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Geometric structures modeled after smooth projective horospherical varieties of Picard number one

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Geometric structures modeled after smooth projective horospherical varieties of Picard number one

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Abstract

Geometric structures modeled after smooth projective horospherical varieties of Picard number one

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Geometric structures modeled after homogeneous manifolds are studied to characterize homogeneous manifolds and to prove the deformation rigidity of them. To generalize these characterizations and deformation rigidity results to quasihomogeneous manifolds, we first study horospherical varieties and geometric structures modeled after horospherical varieties. Using Cartan geometry, we prove that a geometric structure modeled after a smooth projective horospherical variety of Picard number one is locally equivalent to the standard geometric structure when the geometric structure is defined on a Fano manifold of Picard number one.

Keywords : geometric structure · local equivalence · horospherical variety · Cartan geometry · prolongation

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Contents

Chapter 1

Introduction

Let M be a Fano manifold of Picard number one. An irreducible component $\mathcal{K}$ of the space of rational curves on M is called a *minimal dominating component* if the subvariety $\mathcal{K}_x$ which consists of members that pass through x is nonempty and projective for general point $x \in M$. The tangent directions at x of members of $\mathcal{K}_x$ form a subvariety $\mathcal{C}_x$ of $\mathbb{P}T_x(M)$ which is called the *variety of minimal rational tangents at* x. Many techniques can be used to study the projective geometries of $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ which are believed to control the geometry of the manifold M. In this paper, we study geometric structures modeled after horospherical varieties which we expect to get from the variety of minimal rational tangents.

When S is a rational homogeneous manifold of Picard number one, a pair of the automorphism group of the variety of minimal rational tangent $\mathcal{C}_s$ and the linear span D_s of the cone $\widehat{\mathcal{C}}_s \subset T_s(S)$ of $\mathcal{C}_s$ for $s \in S$ corresponds to a geometric structure on S. Ngaiming Mok, Jun-Muk Hwang, and Jaehyun Hong published significant works on the geometric structures modeled after S which arise from the variety of minimal rational tangents. They published works on Hermitian symmetric manifolds and homogeneous contact manifolds in papers [7], [4], and [15], and on the other homogeneous manifolds associated with long simple roots in paper [5].

Theorem 1.0.1 ([5], [7], [4] and [15]). *Let $S = G/P$ where G is a simple Lie group and P is a maximal parabolic subgroup associated with a long root. Let $\mathcal{C}_s \subset \mathbb{P}T_s(S)$ be the variety of minimal rational tangents at a base point $s \in S$. Let M be a Fano manifold of Picard number one and $\mathcal{C}_x$ be the variety of minimal rational tangents at a general point $x \in M$ associated with a minimal dominating component $\mathcal{K}$. Suppose that $\mathcal{C}_s \subset \mathbb{P}T_s(S)$ and $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ are isomorphic as projective subvarieties for a general point $x \in M$. Then M is biholomorphic to S.*

It is natural to ask what happens when we replace rational homogeneous manifolds with quasihomogeneous varieties, especially with smooth projective horospherical varieties of Picard number one. A horospherical variety is a complex normal algebraic variety where a connected complex reductive algebraic group acts with an open orbit isomorphic to a torus bundle over a rational homogeneous manifold. Boris Pasquier classified smooth projective horospherical varieties of Picard number one in his paper [18]. When a smooth projective horospherical variety is homogeneous, it is isomorphic to one of quadrics Q^{2m}, Grassmannians $\mathrm{Gr}(i+1, m+2)$, and spinor varieties $\mathrm{Spin}_{2m+1}/P_{\alpha_m}$. These are all compact irreducible Hermitian symmetric manifolds, and the geometric structures modeled after them were already studied in Theorem 1.0.1.

In this thesis, we will study geometric structures modeled after smooth nonhomogeneous projective horospherical varieties of Picard number one.

Theorem 1.0.2. *Let X be a smooth nonhomogeneous projective horospherical variety of Picard number one. Let M be a Fano manifold of Picard number one. Then any geometric structure on M modeled after X is locally equivalent to the standard geometric structure on X.*

We use Definition 4.3.2 for the definition of a geometric structure modeled after X. We will prove the existence of Cartan connections (Theorem 4.2.2) and use it to prove local equivalence of geometric structures modeled after smooth nonhomogeneous projective horospherical varieties of Picard number one.

Noboru Tanaka ([21]) and Tohru Morimoto ([16]) find the sufficient conditions for the existence of Cartan connections, mainly for geometric structures with certain symmetries, like geometric structures modeled after rational homogenous manifolds. We generalize these conditions for some quasihomogeneous manifolds cases including ours in Theorem 4.2.1. To prove the existence of Cartan connections associated with geometric structures modeled after X, we need to study the Lie algebra $\mathfrak{aut}(X)$ of the automorphism group of X. In particular, it is important to know whether $\mathfrak{g}$ satisfies the prolongation property. When X is a rational homogeneous manifold, Keizo Yamaguchi shows that $\mathfrak{g}$ satisfies the prolongation property by proving that the Lie algebra cohomology space $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ vanishes, where $\mathfrak{m}$ is a nilpotent subalgebra of $\mathfrak{g}$. In this case, $\mathfrak{g}$ is semisimple, and thus we can apply Kostant's harmonic theory on the Lie algebra cohomology spaces. However, in our case, $\mathfrak{g}$ is not semisimple and we cannot apply Kostant's harmonic theory. In this direction, Collen robles and Dennis The ([19]) compute Lie algebra cohomology spaces for some cases, when $\mathfrak{g}$ is not semisimple, by modifying Kostant's harmonic theory. It would be interesting if one can generalize Kostant's harmonic theory fully to the case when $\mathfrak{g}$ is not semisimple. In this thesis, instead of generalizing the whole theory, we reduce the vanishing of Lie algebra cohomology spaces to the vanishing of Lie algebra cohomology spaces associated with the maximal semisimple subalgebra of $\mathfrak{g}$, which now can be computed using Kostant's harmonic theory.

The thesis is organized as follows. In Chapter 2, we review the general theory of Cartan connections. In Chapter 3, we study horospherical varieties. When X is a smooth nonhomogeneous projective horospherical variety of Picard number one, we also study the Lie algebra of the automorphism group of X, and varieties of minimal rational tangents of X. The vanishing of the first generalized Spencer cohomologies of $\mathfrak{aut}(X)$ is proved in Chapter 4. In Chapter 4, we prove the existence of Cartan connections and the local flatness of the geometric structures modeled after X, which proves Theorem 1.0.2.

We work over the complex number field $\mathbb{C}$ without any additional mentioning of a number field. All manifolds, Lie groups and Lie algebras will be understood as complex manifolds, complex Lie groups and complex Lie algebras.

Chapter 2

Geometric structures on filtered manifolds

In this Chapter, we mainly follow the papers of Noboru Tanaka, Tohru Morimoto, and Keizo Yamaguchi([21], [16], and [23]) .

2.1 G_0-structures on filtered manifolds

Definition 2.1.1. Let $\mathfrak{g}$ be a Lie algebra. A *gradation of* $\mathfrak{g}$ is a direct decomposition $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ such that $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$ for any $p, q \in \mathbb{Z}$. A *fundamental graded Lie algebra* is a nilpotent graded Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ generated by $\mathfrak{g}_{-1}$, that is, $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for any $p < -1$.

Definition 2.1.2. Let M be a manifold. A *tangential filtration* $F = \{F^p\}_{p\in\mathbb{Z}_{\leq 0}}$

on M is a sequence of subbundles $F^p = F^pTM$ of the tangent bundle TM of M satisfying the following: i) $F^{p+1} \subset F^p$; ii) $F^0 = 0$ and $\cup F^p = TM$; and iii) $[\mathcal{F}^p, \mathcal{F}^q] \subset \mathcal{F}^{p+q}$ for any $p, q \in \mathbb{Z}_{\leq 0}$ where $\mathcal{F}^{\bullet}$ is the sheaf of sections of $F^{\bullet}$. A manifold M with a tangential filtration $F = \{F^p\}_{p\in\mathbb{Z}_{\leq 0}}$ on M is called a *filtered manifold* and we denote a filtered manifold as (M, F).

The *symbol algebra* $\mathrm{Symb}_x(F) = \bigoplus_{p\in\mathbb{Z}_{\leq 0}} \mathrm{Symb}_x^p(F)$ *of* F *at* $x \in M$ is given by $\mathrm{Symb}_x^p(F) = F_x^pTM/F_x^{p+1}TM$ with a natural bracket induced from the Lie bracket of vector fields. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a fundamental graded Lie algebra with $\dim(\mathfrak{m}) = \dim(M)$. A filtered manifold (M, F) is called *regular of type* $\mathfrak{m}$ if the symbol algebras $\mathrm{Symb}_x(F)$ are all isomorphic to the given fundamental graded Lie algebra $\mathfrak{m}$ for all $x \in M$.

Definition 2.1.3. Let (M, F) be a regular filtered manifold of type $\mathfrak{m}$. Let $\mathscr{R}_x(M, \mathfrak{m})$ be the set of all isomorphisms $r \colon \mathfrak{m} \to \mathrm{Symb}_x(F)$ of graded Lie algebras. Then its structure group $G_0(\mathfrak{m})$ consists of all automorphisms of the graded Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ and $\mathscr{R} := \cup_{x\in M}\mathscr{R}_x(M, \mathfrak{m})$ is a principal $G_0(\mathfrak{m})$-bundle on M. This fiber bundle is called the *frame bundle* of (M, F). Given a Lie subgroup $G_0 \subset G_0(\mathfrak{m})$, a G_0-*structure* on (M, F) is a G_0-subbundle of the frame bundle $\mathscr{R}$. Two G_0-structures on (M_1, F_1) and (M_2, F_2) are *locally equivalent* if there exist two open subsets U_1 of M_1 and U_2 of M_2, and a G_0-bundle isomorphism over the open subsets U_1 and U_2.

Definition 2.1.4. A *differential system* (M,D) on a manifold M is a subbundle D of the tangent bundle TM of M. The subbundle D is *completely integrable* if and only if $[D, D] \subset D$. For a non-integrable differential system D, we consider the *derived system* ∂D of D which is defined, in terms of sections, by

$$\partial\mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$$

where $\mathcal{D}$ denotes the space of sections of D. Moreover, *the k-th weak derived systems* $\partial^{(k)}D$ of D are inductively defined by

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}],$$

where $\partial^{(0)}D = D$ and $\partial^{(k)}\mathcal{D}$ denotes the space of local sections of $\partial^{(k)}D$. A differential system (M,D) is called *regular* if $D^{-(k+1)} := \partial^{(k)}D$ is a subbundle of TM for every integer $k \geq 1$. For a regular differential system (M,D) such that $D^{-\mu} = TM$, we define the associated graded Lie algebra $\mathfrak{m}(x)$ at $x \in M$, which was introduced by Noboru Tanaka in [20]. We put $\mathfrak{g}_{-1}(x) = D^{-1}(x)$, $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$ (for $p < -1$) and

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

Then $\mathfrak{m}(x)$ becomes a fundamental graded Lie algebra which we call *the symbol* *algebra* of (M,D) at $x \in M$. If the symbol algebra $\mathfrak{m}(x)$ is isomorphic to a given fundamental graded Lie algebra $\mathfrak{m}$ for each $x \in M$, then we call (M,D) a *regular differential system of type* $\mathfrak{m}$.

Remark 2.1.5. Let (M, D) be a regular differential system of type $\mathfrak{m}$. A regular tangential filtration (M, F) of type $\mathfrak{m}$ derived from a regular differential system (M, D) of type $\mathfrak{m}$ is given by $F^p = D^p$ for $p < 0$. We just denote (M, D) as a regular filtered manifold of type $\mathfrak{m}$ derived from a regular differential system (M, D) of type $\mathfrak{m}$. A G_0-structure on a regular differential system (M, D) is a G_0-subbundle of the frame bundle $\mathscr{R}$ of the derived regular filtered manifold (M, D).

2.2 Prolongations

Definition 2.2.1. Given a fundamental graded Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$, there exists a unique graded Lie algebra $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p(\mathfrak{m})$ such that

1. $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$.

2. if $z \in \mathfrak{g}_p(\mathfrak{m})$ for $p \geq 0$, satisfies $[z, \mathfrak{m}] = 0$, then $z = 0$.

3. $\mathfrak{g}(\mathfrak{m})$ is the largest graded Lie algebra satisfying conditions 1 and 2.

We call $\mathfrak{g}(\mathfrak{m})$ the *universal prolongation* of $\mathfrak{m}$. Let $\mathfrak{g}_0 \subset \mathfrak{g}_0(\mathfrak{m})$ be a subalgebra. Then the *prolongation* of $(\mathfrak{m}, \mathfrak{g}_0)$ is the largest graded Lie algebra $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0) = \bigoplus_p \mathfrak{g}_p(\mathfrak{m}, \mathfrak{g}_0) \subset \mathfrak{g}(\mathfrak{m})$ such that $\bigoplus_{p<0} \mathfrak{g}_p(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m}$ and $\mathfrak{g}_0(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{g}_0$.

Definition 2.2.2. Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a graded Lie algebra. Assume its negative part $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is a fundamental graded Lie algebra. Define the *coboundary operator* $\partial\colon \mathrm{Hom}(\wedge^q\mathfrak{m}, \mathfrak{g}) \to \mathrm{Hom}(\wedge^{q+1}\mathfrak{m}, \mathfrak{g})$ as follows: for $\phi \in \mathrm{Hom}(\wedge^q\mathfrak{m}, \mathfrak{g})$,

$$\begin{aligned}\partial\phi(z_1 \wedge \cdots \wedge z_{q+1}) = & \sum_{i=1}^{q+1}(-1)^{i+1}[z_i, \phi(z_1 \wedge \cdots \wedge \hat{z}_i \cdots \wedge z_{q+1})] \\ & + \sum_{i<j}(-1)^{i+j}\phi([z_i, z_j] \wedge z_1 \cdots \wedge \hat{z}_i \cdots \wedge \hat{z}_j \cdots \wedge z_{q+1})\end{aligned}$$

where $\hat{z}_i$ means skipping z_i. We denote the induced cochain complex by $(C(\mathfrak{m}, \mathfrak{g}), \partial)$ and the derived space of cohomology by $H(\mathfrak{m}, \mathfrak{g})$.

