &\frac{\partial C}{\partial s}(t, s^*(t, f, m), f, m) = m \end{aligned} \tag{5.1.3}$$

The boundary condition (5.1.3) is called the **smooth pasting** condition.

We introduce a new change of variables

$$z = \frac{f}{m}$$

and transformed value function

$$V(t, s, z) = \frac{C(t, s, f, m)}{m}$$

Then we can rewrite the variational inequalities (5.1.2) as

$$\begin{aligned}
&\min\{\mathcal{L}\,V,\ V-(s-K)^+\}=0\\
&\frac{\partial V}{\partial z}(t,s,1)=V(t,s,1),\ \ V(T,s,z)=(s-K)^+\\
&s>0,\ 0<z<1,\ 0\le t\le T,
\end{aligned}$$

Further, for $0<\epsilon^*(t,z):=s^*(t,f,m)$

$$\begin{aligned}
\mathcal{L}V(t,s,z)=\frac{\partial V}{\partial t}+\frac{\sigma_s^2}{2}s^2\frac{\partial^2 V}{\partial s^2}+\frac{\sigma_f^2}{2}f^2\frac{\partial^2 V}{\partial f^2}+\rho\sigma_s\sigma_f sf\frac{\partial^2 V}{\partial s\partial f}+\delta_s^d s\frac{\partial V}{\partial s}\\
+(r_d-r_f)f\frac{\partial V}{\partial f}-r_dV=0
\end{aligned} \tag{5.1.4}$$

with auxiliary conditions:

$$\begin{aligned}
&V(T,s,z)=\alpha(s):=(s-K)^+\\
&V(t,\epsilon^*(t,z),z)=\epsilon^*(t,z)-K\\
&\frac{\partial V}{\partial s}(t,\epsilon^*(t,z),z)=1\\
&\frac{\partial V}{\partial z}(t,s,1)=V(t,s,1)
\end{aligned} \tag{5.1.5}$$

In terms of the value function $V(t,s,z)$, the stopping region $\bar{\mathcal{S}}$ is given by

$$\begin{aligned}
\bar{\mathcal{S}}:&=\{(t,s,z)\mid 0\le t\le T,\epsilon^*(t,z)<s\}\\
&=\{(t,s,z)\mid V(t,s,z)=s-K,\ 0\le t\le T,\ 0<s<\infty,\ 0<z<1\}
\end{aligned}$$

and the continuation region $\bar{\mathcal{S}}^C$ is given by

$$\begin{aligned}
\bar{\mathcal{S}}^C:&=\{(t,s,z)\mid 0\le t\le T,\epsilon^*(t,z)\ge s\}\\
&=\{(t,s,z)\mid V(t,s,z)>(s-K)^+,\ 0\le t\le T,\ 0<s<\infty,\ 0<z<1\}
\end{aligned}$$

Overall, the value function $V(t,s,z)$ satisfies the following PDE with boundary conditions (5.1.5) :

$$\begin{aligned}
\frac{\partial V}{\partial t}+\frac{\sigma_s^2}{2}s^2\frac{\partial^2 V}{\partial s^2}+\frac{\sigma_f^2}{2}z^2\frac{\partial^2 V}{\partial z^2}+\rho\sigma_s\sigma_f sz\frac{\partial^2 V}{\partial s\partial z}+\delta_s^d s\frac{\partial V}{\partial s}+(r_d-r_f)z\frac{\partial V}{\partial z}-r_dV\\
=\beta(t,s,z)
\end{aligned} \tag{5.1.6}$$

where

$$\beta(t,s,z) = (-(r_d - \delta_s^d)\cdot s + r_d \cdot K)\mathbf{1}_{\{s \geq \epsilon^*(t,z)\}}$$

and on domain $\{(t,s,z) \mid 0 \leq t \leq T,\ 0 < s < \infty,\ 0 < z < 1\}$

Therefore, the value of American maximum exchange rate quanto options satisfy two-dimensional inhomogeneous Black-Scholes equation (5.1.6) with mixed boundary conditions (5.1.5).

5.2 Derivation of analytic solution for two-dimensional inhomogeneous Black-Scholes PDE

In section 5.1, we saw that the price of American maximum exchange rate qunato lookback options are formulated into two-dimensional inhomogeneous Black-Scholes PDE with mixed boundary conditions on restricted domain. In this section, we derive the analytic solution for two-dimensional inhomogeneous Black-Scholes PDE on restricted domain. During the derivation, we mainly use the double Mellin transform, whose basic definition and properties are summarized in the appendix A.2.

5.2.1 Two-dimensional Inhomogeneous Black-Scholes parabolic PDE on Unrestricted Domain

In this subsection, we apply the Mellin transform approach to derive the analytic solution of two-dimensional inhomogeneous Black-Scholes PDE on unrestricted domain. Using the double Mellin transform, we convert the given PDE into a relatively simple ODE whose solution can be explicitly represented. Having found the transformed solution, we invert it to get the original solution by using inverse double Mellin transform and convolution property of the Mellin transform.

Define the two-dimensional Black-Schoels PDE operator $\bar{\mathcal{L}}$ as

$$\bar{\mathcal{L}} = \frac{\partial}{\partial t} + \frac{\sigma_x^2}{2}x^2\frac{\partial^2}{\partial x^2} + \frac{\sigma_y^2}{2}y^2\frac{\partial^2}{\partial y^2} + \rho\sigma_x\sigma_y xy\frac{\partial^2}{\partial x\partial y} + r_x x\frac{\partial}{\partial x} + r_y y\frac{\partial}{\partial y} - r\mathcal{I}$$

Consider the following two-dimensional inhomogeneous PDE problem :

$$\begin{aligned} &\bar{\mathcal{L}}V(t,x,y) = f(t,x,y) \\ &V(T,x,y) = h(x,y) \end{aligned} \tag{5.2.1}$$

with domain $\{(t,x,y) \mid 0 \leq t \leq T,\ 0 < x < \infty,\ 0 < y < \infty\}$.

Let $\hat{V}(t,x^*,y^*)$ is the double Mellin transform of $V(t,x,y)$. Then, by PDE (5.2.1),

$$\begin{aligned} &\frac{d\hat{V}}{dt} + \mathcal{A}(x^*,y^*)\hat{V} = \hat{f}(t,x^*,y^*) \\ &\mathcal{A}(x^*,y^*) = \frac{\sigma_x^2}{2}x^{*2} + \frac{\sigma_y^2}{2}y^{*2} + \rho\sigma_x\sigma_y x^*y^* - (r_x - \frac{\sigma_x^2}{2})x^* - (r_y - \frac{\sigma_y^2}{2})y^* - r \end{aligned} \tag{5.2.2}$$

The solution of (5.2.2) is

$$\hat{V}(t,x^*,y^*) = \hat{h}(x^*,y^*)e^{\mathcal{A}(x^*,y^*)(T-t)} - \int_t^T e^{\mathcal{A}(x^*,y^*)(\eta-t)}\hat{f}(\eta,x^*,y^*)d\eta$$

By inverse double Mellin transform,

$$\begin{aligned} V(t,x,y) = &\frac{1}{(2\pi i)^2}\int_{c_2-i\infty}^{c_2+i\infty}\int_{c_1-i\infty}^{c_1+i\infty}\hat{h}(x^*,y^*)e^{\mathcal{A}(x^*,y^*)(T-t)}x^{-x^*}y^{-y^*}dx^*dy^* \\ &- \int_t^T\left(\int_{c_2-i\infty}^{c_2+i\infty}\int_{c_1-i\infty}^{c_1+i\infty}\hat{f}(\eta,x^*,y^*)e^{\mathcal{A}(x^*,y^*)(\eta-t)}x^{-x^*}y^{-y^*}dx^*dy^*\right)d\eta \end{aligned} \tag{5.2.3}$$

To compute (5.2.3), let us consider

$$\mathcal{G}_{\bar{\mathcal{L}}}(t,x,y) = \frac{1}{(2\pi i)^2}\int_{c_2-i\infty}^{c_2+i\infty}\int_{c_1-i\infty}^{c_1+i\infty} e^{\mathcal{A}(x^*,y^*)(T-t)}x^{-x^*}y^{-y^*}dx^*dy^* \tag{5.2.4}$$

In Yoon and Kim [69],

$$
\begin{aligned}
&\mathcal{G}_{\bar{\mathcal{L}}}(\tau,x,y)\\
&=\exp\left\{-\frac{1}{2}\frac{\left(\frac{\rho\sigma_x\sigma_y\tau}{2}(k_x-1)-\left(r_y-\frac{\sigma_y^2}{2}\right)\tau+\frac{\rho\sigma_y}{\sigma_x}\ln x\right)^2}{(1-\rho^2)\sigma_y^2\tau}-\left(\frac{\sigma_x^2(k_x-1)^2}{8}+r\right)\tau\right\}\cdot\\
&\cdot x^{\frac{1-k_x}{2}}y^{\frac{\frac{\rho\sigma_x\sigma_y\tau}{2}(k_x-1)-\left(r_y-\frac{\sigma_y^2}{2}\right)\tau+\frac{\rho\sigma_y}{\sigma_x}\ln x}{(1-\rho^2)\sigma_y^2\tau}}\frac{e^{-\frac{1}{2}\left(\frac{\ln x}{\sigma_x\sqrt{\tau}}\right)^2}}{\sigma_x\sqrt{2\pi\tau}}\frac{e^{-\frac{1}{2}\left(\frac{\ln y}{\sigma_y\sqrt{(1-\rho^2)\tau}}\right)^2}}{\sigma_y\sqrt{2\pi(1-\rho^2)\tau}}.
\end{aligned}
$$

where $k_x=2r_x/\sigma_x^2,\ k_y=2r_y/\sigma_y^2.$

Let us call $\mathcal{G}_{\bar{\mathcal{L}}}(t,x,y)$ **Green function** of Blak-Scholes PDE operator $\bar{\mathcal{L}}$. By the double Mellin convolution property in Appendix A,

$$
\begin{aligned}
V(t,x,y)&=\int_0^\infty\int_0^\infty h(u,w)\,\mathcal{G}_{\bar{\mathcal{L}}}(T-t,\frac{x}{u},\frac{y}{w})\frac{1}{u}\frac{1}{w}dudw\\
&\quad-\int_t^T\int_0^\infty\int_0^\infty f(\eta,u,w)\,\mathcal{G}_{\bar{\mathcal{L}}}(\eta-t,\frac{x}{u},\frac{y}{w})\frac{1}{u}\frac{1}{w}dudwd\eta
\end{aligned}
$$

5.2.2 Two-dimensional inhomogeneous Black-Scholes parabolic PDE : Dirichlet Boundary Conditions

In this subsection, we derive the analytic solution of two-dimensional inhomogeneous Black-Scholes PDE with Dirichlet boundary conditions. The spatial domain of PDE is restricted to the strip, which means one of the spatial variable is restricted to a finite interval. We use the method of reflection to extend the spatial domian of PDE into unrestricted domain, and then we apply the results of section 5.2.1. Solving the Black-Scholes PDE with Dirichlet boundary conditions is closely related to the valuation of pricing barrier options, and it also enables to derive a closed form solution of European external barrier options(Kwok et al. [49]).

For two-dimensional PDE operator $\bar{\mathcal{L}}$ defined in Section 5.2.1, consider the following PDE problem :

$$\begin{aligned} \bar{\mathcal{L}}V(t,x,y) &= f(t,x,y) \\ V(T,x,y) &= h(x,y) \\ V(T,x,1) &= 0 \end{aligned} \tag{5.2.5}$$

on domain $\{(t,x,y) \mid 0 \le t \le T,\ 0 < x < \infty,\ 0 < y < 1\}$.

To solve (5.2.5), we consider unrestricted domain PDE :

$$\begin{aligned} \bar{\mathcal{L}}\bar{V}(t,x,y) &= f(t,x,y)\mathbf{1}_{\{y<1\}} \\ \bar{V}(T,x,y) &= h(x,y)\mathbf{1}_{\{y<1\}} \end{aligned} \tag{5.2.6}$$

By Section 5.2.1,

$$\begin{aligned} \bar{V}(t,x,y) = &\int_0^\infty \int_0^\infty h(u,w)\mathbf{1}_{\{w<1\}}\, \mathcal{G}_{\bar{\mathcal{L}}}(T-t, \frac{x}{u}, \frac{y}{w})\frac{1}{u}\frac{1}{w}dudw \\ &- \int_t^T \int_0^\infty \int_0^\infty f(\eta,u,w)\mathbf{1}_{\{w<1\}}\, \mathcal{G}_{\bar{\mathcal{L}}}(\eta-t, \frac{x}{u}, \frac{y}{w})\frac{1}{u}\frac{1}{w}dudwd\eta \end{aligned}$$

Lemma 5.2.1 For green function $\mathcal{G}_{\bar{\mathcal{L}}}(t,x,y)$ defined in Section 5.2.1,

$$\mathcal{G}_{\bar{\mathcal{L}}}(t,x,y) = y^{1-k_y}\mathcal{G}_{\bar{\mathcal{L}}}(t, y^{-\frac{2\rho\sigma_x}{\sigma_y}}x, \frac{1}{y})$$

Proof. The proof is similar to the proof of Lemma 2.1.1. □

Theorem 5.2.2 (Two-dimensional inhomogeneous Black-Scholes PDE with **Dirichlet boundary conditions**)
The solution $V(t,x,y)$ of PDE (5.2.5) satisfies the following extended PDE:

$$\begin{aligned} \bar{\mathcal{L}}V(t,x,y) =& f(t,x,y)\mathbf{1}_{\{y<1\}} - y^{1-k_y}f(t, y^{-\frac{2\rho\sigma_x}{\sigma_y}}x, \frac{1}{y})\mathbf{1}_{\{y>1\}} \\ V(T,x,y) =& h(x,y)\mathbf{1}_{\{y<1\}} - y^{1-k_y}h(y^{-\frac{2\rho\sigma_x}{\sigma_y}}x, \frac{1}{y})\mathbf{1}_{\{y>1\}} \end{aligned}$$

