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On a sufficient condition for a Mittag-Leffler function to have real zeros only, and the Pólya-Wiman properties of differential operators

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On a sufficient condition for a Mittag-Leffler function to have real zeros only, and the Pólya-Wiman properties of differential operators

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Abstract

On a sufficient condition for a Mittag-Leffler function to have real zeros only, and the Pólya-Wiman properties of differential operators

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In this dissertation, we study the distribution of zeros of entire functions. First, we study the reality of zeros of Mittag-Leffler functions. If α and β are complex numbers with Re $\alpha > 0$, the Mittag-Leffler function $E_{\alpha,\beta}$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}.$$

One of the most recent results on the zeros of the Mittag-Leffler functions is due to Popov and Sedletskii: if $\alpha > 2$ and $0 < \beta \leq 2\alpha - 1$ or if $\alpha > 4$ and $0 < \beta \leq 2\alpha$ then $E_{\alpha,\beta}(z)$ has only real zeros. We improve the result by showing that if $\alpha \geq 4.07$ and $0 < \beta \leq 3\alpha$ then $E_{\alpha,\beta}(z)$ has only real zeros.

Second, we study the Pólya-Wiman properties of differential operators. Let $\phi(x) = \sum \alpha_n x^n$ be a formal power series with real coefficients and let D denote differentiation. It is shown that "for every real polynomial f there is a positive integer m_0 such that $\phi(D)^m f$ has only real zeros whenever $m \geq m_0$" if and only if "$\alpha_0 = 0$ or $2\alpha_0\alpha_2 - \alpha_1^2 < 0$", and that if ϕ does not represent a Laguerre-Pólya function, then there is a Laguerre-Pólya function f of genus 0 such that for every positive integer m, $\phi(D)^m f$ represents a real entire function having infinitely many nonreal zeros.

Finally, we prove the identity

$$\sup\{\alpha \in \mathbb{R} : e^{\alpha D^2} \cos D \; M^n \text{ has real zeros only}\} = 4{\lambda_n}^{-2},$$

where M^n is the monic monomial of degree n, that is, $M^n(z) = z^n$, and λ_n is the largest zero of the $2n$-th Hermite polynomial H_{2n} given by

$$H_{2n}(z) = (2n)! \sum_{k=0}^{n} \frac{(-1)^k}{k!(2n-2k)!}(2z)^{2n-2k}.$$

Key words: Mittag-Leffler functions, Pólya-Wiman Theorem, zeros of polynomials and entire functions, linear differential operators, Laguerre-Pólya class, Hermite polynomials, De Bruijn-Newman constant
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Contents

Introduction

Let f be an entire function. If there is a positive real number A such that

$$|f(z)| = O(\exp(|z|^A)) \quad as \quad |z| \to \infty, \tag{1}$$

then f is said to be of *finite order*. The order ρ of f is defined to be the greatest lower bound of the set of all positive real numbers A which satisfy (1). If f is of order ρ, $0 < \rho < \infty$, and there is a positive real number B such that

$$|f(z)| = O(\exp(B|z|^\rho)) \quad as \quad |z| \to \infty, \tag{2}$$

then f is said to be of *finite type*. The type τ of f is defined to be the greatest lower bound of the set of all positive real number B which satisfy (2).

It is well known and easy to prove that

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r;f)}{\log r} \quad \text{and} \quad \tau = \limsup_{r\to\infty} \frac{\log M(r;f)}{r^\rho}$$

where $M(r;f) = \max_{|z|=r} |f(z)|$, $r > 0$. The order ρ and type τ of f can also be represented in terms of the Taylor coefficients of f. If $f(z) = \sum a_n z^n$ then

$$\rho = \limsup_{n\to\infty} \frac{n\log n}{\log(1/|a_n|)} \quad \text{and} \quad \tau = \frac{1}{e\rho} \limsup_{n\to\infty} n|a_n|^{\rho/n}. \tag{3}$$

For a proof of (3) see [2].

The *genus* of f is the smallest integer p such that f can be represented in the form

$$f(z) = z^n e^{P(z)} \prod_j \left(1 - \frac{z}{a_j}\right) e^{\frac{z}{a_j} + \frac{1}{2}(\frac{z}{a_j})^2 + \cdots + \frac{1}{p}(\frac{z}{a_j})^p}, \tag{4}$$

where $P(z)$ is a polynomial of degree $\leq p$ and n is a nonnegative integer. Note that if f is of genus p and a_j, $j = 1, 2, \ldots$ are the zeros of f then the convergence of the infinite product in (4) implies that $\sum_{a_j \neq 0} |a_j|^{-p-1} < \infty$.

The order and the genus are closely related, as seen by the following theorem.

Hadamard's Theorem. *The genus p and the order ρ of an entire function satisfy the double inequality*

$$p \leq \rho \leq p + 1.$$

If $\alpha, \beta \in \mathbb{C}$, and $\operatorname{Re} \alpha > 0$, then it is known that the series

$$\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}$$

represents a transcendental entire function of order $1/(\operatorname{Re} \alpha)$ and type 1 [11, Proposition 3.1]. The entire function is denoted by $E_{\alpha,\beta}$ and called a *Mittag-Leffler function*. If $\alpha > 0$, then $E_{\alpha,1}$ is called a classical Mittag-Leffler function. If $\operatorname{Re} \alpha > 1$, then $E_{\alpha,\beta}$ is of order < 1, hence Hadamard's factorization theorem implies that $E_{\alpha,\beta}(z)$ has infinitely many zeros.

Let

$$\mathcal{W} = \{(\alpha, \beta) : \alpha, \beta > 0 \text{ and all the zeros of } E_{\alpha,\beta}(z) \text{ are negative and simple}\}.$$

In 1905, A. Wiman asserted that if $\alpha \geq 2$, then $(\alpha, 1) \in \mathcal{W}$ [33]. Since he only gave some plausible arguments, several mathematicians doubted the validity of Wiman's proof. Later G. Pólya proved that if α is an integer ≥ 2, then $(\alpha, 1) \in \mathcal{W}$ [25]. It follows from an asymptotic formula for $E_{\alpha,\beta}(z)$ that if $0 < \alpha < 2$ and $(\alpha, \beta) \notin \{(1, m) : m = 1, 0, -1, -2, \ldots\}$ then $E_{\alpha,\beta}(z)$ has infinitely many zeros but has only a finite number of real zeros [29, Theorem 2.1.1]; and we have

$$E_{1,m}(z) = z^{1-m} e^z \qquad (m = 1, 0, -1, -2, \ldots).$$

Hence $(1,1) \in \mathcal{W}$ and $(\alpha,\beta) \notin \mathcal{W}$ whenever $\alpha \in (0,1) \cup (1,2)$. In 1997, Ostrovskii and Peresyolkova proved that

$$\mathcal{W} \supset \bigcup_{n=1}^{\infty} \{(\alpha,\beta) : \alpha = 2^n,\ 0 < \beta < 1+\alpha\}.$$

(Especially, if $\alpha = 2$ and $0 < \beta < 3$ then $(\alpha,\beta) \in \mathcal{W}$. After that, Popov and Sedletskii proved that if $\alpha = 2$ and $\beta \geq 3$ then $(\alpha,\beta) \notin \mathcal{W}$ [29].) They also proved that $(\alpha,1),(\alpha,2) \in \mathcal{W}$ for all $\alpha \geq 2$ [22, Theorem 2 and Corollary 3]. In particular, they gave a rigorous proof of Wiman's result.

We put

$$\xi_n = \xi_n(\alpha,\beta) = \pi\left(n + \frac{1}{\alpha}(\beta-1)\right)\csc\left(\frac{\pi}{\alpha}\right) \qquad (n = 1,2,\ldots; \alpha > 2, \beta > 0),$$

and

$$\mathcal{W}_0 = \{(\alpha,\beta) : \alpha > 2, \beta > 0 \text{ and } (-1)^n E_{\alpha,\beta}(-{\xi_n}^{\alpha}) > 0 \text{ for all } n \in \mathbb{N}\}.$$

If $(\alpha,\beta) \in \mathcal{W}_0$ then the intermediate value theorem and the inequality $E_{\alpha,\beta}(0) > 0$ imply that $E_{\alpha,\beta}(z)$ has at least n zeros in $(-{\xi_n}^{\alpha}, 0)$ for every n. On the other hand, Popov and Sedletskii proved that if $\alpha > 2$ then $E_{\alpha,\beta}(z)$ has exactly n zeros (counting multiplicities) in $|z| \leq {\xi_n}^{\alpha}$ for all sufficiently large n [29, Theorem 2.1.4 and Theorem 2.2.2]. Hence we see that $\mathcal{W}_0 \subset \mathcal{W}$. In the same paper, they refined the result of Ostrovskii and Peresyolkova by showing that if $\alpha > 2$ and $0 < \beta \leq 2\alpha - 1$ or if $\alpha \geq 4$ and $0 < \beta \leq 2\alpha$, then $(\alpha,\beta) \in \mathcal{W}_0$, and that if $\alpha > 2$ and $\beta \geq (\log 2)^{-1}\alpha^2 - \alpha + 0.9$, then $(\alpha,\beta) \notin \mathcal{W}$ [29, Theorem 3.1.1 and Theorem 3.1.4]. In Chapter 1, we improve the result of Popov and Sedletskii by showing that if $\alpha \geq 4.07$ and $0 < \beta \leq 3\alpha$ then $(\alpha,\beta) \in \mathcal{W}_0$.

A real entire function is an entire function which takes real values on the real axis. If f is a real entire function, we denote the number of nonreal zeros (counting multiplicities) of f by $Z_C(f)$. (If f is identically equal to 0, we set $Z_C(f) = 0$.) A real entire function f is said to be of *genus* 1^* if it can be expressed in the form

$$f(x) = e^{-\gamma x^2} g(x),$$

where $\gamma \geq 0$ and g is a real entire function of genus at most 1. If f is a real entire function of genus 1^* and $Z_C(f) = 0$, then f is called a *Laguerre-Pólya function* and we write $f \in \mathcal{LP}$. We denote by $\mathcal{LP}^*$ the class of real entire functions f of genus 1^* such that $Z_C(f) < \infty$. It is well known that $f \in \mathcal{LP}$ if and only if there is a sequence $\langle f_n \rangle$ of real polynomials such that $Z_C(f_n) = 0$ for all n and $f_n \to f$ uniformly on compact sets in the complex plane. (See Chapter 8 of [19] and [20, 23, 27].) From this and an elementary argument based on Rolle's theorem, it follows that the classes $\mathcal{LP}$ and $\mathcal{LP}^*$ are closed under differentiation, and that $Z_C(f) \geq Z_C(f')$ for all $f \in \mathcal{LP}^*$. The *Pólya-Wiman theorem* states that for every $f \in \mathcal{LP}^*$ there is a positive integer m_0 such that $f^{(m)} \in \mathcal{LP}$ for all $m \geq m_0$ [6, 7, 14, 17, 26]. On the other hand, it follows from recent results of W. Bergweiler, A. Eremenko and J. Langley that if f is a real entire function, $Z_C(f) < \infty$ and $f \notin \mathcal{LP}^*$, then $Z_C(f^{(m)}) \to \infty$ as $m \to \infty$ [1, 18].

Let ϕ be a formal power series given by

$$\phi(x) = \sum_{n=0}^{\infty} \alpha_n x^n.$$

For convenience we express the n-th coefficient α_n of ϕ as $\phi^{(n)}(0)/n!$ even when the radius of convergence is equal to 0. If f is an entire function and the series

$$\sum_{n=0}^{\infty} \alpha_n f^{(n)}$$

converges uniformly on compact sets in the complex plane, so that it represents an entire function, we write $f \in \operatorname{dom} \phi(D)$ and denote the entire function by $\phi(D)f$. For $m \geq 2$ we denote by $\operatorname{dom} \phi(D)^m$ the class of entire functions f such that $f, \phi(D)f, \ldots, \phi(D)^{m-1}f \in \operatorname{dom} \phi(D)$. It is obvious that if f is a polynomial, then $f \in \operatorname{dom} \phi(D)^m$ for all m. For more general restrictions on the growth of ϕ and f under which $f \in \operatorname{dom} \phi(D)^m$ for all m, see [3, 5].

The following version of the Pólya-Wiman theorem for the operator $\phi(D)$ was established by T. Craven and G. Csordas [5, Theorem 2.4].

Theorem A. *Suppose that ϕ is a formal power series with real coefficients, $\phi'(0) = 0$ and $\phi''(0)\phi(0) < 0$. Then for every real polynomial f there is* *a positive integer m_0 such that all the zeros of $\phi(D)^m f$ are real and simple whenever $m \geq m_0$.*

We also have the following version, which is a consequence of the results in Section 3 of [5].

Theorem B. *Suppose that $\phi \in \mathcal{LP}$ (ϕ represents a Laguerre-Pólya function), $f \in \mathcal{LP}^*$, and that f is of order less than 2. Then $f \in \operatorname{dom}\phi(D)^m$, $\phi(D)^m f \in \mathcal{LP}^*$ and $Z_C(\phi(D)^m f) \geq Z_C(\phi(D)^{m+1} f)$ for all m. Furthermore, if ϕ is not of the form $\phi(x) = ce^{\gamma x}$ with $c \neq 0$, then $Z_C(\phi(D)^m f) \to 0$ as $m \to \infty$.*

In Chpater 2, we complement Theorem A and Theorem B above. Let ϕ be a formal power series with real coefficients and f be a real entire function. If $f \in \operatorname{dom}\phi(D)^m$ for all m and $Z_C(\phi(D)^m f) \to 0$ as $m \to \infty$, then we will say that ϕ (or the corresponding operator $\phi(D)$) has the *Pólya-Wiman property* with respect to f. For instance, if f is a real entire function and $Z_C(f) < \infty$, then the operator D $(= d/dx)$ has the Pólya-Wiman property with respect to f if and only if $f \in \mathcal{LP}^*$. Theorem A gives a sufficient condition for ϕ to have the Pólya-Wiman property with respect to arbitrary real polynomials. In Section 2.2, we prove that this is the case if and only if $\phi(0) = 0$ or $\phi''(0)\phi(0) - \phi'(0)^2 < 0$. In Section 2.3, we prove a strong version of the converse of Theorem B which implies that if ϕ is a formal power series with real coefficients and ϕ does not represent a Laguerre-Pólya function then ϕ does not have the Pólya-Wiman property with respect to some (transcendental) Laguerre-Pólya functions of genus 0

In Chapter 3, we introduce a result on the polynomials all of whose zeros lie in the lower half plane. The result is due to Wall [32] in the case of polynomials with real coefficients and to Frank [10] in the case of polynomial with complex coefficients. By using the Wall-Frank Theorem, we obtain more precise asymptotic results on the distribution of zeros of $\phi(D)^m P(z)$ as $m \to \infty$ than the results obtained in Section 2.4.

A function of growth $(2,0)$ is a real entire function which is at most order 2 and type 0, that is,

$$f(z) = O(\exp(\epsilon|z|^2)) \qquad (|z| \to \infty)$$

for every $\epsilon > 0$. If f is of growth $(2,0)$ then it is known that $f \in \operatorname{dom} e^{\alpha D^2}$ and $e^{\alpha D^2} f$ is of growth $(2,0)$ for every $\alpha \in \mathbb{C}$ [3].

When f is a real entire function of growth $(2,0)$, we define $\lambda(f)$ by

$$\lambda(f) = \sup\{\alpha \in \mathbb{R} : e^{\alpha D^2} f \text{ has real zeros only}\}.$$

Let Ξ denote the Riemann Xi-function:

$$\Xi(t) = \frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) \qquad (s = \frac{1}{2} + it).$$

In [9], N. G. de Bruijn proved that $\lambda(\Xi) \geq -1/8$ and that the Riemann hypothesis is equivalent to the inequality $\lambda(\Xi) \geq 0$. In [21], C. Newman showed that $\lambda(\Xi) < \infty$, and conjectured the opposite inequality $\lambda(\Xi) \leq 0$. The inequality $\lambda(\Xi) \geq -1/8$ has been improved to $\lambda(\Xi) > -1/8$ by Ki, Kim and Lee [16]. The first upper bound was given by Csordas, Norfolk and Varga in 1988 [8]. They denoted $-4\lambda(\Xi)$ by Λ and established $-50 < \Lambda$. In the same paper, they called Λ the de Bruijn-Newman constant. Lower bounds for Λ have been computed by several authors. Recently, Saouter, Gourdon and Demichel have shown that $\Lambda > -1.14541 \times 10^{-11}$ [30].

We extend the notion of the de Bruijn-Newman constant to arbitrary real entire functions of growth $(2,0)$ by calling $-\lambda(f)$ the de Bruijn-Newman constant of f.

For $n = 0, 1, 2, \ldots$ let F_n be the real polynomial defined by

$$F_n(z) = \frac{1}{2}((z+i)^n + (z-i)^n) = (\cos D\ M^n)(z),$$

where M^n is the monic monomial of degree n, that is, $M^n(z) = z^n$. In Chapter 4, we prove that the de Bruijn-Newman constant of the polynomial F_n is $-(2\lambda_n)^{-2}$, where λ_n is the largest zero of the $2n$-th Hermite polynomial H_{2n} given by

$$H_{2n}(z) = (2n)! \sum_{k=0}^{n} \frac{(-1)^k}{k!(2n-2k)!}(2z)^{2n-2k}.$$

Chapter 1

Sufficient condition for a Mittag-Leffler function to have real zeros only

In this chapter, we study the reality of zeros of Mittag-Leffler functions. One of the most recent results is due to Popov and Sedletskii: if $\alpha > 2$ and $0 < \beta \leq 2\alpha - 1$ or if $\alpha > 4$ and $0 < \beta \leq 2\alpha$ then $E_{\alpha,\beta}(z)$ has only real zeros. We improve the result by showing that if $\alpha \geq 4.07$ and $0 < \beta \leq 3\alpha$ then $E_{\alpha,\beta}(z)$ has only real zeros.

1.1 Main result and sketch outline of the proof

If $\alpha, \beta \in \mathbb{C}$, and $\operatorname{Re}\alpha > 0$, then it is known that the series

$$\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}$$

represents a transcendental entire function of order $1/(\operatorname{Re}\alpha)$ and type 1 [11, Proposition 3.1]. The entire function is denoted by $E_{\alpha,\beta}$ and called a *Mittag-Leffler fucntion.* If $\alpha > 0$, then $E_{\alpha,1}$ is called a classical Mittag-Leffler function. If $\operatorname{Re}\alpha > 1$, then $E_{\alpha,\beta}$ is of order < 1, hence Hadamard's factorization theorem implies that $E_{\alpha,\beta}(z)$ has infinitely many zeros.

We put

$$\mathcal{W} = \{(\alpha,\beta) : \alpha,\beta > 0 \text{ and all the zeros of } E_{\alpha,\beta}(z) \text{ are negative and simple}\},$$

and

$$\mathcal{W}_0 = \{(\alpha,\beta) : \alpha > 2, \beta > 0 \text{ and } (-1)^n E_{\alpha,\beta}(-\xi_n{}^\alpha) > 0 \text{ for all } n \in \mathbb{N}\},$$

where

$$\xi_n = \xi_n(\alpha,\beta) = \pi\left(n + \frac{1}{\alpha}(\beta-1)\right)\csc\left(\frac{\pi}{\alpha}\right) \qquad (n = 1, 2, \ldots; \alpha > 2, \beta > 0).$$

If $(\alpha,\beta) \in \mathcal{W}_0$ then the intermediate value theorem and the inequality $E_{\alpha,\beta}(0) > 0$ imply that $E_{\alpha,\beta}(z)$ has at least n zeros in $(-\xi_n{}^\alpha, 0)$ for every n. On the other hand, Popov and Sedletskii proved the following theorem.

Theorem 1.1.1 ([29, Theorem 2.1.4 and Theorem 2.2.2])**.** *If $\alpha > 2$ then $E_{\alpha,\beta}(z)$ has exactly n zeros (counting multiplicities) in $|z| \le \xi_n{}^\alpha$ for all sufficiently large n.*

Hence we see that $\mathcal{W}_0 \subset \mathcal{W}$. In the same paper, they proved that if $\alpha > 2$ and $0 < \beta \le 2\alpha - 1$ or if $\alpha \ge 4$ and $0 < \beta \le 2\alpha$, then $(\alpha,\beta) \in \mathcal{W}_0$. Precisely,

Theorem 1.1.2 ([29, Theorem 3.1.1])**.** *If $\alpha > 2$ and $0 < \beta \le 2\alpha - 1$ or if $\alpha \ge 4$ and $0 < \beta \le 2\alpha$, then all zeros of the function $E_{\alpha,\beta}(z)$ in $\mathbb{C}$ lie on $(-\infty, 0)$ are simple, and if we denote them by $\{z_n(\alpha,\beta)\}_{n\in\mathbb{N}}$ ordered as*

$$z_1(\alpha,\beta) > z_2(\alpha,\beta) > \cdots > z_n(\alpha,\beta) > \cdots,$$

they satisfy the inequalities

$$-\xi_1{}^\alpha < z_1(\alpha,\beta) < -\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)},$$

$$-\xi_n{}^\alpha < z_n(\alpha,\beta) < -\xi_{n-1}{}^\alpha, \quad (n \ge 2).$$

In this chapter, we improve the result of Popov and Sedletskii by the following theorem.