Definition 2.2.3. The cochain complex $(C(\mathfrak{m}, \mathfrak{g}), \partial)$ has the following bigradation (Section 1 of [21] and Section 2.4 of [23]):

$$C^{p,q}(\mathfrak{m}, \mathfrak{g}) = \bigoplus_{j\leq -q} \mathrm{Hom}(\wedge^q_j\mathfrak{m}, \mathfrak{g}_{j+p+q-1}),$$

where

$$\wedge^q_j\mathfrak{m} = \bigoplus_{i_1+\cdots+i_q=j} \mathfrak{g}_{i_1} \wedge \cdots \wedge \mathfrak{g}_{i_q},$$

the i_k are negative. The space of cohomology with the bigradation,

$$H^q(\mathfrak{m}, \mathfrak{g}) = \bigoplus_p H^{p,q}(\mathfrak{m}, \mathfrak{g})$$

is called the *generalized Spencer cohomology* of $(\mathfrak{g}, \mathfrak{m})$.

The following is an effective way to show that a given graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{m}$ (or of $(\mathfrak{m}, \mathfrak{g}_0)$).

Lemma 2.2.4 (Lemma 2.1 of [23]). *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a graded Lie algebra such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. Then $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$ (respectively of $(\mathfrak{m}, \mathfrak{g}_0)$) if and only if the following two conditions hold:*

1. *if $z \in \mathfrak{g}_p$ for $p \geq 0$, satisfies $[z, \mathfrak{m}] = 0$, then $z = 0$.*
2. *$H^{p,1}(\mathfrak{m}, \mathfrak{g}) = 0$ for $p \geq 0$ (respectively, $p \geq 1$).*

2.3 Cartan connections

Definition 2.3.1. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Let H be a connected Lie group with Lie algebra $\mathfrak{h}$ and let $\mathrm{Ad}\colon H \to GL(\mathfrak{g})$ be the adjoint representation of H on $\mathfrak{g}$. A *Cartan connection of type* $(\mathfrak{g}, H)$ on a manifold M is a principal H-bundle $\pi\colon P \to M$ with a $\mathfrak{g}$-valued 1-form ω on P such that

1. $\omega(z^\dagger) = z$ for $z \in \mathfrak{h}$, where $z^\dagger$ denotes the fundamental vector field on P induced by $z \in \mathfrak{h}$;

2. $R_h^*\omega = \mathrm{Ad}(h^{-1})\omega$ for $h \in H$, where $R_h\colon P \to P$ is the right action of $h \in H$ on P;

3. the linear map $\omega_p\colon T_p(P) \to \mathfrak{g}$ is a vector space isomorphism for each $p \in P$.

Two Cartan connections of type $(\mathfrak{g}, H)$, denoted by pairs (P_1, ω_1) on M_1 and (P_2, ω_2) on M_2, are *locally equivalent* if there exist two open subsets U_1 of M_1 and U_2 of M_2, and a biholomorphic map $\phi\colon P_1|_{U_1} \to P_2|_{U_2}$ descending to $U_1 \to U_2$ such that $\phi^*\omega_2 = \omega_1$. A Cartan connection of type $(\mathfrak{g}, H)$ is *locally flat* if it is locally equivalent to the Cartan connection on the principal H-bundle $G \to G/H$ with the Maurer-Cartan form on G, where G is a connected Lie group with the Lie algebra $\mathfrak{g}$ and an inclusion $H \subset G$ as a closed subgroup of G.

Let $\mathfrak{m}$ be a fundamental graded Lie algebra. Let G_0 be a Lie subgroup of $G_0(\mathfrak{m})$ and let $\mathfrak{g}_0$ be the Lie algebra corresponding to G_0. Let $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ be the prolongation of the graded Lie algebra of $(\mathfrak{m}, \mathfrak{g}_0)$. Define $\mathfrak{h}(\mathfrak{m}, \mathfrak{g}_0) := \bigoplus_{p \geq 0} \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)_p$, then $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \mathfrak{h}(\mathfrak{m}, \mathfrak{g}_0)$.

Let $\mathfrak{M}$ be a Lie group with $\mathfrak{m}$ as its Lie algebra. The trivial subbundle $\mathfrak{M} \times G_0$ of $\mathfrak{M} \times G_0(\mathfrak{m})$ is the standard G_0-structure on $(\mathfrak{M}, \mathfrak{m})$. From Theorem

3.6.1 of [16], we obtain a principal $H(\mathfrak{m}, G_0)$-bundle P on $\mathfrak{M}$ with a constant structure function $\hat{c}$ of P (which actually zero in this case, i.e., $\hat{c} = 0$), where $H(\mathfrak{m}, G_0)$ is a Lie group with its Lie algebra $\mathfrak{h}(\mathfrak{m}, \mathfrak{g}_0)$, and the Lie subgroup G_0 is embedded in $H(\mathfrak{m}, G_0)$ as a closed subgroup.

We define a subspace of $\mathrm{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0))$ as

$$F^1 \mathrm{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)) := \{\alpha | \alpha(\mathfrak{g}_i \wedge \mathfrak{g}_j) \subset \bigoplus_{p \geq i+j+1} \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)_p \text{ for } i, j < 0\}.$$

The following Theorem 2.3.2 gives us the sufficient condition of the existence of Cartan connections. For more details, see Chapters 2 and 3 of [16] and Theorem 2.7 of [21].

Theorem 2.3.2 (Definition 3.10.1 and Theorem 3.10.1 of [16])**.** *Let (M, F) be a regular filtered manifold of type $\mathfrak{m}$, and let G_0 be a Lie subgroup of $G_0(\mathfrak{m})$ with Lie algebra $\mathfrak{g}_0$. Suppose there exists a subspace W of $F^1 \mathrm{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0))$ such that*

1. *$F^1 \mathrm{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)) = W \oplus \partial F^1 \mathrm{Hom}(\mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0))$,*

2. *W is stable under the action of $H := H(\mathfrak{m}, G_0)$.*

Then for each G_0-structure on (M, F), we can construct a principal H-bundle $P \to M$ associated with the G_0-structure on (M, F) and obtain a Cartan connection (P, ω) of type $(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0), H)$. Two G_0-structures on (M, F) are (locally) *equivalent when the associated Cartan connections of type* $(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0), H)$ *are (locally) equivalent.*

With the following conditions in Theorem 2.3.3, we see that there exists a H-invariant subspace W of $F^1 \operatorname{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0))$ such that

$$F^1 \operatorname{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)) = W \oplus \partial F^1 \operatorname{Hom}(\mathfrak{m}, \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)).$$

Theorem 2.3.3 (Proposition 3.10.1 of [16])**.** *Let* $\mathfrak{m}$ *be a fundamental graded Lie algebra. Let* G_0 *be a Lie subgroup of* $G_0(\mathfrak{m})$*. Let* $\mathfrak{g}_0$ *be the subalgebra of* $\mathfrak{g}_0(\mathfrak{m})$ *corresponding to* G_0*,* $\mathfrak{g} = \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ *the prolongation of* $(\mathfrak{m}, \mathfrak{g}_0)$*, and* $\mathfrak{h} = \bigoplus_{p \geq 0} \mathfrak{g}_p$ *its non-negative part. Assume that the prolongation* $\mathfrak{g}$ *is finite-dimensional and that there exist a positive definite bilinear form*

$$(,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},$$

a mapping $\tau \colon \mathfrak{h} \to \mathfrak{g}$ *and a mapping* $\tau_0 \colon G_0 \to G_0$ *such that*

1. $(\mathfrak{g}_p, \mathfrak{g}_q) = 0$ *for* $p \neq q$
2. $\tau(\mathfrak{g}_p) \subset \mathfrak{g}_{-p}$ *for* $p \geq 0$*, and* $([A, x], y) = (x, [\tau(A), y])$ *for all* $x, y \in \mathfrak{g}$ *and* $A \in \mathfrak{h}$
3. $(ax, y) = (x, \tau_0(a)y)$ *for* $x, y \in \mathfrak{g}$ *and* $a \in G_0$

Then there exists a full functor from the category of G_0-structures of type $\mathfrak{m}$ to the category of Cartan connections of type $(\mathfrak{g}, H)$, where H is the Lie group with Lie algebra $\mathfrak{h}$.

2.4 Examples

Definition 2.4.1. Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a semisimple graded Lie algebra. Then there exists a unique element $E \in \mathfrak{g}_0$ such that

$$\mathfrak{g}_p = \{X \in \mathfrak{g} \mid [E, X] = pX\} \text{ for } p \in \mathbb{Z}.$$

The element E is called the *characteristic element* of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$.

Gradation and the root space decomposition Let $\mathfrak{g}$ be a semisimple Lie algebra with rank$(\mathfrak{g}) = m$. We take a Cartan subalgebra $\mathfrak{h}$ and a set of simple roots $\triangle = \{\alpha_1, \cdots, \alpha_m\}$ of $\mathfrak{g}$. Let Φ be a set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. The root space decomposition of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$ is the root space for $\alpha \in \Phi$.

We define the *characteristic element* E_{α_i} *associated with* α_i as

$$\alpha_j(E_{\alpha_i}) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i, \end{cases}$$

where $\alpha_i \in \triangle$. Let $I \subset \{1, 2, \cdots, m\}$ be a subset of positive integers. The element $E = \sum_{i \in I} E_{\alpha_i}$ is called the *characteristic element* E *associated with* $\{\alpha_i\}_{i \in I} \subset \triangle$. Then we could construct a gradation $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ which is called a *gradation associated with* E as follows;

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0^+} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \\ \mathfrak{g}_k &= \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_\alpha \\ \mathfrak{g}_{-k} &= \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_{-\alpha} \ (k > 0), \end{aligned}$$

where $\Phi_k^+ = \{\alpha \in \Phi^+ | \alpha(E) = k\}$.

For examples, the gradation of $(A_m, \{\alpha_i\})$ is $\mathfrak{sl}_{m+1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and the gradation of $(A_m, \{\alpha_1, \alpha_2\})$ is $\mathfrak{sl}_{m+1} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$.

Lemma 2.4.2 (Lemma 3.8 of [23]). *Let* $\mathfrak{g} = \oplus_p \mathfrak{g}_p$ *be a simple graded Lie algebra and* $\mathfrak{h}$ *be a Cartan subalgebra. Let* $\triangle$ *be a simple root system such that* $E \in \mathfrak{h}$ *and* $\alpha(E) \geq 0$ *for any* $\alpha \in \triangle$. *The graded Lie subalgebra* $\mathfrak{m} = \oplus_{p<0} \mathfrak{g}_p$ *satisfies* $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ *for* $p < -1$, *if and only if* $\alpha(E) = 0$ *or* 1 *for any*

$\alpha \in \triangle$.

Assume the gradation of Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the gradation associated with E_{α_i} for a simple root α_i. Then, the subalgebra $\bigoplus_{p<0} \mathfrak{g}_p$ is a fundamental graded Lie algebra by Lemma 2.4.2.

Lemma 2.4.3 (From Theorem 5.32 of [23])**.** *Let $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ be a simple graded Lie algebra such that the gradation $\bigoplus_p \mathfrak{g}_p$ is associated with E_{α_i} for a simple root α_i, except when $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ is isomorphic with (A_m, α_1) or (C_m, α_1). Then*

1. *if $z \in \mathfrak{g}_p$ for $p \geq 0$, satisfies $[z, \mathfrak{m}] = 0$, then $z = 0$.*
2. *$H^{p,1}(\mathfrak{m}, \mathfrak{g}) = 0$ for $p \geq 1$.*

In the above Lemma, the property $H^{p,1}(\mathfrak{m}, \mathfrak{g}) = 0$ for $p \geq 1$ is obtained from Kostant's harmonic theory, calculating Laplacian.

Theorem 2.4.4 (Theorem 5.32 of [23])**.** *Let $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ be a simple graded Lie algebra such that $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Then $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ is a prolongation of $\mathfrak{m}$ except for the following three cases.*

1. *$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is of depth 1.*
2. *$\mathfrak{g} = \bigoplus_{-2\leq p\leq 2} \mathfrak{g}_p$ is a contact gradation* ($\dim \mathfrak{g}_{-2} = 1$).
3. *$\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ is isomorphic with $(A_m, \{\alpha_m, \alpha_i\})$ $(1 < i < m)$ or $(C_m, \{\alpha_1, \alpha_m\})$.*

Moreover $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ except when $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ is isomorphic with (A_m, α_1) or (C_m, α_1).

Existence of Cartan connection

Proposition 2.4.5. *Let* $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ *be a simple graded Lie algebra associated with a simple root* α_i *except when* $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ *is isomorphic with* (A_m, α_1) *or* (C_m, α_1)*. Let* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *and* (M, D) *be a regular differential system of type* $\mathfrak{m}$*. Then there exists a Cartan connection of type* $(\mathfrak{g}, H)$ *associated with a given* G_0*-structures on a regular differential system* (M, D) *of type* $\mathfrak{m}$ *such that two* G_0*-structures on* (M, D) *are (locally) equivalent when the associated Cartan connections are (locally) equivalent.*

Proof. The subalgebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ of $\mathfrak{g}$ is fundamental by Lemma 2.4.2 and $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ by Theorem 2.4.4. For a given G_0-structures of type $\mathfrak{m}$, we will apply Theorem 2.3.3 to get a Cartan connection of type $(\mathfrak{g}, H)$ where H is a Lie group with Lie algebra $\bigoplus_{p\geq 0} \mathfrak{g}_p$. We need to show that there exist τ, τ_0 and $(\cdot, \cdot)$ satisfying the conditions of Theorem 2.3.3.

Let $B(\cdot, \cdot)$ be the Cartan-Killing form on $\mathfrak{g}$. Let $e_\alpha \in \mathfrak{g}_\alpha$ be a nonzero vector, there exists a vector $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $B(e_\alpha, e_{-\alpha}) = 2/(\alpha, \alpha)$ and

$$[e_\alpha, e_{-\alpha}] := h_\alpha$$

$$[h_\alpha, e_\alpha] = 2e_\alpha$$

$$[h_\alpha, e_{-\alpha}] = -2e_{-\alpha}.$$

Let $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be an automorphism of $\mathfrak{g}$ such that $\tau(e_\alpha) = -e_{-\alpha}$ and

$\tau(h_\alpha) = -h_\alpha$ for any root $\alpha \in \Phi$. Let $\tau_0 \colon G_0 \to G_0$ be an automorphism of G_0 corresponding to $\tau|_{\mathfrak{g}_0} \colon \mathfrak{g}_0 \to \mathfrak{g}_0$. Define a symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ by

$$(X, Y) \colon = -B(X, \tau(Y)).$$

Then the symmetric bilinear form $(\cdot,\cdot)$ is positive definite on $\mathfrak{g}$. Moreover, $(\mathfrak{g}_p, \mathfrak{g}_q) = 0$ for $p \neq q$, $\tau(\mathfrak{g}_p) \subset \mathfrak{g}_{-p}$ for $p \geq 0$, and $([A, x], y) = (x, [\tau(A), y])$ for all $x, y \in \mathfrak{g}$ and $A \in \mathfrak{h}$ and $(ax, y) = (x, \tau_0(a)y)$ for $x, y \in \mathfrak{g}$ and $a \in G_0$. Hence, there exists a Cartan connection of type $(\mathfrak{g}, H)$ associated with a given G_0-structure on a regular differential system (M, D) of type $\mathfrak{m}$ such that two G_0-structures on (M, D) are (locally) equivalent when the associated Cartan connections are (locally) equivalent. $\square$

Chapter 3

Smooth horospherical varieties of Picard number one

3.1 G-varieties

Definition 3.1.1. An *algebraic group* G is an algebraic variety G with the structure of a group, such that the multiplication map

$$\mu\colon G \times G \to G,\ (g,h) \mapsto gh$$

and the inverse map

$$\iota\colon G \to G,\ g \mapsto g^{-1}$$

are morphisms of varieties.