Proof. By Lemma 5.2.1, $\bar{V}(t,x,y)$ of (5.2.6) is given by

$$
\begin{aligned}
&\bar{V}(t,x,y) \\
&= \int_0^\infty \int_0^\infty h(u,w)\mathbf{1}_{\{w<1\}} \left(\frac{y}{w}\right)^{1-k_y} \mathcal{G}_{\bar{\mathcal{L}}}(T-t, \left(\frac{y}{w}\right)^{-\frac{2\rho\sigma_x}{\sigma_y}} \frac{x}{u}, \frac{w}{y})\frac{1}{u}\frac{1}{w}dudw \\
&- \int_t^T \int_0^\infty \int_0^\infty f(\eta,u,w)\mathbf{1}_{\{w<1\}} \left(\frac{y}{w}\right)^{1-k_y} \mathcal{G}_{\bar{\mathcal{L}}}(T-t, \left(\frac{y}{w}\right)^{-\frac{2\rho\sigma_x}{\sigma_y}} \frac{x}{u}, \frac{w}{y})\frac{1}{u}\frac{1}{w}dudwd\eta
\end{aligned}
$$

and it leads to

$$
\begin{aligned}
&y^{1-k_y}\bar{V}(t, y^{-\frac{2\rho\sigma_x}{\sigma_y}} x, \frac{1}{y}) \\
&= \int_0^\infty \int_0^\infty w^{1-k_y} h(w^{-\frac{2\rho\sigma_x}{\sigma_y}} u, \frac{1}{w})\mathbf{1}_{\{w>1\}}\, \mathcal{G}_{\bar{\mathcal{L}}}(T-t, \frac{x}{u}, \frac{y}{w})\frac{1}{u}\frac{1}{w}dudw \\
&- \int_t^T \int_0^\infty \int_0^\infty w^{1-k_y} f(\eta, w^{-\frac{2\rho\sigma_x}{\sigma_y}} u, \frac{1}{w})\mathbf{1}_{\{w>1\}}\, \mathcal{G}_{\bar{\mathcal{L}}}(\eta-t, \frac{x}{u}, \frac{y}{w})\frac{1}{u}\frac{1}{w}dudwd\eta
\end{aligned}
$$

Then, $\bar{V}^*(t,x,y) := y^{1-k_y}\bar{V}(t, y^{-\frac{2\rho\sigma_x}{\sigma_y}} x, \frac{1}{y})$ is the solution of following PDE:

$$
\begin{aligned}
\bar{\mathcal{L}}\bar{V}^*(t,x,y) &= y^{1-k_y} f(t, y^{-\frac{2\rho\sigma_x}{\sigma_y}} x, \frac{1}{y})\mathbf{1}_{\{y>1\}} \\
\bar{V}^*(T,x,y) &= y^{1-k_y} h(y^{-\frac{2\rho\sigma_x}{\sigma_y}} x, \frac{1}{y})\mathbf{1}_{\{y>1\}}
\end{aligned}
$$

We define $V(t,x,y) = \bar{V}(t,x,y) - \bar{V}^*(t,x,y)$. Then

$$
\begin{aligned}
\bar{\mathcal{L}}V(t,x,y) &= f(t,x,y)\mathbf{1}_{\{y<1\}} - y^{1-k_y} f(t, y^{-\frac{2\rho\sigma_x}{\sigma_y}} x, \frac{1}{y})\mathbf{1}_{\{y>1\}} \\
V(T,x,y) &= h(x,y)\mathbf{1}_{\{y<1\}} - y^{1-k_y} h(y^{-\frac{2\rho\sigma_x}{\sigma_y}} x, \frac{1}{y})\mathbf{1}_{\{y>1\}}
\end{aligned}
$$

and $V(t,x,1) = \bar{V}(t,x,1) - \bar{V}^*(t,x,1) = 0$.
Hence, $V(t,x,y)$ is the solution of PDE (5.2.5) and given by

$$
V(t,x,y) = \bar{V}(t,x,y) - y^{1-k_y}\bar{V}(t, y^{-\frac{2\rho\sigma_x}{\sigma_y}} x, \frac{1}{y})
$$

□

5.2.3 Two-dimensional inhomogeneous Black-Scholes parabolic PDE : Mixed Boundary Conditions

In this subsection, we derive an analytic solution of mixed boundary problem for two-dimensional inhomogeneous Black-Scholes PDE. Using differential operators, we convert the PDE with mixed boundary conditions using differential operators into the PDE with Dirichlet boundary conditions, whose analytic solution is derived in section 5.2.2. Then we apply the theories in section 5.2.2 to derive an analytic solution of mixed boundary problem for two-dimensional inhomogeneous Black-Scholes PDE. The derived solution will be of great importance in section 5.3, where we derive analytic formula for pricing American maximum exchange rate quanto lookback options.

For the PDE operator $\bar{\mathcal{L}}$ defined in Section 5.2.1,

$$\begin{aligned}\bar{\mathcal{L}}V(t,x,y) = \frac{\partial V}{\partial t} + \frac{\sigma_x^2}{2}x^2\frac{\partial^2 V}{\partial x^2} + \frac{\sigma_y^2}{2}y^2\frac{\partial^2 V}{\partial y^2} + \rho\sigma_x\sigma_y xy\frac{\partial^2 V}{\partial x\partial y} + r_x x\frac{\partial V}{\partial x} \\ + r_y y\frac{\partial V}{\partial y} - rV = f(t,x,y)\end{aligned} \tag{5.2.7}$$

with boundary conditions

$$\begin{aligned}&V(T,x,y) = h(x,y) \\ &\frac{\partial V}{\partial y}(t,x,1) = V(t,x,1)\end{aligned} \tag{5.2.8}$$

with domain $\{(t,x,y) \mid 0 \le t \le T,\ 0 < x < \infty,\ 0 < y < 1\}$.

We define differential Operator $\mathcal{D}[\cdot]$ as follows :

$$\mathcal{D}[\cdot] = y\frac{\partial}{\partial y} - \mathcal{I}$$

Then,

$$\begin{aligned}&\mathcal{D}\ [\bar{\mathcal{L}}V(t,x,y)] = \bar{\mathcal{L}}\ \mathcal{D}[V(t,x,y)] \\ &\mathcal{D}\ [V(t,x,1)] = 0\end{aligned}$$

Let $P(t,x,y) = \mathcal{D}[V(t,x,y)]$, $q(x,y) = \mathcal{D}[h(x,y)]$ and $\xi(t,x,y) := \mathcal{D}[f(t,x,y)]$.

$$\begin{aligned}&\bar{\mathcal{L}}P(t,x,y) = \xi(t,x,y) \\ &P(t,x,1) = 0\end{aligned} \tag{5.2.9}$$

with domain $\{(t,x,y) \mid 0 \le t \le T,\ 0 < x < \infty,\ 0 < y < 1\}$.

By Theorem 5.2.2, the solution $P(t,x,y)$ of PDE (5.2.9) is expressed by

$$P(t,x,y) = \bar{P}(t,x,y) - y^{1-k_y}\bar{P}(t, y^{-\frac{2\rho\sigma_x}{\sigma_y}}x, \frac{1}{y})$$

where $\bar{P}(t,x,y)$ satisfies following PDE:

$$\begin{aligned} \bar{\mathcal{L}}\bar{P}(t,x,y) &= \xi(t,x,y)\mathbf{1}_{\{y<1\}} \\ \bar{P}(T,x,y) &= q(x,y)\mathbf{1}_{\{y<1\}} \end{aligned} \tag{5.2.10}$$

with domain $\{(t,x,y) \mid 0 \le t \le T,\ 0 < x < \infty,\ 0 < y < \infty\}$.

Let $\bar{V}(t,x,y) = \mathcal{D}^{-1}[\bar{P}(t,x,y)],\ \bar{V}^*(t,x,y) = \mathcal{D}^{-1}[\bar{P}^*(t,x,y)]$. Then

$$V(t,x,y) = \bar{V}(t,x,y) - \bar{V}^*(t,x,y)$$

By inverse double Mellin transform,

$$\begin{aligned} \hat{P}(t,x^*,y^*) &= -(1+y^*)\hat{V}(t,x^*,y^*) \\ \hat{P}^*(t,x^*,y^*) &= -(1+y^*)\hat{V}^*(t,x^*,y^*) \end{aligned}$$

where $\hat{V}, \hat{P}, \hat{V}^*, \hat{P}^*$ are double Mellin transform of $\bar{V}, \bar{P}, \bar{V}^*, \bar{P}^*$, respectively.

From (5.2.10),

$$\frac{d\hat{P}}{dt}(t,x^*,y^*) + \mathcal{A}(x^*,y^*)\hat{P} = \hat{\xi}(t,x^*,y^*)$$

where $\hat{\xi}$ is the double Mellin transform of $\xi(t,x,y)\mathbf{1}_{\{y<1\}}$. By definition of $\bar{P}, \xi(t,x,y)$,

$$\frac{d\hat{V}}{dt}(t,x^*,y^*) + \mathcal{A}(x^*,y^*)\hat{V} = \hat{f}(t,x^*,y^*)$$

where $\hat{f}$ is the double Mellin transform of $f(t,x,y)\mathbf{1}_{\{y<1\}}$. Hence,

$$\bar{\mathcal{L}}\bar{V}(t,x,y) = f(t,x,y)\mathbf{1}_{\{y<1\}} \tag{5.2.11}$$

By Section 5.2.2, $\bar{P}^*$ is the solution of following PDE :

$$\begin{aligned}
\bar{\mathcal{L}}\bar{P}^*(t,x,y) &= y^{1-k_y}\xi(t, y^{-\frac{2\rho\sigma_x}{\sigma_y}}x, \frac{1}{y})\mathbf{1}_{\{y>1\}} \\
\bar{P}^*(T,x,y) &= y^{1-k_y}q(y^{-\frac{2\rho\sigma_x}{\sigma_y}}x, \frac{1}{y})\mathbf{1}_{\{y>1\}}
\end{aligned}$$

Also,

$$\frac{d\hat{P}^*}{dt}(t,x^*,y^*) + \mathcal{A}(x^*,y^*)\hat{P}^* = \hat{\xi}(t,x^*,-y^* - \frac{2\rho\sigma_x}{\sigma_y}x^* + k_y - 1)$$

and

$$\begin{aligned}
&\hat{\xi}(t,x^*,-y^* - \frac{2\rho\sigma_x}{\sigma_y}x^* + k_y - 1) \\
&= -(1 - y^* - \frac{2\rho\sigma_x}{\sigma_y}x^* + k_y - 1)\hat{f}(t,x^*,-y^* - \frac{2\rho\sigma_x}{\sigma_y}x^* + k_y - 1)
\end{aligned}$$

Since $\hat{P}^*(t,x^*,y^*) = -(1+y^*)\hat{V}^*(t,x^*,y^*)$,

$$\begin{aligned}
&\frac{d\hat{V}^*}{dt}(t,x^*,y^*) + \mathcal{A}(x^*,y^*)\hat{V}^* \\
=&\frac{(1 - y^* - \frac{2\rho\sigma_x}{\sigma_y}x^* + k_y - 1)}{1+y^*}\hat{f}(t,x^*,-y^* - \frac{2\rho\sigma_x}{\sigma_y}x^* + k_y - 1) \\
=& - \hat{f}(t,x^*,-y^* - \frac{2\rho\sigma_x}{\sigma_y}x^* + k_y - 1) + \frac{k_y+1}{1+y^*}\hat{f}(t,x^*,-y^* - \frac{2\rho\sigma_x}{\sigma_y}x^* + k_y - 1) \\
&- \frac{2\rho\sigma_x}{\sigma_y}\frac{x^*}{1+y^*}\hat{f}(t,x^*,-y^* - \frac{2\rho\sigma_x}{\sigma_y}x^* + k_y - 1)
\end{aligned} \tag{5.2.12}$$

Lemma 5.2.3 Let $\hat{Q}(x,y)$ is the double Mellin transform of $Q(x,y)$, i.e. $\hat{Q}(x^*,y^*) = \mathcal{M}_{xy}(Q(x,y);x^*,y^*)$. Then, inverse Mellin transform of $\dfrac{1}{1+y^*}\hat{Q}(x^*,-y^* - \frac{2\rho\sigma_x}{\sigma_y}x^* + k_y - 1)$ is given by

$$\mathcal{M}_{xy}^{-1}\left(\frac{1}{1+y^*}\hat{Q}(x^*,-y^* - \frac{2\rho\sigma_x}{\sigma_y}x^* + k_y - 1)\right) = -y\int_0^y u^{-(1+k_y)}Q(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x, \frac{1}{u})du$$

Proof.

$$
\begin{aligned}
&\mathcal{M}_{xy}^{-1}\left(\frac{1}{1+y^*}\hat{Q}(x^*,-y^*-\frac{2\rho\sigma_x}{\sigma_y}x^*+k_y-1)\right)\\
&=\frac{1}{(2\pi i)^2}\int_{c_2-i\infty}^{c_2+i\infty}\int_{c_1-i\infty}^{c_1+i\infty}\frac{\hat{Q}(x^*,-y^*-\frac{2\rho\sigma_x}{\sigma_y}x^*+k_y-1)}{1+y^*}x^{-x^*}y^{-y^*}dx^*dy^*\\
&=\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\frac{1}{1+y^*}\left(\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\hat{Q}(x^*,-y^*-\frac{2\rho\sigma_x}{\sigma_y}x^*+k_y-1x^{-x^*}dx^*\right)y^{-y^*}dy^*
\end{aligned}
$$

Since,

$$
\begin{aligned}
&\mathcal{M}_y^{-1}(\frac{1}{1+y^*})=-y\mathbf{1}_{\{y>1\}}\\
&\mathcal{M}_y^{-1}\left(\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\hat{Q}(x^*,-y^*-\frac{2\rho\sigma_x}{\sigma_y}x^*+k_y-1)x^{-x^*}dx^*\right)\\
&=\mathcal{M}_{xy}^{-1}\left(\hat{Q}(x^*,-y^*-\frac{2\rho\sigma_x}{\sigma_y}x^*+k_y-1)\right)\\
&=y^{1-k_y)}Q(t,y^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{y})
\end{aligned}
$$

and by Mellin convolution property of single variable,

$$
\begin{aligned}
&\mathcal{M}_{xy}^{-1}\left(\frac{1}{1+y^*}\hat{Q}(x^*,-y^*-\frac{2\rho\sigma_x}{\sigma_y}x^*+k_y-1)\right)\\
&=-\int_0^\infty\left(\frac{y}{u}\right)\mathbf{1}_{\{y\geq u\}}u^{1-k_y}Q(u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})\frac{1}{u}du
\end{aligned}
$$

□

Also, by properties of Mellin transform in Appendix A.2,

$$
\begin{aligned}
&\mathcal{M}_{xy}^{-1}\left(\frac{x^*}{1+y^*}\hat{Q}(x^*,-y^*-\frac{2\rho\sigma_x}{\sigma_y}x^*+k_y-1)\right)\\
&=-x\frac{\partial}{\partial x}\mathcal{M}_{xy}^{-1}\left(\frac{1}{1+y^*}\hat{Q}(x^*,-y^*-\frac{2\rho\sigma_x}{\sigma_y}x^*+k_y-1)\right)\qquad(5.2.13)\\
&=x\frac{\partial}{\partial x}\left(y\int_0^y u^{-(1+k_y)}Q(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du\right)
\end{aligned}
$$

Lemma 5.2.4 Consider the following PDE problem :

$$\begin{aligned}\bar{\mathcal{L}}Q(t,x,y) &= y\mathbf{1}_{\{y>1\}}\int_1^y u^{-(1+k_y)}\xi(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du \\ Q(T,x,y) &= y\mathbf{1}_{\{y>1\}}\int_1^y u^{-(1+k_y)}q(u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du\end{aligned} \tag{5.2.14}$$

Then, the solution $Q(t,x,y)$ of PDE (5.2.14) is given by

$$Q(t,x,y) = y\int_0^y u^{-(1+k_y)}P(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du$$

where $P(t,x,y)$ satisfies $\bar{\mathcal{L}}P(t,x,y) = \xi(t,x,y)\mathbf{1}_{\{y<1\}}$ and $P(T,x,y) = q(x,y)\mathbf{1}_{\{y<1\}}$.