Theorem 1.1.3. *If $\alpha \geq 4.07$ and $0 < \beta \leq 3\alpha$ then $(\alpha, \beta) \in \mathcal{W}$.*

This theorem is an immediate consequence of Theorem 1.1.4 and Theorem 1.1.5 stated below. Let

$$\phi(\alpha, \beta) = \frac{1}{\alpha} {\xi_1}^{1+\alpha-\beta} \exp\left(\xi_1 \cos\left(\frac{\pi}{\alpha}\right)\right) \Gamma(\beta - \alpha).$$

Theorem 1.1.4. *If $\alpha \geq 4$, $2\alpha < \beta \leq 3\alpha$ and $\phi(\alpha, \beta) > 0.51$ then $(\alpha, \beta) \in \mathcal{W}_0$.*

Theorem 1.1.5. *We have $\phi(4.07, 12.21) > 0.512$. Furthermore the function $\alpha \mapsto \phi(\alpha, 3\alpha)$ is increasing on $[4, \infty)$ and for each fixed $\alpha \geq 4$ the function $\beta \mapsto \phi(\alpha, \beta)$ is decreasing on $(2\alpha, 3\alpha]$.*

Remark. Theorem 1.1.5 implies that the inequality $\phi(\alpha, \beta) > 0.512$ holds for $\alpha \geq 4.07$ and $2\alpha < \beta \leq 3\alpha$.

We sketch our proof of Theorem 1.1.4 in this section. The detailed proof is given in Sections 1.2-1.4. Theorem 1.1.5 is proved in Section 1.4.

The main idea of the proof of Theorem 1.1.4 is almost identical with the one given by Popov and Sedletskii [29]. In this section, we describe the differences of the proof of Theorem 1.1.4 in comparison with method of Popov and Sedletskii. From now on, we restrict our attention to the case where $\alpha \geq 4$.

First, Popov and Sedletskii considered the case where $0 < \beta \leq \alpha$. In order to show that $(\alpha, \beta) \in \mathcal{W}_0$, they used the following asymptotic expansion:

Theorem 1.1.6 ([29, Theorem 1.5.4])**.** *For any $\alpha \geq 5/2$, $0 < \beta \leq \alpha$, and $x > 0$, the following representation holds:*

$$E_{\alpha,\beta}(-x^\alpha) = S_{\alpha,\beta}(x) + \omega_{\alpha,\beta}(x),$$

where

$$S_{\alpha,\beta}(x) = \frac{2}{\alpha} x^{1-\beta} \times \sum_{k=1}^{\lfloor \alpha/2 \rfloor} \exp\left(x \cos\left(\frac{(2k-1)\pi}{\alpha}\right)\right) \cos\left(x \sin\left(\frac{(2k-1)\pi}{\alpha}\right) + (2k-1)\pi \frac{1-\beta}{\alpha}\right)$$

and

$$|\omega_{\alpha,\beta}(x)| \leq 0.74 x^{-\beta} \qquad (x > 0).$$

In the case of $\alpha < \beta \le 2\alpha$, they used the identity

$$E_{\alpha,\beta}(z) = \frac{1}{z}\left(E_{\alpha,\beta-\alpha}(z) - \frac{1}{\Gamma(\beta-\alpha)}\right). \tag{1.1}$$

Since $0 < \beta - \alpha \le \alpha$, Theorem 1.1.6 can be applied to (1.1).

In the case of $2\alpha < \beta \le 3\alpha$, we will use the following equality obtained from identity (1.1),

$$E_{\alpha,\beta}(z) = \frac{1}{z^2}E_{\alpha,\beta-2\alpha}(z) - \frac{1}{z^2}\frac{1}{\Gamma(\beta-2\alpha)} - \frac{1}{z}\frac{1}{\Gamma(\beta-\alpha)}. \tag{1.2}$$

Since $0 < \beta - 2\alpha \le \alpha$, we can apply the asymptotic expansion in Theorem 1.1.6 to $E_{\alpha,\beta-2\alpha}(z)$ in equality (1.2).

In Theorem 1.1.6, the first part of the remainder is equal to the product of $2x^{1-\beta}/\alpha$ and the sum

$$\sum_{k=2}^{\lfloor\alpha/2\rfloor} a_k(x)\cos\left(x\sin\left(\frac{(2k-1)\pi}{\alpha}\right) + (2k-1)\pi\frac{1-\beta}{\alpha}\right),$$

where

$$a_k(x) = \exp\left(x\cos\left(\frac{(2k-1)\pi}{\alpha}\right)\right).$$

For fixed $x > 0$ and $k \in \mathbb{N}$, we have

$$\lim_{\alpha\to\infty}\frac{a_{k+1}(x)}{a_k(x)} = 1 \quad \text{uniformly with respect to} \;\; 0 \le x \le o(\alpha^2),$$

and we cannot obtain the required estimate. Thus, in the case of $2\alpha < \beta \le 3\alpha$, we will show that if $\phi(\alpha,\beta) > 0.51$ then $(-1)^n E_{\alpha,\beta}(-{\xi_n}^{\alpha}) > 0$ holds for $n \ge \lfloor\alpha/4\rfloor$. For the notational simplicity, we put

$$R_n = R_n(\alpha,\beta) = \frac{\Gamma(\beta+n\alpha)}{\Gamma(\beta+(n-1)\alpha)} \qquad (n \in \mathbb{N}).$$

With this notation, we will prove the following:

$$(-1)^{n-1}E_{\alpha,\beta}(-R_n) > 0 \qquad (1 \le n \le \lfloor\alpha/4\rfloor), \tag{1.3}$$

$$(-1)^n E_{\alpha,\beta}(-\sqrt{2}R_n) > 0 \qquad (1 \le n < \lfloor \alpha/4 \rfloor), \tag{1.4}$$

and

$$\xi_{n-1}{}^{\alpha} < R_n < \sqrt{2}R_n < \xi_n{}^{\alpha} \qquad (\alpha \ge 8; 1 \le n \le \lfloor \alpha/4 \rfloor) \tag{1.5}$$

(we assume that $\xi_0 = 0$). These inequalities imply that $E_{\alpha,\beta}(z)$ has at least $\lfloor \alpha/4 \rfloor$ zeros in the interval $[-\xi_{\lfloor \alpha/4 \rfloor}{}^{\alpha}, 0]$. It is important that the signs of the function at the points $-R_{\lfloor \alpha/4 \rfloor}$ and $-\xi_{\lfloor \alpha/4 \rfloor}{}^{\alpha}$ are distinct and

$$-\xi_{\lfloor \alpha/4 \rfloor}{}^{\alpha} < -R_{\lfloor \alpha/4 \rfloor} < -\xi_{\lfloor \alpha/4 \rfloor - 1}{}^{\alpha}.$$

Thus we must verify inequality (1.3) for $n = \lfloor \alpha/4 \rfloor$; for $n = \lfloor \alpha/4 \rfloor$, we may omit the proof of inequality (1.4). Then, by Theorem 1.1.1, we complete the proof of Theorem 1.1.4.

In fact, in the case of $\alpha \ge 6$ and $0 < \beta \le 2\alpha$, Popov and Sedletskii chose $\lfloor \alpha/3 \rfloor$ instead of $\lfloor \alpha/4 \rfloor$ and obtained the same result mentioned above (In this case, the condition $\phi(\alpha, \beta) > 0.51$ is not required).

1.2 Sufficient condition to have real zeros only

We denote by ψ the logarithmic derivative of the Γ-function and use the following expansion:

$$\begin{aligned} \psi(z) &= \frac{\Gamma'(z)}{\Gamma(z)} \\ &= -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right) \qquad (z \in \mathbb{C} \setminus \{0, -1, -2, \dots, \}) \end{aligned} \tag{1.6}$$

where γ is the Euler constant. From (1.6), it follows the identity

$$\psi'(z) = \sum_{k=0}^{\infty} (k+z)^{-2}. \tag{1.7}$$

The following lemma will be needed throughout Chapter 1.

Lemma 1.2.1 ([29, Lemma 3.4.2])**.** *The logarithmic derivative of the* Γ*-function satisfies the following estimates:*

$$-(2t-1)^{-1} + \log t < \psi(t) < \log t \qquad (t > 1/2).$$

We first prove some propositions.

Proposition 1.2.2. *If* $x \geq 4$*, then*

$$1 < {\xi_1}^x \frac{\Gamma(x)}{\Gamma(2x)}$$

where $\xi_1 = \xi_1(x, 3x)$*.*

Proof. We put

$$f(x) = {\xi_1}^x \frac{\Gamma(x)}{\Gamma(2x)},$$
$$g(x) = x\log(4x-1) + \log\Gamma(x) - \log\Gamma(2x),$$

and

$$h(x) = \log\pi - \log x - \log\left(\sin\frac{\pi}{x}\right).$$

Then we obtain the equality

$$\log f(x) = g(x) + xh(x).$$

To prove the proposition, it is enough to show that $g(x)$ and $h(x)$ is positive for $x \geq 4$.

First, we take the derivative of $g(x)$,

$$g'(x) = \log(4x-1) + \frac{1}{4x-1} + \psi(x) - 2\psi(2x) + 1.$$

Then by Lemma 1.2.1, we obtain

$$\begin{aligned} g'(x) &\geq \log(4x-1) + \frac{1}{4x-1} - \frac{1}{2x-1} + \log x - 2\log(2x) + 1 \\ &\geq \log\left(1 - \frac{1}{4x}\right) + \frac{1}{4x-1} - \frac{1}{2x-1} \\ &\geq \log\frac{15}{16} - \frac{1}{7} + 1 > 0.7 > 0. \end{aligned}$$

Thus $g(x)$ increases and hence $g(x) \geq g(4) > 4$.

Second, since

$$xh'(x) = -1 + \frac{\pi}{x}\cot\frac{\pi}{x} \leq 0$$

and

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \log\left(\frac{\pi}{x}\csc\frac{\pi}{x}\right) = 0,$$

we have $h(x) \geq 0$.

Therefore, $\log f(x) > 0$ and we obtain $f(x) > 1$. □

Proposition 1.2.3. *If $\alpha \geq 4$ and $2\alpha < \beta \leq 3\alpha$, then*

$$1 < {\xi_1}^{\alpha}\frac{\Gamma(\beta-2\alpha)}{\Gamma(\beta-\alpha)}.$$

Proof. Let

$$f_\alpha(x) = \xi_1(\alpha, x)^{\alpha}\frac{\Gamma(x-2\alpha)}{\Gamma(x-\alpha)} \qquad (2\alpha < x \leq 3\alpha).$$

If $F_\alpha(x)$ denotes the logarithmic derivative of $f_\alpha(x)$, then

$$F_\alpha(x) = \frac{\alpha}{x+\alpha-1} + \psi(x-2\alpha) - \psi(x-\alpha).$$

By (1.7), we obtain

$$\begin{aligned} F'_\alpha(x) &= -\frac{\alpha}{(x+\alpha-1)^2} + \sum_{n=0}^{\infty}\frac{1}{(n+x-2\alpha)^2} - \sum_{n=0}^{\infty}\frac{1}{(n+x-\alpha)^2} \\ &\geq -\frac{\alpha}{(x+\alpha-1)^2} + \int_{x-2\alpha}^{\infty}\frac{1}{t^2}dt - \int_{x-\alpha-1}^{\infty}\frac{1}{t^2}dt \\ &\geq -\frac{\alpha}{(x+\alpha-1)^2} + \frac{1}{x-2\alpha} - \frac{1}{x-\alpha-1} \\ &\geq \frac{-x^2+(5\alpha^2-3\alpha+2)x+(-\alpha^3-5\alpha^2+3\alpha-1)}{(x+\alpha-1)^2(x-2\alpha)(x-\alpha-1)}. \end{aligned} \tag{1.8}$$

Since

$$\frac{5\alpha^2-3\alpha+2}{2} > 3\alpha \qquad (\alpha \geq 4),$$

the numerator of (1.8) has minimum at 2α and

$$-x^2 + (5\alpha^2 - 3\alpha + 2)x + (-\alpha^3 - 5\alpha^2 + 3\alpha - 1) \geq 9\alpha^3 - 15\alpha^2 + 7\alpha - 1.$$

Also, $9\alpha^3 - 15\alpha^2 + 7\alpha - 1$ is increasing on $[4, \infty)$ and has minimum 363 at $\alpha = 4$. Thus $F'_\alpha(x) > 0$ and $F_\alpha(x) \leq F_\alpha(3\alpha)$. By Lemma 1.2.1, we obtain

$$F_\alpha(3\alpha) \leq \frac{\alpha+1}{4\alpha-1} - \log 2 \leq \frac{1}{3} - \log 2 < 0.$$

Hence $f'_\alpha(x) < 0$ and $f_\alpha(x) \geq f_\alpha(3\alpha)$. Therefore, the proof is completed by Proposition 1.2.2. □

Proposition 1.2.4. *Let $\alpha \geq 4$ and $2\alpha < \beta \leq 3\alpha$. If we put*

$$f_{\alpha,\beta}(x) = x^{1-\beta} \exp\left(x \cos \frac{\pi}{\alpha}\right),$$

then $f_{\alpha,\beta}(x)$ is increasing on $[\xi_1, \infty)$.

Proof. Since

$$f'_{\alpha,\beta}(x) = x^{-\beta} \cos\left(\frac{\pi}{\alpha}\right) \exp\left(x \cos \frac{\pi}{\alpha}\right) \left(x - (\beta - 1) \sec \frac{\pi}{\alpha}\right),$$

$f_{\alpha,\beta}(x)$ increases for $x > (\beta - 1)\sec(\pi/\alpha)$. It is enough to show that $\xi_1 > (\beta - 1)\sec(\pi/\alpha)$, i.e. $\pi(\alpha + \beta - 1) - (\beta - 1)\alpha \tan(\pi/\alpha) > 0$. To prove the inequality, for each $\alpha \geq 4$, we put

$$g_\alpha(x) = \pi(x + \alpha - 1) - (x - 1)\alpha \tan \frac{\pi}{\alpha} \qquad (2\alpha < x \leq 3\alpha).$$

Since $g'_\alpha(x) = \pi - \alpha \tan(\pi/\alpha) < 0$, we have $g_\alpha(x) \geq g_\alpha(3\alpha)$. Also,

$$\begin{aligned} g_\alpha(3\alpha) &= \pi(4\alpha - 1) - (3\alpha - 1)\alpha \tan \frac{\pi}{\alpha} \\ &\geq \pi(4\alpha - 1) - 4(3\alpha - 1) \\ &\geq (4\pi - 12)\alpha + (4 - \pi) \geq 15\pi - 44 > 0. \end{aligned}$$

Therefore, $g_\alpha(x) > 0$ for $2\alpha < x \leq 3\alpha$, which proves the proposition. □

Proposition 1.2.5. *If we put*

$$f(x) = \xi_1(x, 2x)^{-x}\Gamma(x)$$

then $f(x)$ *is decreasing on* $[4, \infty)$.

Proof. We put

$$g(x) = -\log\left(3 - \frac{1}{x}\right)$$

and

$$h(x) = \psi(x) - \frac{1}{3x-1} + \log\sin\frac{\pi}{x} - \log\pi.$$

Then we obtain

$$\frac{f'(x)}{f(x)} = g(x) + h(x) - \frac{\pi}{x}\cot\frac{\pi}{x}.$$

Since

$$-\log\left(3 - \frac{1}{x}\right) \leq -\log\frac{11}{4} < 0,$$

we have $g(x) < 0$. Now, it remains to prove $h(x) \leq 0$. By Lemma 1.2.1, we obtain

$$h(x) \leq \log\left(\frac{x}{\pi}\sin\frac{\pi}{x}\right) - \frac{1}{3x-1}.$$

Since

$$\frac{x}{\pi}\sin\frac{\pi}{x} \quad \text{and} \quad -\frac{1}{3x-1}$$

is increasing on $[4, \infty]$ and

$$\lim_{x\to\infty}\left(\log\left(\frac{x}{\pi}\sin\frac{\pi}{x}\right) - \frac{1}{3x-1}\right) = 0,$$

we have $h(x) \leq 0$. Therefore, $f'(x) < 0$. □

Proposition 1.2.6. *If* $\alpha \geq 4$ *and* $2\alpha < \beta \leq 3\alpha$, *then*

$$1 + 0.74\ {\xi_1}^{\alpha-\beta}\ \Gamma(\beta - \alpha) < 1.0002.$$

Proof. For $\alpha \geq 4$, let

$$f_\alpha(x) = \log(\xi_1(\alpha, x)^{\alpha - x}\ \Gamma(x-\alpha)) \qquad (2\alpha < x \leq 3\alpha).$$

Then we have

$$f'_\alpha(x) = \psi(x-\alpha) - \log(x+\alpha-1) - \frac{x-\alpha}{x+\alpha-1} - \log\left(\frac{\pi}{\alpha}\csc\frac{\pi}{\alpha}\right).$$

By Lemma 1.2.1, we obtain

$$f'_\alpha(x) \leq \log\left(\frac{x-\alpha}{x+\alpha-1}\right) - \frac{x-\alpha}{x+\alpha-1} - \log\left(\frac{\pi}{\alpha}\csc\frac{\pi}{\alpha}\right).$$

Since $\log t < t$ and $\log(t \csc t) \geq 0$ for $t > 0$, $f'_\alpha(x) \leq 0$. Hence, by Proposition 1.2.5, we get

$$f_\alpha(x) \leq f_\alpha(2\alpha) \leq f_4(8).$$

Therefore,

$$1 + 0.74\exp(f_\alpha(\beta)) \leq 1 + 0.74\exp(f_4(8)) < 1.0002.$$

□

Now, we prove that if $n \geq \lfloor \alpha/4 \rfloor$ and $\phi(\alpha, \beta) > 0.51$ then $(-1)^n E_{\alpha,\beta}(-{\xi_n}^\alpha) > 0$ holds.

From the following relations,

$$\xi_n(\alpha,\beta) = \xi_{n+2}(\alpha, \beta - 2\alpha) = \xi_n \qquad (n \geq 1),$$

and

$$(\alpha, \beta - 2\alpha) \in \mathcal{W}_0,$$

we obtain

$$(-1)^n\ E_{\alpha,\beta-2\alpha}(-{\xi_n}^\alpha) > 0.$$

To prove the theorem, we will find a condition which implies the following equalities:

$$\text{(1.9)} \qquad \operatorname{sgn}\ E_{\alpha,\beta}(-{\xi_n}^\alpha) = \operatorname{sgn}\ E_{\alpha,\beta-2\alpha}(-{\xi_n}^\alpha) = (-1)^n \qquad (n \geq \lfloor \alpha/4 \rfloor).$$

For this, we use the identity (1.2). Since $0 < \beta - 2\alpha \le \alpha$, by Theorem 1.1.6, we obtain for $x > 0$,

$$E_{\alpha,\beta-2\alpha}(-x^\alpha) = S_{\alpha,\beta-2\alpha}(x) + \omega_{\alpha,\beta-2\alpha}(x), \qquad |\omega_{\alpha,\beta-2\alpha}(x)| \le 0.74x^{-(\beta-2\alpha)}.$$

We put

$$L(\xi_n) = {\xi_n}^{-2\alpha} S_{\alpha,\beta-2\alpha}(\xi_n)$$

and

$$R(\xi_n) = {\xi_n}^{-2\alpha}\omega_{\alpha,\beta-2\alpha}(\xi_n) - {\xi_n}^{-2\alpha}\frac{1}{\Gamma(\beta-2\alpha)} + {\xi_n}^{-\alpha}\frac{1}{\Gamma(\beta-\alpha)}.$$

Then

$$E_{\alpha,\beta}(-{\xi_n}^\alpha) = L(\xi_n) + R(\xi_n).$$

If we show

$$|R(\xi_n)| < |L(\xi_n)| \qquad (n \ge \lfloor \alpha/4 \rfloor), \tag{1.10}$$

then sgn $E_{\alpha,\beta}(-{\xi_n}^\alpha)$ is determined by $L(\xi_n)$.