Definition 3.1.2. A *G-variety* is an algebraic variety X equipped with an action of the algebraic group G, where the action

$$\alpha\colon G \times X \to X,\ (g, x) \mapsto g.x$$

is a morphism of varieties.

Given a G-variety X and a point $x \in X$, the orbit $G.x \subset X$ is the set of all $g.x$, where $g \in G$. The isotropy group $G_x \subset G$ is the set of those $g \in G$ such that $g.x = x$.

Example 3.1.3. A torus $T = (\mathbb{C}^*)^n$ with self T-action is a T-variety.

Example 3.1.4. Let $\{e_1, \cdots, e_{2m}\}$ be a basis of the vector space $\mathbb{C}^{2m}$. Let ω be the non-degenerate skew-form given by

$$\omega(e_i, e_j) = \delta_{i,2m+1-j},\ \text{for all } 1 \leq i, j \leq 2m.$$

The symplectic Grassmannian $G_\omega(k, 2m)$ is

$$G_\omega(k, 2m)\colon = \{V | V \subset \mathbb{C}^{2m}, \dim V = k, V \text{ isotropic with respect to } \omega\}.$$

Then $G_\omega(k, 2m) = \mathrm{Sp}_{2m} / P_k$ is a projective Sp_{2m}-variety, where $P_k \subset \mathrm{Sp}_{2m}$ is the parabolic subgroup which stabilizes the subspace $E_k = \langle e_1, \cdots, e_k \rangle$ of

$\mathbb{C}^{2m}$.

3.2 Classifications

Let L be a connected reductive algebraic group.

Definition 3.2.1.

1. An L-homogeneous space X is called a *horospherical homogeneous space* if X is isomorphic to a homogeneous space L/H such that H contains a maximal unipotent subgroup U of L. Then X is isomorphic to an n-dimensional torus $(\mathbb{C}^*)^n$ bundle over a rational homogeneous space. We call X is of *rank n*.

2. An L-variety X is called a *horospherical variety* if X is the embedding of a horospherical homogeneous spaces L/H, that is, X has an open L-orbit which is isomorphic to L/H.

Example 3.2.2. Let B is the set of upper triangular matrices of SL_2. The homogeneous space SL_2/B is a horospherical homogeneous space of rank 0, which is isomorphic to the projective space $\mathbb{P}^1$. B contains U the set of upper triangular matrices of SL_2 such that all diagonal components are the unit 1. The homogeneous space SL_2/U is a horospherical homogeneous space of rank 1, which is isomorphic to the space $\mathbb{C}^2 - \{0\}$. The SL_2-varieties $\mathbb{C}^2$, $\mathbb{P}^2 - \{0\}$, $\mathbb{P}^2$, blow-up of 0 in $\mathbb{C}^2$ and blow-up of 0 in $\mathbb{P}^2$ are horospherical varieties with one open SL_2/U orbit for each. The SL_2-variety $\mathbb{P}^2$ is the unique smooth projective horospherical variety of Picard number one among these SL_2-varieties.

Example 3.2.3. Let $\{e_0, e_1, \cdots, e_{2m}\}$ be a basis of the vector space $\mathbb{C}^{2m+1}$. Let ω be a skew-form given by

$$\omega(e_i, e_j) = \delta_{i,2m+1-j}, \text{ for all } 1 \leq i, j \leq 2m.$$

Then $\omega(e_0, e_j) = 0$ for all $1 \leq j \leq 2m$. The odd symplectic Grassmannian $G_\omega(k, 2m+1)$ is

$$G_\omega(k, 2m+1) := \{V | V \subset \mathbb{C}^{2m+1}, \dim V = k, V \text{ isotropic with respect to } \omega\}.$$

This could be realized as the closure of Sp_{2m}-orbit at $e_1 \wedge \cdots \wedge e_{k-1} \wedge (e_0 + e_k)$;

$$G_\omega(k, 2m+1) = \overline{\mathrm{Sp}_{2m}.[e_1 \wedge \cdots \wedge e_{k-1} \wedge (e_0 + e_k)]} \subset \mathbb{P}(\wedge^k \mathbb{C}^{2m+1}).$$

The isotropic subgroup of Sp_{2m} at $e_1 \wedge \cdots \wedge e_{k-1} \wedge (e_0 + e_k)$ is $P_{k-1} \cap P_k$, where $P_l \subset \mathrm{Sp}_{2m}$ is the parabolic subgroup which stabilizes the subspace $E_l = \langle e_1, \cdots, e_l \rangle$ of $\mathbb{C}^{2m+1}$. The open Sp_{2m}-orbit is isomorphic to a $\mathbb{C}^*$-bundle over $\mathrm{Sp}_{2m}/(P_{k-1} \cap P_k)$, where $\mathbb{C}^*$-action is given by

$$\lambda.[e_1 \wedge \cdots \wedge e_{k-1} \wedge (e_0 + e_k)] = [e_1 \wedge \cdots \wedge e_{k-1} \wedge (e_0 + \lambda e_k)] \in \mathbb{P}(\wedge^k \mathbb{C}^{2m+1}).$$

The variety $G_\omega(k, 2m+1)$ is a horospherical Sp_{2m}-variety of rank 1.

Theorem 3.2.4 (Theorem 0.1 and Theorem 1.11 of [18]). *Let X be a smooth nonhomogeneous projective horospherical L-variety with Picard number one. Let π_i be a i-th fundamental weight of L-representation space. Then X is horospherical of rank one and the automorphism group of X is a connected non-reductive linear algebraic group G, acting with exactly two orbits. Moreover, X is uniquely determined by its two closed L-orbits Y and Z, which are isomorphic to L/P_α and L/P_β, respectively. The variety $X = (L, \alpha, \beta)$ is one of the triples, with the group G, of the following list:*

1. $(B_m, \alpha_{m-1}, \alpha_m)$ *for* $m \geq 3$ *and* $(\mathrm{SO}_{2m+1} \times \mathbb{C}^*) \ltimes V(\pi_m)$
2. $(B_3, \alpha_1, \alpha_3)$ *and* $(\mathrm{SO}_7 \times \mathbb{C}^*) \ltimes V(\pi_3)$
3. $(C_m, \alpha_{i+1}, \alpha_i)$ *for* $m \geq 2$, $i \in \{1, \ldots, m-1\}$ *and* $((\mathrm{Sp}_{2m} \times \mathbb{C}^*)/\{\pm 1\}) \ltimes V(\pi_1)$
4. $(F_4, \alpha_2, \alpha_3)$ *where* α_2 *is a long root and* $(F_4 \times \mathbb{C}^*) \ltimes V(\pi_4)$
5. $(G_2, \alpha_2, \alpha_1)$ *and* $(G_2 \times \mathbb{C}^*) \ltimes V(\pi_1)$

Here, P_{α_i} is the maximal parabolic subgroup of L associated with the simple root α_i, and $V(\pi_i)$ is the irreducible L-representation with the highest weight π_i.

For a given $X = (L, \alpha, \beta)$, there are irreducible L-representations $V(\pi_\alpha)$ and $V(\pi_\beta)$. Let v_α be the highest weight vector of $V(\pi_\alpha)$ and v_β be the highest weight vector of $V(\pi_\beta)$. X is the orbit closure of $L.[v_\alpha+v_\beta] \subset \mathbb{P}(V(\pi_\alpha)\oplus V(\pi_\beta))$ (Section 1.3 of [18]). Hence, X has three orbits under the action of L: one open orbit isomorphic to a torus $\mathbb{C}^*$-bundle over $L/(P_\alpha \cap P_\beta)$, and two closed orbits Y and Z which are isomorphic to L/P_α and L/P_β, respectively.

Let G be the automorphism group of X. According to Lemma 1.15 of [18], the closed L-orbit Z is stable under the G-action. Let $\tilde{X}$ be the blowing-up of X along Z. Then $G = \mathrm{Aut}\,\tilde{X}$. According to the proof of Lemma 1.17 of [18], $\tilde{X}$ is a projective bundle over the L-orbit Y and $U \subset G$ acts on $\tilde{X}$ by translation on the fibers of $\tilde{X} \to Y \cong L/P_\alpha$. And $G = (L \times \mathbb{C}^*)/C \ltimes U$, where U is a L-representation space and C is a centralizer.

Example 3.2.5. From the result of I.Mihai in the paper [14], the automorphism group of the odd symplectic Grassmannian $G_\omega(k, 2m+1)$ is equal to $((\mathrm{Sp}_{2m} \times \mathbb{C}^*)/\{\pm 1\} \ltimes V(\pi_1)$.

Let $G = \mathrm{Aut}(G_\omega(k, 2m+1))$. We also see that G act on $G_\omega(k, 2m+1)$ with two orbits

$$\begin{aligned} X_0 &= \{V \in G_\omega(k, 2m+1) | e_0 \in V\} \\ X_1 &= \{V \in G_\omega(k, 2m+1) | e_0 \notin V\}. \end{aligned}$$

Moreover, X_0 is a closed G-orbit isomorphic to the symplectic Grassmannian $G_\omega(k-1,2m)$, where $G_\omega(k-1,2m)$ is isomorphic to a closed Sp_{2m}-orbit Sp_{2m}/P_{k-1}. The orbit X_1 is an open G-orbit isomorphic to the dual of the tautological bundle over the symplectic Grassmannian $G_\omega(k,2m)$, where the symplectic Grassmannian $G_\omega(k,2m)$ is isomorphic to a closed Sp_{2m}-orbit Sp_{2m}/P_k.

In the list of above Theorem 3.2.4, the horospherical varieties $(C_m, \alpha_{i+1}, \alpha_i)$ are the odd symplectic Grassmannian $G_\omega(i+1, 2m+1)$ for $m \geq 2$, $i \in \{1, \ldots, m-1\}$.

3.3 Lie algebras of the automorphism groups

Proposition 3.3.1. *Let $X = (L, \alpha, \beta)$ be a smooth nonhomogeneous projective horospherical variety of Picard number one. Let $\mathfrak{g}$ be the Lie algebra of the automorphism group of X. Then, we have the followings:*

1. *The Lie algebra $\mathfrak{g}$ is a semidirect product of $(\mathfrak{l} + \mathbb{C})$ and an irreducible $\mathfrak{l}$-representation U where $\mathfrak{l}$ is a semisimple Lie algebra, i.e., $\mathfrak{g} = (\mathfrak{l}+\mathbb{C}) \rhd U$.*

2. *There exist two irreducible L-representations V_α and V_β such that $\mathfrak{l} \subset \mathrm{End}(V_\alpha)$, $\mathbb{C} \simeq \mathbb{C}I \subset \mathrm{End}(V_\beta)$, and $U \subset \mathrm{End}(V_\alpha, V_\beta)$. Hence, we regard $\mathfrak{g}$ as a Lie subalgebra via the inclusion $i\colon \mathfrak{g} \hookrightarrow \mathfrak{gl}(V) = \mathrm{End}\, V$ where*

$V = V_\alpha \oplus V_\beta$. In particular, we could write an element Z of $\mathfrak{g}$ as

$$\mathbf{Z} = \begin{pmatrix} z \in \mathfrak{l} & 0 \\ u \in U & c \in \mathbb{C}I \end{pmatrix} \in \mathrm{End}(V) = \mathfrak{gl}(V).$$

3. *Let $*$ be an operator on $\mathfrak{gl}(V)$ given by $z^* = \bar{z}^t$ for $z \in \mathfrak{gl}(V)$. Let τ be an operator defined by $\tau(z) = -z^*$ for $z \in \mathfrak{gl}(V)$. Let $(.,.)$ be the Cartan-Killing form on $\mathfrak{gl}(V)$. We define an inner product $\{\cdot,\cdot\}$ by $\{z,y\} = (z,y^*) = -(z,\tau(y))$ for $z,y \in \mathfrak{gl}(V)$. Then a restricted inner product $\{\cdot,\cdot\}$ is a positive definite Hermitian inner product on $\mathfrak{g}$.*

Proof. 1. It is from Theorem 1.11 of [18].

2. It is from the proof of Theorem 1.1 of [18]. Since X is the orbit closure of $L.[v_\alpha + v_\beta] \subset \mathbb{P}(V(\pi_\alpha) \oplus V(\pi_\beta))$, let $V_\alpha = V(\pi_\alpha)$ and $V_\beta = V(\pi_\beta)$.

3. If we take two elements $\mathbf{Z_1}$ and $\mathbf{Z_2}$ in $\mathfrak{g}$,

$$\mathbf{Z_1} = \begin{pmatrix} z_1 \in \mathfrak{l} & 0 \\ u_1 \in U & c_1 \in \mathbb{C} \end{pmatrix} \text{ and } \mathbf{Z_2} = \begin{pmatrix} z_2 \in \mathfrak{l} & 0 \\ u_2 \in U & c_2 \in \mathbb{C} \end{pmatrix}.$$

Then

$$\mathbf{Z_1}\mathbf{Z_2^*} = \begin{pmatrix} z_1 z_2^* & z_1 u_2^* \\ u_1 z_2^* & u_1 u_2^* + c_1 c_2^* \end{pmatrix}.$$

By page 271 of the book [11], we see

$$Tr\,\mathrm{ad}\,\mathbf{X}\,\mathrm{ad}\,\mathbf{Y} = 2nTr(\mathbf{X}\mathbf{Y}) - 2Tr(\mathbf{X})Tr(\mathbf{Y})$$

for $\mathbf{X}, \mathbf{Y} \in \mathfrak{gl}(V)$.