Proof. Let $P^*(t,x,y) = y^{1-k_y}P(t,y^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{y})$. Then, by Section 5.2.2,

$$\bar{\mathcal{L}}P^*(t,x,y) = y^{1-k_y}\xi(t,y^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{y})\mathbf{1}_{\{y>1\}}$$

By double Mellin transform,

$$\frac{d\hat{P}^*}{dt} + \mathcal{A}(x^*,y^*)\hat{P}^* = \hat{\xi}(t,x^*,-y^*-\frac{2\rho\sigma_x}{\sigma_y}x^*+k_y-1)$$

where $\hat{P}^*$ and $\hat{\xi}$ are the double Mellin transform of $P^*(t,x,y)$ and $\xi(t,x,y)\mathbf{1}_{\{y<1\}}$, respectively.

By Lemma 5.2.3,

$$\begin{aligned}&\mathcal{M}_{xy}^{-1}\left(-\frac{1}{1+y^*}\hat{\xi}(t,x^*,-y^*-\frac{2\rho\sigma_x}{\sigma_y}x^*+k_y-1)\right) \\ &= y\mathbf{1}_{\{y>1\}}\int_0^\infty \frac{1}{u}\mathbf{1}_{\{y\geq u\}}u^{1-k_y}\xi(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})\frac{1}{u}du\end{aligned}$$

Hence

$$\hat{Q}(t,x^*,y^*) = -\frac{1}{1+y^*}\hat{P}^*(t,x^*,y^*) = -\frac{1}{1+y^*}\hat{P}(t,x^*,-y^*-\frac{2\rho\sigma_x}{\sigma_y}x^*+k_y-1)$$

By Lemma 5.2.3, we have proved the desired result.

□

By Lemma 5.2.3 and (5.2.12), (5.2.13),

$$\begin{aligned}\bar{\mathcal{L}}\bar{V}^*(t,x,y) =& -y^{1-k_y}f(t,y^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{y})\mathbf{1}_{\{y>1\}}\\ &-(k_y+1)y\mathbf{1}_{\{y>1\}}\int_1^y u^{-(1+k_y)}f(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du \\ &-\frac{2\rho\sigma_x}{\sigma_y}xy\mathbf{1}_{\{y>1\}}\int_1^y u^{-(1+k_y+\frac{2\rho\sigma_x}{\sigma_y})}f_x(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du\end{aligned} \tag{5.2.15}$$

In (5.2.15), by integral by parts,

$$\begin{aligned}&k_y\int_1^y u^{-(1+k_y)}f(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du\\ =&-y^{1-k_y}f(t,y^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{y})+f(t,x,1)+\int_1^y u^{-k_y}\frac{\partial}{\partial u}\left(f(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})\right)du\\ =&-y^{1-k_y}f(t,y^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{y})+f(t,x,1)-\frac{2\rho\sigma_x}{\sigma_y}x\int_1^y u^{-(1+k_y+\frac{2\rho\sigma_x}{\sigma_y})}f_x(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du\\ &+\int_1^y u^{-k_y}\cdot-\frac{1}{u^2}\cdot f_y(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du\end{aligned} \tag{5.2.16}$$

From (5.2.11),(5.2.15) and (5.2.16),

$$\begin{aligned}\bar{\mathcal{L}}V(t,x,y) =& f(t,x,y)\mathbf{1}_{\{y<1\}}+yf(t,x,1)\mathbf{1}_{\{y>1\}}+y\mathbf{1}_{\{y>1\}}\int_1^y u^{-(1+k_y)}f(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du\\ &-y\mathbf{1}_{\{y>1\}}\int_1^y u^{-(k_y+2)}f_y(t,u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du\end{aligned}$$

By same procedure,

$$\begin{aligned}V(T,x,y) =& h(x,y)\mathbf{1}_{\{y<1\}}+yh(x,1)\mathbf{1}_{\{y>1\}}+y\mathbf{1}_{\{y>1\}}\int_1^y u^{-(1+k_y)}h(u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du\\ &-y\mathbf{1}_{\{y>1\}}\int_1^y u^{-(k_y+2)}h_y(u^{-\frac{2\rho\sigma_x}{\sigma_y}}x,\frac{1}{u})du\end{aligned}$$

Hence, we proved the following Theorem.

Theorem 5.2.5 (Two-dimensional inhomogeneous Black-Scholes PDE with **mixed boundary conditions**)
The solution $V(t,x,y)$ of PDE (5.2.8) with operator (5.2.7) satisfies the

following extended PDE:

$$
\begin{aligned}
\bar{\mathcal{L}}V(t,x,y) =& f(t,x,y)\mathbf{1}_{\{y<1\}} + yf(t,x,1)\mathbf{1}_{\{y>1\}} + y\mathbf{1}_{\{y>1\}}\int_1^y u^{-(1+k_y)} f(t, u^{-\frac{2\rho\sigma_x}{\sigma_y}}x, \frac{1}{u})du \\
& - y\mathbf{1}_{\{y>1\}}\int_1^y u^{-(k_y+2)} f_y(t, u^{-\frac{2\rho\sigma_x}{\sigma_y}}x, \frac{1}{u})du \\
V(T,x,y) =& h(x,y)\mathbf{1}_{\{y<1\}} + yh(x,1)\mathbf{1}_{\{y>1\}} + y\mathbf{1}_{\{y>1\}}\int_1^y u^{-(1+k_y)} h(u^{-\frac{2\rho\sigma_x}{\sigma_y}}x, \frac{1}{u})du \\
& - y\mathbf{1}_{\{y>1\}}\int_1^y u^{-(k_y+2)} h_y(u^{-\frac{2\rho\sigma_x}{\sigma_y}}x, \frac{1}{u})du
\end{aligned}
$$

5.3 Analytic Pricing of American Maximum Exchange-Rate Quanto Lookback Options

In section 5.1, we derived the Black-Scholes PDE with mixed boundary conditions satisfied by the value of American maximum exchange rate quanto options. Using the technique developed throughout the section 5.2 to solve Black-Scholes PDE with mixed boundary conditions, we now derive analytic formula for American maximum exchange rate quanto options.

Since $\alpha(s) = (s-K)^+$ and $\beta(t,s,z) = (-(r_d - \delta_s^d)\cdot s + r_d \cdot K)\mathbf{1}_{\{s \geq \epsilon^*(t,z)\}}$ in (5.1.6), by Theorem 5.2.5, the value function $V(t,s,z)$ satisfies the following PDE :

$$\begin{aligned}\mathcal{L}V(t,s,z) =& \beta(t,s,z)\mathbf{1}_{\{z<1\}} + z\beta(t,s,1)\mathbf{1}_{\{z>1\}}\\ &+ z\mathbf{1}_{\{z>1\}}\int_1^z u^{-(1+k_f)}\beta(t,u^{-\frac{2\rho\sigma_s}{\sigma_f}}s,\frac{1}{u})du\end{aligned}$$

$$\begin{aligned}V(T,s,z) =& \alpha(s)\mathbf{1}_{\{z<1\}} + z\alpha(s)\mathbf{1}_{\{z>1\}}\\ &+ z\mathbf{1}_{\{z>1\}}\int_1^z u^{-(1+k_f)}\alpha(u^{-\frac{2\rho\sigma_s}{\sigma_f}}s)du\end{aligned}$$

where $k_f = 2(r_d - r_f)/\sigma_f^2$.

Let

$$V(t,s,z) = V_E(t,s,z) + V_P(t,s,z)$$

where $V_E(t,s,z)$ and $V_P(t,s,z)$ satisfy following PDEs:

$$\begin{aligned}\mathcal{L}V_E(t,s,z) =& 0\\ V_E(T,s,z) =& \alpha(s)\mathbf{1}_{\{z<1\}} + z\alpha(s)\mathbf{1}_{\{z>1\}}\\ &+ z\mathbf{1}_{\{z>1\}}\int_1^z u^{-(1+k_f)}\alpha(u^{-\frac{2\rho\sigma_s}{\sigma_f}}s)du\end{aligned} \tag{5.3.1}$$

and

$$\begin{aligned}\mathcal{L}V_P(t,s,z) =& \beta(t,s,z)\mathbf{1}_{\{z<1\}} + z\beta(t,s,1)\mathbf{1}_{\{z>1\}}\\ &+ z\mathbf{1}_{\{z>1\}}\int_1^z u^{-(1+k_f)}\beta(t,u^{-\frac{2\rho\sigma_s}{\sigma_f}}s,\frac{1}{u})du\\ V_P(T,s,z) =& 0\end{aligned} \tag{5.3.2}$$

Define $C_E(t,s,f,m) = m\cdot V_E(t,s,\frac{f}{m})$, $C_P(t,s,f,m) = m\cdot V_P(t,s,\frac{f}{m})$. Then, C_E is the value of the European maximum exchange rate quanto lookback options with terminal payoff $m \cdot (s-K)^+$ and $C_P(t,s,f,m)$ is the early exercise premium of American maximum exchange rate quanto lookback options. Clearly,

$$C(t,s,f,m) = C_E(t,s,f,m) + C_P(t,s,f,m)$$

We decompose the value of American maximum exchange rate quanto options into their European part and the early exercise premium part as above. Now we derive the analytic solution for each part separately.

5.3.1 European Maximum Exchange Rate Quanto Lookback Options

Theorem 5.3.1 The price of European maximum exchange rate quanto lookback option, $C_E(t,s,f,m)$, is given by

$$\begin{aligned}
&C_E(t,s,f,m)\\
&= mse^{-(r_d-\delta_s^d)\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K})+\sigma_s\sqrt{\tau}, -d_2(\tau,\frac{f}{m})-\rho\sigma_s\sqrt{\tau}, -\rho\right)\\
&\ - Km\cdot e^{-r_d\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K}), -d_2(\tau,\frac{f}{m}), -\rho\right)\\
&+ sfe^{-q\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K})+(\sigma_s+\rho\sigma_f)\sqrt{\tau}, d_2(\tau,\frac{f}{m})+(\rho\sigma_s+\sigma_f)\sqrt{\tau}, \rho\right)\\
&- Kf\cdot e^{-r_f\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K})+\rho\sigma_f\sqrt{\tau}, d_2(\tau,\frac{f}{m})+\sigma_f\sqrt{\tau}, \rho\right)\\
&+ f\int_0^{f/m} u^{-(1+k_f)}\left[se^{-(r_d-\delta_s^d)\tau}\mathcal{N}_2\left(d_1(\tau,\frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}s}{K})+\sigma_s\sqrt{\tau}, -d_2(\tau,\frac{1}{u})-\rho\sigma_s\sqrt{\tau}, -\rho\right)\right.\\
&\left. -Ke^{-r_d\tau}\mathcal{N}_2\left(d_1(\tau,\frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}s}{K}), -d_2(\tau,\frac{1}{u}), -\rho\right)\right] du
\end{aligned} \tag{5.3.3}$$

where $\tau = T-t$ and $d_1(t,s) = \dfrac{\log s + \left(\delta_s^d - \frac{\sigma_s^2}{2}\right)t}{\sigma_s\sqrt{t}}$, $d_2(t,z) = \dfrac{\log z + \left(r_d - r_f - \frac{\sigma_f^2}{2}\right)t}{\sigma_f\sqrt{t}}$.
$\mathcal{N}_2$ is the bivariate normal cumulative distribution defined by

$$\mathcal{N}_2(x_1,x_2,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2} e^{-\frac{1}{2(1-\rho^2)}(p^2+q^2-2\rho pq)}dpdq$$

Proof. In (5.3.1), let $V_E(t,s,z) = V_E^1(t,s,z) + V_E^2(t,s,z) + V_E^3(t,s,z)$ and

V_E^1, V_E^2 and V_E^3 satisfy

$$\mathcal{L}V_E^1(t,s,z) = 0,\ V_E^1(T,s,z) = \alpha(s)\mathbf{1}_{\{z<1\}}$$
$$\mathcal{L}V_E^2(t,s,z) = 0,\ V_E^2(T,s,z) = z\alpha(s)\mathbf{1}_{\{z>1\}}$$
$$\mathcal{L}V_E^3(t,s,z) = 0,\ V_E^3(T,s,z) = z\mathbf{1}_{\{z>1\}}\int_1^z u^{-(1+k_f)}\alpha(u^{-\frac{2\rho\sigma_s}{\sigma_f}}s)du$$

Then, by double Mellin transform approach in Section 5.2.1,

$$\begin{aligned}V_E^1(t,s,z) &= \int_0^\infty\int_0^\infty \alpha(u)\mathbf{1}_{\{w<1\}}\mathcal{G}_L(T-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudw\\ &= \int_0^1\int_K^\infty u\cdot\mathcal{G}_L(T-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudw - K\int_0^1\int_K^\infty \mathcal{G}_L(T-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudw\end{aligned}$$

where $\mathcal{G}_\mathcal{L}$ is green function of Black-Scholes PDE operator $\mathcal{L}$.