On the other hand, by Theorem 1.1.6, we obtain

$$L(\xi_n) = \frac{2}{\alpha}{\xi_n}^{1-\beta}\left((-1)^n \exp\left(\xi_n \cos\left(\frac{\pi}{\alpha}\right)\right) + \sum_{k=2}^{\lfloor \alpha/2 \rfloor} s_k\right),$$

where

$$s_k = \exp\left(\xi_n \cos\left(\frac{(2k-1)\pi}{\alpha}\right)\right)\cos\left(\xi_n \sin\left(\frac{(2k-1)\pi}{\alpha}\right) + (2k-1)\pi\frac{1-\beta}{\alpha}\right).$$

If we put

$$a_k = \exp\left(\xi_n \cos\left(\frac{(2k-1)\pi}{\alpha}\right)\right),$$

then

$$|L(\xi_n)| \ge \frac{2}{\alpha}{\xi_n}^{1-\beta}\left(a_1 - \sum_{k=2}^{\lfloor \alpha/2 \rfloor} a_k\right).$$

If $n \geq \lfloor \alpha/4 \rfloor$ and $k \geq 1$, then

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \exp\left(-2\xi_n \sin\left(\frac{\pi}{\alpha}\right)\sin\left(\frac{2k\pi}{\alpha}\right)\right)\\ &= \exp\left(-2\pi\left(n+\frac{1}{\alpha}(\beta-1)\right)\sin\left(\frac{2k\pi}{\alpha}\right)\right)\\ &\leq \exp\left(-2\pi\left(\frac{\alpha}{4}+1-\frac{1}{\alpha}\right)\sin\left(\frac{2\pi}{\alpha}\right)\right)\\ &\leq \exp\left(-2\pi\left(\frac{\alpha}{4}+1-\frac{1}{\alpha}\right)\frac{4}{\alpha}\right)\\ &\leq \exp\left(-8\pi\left(\frac{1}{4}+\frac{1}{\alpha}-\frac{1}{\alpha^2}\right)\right)\\ &\leq \exp(-2\pi).\end{aligned}$$

Thus we have

$$\begin{aligned}a_1 - \sum_{k=2}^{\lfloor \alpha/4 \rfloor} a_k &\geq a_1 - \sum_{k=2}^{\infty} a_k\\ &\geq \frac{1-2\exp(-2\pi)}{1-\exp(-2\pi)} a_1 > 0.\end{aligned}$$

Hence, we obtain

$$\operatorname{sgn}\ L(\xi_n) = (-1)^n \tag{1.11}$$

and

$$|L(\xi_n)| \geq 2 \cdot \frac{1-2\exp(-2\pi)}{1-\exp(-2\pi)} \cdot \frac{1}{\alpha}{\xi_n}^{1-\beta} \exp\left(\xi_n \cos\frac{\pi}{\alpha}\right).$$

By Proposition 1.2.4,

$$|L(\xi_n)| \geq 2 \cdot \frac{1-2\exp(-2\pi)}{1-\exp(-2\pi)} \cdot \frac{1}{\alpha}{\xi_1}^{1-\beta} \exp\left(\xi_1 \cos\frac{\pi}{\alpha}\right).$$

Now,

$$|R(\xi_n)| \leq {\xi_n}^{-\alpha}\left|\frac{1}{\Gamma(\beta-\alpha)} - {\xi_n}^{-\alpha}\frac{1}{\Gamma(\beta-2\alpha)}\right| + {\xi_n}^{-2\alpha}|\omega_{\alpha,\beta-2\alpha}(\xi_n)|.$$

By Proposition 1.2.3 and Theorem 1.1.6,

$$\begin{aligned}|R(\xi_n)| &\le {\xi_n}^{-\alpha}\left(\frac{1}{\Gamma(\beta-\alpha)} - {\xi_n}^{-\alpha}\frac{1}{\Gamma(\beta-2\alpha)} + 0.74{\xi_n}^{-\beta+\alpha}\right)\\ &\le {\xi_n}^{-\alpha}\left(\frac{1}{\Gamma(\beta-\alpha)} + 0.74{\xi_n}^{-\beta+\alpha}\right).\end{aligned}$$

Since $\xi_n \ge \xi_1$ $(n \in \mathbb{N})$ and $\alpha - \beta \le 0$,

$$\begin{aligned}|R(\xi_n)| &\le {\xi_1}^{-\alpha}\left(\frac{1}{\Gamma(\beta-\alpha)} + 0.74{\xi_1}^{-\beta+\alpha}\right)\\ &\le {\xi_1}^{-\alpha}\frac{1}{\Gamma(\beta-\alpha)}\left(1 + 0.74{\xi_1}^{-\beta+\alpha}\Gamma(\beta-\alpha)\right).\end{aligned}$$

From Proposition 1.2.6, we obtain

$$|R(\xi_n)| \le 1.0002{\xi_1}^{-\alpha}\frac{1}{\Gamma(\beta-\alpha)}.$$

If (α,β) satisfies the inequality

$$1.0002{\xi_1}^{-\alpha}\frac{1}{\Gamma(\beta-\alpha)} \le 2\cdot\frac{1-2\exp(-2\sqrt{2}\pi)}{1-\exp(-2\sqrt{2}\pi)}\cdot\frac{1}{\alpha}{\xi_1}^{1-\beta}\exp\left(\xi_1\cos\frac{\pi}{\alpha}\right),$$

i.e.,

$$0.51 \le \frac{1}{\alpha}{\xi_1}^{1-\beta}\exp\left(\xi_1\cos\frac{\pi}{\alpha}\right)\Gamma(\beta-\alpha),$$

then (1.10) holds and (1.10) and (1.11) imply (1.9).

In the next section, we will prove (1.3) through (1.5).

1.3 Proof of Theorem 1.1.4 in the case $n < \lfloor\alpha/4\rfloor$

The following argument is almost identical with the one given in [29, pp. 294-305]. But there are only two differences that $\lfloor\alpha/3\rfloor$ is replaced by $\lfloor\alpha/4\rfloor$ and the upper bound of β/α is changed from 2 to 3. We include it here for the readers convenience.

Lemma 1.3.1 ([29, Lemma 3.4.5])**.** *For any $a, b > 0$, $a < b$, we have the inequality*

$$\frac{\log\Gamma(b) - \log\Gamma(a)}{b-a} < \log\left(\frac{a+b}{2}\right).$$

If $2 \le a < b \le 2a$, then

$$-\frac{2(b-a)^2}{3(a+b)^2} + \psi\left(\frac{a+b}{2}\right) < \frac{\log\Gamma(b) - \log\Gamma(a)}{b-a}.$$

If $4 \le \alpha < 8$, we have proved that $(-1)^n E_{\alpha,\beta}(-{\xi_n}^{\alpha}) > 0$ for all $n \ge \lfloor \alpha/4 \rfloor = 1$ in Section 1.2. So, we restrict $\alpha \ge 8$. We have divided the proof into a sequence of propositions.

Proposition 1.3.2. *If $\alpha \ge 8$, $2\alpha < \beta \le 3\alpha$, and $1 \le n \le \lfloor \alpha/4 \rfloor$, then*

$${\xi_{n-1}}^{\alpha} < R_n < \sqrt{2}R_n < {\xi_n}^{\alpha}.$$

Proof. We obtain the following inequalities by taking logarithm:

$$\begin{aligned}\log\left(\frac{\pi}{\alpha}\csc\left(\frac{\pi}{\alpha}\right)\right) &+ \log(\alpha(n-1)+\beta-1)\\ &< \frac{1}{\alpha}(\log\Gamma(\beta+n\alpha) - \log\Gamma(\beta+(n-1)\alpha)) \qquad (2 \le n \le \lfloor \alpha/4 \rfloor)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\alpha}\log 2 &+ \frac{1}{\alpha}(\log\Gamma(\beta+n\alpha) - \log\Gamma(\beta+(n-1)\alpha))\\ &< \log\left(\frac{\pi}{\alpha}\csc\left(\frac{\pi}{\alpha}\right)\right) + \log(\alpha n+\beta-1) \qquad (1 \le n \le \lfloor \alpha/4 \rfloor).\end{aligned}$$

We simplify the above inequalities by using the estimate

$$0 < \log\left(\frac{\pi}{\alpha}\csc\left(\frac{\pi}{\alpha}\right)\right) < \frac{2}{\alpha^2} \qquad (\alpha \ge 2),$$

and Lemma 1.3.1 taking

$$a = \beta + \alpha(n-1), \quad b = \beta + n\alpha.$$

We arrive at the proof of the inequalities

$$\begin{aligned}\frac{2}{\alpha^2} &+ \log(\alpha(n-1)+\beta-1) \\ &< -\frac{1}{6(n-(1/2)+(\alpha/\beta))^2} + \psi\left(\beta+\alpha\left(n-\frac{1}{2}\right)\right) \qquad (2 \le n \le \lfloor \alpha/4 \rfloor)\end{aligned} \tag{1.12}$$

and

$$\frac{1}{2\alpha}\log 2 + \log\left(\beta+\alpha\left(n-\frac{1}{2}\right)\right) < \log(\alpha n+\beta-1) \qquad (1 \le n \le \lfloor \alpha/4 \rfloor).$$

By Lemma 1.2.1, we obtain the following inequalities:

$$\begin{aligned}\frac{2}{\alpha^2}+\frac{1}{6((\alpha/\beta)+n-(1/2))^2} &+ \frac{1/\alpha}{2((\beta/\alpha)+n-(1/2))-(1/\alpha)} \\ &< \log\left(\frac{(\beta/\alpha)+n-(1/2)}{(\beta/\alpha)+n-1-(1/\alpha)}\right) \qquad (2 \le n \le \lfloor \alpha/4 \rfloor)\end{aligned}$$

and

$$\frac{1}{2\alpha}\log 2 < \log\left(\frac{(\beta/\alpha)+n-(1/\alpha)}{(\beta/\alpha)+n-(1/2)}\right) \qquad (1 \le n \le \lfloor \alpha/4 \rfloor).$$

If we put $y = (\beta/\alpha)+n-(1/2)$, then we obtain

$$\frac{2}{\alpha^2}+\frac{1}{6y^2}+\frac{1/\alpha}{2y-(1/\alpha)} < \log\left(\frac{y}{y-(1/2)-(1/\alpha)}\right) \quad (2 \le n \le \lfloor \alpha/4 \rfloor) \tag{1.13}$$

and

$$\frac{1}{2\alpha}\log 2 < \log\left(\frac{y+(1/2)-(1/\alpha)}{y}\right) \qquad (1 \le n \le \lfloor \alpha/4 \rfloor). \tag{1.14}$$

Since $(y+(1/2)-(1/\alpha))/y$ is decreasing on the ray $0 < y < +\infty$, it suffices to prove inequality (1.14) for the maximal value of y, i.e.,

$$\frac{1}{2\alpha}\log 2 < \log\left(1+\frac{(1/2)-(1/\alpha)}{n+(5/2)}\right) \qquad (n = \lfloor \alpha/4 \rfloor).$$

From the fact that $\log(1+t) > 0.9t$ for $0 < t \le 0.2$ and the inequality

$$\frac{\log 2}{2}\frac{n}{\alpha} < \frac{\log 2}{8} < 0.09,$$

we obtain

$$n\log\left(1+\frac{(1/2)-(1/\alpha)}{n+(5/2)}\right)>\frac{0.9n((1/2)-(1/\alpha))}{n+(5/2)}$$
$$>\frac{(27/80)n}{n+(5/2)}>\frac{27}{280}>0.09>\frac{\log 2}{2}\frac{n}{\alpha}.$$

Thus (1.14) is proved.

To prove (1.13), we use the estimate

$$\log\frac{y}{y-2h}>\frac{2h}{y-h}\qquad(0<h<y/2).$$

Then we obtain

$$\log\left(\frac{y}{y-(1/2)-(1/\alpha)}\right)>\frac{1+(2/\alpha)}{2y-(1/2)-(1/\alpha)}>\frac{1+(2/\alpha)}{2y-(1/\alpha)}.$$

Now, it remains to prove the inequality

$$\frac{2}{\alpha^2}+\frac{1}{6y^2}<\frac{1+(1/\alpha)}{2y-(1/\alpha))},$$

i.e.,

$$\frac{2}{\alpha}\left(\frac{y}{\alpha}\right)+\frac{1}{6y}<\frac{1+(1/\alpha)}{2-(1/(\alpha y))}. \tag{1.15}$$

Since

$$\frac{y}{\alpha}\le\frac{1}{\alpha}\left(\frac{\alpha}{4}+\frac{5}{2}\right)\le\frac{1}{4}+\frac{5}{16}=\frac{9}{16}\qquad(y\ge 5/2),$$

we have

$$\frac{9}{8\alpha}+\frac{1}{15}<\frac{9}{64}+\frac{1}{15}<\frac{1}{2}<\frac{1+(1/\alpha)}{2}\qquad(\alpha\ge 8).$$

Thus (1.15) is valid, and the proof of (1.5) is complete.

□

In order to prove (1.3) and (1.4), we need the following results.

Proposition 1.3.3 ([29, Corollary 3.3.1])**.** *For any $\beta>0$ and $\alpha>0$, $E_{\alpha,\beta}(z)$ is positive on $[-R_1,\infty)$.*

Lemma 1.3.4 ([29, Lemma 3.9.1])**.** *For any α, $\beta > 0$, and $n \in \mathbb{N}$, the following inequality holds:*

$$\frac{R_n}{R_{n+1}} < \exp\left(-\frac{\alpha}{n + (\beta/\alpha)}\right).$$

Until the end of this section, we use the notation

$$A_k = \frac{1}{\Gamma(\beta + k\alpha)},$$

so that

$$R_n = \frac{A_{n-1}}{A_n}, \qquad E_{\alpha,\beta}(-x) = \sum_{k=0}^{\infty}(-1)^k A_k x^k.$$

Also, we omit the arguments α and β in the notation of A_k.

Proposition 1.3.5. *If $\alpha \geq 8$ and $2\alpha < \beta \leq 3\alpha$ then $E_{\alpha,\beta}(-R_2) < 0$.*

Proof. We have

$$E_{\alpha,\beta}(-R_2) = A_0 - A_1 R_2 + A_2 {R_2}^2 - A_3 {R_2}^3 + A_4 {R_2}^4 + \sum_{k=5}^{\infty}(-1)^k A_k {R_2}^k. \tag{1.16}$$

The last term on the right-hand side of (1.16) is negative; this follows from the fact that the sequence $\{A_k {R_2}^k\}_{k=5}^{\infty}$ is decreasing, which is equivalent to the inequality

$$R_2 < R_k \qquad (k \geq 6). \tag{1.17}$$

From Lemma 1.3.4, we know that $\{R_k\}$ is increasing sequence. Thus (1.17) is valid. Since $R_2 = A_1/A_2$, we have $A_2 {R_2}^2 - A_1 R_2 = 0$. Therefore,

$$E_{\alpha,\beta}(-R_2) < A_0 - A_3 {R_2}^3 + A_4 {R_2}^4. \tag{1.18}$$

Inequality (1.18) can be rewritten in the form

$${A_3}^{-1} {R_2}^{-3} E_{\alpha,\beta}(-R_2) < -1 + (A_0/A_3){R_2}^{-3} + (A_4/A_3)R_2.$$

If we put

$$B = \left(\frac{A_0}{A_3}\right) R_2{}^{-3} = A_0 A_1{}^{-3} A_2{}^3 A_3{}^{-1},$$

then we obtain

$$A_3{}^{-1} R_2{}^{-3} E_{\alpha,\beta}(-R_2) < -1 + B + R_2/R_4$$

and

$$\log B = \log\Gamma(\beta + 3\alpha) - 3\log\Gamma(\beta + 2\alpha) + 3\log\Gamma(\beta + \alpha) - \log\Gamma(\beta).$$

By the mean value theorem applied to the third difference of the function $\log\Gamma(z)$ at the point β with step α, for some $\xi \in (\beta, \beta + 3\alpha)$, we obtain

$$\log B = \alpha^3 \psi''(\xi) = -2\alpha^3 \sum_{k=0}^{\infty} (k + \xi)^{-3}.$$

This implies

$$\begin{aligned} \log B &< -2\alpha^3 \sum_{k=0}^{\infty} (k + \beta + 3\alpha)^{-3} \\ &< -2\alpha^3 \int_0^{\infty} (t + \beta + 3\alpha)^{-3} dt \\ &= -\alpha^3 (\beta + 3\alpha)^{-2} = -\alpha \left(\frac{\beta}{\alpha} + 3\right)^{-2}. \end{aligned} \tag{1.19}$$

Since $\beta/\alpha \leq 3$ and $\alpha \geq 8$, we obtain

$$B < \exp\left(-\frac{2}{9}\right). \tag{1.20}$$

By Lemma 1.3.4, we have

$$\frac{R_2}{R_3} < \exp\left(-\frac{8}{5}\right) \quad \text{and} \quad \frac{R_3}{R_4} < \exp\left(-\frac{4}{3}\right).$$

Therefore,

$$\frac{R_2}{R_4} = \frac{R_2}{R_3}\frac{R_3}{R_4} < \exp\left(-\frac{44}{15}\right). \tag{1.21}$$

From (1.20) and (1.21), we obtain

$$A_3{}^{-1}R_2{}^{-3}E_{\alpha,\beta}(-R_2) < -1 + \exp\left(-\frac{2}{9}\right) + \exp\left(-\frac{44}{15}\right) < 0,$$

i.e., $E_{\alpha,\beta}(-R_2) < 0$, which was required. □

Proposition 1.3.6. *If $\alpha \geq 8$ and $2\alpha < \beta \leq 3\alpha$ then $E_{\alpha,\beta}(-\sqrt{2}R_1) < 0$ and $E_{\alpha,\beta}(-\sqrt{2}R_2) > 0$.*

Proof. We have

$$E_{\alpha,\beta}(-\sqrt{2}R_1) = A_0 - A_1\sqrt{2}R_1 + 2A_2R_1{}^2 + \sum_{k=3}^{\infty}(-1)^k A_k(\sqrt{2}R_1)^k. \tag{1.22}$$

The last term on the right-hand side of (1.22) is negative; this follows from the fact that the sequence $\{A_k\sqrt{2}R_1{}^k\}_{k=3}^{\infty}$ is decreasing, which is equivalent to the inequality

$$\sqrt{2}R_1 < R_k \qquad (k \geq 4). \tag{1.23}$$

From Lemma 1.3.4, we obtain

$$\frac{R_1}{R_k} < \frac{R_1}{R_2} < \exp\left(-\frac{\alpha}{1 + (\beta/\alpha)}\right).$$

Since $\beta/\alpha \leq 3$ and $\alpha \geq 8$,

$$\frac{R_1}{R_2} < \exp(-2) < 0.14 < \frac{1}{\sqrt{2}}$$

and (1.23) is valid.

Since the last term in the right hand side of (1.22) is negative, we obtain

$$E_{\alpha,\beta}(-\sqrt{2}R_1) < A_0 - A_1\sqrt{2}R_1 + 2A_2R_1{}^2 = A_0 - \sqrt{2}A_0 + 2A_2(A_0/A_1)^2. \tag{1.24}$$

Multiplying both sides of (1.24) by $A_0{}^{-1} = \Gamma(\beta)$, we obtain

$$\Gamma(\beta)E_{\alpha,\beta}(-\sqrt{2}R_1) < 1 - \sqrt{2} + 2(R_1/R_2) < 1 - \sqrt{2} + 0.28 < 0,$$

i.e., $E_{\alpha,\beta}(-\sqrt{2}R_1) < 0$.

Second, we have

$$E_{\alpha,\beta}(-\sqrt{2}R_2) = A_0 - \sqrt{2}A_1R_2 + 2A_2{R_2}^2 - 2\sqrt{2}A_3{R_2}^3 + \sum_{k=4}^{\infty}(-1)^k A_k\left(\sqrt{2}R_2\right)^k.$$

The sequence $\{A_k(\sqrt{2}R_2)^k\}_{k=4}^{\infty}$ is decreasing since the ratio of its elements with numbers k and $k+1$ is equal to $\sqrt{2}R_2/R_{k+1} \le \sqrt{2}R_2/R_4$ and is less than 1 by (1.21). Therefore, the sum

$$\sum_{k=4}^{\infty}(-1)^k A_k\left(\sqrt{2}R_2\right)^k$$

is positive and we obtain the inequality

$$\frac{1}{2}{A_2}^{-1}{R_2}^{-2}E_{\alpha,\beta}(-\sqrt{2}R_2) > 1 - \frac{1}{\sqrt{2}} - \sqrt{2}\frac{R_2}{R_3}.$$

In this case, by the restriction $n \le \lfloor \alpha/4 \rfloor - 1$, $\alpha \ge 12$ holds. Thus by Lemma 1.3.4, we have

$$\frac{R_2}{R_3} \le \exp\left(-\frac{12}{5}\right).$$

Therefore,

$$\frac{1}{2}{A_2}^{-1}{R_2}^{-2}E_{\alpha,\beta}(-\sqrt{2}R_2) > 1 - \frac{1}{\sqrt{2}} - \sqrt{2}\exp\left(-\frac{12}{5}\right) > 0,$$

i.e.,

$$E_{\alpha,\beta}(-\sqrt{2}R_2) > 0.$$

□

Proposition 1.3.7. *If $\alpha \ge 12$, $2\alpha < \beta \le 3\alpha$, and $3 \le n \le \lfloor \alpha/4 \rfloor$, then*

$$(-1)^n E_{\alpha,\beta}(-\sqrt{2}R_n) > 0.$$

Proof. We express $(-1)^n\ E_{\alpha,\beta}(-\sqrt{2}R_n)$ as

$$(-1)^n\ E_{\alpha,\beta}(-\sqrt{2}R_n) = S_{n,0}-A_{n-1}(\sqrt{2}R_n)^{n-1}+A_n(\sqrt{2}R_n)^n-A_{n+1}(\sqrt{2}R_n)^{n+1}+S_{n,1},$$

where

$$S_{n,0} = \sum_{k=0}^{n-2}(-1)^{k-n}A_k(\sqrt{2}R_n)^k \text{ and } S_{n,1} = \sum_{k=n+2}^{\infty}(-1)^{k-n}A_k(\sqrt{2}R_n)^k.$$

We prove that the sums $S_{n,0}$ and $S_{n,1}$ are positive. Since they are alternating and the terms with numbers $k = n\pm 2$ are positive, it suffices to prove that the sequence $A_k(\sqrt{2}R_n)^k$ increases for $0 \le k \le n-2$ and decreases for $k \ge n+2$. The ratio of the elements of this sequence with numbers $k+1$ and k is equal to

$$d_k = \frac{A_{k+1}(\sqrt{2}R_n)^{k+1}}{A_k(\sqrt{2}R_n)^k} = \sqrt{2}\frac{R_n}{R_{k+1}}. \tag{1.25}$$

By Lemma 1.3.4, we obtain

$$\frac{R_n}{R_{n+1}} < \exp\left(-\frac{\alpha}{n+3}\right) = \exp\left(-\frac{1}{(n/\alpha)+(3/\alpha)}\right) \le \exp(-2). \tag{1.26}$$

The equality (1.25) with (1.26) and the fact that R_k increases imply the estimates

$$d_k > \sqrt{2} > 1 \quad (0 \le k \le n-2) \quad \text{and} \quad d_k \le \sqrt{2}\frac{R_n}{R_{n+1}} < \sqrt{2}e^{-2} < 1 \quad (k \ge n+2),$$

which prove the required assertion. Since $S_{n,0},\ S_{n,1} > 0$, we have

$$(-1)^n\ E_{\alpha,\beta}(-\sqrt{2}R_n) > -A_{n-1}(\sqrt{2}R_n)^{n-1} + A_n(\sqrt{2}R_n)^n - A_{n+1}(\sqrt{2}R_n)^{n+1}. \tag{1.27}$$

By dividing both sides of (1.27) by $A_n(\sqrt{2}R_n)^n$, we obtain

$$(-1)^n {A_n}^{-1}(\sqrt{2}R_n)^{-n}\ E_{\alpha,\beta}(-\sqrt{2}R_n) > 1-\frac{1}{\sqrt{2}}-\sqrt{2}\frac{R_n}{R_{n+1}} > 1-\frac{1}{\sqrt{2}}-\sqrt{2}e^{-2} > 0.$$

Thus Proposition 1.3.7 is completely proved. □

Proposition 1.3.8. *Let*

$$v_\nu(x) = A_{n+\nu}x^{n+\nu} - A_{n-\nu-1}x^{n-\nu-1}.$$

Then, the following inequalities hold:

$$0 < v_{\nu+1}(R_n) < v_\nu(R_n) \qquad (1 \le \nu \le n-2;\ 3 \le n \le \lfloor \alpha/4 \rfloor), \tag{1.28}$$

$$A_{2n}{R_n}^{2n} < v_{n-1}(R_n) \quad (3 \le n \le \lfloor \alpha/4 \rfloor). \tag{1.29}$$

We will prove the Proposition 1.3.8 in the next section.