Since the semisimple Lie algebra $\mathfrak{l}$ in $\mathfrak{gl}(V)$ is contained in $\mathfrak{sl}(V)$ which is the traceless subalgebra of $\mathfrak{gl}(V)$,

$$\begin{aligned}\{\mathbf{Z_1}, \mathbf{Z_2}\} &= 2nTr(\mathbf{Z_1}\mathbf{Z_2}^*) - 2Tr(\mathbf{Z_1})Tr(\mathbf{Z_2}^*) \\ &= 2nTr(z_1 {z_2}^*) + 2nTr(u_1 u_2^*) + 2n_\alpha n_\beta c_1 \cdot c_2^*\end{aligned}$$

where $n = \dim(V)$, $n_\alpha = \dim(V_\alpha)$ and $n_\beta = \dim(V_\beta)$. Hence, $\{\cdot, \cdot\}$ is a positive definite Hermitian inner product on $\mathfrak{g}$.

□

Remark 3.3.2. We rescale the Hermitian inner product on $\mathfrak{g}$ divide by $2n$ for $n = \dim(V)$ (respectively, rescale the Cartan-Killing form). That is,

$$\{\mathbf{Z_1}, \mathbf{Z_2}\} = Tr(z_1 {z_2}^*) + Tr(u_1 u_2^*) + \frac{n_\alpha n_\beta}{n} c_1 \cdot c_2^*.$$

Then for $E_{ij} \in V_\alpha^* \otimes V_\beta$ which is zero except ij-component or if we write a unit column vector e_i in j-th entry, we see $\{E_{ij}, E_{kl}\} = Tr(E_{ij}, E_{kl}^*) = \delta_{jl} e_i \cdot e_k^* = \delta_{ik}\delta_{jl}$.

3.4 Gradations

Let $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ be a semisimple graded Lie algebra of rank$(\mathfrak{l}) = m$. We choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{l}_0$ and let $\triangle = \{\alpha_1, \cdots, \alpha_m\}$ be the system of simple roots of $\mathfrak{l}$ with respect to $\mathfrak{h}$. We will consider the Lie algebra $\mathfrak{l}$ which has a gradation $\bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ associated with α_i as we see in section 2.4.

Example 3.4.1. Let L be a semisimple Lie group. Let P_{α_i} be a maximal parabolic subgroup of L associated with a simple root α_i. Let $\mathfrak{l}$ be the semisimple Lie algebra of L. The Lie algebra $\mathfrak{l}$ has a gradation $\bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ associated with the root α_i. The tangent space of the homogeneous space L/P_{α_i} at each point is identified with $\bigoplus_{p<0} \mathfrak{l}_p$, which is a fundamental graded Lie subalgebra of $\mathfrak{l}$.

Proposition 3.4.2. *Let X be a smooth nonhomogeneous projective horospherical variety (L, α, β) of Picard number one. Let $G = \mathrm{Aut}(X)$ and let $\mathfrak{g} = (\mathfrak{l} + \mathbb{C}) \rhd U$ be the corresponding Lie algebra. Then we could give a gradation of $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ such that the graded Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is identified with the tangent space of X at a point x where x is in the open G-orbit.*

More precisely, let $\mathfrak{l}_k$ and U_k be eigenspaces that have eigenvalue k under the action of $E_X := E_\alpha$. Then

$$\mathfrak{l} = \bigoplus_{k=-\mu(\mathfrak{l})}^{\mu(\mathfrak{l})} \mathfrak{l}_k \text{ and } U = \bigoplus_{k=-\mu(U)}^{\mu(U)} U_k,$$

where $\mu(\mathfrak{l})$ and $\mu(U)$ are the largest numbers among the nonzero eigenvalues *of the action of E_X on $\mathfrak{l}$ and on U, respectively. Now give the gradation on $\mathfrak{g}$ by shifting the above decompositions as follows:*

$$\begin{aligned}
\mathfrak{g}_{-p} &= \mathfrak{l}_{-p} \textit{ for } p > 1 \\
\mathfrak{g}_{-1} &= \mathfrak{l}_{-1} + U_{-\mu(U)} \\
\mathfrak{g}_{0} &= (\mathfrak{l}_0 + \mathbb{C}) \rhd U_{-\mu(U)+1} \\
\mathfrak{g}_{p} &= \mathfrak{l}_{p} + U_{-\mu(U)+p+1} \textit{ for } p \geq 1.
\end{aligned}$$

Let $\mathfrak{l}_- = \bigoplus_{p<0} \mathfrak{l}_p$ and $U_- = U_{-\mu(U)}$ and let $\mathfrak{m} = \mathfrak{l}_- + U_-$. Also, let $U_{\geq 0} = U_{-\mu(U)+1} + \cdots + U_{\mu(U)}$. Then

$$T_xX \cong \mathfrak{m} = \mathfrak{l}_- + U_- = \bigoplus_{p<0} \mathfrak{g}_p.$$

Lemma 3.4.3. *We decompose the space $\mathfrak{l}$ (and U) to the eigenspaces $\mathfrak{l}_k$ (and U_k) that have eigenvalue k under the action of $E_X := E_\alpha$ as follows:*

1. *$(B_m, \alpha_{m-1}, \alpha_m)$, $m > 2$ where $U = V(\pi_m)$; let $E_X = E_{\alpha_{m-1}}$ and then*

$$\mathfrak{l}_{-2} + \mathfrak{l}_{-1} + \mathfrak{l}_0 + \mathfrak{l}_1 + \mathfrak{l}_2,$$

$$U_{-\frac{m-1}{2}} + U_{-\frac{m-1}{2}+1} + \cdots + U_{\frac{m-1}{2}-1} + U_{\frac{m-1}{2}},$$

and $\dim U_{-\frac{m-1}{2}} = 2$.

2. $(B_3, \alpha_1, \alpha_3)$ *where* $U = V(\pi_3)$*; let* $E_X = E_{\alpha_1}$ *and then*

$$\mathfrak{l}_{-1} + \mathfrak{l}_0 + \mathfrak{l}_1,\ U_{-\frac{1}{2}} + U_{\frac{1}{2}},$$

and $\dim U_{-\frac{1}{2}} = 4$.

3. $(C_m, \alpha_m, \alpha_{m-1})$ *where* $U = V(\pi_1)$*; let* $E_X = E_{\alpha_m}$ *and then*

$$\mathfrak{l}_{-1} + \mathfrak{l}_0 + \mathfrak{l}_1,\ U_{-\frac{1}{2}} + U_{\frac{1}{2}},$$

and $\dim U_{-\frac{1}{2}} = m$.

4. $(C_m, \alpha_{i+1}, \alpha_i)$, $m > 2$, $i = 1, \ldots, m-2$ *where* $U = V(\pi_1)$*; let* $E_X = E_{\alpha_{i+1}}$ *and then*

$$\mathfrak{l}_{-2} + \mathfrak{l}_{-1} + \mathfrak{l}_0 + \mathfrak{l}_1 + \mathfrak{l}_2,\ U_{-1} + U_0 + U_1,$$

and $\dim U_{-1} = i + 1$.

5. $(F_4, \alpha_2, \alpha_3)$ *where* α_2 *is a long root and* $U = V(\pi_4)$*; let* $E_X = E_{\alpha_2}$ *and then*

$$\mathfrak{l}_{-3} + \mathfrak{l}_{-2} + \mathfrak{l}_{-1} + \mathfrak{l}_0 + \mathfrak{l}_1 + \mathfrak{l}_2 + \mathfrak{l}_3,$$

$$U_{-2} + U_{-1} + U_0 + U_1 + U_2,$$

and $\dim U_{-2} = 3$.

6. $(G_2, \alpha_2, \alpha_1)$ *where* $U = V(\pi_1)$*; let* $E_X = E_{\alpha_2}$ *and then*

$$\mathfrak{l}_{-2} + \mathfrak{l}_{-1} + \mathfrak{l}_0 + \mathfrak{l}_1 + \mathfrak{l}_2,\ U_{-1} + U_0 + U_1,$$

and $\dim U_{-1} = 2$.

Furthermore, $\mathfrak{l}_k$ *and* U_k *are irreducible* $\mathfrak{l}_0$*-representations.*

Proof. It was calculated with basis elements from [22] or [17]. □

Proof of Proposition 3.4.2. Let $\tilde{X}$ be the blowing-up of X along Z. Since the open G-orbit of X is isomorphic to the open G-orbit of $\tilde{X}$, it is enough to show that $T_x\tilde{X}$ is identified with $\mathfrak{m} = \mathfrak{l}_- + U_-$ for any x which is in the open G-orbit of $\tilde{X}$. Hence, we assume that x is in the open G-orbit of $\tilde{X}$.

Remember that $G = (L \times \mathbb{C}^*)/C \ltimes U$, which is listed in Theorem 3.2.4, where C acts trivially and $\tilde{X}$ is a projective bundle over the L-orbit Y such that U acts by translation on the fibers. So we choose $E_X = E_\alpha$ as the characteristic element of $\mathfrak{l}$ associated with a root α. The tangent directions of L-action at x are naturally identified with $\mathfrak{l}_-$, and the other tangent directions are contained in U.

According to Lemma 3.4.3, U_k is an irreducible $\mathfrak{l}_0$-module. We see that $[\mathfrak{l}_{-1}, U_k] = U_{k-1}$, and hence if the tangent space of X at x contains U_k, it must contain U_{k-1}. We can easily check that the dimension $\dim X = \dim L/(P_\alpha \cap$

$P_\beta) + 1$ equals $\dim L/P_\alpha + \dim U_-$ in all cases. Hence, the tangent space $T_x X$ at x is identified with $\mathfrak{m} = \mathfrak{l}_- + U_-$. □

Lemma 3.4.4. *Let $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ be a graded Lie algebra given in Proposition 3.4.2.*

1. *$\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental, i.e., $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$.*

2. *if $z \in \mathfrak{g}_p$ for $p \geq 0$, satisfies $[z, \mathfrak{g}_{-1}] = 0$, then $z = 0$.*

3. *for any nonzero vector $u \in U_0$, the dimension of the subspace $\mathfrak{l}_{-1}.u \subset U_-$ is more than or equal to 2.*

Proof. 1. We have a gradation of $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ associated with α, which satisfies $\mathfrak{l}_p = [\mathfrak{l}_{p+1}, \mathfrak{l}_{-1}]$ for $p < -1$, and we have $[U_-, \mathfrak{l}_{-1}] = 0$. Hence, $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is a fundamental graded Lie algebra.

2 and 3. By Lemma 2.4.3, if $z \in \bigoplus_{p\geq 0} \mathfrak{l}_p$, then $[z, \mathfrak{g}_{-1}] = 0$ implies $z = 0$. We want to show that for $z \in U_{\geq 0}$, if $[z, \mathfrak{l}_{-1}] = 0$ then $z = 0$. Also, we want to show that for any nonzero vector $u \in U_0$, the dimension of the subspace $\mathfrak{l}_{-1}.u \subset U_-$ is more than or equal to 2.

The action

$$\begin{aligned} \mathfrak{l}_{-1} \times U_0 &\rightarrow U_- \\ (l, u) &\mapsto [l, u] = l.u \end{aligned}$$

and the actions

$$\begin{aligned} \mathfrak{l}_{-1} \times U_k &\rightarrow U_{k-1} \\ (l, u) &\mapsto [l, u] = l.u \end{aligned}$$

are described as follows up to scalar, which is from weights and weight diagrams([3]) of the irreducible $\mathfrak{l}_0$-representations $\mathfrak{l}_k$ and U_k. Let $R\omega(T)$ is the irreducible representation of type T with the highest weight ω.

1. $(B_m, \alpha_{m-1}, \alpha_m)$, $m > 2$ where $U = V(\pi_m)$; Let $R\pi_1(A_1) = W$ be the standard representation of A_1 with $\dim W = 2$ and $W^* = W$. Let $R\pi_1(A_{m-2}) = Q$ be the standard representation of A_{m-2} with $\dim Q = m-1$. Then

$$\mathfrak{l}_{-1} = R\pi_1(A_{m-2})^* \otimes R2\pi_1(A_1)^* = Q^* \otimes \mathrm{Sym}^2 W^*$$

$$U_- = R\pi_1(A_1)^* = W^*$$

$$U_0 = R\pi_1(A_{m-2}) \otimes R\pi_1(A_1) = Q \otimes W$$

$$U_1 = R\pi_2(A_{m-2}) \otimes R\pi_1(A_1) = \wedge^2 Q \otimes W$$

$$U_2 = R\pi_3(A_{m-2}) \otimes R\pi_1(A_1) = \wedge^3 Q \otimes W$$

$$\vdots$$

For $w_1, w_2 \in W$ such that $W = \langle w_1, w_2 \rangle$ and $q \in Q$, the action $\mathfrak{l}_{-1} \times U_0 \rightarrow$

U_- is given as

$$\begin{array}{rcl}(Q^* \otimes \mathrm{Sym}^2 W^*) \times (Q \otimes W) & \to & W^* \\ (q^* \otimes w_1^* \odot w_2^*, q \otimes w_1) & \mapsto & q^*(q) w_1^* \odot w_2^*(w_1) = w_2^* \\ (q^* \otimes w_1^* \odot w_1^*, q \otimes w_1) & \mapsto & q^*(q) w_1^* \odot w_1^*(w_1) = 2w_1^*.\end{array}$$

For $w_1, w_2 \in W$ such that $W = \langle w_1, w_2 \rangle$ and $q_1, \cdots q_k \in Q$, the action $U_{-1} \times U_{k-1} \to U_{k-2}$ is given as

$$\begin{array}{rcl}(Q^* \otimes \mathrm{Sym}^2 W^*) \times (\wedge^k Q \otimes W) & \to & Q \otimes W^* = Q \otimes W \\ (q_1^* \otimes w_1^* \odot w_2^*, q_1 \wedge \cdots \wedge q_k \otimes w_1) & \mapsto & q_1^*(q_1) q_2 \wedge \cdots \wedge q_k \otimes w_1^* \odot w_2^*(w_1) \\ & = & q_1^*(q_1) q_2 \wedge \cdots \wedge q_k \otimes w_2^* \\ (q_1^* \otimes w_1^* \odot w_1^*, q_1 \wedge \cdots \wedge q_k \otimes w_1) & \mapsto & q_1^*(q_1) q_2 \wedge \cdots \wedge q_k \otimes w_1^* \odot w_1^*(w_1) \\ & = & q_1^*(q_1) q_2 \wedge \cdots \wedge q_k \otimes 2w_1^*.\end{array}$$