By Section 5.2.1, for PDE operator $\mathcal{L}$ defined (5.1.6), green function $\mathcal{G}_\mathcal{L}$ is given by

$$\begin{aligned}&\mathcal{G}_\mathcal{L}(\tau,x,y) =\\ &\exp\left\{-\frac{1}{2}\frac{\left(\frac{\rho\sigma_s\sigma_f\tau}{2}(k_s-1)-\left(r_d-r_f-\frac{\sigma_f^2}{2}\right)\tau+\frac{\rho\sigma_f}{\sigma_s}\ln x\right)^2}{(1-\rho^2)\sigma_f^2\tau}-\left(\frac{\sigma_s^2(k_s-1)^2}{8}+r_d\right)\tau\right\}\\ &\cdot x^{\frac{1-k_s}{2}}y^{\frac{\frac{\rho\sigma_s\sigma_f\tau}{2}(k_s-1)-\left(r_d-r_f-\frac{\sigma_f^2}{2}\right)\tau+\frac{\rho\sigma_f}{\sigma_s}\ln x}{(1-\rho^2)\sigma_f^2\tau}}\frac{e^{-\frac{1}{2}\left(\frac{\ln x}{\sigma_s\sqrt{\tau}}\right)^2}}{\sigma_s\sqrt{2\pi\tau}}\frac{e^{-\frac{1}{2}\left(\frac{\ln y}{\sigma_f\sqrt{(1-\rho^2)\tau}}\right)^2}}{\sigma_f\sqrt{2\pi(1-\rho^2)\tau}}.\end{aligned}\tag{5.3.4}$$

where $k_s := 2\delta_s^d/\sigma_s^2$.

Let $\tau = T-t$.

By Lemma B.1 in Appendix B,

$$\int_0^1\int_K^\infty u\cdot\mathcal{G}_L(T-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudw = se^{-(r_d-\delta_s^d)\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K})+\sigma_s\sqrt{\tau},-d_2(\tau,z)-\rho\sigma_s\sqrt{\tau},-\rho\right)$$
$$\int_0^1\int_K^\infty \mathcal{G}_L(T-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudw = e^{-r_d\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K}),-d_2(\tau,z),-\rho\right)$$

Hence,

$$\begin{aligned}V_E^1(t,s,z) &= se^{-(r_d-\delta_s^d)\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K})+\sigma_s\sqrt{\tau},-d_2(\tau,z)-\rho\sigma_s\sqrt{\tau},-\rho\right)\\ &- K\cdot e^{-r_d\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K}),-d_2(\tau,z),-\rho\right)\end{aligned}\tag{5.3.5}$$

Similarly,

$$\begin{aligned}
&V_E^2(t,s,z)\\
&=\int_0^\infty\int_0^\infty v\alpha(u)\mathbf{1}_{\{v>1\}}\mathcal{G}_L(T-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudw\\
&=\int_1^\infty\int_K^\infty uv\cdot\mathcal{G}_L(T-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudw-K\int_1^\infty\int_K^\infty v\cdot\mathcal{G}_L(T-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudw \qquad (5.3.6)\\
&=sze^{-q\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K})+(\sigma_s+\rho\sigma_f)\sqrt{\tau},d_2(\tau,z)+(\rho\sigma_s+\sigma_f)\sqrt{\tau},\rho\right)\\
&-K\cdot ze^{-r_f\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K})+\rho\sigma_f\sqrt{\tau},d_2(\tau,z)+\sigma_f\sqrt{\tau},\rho\right)
\end{aligned}$$

By Lemma 5.2.4 in Section 5.2,

$$\begin{aligned}
&V_E^3(t,s,z)\\
&=z\int_0^z u^{-(1+k_f)}V_E^1(t,u^{-\frac{2\rho\sigma_s}{\sigma_f}}s,\frac{1}{u})du\\
&=zse^{-(r_d-\delta_s^d)\tau}\int_0^z u^{-(1+k_f\frac{2\rho\sigma_s}{\sigma_f})}\mathcal{N}_2\left(d_1(\tau,\frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}s}{K})+\sigma_s\sqrt{\tau},-d_2(\tau,\frac{1}{u})\right. \qquad (5.3.7)\\
&\left.-\rho\sigma_s\sqrt{\tau},-\rho\right)du\\
&-Ke^{-r_d\tau}z\int_0^z u^{-(1+k_f)}\mathcal{N}_2\left(d_1(\tau,\frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}s}{K}),-d_2(\tau,\frac{1}{u}),-\rho\right)du
\end{aligned}$$

By (5.3.5),(5.3.6) and (5.3.7),

$$\begin{aligned}
&V_E(t,s,z)=se^{-(r_d-\delta_s^d)\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K})+\sigma_s\sqrt{\tau},-d_2(\tau,z)-\rho\sigma_s\sqrt{\tau},-\rho\right)\\
&-K\cdot e^{-r_d\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K}),-d_2(\tau,z),-\rho\right)\\
&+sze^{-q\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K})+(\sigma_s+\rho\sigma_f)\sqrt{\tau},d_2(\tau,z)+(\rho\sigma_s+\sigma_f)\sqrt{\tau},\rho\right)\\
&-K\cdot ze^{-r_f\tau}\mathcal{N}_2\left(d_1(\tau,\frac{s}{K})+\rho\sigma_f\sqrt{\tau},d_2(\tau,z)+\sigma_f\sqrt{\tau},\rho\right)\\
&+zse^{-(r_d-\delta_s^d)\tau}\cdot\int_0^z u^{-(1+k_f+\frac{2\rho\sigma_s}{\sigma_f})}\mathcal{N}_2\left(d_1(\tau,\frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}s}{K})+\sigma_s\sqrt{\tau},-d_2(\tau,\frac{1}{u})-\rho\sigma_s\sqrt{\tau},-\rho\right)du\\
&-Ke^{-r_d\tau}z\cdot\int_0^z u^{-(1+k_f)}\mathcal{N}_2\left(d_1(\tau,\frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}s}{K}),-d_2(\tau,\frac{1}{u}),-\rho\right)du
\end{aligned}$$

Since $C_E(t,s,f,m)=m\cdot V_E(t,s,\frac{f}{m})$, we have showed desired result.

□

We note that the formula we derived for European maximum exchange rate quanto options in theorem 5.3.1 is exactly the same as the one derived in Dai et al. [13], which utilized the joint density function of the extreme values and the terminal values of stock price and exchange rate.

5.3.2 American Maximum Exchange Rate Quanto Lookback Options

Dai et al. [11] proved several properties for the free boundary of American maximum exchange rate quanto lookback options. However, there are no known analytic formulas for the free boundary so far. In this subsection, we derive the analytic representation of early exercise premium for American maximum exchange rate quanto lookback options using the double Mellin transform techniques. We also draw the integral equation satisfied by the free boundary.

Theorem 5.3.2 (Premium decomposition of American maximum exchange rate quanto lookback options)
The price $C(t,s,f,m)$ of the American maximum exchange rate quanto lookback option defined in Section 5.1 is given by

$$C(t,s,f,m) = C_E(t,s,f,m) + C_P(t,s,f,m)$$

where $C_E(t,s,f,m)$ defined in Theorem 5.3.1 and $V_P(t,s,z) = C_P(t,s,f,m)/m$, $(z = f/m)$ is given by

$$\begin{aligned}
&V_P(t,s,z)\\
&= (r_d-\delta_s^d)s\int_t^T\int_0^1 \frac{e^{-(r_d-\delta_s^d)(\eta-t)}}{w}\phi_1(\eta-t,\frac{s}{\epsilon^*(\eta,w)},\frac{z}{w})dwd\eta\\
&- r_dK\cdot\int_t^T\int_0^1 \frac{e^{-r_d(\eta-t)}}{w}\phi_2(\eta-t,\frac{s}{\epsilon^*(\eta,w)},\frac{z}{w})dwd\eta\\
&+ (r_d-\delta_s^d)sz\cdot\int_t^T e^{-q(\eta-t)}\mathcal{N}_2\left(d_1(\eta-t,\frac{s}{\epsilon^*(\eta,1)})+(\sigma_s+\rho\sigma_f)\sqrt{\eta-t}, d_2(\eta-t,z)\right.\\
&\quad\left.+(\rho\sigma_s+\sigma_f)\sqrt{\eta-t},\rho\right)d\eta\\
&- r_dKz\cdot\int_t^T e^{-r_f(\eta-t)}\mathcal{N}_2\left(d_1(\eta-t,\frac{s}{\epsilon^*(\eta,1)})+\rho\sigma_f\sqrt{\eta-t}, d_2(\eta-t,z)+\sigma_f\sqrt{\eta-t},\rho\right)d\eta\\
&+ (r_d-\delta_s^d)sz\int_t^T\int_0^z\int_0^1 u^{-(1+k_f+\frac{2\rho\sigma_s}{\sigma_f})}\frac{e^{-(r_d-\delta_s^d)(\eta-t)}}{w}\phi_1(\eta-t,\frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}s}{\epsilon^*(\eta,w)},\frac{1}{uw})dwdud\eta\\
&- r_dK\cdot z\int_t^T\int_0^z\int_0^1 u^{-(1+k_f)}\frac{e^{-r_d(\eta-t)}}{w}\phi_2(\eta-t,\frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}s}{\epsilon^*(\eta,w)},\frac{1}{uw})dwdud\eta
\end{aligned} \tag{5.3.8}$$

and the free boundary $\epsilon^*(t,z) = s^*(t,f,m)$ satisfies the following integral equation.

$$
\begin{aligned}
&\epsilon^*(t,z) - K \\
&= \epsilon^*(t,z) e^{-(r_d-\delta_s^d)\tau} \mathcal{N}_2\left(d_1(\tau, \frac{\epsilon^*(t,z)}{K}) + \sigma_s\sqrt{\tau}, -d_2(\tau,z) - \rho\sigma_s\sqrt{\tau}, -\rho\right) \\
&\quad - K\cdot e^{-r_d\tau}\mathcal{N}_2\left(d_1(\tau, \frac{\epsilon^*(t,z)}{K}), -d_2(\tau,z), -\rho\right) \\
&+ \epsilon^*(t,z) z e^{-q\tau}\mathcal{N}_2\left(d_1(\tau, \frac{\epsilon^*(t,z)}{K}) + (\sigma_s + \rho\sigma_f)\sqrt{\tau}, d_2(\tau,z) + (\rho\sigma_s + \sigma_f)\sqrt{\tau}, \rho\right) \\
&- K\cdot z e^{-r_f\tau}\mathcal{N}_2\left(d_1(\tau, \frac{\epsilon^*(t,z)}{K}) + \rho\sigma_f\sqrt{\tau}, d_2(\tau,z) + \sigma_f\sqrt{\tau}, \rho\right) \\
&+ \epsilon^*(t,z) z e^{-(r_d-\delta_s^d)\tau}\cdot\int_0^z u^{-(1+k_f+\frac{2\rho\sigma_s}{\sigma_f})}\mathcal{N}_2\left(d_1(\tau, \frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}\epsilon^*(t,z)}{K}) + \sigma_s\sqrt{\tau}, -d_2(\tau, \frac{1}{u})\right. \\
&\left. -\rho\sigma_s\sqrt{\tau}, -\rho\right) du \\
&- Kze^{-r_d\tau}\cdot\int_0^z u^{-(1+k_f)}\mathcal{N}_2\left(d_1(\tau, \frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}\epsilon^*(t,z)}{K}), -d_2(\tau, \frac{1}{u}), -\rho\right) du \\
&+ (r_d - \delta_s^d)\epsilon^*(t,z)\int_t^T\int_0^1 \frac{e^{-(r_d-\delta_s^d)(\eta-t)}}{w}\phi_1(\eta - t, \frac{\epsilon^*(t,z)}{\epsilon^*(\eta,w)}, \frac{z}{w}) dw d\eta \\
&- r_d K\cdot\int_t^T\int_0^1 \frac{e^{-r_d(\eta-t)}}{w}\phi_2(\eta - t, \frac{\epsilon^*(t,z)}{\epsilon^*(\eta,w)}, \frac{z}{w}) dw d\eta \\
&+ (r_d - \delta_s^d)\epsilon^*(t,z) z\cdot\int_t^T e^{-q(\eta-t)}\mathcal{N}_2\left(d_1(\eta - t, \frac{\epsilon^*(t,z)}{\epsilon^*(\eta,1)}) + (\sigma_s + \rho\sigma_f)\sqrt{\eta - t}, d_2(\eta - t, z)\right. \\
&\left. + (\rho\sigma_s + \sigma_f)\sqrt{\eta - t}, \rho\right) d\eta \\
&- r_d K z\cdot\int_t^T e^{-r_f(\eta-t)}\mathcal{N}_2\left(d_1(\eta - t, \frac{\epsilon^*(t,z)}{\epsilon^*(\eta,1)}) + \rho\sigma_f\sqrt{\eta - t}, d_2(\eta - t, z) + \sigma_f\sqrt{\eta - t}, \rho\right) d\eta \\
&+ (r_d - \delta_s^d)\epsilon^*(t,z) z\cdot\int_t^T\int_0^z\int_0^1 u^{-(1+k_f+\frac{2\rho\sigma_s}{\sigma_f})}\frac{e^{-(r_d-\delta_s^d)(\eta-t)}}{w} \\
&\times\phi_1(\eta - t, \frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}\epsilon^*(t,z)}{\epsilon^*(\eta,w)}, \frac{1}{uw}) dw du d\eta \\
&- r_d K z\cdot\int_t^T\int_0^z\int_0^1 u^{-(1+k_f)}\frac{e^{-r_d(\eta-t)}}{w}\phi_2(\eta - t, \frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}\epsilon^*(t,z)}{\epsilon^*(\eta,w)}, \frac{1}{uw}) dw du d\eta
\end{aligned}
\tag{5.3.9}
$$

where

$$\phi_1(t,x,y) := \frac{e^{-\frac{1}{2}\left(d_2(t,y) + \rho\sigma_s\sqrt{t}\right)^2}}{\sigma_f\sqrt{2\pi t}}\mathcal{N}\left(\frac{d_1(t,x) + \sigma_s\sqrt{t}(1-\rho^2) - \rho\cdot d_2(t,y)}{\sqrt{(1-\rho^2)}}\right)$$

$$\phi_2(t,x,y) := \frac{e^{-\frac{1}{2}(d_2(t,y))^2}}{\sigma_f\sqrt{2\pi t}}\mathcal{N}\left(\frac{d_1(t,x) - \rho\cdot d_2(t,y)}{\sqrt{(1-\rho^2)}}\right)$$

and $\tau = T - t$, $d_1(\cdot,\cdot)$, $d_2(\cdot,\cdot)$ are defined in Theorem 6.3.1.