Proposition 1.3.9. *If $\alpha \ge 12$, $2\alpha < \beta \le 3\alpha$, and $3 \le n \le \lfloor \alpha/4 \rfloor$ then*

$$(-1)^{n-1}E_{\alpha,\beta}(-R_n) > 0.$$

Proof. We have

$$E_{\alpha,\beta}(-R_n) = \sum_{k=0}^{n-2}(-1)^k A_k {R_n}^k + (-1)^{n-1}(A_{n-1}{R_n}^{n-1} - A_n {R_n}^n) + \sum_{k=n+1}^{\infty}(-1)^k A_k {R_n}^k. \tag{1.30}$$

By grouping in (1.30) terms with numbers $k = n - \nu - 1$ and $k = n + \nu$ $(1 \le \nu \le n-1)$, and using the equality

$$A_{n-1}{R_n}^{n-1} - A_n {R_n}^n = A_n {R_n}^{n-1}\left(\frac{A_{n-1}}{A_n} - R_n\right) = 0,$$

we obtain

$$E_{\alpha,\beta}(-R_n) = (-1)^{n+1}\sum_{\nu=1}^{n-1}(-1)^{\nu-1}v_\nu(R_n) + \sum_{k=2n}^{\infty}(-1)^k A_k {R_n}^k,$$

where

$$v_\nu(x) = A_{n+\nu}x^{n+\nu} - A_{n-\nu-1}x^{n-\nu-1}.$$

Then,

$$(-1)^{n-1}E_{\alpha,\beta}(-R_n) = \sum_{\nu=1}^{n-1}(-1)^{\nu-1}v_\nu(R_n) + (-1)^{n-1}\sum_{k=2n}^{\infty}(-1)^k A_k {R_n}^k. \tag{1.31}$$

By using the Proposition 1.3.8, we can prove that the sums in the right hand side of (1.31) are positive. Indeed, by (1.28), the absolute value of terms in the alternating sum

$$\sum_{\nu=1}^{n-1}(-1)^{\nu-1}v_\nu(R_n)$$

decrease and the first them is positive. Therefore,

$$\sum_{\nu=1}^{n-1}(-1)^{\nu-1}v_\nu(R_n) > 0 \qquad (3 \le n \le \lfloor \alpha/4 \rfloor).$$

This immediately implies the required assertion for odd n since

$$\sum_{k=2n}^{\infty}(-1)^k A_k {R_n}^k$$

is also an alternating series which has terms with decreasing absolute value and the first term is positive. If n is even, then

$$\sum_{\nu=1}^{n-2}(-1)^{\nu-1}v_\nu(R_n) > 0 \quad \text{and} \quad (-1)^{n-1}\sum_{k=2n+1}^{\infty}(-1)^k A_k {R_n}^k > 0$$

by the same reasoning as above. Furthermore, by (1.29)

$$(-1)^{n-2}v_{n-1}(R_n) + (-1)^{3n+1}A_{2n}{R_n}^{2n} = v_{n-1}(R_n) - A_{2n}{R_n}^{2n} > 0,$$

and we also obtain that (1.31) is positive. □

Hence, (1.3) and (1.4) are proved by Propositions 1.3.3-1.3.9. Consequently, (1.5) is proved by Proposition 1.3.2 and the proof of Theorem 1.1.4 is completed.

1.3.1 Proof of Proposition 1.3.8

In this section, we will prove Proposition 1.3.8. The proof will be divided into three propositions.

We let

$$v_\nu(x) = A_{n+\nu}x^{n+\nu} - A_{n-\nu-1}x^{n-\nu-1}.$$

Proposition 1.3.10. *If* $1 \leq \nu \leq n-1$ *then* $v_\nu(R_n) > 0$.

Proof. From the identities

$$\frac{A_{n-\nu-1}}{A_{n+\nu}} = \prod_{p=n-\nu}^{n+\nu} R_p = R_n \prod_{j=1}^{\nu} (R_{n-j} R_{n+j}), \tag{1.32}$$

we obtain

$$\begin{aligned} 0 < v_\nu(R_n) &\iff \frac{A_{n-\nu-1}}{A_{n+\nu}} < {R_n}^{2\nu+1} \\ &\iff \prod_{j=1}^{\nu} (R_{n-j} R_{n+j}) {R_n}^{2\nu} \\ &\iff \sum_{j=1}^{\nu} (\log R_{n+j} - 2\log R_n + \log R_{n-j}) < 0; \end{aligned}$$

the last inequality follows from the concavity of the sequence $\{\log R_n\}$ (the concavity follows from $\psi'' < 0$). ☐

We next show that

$$v_{\nu+1}(R_n) < v_\nu(R_n), \tag{1.33}$$

which is equivalent to the inequality

$$A_{n-\nu-1}{R_n}^{n-\nu-1} - A_{n-\nu-2}{R_n}^{n-\nu-2} < A_{n+\nu}{R_n}^{n+\nu} - A_{n+\nu+1}{R_n}^{n+\nu+1}.$$

By applying (1.32), we can rewrite (1.33) in the form

$$\left(1 - \frac{R_{n-\nu-1}}{R_n}\right) \prod_{j=1}^{\nu} (R_{n+j}{R_n}^{-2} R_{n-j}) < 1 - \frac{R_n}{R_{n+\nu+1}}. \tag{1.34}$$

For notational simplicity, we put

$$a_{n,\nu} = \frac{R_{n-\nu-1}}{R_n}, \quad b_{n,\nu} = \frac{R_n}{R_{n+\nu+1}}, \quad \text{and} \quad u_{n,j} = R_{n+j}{R_n}^{-2}R_{n-j},$$

so that (1.34) takes the form

$$(1 - a_{n,\nu}) \prod_{j=1}^{\nu} u_{n,j} < 1 - b_{n,\nu}.$$

Dividing by $1 - b_{n,\nu}$ and using the identity

$$(1-a)(1-b)^{-1} = 1 + b(1-b)^{-1}(1 - a/b),$$

we obtain

$$(1 + b_{n,\nu}(1 - b_{n,\nu})^{-1}(1 - u_{n,\nu+1}) \prod_{j=1}^{\nu} u_{n,j} < 1, \tag{1.35}$$

which is equivalent to (1.34). Note that by the concavity of the sequence $\{\log R_n\}$, we have $u_{n,j} < 1$. To prove (1.33), we need the inequality

$$(1 + b_{n,\nu}(1 - b_{n,\nu})^{-1})u_{n,1} < 1, \tag{1.36}$$

which is even stronger than (1.35). To prove this, we introduce some lemmas.

Lemma 1.3.11. *Let $h > 0$, $x \in \mathbb{R}$, and $I = [x - 2h, x + 2h]$. And let $g(I) \subset \mathbb{R}$, $g \in C^{(4)}(I)$, and $g^{(4)}$ be positive and decrease on I. Then the following inequality holds:*

$$\begin{aligned} 3g(x+2h) - 10g(x+h) + 12g(x) &- 6g(x-h) + g(x-2h) \\ &< 2h^3 g^{(3)}(x) + h^4 \left(\frac{7}{4} g^{(4)}(x) + \frac{2}{3} g^{(4)}(x - 2h) \right). \end{aligned}$$

Proof. See [29, p.299] □

Lemma 1.3.12. *If $\alpha \geq 12$, $2\alpha < \beta \leq 3\alpha$, and $m \geq 5$, then*

$${u_{m+1,1}}^3 < u_{m,1}.$$

Proof. We have

$$\begin{aligned} \frac{{u_{m+1}}^3}{u_m} &= \left(\frac{R_{m+2} R_m}{{R_{m+1}}^2} \right)^3 \frac{{R_m}^2}{R_{m-1} R_{m+1}} \\ &= \left(\frac{A_{m+1}}{A_{m+2}} \right)^3 \left(\frac{A_m}{A_{m+1}} \right)^{-7} \left(\frac{A_{m-1}}{A_m} \right)^5 \left(\frac{A_{m-2}}{A_{m-1}} \right)^{-1}, \end{aligned} \tag{1.37}$$

where $A_k = 1/\Gamma(\beta + k\alpha)$. Taking the logarithm of both sides of (1.37), we obtain

$$\log\left(\frac{{u_{m+1}}^3}{u_m}\right) = 3\log\Gamma(\beta+(m+2)\alpha) - 10\log\Gamma(\beta+(m+1)\alpha) + 12\log\Gamma(\beta+m\alpha) - 6\log\Gamma(\beta+(m-1)\alpha) + \log\Gamma(\beta+(m-2)\alpha).$$

We take $g(t) = \log\Gamma(t)$, $x = \beta + m\alpha$, and $h = \alpha$. Since $g'(t) = \psi(t)$,

$$g^{(4)} = \psi^{(3)}(t) = 6\sum_{k=0}^{\infty}(k+t)^{-4}$$

is positive and decrease. Thus, by Lemma 1.3.11, we obtain

$$\log\left(\frac{{u_{m+1}}^3}{u_m}\right) < 2\alpha^3\psi''(x) + \alpha^4\left(\frac{7}{4}\psi^{(3)}(x) + \frac{2}{3}\psi^{(3)}(x-2\alpha)\right).$$

Since

$$\psi''(x) = \sum_{k=0}^{\infty}\frac{-2}{(k+x)^3} < -2\int_x^{\infty}\frac{1}{t^3}dt = -\frac{1}{x^2} \qquad (x>0)$$

and

$$\psi^{(3)}(x) = \sum_{k=0}^{\infty}\frac{6}{(k+x)^3} < 6\int_{x-1}^{\infty}\frac{1}{t^4}dt = \frac{2}{(x-1)^3} \qquad (x>1),$$

we obtain

$$\log\left(\frac{{u_{m+1}}^3}{u_m}\right) < -2\frac{\alpha^3}{x^2} + \alpha^4\left(\frac{7}{2(x-1)^3} + \frac{4}{3(x-2\alpha-1)^3}\right).$$

To complete the proof, we must show that the last expression is negative, i.e.,

$$\frac{7}{2(x-1)^3} + \frac{4}{3(x-2\alpha-1)^3} < \frac{2}{\alpha x^2}. \tag{1.38}$$

(1.38) can be rewritten in the form

$$\frac{7}{2}\left(\frac{\alpha}{x-1}\right)\left(\frac{x}{x-1}\right)^2 + \frac{3}{4}\left(\frac{\alpha}{x-2\alpha-1}\right)\left(\frac{x}{x-2\alpha-1}.\right)^2 < 2. \tag{1.39}$$

The function $t/(t-a)$ $(a > 0)$ is decreasing on $t > a$, and by the condition $m \geq 5$, the inequality $x = \beta + m\alpha \geq (m+2)\alpha \geq 7\alpha \geq 84$ holds. Hence, the following estimates hold:

$$\frac{1}{\alpha}(x-1) = \frac{x}{\alpha} - \frac{1}{\alpha} \geq 7 - \frac{1}{12} = \frac{83}{12},$$

$$\frac{1}{\alpha}(x-2\alpha-1) \geq 7 - 2 - \frac{1}{12} = \frac{59}{12},$$

$$\frac{x}{x-1} < \frac{84}{83},$$

and

$$\frac{x}{x-2\alpha-1} < \frac{7\alpha}{5\alpha-1} = \frac{7}{5-(1/\alpha)} \leq \frac{7}{5-(1/12)} \leq \frac{84}{59}.$$

This implies that the left-hand side of (1.39) does not exceed

$$\frac{7}{2}\left(\frac{12}{83}\right)\left(\frac{84}{83}\right)^2 + \frac{4}{3}\left(\frac{12}{59}\right)\left(\frac{84}{59}\right)^2 < 2.$$

The lemma is proved. □

Proposition 1.3.13. *Let $\alpha \geq 12$ and $2\alpha < \beta \leq 3\alpha$. If $3 \leq n \leq \lfloor \alpha/4 \rfloor$ and $1 \leq \nu \leq n-2$, the inequality (1.35) holds, i.e.,*

$$(1 + b_{n,\nu}(1-b_{n,\nu})^{-1}(1-u_{n,\nu+1})\prod_{j=1}^{\nu} u_{n,j} < 1.$$

Proof. The proof will be divided into three cases.

Case 1. $3 \leq n \leq 12$ and $1 \leq \nu \leq n-2$.

Case 2. $13 \leq n \leq \lfloor \alpha/4 \rfloor$ and $(n/2) - 1 \leq \nu \leq n-2$.

Case 3. $13 \leq n \leq \lfloor \alpha/4 \rfloor$ and $1 \leq \nu < (n/2) - 1$.

In the Cases 1 and 2, we will prove (1.36) instead of (1.35).

We first consider Case 1. Since

$$u_{n,1} = R_{n+1}{R_n}^{-2}R_{n-1} = {A_{n+1}}^{-1}{A_n}^{3}{A_{n-1}}^{-3}A_{n-2},$$

we have

$$\log u_{n,1} = \log\Gamma(\beta+(n+1)\alpha) - 3\log\Gamma(\beta+n\alpha) + 3\log\Gamma(\beta+(n-1)\alpha) - \log\Gamma(\beta+(n-2)\alpha).$$

Then, we obtain that for some $\xi \in ((n-2)\alpha, (n+1)\alpha)$,

$$\log u_{n,1} = \alpha^3(\log\Gamma(\beta+z))^{(3)}|_{z=\xi} = \alpha^3\psi''(\beta+\xi) = -2\alpha^3\sum_{k=0}^{\infty}(k+\beta+\xi)^{-3}.$$

This implies the estimate
(1.40)

$$\begin{aligned}\log u_{n,1} &< -2\alpha^3\sum_{k=0}^{\infty}(k+\beta+(n+1)\alpha)^{-3}\\ &< -2\alpha^3\int_0^{\infty}(t+\beta+(n+1)\alpha)^{-3}dt = -\alpha\left(\frac{\beta}{\alpha}+n+1\right)^{-2} \qquad (n\in\mathbb{N}).\end{aligned}$$

Since $\beta/\alpha \le 3$ and $\alpha \ge 12$, we obtain

$$u_{n,1} < \exp\left(-\frac{3}{64}\right) < \exp\left(-\frac{1}{25}\right) \qquad (n \le 12). \tag{1.41}$$

By Lemma 1.3.4, for any $n,\ \nu \in \mathbb{N}$, we have

$$b_{n,\nu} = \frac{R_n}{R_{n+\nu+1}} \le \frac{R_n}{R_{n+2}} = \frac{R_n}{R_{n+1}}\frac{R_{n+1}}{R_{n+2}} \le \exp\left(-\frac{1}{(n/\alpha)+(3/\alpha)} - \frac{1}{(n/\alpha)+(4/\alpha)}\right).$$

Since $n/\alpha \le 1/4$ and $\alpha \ge 12$, we have

$$b_{n,\nu} < \exp\left(-\frac{26}{7}\right) < \frac{1}{26}.$$

This implies

$$b_{n,\nu}(1-b_{n,\nu})^{-1} < \frac{1}{25} \qquad (n \le \lfloor \alpha/4 \rfloor,\ \alpha \ge 12,\ 1 \le \nu \le n-2). \tag{1.42}$$

From (1.41) and (1.42), we conclude that if $3 \le n \le 12$, then the left-hand side of (1.36) does not exceed

$$\frac{26}{25}\exp\left(-\frac{1}{25}\right) < 1.$$

This finishes the proof in Case 1.

In Case 2, by Lemma 1.3.4. we obtain

$$\begin{aligned} b_{n,\nu} = \prod_{j=0}^{\nu}\left(\frac{R_{n+j}}{R_{n+j+1}}\right) &< \exp\left(-\alpha\sum_{j=0}^{\nu}\frac{1}{n+j+(\beta/\alpha)}\right) \le \exp\left(-\alpha\cdot\frac{\nu+1}{n+\nu+(\beta/\alpha)}\right) \\ &\le \exp\left(-\alpha\cdot\frac{\nu+1}{n+\nu+3}\right) \le \exp\left(-\alpha\cdot\frac{n}{4n+2}\right) \le \exp\left(-\frac{\alpha}{5}\right). \end{aligned}$$

Since $x/(1-x) < 2x$ $(0 < x < (1/2))$ and $x < \exp(x/5)$ $(x \ge 52)$, the following estimate hold:

$$b_{n,\nu}(1-b_{n,\nu})^{-1} < 2\exp\left(-\frac{\alpha}{5}\right) < \frac{2}{\alpha}. \tag{1.43}$$

Now, from (1.40) we obtain

$$u_{n,1} < \exp\left(-\frac{\alpha}{(n+4)^2}\right) = \exp\left(-\frac{1}{\alpha((n/\alpha)+(4/\alpha))^2}\right).$$

Recall that $n/\alpha \le 1/4$ and $\alpha \ge 52$; then $((n/\alpha)+(4/\alpha))^2 \le 289/52$ and hence $u_{n,1} < \exp(-2/\alpha)$. Therefore,

$$(1+b_{n,\nu}(1-b_{n,\nu})^{-1})u_{n,1} < \left(1+\frac{2}{\alpha}\right)\exp\left(-\frac{2}{\alpha}\right) < 1.$$

This completes the proof in Case 2.

Finally, we consider Case 3. From the inequality (1.42), we obtain the following inequality

$$\left(1+\frac{1}{25}(1-u_{n,\nu+1})\right)\prod_{j=1}^{\nu}u_{n,j} < 1. \tag{1.44}$$

We will prove (1.44) by using the estimate (See [29, p.304])

$$u_{n,\nu}{}^{6} < u_{n,\nu+1} \qquad (1 \leq \nu < \frac{1}{2}n - 1; n \geq 13). \tag{1.45}$$

(Note that Lemma 1.3.12 is the key to prove inequality (1.45).) It allows one to replace inequalities (1.44) by stronger inequalities

$$\left(1 + \frac{1}{25}(1 - u_{n,\nu}{}^{6})\right) u_{n,\nu} < 1. \tag{1.46}$$

It is enough to prove (1.46). Consider the function

$$f(t) = \left(1 + \frac{1}{25}(1 - t^{6})\right) t \qquad (0 < t < 1).$$

Then $f(t)$ is increasing on $[0, 1]$, and since $f(1) = 1$, we see that $f(t) < 1$ for all $t \in (0, 1)$. From this and the fact $u_{n,\nu} < 1$, we obtain (1.46). Thus we obtain the desired result in the last case too. □

Proposition 1.3.14. *Let* $\alpha \geq 12$ *and* $2\alpha < \beta \leq 3\alpha$. *If* $3 \leq n \leq \lfloor \alpha/4 \rfloor$, *then*

$$A_{2n} R_n{}^{2n} < v_{n-1}(R_n).$$

Proof. By the definition of $v_\nu(x)$, we have

$$A_{2n} R_n{}^{2n} < A_{2n-1} R_n{}^{2n-1} - A_0.$$

And we obtain

$$\left(\frac{A_0}{A_{2n-1}}\right) R_n{}^{1-2n} + \left(\frac{A_{2n}}{A_{2n-1}}\right) R_n < 1,$$

which can be rewritten in the following form:

$$\prod_{j=1}^{n-1} u_{n,j} + \frac{R_n}{R_{2n}} < 1. \tag{1.47}$$

Representing $u_{n,j}$ by formula

$$u_{n,j} = u_{n,1}{}^{j} \prod_{k=1}^{j-1} (u_{n-k,1} u_{n+k,1})^{j-k},$$

omitting factors less than 1, and using the fact that $\{R_k\}$ is increasing, we strengthen inequality (1.47):

$$u_{n,1}^{\frac{n(n-1)}{2}} + \frac{R_n}{R_{n+1}} < 1. \tag{1.48}$$

To obtain an upper estimate of the left-hand side of (1.48) (we denote it by U_n), we use inequalities (1.26) and (1.40). We have

$$U_n < \exp(-2)+\exp\left(-\frac{\alpha n(n-1)}{2((\beta/\alpha)+n+1)^2}\right) < \exp(-2)+\exp\left(-\frac{\alpha n(n-1)}{2(n+4)^2}\right).$$

Since

$$\frac{n(n-1)}{(n+4)^2} > \frac{6}{49} \qquad (n \geq 3)$$

and $\alpha \geq 12$ we have

$$U_n < \exp(-2) + \exp\left(-\frac{36}{49}\right) < 1,$$

which was required. □

Therefore, Propositions 1.3.10-1.3.14 complete the proof of Proposition 1.3.8.