2. $(B_3, \alpha_1, \alpha_3)$ where $U = V(\pi_3)$; Let W be the spin representation of B_2 with $\dim W = 4$ and $W = W^*$. Let V be the standard representation of

B_2 with $V^* = V$ and $\dim V = 5$. Then

$$\mathfrak{l}_{-1} = R\pi_1(B_2)^* = V^*$$

$$U_- = R\pi_2(B_2) = W^*$$

$$U_0 = R\pi_2(B_2) = W.$$

The action $\mathfrak{l}_{-1} \times U_0 \to U_-$ is given by the following: for a basis $\{w_1, w_2, w_3, w_4\}$ of W and a basis $\{v_1, v_2, v_3, v_4, v_5\}$ of V, $V^* \times W \to W^*$ is

	v_1^*	v_2^*	v_3^*	v_4^*	v_5^*
w_1	w_4^*	w_3^*	w_2^*	·	·
w_2	w_3^*	·	w_1^*	w_4^*	·
w_3	w_2^*	w_1^*	·	·	w_4^*
w_4	w_1^*	·	·	w_3^*	w_2^*

3. $(C_m, \alpha_m, \alpha_{m-1})$ where $U = V(\pi_1)$; Let W be the standard representation of A_{m-1} with $\dim W = m$. Then

$$\mathfrak{l}_{-1} = R2\pi_1(A_{m-1})^* = \mathrm{Sym}^2 W^*$$

$$U_- = R\pi_1(A_{m-1})^* = W^*$$

$$U_0 = R\pi_1(A_{m-1}) = W.$$

For the orthonormal basis $w_i, w_j, w_k \in W$, we see

$$\begin{aligned}\operatorname{Sym}^2 W^* \times W &\to W^* \\ (w_i^* \odot w_j^*, w_k) &\mapsto (w_i^* \odot w_j^*)(w_k) = \delta_{jk} w_i^* + \delta_{ik} w_j^*.\end{aligned}$$

4. $(C_m, \alpha_{i+1}, \alpha_i)$, $m > 2$, $i = 1, \ldots, m-2$ where $U = V(\pi_1)$; Let W be the standard representation of A_i with $\dim W = i$ and let Q be the standard representation of C_{m-i-1} with $\dim Q = 2m - 2i - 1$. Then

$$\mathfrak{l}_{-1} = R\pi_1(A_i)^* \otimes R\pi_1(C_{m-i-1})^* = W^* \otimes Q^*$$
$$U_- = R\pi_1(A_i)^* = W^*$$
$$U_0 = R\pi_1(C_{m-i-1}) = Q$$
$$U_1 = R\pi_1(A_i) = W.$$

The action $\mathfrak{l}_{-1} \times U_0 \to U_-$ is given as, for $q \in Q$ and $w \in W$,

$$\begin{aligned}(W^* \otimes Q^*) \times Q &\to W^* \\ (w^* \otimes q^*, q) &\mapsto w^* q^*(q).\end{aligned}$$

By the action $U_{-1} \times U_1 \to U_0$ is given as, for $q \in Q$ and $w \in W$,

$$\begin{aligned} (W^* \otimes Q^*) \times W &\to Q^* = Q \\ (w^* \otimes q^*, w) &\mapsto w^*(w)q^*. \end{aligned}$$

5. $(F_4, \alpha_2, \alpha_3)$ where α_2 is a long root and $U = V(\pi_4)$; Let W be the standard representation of A_1 with $\dim W = 2$ and $W^* = W$ and let V be the standard representation of A_2 with $\dim V = 3$. Then

$$U_{-1} = R2\pi_1(A_2) \otimes R\pi_1(A_1) = \mathrm{Sym}^2 V \otimes W$$

$$U_{-} = R\pi_1(A_2) = V$$

$$U_0 = R\pi_1(A_2)^* \otimes R\pi_1(A_1)^* = V^* \otimes W^*$$

$$U_1 = R\,\mathrm{Ad}(A_2)$$

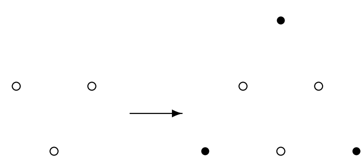
$$U_2 = R\pi_1(A_2) \otimes R\pi_1(A_1) = V \otimes W$$

$$U_3 = R\pi_1(A_2)^* = V^*$$

By the action $l_{-1} \times U_0 \to U_-$, for $v_1, v_2, v_3 \in V$ and $w \in W$,

$$\begin{aligned}
(\mathrm{Sym}^2 V \otimes W) \times (V^* \otimes W^*) &\to V \\
(v_1 \odot v_1 \otimes w, v_1^* \otimes w^*) &\mapsto v_1 \odot v_1(v_1^*) w(w^*) = 2v_1 \\
(v_1 \odot v_2 \otimes w, v_1^* \otimes w^*) &\mapsto v_1 \odot v_2(v_1^*) w(w^*) = v_2 \\
(v_1 \odot v_3 \otimes w, v_1^* \otimes w^*) &\mapsto v_1 \odot v_3(v_1^*) w(w^*) = v_3.
\end{aligned}$$

We have the embedding of the weight diagram of V^* to the weight diagram of $\mathrm{Sym}^2 V$:



Under the identification of V^* and a subspace of $\mathrm{Sym}^2 V$ by the embedding of the weight diagram of V^* to the weight diagram of $\mathrm{Sym}^2 V$, for $v_1, v_2, v_3 \in V$,

$$\begin{aligned}
v_1 \odot v_2 &= v_3^* \\
v_2 \odot v_3 &= v_1^* \\
v_3 \odot v_1 &= v_2^*.
\end{aligned}$$

The action $\mathfrak{l}_{-1} \times U_1 \to U_0$ is

$$(\mathrm{Sym}^2 V \otimes W) \times R\,\mathrm{Ad}(A_2) \quad \to \quad V^* \otimes W,$$

which reduces to

$$\mathrm{Sym}^2 V \times R\,\mathrm{Ad}(A_2) \quad \to \quad V^*.$$

The action $\mathfrak{l}_{-1} \times U_2 \to U_1$ is

$$(\mathrm{Sym}^2 V \otimes W) \times (V \otimes W^*) \quad \to \quad R\,\mathrm{Ad}(A_2),$$

which reduces to

$$\mathrm{Sym}^2 V \times V \quad \to \quad R\,\mathrm{Ad}(A_2).$$

The reduced action of $\mathfrak{l}_{-1} \times U_2 \to U_1$ is given as, for $i \neq j \neq k$,

$$\begin{aligned} \mathrm{Sym}^2 V \times V &\to R\,\mathrm{Ad}(A_2) \subset V^* \otimes V \\ (v_i \odot v_i, v_i) &\mapsto 0 \\ (v_i \odot v_j, v_i) &\mapsto \delta_{ijk} v_k^* \otimes v_i. \end{aligned}$$

By the action $\mathfrak{l}_{-1} \times U_3 \to U_2$, for $v_1, v_2, v_3 \in V$ and $w \in W$,

$$\begin{aligned}
(\mathrm{Sym}^2 V \otimes W) \times V^* &\to V \otimes W \\
(v_1 \odot v_1 \otimes w, v_1^*) &\mapsto v_1 \odot v_1(v_1^*) \otimes w = v_1 \otimes w \\
(v_1 \odot v_2 \otimes w, v_1^*) &\mapsto v_1 \odot v_2(v_1^*) \otimes w = v_2 \otimes w \\
(v_1 \odot v_3 \otimes w, v_1^*) &\mapsto v_1 \odot v_3(v_1^*) \otimes w = v_3 \otimes w.
\end{aligned}$$

6. $(G_2, \alpha_2, \alpha_1)$ where $U = V(\pi_1)$; Let W be the standard representation of A_1 with $\dim W = 2$ and $W^* = W$. Then

$$\mathfrak{l}_{-1} = R3\pi_1(A_1)^* = \mathrm{Sym}^3 W^*$$

$$U_- = R\pi_1(A_1)^* = W^*$$

$$U_0 = R2\pi_1(A_1) = \mathrm{Sym}^2 W$$

$$U_1 = R\pi_1(A_1) = W.$$

By the action $\mathfrak{l}_{-1} \times U_0 \to U_-$, for $w_1, w_2 \in W$,

$$\begin{aligned}
\operatorname{Sym}^3 W^* \times \operatorname{Sym}^2 W &\to W^* \\
(w_1^* \odot w_2^* \odot w_1^*, w_1 \otimes w_2) &\mapsto w_1^* \odot w_2^* \odot w_1^*(w_1 \otimes w_2) = w_1^* \\
(w_1^* \odot w_2^* \odot w_2^*, w_1 \otimes w_2) &\mapsto w_1^* \odot w_2^* \odot w_2^*(w_1 \otimes w_2) = w_2^* \\
(w_1^* \odot w_1^* \odot w_1^*, w_1 \otimes w_1) &\mapsto w_1^* \odot w_1^* \odot w_1^*(w_1 \otimes w_1) = w_1^* \\
(w_1^* \odot w_1^* \odot w_2^*, w_1 \otimes w_1) &\mapsto w_1^* \odot w_1^* \odot w_2^*(w_1 \otimes w_1) = w_2^*.
\end{aligned}$$

By the action $\mathfrak{l}_{-1} \times U_1 \to U_0$, for $w_1, w_2 \in W$,

$$\begin{aligned}
\operatorname{Sym}^3 W^* \times W &\to \operatorname{Sym}^2 W^* = \operatorname{Sym}^2 W \\
(w_1^* \odot w_2^* \odot w_1^*, w_1) &\mapsto w_1^* \odot w_2^* \odot w_1^*(w_1) = w_1^* \odot w_2^* \\
(w_1^* \odot w_2^* \odot w_2^*, w_1) &\mapsto w_1^* \odot w_2^* \odot w_2^*(w_1) = w_2^* \odot w_2^* \\
(w_1^* \odot w_1^* \odot w_1^*, w_1) &\mapsto w_1^* \odot w_1^* \odot w_1^*(w_1) = w_1^* \odot w_1^*.
\end{aligned}$$

Hence, for $z \in U_{\geq 0}$, if $[z, \mathfrak{l}_{-1}] = 0$ then $z = 0$. And for a nonzero vector $u \in U_0$, the dimension of the subspace $\mathfrak{l}_{-1}.u \subset U_-$ is more than or equal to 2. □

3.5 Varieties of minimal rational tangents

The two papers [8] and [9] are main references to get varieties of minimal rational tangents of smooth nonhomogeneous projective horospherical varieties of Picard number one. We see the paper [6] for basic concepts of varieties of minimal rational tangents.

Lemma 3.5.1. *Let X be a smooth nonhomogeneous projective horospherical variety (L, α, β) of Picard number one. Let X_o be the open* $\mathrm{Aut}(X)$*-orbit.*

1. $(B_m, \alpha_{m-1}, \alpha_m)$*,* $m > 2$ *where* $U = V(\pi_m)$*; Let* $R\pi_1(A_1) = W$ *be the standard representation of* A_1 *with* $\dim W = 2$ *and* $W^* = W$*. Let* $R\pi_1(A_{m-2}) = Q$ *be the standard representation of* A_{m-2} *with* $\dim Q = m-1$*. Then*

$$\mathfrak{l}_{-2} = R\pi_2(A_{m-2})^* = \wedge^2 Q^*$$

$$\mathfrak{l}_{-1} = R\pi_1(A_{m-2})^* \otimes R2\pi_1(A_1)^* = Q^* \otimes \mathrm{Sym}^2 W^*$$

$$U_- = R\pi_1(A_1)^* = W^*.$$

The Chern number of TX_o *is* $2 + m$*.*

2. $(B_3, \alpha_1, \alpha_3)$ *where* $U = V(\pi_3)$*; Let* W *be the spin representation of* B_2 *with* $\dim W = 4$ *and* $W = W^*$*. Let* V *be the standard representation of*

B_2 with $V^* = V$ and $\dim V = 5$. Then

$$\mathfrak{l}_{-1} = R\pi_1(B_2)^* = V^*$$
$$U_- = R\pi_2(B_2) = W^*.$$

The Chern number of TX_o is $2 + 5$.

3. *$(C_m, \alpha_m, \alpha_{m-1})$ where $U = V(\pi_1)$; Let W be the standard representation of A_{m-1} with $\dim W = m$. Then*

$$\mathfrak{l}_{-1} = R2\pi_1(A_{m-1})^* = \operatorname{Sym}^2 W^*$$
$$U_- = R\pi_1(A_{m-1})^* = W^*.$$

The Chern number of TX_o is $2 + m$.

4. *$(C_m, \alpha_{i+1}, \alpha_i)$, $m > 2$, $i = 1, \ldots, m-2$ where $U = V(\pi_1)$; Let W be the standard representation of A_i with $\dim W = i$ and let Q be the standard representation of C_{m-i-1} with $\dim Q = 2m - 2i - 1$. Then*

$$\mathfrak{l}_{-2} = R2\pi_1(A_i)^* = \operatorname{Sym}^2 W^*$$
$$\mathfrak{l}_{-1} = R\pi_1(A_i)^* \otimes R\pi_1(C_{m-i-1})^* = W^* \otimes Q^*$$
$$U_{-1} = R\pi_1(A_i) = W^*.$$

The Chern number of TX_o *is* $2 + 2m - (i + 1)$.

5. $(F_4, \alpha_2, \alpha_3)$ *where* α_2 *is a long root and* $U = V(\pi_4)$*; Let* W *be the standard representation of* A_1 *with* $\dim W = 2$ *and* $W^* = W$ *and let* V *be the standard representation of* A_2 *with* $\dim V = 3$*. Then*

$$\mathfrak{l}_{-3} = R\pi_1(A_1)^* = W$$

$$\mathfrak{l}_{-2} = R2\pi_1(A_2)^* = \operatorname{Sym}^2 V$$

$$\mathfrak{l}_{-1} = R2\pi_1(A_2)^* \otimes R\pi_1(A_1)^* = \operatorname{Sym}^2 V \otimes W$$

$$U_- = R\pi_1(A_2) = V^*.$$

The Chern number of TX_o *is* $2 + 4$.

6. $(G_2, \alpha_2, \alpha_1)$ *where* $U = V(\pi_1)$*; Let* W *be the standard representation of* A_1 *with* $\dim W = 2$ *and* $W^* = W$*. Then*

$$\mathfrak{l}_{-2} = R\pi_2(A_1)^* = \wedge^2 W^*$$

$$\mathfrak{l}_{-1} = R3\pi_1(A_1)^* = \operatorname{Sym}^3 W^*$$

$$U_- = R\pi_1(A_1) = W.$$

The Chern number of TX_o *is* $2 + 2$.