Proof. In (5.3.2), let $V_P(t,s,z) = V_P^1(t,s,z) + V_P^2(t,s,z) + V_E^3(t,s,z)$ and V_P^1, V_P^2, V_P^3 are solution of following PDEs.

$$\mathcal{L}V_P^1(t,s,z) = \beta(t,s,z)\mathbf{1}_{\{z<1\}}, \quad V_P^1(T,s,z) = 0,$$
$$\mathcal{L}V_P^2(t,s,z) = z\beta(t,s,1)\mathbf{1}_{\{z>1\}} \quad V_P^2(T,s,z) = 0$$
$$\mathcal{L}V_P^3(t,s,z) = z\mathbf{1}_{\{z>1\}}\int_1^z u^{-(1+k_f)}\beta(t,u^{-\frac{2\rho\sigma_s}{\sigma_f}}s,\frac{1}{u})du, \quad V_P^3(T,s,z) = 0$$

By Section 5.2 and direct computation,

$$\begin{aligned}
&V_P^1(t,s,z)\\
&= -\int_t^T\int_0^\infty\int_0^\infty \beta(\eta,u,w)\mathbf{1}_{\{w<1\}}\cdot\mathcal{G}_{\mathcal{L}}(\eta-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudwd\eta\\
&= -\int_t^T\int_0^1\int_{\epsilon^*(\eta,w)}^\infty (-(r_d-\delta_s^d)\cdot u + r_d\cdot K)\cdot\mathcal{G}_{\mathcal{L}}(\eta-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudwd\eta\\
&= (r_d-\delta_s^d)\int_t^T\int_0^1\int_{\epsilon^*(\eta,w)}^\infty u\cdot\mathcal{G}_{\mathcal{L}}(\eta-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudwd\eta\\
&\quad - r_dK\cdot\int_t^T\int_0^1\int_{\epsilon^*(\eta,w)}^\infty \mathcal{G}_{\mathcal{L}}(\eta-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudwd\eta
\end{aligned}$$

Using Lemma B.2 in Appendix B,

$$\begin{aligned}
&\int_t^T\int_0^1\int_{\epsilon^*(\eta,w)}^\infty u\cdot\mathcal{G}_{\mathcal{L}}(\eta-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudwd\eta\\
&= \int_t^T\int_0^1 \frac{se^{-(r_d-\delta_s^d)(\eta-t)}}{\sigma_f\sqrt{2\pi(\eta-t)}}\frac{1}{w}e^{-\frac{1}{2}\left(d_2(\eta-t,\frac{z}{w})+\rho\sigma_s\sqrt{\eta-t}\right)^2}\\
&\times\mathcal{N}\left(\frac{d_1(\eta-t,\frac{s}{\epsilon^*(\eta,w)})+\sigma_s\sqrt{\eta-t}-\rho\left(d_2(\eta-t,\frac{z}{w})+\rho\sigma_s\sqrt{\eta-t}\right)}{\sqrt{1-\rho^2}}\right)dwd\eta
\end{aligned}$$

and

$$\begin{aligned}
&\int_t^T\int_0^1\int_{\epsilon^*(\eta,w)}^\infty \mathcal{G}_{\mathcal{L}}(\eta-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudwd\eta\\
&= \int_t^T\int_0^1 \frac{e^{-r_d(\eta-t)}}{\sigma_f\sqrt{2\pi(\eta-t)}}\frac{1}{w}e^{-\frac{1}{2}\left(d_2(\eta-t,\frac{z}{w})\right)^2}\mathcal{N}\left(\frac{d_1(\eta-t,\frac{s}{\epsilon^*(\eta,w)})-\rho\left(d_2(\eta-t,\frac{z}{w})\right)}{\sqrt{1-\rho^2}}\right)dwd\eta
\end{aligned}$$

where $d_1(t,x)$ and $d_2(t,y)$ are defined in Theorem 5.3.1.

Hence,

$$\begin{aligned}V_P^1(t,s,z) = (r_d-\delta_s^d)s\cdot\int_t^T\int_0^1 \frac{e^{-(r_d-\delta_s^d)(\eta-t)}}{w}\phi_1(\eta-t,\frac{s}{\epsilon^*(\eta,w)},\frac{z}{w})dwd\eta \\ -r_dK\cdot\int_t^T\int_0^1\frac{e^{-r_d(\eta-t)}}{w}\phi_2(\eta-t,\frac{s}{\epsilon^*(\eta,w)},\frac{z}{w})dwd\eta\end{aligned} \tag{5.3.10}$$

where

$$\phi_1(t,x,y) := \frac{e^{-\frac{1}{2}\left(d_2(t,y)+\rho\sigma_s\sqrt{t}\right)^2}}{\sigma_f\sqrt{2\pi t}}\mathcal{N}\left(\frac{d_1(t,x)+\sigma_s\sqrt{t}(1-\rho^2)-\rho\cdot d_2(t,y)}{\sqrt{(1-\rho^2)}}\right)$$
$$\phi_2(t,x,y) := \frac{e^{-\frac{1}{2}(d_2(t,y))^2}}{\sigma_f\sqrt{2\pi t}}\mathcal{N}\left(\frac{d_1(t,x)-\rho\cdot d_2(t,y)}{\sqrt{(1-\rho^2)}}\right)$$

Since,

$$\begin{aligned}V_P^2(t,s,z) &= -\int_t^T\int_0^\infty\int_0^\infty w\cdot\beta(\eta,u,1)\mathbf{1}_{\{w>1\}}\cdot\mathcal{G}_{\mathcal{L}}(\eta-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudwd\eta \\ &= -\int_t^T\int_1^\infty\int_{\epsilon^*(\eta,1)}^\infty(-(r_d-\delta_s^d)\cdot u+r_d\cdot K)\cdot w\cdot\mathcal{G}_{\mathcal{L}}(\eta-t,\frac{s}{u},\frac{z}{w})\frac{1}{u}\frac{1}{w}dudwd\eta\end{aligned}$$

by using Lemma B.1 in Appendix B,

$$\begin{aligned}&V_P^2(t,s,z) \\ &= (r_d-\delta_s^d)sz\cdot\int_t^T e^{-q(\eta-t)}\mathcal{N}_2\Big(d_1(\eta-t,\frac{s}{\epsilon^*(\eta,1)})+(\sigma_s+\rho\sigma_f)\sqrt{\eta-t},d_2(\eta-t,z) \\ &+(\rho\sigma_s+\sigma_f)\sqrt{\eta-t},\rho\Big)\,d\eta \\ &-r_dKz\cdot\int_t^T e^{-r_f(\eta-t)}\mathcal{N}_2\left(d_1(\eta-t,\frac{s}{\epsilon^*(\eta,1)})+\rho\sigma_f\sqrt{\eta-t},d_2(\eta-t,z)+\sigma_f\sqrt{\eta-t},\rho\right)d\eta\end{aligned}$$

Also, by Theorem 5.2.4 in Section 5.2,

$$\begin{aligned}
&V_P^3(t,s,z)\\
&= z\int_0^z u^{-(1+k_f)} V_P^1(t, u^{-\frac{2\rho\sigma_s}{\sigma_f}} s, \frac{1}{u})du\\
&= (r_d-\delta_s^d)sz\int_t^T\int_0^z\int_0^1 u^{-(1+k_f+\frac{2\rho\sigma_s}{\sigma_f})}\frac{e^{-(r_d-\delta_s^d)(\eta-t)}}{w}\\
&\times\phi_1(\eta-t, \frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}s}{\epsilon^*(\eta,w)}, \frac{1}{uw})dwdud\eta\\
&-r_dK\cdot z\int_t^T\int_0^z\int_0^1 u^{-(1+k_f)}\frac{e^{-r_d(\eta-t)}}{w}\phi_2(\eta-t, \frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}s}{\epsilon^*(\eta,w)}, \frac{1}{uw})dwdud\eta
\end{aligned} \tag{5.3.11}$$

By (5.3.10), (5.3.11) and (5.3.12),

$$\begin{aligned}
&V_P(t,s,z)\\
&= (r_d-\delta_s^d)s\int_t^T\int_0^1\frac{e^{-(r_d-\delta_s^d)(\eta-t)}}{w}\phi_1(\eta-t, \frac{s}{\epsilon^*(\eta,w)}, \frac{z}{w})dwd\eta\\
&-r_dK\cdot\int_t^T\int_0^1\frac{e^{-r_d(\eta-t)}}{w}\phi_2(\eta-t, \frac{s}{\epsilon^*(\eta,w)}, \frac{z}{w})dwd\eta\\
&+(r_d-\delta_s^d)sz\cdot\int_t^T e^{-q(\eta-t)}\mathcal{N}_2\left(d_1(\eta-t, \frac{s}{\epsilon^*(\eta,1)})+(\sigma_s+\rho\sigma_f)\sqrt{\eta-t}, d_2(\eta-t,z)\right.\\
&\left.+(\rho\sigma_s+\sigma_f)\sqrt{\eta-t}, \rho\right)d\eta\\
&-r_dKz\cdot\int_t^T e^{-r_f(\eta-t)}\mathcal{N}_2\left(d_1(\eta-t, \frac{s}{\epsilon^*(\eta,1)})+\rho\sigma_f\sqrt{\eta-t}, d_2(\eta-t,z)+\sigma_f\sqrt{\eta-t}, \rho\right)d\eta\\
&+(r_d-\delta_s^d)sz\cdot\int_t^T\int_0^z\int_0^1 u^{-(1+k_f+\frac{2\rho\sigma_s}{\sigma_f})}\frac{e^{-(r_d-\delta_s^d)(\eta-t)}}{w}\phi_1(\eta-t, \frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}s}{\epsilon^*(\eta,w)}, \frac{1}{uw})dwdud\eta\\
&-r_dKz\cdot\int_t^T\int_0^z\int_0^1 u^{-(1+k_f)}\frac{e^{-r_d(\eta-t)}}{w}\phi_2(\eta-t, \frac{u^{-\frac{2\rho\sigma_s}{\sigma_f}}s}{\epsilon^*(\eta,w)}, \frac{1}{uw})dwdud\eta
\end{aligned}$$

By **smooth pasting** condition in (5.1.3),

$$\epsilon^*(t,z)-K = V_C(t,\epsilon^*(t,z),z)+V_P(t,\epsilon^*(t,z),z)$$

we have proved the Theorem 5.3.2.

□

Therefore, we derive the analytic formula for American maximum exchange rate quanto options. Now, we state important two properties of the free boundary of such options without proof. Those who are interested in proof can refer to Dai et al. [13].

Theorem 5.3.3 (Properties of the free boundary $\epsilon^*(t,z)$)

(1) At time close to expiry, $t \to T^-$,

$$\epsilon^*(T^-, z) = \begin{cases} \max\left(1, \dfrac{r_d}{r_d - \delta_s^d}\right) K & \text{for} \quad r_d > \delta_s^d \\ +\infty & \text{for} \quad r_d \le \delta_s^d \end{cases}$$

(2) $\epsilon^*(t,z)$ is monotonically increasing with respect to z and monotonic decreasing with respect to t.

5.4 Numerical Results

In this section, we present numerical solutions for the integral equation (5.3.9) satisfied by the free boundary of American maximum exchange quanto options. Since the free boundary $\epsilon^*(t,z)$ depends on two variables t and z, numerical algorithms for solving the integral equation (5.3.9) have large time complexity. There has been numerical approaches for solving the integral equation with two underlying assets. For American spread call options, Charella [8] suggested a numerical method which approximate $\log \epsilon^*(t,z)$ to $a(t) + b(t)z$. However, In Dai et al. [11], the free boundary for American maximum exchange rate qunato options does not behave like exponential functions of z variable. Therefore Charella's numerical approximation in [8] is not possible for our case. Instead, we extend the idea of simple iterative method proposed by I.Kim et al. [44] to solve the integral equation (5.3.9) numerically. Then using the free boundary solution and adding the option valuation formula (5.3.3) and (5.3.8) together, we compute the desired option values.

5.4.1 An iterative method

In this subsection, we first describe an iterative method for solving the integral equation (5.3.9) involving the free boundary of American maximum exchange quanto options. As mentioned earlier, there is a simple iterative method for the valuation of American options proposed by I.Kim et al. [44], which dealt with solving the integral equation when the free boundary only depends on a single variable. Extending I.Kim's idea, we develop the iterative method for the integral equation which depends on two variables.

Algorithm : Iterative method for valuing American quanto lookback options

---

Step 0: Discretize the domain of free boundary $\{(\tau, z)|0 \leq \tau \leq T, 0 \leq z \leq 1\}$.
Set $n+1$ and $m+1$ to be the number of nodes for time to expiration and for z-variable,
respectively.

Step 1: Approximate free boundary $\epsilon^*(\tau, z)$ by solving the integral equation using an iterative method.

- **Step 1-1**: Set the value $B^0_{i,j} = K \max\left(1, \dfrac{r_d}{r_d - \delta^d_s}\right)$ for $0 \leq i \leq n, 0 \leq j \leq m$.
- **Step 1-2**: Set the value $\{B^k_{i,j}\}_{0\leq i\leq n, 0\leq j\leq m}(k = 1, 2, 3...)$ as the right hand side of the integral equation (5.3.9) computed from the set of old data $\{B^{k-1}_{i,j}\}_{\leq i\leq n, 0\leq j\leq m}$.
- **Step 1-3**: Repeat Step 1-2 until sufficient accuracy for the free boundary is obtained. Take $B^k_{i,j}$ as an approximation of $\epsilon^*(\tau_i, z_j)$.

Step 2: Approximate the value of American maximum exchange rate quanto lookback option $C(t, s, f, m)$

- **Step 2-1**: Compute the European term (5.3.3) with given parameters.
- **Step 2-2**: Compute the early exercise premium term (5.3.8) with the free boundary data $\epsilon^*(\tau_i, z_j)$ gotten in step 1.
- **Step 2-3**: Compute $C(t, s, f, m)$ by adding two terms in step 2-1, 2-2.