1.4 Proof of Theorem 1.1.5

We first prove that $\alpha \mapsto \phi(\alpha, 3\alpha)$ is increasing on $[4, \infty)$. Let $f(\alpha) = \phi(\alpha, 3\alpha)$. Then the logarithmic derivative of $f(\alpha)$ can be expressed as follows:

$$\frac{f'(\alpha)}{f(\alpha)} = 2f_1(\alpha) + 2f_2(\alpha) + f_3(\alpha) + 2f_4(\alpha)$$

where

$$f_1(\alpha) = -\frac{\pi}{\alpha}\cot\frac{\pi}{\alpha} + 1,$$

$$f_2(\alpha) = -\log(4\alpha - 1) + \frac{1}{4\alpha - 1} + \psi(2\alpha) + \log 2,$$

$$f_3(\alpha) = -\frac{\pi^2}{\alpha^3}\csc^2\frac{\pi}{\alpha} + 4\frac{\pi^2}{\alpha^2}\csc^2\frac{\pi}{\alpha} - 1.8\log\pi + \log 2 - 2.38,$$

and

$$f_4(\alpha) = \log\alpha - \frac{1}{\alpha} + \log\sin\frac{\pi}{\alpha} + \frac{\pi}{\alpha^2}\cot\frac{\pi}{\alpha} - 0.1\log\pi - 1.5\log 2 + 0.19.$$

In order to prove that $f' \geq 0$, we will show that f_1, f_2, f_3 and $f_4 \geq 0$ for $\alpha \geq 4$.

Since $\tan t > t$ for all $t \in (0, \pi/4]$, we have $f_1(\alpha) > 0$. And by Lemma 1.2.1, we obtain

$$f_2(\alpha) \geq -\log\left(2 - \frac{1}{2\alpha}\right) + \log 2 \geq 0.$$

In the case of f_3 and f_4, we put $\pi/\alpha = t$. Then we obtain

$$f_3\left(\frac{\pi}{t}\right) = \left(\frac{t}{\sin t}\right)^2\left(4 - \frac{t}{\pi}\right) - 1.8\log\pi - \log 2 - 2.38 \qquad (0 < t \leq \frac{\pi}{4}).$$

Since $t/(\sin t)$ is increasing on $(0, \pi/4]$ and approaches to 1 as $t \to 0$,

$$f_3\left(\frac{\pi}{t}\right) \geq \frac{15}{4} - 1.8\log\pi + \log 2 - 2.38 > 0.002.$$

And we also have

$$f_4\left(\frac{\pi}{t}\right) = -\log t + \log\sin t - \frac{t}{\pi} + \frac{t^2}{\pi}\cot t + 0.9\log\pi - 1.5\log 2 + 0.19 \qquad (0 < t \leq \frac{\pi}{4}).$$

If we write $f_4(\pi/t) = g(t)$, we obtain

$$g'(t) = \left(\cot t - \frac{1}{t}\right) + \frac{2}{\pi}t\cot t - \frac{1}{\pi}\left(\frac{t}{\sin t}\right)^2 - \frac{1}{\pi}.$$

Since $t\cot t < 1$ and $t/(\sin t)$ is increasing on $(0, \pi/4]$, we have $g'(t) \leq 0$. Thus $g(t) \geq g(\pi/4) > 0.02$.

Therefore, $f'(\alpha) > 0$, which completes the first part of proof.

We next show that for each fixed $\alpha \geq 4$ the function $\beta \mapsto \phi(\alpha, \beta)$ is decreasing on $(2\alpha, 3\alpha]$. For each fixed $\alpha \geq 4$, let

$$h_\alpha(x) = \phi(\alpha, x) \qquad (2\alpha < x \leq 3\alpha).$$

Then we have

$$\frac{h'_\alpha(x)}{h_\alpha(x)} = k_\alpha(x) + c(\alpha)$$

where

$$k_\alpha(x) = \frac{2\alpha}{x+\alpha-1} - \log(x+\alpha-1) + \psi(x-\alpha)$$

and

$$c(\alpha) = \log\left(\frac{\alpha}{\pi}\sin\frac{\pi}{\alpha}\right) + \frac{\pi}{\alpha}\cot\frac{\pi}{\alpha} - 1.$$

By Lemma 1.2.1, we obtain

$$k_\alpha(x) \le \frac{1}{x+\alpha-1} + \frac{2\alpha-1}{x+\alpha-1} + \log\left(1 - \frac{2\alpha-1}{x+\alpha-1}\right).$$

Since $t + \log(1-t)$ is decreasing on $(0,1)$ and

$$0 < \frac{2\alpha-1}{4\alpha-1} < \frac{2\alpha-1}{x+\alpha-1} \le \frac{2\alpha-1}{3\alpha-1} < 1,$$

we have

$$\begin{aligned} k_\alpha(x) &\le \frac{1}{3\alpha-1} + \frac{2\alpha-1}{4\alpha-1} + \log\left(1 - \frac{2\alpha-1}{4\alpha-1}\right) \\ &\le \frac{1}{11} + \frac{1}{2} + \log\left(\frac{8}{15}\right) < -0.03. \end{aligned}$$

Now, we put $\alpha = \pi/t$, so that we obtain

$$c(\alpha) = c\left(\frac{\pi}{t}\right) = \log\left(\frac{\sin t}{t}\right) + t\cot t - 1 \qquad (0 < t \le \frac{\pi}{4}).$$

Since $(\sin t)/t$ is decreasing on $(0, \pi/4]$ and approaches the limit 1 as $t \to 0$, we have $\log((\sin t)/t) \le 0$. Thus

$$c\left(\frac{\pi}{t}\right) \le t\cot t - 1 \le 0.$$

Therefore, $h'_\alpha(x) < 0$ and the second part of proof is completed.

Lastly, we can compute $\phi(4.07, 12.21) > 0.512$, which completes the proof.

Chapter 2

Pólya-Wiman properties of differential operators

Let $\phi(x) = \sum \alpha_n x^n$ be a formal power series with real coefficients, and let D denote differentiation. In this chapter, we will show that "for every real polynomial f there is a positive integer m_0 such that $\phi(D)^m f$ has only real zeros whenever $m \geq m_0$" if and only if "$\alpha_0 = 0$ or $2\alpha_0\alpha_2 - \alpha_1^2 < 0$", and that if ϕ does not represent a Laguerre-Pólya function, then there is a Laguerre-Pólya function f of genus 0 such that for every positive integer m, $\phi(D)^m f$ represents a real entire function having infinitely many nonreal zeros.

2.1 Pólya-Wiman property

A real entire function is an entire function which takes real values on the real axis. If f is a real entire function, we denote the number of nonreal zeros (counting multiplicities) of f by $Z_C(f)$. (If f is identically equal to 0, we set $Z_C(f) = 0$.) A real entire function f is said to be of *genus* 1^* if it can be expressed in the form

$$f(x) = e^{-\gamma x^2} g(x),$$

where $\gamma \geq 0$ and g is a real entire function of genus at most 1. If f is a real entire function of genus 1^* and $Z_C(f) = 0$, then f is called a *Laguerre-Pólya* *function* and we write $f \in \mathcal{LP}$. We denote by $\mathcal{LP}^*$ the class of real entire functions f of genus 1^* such that $Z_C(f) < \infty$. It is well known that $f \in \mathcal{LP}$ if and only if there is a sequence $\langle f_n \rangle$ of real polynomials such that $Z_C(f_n) = 0$ for all n and $f_n \to f$ uniformly on compact sets in the complex plane. (See Chapter 8 of [19] and [20, 23, 27].) From this and an elementary argument based on Rolle's theorem, it follows that the classes $\mathcal{LP}$ and $\mathcal{LP}^*$ are closed under differentiation, and that $Z_C(f) \geq Z_C(f')$ for all $f \in \mathcal{LP}^*$. The *Pólya-Wiman theorem* states that for every $f \in \mathcal{LP}^*$ there is a positive integer m_0 such that $f^{(m)} \in \mathcal{LP}$ for all $m \geq m_0$ [6, 7, 14, 17, 26]. On the other hand, it follows from recent results of W. Bergweiler, A. Eremenko and J. Langley that if f is a real entire function, $Z_C(f) < \infty$ and $f \notin \mathcal{LP}^*$, then $Z_C(f^{(m)}) \to \infty$ as $m \to \infty$ [1, 18].

Let ϕ be a formal power series given by

$$\phi(x) = \sum_{n=0}^{\infty} \alpha_n x^n.$$

For convenience we express the n-th coefficient α_n of ϕ as $\phi^{(n)}(0)/n!$ even when the radius of convergence is equal to 0. If f is an entire function and the series

$$\sum_{n=0}^{\infty} \alpha_n f^{(n)}$$

converges uniformly on compact sets in the complex plane, so that it represents an entire function, we write $f \in \operatorname{dom} \phi(D)$ and denote the entire function by $\phi(D)f$. For $m \geq 2$ we denote by $\operatorname{dom} \phi(D)^m$ the class of entire functions f such that $f, \phi(D)f, \ldots, \phi(D)^{m-1}f \in \operatorname{dom} \phi(D)$. It is obvious that if f is a polynomial, then $f \in \operatorname{dom} \phi(D)^m$ for all m. For more general restrictions on the growth of ϕ and f under which $f \in \operatorname{dom} \phi(D)^m$ for all m, see [3, 5].

The following version of the Pólya-Wiman theorem for the operator $\phi(D)$ was established by T. Craven and G. Csordas.

Theorem 2.1.1 ([5, Theorem 2.4])**.** *Suppose that ϕ is a formal power series with real coefficients, $\phi'(0) = 0$ and $\phi''(0)\phi(0) < 0$. Then for every real polynomial f there is a positive integer m_0 such that all the zeros of $\phi(D)^m f$ are real and simple whenever $m \geq m_0$.*

Remark. The assumption implies that $\phi(0) \neq 0$. On the other hand, if $\phi(0) = 0$ and f is a real polynomial, then it is trivial to see that $Z_C(\phi(D)^m f) \to 0$ as $m \to \infty$. (Recall that we have set $Z_C(f) = 0$ if f is identically equal to 0.)

We also have the following version, which is a consequence of the results in Section 3 of [5].

Theorem 2.1.2. *Suppose that $\phi \in \mathcal{LP}$ (ϕ represents a Laguerre-Pólya function), $f \in \mathcal{LP}^*$, and that f is of order less than 2. Then $f \in \operatorname{dom} \phi(D)^m$, $\phi(D)^m f \in \mathcal{LP}^*$ and $Z_C(\phi(D)^m f) \geq Z_C(\phi(D)^{m+1} f)$ for all m. Furthermore, if ϕ is not of the form $\phi(x) = ce^{\gamma x}$ with $c \neq 0$, then $Z_C(\phi(D)^m f) \to 0$ as $m \to \infty$.*

Remarks. (1) If $\phi(x) = ce^{\gamma x}$, then

$$\phi(D)f(x) = \sum_{n=0}^{\infty} \frac{c\gamma^n}{n!} f^{(n)}(x) = cf(x+\gamma)$$

for every entire function f. Hence $Z_C(\phi(D)^m f) = Z_C(f)$ for all m whenever $c, \gamma \in \mathbb{R}$, $c \neq 0$ and f is a real entire function. We also remark that $\phi(x) = ce^{\gamma x}$ with $c \neq 0$ if and only if $\phi(0) \neq 0$ and $\phi^{(n)}(0)\phi(0)^{n-1} - \phi'(0)^n = 0$ for all n.

(2) From [5, Lemma 3.2], [15, Theorem 2.3] and the arguments given in [3], it follows that the restriction "f is of order less than 2" can be weakened to "ϕ or f is of genus at most 1". See also [5, Theorem 3.3].

(3) In the case where ϕ is of genus 2, that is, ϕ is of the form $\phi(x) = e^{-\gamma x^2}\psi(x)$, where $\gamma > 0$ and $\psi \in \mathcal{LP}$ is of genus at most 1, we have the following stronger result: If f is a real entire function of genus at most 1, and if the imaginary parts of the zeros of f are uniformly bounded, then $f \in \operatorname{dom} \phi(D)^m$ and $Z_C(\phi(D)^m f) \geq Z_C(\phi(D)^{m+1} f)$ for all m, and $Z_C(\phi(D)^m f) \to 0$ as $m \to \infty$, even when f has infinitely many nonreal zeros. See [3], [5, Lemma3.2], [9, Theorems 9a, 13 and 14] and [15, Theorem 2.3].

In this chapter, we complement Theorem 2.1.1 and 2.1.2 above. Let ϕ be a formal power series with real coefficients and f be a real entire function. If $f \in \operatorname{dom} \phi(D)^m$ for all m and $Z_C(\phi(D)^m f) \to 0$ as $m \to \infty$, then we will say that ϕ (or the corresponding operator $\phi(D)$) has the *Pólya-Wiman property* with respect to f. For instance, if f is a real entire function and $Z_C(f) < \infty$, then the operator D $(= d/dx)$ has the Pólya-Wiman property with respect to f if and only if $f \in \mathcal{LP}^*$.

2.2 Pólya-Wiman property with respect to real polynomials

Theorem 2.1.1 gives a sufficient condition for ϕ to have the Pólya-Wiman property with respect to arbitrary real polynomials. The following two theorems imply that this is the case if and only if $\phi(0) = 0$ or $\phi''(0)\phi(0) - \phi'(0)^2 < 0$.

Theorem 2.2.1. *Suppose that ϕ is a formal power series with real coefficients, $\phi(0) \neq 0$ and $\phi''(0)\phi(0) - \phi'(0)^2 < 0$. Then for every real polynomial f there is a positive integer m_0 such that all the zeros of $\phi(D)^m f$ are real and simple whenever $m \geq m_0$.*

Theorem 2.2.2. *Suppose that ϕ is a formal power series with real coefficients, $\phi(0) \neq 0$, $\phi''(0)\phi(0) - \phi'(0)^2 \geq 0$, ϕ is not of the form $\phi(x) = ce^{\gamma x}$ with $c \neq 0$, f is a real polynomial, and that*

$$\deg f \geq \min\{n \geq 2 : \phi^{(n)}(0)\phi(0)^{n-1} - \phi'(0)^n \neq 0\}.$$

Then there is a positive integer m_0 such that $Z_C(\phi(D)^m f) > 0$ for all $m \geq m_0$.

If $\phi \in \mathcal{LP}$ is not of the form $\phi(x) = ce^{\gamma x}$ with $c \neq 0$, then it is easy to see that $\phi(0) = 0$ or $\phi''(0)\phi(0) - \phi'(0)^2 < 0$ (for a proof, see [4, 13]); hence Theorem 2.1.2 as well as Theorem 2.2.1 implies that ϕ has the Pólya-Wiman property with respect to arbitrary real polynomials.

Theorems 2.2.1 and 2.2.2 are almost immediate consequences of Theorems 2.2.3 and 2.2.4 below, which are proved by refining the arguments of Craven and Csordas given in Section 2 of [5].

For notational clarity, we denote the monic monomial of degree d by M^d, that is, $M^d(x) = x^d$. With this notation, we have

$$\left(\exp\left(\beta D^p\right) M^d\right)(x) = \sum_{k=0}^{\lfloor d/p \rfloor} \frac{d!\beta^k}{k!(d-pk)!} x^{d-pk} \qquad (\beta \in \mathbb{C};\ d, p = 1, 2, \dots).$$

Theorem 2.2.3. *Suppose that ϕ is a formal power series with complex coefficients, $\phi(0) = 1$, ϕ is not of the form $\phi(x) = e^{\gamma x}$,*

$$p = \min\{n : n \geq 2 \text{ and } \phi^{(n)}(0) \neq \phi'(0)^n\},$$

$\alpha = \phi'(0)$ and $\beta = \left(\phi^{(p)}(0) - \phi'(0)^p\right)/p!$. Suppose also that f is a monic complex polynomial of degree d, and $f_1, f_2, \dots$ are given by

$$f_m(x) = m^{-d/p} \left(\phi(D)^m f\right)\left(m^{1/p} x - m\alpha\right). \tag{2.1}$$

Then $f_m \to \exp\left(\beta D^p\right) M^d$ uniformly on compact sets in the complex plane.

Theorem 2.2.4. *Suppose that d and p are positive integers, $p \geq 2$, $q = \lfloor d/p \rfloor$ and $r = d - pq$.*

(1) If $q = 0$ $(d < p)$, then $\exp\left(-D^p\right) M^d = M^d$.

(2) If $q \geq 1$, then $\exp\left(-D^p\right) M^d$ has exactly q distinct positive zeros; and if we denote them by $\rho_1, \dots, \rho_q$, then

$$\left(\exp\left(-D^p\right) M^d\right)(x) = x^r \prod_{j=1}^{q} \prod_{k=0}^{p-1} \left(x - e^{2k\pi i/p} \rho_j\right).$$

Remark. The d-th Hermite polynomial H_d is given by

$$H_d(x) = \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{(-1)^k d!}{k!(d-2k)!} (2x)^{d-2k}.$$

Thus we have $\left(\exp\left(-D^2\right) M^d\right)(x) = H_d(x/2)$ for all d, and Theorem 2.2.4 implies the well known fact that all the zeros of the Hermite polynomials are real and simple.

Corollary 2.2.5. *If $\beta > 0$, then all the zeros of* $\exp(-\beta D^2) M^d$ *are real and simple, and* $\exp(\beta D^2) M^d$ *has exactly* $2\lfloor d/2 \rfloor$ *distinct purely imaginary zeros; and if* $\beta \neq 0$ *and* $3 \leq p \leq d$*, then* $\exp(\beta D^p) M^d$ *has nonreal zeros.*

This corollary is an immediate consequence of Theorem 2.2.4 and the following relations which are trivially proved: If $\beta > 0$ and $\rho^p = -1$, then

$$\left(\exp(-\beta D^p) M^d\right)(x) = \beta^{d/p}\left(\exp(-D^p) M^d\right)\left(\frac{x}{\beta^{1/p}}\right)$$

and

$$\left(\exp(\beta D^p) M^d\right)(x) = \left(\rho\beta^{1/p}\right)^d \left(\exp(-D^p) M^d\right)\left(\frac{x}{\rho\beta^{1/p}}\right).$$

Proof of Theorem 2.2.1. Let f be a (nonconstant) real polynomial. Since multiplication by a nonzero constant does not change the zeros of a polynomial, we may assume that f is monic and $\phi(0) = 1$. Let $d = \deg f$, $\alpha = \phi'(0)$, $\beta = (\phi''(0) - \phi'(0)^2)/2$, and $f_1, f_2, \ldots$ be given by

$$f_m(x) = m^{-d/2}\left(\phi(D)^m f\right)(m^{1/2}x - m\alpha). \tag{2.2}$$

Then $\beta < 0$, and Theorem 2.2.3 implies that $f_m \to \exp(\beta D^2) M^d$ uniformly on compact sets in the complex plane. We have $\deg f_m = d = \deg(\exp(\beta D^2) M^d)$ for all m; and since $\beta < 0$, the corollary to Theorem 2.2.4 implies that all the zeros of $\exp(\beta D^2) M^d$ are real and simple. Hence the intermediate value theorem implies that there is a positive integer m_0 such that all the zeros of f_m are real and simple whenever $m \geq m_0$, and (2.2) shows that the same holds for $\phi(D)^m f$. □

Proof of Theorem 2.2.2. Again, we may assume that f is monic and $\phi(0) = 1$. Let $d = \deg f$, and let p, α, β and the polynomials $f_1, f_2, \ldots$ be as in Theorem 2.2.3. We have $\beta \neq 0$; and in the case where $p = 2$ we must have $\beta > 0$, because we are assuming that $\phi''(0) - \phi'(0)^2 \geq 0$. Hence the corollary to Theorem 2.2.4 implies that $Z_C\left(\exp(\beta D^p) M^d\right) > 0$. By Theorem 2.2.3, $f_m \to \exp(\beta D^p) M^d$ uniformly on compact sets in the complex plane. Hence Rouche's theorem implies that there is a positive integer m_0 such that $Z_C(f_m) > 0$ whenever $m \geq m_0$, and (2.1) shows that the same holds for $\phi(D)^m f$. □

2.2.1 Proof of Theorem 2.2.3

In order to prove Theorem 2.2.3, we need some preliminaries. Let $\mathbb{C}[x]$ denote the (complex) vector space of complex polynomials, let $\mathbb{C}[x]^d$ denote the $(d+1)$-dimensional subspace of $\mathbb{C}[x]$ whose members are complex polynomials of degree $\leq d$, and let $\| \ \|_\infty$ denote the norm on $\mathbb{C}[x]$ defined by

$$\|f\|_\infty = \max\{|f^{(k)}(0)/k!| : 0 \leq k \leq \deg f\}.$$

Note that if $\langle f_m \rangle$ is a sequence of polynomials in $\mathbb{C}[x]^d$, then $\|f_m\|_\infty \to 0$ if and only if $f_m \to 0$ uniformly on compact sets in the complex plane. When ϕ is a formal power series (with complex coefficients) and d is a nonnegative integer, we denote the *operator norm* of $\phi(D)|_{\mathbb{C}[x]^d}$ with respect to $\| \ \|_\infty$ by $\|\phi(D)\|_d$, that is,

$$\|\phi(D)\|_d = \sup\{\|\phi(D)f\|_\infty : f \in \mathbb{C}[x]^d \text{ and } \|f\|_\infty \leq 1\}.$$

If we denote the d-th partial sum of ϕ by $\phi|_d$, that is,

$$\phi|_d(x) = \sum_{k=0}^{d} \frac{\phi^{(k)}(0)}{k!} x^k,$$

then the restriction of $\phi(D)$ to $\mathbb{C}[x]^d$ is completely determined by $\phi|_d$. Hence there are positive constants A_d and B_d such that

$$A_d\|\phi(D)\|_d \leq \|\phi|_d\|_\infty \leq B_d\|\phi(D)\|_d$$

for all ϕ.