Proof. For a smooth nonhomogeneous projective horospherical variety $X = (L, \alpha, \beta)$, we can see $\mathfrak{g}_- = \mathfrak{l}_- + U_-$ as $\mathfrak{l}_0$-representations as above decompositions (Proposition 3.4.2). Let $\mathfrak{h}(\mathfrak{l})$ be a Cartan subalgebra of $\mathfrak{l}$ and let $\{\gamma\}$ be a set of weights and roots of $\mathfrak{g}_-$. We can calculate signature $<\gamma, \alpha>= \gamma(H_\alpha)$, where $H_\alpha \in \mathfrak{h}(\mathfrak{l})$ is the coroot of the simple root α. Let $\mathfrak{s}_\alpha \subset \mathfrak{g}$ be the subalgebra isomorphic to $\mathfrak{sl}_2$ such that $\mathfrak{s}_\alpha \cap \mathfrak{h}(\mathfrak{l}) = \mathbb{C}H_\alpha$. Let $S_\alpha \subset G$ be the Lie subgroup corresponding to Lie algebra $\mathfrak{s}_\alpha$. The orbit of $o \in G/H \subset X$ under the subgroup $S_\alpha \subset G$ action is a rational curve C_α. By Grothendieck theorem, the Chern number of TX restricted to C_α is the sum of signatures $\sum_\gamma <\gamma, \alpha>$. □

Proposition 3.5.2. *Let X be a smooth nonhomogeneous projective horospherical variety (L, α, β) of Picard number one. Then the varieties of minimal rational tangents (VMRT) $\mathcal{C}_o \subset \mathbb{P}(T_oX)$ at $o \in X$ are followings:*

1. *$(B_m, \alpha_{m-1}, \alpha_m)$, $m > 2$ where $U = V(\pi_m)$; Let $R\pi_1(A_1) = W$ be the standard representation of A_1 with $\dim W = 2$ and $R\pi_1(A_{m-2}) = Q$ be the standard representation of A_{m-2} with $\dim Q = m-1$. Then the variety of minimal rational tangents at $o \in X$ is the closure*

$$\overline{\mathfrak{l}_0.(q \otimes w^2 + w)} \subset \mathbb{P}(Q \otimes \mathrm{Sym}^2 W + W) = \mathbb{P}(\mathfrak{g}_{-1})$$

   of $\mathfrak{l}_0$-orbit of $q \otimes w^2 + w$, where q is a highest weight vector of Q and w is a highest weight vector of W. The dimension of VMRT is m.

2. *$(B_3, \alpha_1, \alpha_3)$ where $U = V(\pi_3)$; Let W be the spin representation of B_2*

with $\dim W = 4$ *and let* V *be the standard representation with* $\dim V = 5$. *Then the variety of minimal rational tangents at* $o \in X$ *is the closure*

$$\overline{\mathfrak{l}_0.(v+w)} \subset \mathbb{P}(V+W) = \mathbb{P}(\mathfrak{g}_{-1}),$$

of $\mathfrak{l}_0$*-orbit of* $v+w$*, where* v *is a highest weight vector of* V *and* w *is a highest weight vector of* W*. The dimension of VMRT is* 5.

3. $(C_m, \alpha_m, \alpha_{m-1})$ *where* $U = V(\pi_1)$*; Let* W *be the standard representation of* A_{m-1} *with* $\dim W = m$*. Then the variety of minimal rational tangents at* $o \in X$ *is the closure*

$$\overline{\mathfrak{l}_0.(w^2+w)} \subset \mathbb{P}(\operatorname{Sym}^2 W + W) = \mathbb{P}(\mathfrak{g}_{-1}),$$

of $\mathfrak{l}_0$*-orbit* w^2+w*, where* w *is the highest weight vector of* W*. The dimension of VMRT is* m.

4. $(C_m, \alpha_{i+1}, \alpha_i)$*,* $m > 2$*,* $i = 1, \ldots, m-2$ *where* $U = V(\pi_1)$*; Let* W *be the standard representation of* A_i *with* $\dim W = i+1$ *and let* Q *be the standard representation of* C_{m-i-1} *with* $\dim Q = 2m-2i-2$*. Then the variety of minimal rational tangents at* $o \in X$ *is the closure*

$$\overline{\mathfrak{l}_0.(w\otimes q + w^2 + w)} \subset \mathbb{P}(W\otimes Q + \operatorname{Sym}^2 W + W) = \mathbb{P}(\mathfrak{g}_{-1} + \mathfrak{g}_{-2}),$$

of $\mathfrak{l}_0$-orbit of $w \otimes q + w^2 + w$, where q is a highest weight vector of Q and w is a highest weight vector of W. The dimension of VMRT is $2m - (i+1)$.

5. *$(F_4, \alpha_2, \alpha_3)$ where α_2 is a long root and $U = V(\pi_4)$; Let W be the standard representation of A_1 with $\dim W = 2$ and let V be the standard representation of A_2 with $\dim V = 3$. Then the variety of minimal rational tangents at $o \in X$ is the closure*

$$\overline{\mathfrak{l}_0.(v^2 \otimes w + v)} \subset \mathbb{P}(\mathrm{Sym}^2 V \otimes W + V) = \mathbb{P}(\mathfrak{g}_{-1}),$$

of $\mathfrak{l}_0$-orbit $v^2 \otimes w + v$, where v is a highest weight vector of V and w is a highest weight vector of W. The dimension of VMRT is 4.

6. *$(G_2, \alpha_2, \alpha_1)$ where $U = V(\pi_1)$; Let W be the standard representation of A_1 with $\dim W = 2$. Then the variety of minimal rational tangents at $o \in X$ is the closure*

$$\overline{\mathfrak{l}_0.(w^3 + w)} \subset \mathbb{P}(\mathrm{Sym}^3 W + W) = \mathbb{P}(\mathfrak{g}_{-1}),$$

of $\mathfrak{l}_0$-orbit $w^3 + w$, where w is a highest weight vector of W. The dimension of VMRT is 2.

To prove this proposition, we need a lemma that follows;

Lemma 3.5.3 (Lemma 1.4 of [9])**.** *Let $C \subset M$ be a free rational curve on complex manifold. Suppose there exists a point $P \in C$ and an m-dimensional family of deformations of C fixing P such that the members of the family are all distinct rational curves. Then $-K_M \cdot C \geq 2 + m$.*

Proof of Proposition 3.5.2. By Lemma 3.5.3, the dimension of the variety of minimal rational tangents at $o \in X$ is equal or less then the Chern number minus 2. We have $[(\mathfrak{l}_0 + \mathbb{C}) \rhd U_0, U_-] \subset U_-$, $[(\mathfrak{l}_0 + \mathbb{C}) \rhd U_0, \mathfrak{l}_{-1}] \subset \mathfrak{l}_{-1} + U_-$, $[(\mathfrak{l}_0 + \mathbb{C}) \rhd U_0, \mathfrak{l}_{-2}] \subset \mathfrak{l}_{-2}$ and $[\mathfrak{l}_1 + U_1, \mathfrak{l}_{-2}] \subset \mathfrak{l}_{-1} + U_-$. Hence, if there exist a vector $v \in U_-$ such that dimension of the $\mathfrak{l}_0$-orbit closure $\overline{\mathfrak{l}_0.v} \subset \mathbb{P}(U_-)$ is the Chern number minus 2, then the variety of minimal rational tangents is that $\mathfrak{l}_0$-orbit closure $\overline{\mathfrak{l}_0.v} \subset \mathbb{P}(U_-)$. Suppose not, and if there exist vectors $v \in U_-$ and $w \in \mathfrak{l}_{-1}$ such that dimension of the $\mathfrak{l}_0$-orbit closure $\overline{\mathfrak{l}_0.(v+w)} \subset \mathbb{P}(\mathfrak{l}_{-1}+U_-)$ is the Chern number minus 2, then the variety of minimal rational tangents is that $\mathfrak{l}_0$-orbit closure $\overline{\mathfrak{l}_0.(v+w)} \subset \mathbb{P}(\mathfrak{l}_{-1}+U_-)$. Continuing this until getting the dimension of $\mathfrak{l}_0$-orbit closure is Chern number minus 2. The conclusion follows from Lemma 3.5.1. □

Chapter 4

Existence of Cartan connections

4.1 Prolongations

Let X be a smooth nonhomogeneous projective horospherical variety (L, α, β) of Picard number one. Let $G = \mathrm{Aut}(X)$ and let $\mathfrak{g} = (\mathfrak{l} + \mathbb{C}) \rhd U$ be the corresponding Lie algebra. By Proposition 3.4.2, we could give a gradation of $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ such that the graded Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is identified with the tangent space of X at a point x where x is in the open G-orbit.

Proposition 4.1.1. *Let* $\mathfrak{g} = (\mathfrak{l} + \mathbb{C}) \rhd U$ *and* $\mathfrak{m} = \mathfrak{l}_- + U_-$. *Assume that*

1. *For* $z \in \mathfrak{l}_{\geq 0} + U_{\geq 0}$, *if* $[\mathfrak{l}_-, z] = 0$, *than* $z = 0$.

2. *For any vector* $u \in U_0$, *if the dimension of the subspace* $\mathfrak{l}_{-1}.u \subset U_-$ *is less then or equal to* 1, *then* $u = 0$.

If $H^{p,1}(\mathfrak{l}_-, \mathfrak{l}) = 0$ *and* $H^{p,1}(\mathfrak{l}_-, U) = 0$ *for* $p > 0$*, then* $H^{p,1}(\mathfrak{m}, \mathfrak{g}) = 0$ *for* $p > 0$*.*

Proof. Let $p \in \mathfrak{m}' \otimes \mathfrak{g}$ be such that $\partial p = 0$. Write $p = p_{\mathfrak{l}} + p_{\mathbb{C}} + p_U$, where $p_{\mathfrak{l}} \in \mathfrak{m}' \otimes \mathfrak{l}$, $p_{\mathbb{C}} \in \mathfrak{m}' \otimes \mathbb{C}$ and $p_U \in \mathfrak{m}' \otimes U$. Then for any $X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-} \in \mathfrak{l}_-$ and $X^{U_-}, Y^{U_-} \in U_-$, we have

$$\begin{aligned}
0 &= \partial p(X^{\mathfrak{l}_-} + X^{U_-}, Y^{\mathfrak{l}_-} + Y^{U_-}) \\
&= (X^{\mathfrak{l}_-} + X^{U_-})p(Y^{\mathfrak{l}_-} + Y^{U_-}) - (Y^{\mathfrak{l}_-} + Y^{U_-})p(X^{\mathfrak{l}_-} + X^{U_-}) - p([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}]) \\
&\qquad \text{because } [\mathfrak{l}_- + U_-, U_-] = 0 \\
&= \left\{X^{\mathfrak{l}_-} p_{\mathfrak{l}+\mathbb{C}}(Y^{\mathfrak{l}_-} + Y^{U_-}) - Y^{\mathfrak{l}_-} p_{\mathfrak{l}+\mathbb{C}}(X^{\mathfrak{l}_-} + X^{U_-}) - p_{\mathfrak{l}+\mathbb{C}}([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}])\right\} \\
&\quad + \left\{X^{U_-} p_{\mathfrak{l}+\mathbb{C}}(Y^{\mathfrak{l}_-} + Y^{U_-}) - Y^{U_-} p_{\mathfrak{l}+\mathbb{C}}(X^{\mathfrak{l}_-} + X^{U_-})\right. \\
&\quad \left. + X^{\mathfrak{l}_-} p_U(Y^{\mathfrak{l}_-} + Y^{U_-}) - Y^{\mathfrak{l}_-} p_U(X^{\mathfrak{l}_-} + X^{U_-}) - p_U([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}])\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
0 &\overset{(*)}{=} X^{\mathfrak{l}_-} p_{\mathfrak{l}}(Y^{\mathfrak{l}_-} + Y^{U_-}) - Y^{\mathfrak{l}_-} p_{\mathfrak{l}}(X^{\mathfrak{l}_-} + X^{U_-}) - p_{\mathfrak{l}}([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}]) \\
0 &\overset{(\diamondsuit)}{=} X^{U_-} p_{\mathfrak{l}}(Y^{\mathfrak{l}_-} + Y^{U_-}) - Y^{U_-} p_{\mathfrak{l}}(X^{\mathfrak{l}_-} + X^{U_-}) \\
&\quad + X^{U_-} p_{\mathbb{C}}(Y^{\mathfrak{l}_-} + Y^{U_-}) - Y^{U_-} p_{\mathbb{C}}(X^{\mathfrak{l}_-} + X^{U_-}) \\
&\quad + X^{\mathfrak{l}_-} p_U(Y^{\mathfrak{l}_-} + Y^{U_-}) - Y^{\mathfrak{l}_-} p_U(X^{\mathfrak{l}_-} + X^{U_-}) - p_U([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}])
\end{aligned}$$

and

$$0 \quad = \quad p_{\mathbb{C}}([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}]). \tag{4.1.2}$$

Put $X^{U_-} = Y^{U_-} = 0$ into (*) to get

$$X^{\mathfrak{l}_-} p_{\mathfrak{l}}(Y^{\mathfrak{l}_-}) - Y^{\mathfrak{l}_-} p_{\mathfrak{l}}(X^{\mathfrak{l}_-}) - p_{\mathfrak{l}}([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}]) = 0. \tag{4.1.3}$$

Put $Y^{\mathfrak{l}_-} = 0$ into (*) to get

$$X^{\mathfrak{l}_-} p_{\mathfrak{l}}(Y^{U_-}) = 0. \tag{4.1.4}$$

Put $Y^{\mathfrak{l}_-} = 0$ and $X^{U_-} = 0$ in ($\diamondsuit$) to get

$$X^{\mathfrak{l}_-} p_U(Y^{U_-}) - Y^{U_-} p_{\mathfrak{l}}(X^{\mathfrak{l}_-}) - Y^{U_-} p_{\mathbb{C}}(X^{\mathfrak{l}_-}) = 0. \tag{4.1.5}$$

Put $Y^{\mathfrak{l}_-} = 0$ and $X^{\mathfrak{l}_-} = 0$ in ($\diamondsuit$) to get

$$X^{U_-} p_{\mathfrak{l}}(Y^{U_-}) - Y^{U_-} p_{\mathfrak{l}}(X^{U_-}) + X^{U_-} p_{\mathbb{C}}(Y^{U_-}) - Y^{U_-} p_{\mathbb{C}}(X^{U_-}) = 0. \tag{4.1.6}$$

Put $X^{U_-} = Y^{U_-} = 0$ in ($\diamondsuit$) to get

$$X^{\mathfrak{l}_-} p_U(Y^{\mathfrak{l}_-}) - Y^{\mathfrak{l}_-} p_U(X^{\mathfrak{l}_-}) - p_U([X^{\mathfrak{l}_-}, Y^{\mathfrak{l}_-}]) = 0. \tag{4.1.7}$$

In (4.1.4), since $X^{\mathfrak{l}_-} \in \mathfrak{l}_-$ is arbitrary, by the assumption 1, we have

$$p_{\mathfrak{l}_{\geq 0}}(Y^{U_-}) = 0, \tag{4.1.8}$$

where $p_{\mathfrak{l}_{\geq 0}} \in \bigoplus_{p \geq 0} \mathfrak{l}_p$.