The algorithm consists of the following steps. We first discretize the domain of free boundary $\{(\tau, z)|0 \leq \tau \leq T, 0 \leq z \leq 1\}$. Set $n+1$ and $m+1$ to be the number of nodes for time to expiration and for z-variable, respectively. We begin with the grid function $B^0_{i,j} = K \max\left(1, \dfrac{r_d}{r_d - \delta^d_s}\right)$ which is an initial guess for the free boundary. Then we use $\{B^0_{i,j}\}_{0\leq i\leq n, 0\leq j\leq m}$ to the right hand side of the integral equation (5.3.9) to get an updated grid function $\{B^1_{i,j}\}_{0\leq i\leq n, 0\leq j\leq m}$. For this update, we have to compute the integral terms on right hand side of the equation (5.3.9). Any method of numerical integration is possible, e.g. the trapezoid rule. we suggest that due to the sensitivity of the integrand, the space grid for numerical integration should be sufficiently small in a neighborhood of $w = z$. Updating process is repeated until sufficient convergence is obtained.

To obtain option values, we proceed with the following two steps. We first compute the European term (5.3.3) with given state variables. Then we compute the early exercise premium term (5.3.8) using the free boundary data gotten by an iterative method. During the computation, all terms involving numerical integration can be safely treated with the trapezoid rule again.

5.4.2 Forward shooting grid method for two-state model

In this section, we explain a numerical method for valuing American maximum exchange rate quanto lookback option. The method is the extension of the forward shooting method for a single asset suggested by Baraquand et al. [1] to two assets. From (5.1.1), we have over $[t, t+\Delta t]$,

$$\begin{aligned} \ln S(t+\Delta t) &= \ln S(t) + \xi_s(t) \\ \ln F(t+\Delta t) &= \ln F(t) + \xi_f(t) \end{aligned} \tag{5.4.1}$$

where ξ_s is a normal random variable with mean $\left(\delta_s^d - \frac{{\sigma_s}^2}{2}\right)\Delta t$ and variance ${\sigma_s}^2\Delta t$, ξ_f is a normal random variable with mean $\left(r_d - r_f - \frac{{\sigma_s}^2}{2}\right)\Delta t$ and variance ${\sigma_f}^2\Delta t$. Also, the correlation between ξ_s and ξ_f is ρ.

Following the same process in Kamrad et al. [42], we approximate the joint bivariate normal processes $\{\xi_s, \xi_f\}$ by a pair of joint discrete random variables $\{\xi_s^a, \xi_f^a\}$ with the following discrete distribution.

ξ_s^a	ξ_f^a	Probability
v_s	v_f	p_1
v_s	-v_f	p_2
-v_s	-v_f	p_3
-v_s	v_f	p_4
0	0	p_5

Table 5.1: Joint discrete random variables $\{\xi_s^a, \xi_f^a\}$.

where $v_s = \lambda_s\sigma_s\sqrt{\Delta t}$ and $v_f = \lambda_f\sigma_f\sqrt{\Delta t}$, and $\sum_{k=1}^{5} p_k = 1$.

For the convergence of the discretized distribution to the true distribution as $\Delta t \to 0$, the first and second moment generating functions for the joint discrete random variables $\{\xi_s, \xi_f\}$ should be equal to the true moment generating functions. This yields four equations as follows.

$$\begin{aligned}
\mathbb{E}\left[\xi_s^a\right] &= v_s(p_1+p_2-p_3-p_4) = \left(\delta_s^d - \frac{{\sigma_s}^2}{2}\right)\Delta t \\
\mathbb{E}\left[\xi_f^a\right] &= v_f(p_1-p_2-p_3+p_4) = \left(r_d - r_f - \frac{{\sigma_s}^2}{2}\right)\Delta t \\
Var\left[\xi_s^a\right] &= v_s^2(p_1+p_2-p_3-p_4) = {\sigma_s}^2\Delta t + O(\Delta t) \\
Var\left[\xi_f^a\right] &= v_f^2(p_1-p_2-p_3+p_4) = {\sigma_f}^2\Delta t + O(\Delta t)
\end{aligned} \tag{5.4.2}$$

The four equations above are not indepedent, which means we need another equation for p_1, p_2, p_3, p_4 to determine their values. By considering the covariance term, we can get a new equation for p_1, p_2, p_3, p_4.

$$\mathbb{E}\left[\xi_s^a \xi_f^a\right] = v_s v_f(p_1-p_2+p_3-p_4) = \rho\sigma_s\sigma_f t + O(\Delta t) \tag{5.4.3}$$

From (5.4.2),(5.4.3) and $\sum_{k=1}^{5} p_k = 1$, the solution is

$$\begin{aligned}
p_1 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\delta_s^d - {\sigma_s}^2}{\sigma_s} + \frac{r_d - r_f - {\sigma_f}^2}{\sigma_f}\right) + \frac{\rho}{\lambda^2}\right] \\
p_2 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\delta_s^d - {\sigma_s}^2}{\sigma_s} - \frac{r_d - r_f - {\sigma_f}^2}{\sigma_f}\right) - \frac{\rho}{\lambda^2}\right] \\
p_3 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\delta_s^d - {\sigma_s}^2}{\sigma_s} - \frac{r_d - r_f - {\sigma_f}^2}{\sigma_f}\right) + \frac{\rho}{\lambda^2}\right] \\
p_4 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\delta_s^d - {\sigma_s}^2}{\sigma_s} + \frac{r_d - r_f - {\sigma_f}^2}{\sigma_f}\right) - \frac{\rho}{\lambda^2}\right] \\
p_5 &= 1 - \frac{1}{\lambda^2}, \qquad where \quad \lambda = \lambda_s = \lambda_f \geq 1
\end{aligned} \tag{5.4.4}$$

Note that for any value of $\lambda \geq 1$ generates an admissible probability set $\{p_i\}_{1\leq i\leq 5}$. Kamrad et al. [42] reported that the accuracy and convergence speed for discrete bivariate model varies with the parameter λ, but the results are quite satisfactory for λ that force p_5 to be near 1/5.

Now, we explain how FSG method can be applied to value our American maximum exchange rate quanto lookback options. We discretize the time by $t_0 = 0, t_1 = \Delta t, ..., t_N = N\Delta t = T$. From (5.4.1), the value of underlying asset s(resp. f) can move up by a factor of $u_s = e^{\lambda\sigma_s\sqrt{\Delta t}}$(resp. $u_f = e^{\lambda\sigma_f\sqrt{\Delta t}}$) or they can move down by a factor of $1/u_d$ (resp. $1/u_f$) at each node of the tree. Using this, we can quantize all possible states for two underlying assets s, f and the path-dependent maximum exchange rate m as follows.

When evolved with $n \leq N$ timesteps, the possible state for s and f are given by $s_i = s_0 \cdot u_s^i$ $(-n \leq i \leq n)$, $f_j = f_0 \cdot u_f^j$ $(-n \leq j \leq n)$. Similarly, the possible state for m is given by $m_k = m_0 \cdot u_f^k$ $(0 \leq k \leq n)$. Let $V_{i,j,k}^n$ denote the value of American maximum exchange rate quanto lookback options with $s = s_i, f = f_j, m = m_k$. We first initialize all values of $V_{i,j,k}^n$ by $\tilde{V}_{i,j,k}^n = \max(m, f_k) \cdot \max(s_i - K, 0)$, which is the exact payoff of maximum exchange rate quanto option when the option holder decides to exercise the option at such node of the tree.

After the initialization, we proceed from the maturity back in time to decide the value of American maximum exchange rate quanto option. Using the approximated discrete distribution in Table 5.1 with the probability (5.4.4), we can get the expected payoff $\hat{V}_{i,j,k}^n$ by discounting the possible payoff at time $n+1$ as follows.

$$\begin{aligned}\hat{V}_{i,j,k}^n = e^{-r_d\Delta t}\Big[&p_1V_{i+1,j+1,\max(j+1,k)}^{n+1} + p_2V_{i+1,j-1,\max(j-1,k)}^{n+1} + p_3V_{i-1,j-1,\max(j-1,k)}^{n+1}\\ &+p_4V_{i-1,j+1,\max(j+1,k)}^{n+1} + p_5V_{i,j,k}^{n+1}\Big]\end{aligned} \tag{5.4.5}$$

Then **the maximum of expected payoff $\hat{V}_{i,j,k}^n$ and the current payoff $\tilde{V}_{i,j,k}^n$ is the value of** $V_{i,j,k}^n$. This process is repeated until we reach to $n = 0$.

5.4.3 Implications

In this section, we solve the integral equation (5.3.9) numerically using the iterative method described in section 5.4.1. Also, we compare solutions with the benchmark solutions which are computed by using the forward grid shooting method described in section 5.4.2.

We simulate the following four different state variables.

Case 1 and 2 deal with the situation where the domestic risk-netural interest rate r_d equals the foreign risk-netural interest rate r_f, while case 3 and 4 deal with the situation where the domestic risk-netural interest rate r_d is less(resp. greater) than the foreign risk-netural interest rate r_f. We also simulate the case of no dividends in case 3. For each given parameters, by Theorem 5.3.3, there is an admissible range of ρ for the free boundary of

Case ♯	s	K	r_d	r_f	q	σ_s	σ_f	chosen ρ value	admissible range of $\rho(\geq)$
1	1	1	0.05	0.05	0.02	0.2	0.2	0.5	-0.5
2	1	1	0.05	0.05	0.02	0.2	0.2	0.8	-0.5
3	1	1	0.01	0.05	0	0.4	0.4	0.5	0.25
4	1	1	0.05	0.01	0.02	0.2	0.2	-0.1	-1

Table 5.2: Parameter values for numerical simulations.

options to be exist.

For each case, the iterative method computes the value function $V(t,s,z)$ at each grid point $(t_i, z_j) \in [0,1] \times [0,1]$. Since the free boundary $\epsilon^*(t,z)$ is sensitive near $z = 1$, we used adaptive grid for z-variable. For our simulation we use 30×46 adaptive grid system, where the regular grid with $\Delta t = 1/30$ is used for t variable, and the adaptive grid with $\Delta z = 0.1$ for $0 \leq z \leq 0.6$, $\Delta z = 0.01$ for $0.6 \leq z \leq 1$ is used for z variable. To demonstrate the accuracy of the iterative method, we compute benchmark solutions using forward grid shooting binary tree method for $z = \frac{f}{m}$ equal to 0.9,1, and for $t = 0.1, 0.5, 1$. The following Table 5.3∼5.6 summarizes the option values for case 1∼4.

T	f/m	Binomial (100)	Binomial (500)	Binomial (1500)	Iterative Method
0.1	1	0.02744	0.02750	0.02752	0.02754
	0.9	0.02576	0.02578	0.02578	0.02579
0.5	1	0.06724	0.06753	0.06763	0.06776
	0.9	0.06156	0.06174	0.06180	0.06194
1	1	0.10093	0.10154	0.10175	0.10204
	0.9	0.09198	0.09233	0.09248	0.09279

Table 5.3: American quanto lookback option values $C(t,s,f,m)/m$ with $s=1$, $K=1$, $r_d=r_f=0.05$, $q=0.02$, $\sigma_s=\sigma_f=0.2$ and $\rho=0.5$.

T	f/m	Binomial (100)	Binomial (500)	Binomial (1500)	Iterative Method
0.1	1	0.02722	0.02727	0.02729	0.02735
	0.9	0.02524	0.02526	0.02527	0.02531
0.5	1	0.06606	0.06634	0.06644	0.06659
	0.9	0.05992	0.06012	0.06020	0.06038
1	1	0.09850	0.09908	0.09929	0.09952
	0.9	0.08904	0.08951	0.08968	0.08997

Table 5.4: American quanto lookback option values $C(t,s,f,m)/m$ with $s=1$, $K=1$, $r_d=r_f=0.05$, $q=0.02$, $\sigma_s=\sigma_f=0.2$ and $\rho=0.8$.

T	f/m	Binomial (100)	Binomial (500)	Binomial (1500)	Iterative Method
0.1	1	0.05605	0.05626	0.05634	0.05668
	0.9	0.05148	0.05160	0.05164	0.05184
0.5	1	0.14123	0.14239	0.14280	0.14370
	0.9	0.12820	0.12909	0.12941	0.12998
1	1	0.21672	0.21928	0.22017	0.22177
	0.9	0.19626	0.19833	0.19907	0.20018

Table 5.5: American quanto lookback option values $C(t, s, f, m)/m$ with $s = 1$, $K = 1$, $r_d = 0.01$, $r_f = 0.05$, $q = 0$, $\sigma_s = \sigma_f = 0.4$ and $\rho = 0.5$.

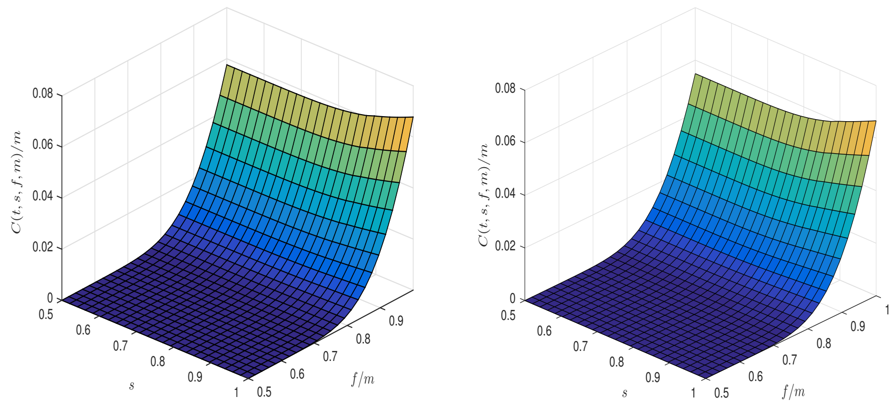
T	f/m	Binomial (100)	Binomial (500)	Binomial (1500)	Iterative Method
0.1	1	0.02595	0.02601	0.02602	0.02610
	0.9	0.02488	0.02489	0.02489	0.02494
0.5	1	0.05960	0.05985	0.05993	0.06006
	0.9	0.05557	0.05567	0.05570	0.05574
1	1	0.08549	0.08598	0.08615	0.08624
	0.9	0.07898	0.07923	0.07932	0.07929

Table 5.6: American quanto lookback option values $C(t, s, f, m)/m$ with $s = 1$, $K = 1$, $r_d = 0.05$, $r_f = 0.01$, $q = 0.02$, $\sigma_s = \sigma_f = 0.2$ and $\rho = -0.1$.

From Table 5.3∼5.6, the option value calculated from binomial tree model with forward shooting grid method increases as the tree depth becomes large. Since it is known that the binomial tree model with forward shooting grid method converges to a correct option value as the tree depth goes to infinity, we use the solution from Binomial tree model with tree depth $n = 1500$ as a benchmark. In all cases, the relative error between the iterative solution and the benchmark are less than 0.73% (this maximum error occurs when $T = z = 1$, most sensitive grid points), which demonstrates the accuracy of the iterative method.