For $c \neq 0$ we define the *dilation operator* Δ_c by

$$(\Delta_c f)(x) = f(cx).$$

It is then easy to see that

$$\Delta_c(\phi(D)f) = \phi(c^{-1}D)(\Delta_c f) \qquad (c \neq 0), \tag{2.3}$$

whenever ϕ is a formal power series and $f \in \operatorname{dom} \phi(D)$.

Proof of Theorem 2.2.3. Let $r = \max\{p, d\}$. If $\tilde{\phi}$ is a formal power series and $\tilde{\phi}|_r = \phi|_r$, then $\tilde{\phi}$ satisfies the identical assumptions in the theorem that are satisfied by ϕ, and we have $\tilde{\phi}(D)^m f = \phi(D)^m f$ for all m. In other words, the theorem is about the first $r+1$ coefficients of ϕ only, and the coefficients $\phi^{(n)}(0)/n!$, $n > r$, are irrelevant to the theorem. For this reason, we may assume that $\phi^{(n)}(0) = 0$ for all $n > r$. Then there is a neighborhood U of 0 in the complex plane and there is an analytic function ψ in U such that

$$\log \phi(x) = \alpha x + \beta x^p + x^{p+1}\psi(x) \qquad (x \in U).$$

We substitute $m^{-1/p}x$ for x and multiply both sides by m to obtain

$$m \log \phi\left(m^{-1/p}x\right) = m^{1-\frac{1}{p}}\alpha x + \beta x^p + m^{-1/p}x^{p+1}\psi\left(m^{-1/p}x\right) \qquad (x \in m^{1/p}U).$$

If we put

$$\exp\left(-m^{1-\frac{1}{p}}\alpha x\right)\phi\left(m^{-1/p}x\right)^m - \exp\left(\beta x^p\right) = R_m(x),$$

then R_m is an entire function and we have

$$R_m(x) = \exp\left(\beta x^p\right)\left(\exp\left(m^{-1/p}x^{p+1}\psi\left(m^{-1/p}x\right)\right) - 1\right) \qquad (x \in m^{1/p}U).$$

It is then clear that

$$\sup_{|x|\le R} |R_m(x)| = O\left(m^{-1/p}\right) \qquad (m \to \infty)$$

for every $R > 0$, and this implies that

$$\left\|\exp\left(-m^{1-\frac{1}{p}}\alpha D\right)\phi\left(m^{-1/p}D\right)^m - \exp\left(\beta D^p\right)\right\|_d = O\left(m^{-1/p}\right) \tag{2.4}$$

as $m \to \infty$. Since f is monic and of degree d, it follows that

$$\left\|m^{-d/p}\Delta_{m^{1/p}}f - M^d\right\|_\infty = O\left(m^{-1/p}\right) \qquad (m \to \infty). \tag{2.5}$$

It is easy to see that (2.1) is equivalent to

$$f_m = m^{-d/p}\exp\left(-m^{1-\frac{1}{p}}\alpha D\right)\Delta_{m^{1/p}}\left(\phi(D)^m f\right);$$

and (2.3) implies that the right hand side is equal to

$$\exp\left(-m^{1-\frac{1}{p}}\alpha D\right)\phi\left(m^{-1/p}D\right)^m\left(m^{-d/p}\Delta_{m^{1/p}}f\right).$$

Therefore we have

$$\left\|f_m - \exp\left(\beta D^p\right)M^d\right\|_\infty = O\left(m^{-1/p}\right) \qquad (m\to\infty),$$

by (2.4), (2.5) and the triangle inequality. This proves the theorem. □

2.2.2 Proof of Theorem 2.2.4

As we shall see soon, Theorem 2.2.4 is a consequence of a known result on Jensen polynomials and the fact that all the zeros of the classical Mittag-Leffler functions $E_{p,1}$, $p = 1, 2, \ldots$, are negative and simple. The following is a simplified version of [5, Proposition 4.1].

Proposition 2.2.6. *Suppose that $\phi \in \mathcal{LP}$, q is a positive integer and f is given by*

$$f(x) = \sum_{k=0}^{q}\binom{q}{k}\phi^{(k)}(0)x^k.$$

Suppose also that $\phi(0) \neq 0$ and ϕ is not of the form $\phi(x) = p(x)e^{\alpha x}$, where p is a polynomial and $\alpha \neq 0$. Then all the zeros of f are real and simple.

Remark. The polynomial f is called the q-th *Jensen polynomial* associated with ϕ.

For positive integers p and q, let $J_{(p,q)}$ denote the q-th Jensen polynomial associated with the classical Mittag-Leffler function $E_{p,1}$:

$$J_{(p,q)}(x) = \sum_{k=0}^{q}\binom{q}{k}E_p^{(k)}(0)x^k = \sum_{k=0}^{q}\frac{q!x^k}{(q-k)!(pk)!}.$$

Proposition 2.2.7. *The zeros of $J_{(p,q)}$ are all negative and simple for $p = 2, 3, \ldots$ and for $q = 1, 2, \ldots$.*

Proof. Suppose that $p \geq 2$ and $q \geq 1$. Then $E_{p,1}$ is of order $\leq 1/2$, hence it is not of the form $E_{p,1}(x) = p(x)e^{\alpha x}$ where p is a polynomial and $\alpha \neq 0$; and we have $E_{p,1}(0) = 1 \neq 0$. Since $E_{p,1} \in \mathcal{LP}$, Proposition 2.2.6 implies that all the zeros of $J_{(p,q)}$ are real and simple. Finally, they are all negative, because the coefficients of $J_{(p,q)}$ are all positive. □

Proof of Theorem 2.2.4. We have $d = pq + r$, $0 \leq r \leq p - 1$ and

$$\left(\exp\left(-D^p\right) M^d\right)(x) = x^r \sum_{k=0}^{q} \frac{(-1)^k d!}{k!(d-pk)!} x^{p(q-k)}.$$

The right hand side is of the form $x^r f(x^p)$, where f is a monic polynomial of degree q and $f(0) = (-1)^q d!/(q!r!) \neq 0$. From this, we see that (1) is trivial, $\exp\left(-D^p\right) M^d$ has exactly r zeros at the origin, and that the second assertion of (2) follows from the first one. If $a \neq 0$ is a zero of $\exp\left(-D^p\right) M^d$, then so are $e^{2k\pi i/p}a$, $k = 0, 1, \ldots, p-1$, and they are distinct. Since $\exp\left(-D^p\right) M^d$ has exactly $d = pq + r$ zeros in the whole plane and has exactly r zeros at the origin, it follows that $\exp\left(-D^p\right) M^d$ has at most q distinct positive zeros. Hence it is enough to show that if $q \geq 1$, then $\exp\left(-D^p\right) M^d$ has (at least) q distinct positive zeros.

Suppose that $q \geq 1$. We first consider the case where d is a multiple of p. In this case, we have $d = pq$, $r = 0$ and

$$\begin{aligned}
\left(\exp\left(-D^p\right) M^d\right)(x) &= \sum_{k=0}^{q} \frac{(-1)^k (pq)!}{k!(p(q-k))!} x^{p(q-k)} \\
&= \sum_{k=0}^{q} \frac{(-1)^{q-k} (pq)!}{(q-k)!(pk)!} x^{pk} \\
&= (-1)^q \frac{(pq)!}{q!} \sum_{k=0}^{q} \frac{q!}{(q-k)!(pk)!} (-x^p)^k \\
&= (-1)^q \frac{(pq)!}{q!} J_{(p,q)}(-x^p).
\end{aligned}$$

Since $p \geq 2$, Proposition 2.2.7 implies that all the zeros of $J_{(p,q)}$ are negative and simple. Hence $\exp\left(-D^p\right) M^d$ has exactly q (=deg $J_{(p,q)}$) distinct positive zeros.

Finally, the result for the remaining case follows from an inductive argument based on Rolle's theorem, because $\left(\exp\left(-D^p\right) M^{pq+r}\right)(0) = 0$ for $1 \leq r \leq p-1$,

$$\exp\left(-D^p\right) M^d = \frac{1}{d+1} D\left(\exp\left(-D^p\right) M^{d+1}\right),$$

and $\exp\left(-D^p\right) M^{p(q+1)}$ has exactly $q+1$ distinct positive zeros. □

2.2.3 Laguerre-Pólya class and Pólya-Wiman property with respect to real polynomials

In this section, we establish the following proposition.

Proposition 2.2.8. *Let ϕ be a formal power series with real coefficients. Then the following hold:*

(1) If $\phi \in \mathcal{LP}$, then $Z_C(\phi(D)f) \leq Z_C(f)$ for every $f \in \mathbb{R}[x]$.

(2) If $\phi \notin \mathcal{LP}$, then for every positive integer m there is an $f \in \mathbb{R}[x]$ such that $Z_C(\phi(D)^m f) = 0$ but $Z_C(\phi(D)^{m+1} f) > 0$.

Let ϕ be a formal power series with real coefficients. For $n = 1, 2, \ldots$ we define the polynomial $J_{(\phi,n)}$ by

$$J_{(\phi,n)}(x) = \sum_{k=0}^{n} \binom{n}{k} \phi^{(k)}(0) x^k.$$

Thus $J_{(\phi,n)}$ may be called the n-th Jensen polynomial associated with the formal power series ϕ. From

$$\left(\phi(D) M^n\right)(x) = \sum_{k=0}^{n} \frac{\phi^{(k)}(0)}{k!} \left(\frac{n!}{(n-k)!} x^{n-k}\right) = \sum_{k=0}^{n} \binom{n}{k} \phi^{(k)}(0) x^{n-k},$$

we see that $Z_C(J_{(\phi,n)}) = Z_C(\phi(D) M^n)$ for all n.

The following characterization of the class $\mathcal{LP}$ was established by Pólya and Schur [27].

Theorem 2.2.9. *We have $\phi \in \mathcal{LP}$ if and only if $Z_C(J_{(\phi,n)}) = 0$ for all n.*

Corollary 2.2.10. *We have $\phi \in \mathcal{LP}$ if and only if $Z_C(\phi(D)M^n) = 0$ for all n.*

Remark. Since $Z_C(Df) \leq Z_C(f)$ for every $f \in \mathbb{R}[x]$, and since $D\phi(D)M^n = n\phi(D)M^{n-1}$, we see that if $\phi \notin \mathcal{LP}$, then there is a positive integer n_0 such that $Z_C(\phi(D)M^n) > 0$ for all $n \geq n_0$.

The following two results are easily proved. (See, for instance, Problem 62 in Part V of [28] and Section 3 of [27], respectively.)

The Hermite-Poulain Theorem. *If ϕ is a real polynomial whose zeros are all real and f is a real polynomial, then $Z_C(\phi(D)f) \leq Z_C(f)$.*

Proposition 2.2.11. *For each fixed k we have*

$$\lim_{n\to\infty} \left(\Delta_{1/n} J_{(\phi,n)}\right)^{(k)}(0) = \phi^{(k)}(0).$$

As a consequence, we have

$$\lim_{n\to\infty} \left\| J_{(\phi,n)}(n^{-1}D)f - \phi(D)f \right\|_\infty = 0$$

for every polynomial f.

If $\phi(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ and $\alpha_0 \neq 0$, then the reciprocal ϕ^{-1} of ϕ is given as $\phi^{-1}(x) = \sum_{k=0}^{\infty} \beta_k x^k$, where the coefficients $\beta_0, \beta_1, \beta_2, \ldots$ are defined successively by

$$\beta_0 = \alpha_0^{-1} \quad \text{and} \quad \beta_n = -\alpha_0^{-1} \sum_{k=1}^{n} \alpha_k \beta_{n-k} \qquad (n = 1, 2, \ldots).$$

In this case, we have

$$\alpha_0\beta_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n} \alpha_k \beta_{n-k} = 0 \qquad (n = 1, 2, \ldots);$$

hence $\phi(D)\left(\phi^{-1}(D)f\right) = \phi^{-1}(D)\left(\phi(D)f\right) = f$ for every polynomial f.

Proof of Proposition 2.2.8. To prove (1), suppose that $\phi \in \mathcal{LP}$ and $f \in \mathbb{R}[x]$. We may assume that $\phi(D)f$ is not identically equal to 0. Then Proposition 2.2.11 implies that

$$Z_C(\phi(D)f) \leq \liminf_{n\to\infty} Z_C(J_{(\phi,n)}(n^{-1}D)f).$$

Since $\phi \in \mathcal{LP}$, Theorem 2.2.9 implies that $J_{(\phi,n)} \in \mathbb{R}[x]_0$ for all n, hence we have

$$Z_C(J_{(\phi,n)}(n^{-1}D)f) \leq Z_C(f) \qquad (n = 1, 2, \dots),$$

by the Hermite-Poulain theorem. This proves (1).

To prove (2), suppose that $\phi \notin \mathcal{LP}$. Then there is a positive integer d such that $Z_C(\phi(D)M^d) > 0$, by the Corollary 2.2.10. In particular, $\phi^{(k)}(0) \neq 0$ for some k, and hence there is a nonnegative integer r and there is a formal power series ψ such that $\phi(x) = x^r\psi(x)$ and $\psi(0) \neq 0$.

Let m be a positive integer. If we put $f = \psi^{-1}(D)^m M^{d+mr}$, then f is a real polynomial of degree $d + mr$ and we have

$$\phi(D)^m f = D^{mr}\psi(D)^m\psi^{-1}(D)^m M^{d+mr} = \frac{(d+mr)!}{d!}M^d,$$

hence $Z_C(\phi(D)^m f) = 0$, but

$$\phi(D)^{m+1} f = \frac{(d+mr)!}{d!}\phi(D)M^d$$

has a nonreal zero. This proves (2). □

We have introduced the reciprocal of a formal power series above.

Proposition 2.2.12. *Suppose that ϕ is a formal power series with real coefficients and $\phi(0) \neq 0$. Then each of the following implies the other two:*

(1) ϕ^{-1} has the Pólya-Wiman property with respect to arbitrary real polynomials.

(2) $\phi(0)\phi''(0) - \phi'(0)^2 > 0$.

(3) For every $f \in \mathbb{R}[x]$ the sequence $\langle Z_C(\phi(D)^m f)\rangle$ converges to $2\lfloor(\deg f)/2\rfloor$.

Remark. Note that a real polynomial of degree d can have at most $2\lfloor d/2 \rfloor$ nonreal zeros and (since $\phi(0) \neq 0$) we have $\deg \phi(D)^m f = \deg f$ for all m.

Proof. The equivalence (1)⇔(2) is a consequence of Theorem 2.1.2 and 2.2.1 and a simple calculation; and the implication (2)⇒(3) follows from Theorem 2.2.3 and Corollary 2.2.5.

To prove (3)⇒(2), suppose that (2) does not hold, that is, $\phi(0)\phi''(0) - \phi'(0)^2 \leq 0$. If $\phi(0)\phi''(0) - \phi'(0)^2 < 0$, then Theorem 2.2.1 implies that the sequence $\langle Z_C(\phi(D)^m f)\rangle$ converges to 0 for every $f \in \mathbb{R}[x]$; and if $\phi(0)\phi''(0) - \phi'(0)^2 = 0$, then for every $f \in \mathbb{R}[x]$ of degree ≤ 2 we have

$$\left(\phi(D)^m f\right)(x) = \phi(0)^m f\left(x + \phi(0)^{-1}\phi'(0)m\right) \qquad (m = 1, 2, \ldots).$$

Hence it is clear that (3) does not hold. □

The following is the reciprocal version of Proposition 2.2.8.

Proposition 2.2.13. *Suppose that ϕ is a formal power series with real coefficients and $\phi(0) \neq 0$. Then the following hold:*

(1) *If $\phi^{-1} \in \mathcal{LP}$, then $Z_C(f) \leq Z_C(\phi(D)f)$ for all $f \in \mathbb{R}[x]$.*

(2) *If $\phi^{-1} \notin \mathcal{LP}$, then for every positive integer m there is an $f \in \mathbb{R}[x]$ such that $Z_C(\phi(D)^m f) > 0$ but $Z_C(\phi(D)^{m+1} f) = 0$.*

Proof. If $\phi^{-1} \in \mathcal{LP}$ and $f \in \mathbb{R}[x]$, then (1) of Proposition 2.2.8 implies that

$$Z_C(f) = Z_C(\phi^{-1}(D)\phi(D)f) \leq Z_C(\phi(D)f),$$

hence (1) is proved.

To prove (2), suppose that $\phi^{-1} \notin \mathcal{LP}$. Then there is a positive integer d such that $Z_C(\phi^{-1}(D)M^d) > 0$, by the Corollary 2.2.10. Let m be a positive integer. If we put $f = \phi^{-1}(D)^{m+1}M^d$, then f is a real polynomial of degree d, $Z_C(\phi(D)^m f) = Z_C(\phi^{-1}(D)M^d) > 0$, but $Z_C(\phi(D)^{m+1} f) = Z_C(M^d) = 0$. Hence the result follows.

□

2.3 Pólya-Wiman property with respect to Laguerre -Pólya functions of genus 0

There are plenty of formal power series ϕ with real coefficients which satisfy $\phi(0) = 0$ or $\phi''(0)\phi(0) - \phi'(0)^2 < 0$, but do not represent Laguerre-Pólya functions. The following theorem, which is a strong version of the converse of Theorem 2.1.2, implies that if ϕ is one of such formal power series, then ϕ does not have the Pólya-Wiman property with respect to some (transcendental) Laguerre-Pólya function of genus 0, although it has the property with respect to arbitrary real polynomials.

Theorem 2.3.1. *Suppose that ϕ is a formal power series with real coefficients and ϕ does not represent a Laguerre-Pólya function. Then there is a Laguerre-Pólya function f of genus 0 such that $f \in \operatorname{dom}\phi(D)^m$ and $Z_C(\phi(D)^m f) = \infty$ for all positive integers m.*

Theorem 2.3.1 is a consequence of Pólya's characterization of the class $\mathcal{LP}$ given in [24, 27] and a diagonal argument.

Let ϕ be a formal power series. First of all, we need to find a sufficient condition for an entire function f to be such that $f \in \operatorname{dom}\phi(D)^m$ and $\phi(D)^m f$ is not identically equal to 0 for all positive integers m. Let $\langle C_n\rangle$ be a sequence of positive numbers. If $|\phi^{(n)}(0)| < C_n$ for all n, we write $\phi \ll \langle C_n\rangle$. More generally, if there are constants c and d such that $c > 0$, $d \geq 0$ and $\phi \ll \langle c(1+n)^d C_n\rangle$, then we will write $\phi \prec \langle C_n\rangle$.

Lemma 2.3.2. *Suppose that $\langle B_n\rangle$ is an increasing sequence of positive numbers,*

$$B_m B_n \leq B_0 B_{m+n} \qquad (m, n = 0, 1, 2, \dots), \tag{2.6}$$

ϕ and ψ are formal power series, $\phi, \psi \prec \langle n!B_n\rangle$, f is an entire function, and that $f \prec \langle (n!B_n)^{-1}\rangle$. Then $\phi\psi \prec \langle n!B_n\rangle$, $f \in \operatorname{dom}\phi(D)$, $\phi(D)f \prec \langle (n!B_n)^{-1}\rangle$ and $\psi(D)(\phi(D)f) = (\phi\psi)(D)f$.

Proof. Suppose that a, b are nonnegative constants, $\phi \ll \langle (1+n)^a n! B_n \rangle$ and $\psi \ll \langle (1+n)^b n! B_n \rangle$. Then

$$\begin{aligned}\left|(\phi\psi)^{(n)}(0)\right| &\le \sum_{k=0}^{n} \binom{n}{k} \left|\phi^{(k)}(0)\right| \left|\psi^{(n-k)}(0)\right| \\ &< B_0 (1+n)^{a+b+1} n! B_n \qquad (n = 0, 1, 2, \dots),\end{aligned}$$

hence $\phi\psi \prec \langle n! B_n \rangle$.