In (4.1.6), by (4.1.8), we see

$$X^{U_-} p_{\mathbb{C}}(Y^{U_-}) - Y^{U_-} p_{\mathbb{C}}(X^{U_-}) = 0. \tag{4.1.9}$$

This is also valid for the two linearly independent vector X^{U_-} and Y^{U_-}. Hence,

$$p_{\mathbb{C}}(X^{U_-}) = 0. \tag{4.1.10}$$

Let $\partial_{\mathfrak{l}_-}$ denote the restriction of ∂ to $\wedge \mathfrak{l}'_- \otimes \mathfrak{g}$. Then, by (4.1.3) and (4.1.7), we have $\partial_{\mathfrak{l}_-}(p_{\mathfrak{l}} + p_U)|_{\mathfrak{l}_-} = 0$.

By hypothesis, $H^{p,1}(\mathfrak{l}_-, \mathfrak{l}) = 0$ and $H^{p,1}(\mathfrak{l}_-, U) = 0$. Then there is $q = q_{\mathfrak{l}} + q_U$, where $q_{\mathfrak{l}} \in \mathfrak{l}$ and $q_U \in U$ such that

$$\partial_{\mathfrak{l}_-} q = (p_{\mathfrak{l}} + p_U)|_{\mathfrak{l}_-}. \tag{4.1.11}$$

We will show that $\partial q = p_{\mathfrak{l}} + p_U + p_{\mathbb{C}}$. By (4.1.11), (4.1.2) and (4.1.10), it suffices to show that $\partial q(X^{U_-}) = (p_{\mathfrak{l}} + p_U)(X^{U_-})$ for all $X^{U_-} \in U_-$ and

$p_{\mathbb{C}}(X^{\mathfrak{l}_-}) = 0$ for all $X^{\mathfrak{l}_-} \in \mathfrak{l}_{-1}$.

In (4.1.5), we have

$$\begin{aligned} Y^{U_-} p_{\mathbb{C}}(X^{\mathfrak{l}_-}) &= X^{\mathfrak{l}_-} p_U(Y^{U_-}) - Y^{U_-} p_{\mathfrak{l}}(X^{\mathfrak{l}_-}) \\ &= X^{\mathfrak{l}_-} p_U(Y^{U_-}) - Y^{U_-} X^{\mathfrak{l}_-} q_{\mathfrak{l}} \text{ because } p_{\mathfrak{l}}(X^{\mathfrak{l}_-}) = \partial q_{\mathfrak{l}}(X^{\mathfrak{l}_-}) \\ &= X^{\mathfrak{l}_-} p_U(Y^{U_-}) - X^{\mathfrak{l}_-} Y^{U_-} q_{\mathfrak{l}} \text{ because } [\mathfrak{l}_- + U_-, U_-] = 0 \\ &= X^{\mathfrak{l}_-}(p_U(Y^{U_-}) - Y^{U_-} q_{\mathfrak{l}}). \end{aligned}$$

For $X^{\mathfrak{l}_-} \in \{X^{\mathfrak{l}_-} \in \mathfrak{l}_{-1} | p_{\mathbb{C}}(X^{\mathfrak{l}_-}) = 0\}$,

$$0 = X^{\mathfrak{l}_-}(p_U(Y^{U_-}) - Y^{U_-} q_{\mathfrak{l}}). \tag{4.1.12}$$

Since $\{X^{\mathfrak{l}_-} \in \mathfrak{l}_{-1} | p_{\mathbb{C}}(X^{\mathfrak{l}_-}) = 0\} \subset \mathfrak{l}_{-1}$ is a hyperplane or $\mathfrak{l}_-$, by the assumption 2,

$$0 = (p_U(Y^{U_-}) - Y^{U_-} q_{\mathfrak{l}})|_{U_0}. \tag{4.1.13}$$

Hence, for any $X^{\mathfrak{l}_-} \in \mathfrak{l}_{-1}$, we see

$$p_{\mathbb{C}}(X^{\mathfrak{l}_-}) = 0. \tag{4.1.14}$$

Equation (4.1.5) becomes

$$X^{\mathfrak{l}_-} p_U(Y^{U_-}) = Y^{U_-} p_{\mathfrak{l}}(X^{\mathfrak{l}_-}) = Y^{U_-} \partial q_{\mathfrak{l}}(X^{\mathfrak{l}_-}). \tag{4.1.15}$$

Write $\partial q = (\partial q)_{\mathfrak{l}} + (\partial q)_{\mathbb{C}} + (\partial q)_U$ where $(\partial q)_{\mathfrak{l}} \in \mathfrak{l}$, $(\partial q)_{\mathbb{C}} \in \mathbb{C}$ and $(\partial q)_U \in U$. Then

$$\begin{aligned} \partial q(X^{\mathfrak{l}_-}) &= X^{\mathfrak{l}_-} q &= X^{\mathfrak{l}_-} q_{\mathfrak{l}} + X^{\mathfrak{l}_-} q_{U_-} \\ \partial q(X^{U_-}) &= X^{U_-} q &= X^{U_-} q_{\mathfrak{l}}. \end{aligned}$$

Thus

$$\begin{aligned} (\partial q)_{\mathbb{C}} &= 0 \\ (\partial q)_{\mathfrak{l}}(X^{\mathfrak{l}_-}) &= X^{\mathfrak{l}_-} q_{\mathfrak{l}} \\ (\partial q)_{\mathfrak{l}}(X^{U_-}) &= 0 \\ (\partial q)_U(X^{\mathfrak{l}_-}) &= X^{\mathfrak{l}_-} q_U \\ (\partial q)_U(X^{U_-}) &= X^{U_-} q_{\mathfrak{l}}. \end{aligned}$$

In particular, this implies

$$\begin{aligned} X^{\mathfrak{l}_-}(\partial q)_U(X^{U_-}) &= X^{\mathfrak{l}_-}X^{U_-}(q_{\mathfrak{l}}+q_{\mathbb{C}}) \\ &= X^{U_-}X^{\mathfrak{l}_-}(q_{\mathfrak{l}}+q_{\mathbb{C}}) \text{ because } [\mathfrak{l}_-,U_-]=0 \\ &= X^{U_-}X^{\mathfrak{l}_-}q_{\mathfrak{l}} \\ &= X^{U_-}(\partial q)_{\mathfrak{l}}(X^{\mathfrak{l}_-}). \end{aligned}$$

Hence, by (4.1.15),

$$X^{\mathfrak{l}_-}(\partial q)_U(X^{U_-}) = X^{\mathfrak{l}_-}p_U(X^{U_-}).$$

Since $X^{\mathfrak{l}_-} \in \mathfrak{l}_-$ is arbitrary, by the assumption 1, we have

$$(\partial q)_{U_{\geq 0}}(X^{U_-}) = p_{U_{\geq 0}}(X^{U_-}), \tag{4.1.16}$$

where $(\partial q)_{U_{\geq 0}} \in \bigoplus_{p\geq 0} U_p$ and $p_{U_{\geq 0}} \in \bigoplus_{p\geq 0} U_p$.

It follows that $\partial q_{\geq 0} = p_{\geq 0} \in \bigoplus_{p\geq 0} \mathfrak{g}_p$. Therefore, $H^{p,1}(\mathfrak{m},\mathfrak{g}) \subset \mathfrak{m}^*_{-1} \otimes \mathfrak{g}_{p-1}$ vanishes for any $p > 0$. $\square$

Lemma 4.1.17. $H^{p,1}(\mathfrak{l}_-, U) = 0$ *for* $p > 0$.

Proof. Let $x_\alpha \in \mathfrak{l}$ be the root vector associate with simple root α. Let σ_α be the simple reflection associated with α. Let λ be the highest weight of irreducible representation U of $\mathfrak{l}$. Then $-\lambda$ is the lowest weight of U. Let $u_{-\sigma_\alpha(\lambda)} \in U$ be the weight vector of weight $-\sigma_\alpha(\lambda)$.

By Theorem 5.15 of [13], we have

$$H^1(\mathfrak{l}_-, U) = \mathcal{H}^{\xi_\sigma},$$

where $\mathcal{H}^{\xi_\sigma}$ is the irreducible $\mathfrak{l}_0$-module with lowest weight vector $x_\alpha^* \otimes u_{-\sigma_\alpha(\lambda)}$ of weight $\xi_\sigma = -(\sigma_\alpha(\lambda) + \alpha)$.

Since $x_\alpha^* \otimes u_{-\sigma_\alpha(\lambda)} \in \mathfrak{l}_{-1}^* \otimes U_-$ and $\mathcal{H}^{\xi_\sigma} = \mathfrak{l}_{-1}^* \otimes U_- = H^{0,1}(\mathfrak{l}_-, U)$, we see $H^{p,1}(\mathfrak{l}_-, U) = 0$ for $p > 0$. □

Theorem 4.1.18. *Let X be a smooth nonhomogeneous projective horospherical variety (L, α, β) of Picard number one. Let $G = \mathrm{Aut}(X)$ and let $\mathfrak{g} = (\mathfrak{l} + \mathbb{C}) \rhd U$ be the corresponding Lie algebra. By Proposition 3.4.2, we could give a gradation of $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ such that the graded Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is identified with the tangent space of X at a point x where x is in the open G-orbit. Then $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$.*

Proof. By Theorem 2.4.3 and Lemma 4.1.17, 2, 3 of Lemma 3.4.4 and Proposition 4.1.1, for $p > 0$, $H^{p,1}(\mathfrak{m}, \mathfrak{g}) = 0$. Then, by 1, 2 of Lemma 3.4.4 and Lemma 2.2.4, $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$. □

4.2 Existence of Cartan connections

We will show the existence of Cartan Connections. Theorem 4.2.1 is generalization of Theorem 2.3.3 (when $\tilde{\mathfrak{g}} = \mathfrak{g}$). We will apply it to the Lie algebras of the automorphism groups of smooth nonhomogeneous projective horospherical varieties of Picard number one.

Theorem 4.2.1. *Let $\mathfrak{m}$ be a fundamental graded Lie algebra. Let G_0 be a Lie subgroup of $G_0(\mathfrak{m})$ and let $\mathfrak{g}_0$ be the Lie algebra corresponding to G_0. Let $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ and $\mathfrak{h}(\mathfrak{m}, \mathfrak{g}_0) = \bigoplus_{i\geq 0} \mathfrak{g}_i$ be its non-negative part. Let $H(\mathfrak{m}, G_0)$ be the Lie group with its Lie algebra $\mathfrak{h}(\mathfrak{m}, \mathfrak{g}_0)$. Assume the prolongation $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is finite-dimensional. We also assume that there exists a graded Lie algebra $\tilde{\mathfrak{g}}$ which contains $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ as a Lie subalgebra, an symmetric bilinear form $(.,.)$ on $\tilde{\mathfrak{g}}$, and a map $\tau\colon \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ satisfying*

1. *the gradation of $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ could be extended to $\tilde{\mathfrak{g}}$;*

2. *$\{.,.\} := -(.,\tau.)$ is a positive definite Hermitian inner product on $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ and $\{\mathfrak{g}_i, \mathfrak{g}_j\} = 0$ if $i \neq j$;*

3. *$\tau(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0))$ is also graded Lie algebra and $\tau(\mathfrak{g}_i) \subset \tilde{\mathfrak{g}}_{-i}$ for $i \geq 0$;*

4. *$\{[A, x], y\} = -\{x, [\tau(A), y]\}$ for $A \in \mathfrak{g}$, $x, y \in \tilde{\mathfrak{g}}$;*

5. *there is $\tau_0\colon G_0 \to \tilde{G}_0$ such that $(ax, y) = (x, \tau_0(a)y)$ for $x, y \in \tilde{\mathfrak{g}}$ and $a \in G_0$*

Then for each G_0-structure on (M, F) of type $\mathfrak{m}$, we can construct a Cartan connection (P, ω) of type $(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0), H(\mathfrak{m}, G_0))$ so that two G_0-structures on (M, F) are (locally) equivalent when the associated Cartan connections of type $(\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0), H(\mathfrak{m}, G_0))$ are (locally) equivalent.

Proof. We simplify $\mathfrak{g} = \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$, $\mathfrak{h} = \mathfrak{h}(\mathfrak{m}, \mathfrak{g}_0)$, and $H = H(\mathfrak{m}, G_0)$. For proof, it is enough to show that there exists a subspace W of $F^1 \operatorname{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g})$, which satisfies the conditions of Theorem 2.3.2.

We extend the Hermitian inner product $\{.,.\}$ to $\operatorname{Hom}(\wedge^{\cdot}\mathfrak{m}, \mathfrak{g})$. We identify $\wedge^{\cdot}\mathfrak{m}'$ and $\wedge^{\cdot}\tau(\mathfrak{m})$ by defining a map $\eta\colon\ \wedge^{\cdot}\ \mathfrak{m}' \to \wedge^{\cdot}\tau(\mathfrak{m}) \subset \wedge^{\cdot}\tilde{\mathfrak{g}}$ such that $\{\eta(f), \tau(z)\} = -(\eta(f), z) = -f(z)$ for $f \in \wedge^{\cdot}\mathfrak{m}'$ $z \in \wedge^{\cdot}\mathfrak{m}$. Let ∂^* be the formal adjoint of ∂ with respect to the extended Hermitian inner product $\{.,.\}$ on $\operatorname{Hom}(\wedge^{\cdot}\mathfrak{m}, \mathfrak{g})$.