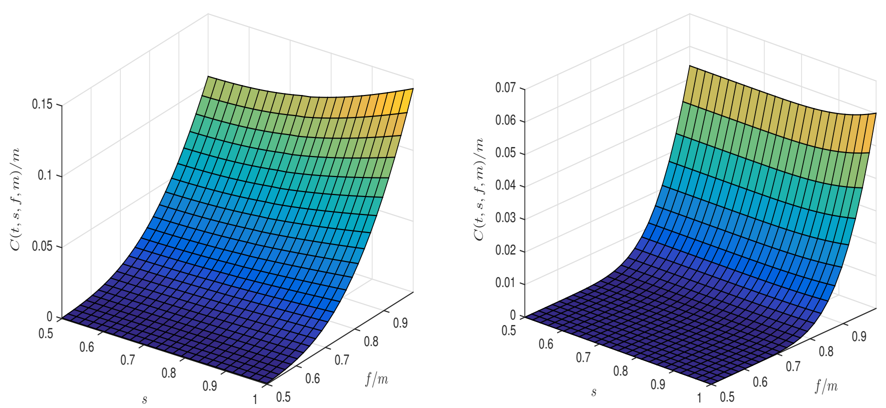
The following Figure 5.1 presents a surface plot of option values $C(t, s, f, m)/m$ for $K = 1, T = 0.5$ in a domain $0.5 \le s \le 1$, $0.5 \le f/m \le 1$.
From Figure 5.1, it is apparent that the option value increases as the price of underlying asset increases. Also, it can be observed that the option value increases as the current exchange rate is close to the realized maximum exchange rate.

The following Figure 5.2 presents a plot of free boundary $s^*(\tau, f, m)$ versus f/m for three different time to maturity on an interval $0.5 \le f/m \le 1$. From Figure 5.2, we can observe the followings. First, the free boundary increases as f/m increases. Second, the free boundary increases as the time to maturity increases. Finally, the critical value of f/m for which the free boundary starts to increase rapidly decreases as the time to maturity increases. All of these observations are reasonable since the higher value of f/m means it is much likely to have a higher value of exchange rate later on. Therefore, the option holder restrain from exercising the option prematurely.



(a) $r_d = r_f = 0.05,\ q = 0.02,\ \sigma_s = \sigma_f = 0.2$ and $\rho = 0.5$.

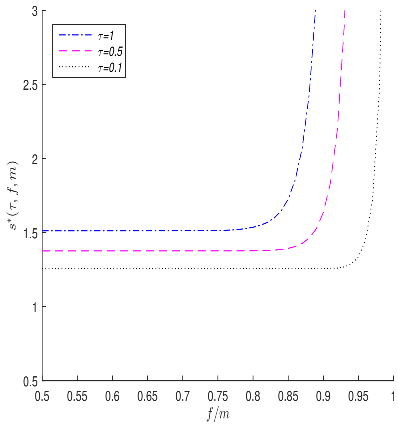
(b) $r_d = r_f = 0.05,\ q = 0.02,\ \sigma_s = \sigma_f = 0.2$ and $\rho = 0.8$.



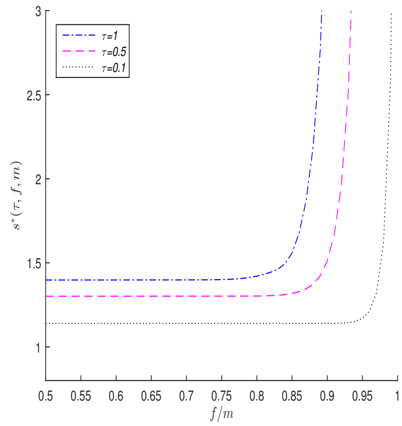
(c) $r_d = 0.01,\ r_f = 0.05,\ q = 0,\ \sigma_s = \sigma_f = 0.4$ and $\rho = 0.5$.

(d) $r_d = 0.05,\ r_f = 0.01,\ q = 0.02,\ \sigma_s = \sigma_f = 0.2$ and $\rho = -0.1$.

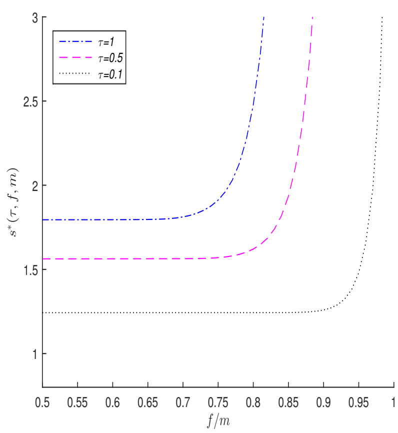
Figure 5.1: option values $C(t,s,f,m)/m$ for $K = 1,\ T = 0.5$



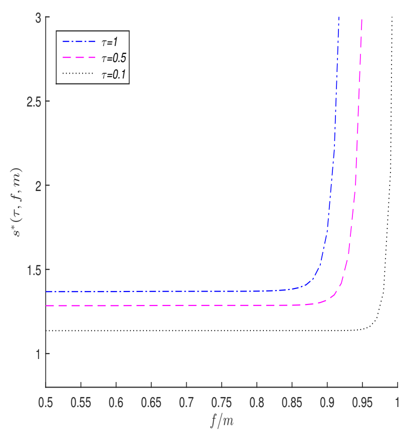
(a) $r_d = r_f = 0.05,\ q = 0.02,\ \sigma_s = \sigma_f = 0.2$ and $\rho = 0.5$.



(b) $r_d = r_f = 0.05,\ q = 0.02,\ \sigma_s = \sigma_f = 0.2$ and $\rho = 0.8$.



(c) $r_d = 0.01,\ r_f = 0.05,\ q = 0,\ \sigma_s = \sigma_f = 0.4$ and $\rho = 0.5$.



(d) $r_d = 0.05,\ r_f = 0.01,\ q = 0.02,\ \sigma_s = \sigma_f = 0.2$ and $\rho = -0.1$.

Figure 5.2: free boundary of $s^*(t, f, m)$ for $K = 1$.

5.5 Summary

In this chapter, we provide an analytic formula for American maximum exchange rate quanto lookback options. The option value can be formulated into a solution of two dimensional inhomogeneous Black-Scholes equation with mixed boundary conditions. Our approach is to solve general two dimensional inhomogeneous Black-Scholes equation with mixed boundary conditions using double Mellin transform techniques. Also, we draw out integral equations for the free boundary of American maximum exchange rate quanto lookback options. We verify that the derived integral equation is correct by numerically solving the equation using an iterative method and comparing the result with a solution from standard binomial tree method.

The iterative method used in this chapter has many advantages. it is very simple and it enables us to solve very complicated integral equations numerically. However, it is computationally expensive and there is no theory for the convergence of the method. We will continue to research on improving computational efficiency as well as developing mathematical theories for our method.

Mixed boundary problems frequently arise in option pricing problems involving maximum or minimum process of underlying assets. Especially for multi-asset pricing, presence of the correlation between underlying assets makes the option pricing problem more complex, therefore it is not easy to derive analytic pricing formula in general. However, our Mellin transform based methodology is distinguished from existing methods in that it gives an analytical representation for the solution of general inhomogeneous Black–Scholes equation with mixed boundary conditions. Since we theoretically present the general solution of Black–Scholes equation with mixed boundary conditions, our methodology is advantageous to derive analytic pricing formula for a variety of option pricing problems.

Bibliography

[1] Barraquand J, Pudet T., 1994, Pricing of American path-dependent contingent claims, Mathematical Finance, 6, 17–51.

[2] Bertrand, J., Bertrand, P., Ovarlez, J., 2000, "The Mellin transform" in The Transforms and Applications Handbook, D. Alexander, (Ed.), second ed., CRC Press, Boca Raton, FL.

[3] Broadie, M., Detemple, J., 1997, Valuation of American Options on Multiple Assets, Mathematical Finance, 7, 241–286.

[4] Buchen, P. 2012, An Introduction to Exotic Option Pricing, CRC Press, Boca Raton, FL.

[5] Carmona, R., Durrleman, V., 2003, Pricing and hedging spread options. SIAM Rev., 45(4), 627–685.

[6] Chiarella, C., Ziveyi, J., 2014, Pricing American options written on two underlying assets. Quantitative Finance, 14(3), 409–426.

[7] Conze, A., Viswanathan, R., 1991, Path dependent options: The case of lookback options, J. Finance, 46, 1893–1907.

[8] Cox, J., Ross, S., Rubinstein, M., 1979, Option pricing: A simplified approach, Journal of Financial Economics, 3, 229–263.

[9] Dai, M., 2000, A closed-form solution for perpetual American floating strike lookback options, J. Comput. Finance, 4, 63–68.

[10] Dai, M, Kwok, Y.K., 2004, Knock-in American options, J. Futures Markets, 24, 179–192.

[11] Dai, M. Wong, H.Y., Kwok, Y.K., 2004, Quanto lookback options, Math. Finance, 14, 445–467.

[12] Dai, M, Kwok, Y.K., 2005, American options with lookback payoff, SIAM J. Appl. Math., 66, 206–227.

[13] Dai, M, Kwok, Y.K., 2006, Characterization of optimal stopping regions of American Asian and lookback options, Math. Finance, 16, 63–82.

[14] Dravid, A., Richardson, M., Sun, T.S., 1993, Pricing foreign index contingent claims: An application to Nikkei index warrants. Journal of Derivatives, 33–51.

[15] Ekström, E., 2004, Russian options with a finite time horizon, J. Appl. Probab., 41, 313–326.

[16] Erdlyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G., 1954, Tables of Integral Transforms, Vols. 1–2, McGraw-Hill, New York.

[17] Evans, L.C., 1998, Partial Differential Equations, Graduate Studies in Mathematics, Vol 60, American Mathematical Society.

[18] Frontczak, R., Schobel, R., 2010, On modified Mellin transforms, Gauss-Laguerre quadrature, and the valuation of American call options, Journal of Computational and Applied Mathematics, 234(5), 1559–1571.

[19] Gerber, H., Shiu, E., 2003, Pricing dynamic investment fund protection. North American Actuarial Journal, 4(2), 60–77.

[20] Gerber, H., Shiu, E., 2003, Pricing lookback options and dynamic guarantees, North American Actuarial Journal, 7, 48–67.

[21] Geman, H., Yor, M., 1996, Pricing and hedging double-barrier options: A probabilistic approach, Math. Finance, 6, 365–378.

[22] Geske, R., Johnson, H. E., 1984, The American Put Option Valued Analytically, Journal of Finance, 2, 1511–1524.

[23] Goldman, M.B., Sosin, H.B., Gatto, M.A., 1979, Path dependent options: buy at the low, sell at the high, J. Finance, 34, 1111–1127.

[24] Hassan, E., Adem, K., 2007, A note on Mellin transform and partial differential equations, International Journal of Pure and Applied Mathematics, 34(4), 457–467.

[25] Haug, E. G, 1997, The Complete Guide to Option Pricing Formulas., McGraw-Hill.

[26] Heynen, R. C., Kat, H.M., 1994, Partial barrier options. Journal of Financial Engineering, 3, 253-274.

[27] Huang, J.Z., Subrahmanyam, M.G., Yu, G.G., 1996, Pricing and hedging American options: A recursive integration method, Rev. Financ. Stud., 9(1), 277–300.

[28] Ingersoll, J.E., 1998, Approximating American options and other financial contracts using barrier derivatives, J. Comput.Finance, 2, 85–112.

[29] Jeon, J., Han, H., Kang, M., 2015, Valuing American floating strike lookback option and Neumann problem for inhomogeneous Black-Scholes equation, submitted.

[30] Jeon, J., Kang, M., Yoon, J.H.,2015, Pricing vulnerable path-dependent options using integral transforms, submitted.

[31] Jeon, J., Yoon, J., 2016, A closed-form solution for lookback options using Mellin transform Approach, to be appear East Asian Mathematical Journal.

[32] Jeon, J.K., Han, H.J., Kim, H.U., Kang, M.J., 2016a , An integral equation representation approach for valuing Russian options with a finite time horizon, Communications in Nonlinear Science and Numerical Simulation, 36, 496–516.

[33] Jeon, J.K., Yoon, J.H., Kang, M.J., 2016b, Valuing Vulnerable Geometric Asian Options, Computers and Mathematics with Applications, 71, 676–691.

[34] Jiang, L., 2003, Mathematical Modeling and Methods of Option Pricing, Higher Education Press.

[35] Jiang, L., Dai, M., 2004, Convergence of binomial tree methods for European/American path-dependent options, SIAM J. Numer. Anal. 42, 1094–1109.

[36] Johnson, H., Stulz, R., 1987, The pricing of options with default risk, Journal of Finance, 42, 267–280.

[37] Johnson, H., 1987, Options on the Maximum or the Minimum of Several Assets, J. Financial Quant. Anal., 22, 227–283.

[38] Ju, N., 1998, Pricing an American option by approximating its early exercise boundary as a multipiece exponential function, Rev. Financ. Stud. 11, 627–646.

[39] Jun, D., Ku, H., 2012, Cross a barrier to reach barrier options, J. Math. Anal. Appl., 389, 968–978.

[40] Jun, D., Ku, H., 2013, Pricing chained options with curved barriers, Math. Finance, 23, 763–776.

[41] Jun, D., Ku, H., 2015, Analytic Solutions for American Barrier Options with Two Barrier , Journal of Mathematical Analysis and Applications, 422, 408–422.

[42] Kamrad, B., Ritchken, P., 1991, Multinomial approximating models for options with k state variables, Mgmt Sci., 37, 1640–1652.

[43] Kim,I., 1990, The analytic valuation of American options, Rev. Finan. Stud. 3(4), 547–572.

[44] Kim, I., Jang, B., Kim, K., 2013, A simple iterative method for the valuation of American options, Quant. Finance, 13, 885–895.

[45] Kimura, T., 2008, Valuing finite-lived Russian options, European Journal of Operational Research, 189(2), 363–374.

[46] Kimura T., 2009, American continuous-installment options: Valuation and premium decomposition, SIAM J. Appl. Math., 70, 803–824.

[47] Kimura T., 2010, Alternative randomization for valuing American options, Asia-Pac. J. Oper. Res., 27, 167–187.