Now suppose that c is a nonnegative constant, $f \ll \langle (1+n)^c (n! B_n)^{-1} \rangle$, $R > 0$, and $|x| \le R$. Then

$$\left| \frac{\phi^{(n)}(0) f^{(n+k)}(0) x^k}{n! k!} \right| \le \frac{B_0 (1+n)^{a+c} (1+k)^c R^k}{n! (k!)^2 B_k}, \tag{2.7}$$

and we have

$$\sum_{n,k \ge 0} \frac{B_0 (1+n)^{a+c} (1+k)^c R^k}{n! (k!)^2 B_k} \le \sum_{n=0}^{\infty} \frac{(1+n)^{a+c}}{n!} \sum_{k=0}^{\infty} \frac{(1+k)^c R^k}{(k!)^2} < \infty.$$

Hence the double series

$$\sum_{n,k \ge 0} \frac{\phi^{(n)}(0) f^{(n+k)}(0) x^k}{n! k!}$$

converges absolutely and uniformly on compact sets in the complex plane. As a consequence, the series

$$\sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{n!} f^{(n)}$$

converges uniformly on compact sets in the complex plane, that is, $f \in \operatorname{dom} \phi(D)$. Furthermore, the absolute convergence of the double series implies that

$$\phi(D) f(x) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0) f^{(n+k)}(0)}{n!} \frac{x^k}{k!} \qquad (x \in \mathbb{C}),$$

from which we obtain

$$(\phi(D) f)^{(k)}(0) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0) f^{(n+k)}(0)}{n!} \qquad (k = 0, 1, 2, \dots),$$

and the assumptions imply that

$$\left|(\phi(D)f)^{(k)}(0)\right| < \frac{B_0(1+k)^c}{k!B_k}\sum_{n=0}^{\infty}\frac{(1+n)^{a+c}}{n!} \qquad (k=0,1,2,\dots),$$

hence we have $\phi(D)f \prec \langle (n!B_n)^{-1}\rangle$.

Finally, an estimate which is similar to (2.7) shows that the triple series

$$\sum_{m,n,k\geq 0}\frac{\psi^{(m)}(0)\phi^{(n)}(0)f^{(m+n+k)}(0)x^k}{m!n!k!}$$

converges absolutely for every $x \in \mathbb{C}$, hence the last assertion follows. □

Corollary 2.3.3. *Suppose that* ϕ, ψ, f *and* $\langle B_n\rangle$ *are as in Lemma 2.3.2,* μ *is a nonnegative integer,* $\phi(x)\psi(x) = x^\mu$*, and that* f *is transcendental. Then* $f \in \operatorname{dom}\phi(D)^m$ *and* $\phi(D)^m f$ *is not identically equal to* 0 *for all positive integers* m.

Proof. An inductive argument shows that $f \in \operatorname{dom}\phi(D)^m$, $\phi(D)^m f \in \operatorname{dom}\psi(D)^m$, and that $\psi(D)^m(\phi(D)^m f) = f^{(m\mu)}$ for all m. Since f is transcendental, $f^{(m\mu)}$ is not identically equal to 0 for all m, hence the same is true for $\phi(D)^m f$. □

Lemma 2.3.4. *Suppose that* $\langle B_n\rangle$ *and* ϕ *are as in Lemma 2.3.2,* f *is an entire function,* $\langle f_N\rangle$ *is a sequence of entire functions,* $f_N \ll \langle (n!B_n)^{-1}\rangle$ *for all* N*, and that* $f_N \to f$ *as* $N \to \infty$ *uniformly on compact sets in the complex plane. Then* $f_1, f_2, \dots, f \in \operatorname{dom}\phi(D)$ *and* $\phi(D)f_N \to \phi(D)f$ *as* $N \to \infty$ *uniformly on compact sets in the complex plane.*

Proof. First of all, Lemma 2.3.2 implies that $f_N \in \operatorname{dom}\phi(D)$ for all N. Since $f_N \to f$ uniformly on compact sets in the complex plane, and since

$$|f_N^{(n)}(0)| < (n!B_n)^{-1} \qquad (N=1,2,\dots;\ n=0,1,2,\dots),$$

it follows that

$$|f^{(n)}(0)| \leq (n!B_n)^{-1} \qquad (n=0,1,2,\dots),$$

hence $f \in \operatorname{dom} \phi(D)$, by Lemma 2.3.2.

To prove the uniform convergence on compact sets in the complex plane, let $R > 0$ and $\epsilon > 0$ be arbitrary. Suppose that a is a nonnegative constant and $\phi \ll \langle (1+n)^a n! B_n \rangle$. If we put

$$b = \sum_{k=0}^{\infty} \frac{B_0 R^k}{(k!)^2 B_k},$$

then it is easy to see that

$$|f_N^{(n)}(x)| < \frac{b}{n! B_n} \qquad (|x| \le R;\ N = 1, 2, \dots;\ n = 0, 1, 2, \dots),$$

and that

$$|f^{(n)}(x)| \le \frac{b}{n! B_n} \qquad (|x| \le R;\ n = 0, 1, 2, \dots).$$

Let ν be a positive integer such that

$$b \sum_{n=\nu+1}^{\infty} \frac{(1+n)^a}{n!} < \epsilon.$$

Then there is a positive integer N_0 such that

$$\left| \sum_{n=0}^{\nu} \frac{\phi^{(n)}(0)}{n!} \left(f_N^{(n)}(x) - f^{(n)}(x) \right) \right| < \epsilon \qquad (|x| \le R;\ N \ge N_0),$$

because $f_N \to f$ uniformly on compact sets in the complex plane.

Now, suppose that $|x| \le R$ and $N \ge N_0$. Then we have

$$|\phi(D) f_N(x) - \phi(D) f(x)| \le \left| \sum_{n=0}^{\nu} \frac{\phi^{(n)}(0)}{n!} \left(f_N^{(n)}(x) - f^{(n)}(x) \right) \right| + 2 \sum_{n=\nu+1}^{\infty} \frac{|\phi^{(n)}(0)|}{n!} \frac{b}{n! B_n} < 3\epsilon.$$

This completes the proof. □

Corollary 2.3.5. *Under the same assumptions as in Lemma 2.3.4, $\phi(D)^m f_N \to \phi(D)^m f$ as $N \to \infty$ uniformly on compact sets in the complex plane for every positive integer m.*

Proof. Lemma 2.3.2 implies that $\phi^m \prec \langle n!B_n \rangle$ for all positive integers m. □

We denote the open disk with center at a and radius r by $D(a;r)$, and its closure by $\bar{D}(a;r)$. For a complex constant c we define the *translation operator* T^c by $(T^c f)(x) = f(x+c)$. It is clear that if f is a monic polynomial of degree d, then $c^{-d}T^c f \to 1$ as $|c| \to \infty$ uniformly on compact sets in the complex plane. This observation leads to the following:

Lemma 2.3.6. *Suppose that ϕ is a formal power series, f and g are polynomials, $a_1, \ldots, a_N$ are zeros of $\phi(D)f$, b is a zero of $\phi(D)g$, and that neither $\phi(D)f$ nor $\phi(D)g$ is identically equal to 0. Then for every $c \in \mathbb{C}$ the polynomial $\phi(D)(fT^c g)$ is not identically equal to 0, and for every $\epsilon > 0$ there is an $R > 0$ such that if $|c| > R$, then $\phi(D)(fT^c g)$ has a zero in each of the disks $D(a_1;\epsilon), \ldots, D(a_N;\epsilon)$ and $D(b-c;\epsilon)$.*

Proof. The assumptions imply that neither f nor g is identically equal to 0. In particular, we have $\deg(fT^c g) \geq \deg f$, hence the first assertion follows, because $\phi(D)f$ is not identically equal to zero.

Let $\epsilon > 0$. We first observe that if c is a constant, then $\phi(D)(fT^c g)$ has a zero in $D(b-c;\epsilon)$ if and only if $\phi(D)(gT^{-c}f)$ has a zero in $D(b;\epsilon)$. Since neither f nor g is identically equal to 0, we may assume that f and g are monic. Then $c^{-\deg g} fT^c g \to f$ and $(-c)^{-\deg f} gT^{-c} f \to g$ as $|c| \to \infty$ uniformly on compact sets in the complex plane. Hence there is an $R > 0$ such that if $|c| > R$, then $\phi(D)(fT^c g)$ has a zero in each of the disks $D(a_1;\epsilon), \ldots, D(a_N;\epsilon)$ and $\phi(D)(gT^{-c}f)$ has a zero in $D(b;\epsilon)$. □

The following characterization of the class $\mathcal{LP}$ given in [24, 27] will play a crucial role in the proof of Theorem 2.3.1.

Theorem (Pólya). *Let ϕ be a formal power series with real coefficients. Then $\phi \in \mathcal{LP}$ if and only if $Z_C(\phi(D)M^d) = 0$ for all positive integers d.*

Corollary 2.3.7. *Suppose that ϕ is a formal power series with real coefficients and ϕ does not represent a Laguerre-Pólya function. Then there is a positive integer d_0 such that $Z_C(\phi(D)M^d) > 0$ for all $d \geq d_0$.*

Proof. By Pólya's theorem, there is a positive integer d_0 such that $Z_C(\phi(D)M^{d_0}) > 0$, and Rolle's theorem implies that if $Z_C(\phi(D)M^{d+1}) = 0$, then $Z_C(\phi(D)M^d) = 0$. □

Proof of Theorem 2.3.1. We will construct a sequence $\langle d(k)\rangle$ of positive integers and a sequence $\langle\gamma(k)\rangle$ of positive numbers such that $\sum_{k=1}^{\infty} d(k)\gamma(k) < \infty$ and the entire function f represented by

$$f(x) = \prod_{k=1}^{\infty} (1 + \gamma(k)x)^{d(k)}$$

has the desired property.

Since ϕ does not represent a Laguerre-Pólya function, it follows that neither does the formal power series ϕ^m for every positive integer m. Hence the corollary to Pólya's theorem implies that there is an increasing sequence $\langle d(m)\rangle$ of positive integers such that $Z_C(\phi(D)^m M^{d(m)}) > 0$ for all positive integers m. Since $\langle d(m)\rangle$ is increasing, we have $Z_C(\phi(D)^m M^{d(k)}) > 0$ whenever $m \leq k$. For each pair (m, k) of positive integers with $m \leq k$ choose a nonreal zero of $\phi(D)^m M^{d(k)}$ in the upper half plane, denote it by $a(m, k)$ and set $r(m, k) = Ia(m, k)/2$. It is obvious that $r(m, k) > 0$, and that $\bar{D}(a(m, k) - \gamma; r(m, k)) \cap \mathbb{R} = \emptyset$ for all $\gamma \in \mathbb{R}$. The assumption also implies that $\phi^{(n)}(0) \neq 0$ for some n, hence there is a nonnegative integer μ and there is a formal power series ψ such that $\phi(x)\psi(x) = x^\mu$. Choose an increasing sequence $\langle A_n\rangle$ of positive numbers such that $\phi, \psi \ll \langle A_n\rangle$, and define $\langle B_n\rangle$ by $B_0 = A_0$, $B_1 = A_1$ and

$$B_{n+1} = \max\left[\{A_{n+1}\} \cup \{B_0^{-1} B_k B_{n+1-k} : k = 1, \dots, n\}\right] \qquad (n = 1, 2, \dots).$$

It is clear that $\langle B_n\rangle$ is an increasing sequence of positive numbers, $\langle B_n\rangle$ satisfies (2.6), and that $\phi, \psi \prec \langle n! B_n\rangle$.

For $k = 1, 2, \dots$ and for $\gamma > 0$ define $g_{k,\gamma}$ by

$$g_{k,\gamma}(x) = (1 + \gamma x)^{d(k)},$$

that is, $g_{k,\gamma} = \gamma^{d(k)} T^{1/\gamma} M^{d(k)}$. From the definition, it follows that $g_{k,\gamma}$ is a real polynomial of degree $d(k)$, $g_{k,\gamma}(0) = 1$, $\phi(D)^m g_{k,\gamma}$ is not identically equal to 0 for $1 \leq m \leq k$,

$$(\phi(D)^m g_{k,\gamma}) \left(a(m,k) - \gamma^{-1}\right) = 0 \qquad (1 \leq m \leq k), \tag{2.8}$$

and that $g_{k,\gamma} \to 1$ as $\gamma \to 0$ uniformly on compact sets in the complex plane.

Since $g_{1,\gamma}(0) = 1 < 2$ and $g_{1,\gamma}$ is a polynomial of degree $d(1)$ for every $\gamma > 0$, and since $g_{1,\gamma} \to 1$ as $\gamma \to 0$ uniformly on compact sets in the complex plane, there is a positive number $\gamma(1)$ such that $g_{1,\gamma(1)} \ll \langle 2B_0(n!B_n)^{-1}\rangle$. From the definition, the polynomial $\phi(D)g_{1,\gamma(1)}$ is not identically equal to 0, and from (3.3) we have $\left(\phi(D)g_{1,\gamma(1)}\right)(a(1,1) - \gamma(1)^{-1}) = 0$. Now suppose that $\gamma(1), \ldots, \gamma(N)$ are positive numbers,

$$\prod_{k=1}^{N} g_{k,\gamma(k)} \ll \langle 2B_0(n!B_n)^{-1}\rangle, \tag{2.9}$$

and that for each $m \in \{1, \ldots, N\}$ the closures of the disks

$$D\left(a(m,k) - \gamma(k)^{-1}; r(m,k)\right) \qquad (m \leq k \leq N) \tag{2.10}$$

are mutually disjoint and the polynomial $\phi(D)^m \left(\prod_{k=1}^{N} g_{k,\gamma(k)}\right)$ has a zero in each of these disks. Suppose also that the polynomials $\phi(D)^m \left(\prod_{k=1}^{N} g_{k,\gamma(k)}\right)$, $m = 1, \ldots, N$ are not identically equal to 0. Since $\prod_{k=1}^{N} g_{k,\gamma(k)}$ is a polynomial, $g_{N+1,\gamma}$ is a polynomial of degree $d(N+1)$ for every $\gamma > 0$, and since $g_{N+1,\gamma} \to 1$ as $\gamma \to 0$ uniformly on compact sets in the complex plane, (2.9) implies that there is a $\delta > 0$ such that

$$\left(\prod_{k=1}^{N} g_{k,\gamma(k)}\right) g_{N+1,\gamma} \ll \langle 2B_0(n!B_n)^{-1}\rangle \qquad (0 < \gamma < \delta). \tag{2.11}$$

From Lemma 2.3.6, it follows that for each $m \in \{1, \ldots, N\}$ there is an $R_m > 0$ such that if $|c| > R_m$, then $\phi(D)^m \left(\left(\prod_{k=1}^{N} g_{k,\gamma(k)}\right) T^c M^{d(N+1)}\right)$ has a zero in each of the disks given in (2.10) and also has a zero in the disk $D\left(a(m,N+1) - c; r(m,N+1)\right)$, because $\phi(D)^m \left(\prod_{k=1}^{N} g_{k,\gamma(k)}\right)$ has a zero in each of the disks given in (2.10) and $\left(\phi(D)^m M^{d(N+1)}\right)(a(m,N+1)) = 0$. By taking R_m sufficiently large, we may assume that

$$\bar{D}\left(a(m,k)-\gamma(k)^{-1};r(m,k)\right)\cap\bar{D}\left(a(m,N+1)-c;r(m,N+1)\right)=\emptyset$$

for $|c| > R_m$ and for $m \le k \le N$. Since $\phi(D)^{N+1}M^{d(N+1)}$ has a zero at $a(N+1,N+1)$ and $r(N+1,N+1) > 0$, Lemma 2.3.6 implies that there is an $R_{N+1} > 0$ such that if $|c| > R_{N+1}$, then $\phi(D)^{N+1}\left(\left(\prod_{k=1}^{N} g_{k,\gamma(k)}\right)T^c M^{d(N+1)}\right)$ has a zero in $D\left(a(N+1,N+1)-c;r(N+1,N+1)\right)$. Let $\gamma(N+1)$ be such that $0 < \gamma(N+1) < \min\{\delta, R_1^{-1},\dots,R_N^{-1},R_{N+1}^{-1}\}$. Then (2.11) implies that

$$\prod_{k=1}^{N+1} g_{k,\gamma(k)} \ll \langle 2B_0(n!B_n)^{-1}\rangle.$$

The construction shows that for each $m \in \{1,\dots,N+1\}$ the closures of the disks

$$D\left(a(m,k)-\gamma(k)^{-1};r(m,k)\right) \qquad (m \le k \le N+1)$$

are mutually disjoint and the polynomial $\phi(D)^m\left(\prod_{k=1}^{N+1} g_{k,\gamma(k)}\right)$ has a zero in each of these disks. Finally, the polynomials $\phi(D)^m\left(\prod_{k=1}^{N+1} g_{k,\gamma(k)}\right)$, $m = 1,\dots,N+1$, are not identically equal to 0, by Lemma 2.3.6.

By induction, this process produces a sequence $\langle\gamma(k)\rangle$ of positive numbers which has the following properties:

(1) For each positive integer N we have

$$\prod_{k=1}^{N} g_{k,\gamma(k)} \ll \langle 2B_0(n!B_n)^{-1}\rangle.$$

(2) For each positive integer m the closed disks

$$\bar{D}\left(a(m,k)-\gamma(k)^{-1};r(m,k)\right) \qquad (k = m, m+1, m+2, \dots) \tag{2.12}$$

are mutually disjoint.

(3) For each positive integer m the polynomial $\phi(D)^m \left(\prod_{k=1}^{N} g_{k,\gamma(k)}\right)$ has a zero in each of the disks given in (2.10), whenever $N \geq m$.

For $N = 1, 2, \ldots$ we set $f_N = \prod_{k=1}^{N} g_{k,\gamma(k)}$, that is,

$$f_N(x) = \prod_{k=1}^{N} (1 + \gamma(k)x)^{d(k)}.$$

From (1), it follows that

$$0 \leq f_N^{(n)}(0) < 2B_0(n!B_n)^{-1} \qquad (N = 1, 2, \ldots;\ n = 0, 1, 2, \ldots). \tag{2.13}$$

In particular, we have

$$\sum_{k=1}^{N} d(k)\gamma(k) = f_N'(0) < 2B_0/B_1 \qquad (N = 1, 2, \ldots),$$

hence the infinite product $\prod_{k=1}^{\infty} (1 + \gamma(k)x)^{d(k)}$ represents an entire function of genus 0. Let f denote the entire function. It is then obvious that f is transcendental, $f \in \mathcal{LP}$, $f_N \to f$ uniformly on compact sets in the complex plane, and that

$$0 < f^{(n)}(0) \leq 2B_0(n!B_n)^{-1} \qquad (n = 0, 1, 2, \ldots).$$

To complete the proof, let m be a positive integer. From the corollary to Lemma 2.3.2, it follows that $f \in \operatorname{dom} \phi(D)^m$ and $\phi(D)^m f$ is not identically equal to 0; and from (2.13) and the corollary to Lemma 2.3.4, we see that $\phi(D)^m f_N \to \phi(D)^m f$ as $N \to \infty$ uniformly on compact sets in the complex plane. Furthermore, $\phi(D)^m f_N$ has a zero in each of the disks given in (2.10) whenever $N \geq m$. Hence $\phi(D)^m f$ has a zero in each of the closed disks given in (2.12) which are mutually disjoint and do not intersect the real axis. Therefore $Z_C(\phi(D)^m f) = \infty$.

□

2.4 Asymptotic behavior of distribution of zeros of $\phi(D)^m f$ as $m \to \infty$

In this section, we conclude chpater 2 with some consequences of Theorems 2.2.3 and 2.2.4 on the asymptotic behavior of the distribution of zeros of $\phi(D)^m f$ as $m \to \infty$, in the case where the coefficients of ϕ are complex numbers and f is a complex polynomial. When f is an entire function, we denote its zero set by $\mathcal{Z}(f)$, that is, $\mathcal{Z}(f) = \{z \in \mathbb{C} : f(z) = 0\}$, and for $a \in \mathcal{Z}(f)$ the multiplicity by $\mathrm{m}(a, f)$.

Let ϕ, p, α, β, f, d and $f_1, f_2, \ldots$ be as in Theorem 2.2.3. Then $\beta \neq 0$ and $f_m \to \exp(\beta D^p)M^d$ uniformly on compact sets in the complex plane. We also have

$$\mathcal{Z}(\phi(D)^m f) = -m\alpha + m^{1/p}\mathcal{Z}(f_m) \tag{2.14}$$

and

$$\mathrm{m}(a, \phi(D)^m f) = \mathrm{m}(m^{-1/p}(a + m\alpha), f_m) \tag{2.15}$$

for all $a \in \mathcal{Z}(\phi(D)^m f))$. Let $\epsilon > 0$ be so small that the disks $D(b; \epsilon)$, $b \in \mathcal{Z}\left(\exp(\beta D^p)M^d\right)$, are mutually disjoint. Then Rouche's theorem implies that there is a positive integer m_0 such that

$$\sum_{c \in D(b;\epsilon) \cap \mathcal{Z}(f_m)} \mathrm{m}(c, f_m) = \mathrm{m}\left(b, \exp(\beta D^p)M^d\right) \tag{2.16}$$

holds for all $b \in \mathcal{Z}\left(\exp(\beta D^p)M^d\right)$ for all and $m \geq m_0$. As a consequence, we have

$$\mathcal{Z}(f_m) \subset D(0; \epsilon) + \mathcal{Z}\left(\exp(\beta D^p)M^d\right) \tag{2.17}$$

for all $m \geq m_0$. Let γ be a complex number such that $\gamma^p = -\beta$. Then $\gamma \neq 0$ and we have

$$\mathcal{Z}\left(\exp(\beta D^p)M^d\right) = \gamma \mathcal{Z}\left(\exp(-D^p)M^d\right),$$

because

$$\left(\exp(\beta D^p)M^d\right)(x) = \gamma^d \left(\exp(-D^p)M^d\right)(x/\gamma).$$

Now, (2.14) and (2.17) imply that

$$\mathcal{Z}(\phi(D)^m f) \subset -m\alpha + m^{1/p}\left(D(0;\epsilon) + \gamma\mathcal{Z}\left(\exp(-D^p)M^d\right)\right) \tag{2.18}$$

holds for all $m \geq m_0$.