Then we have the direct sum decomposition:

$$\operatorname{Hom}(\wedge^p\mathfrak{m}, \mathfrak{g}) = \partial \operatorname{Hom}(\wedge^{p-1}\mathfrak{m}, \mathfrak{g}) \oplus \operatorname{Ker} \partial^*.$$

Let us show that $\operatorname{Ker}\partial^*$ is an invariant subspace by the action of H. Let ρ be the representation of $H \subset G$ on $\operatorname{Hom}(\wedge^{\cdot}\mathfrak{m}, \mathfrak{g})$ and ρ_* be the corresponding adjoint representation of $\mathfrak{h} \subset \mathfrak{g}$ on $\operatorname{Hom}(\wedge^{\cdot}\mathfrak{m}, \mathfrak{g})$. Since any element $a \in H$ is written as

$$a = a_0 \cdot \exp(A)$$

with $a_0 \in G_0$, $A \in F^1\mathfrak{h} := \bigoplus_{i>0} \mathfrak{g}_i$, it suffices to show that

$$(1) \quad \partial^* \circ \rho(a_0) = \rho(a_0) \circ \partial^* \text{ for } a_0 \in G_0$$

$$(2) \quad \partial^* \circ \rho_*(A) = \rho_*(A) \circ \partial^* \text{ for } A \in \mathfrak{h}.$$

Generally, we have

$$\partial \circ \rho(a_0) = \rho(a_0) \circ \partial \text{ for } a_0 \in G_0.$$

For the adjoint representation $\tilde{\lambda}$ of $\tilde{\mathfrak{m}} + \tilde{\mathfrak{g}}_0 = \bigoplus_{i\leq 0} \tilde{\mathfrak{g}}_i$ on $\mathrm{Hom}(\wedge^{\cdot}\tilde{\mathfrak{m}}, \tilde{\mathfrak{g}})$, we see

$$\partial \circ \tilde{\lambda}(B) = \tilde{\lambda}(B) \circ \partial \text{ for } B \in \bigoplus_{i\leq 0} \tilde{\mathfrak{g}}_i.$$

Since $\tau(\mathfrak{h}) \subset \bigoplus_{i\leq 0} \tilde{\mathfrak{g}}_i$, for the adjoint representation λ of $\tau(\mathfrak{h})$ on $\mathrm{Hom}(\wedge^{\cdot}\mathfrak{m}, \mathfrak{g}) = \wedge^{\cdot}\tau(\mathfrak{m}) \otimes \mathfrak{g}$,

$$\partial \circ \lambda(B) = \lambda(B) \circ \partial \text{ for } B \in \tau(\mathfrak{h}).$$

It follows that for $A \in \mathfrak{h}$ and $\phi, \psi \in \mathrm{Hom}(\wedge^{\cdot}\mathfrak{m}, \mathfrak{g})$,

$$\begin{aligned}\{\partial^* \circ \rho_*(A)\phi, \psi\} &= \{\rho_*(A)\phi, \partial\psi\} \\ &= -\{\phi, \lambda(\tau A)\partial\psi\} \\ &= -\{\phi, \partial\lambda(\tau A)\psi\} \\ &= -\{\partial^*\phi, \lambda(\tau A)\psi\} = \{\rho_*(A)\partial^*\phi, \psi\},\end{aligned}$$

which shows (2).

Similarly, we can verify (1). So $\mathrm{Ker}\,\partial^*$ is an invariant subspace by the action of H.

If we set $W = \mathrm{Ker}\,\partial^* \cap F^1\,\mathrm{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g})$, the proof is completed by Theorem 2.3.2. □

Theorem 4.2.2. *Let X be a smooth nonhomogeneous projective horospherical variety (L, α, β) of Picard number one. Let $G = \mathrm{Aut}(X)$ and let $\mathfrak{g} = (\mathfrak{l}+\mathbb{C}) \rhd U$ be the corresponding Lie algebra. As in Proposition 3.4.2, we give a gradation on the Lie algebra $\mathfrak{g}$. $H \subset G$ is the Lie subgroup associated with $\mathfrak{h} = \bigoplus_{i\geq 0}\mathfrak{g}_i$. Let $\mathfrak{m} = \bigoplus_{p<0}\mathfrak{g}_p$ and let G_0 be the Lie subgroup of G corresponding to $\mathfrak{g}_0$. Let (M, F) be a regular filtered manifold of type $\mathfrak{m}$. Then, for a given G_0-structure on (M, F), there exists a Cartan connection of type $(\mathfrak{g}, H)$ so that two G_0-structures on (M, F) are (locally) equivalent when the associated Cartan connections of type $(\mathfrak{g}, H)$ are (locally) equivalent.*

Proof. We will apply Theorem 4.2.1 to $\mathfrak{g} = (\mathfrak{l} + \mathbb{C}) \rhd U$. By Lemma 3.4.4 (1), $\mathfrak{m}$ is a fundamental graded Lie algebra. By Theorem 4.1.18, the Lie algebra $\mathfrak{g}$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$. Let $\tilde{\mathfrak{g}} = \mathfrak{gl}(V)$ which contains $\mathfrak{g}$ and $\mathfrak{g}^*$. Now we consider Proposition 3.3.1. The Cartan-Killing form on $\mathfrak{gl}(V)$ is a symmetric bilinear form $(\cdot, \cdot)$ such that the restricted inner product $\{\cdot, \cdot\}$ on $\mathfrak{g}$ is a positive definite Hermitian inner product. This proves condition 2. We could give a gradation on $\mathfrak{gl}(V)$ by the element E_X, since V is a representation space of $\mathfrak{l}$. And shift the gradation on $V_\alpha \otimes V_\beta^*$ and $V_\alpha^* \otimes V_\beta$ to make it be the extended gradation of the shifted gradation of U and U^*, which proves condition 1. Then the shifted gradation has also symmetry with respect to τ, which proves condition 3. More precisely, we have $\tau(E_X) = -E_X$ and for $x, y \in \mathfrak{g}$, $[\tau(x), \tau(y)] = \tau([x, y])$. Hence, from $[E, \tau(z)] = -[\tau(E), \tau(z)] = -\tau([E, z]) = -i\tau(z)$ for $z \in \mathfrak{g}_i$, we have $\tau(\mathfrak{g}_i) \subset \tilde{\mathfrak{g}}_{-i}$ for $i \geq 0$. Since the Killing-form on $\mathfrak{gl}(V)$ itself is an ad-invariant symmetric bilinear form $(\cdot, \cdot)$, the remaining conditions 4 and 5 are clear. □

4.3 Geometric structures modeled after horospherical varieties

Let $X = (L, \alpha, \beta)$ be a smooth nonhomogeneous projective horospherical variety of Picard number one. Let $\mathfrak{g}$ be the Lie algebra of the automorphism group of X. Let X_0 be the open orbit of X with respect to $G = \mathrm{Aut}(X)$. We recall from Proposition 3.4.2 that there is a gradation on $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ such that $\mathfrak{m} = \bigoplus_{i<0} \mathfrak{g}_p$ is fundamental and $\iota : T_xX_0 \simeq \mathfrak{m}$ for a base point $x \in X_0$. Let $G_0 \subset G_0(\mathfrak{m})$ be the Lie subgroup corresponding to $\mathfrak{g}_0$.

Definition 4.3.1. Let (X_0, E) be the regular differential system of type $\mathfrak{m}$ derived from the subbundle E of TX_0, where E_x corresponds to $\mathfrak{g}_{-1}$ under the identification $T_xX_0 \simeq \mathfrak{m}$ for a base point $x \in X_0$. Let $\mathscr{R}$ be the frame bundle of (X_0, E). Then $\mathscr{R}$ is isomorphic to $G \times_H G_0(\mathfrak{m})$. The G_0-subbundle $\mathscr{P}$ of $\mathscr{R}$, which is isomorphic to G_0-subbundle $G \times_H G_0$ of $G \times_H G_0(\mathfrak{m})$, is a G_0-structure on (X_0, E). We call the G_0-structure on (X_0, E) *the standard geometric structures* on X.

Since X is a Fano manifold, it is uniruled. By Theorem 1.12 of [18], the closed G-orbit has a codimension of at least two.

Definition 4.3.2. Let M be a projective manifold. There is a subbundle D of $T(M)$, which is defined outside of a subvariety $\mathrm{Sing}(D)$ of M. Suppose there exists a connected Zariski open subset M_0 of $M - \mathrm{Sing}(D)$ such that

1. $\mathrm{Sing}(D)$ has a codimension of at least two, and

2. (M_0, D) is a regular differential system of type $\mathfrak{m}$.

A G_0-structure on (M_0, D) is called a *geometric structure on M modeled after X*.

Two geometric structures of M_1 and M_2 modeled after X are *locally equivalent* if the G_0-structure on $((M_1)_0, D_1)$ and the G_0-structure on $((M_2)_0, D_2)$ are locally equivalent in the sense of Definition 2.1.3. A geometric structure modeled after X is *locally flat* if it is locally equivalent to the standard geometric structure on X.

The next proposition is proved in [10], getting the essence of Theorem 4.1 in [2].

Proposition 4.3.3 (Proposition 2.9 of [10])**.** *Let M be a manifold. Assume that there exists a non-constant holomorphic map $f\colon \mathbb{P}^1 \to M$ such that $f^*T(M)$ is a positive vector bundle, i.e., $f^*T(M) \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$ where $a_i \geq 1$ and $n = \dim M$. Let $M_0 \subset M$ be a connected Zariski open subset. Let $H \subset G$ be a closed connected subgroup of a connected Lie group G with Lie algebra $\mathfrak{g}$. Then any Cartan connection on M_0 of type $(\mathfrak{g}, H)$ is locally flat.*

Proof. Given a Cartan connection ω on a principal H-bundle $P \to M_0$, we can associate a principal G-bundle $\tilde{P} \to M_0$ with an Ehresmann connection $\tilde{\omega}$ as Section 3 of [2]. For a curve $f\colon \mathbb{P}^1 \to M$ with positive $f^*T(M)$, we see $f^*K(\tilde{\omega}) = 0$, where $K(\tilde{\omega})$ is the curvature of the connection $\tilde{\omega}$. We could see the vanishing of that curvature along a curve with positive tangents in the proof of Theorem 3.1 in the paper [1].

By assumption, there is a non-constant holomorphic map $f\colon \mathbb{P}^1 \to M$ such that $f^*T(M)$ is a positive vector bundle. Then there exist a family of holomorphic maps

$$\{f_t\colon \mathbb{P}^1 \to M | t \in \triangle^k, f_t^*T(M) \text{ positive}\},$$

parametrized by a polydisc $\triangle^k$, for some $k > 0$ such that the union of their images $\cup_{t\in\triangle^k} f_t(\mathbb{P}^1)$ contain a nonempty open subset U of M. For a nonempty open set $U \cap M_0$, the curvature $K(\tilde{\omega})$ vanishes on $U \cap M_0$ as above, hence vanishes on the whole space M_0. Hence, $\tilde{\omega}$ is locally flat on M_0, which implies ω is locally flat on M_0. □

The following is from Proposition 7.9 of [10], which is well-known from Proposition II.3.7 and Theorem IV.3.7 of [12].

Proposition 4.3.4. *Let M be a uniruled projective manifold of Picard number one. Then for any subvariety $Z \subset M$ of codimension two, there exists $f : \mathbb{P}^1 \to M$ with $f(\mathbb{P}^1) \cap Z = \emptyset$ and $f^*T(M)$ is positive.*

Theorem 4.3.5. *Let X be a smooth nonhomogeneous projective horospherical variety of Picard number one. Let M be a Fano manifold of Picard number one. Then any geometric structure on M modeled after X is locally equivalent to the standard geometric structure on X.*

Proof. The variety X is a smooth nonhomogeneous projective horospherical variety (L, α, β) of Picard number one. The Lie group G is $\mathrm{Aut}(X)$ and $\mathfrak{g}$ is the Lie algebra of G. Then there is a gradation of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ such that

$\mathfrak{m} = \bigoplus_{i<0} \mathfrak{g}_p$ is fundamental and the tangent space $T_x X_0$ is isomorphic to $\mathfrak{m}$ for each $x \in X_0$ by Proposition 3.4.2. By Theorem 4.1.18, the graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$.

Let M be a Fano manifold of Picard number one. M is a uniruled projective manifold. Let G_0-structure on (M_0, D) be a geometric structure on M modeled after X where D is a subbundle of $T(M)$ with singularity $\mathrm{Sing}(D)$ and M_0 is a connected Zariski open subset in $M - \mathrm{Sing}(D)$. Then the regular filtered manifold (M_0, D) of type $\mathfrak{m}$ admits a Cartan connection on M_0 of type $(\mathfrak{g}, H)$ by Theorem 4.2.2.

Since the subvariety $\mathrm{Sing}(D)$ has codimension of at least two in M, by Proposition 4.3.4, there is a rational curve $f\colon \mathbb{P}^1 \to M$ such that $f(\mathbb{P}^1) \cap \mathrm{Sing}(D) = \emptyset$ and $f^*T(M)$ is positive. We apply Proposition 4.3.3 to $M_0 \subset M - \mathrm{Sing}(D)$; thus, the Cartan connection on M_0 of type $(\mathfrak{g}, H)$ is locally flat.

To conclude, a geometric structure on M modeled after X is locally equivalent to the standard geometric structure on X. $\square$

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국문초록

준동형다양체의 기하구조

일반적으로 동질다양체의 기하구조의 연구를 통해 동질다양체의 요건이나 변형불변성을 증명한다. 이와같이 다양체를 특정하거나 다양체의 변형불변성을 증명하는 문제를 준동질다양체로 확장하기 위해, 우선 호로스피리컬다양체와 호로스피리컬다양체를 모델로 한 기하구조를 연구한다. 본 논문에서는 카르탄기하를 이용하여 피카드 수가 1이고 특이점이 없는 프로젝티브 호로스피리컬 다양체를 모델로 한 모든 기하구조가 (피카드 수가 1인 파노다양체 위에서는) 표준기하구조와 국소적으로 동일하다는 것을 보인다.

호로스피리컬 다양체를 모델로 한 기하구조와 연관된 카르탄커넥션의의 존재성을 증명하기 위해, 코스탄트의 하모닉이론과 호로스피리컬 다양체에에 작용하는그룹의 리대수(세미심플이 아닌)를 연구한다. 이러한 세미심플이 아닌 리대수에 대한 코스탄트 하모닉이론을 완전히 일반화하는 것도 흥미롭다. 이 논문에서는, 코스탄트 하모닉이론을 완전히 일반화하는 것 대신, 리대수 안에 포함되는 가장 큰 세미심플 리대수에 연관된 코호몰로지 공간이 없음을 보인다.

주요어 : 기하구조 · 국소적 동일성 · 호로스피리컬다양체 · 카르탄기하 · 연장

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