[48] Kimura T., 2011, American fractional lookback options: valuation and premium decomposition., SIAM J Appl Math, 71, 517–39.

[49] Kwok, Y. K., Wu, L., Yu, H., 1998, Pricing Multi-Asset Options with an External Barrier, Int. J. Theor. Appl. Finance, 1(4), 523–541.

[50] Kwok, Y.K., Wong, H. Y., 2000, Currency-Translated Foreign Equity Options with Path Dependent Features and Their Multi-asset Extension, Int. J. Theor. Appl. Finance, 3, 257–278.

[51] Kwok, Y.K., 2008, Mathematical Models of Financial Derivatives, second ed., Springer-Verlag, Berlin.

[52] Lai, T.L., Lim, T.W., 2004, Exercise regions and effective valuation of American lookback options, Mathematical Finance, 14, 249–269.

[53] Lai, T.L., Lim, T.W., 2004, Efficient Valuation of American Floating Strike Lookback Options, Technical report, Department of Statistics, Stanford University, Stanford, CA.

[54] Lee, H., 2003, Pricing equity-indexed annuities with path-dependent options, Insurance Math. Econom., 33, 677–690.

[55] Li, A., 1999, The pricing of double barrier options and their variations. Advances in Futures and Options Research, 10, 17–41.

[56] Panini, R., Srivastav, R.P., 2004, Option pricing with Mellin transforms, Mathematical and Computer Modelling, 40, 43–56.

[57] Panini, R. and Srivastav, R.P., 2005, Pricing perpetual options using Mellin transforms, Applied Mathematics Letters, 18(4), 471–474.

[58] Peskir, G., Shiryaev, A., 2006, Optimal Stopping and Free-Boundary Problems, Birkhuser, Basel, Switzerland.

[59] Peskir, G., 2005, The Russian option: Finite horizon, Finance Stoch., 9, 251–267.

[60] Pfeffer, D., 2001, Sequential barrier options, Algo Res. Quarterly, 4(3), 65–73.

[61] Reiner, E., Rubinstein, M., 1991, Breaking down the barriers, Risk, 4, 428–35.

[62] Rich, D., 1994, The Mathematical Foundations of Barrier Option Pricing Theory, Adv. Futures Opt. Res., 7, 267–312.

[63] Shepp, L.A., Shiryaev, A.N., 1993, The Russian option: Reduced regret, Ann. Appl. Probab., 3, 631–640.

[64] Sneddon, I.N., 1972, The Use of Integral Transforms, McGraw-Hill, New York.

[65] Tsao, C.Y., Liu, C.C., 2012, Asian options with credit risks : Pricing and sensitivity analysis, Emerging Markets Finance and Trade, 48(3), 96–115.

[66] Wilmott, P., Dewynne, J., Howison, J., 1993, Option pricing: Mathematical models and computation, Oxford Financial Press, Oxford.

[67] Wong H.Y., Chan C.M., 2008, Turbo warrants under stochastic volatility. Quant Financ 2008, 8(7), 739–51.

[68] Yoon, J.H., 2014, Mellin transform method for European option pricing with Hull-White stochastic interest rate, Journal of Applied Mathematics, Volume 2014, Article ID 759562, 7 page.

[69] Yoon, J.H., Kim, J.H., 2015, The pricing of vulnerable options with double Mellin transforms, J. Math. Anal. Appl., 422(2), 838–857.

[70] Yu, H., Kwok, Y.K., Wu, L., 2001, Early exercise policies of American floating and fixed strike lookback options, Nonlinear Anal., 47, 4591–4602.

Appendix A

Basic Properties of Mellin Transforms

We summarize the definition and basic properties of the Mellin transform for those who are unfamiliar with the Mellin transform. Also, Most properties of the double Mellin transform are almost same as those of the single Mellin transform. Those who are interested in the Mellin transform can refer to Bertrand et al. [2], Erdlyi et al. [19], Sneddon [74], for further details.

A.1 Properties of Mellin transform

Definition A.1.1 (Definition of Mellin transform and inverse Mellin transform)
Let $g(x)$ be a *locally integrable function* on $(0, \infty)$. Then, the *Mellin transform* $\mathcal{M}(g(x), w)$ of $g(x)$ is defined by

$$\mathcal{M}_x(g(x); w) := \hat{g}(w) = \int_0^\infty g(x) x^{w-1} dx, \quad w \in \mathbb{C} \tag{A.1.1}$$

and if this integral converges for $a < Re(w) < b$ and $a < c < b$, then the inverse of the Mellin transform is given by

$$g(x) = \mathcal{M}_x^{-1}\left(\hat{g}(w)\right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{g}(w) x^{-w} dw. \tag{A.1.2}$$

Proposition A.1.1 (Convolution property of Mellin transform)
Let $g(x)$ and $h(x)$ be *locally integrable functions* on $(0, \infty)$. For $a < w < b$, let the Mellin transform $\hat{g}(w)$ and $\hat{h}(w)$ exist. Then, the *Mellin convolution*

is given by the inverse Mellin transform of $\hat{g}(w)\hat{h}(w)$ as follows:

$$\begin{aligned} g(x) \vee h(x) :&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{g}(w)\hat{h}(w)x^{-w}dw, \quad a < c < b \\ &= \int_0^\infty g\left(\frac{x}{u}\right) h(u)\frac{du}{u}. \end{aligned} \tag{A.1.3}$$

Proposition A.1.2 (Inverse Mellin transform of exponential function)
For α with $Re(\alpha) > 0$, let $\hat{g}(w) = e^{\alpha(w+\beta)^2}$. Then, the inverse Mellin transform of $g(x)$ is given by

$$\mathcal{M}_x^{-1}\left(\hat{g}(w); x\right) = \frac{1}{2}(\pi\alpha)^{-\frac{1}{2}}x^{\beta}e^{-\frac{1}{4\alpha}(\log x)^2}$$

Proposition A.1.3 (Basic Properties of Mellin transform)
Suppose that there exists a Mellin transform of $g(x)$ and let $\hat{g}(w)$ be the Mellin transform of $g(x)$.

(1) For constant α, $\mathcal{M}(x^{\alpha}g(x); w) = \hat{g}(w+\alpha)$;

(2) For constant α,

$$\mathcal{M}_x(g(x^{\alpha}); w) = \begin{cases} \dfrac{1}{\alpha}\hat{g}\left(\dfrac{w}{\alpha}\right) & \text{for} \quad \alpha > 0 \\ \\ -\dfrac{1}{\alpha}\hat{g}\left(\dfrac{w}{\alpha}\right) & \text{for} \quad \alpha < 0 \end{cases}$$

(3) For positive integer n,

$$\mathcal{M}_x\left(\left(x\frac{\partial}{\partial x}\right)^n g(x); w\right) = (-w)^n\hat{g}(w)$$

and

$$\mathcal{M}_x\left(\left(\frac{\partial}{\partial x}x\right)^n g(x); w\right) = (1-w)^n\hat{g}(w)$$

(4) For constant α, β, define

$$g(x) = \begin{cases} x^\beta & \text{for} \quad x < \alpha \\ 0 & \text{for} \quad x > \alpha \end{cases}$$

Then,

$$\hat{g}(w) = \alpha^{(w+\beta)} \frac{1}{w+\beta},$$

where $Re(w) > -Re(\beta)$.

A.2 Properties of double Mellin transform

Definition A.2.1 (Definition of double Mellin transform and inverse double Mellin transform)
Let $g(x, y)$ be *locally integrable function* on $\mathbb{R}_+ \times \mathbb{R}_+$. Then the **double Mellin transform** $\mathcal{M}_{xy}(g(x, y), x^*, y^*)$ of $g(x, y)$ is defined by

$$\mathcal{M}_{xy}(g(x,y), x^*, y^*) := \hat{g}(x^*, y^*) = \int_0^\infty \int_0^\infty g(x,y) x^{x^*-1} y^{y^*-1} dxdy, \quad x^*, y^* \in \mathbb{C} \tag{A.2.1}$$

and if this integral converges for $a_1 < Re(x^*) < b_1,\ a_2 < Re(y^*) < b_2$, then for $a_1 < c_1 < b_1,\ a_2 < c_2 < b_2$, the inverse double Mellin transform is given by

$$f(x,y) = \mathcal{M}_{xy}^{-1}(\hat{g}\,(x^*, y^*)) = \frac{1}{(2\pi i)^2} \int_{c_2 - i\infty}^{c_2 + i\infty} \int_{c_1 - i\infty}^{c_1 + i\infty} \hat{g}(x^*, y^*) x^{-x^*} y^{-y^*} dx^* dy^* \tag{A.2.2}$$

Proposition A.2.1 (Convolution property of Double Mellin transform)
Let $f(x, y)$ and $g(x, y)$ be *locally integrable functions* on $\mathbb{R}_+ \times \mathbb{R}_+$. For $a_1 < Re(x^*) < b_1,\ a_2 < Re(y^*) < b_2$, let the double Mellin transform $\hat{f}(x^*, y^*)$ and $\hat{g}(x^*, y^*)$ exist. Then, the *double Mellin convolution* is given by the inverse double Mellin transform of $\hat{f}(x^*, y^*) \cdot \hat{g}(x^*, y^*)$ as follows:

$$f(x,y) \vee g(x,y) := \int_0^\infty \int_0^\infty f(u,w) \cdot g(\frac{x}{u}, \frac{y}{w}) \frac{1}{u} \frac{1}{w} dudw \tag{A.2.3}$$

Proof. See Hassan and Adam [29]. □

For positive integer n,

$$\begin{aligned}
\mathcal{M}_{xy}\left(\left(x\frac{\partial}{\partial x}\right)^n f(x,y), x^*, y^*\right) &= (-x^*)^n \hat{f}(x^*, y^*) \\
\mathcal{M}_{xy}\left(\left(y\frac{\partial}{\partial y}\right)^n f(x,y), x^*, y^*\right) &= (-y^*)^n \hat{f}(x^*, y^*)
\end{aligned} \tag{A.2.4}$$

Appendix B

Some useful lemmas

Lemma B.1 For constant $A, B > 0$ and $\tau = T - t$,

$$\begin{aligned}
\int_0^B \int_A^\infty \mathcal{G}_{\bar{\mathcal{L}}}(\tau, \frac{x}{u}, \frac{y}{w}) \frac{1}{u} \frac{1}{w} du dw =& e^{-r\tau} \mathcal{N}_2 \left(d_1(\tau, \frac{x}{A}), -d_2(\tau, \frac{y}{B}), -\rho \right) \\
\int_0^B \int_A^\infty u \cdot \mathcal{G}_{\bar{\mathcal{L}}}(\tau, \frac{x}{u}, \frac{y}{w}) \frac{1}{u} \frac{1}{w} du dw =& x e^{-(r-r_x)\tau} \mathcal{N}_2 \left(d_1(\tau, \frac{x}{A}) + \sigma_x \sqrt{\tau}, -d_2(\tau, \frac{y}{B}) - \rho \sigma_x \sqrt{\tau}, -\rho \right) \\
\int_B^\infty \int_A^\infty w \cdot \mathcal{G}_{\bar{\mathcal{L}}}(\tau, \frac{x}{u}, \frac{y}{w}) \frac{1}{u} \frac{1}{w} du dw =& y e^{-(r-r_y)\tau} \mathcal{N}_2 \left(d_1(\tau, \frac{x}{A}) + \rho \sigma_y \sqrt{\tau}, d_2(\tau, \frac{y}{B}) + \sigma_y \sqrt{\tau}, \rho \right) \\
\int_B^\infty \int_A^\infty uw \cdot \mathcal{G}_{\bar{\mathcal{L}}}(\tau, \frac{x}{u}, \frac{y}{w}) \frac{1}{u} \frac{1}{w} du dw =& xy e^{-(r-r_x-r_y-\rho\sigma_x\sigma_y)\tau} \mathcal{N}_2 \Big(d_1(\tau, \frac{x}{A}) + (\sigma_x + \rho\sigma_y)\sqrt{\tau}, d_2(\tau, \frac{y}{B}) \\
& + (\rho\sigma_x + \sigma_y)\sqrt{\tau}, \rho \Big)
\end{aligned}$$

where $\mathcal{N}_2$ is *bivariate normal cumulative distribution* and

$$d_1(t,x) = \frac{\log x + \left(r_x - \frac{\sigma_x^2}{2} \right) t}{\sigma_x \sqrt{t}}, \; d_2(t,y) = \frac{\log y + \left(r_y - \frac{\sigma_y^2}{2} \right) t}{\sigma_y \sqrt{t}}$$

Proof. The computations are approximately the same as Theorem 1 in Yoon and Kim (2015). By using changes of variables and the methods of undetermined coefficients, we derive the desired results. □

Lemma B.2 For any real number a, b and $-1 \leq \rho \leq 1$,

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^b \int_{-\infty}^a e^{-\frac{1}{2(1-\rho^2)}(p^2+q^2-2\rho pq)} dp dq = \int_{-\infty}^b n(q) \cdot \mathcal{N}\left(\frac{a - \rho \cdot q}{\sqrt{1-\rho^2}} \right) dq$$

where n is the probability density function of the standard normal distribution and $\mathcal{N}$ is the cumulative standard normal distribution.

국문초록

이 학위 논문에서는 경로에 의존하는 미국형 옵션에 대한 해석적 해에 관하여 연구한다. 미국형 옵션은 만기전에 언제든지 옵션의 권리를 행사할 수 있기 때문에 자유 경계 문제로 분류된다. 미국형 체인드 옵션의 가격 함수가 만족하는 적분 방정식을 Girsanov's 정리와 reflection 원리를 이용하여 구하고 수치적인 근사값을 구한다.

미국형 룩백 옵션과 유한 만기를 가지는 러시안 옵션이 만족하는 각각의 편미분 방정식을 도출한다. 미국형 룩백 옵션은 노이만 경계 조건, 러사인 옵션은 혼합 경계 조건을 가지게 되고 멜린 적분 변환 방법을 이용하여 각각의 해석적 해를 도출한다.

마지막으로, 미국형 환율 룩백 옵션에 관한 가격 결정 연구를 하였다. 미국형 환율 룩백 옵션은 3차원 Parabolic PDE 로 변환되며 이중 멜린 적분 변환으로 적분 방정식을 도출하고 수치적인 방법으로 해를 구한다.

주요어휘: 옵션 가격결정, 경로의존형 옵션, 미국형 옵션, 적분 방정식, 자유 경계 문제, 멜린 변환
학번: 2009-22894