With the aid of Theorem 2.2.4, the above results give us some information on the zeros of $\phi(D)^m f$ for large values of m. From Theorem 2.2.4, it follows that

$$\mathcal{Z}\left(\exp(-D^p)M^d\right) \subset S_p,$$

where

$$S_p = \bigcup_{k=0}^{p-1}\left\{re^{2k\pi i/p} : r \geq 0\right\}.$$

It also follows from Theorem 2.2.4 that if $d \equiv 0 \text{ or } 1 \mod p$, then all the zeros of $\exp(-D^p)M^d$ are simple. Hence (2.18) implies that for every $\epsilon > 0$ there is a positive integer m_0 such that

$$\mathcal{Z}(\phi(D)^m f) \subset -m\alpha + N(0, m^{1/p}\epsilon) + \gamma S_p$$

for all $m \geq m_0$, and (2.14) through (2.16) imply that if $d \equiv 0 \text{ or } 1 \mod p$, then all the zeros of $\phi(D)^m f$ are simple whenever m becomes sufficiently large.

Chapter 3

Asymptotic behavior of distribution of the zeros of a one-parameter family of polynomials

Let $\phi(z) = \sum_{k=0}^{\infty} a_k z^k/k!$ be a real power series with $a_0 = 1$ and $a_1 = 0$. In this chapter, when P is a polynomial of degree at least two, the asymptotic behavior of distribution of the zeros of $\phi(D)^m P(z)$ for $m \to \infty$ is described, where D denotes differentiation.

3.1 Asymptotic behavior of distribution of the zeros of $\phi(D)^m f$ as $m \to \infty$

Let P be an arbitrary polynomial of degree d with leading coefficient α. If $z_1, z_2, \ldots, z_d$ are zeros of P then the arithmetic mean of zeros $\mathcal{A}_p$ is given by $\frac{1}{d}\sum_{k=1}^{d} z_k$. We consider the polynomial $\frac{1}{\alpha}P(z + i \text{ Im } \mathcal{A}_p) = \sum_{k=0}^{d} \alpha_k z^{d-k}$, $\alpha_0 = 1$. By translation, it is clear that α_1 is real. If $\alpha_1, \ldots, \alpha_{\mu-1}$ are all real and α_μ is the first nonreal coefficient, then we define $I_{\mathcal{A}_p} = \mu$, $(2 \leq \mu \leq d)$. If there is no such a μ, we can apply Theorem 2.1.1 to $P(z + i \text{ Im } \mathcal{A}_p)$. Let

$W_H(P) = \sup\{|\mathrm{Im}(\mathcal{A}_p - z)| : P(z) = 0\}$. Then we obtain the following result.

Theorem 3.1.1. *Let $\phi(z) = \sum_{k=0}^{\infty} a_k z^k/k!$ be a real power series with $a_0 = 1$, $a_1 = 0$, and $a_2 < 0$. Let $P(z) = \sum_{k=0}^{d} \alpha_k z^{d-k}$ be a polynomial of degree at least two and $I_{\mathcal{A}_p} = \mu$, $(2 \leq \mu \leq d)$. Then, for positive integer m,*

$$\overline{\lim_{m\to\infty}}\, m^{(\mu-1)/2} W_H(\phi(D)^m P(z)) < \infty,$$

$$\overline{\lim_{m\to\infty}}\, m^{\delta} W_H(\phi(D)^m P(z)) = \infty, \quad \forall \delta > \frac{\mu - 1}{2}\ .$$

Let $H_n(z)$ be the nth Hermite polynomial defined by

$$H_n(z) = (-1)^n e^{z^2} D^n e^{-z^2}.$$

It is known that $H_n(z)$ has only real and simple zeros. From this, the next theorem gives more specific result on the distribution of zeros of $\phi(D)^m P(z)$ for sufficiently large m.

Theorem 3.1.2. *Let $\phi(z) = \sum_{k=0}^{\infty} a_k z^k/k!$ be a real power series with $a_0 = 1$, $a_1 = 0$, and $a_2 < 0$. Let $P(z) = \sum_{k=0}^{d} \alpha_k z^{d-k}$ be a polynomial of degree at least two and $I_{\mathcal{A}_p} = \mu$, $(2 \leq \mu \leq d)$. Let $\rho_1, \rho_2, \ldots, \rho_d$ be distinct zeros of $H_d(z)$ and $r := \min\{\frac{|\rho_i - \rho_j|}{2} : i \neq j\}$. Then for every $\epsilon > 0$, there is a positive integer m_0 such that each open square $\{z : |\mathrm{Im}\ (z - \mathcal{A}_p)| < \epsilon, |\mathrm{Re}\ z - \sqrt{-2a_2 m}\rho_j| < r\sqrt{-2a_2 m}\}$ contains only one zero of $\phi(D)^m P(z)$ for all $m \geq m_0$ and $j = 1, 2, \ldots, d$.*

3.2 Zeros of polynomials with complex coefficients

We begin with this section by introducing of Wall-Frank Theorem which will be used in our proof of the Theorem 3.1.1. H. S. Wall proved theorem on the zeros of polynomial with real coefficients [32] and then E. Frank extended the result to polynomial with complex coefficients [10]. In original papers, the theorem is related to necessary and sufficient condition that a polynomial shall have only zeros with negative real parts. But in this section, we reformulate the necessary and sufficient condition that a polynomial shall have only zeros with negative imaginary parts.

The Wall-Frank Theorem. *Let* $P(z) = \sum_{k=0}^{d} \alpha_k z^{d-k}$ *with* $\alpha_0 = 1$. *And let*

$$Q_0 = \sum_{k=0}^{d} (-i)^k (\mathrm{Re}\ \alpha_k) z^{d-k} \quad \text{and} \quad Q_1 = \sum_{k=1}^{d} (-i)^{k-1} (\mathrm{Im}\ \alpha_k) z^{d-k}. \tag{3.1}$$

Then all the zeros of $P(z)$ *have negative imaginary part if and only if the quotient* Q_1/Q_0 *can be written in the continued fraction,*

$$\frac{Q_1}{Q_0} = \cfrac{1}{r_1 z + s_1 + \cfrac{1}{r_2 z + s_2 + \cfrac{1}{\cdots + \cfrac{1}{r_d z + s_d}}}} \tag{3.2}$$

with $r_j > 0$ *and* s_j *is pure imaginary or zero for* $1 \le j \le d$.

The problem of determining $r_j,\ s_j$ is equivalent to the problem of deriving polynomials Q_j of degree $d - j$ which are connected with Q_0 and Q_1 by recurrence relations

$$\begin{cases} Q_{j+1} = Q_{j-1} - (r_j z + s_j) Q_j, \quad (j = 1, \ldots, d) \\ Q_{d+1} = 0\ . \end{cases} \tag{3.3}$$

For convenience of notation, we denote the coefficient of z^{d-k} of Q_j by $e_{j,k}$. Then by (3.3), we obtain the following formulas :

$$\begin{cases} e_{j+1,k} = e_{j-1,k} - r_j e_{j,k+1} - s_j e_{j,k} \\ r_{j+1} = e_{j,j}/e_{j+1,j+1} \\ s_{j+1} = (e_{j,j+1} - r_{j+1} e_{j+1,j+2})/e_{j+1,j+1} \\ e_{j,k} = 0 \qquad \text{if } k < j \text{ or } k > d \qquad . \end{cases} \tag{3.4}$$

Thus $r_1, \ldots, r_d$ are determined completely by Q_0, Q_1, and (3.4). Since real and pure imaginary coefficients appear alternatively in Q_0 and Q_1, we can check easily that $e_{j,k}$ is pure imaginary or zero (resp. real) if $j + k$ is odd (resp. even). Accordingly, s_j is a pure imaginary or zero.

3.3 Proofs of Theorem 3.1.1 and Theorem 3.1.2

In the proof of Theorem 3.1.1, we will use the following lemma.

Lemma 3.3.1 ([5, Lemma 2.1 and 2.2]). *Let*

$$\phi(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k \qquad (a_0 = 1,\ a_1 = 0)$$

be a real power series and let

$$\phi(z)^m = \sum_{k=0}^{\infty} \frac{b_k}{k!} z^k \qquad (b_k = b_k(m),\ m = 1, 2, 3, \ldots). \tag{3.5}$$

Then

$$b_0 = 1,\ b_k = \frac{1}{k} \sum_{j=1}^{k} \binom{k}{j} [\ j(m+1) - k\ ] a_j b_{k-j} \qquad (k = 1, 2, \ldots)$$

and

$$b_{2k-1} = O(m^{k-1})\ ,\quad b_{2k} = \frac{(2k)!}{k!} \left(\frac{a_2}{2}\right)^k m^k + o(m^k) \qquad (k = 1, 2, \ldots;\ m \to \infty).$$

To prove the Theorem 3.1.1 we need some preparations. For simplicity of expression, we put $E_{j,k}$ for $j \geq 0$ as follows;

(3.6)

$$E_{j,k} = \begin{cases} \dfrac{k!}{(k/2)!} \left(-\dfrac{a_2}{2}\right)^{\frac{k}{2}} \dbinom{d}{k} \displaystyle\prod_{l=1}^{j/2} \dfrac{k+2l-j}{d-2l+2} & \text{if } j, k \text{ are even,} \\ \dfrac{k!}{((k-1)/2)!} \left(-\dfrac{a_2}{2}\right)^{\frac{k-1}{2}} \dbinom{d}{k} \displaystyle\prod_{l=1}^{(j-1)/2} \dfrac{k+2l-j}{d-2l+1} & \text{if } j, k \text{ are odd,} \\ 0 & \text{if } k < j \text{ or } k > d. \end{cases}$$

In (3.6), if $j = 0$ or 1, then empty product represents a unity. And let

$$R_{j+1} = E_{j,j}/E_{j+1,j+1}\ . \tag{3.7}$$

Then we check at once that

$$(3.8)\qquad \begin{cases} E_{j+2,k} = E_{j,k} - R_{j+1}E_{j+1,k+1} \\ E_{d,d} = E_{d-2,d}. \end{cases}$$

Using (3.6) and (3.7), for positive integer m, we put $f_m(j,k)$, $g_m(j,k)$, and $R_m(j)$ as follows;

(3.9)

$$f_m(j,k) = \begin{cases} E_{j,k}\ m^{\frac{k}{2}} + o(m^{\frac{k}{2}}) & \text{if } k+j \in 2\mathbb{Z} \\ A_{j,k}\ m^{\frac{k-1}{2}} + o(m^{\frac{k-1}{2}}) & \text{if } k+j \notin 2\mathbb{Z} \end{cases} \qquad (m \to \infty),$$

$$g_m(j,k) = \begin{cases} C\ E_{j,k}\ m^{\frac{k}{2}} + o(m^{\frac{k}{2}}) & \text{if } k+j \in 2\mathbb{Z} \\ A_{j,k}\ m^{\frac{k}{2}} + o(m^{\frac{k}{2}}) & \text{if } k+j \notin 2\mathbb{Z} \end{cases} \qquad (C > 0\ ;\ m \to \infty),$$

where $A_{j,k}$ is an appropriate pure imaginary constant or may be zero, and

$$(3.10)\qquad R_m(j) = R_j\ m^{-\frac{1}{2}} + o(m^{-\frac{1}{2}}) \qquad (m \to \infty).$$

If $E_{j,k}$ and R_j in (3.9), (3.10) are replaced by $E_{j,k}+O(C^{-1})$, and $R_j+O(C^{-1})$ $(C \to \infty)$, then we write $f^*_m(j,k),\ g^*_m(j,k),$ and $R^*_m(j)$ instead of $f_m(j,k),\ g_m(j,k),$ and $R_m(j)$. And if $O(C^{-1})$ is replaced by $O(C^{-2})$ then we use double star **.

Proof of Theorem 3.1.1. There is no loss of generality in assuming $\alpha_0 = 1$ and $\alpha_1 = 0$. Then $\mathcal{A}_p = 0$ and Im $\alpha_\mu \neq 0$. Let $\phi(z)^m$ be the form (3.5). And consider the polynomial

$$(3.11)\qquad P_m(z) = \phi(D)^m P(z).$$

For positive numbers C and δ, we can write

$$(3.12)\qquad P_m(z + iCm^{-\delta}) = \sum_{k=0}^{d} \gamma_k z^{d-k}$$

where $\gamma_0, \dots, \gamma_d$ are given by

$$(3.13)\qquad \gamma_k = \sum_{\nu=0}^{k} b_\nu \beta_{k-\nu} \binom{d-k+\nu}{\nu}, \qquad \beta_l = \sum_{\tau=0}^{l} \alpha_\tau \binom{d-\tau}{l-\tau} (iCm^{-\delta})^{l-\tau}.$$

For polynomial (3.12), Q_0 and Q_1 are obtained in the manner indicated in (3.1). Let $e_{0,k}$ and $e_{1,k}$ be coefficients of z^{d-k} of Q_0 and Q_1, respectively. Then by (3.4), we get the continued fraction (3.2). It suffices to show that there is a $C_0 > 0$ such that all r_j in (3.2) are positive for $C > C_0$ as $m \to \infty$. Throughout the proof, we use the induction on j. From (3.1), (3.13) and Lemma 3.3.1, we see that

$$e_{0,k} = f_m(0,k), \qquad (e_{0,0} = 1,\ e_{0,1} = 0). \tag{3.14}$$

Let $\mu > 2$, $\delta = (\mu - 1)/2$ and $k \leq \mu - 2$. Then we have

$$e_{1,k} = Cm^{-\delta-\frac{1}{2}} f_m(1,k)(e_{1,2} = 0), \quad r_1 = C^{-1}m^{\delta+\frac{1}{2}}R_m(1), \quad s_1 = 0. \tag{3.15}$$

In fact, $A_{0,k}$ and $A_{1,k}$ of (3.9), which are related to (3.14) and (3.15) respectively, are both independent of C. So, from now on, we assume that $A_{j,k}$ is independent of C for all j and k. (But we need not know its exact value.) This condition is essential to the proof. To simplify notation, we put

$$S_m(j) = S_j\ m^{-\frac{1}{2}} + o(m^{-\frac{1}{2}}) \qquad (m \to \infty).$$

Here, S_j is independent of C. Then by (3.4) and (3.8), if $j \leq \mu/2$ and $k+j < \mu$, we get

$$e_{j,k} = \begin{cases} f_m(j,k) & \text{if } j \in 2\mathbb{Z} \\ C\ m^{-\delta-\frac{1}{2}} f_m(j,k) & \text{if } j \notin 2\mathbb{Z}, \end{cases} \qquad r_j = \begin{cases} C\ m^{-\delta-\frac{1}{2}} R_m(j) & \text{if } j \in 2\mathbb{Z} \\ C^{-1}\ m^{\delta+\frac{1}{2}} R_m(j) & \text{if } j \notin 2\mathbb{Z}. \end{cases} \tag{3.16}$$

And for $j < \mu/2$,

$$s_j = C\ m^{-\delta-\frac{1}{2}} S_m(j)\ (j \in 2\mathbb{Z}), \quad s_j = C^{-1}\ m^{\delta+\frac{1}{2}} S_m(j)\ (j \notin 2\mathbb{Z}). \tag{3.17}$$

Assume that μ is even. If $k \geq \mu - 1$ then $e_{1,k} = m^{-\frac{\mu}{2}} g_m(j,k)$. From this, if $j \leq \mu/2$ and $j + k \geq \mu$ then

$$e_{j,k} = \begin{cases} C^{-1}\ g_m(j,k) & \text{if } j \in 2\mathbb{Z}, \\ m^{-\frac{\mu}{2}} g_m(j,k) & \text{if } j \notin 2\mathbb{Z}. \end{cases} \tag{3.18}$$

If $j = \mu/2$ then

$$s_j = \begin{cases} m^{\frac{1-\mu}{2}} S_m(j) & \text{if } j \in 2\mathbb{Z}, \\ C^{-2}\ m^{\frac{\mu+1}{2}} S_m(j) & \text{if } j \notin 2\mathbb{Z}. \end{cases} \tag{3.19}$$

Hence, for $j > \mu/2$, $e_{j,k}$ is the form in (3.18) with $g_m(j,k)$ replaced by $g_m^{**}(j,k)$. And r_j is the form in (3.16) with $R_m(j)$ replaced by $R_m^{**}(j)$, s_j is same as (3.19).

If μ is odd, $e_{1,k} = C\ m^{-\frac{\mu}{2}} f_m^*(1,k)$, for $k \geq \mu - 1$. (Here, $A_{1,k}$ in (3.9) related to $f_m^*(1,k)$ depends on C. But it does not affect the result.) And for $j + k \geq \mu$, $e_{j,k}$ and r_j are the same as the form (3.16) replaced by $f_m^*(j,k)$, $R_m^*(j)$.

If $\mu = 2$ then $e_{1,k} = m^{-1} g_m(j,k)$, $r_1 = C^{-1} m R_m(1)$, and $s_1 = C^{-2} m^{\frac{3}{2}} S_m(1)$. Thus for $j > 1$, we can apply the above result to $e_{j,k},\ r_j,$ and s_j.

In any case, we can find $C_0 > 0$ such that $r_j > 0$ for all $C > C_0$ as $m \to \infty$. Thus by Wall-Frank Theorem, all the zeros of $P_m(z)$ are in the $\{z; \text{Im } z < C\ m^{-\frac{\mu-1}{2}}\}$ as $m \to \infty$. And if we consider $P_m(-z)$ then we can obtain the same result with $P_m(z)$ and the first part of proof is completed.

Next, suppose that $\delta > (\mu-1)/2$ in (3.12). Let $\mu \geq 2$ and $\delta \leq \mu/2$. We get (3.15) for $k \leq \mu - 1$. But, if $\mu = 2$, s_1 in (3.15) is removed. If μ is even then

$$e_{1,\mu} = (-1)^{\frac{\mu}{2}} \text{ Im } \alpha_\mu i + o(1), \quad e_{1,\mu+1} = O(m^{-\delta + \frac{k-1}{2}}),\ (m \to \infty),$$

if μ is odd then

$$e_{1,\mu} = (-1)^{\frac{\mu-1}{2}} \text{ Im } \alpha_\mu + o(1),\ e_{1,\mu+1} = O(1),\ e_{1,\mu+2} = O(m),\ (m \to \infty).$$

By (3.4) and induction on j, if $j + k \leq \mu$ and $j \leq \mu/2$ then we obtain (3.16) and (3.17). If $j + k = \mu + 1$, then there exists a nonzero constant $T_{j,k}$ such that

$$e_{j,k} = \begin{cases} T_{j,k}\ m^{\delta+1-\frac{j}{2}} + o(m^{\delta+1-\frac{j}{2}}) & \text{if } j \in 2\mathbb{Z} \\ T_{j,k}\ m^{-\frac{j-1}{2}} + o(m^{-\frac{j-1}{2}}) & \text{if } j \notin 2\mathbb{Z}. \end{cases} \quad (m \to \infty) \tag{3.20}$$

We first do the case of even μ. If $j + k = \mu + 2$ and $j \leq \mu/2$ then

$$e_{j,k} = O(m^{\frac{k}{2}})\ (j \in 2\mathbb{Z}), \quad e_{j,k} = O(m^{-\delta + \frac{k-1}{2}})\ (j \notin 2\mathbb{Z}), \quad (m \to \infty).$$

Let $j = \mu/2$. Then the leading term of s_j is determined by the second term of s_j in (3.4), thereby leading term of $e_{j+1,j+1}$ is determined by

$$-s_j e_{j,j+1} = \frac{r_j}{e_{j,j}}(e_{j,j+1})^2.$$

From (3.20), $e_{j,j+1}$ is nonzero pure imaginary. Hence, for all $C > 0$, we obtain $e_{j+1,j+1} < 0$ as $m \to \infty$.

In the case of odd μ, if $j + k = \mu + 2$ and $j \leq \mu/2$ then

$$e_{j,k} = O(m^{\delta+1-\frac{j}{2}})\ (j \in 2\mathbb{Z}), \quad e_{j,k} = O(m^{-\frac{j-1}{2}})\ (j \notin 2\mathbb{Z}), \quad (m \to \infty),$$

and if $j + k = \mu + 3$ and $j \leq \mu/2$ then

$$e_{j,k} = O(m^{\delta-\frac{j}{2}+2})\ (j \in 2\mathbb{Z}), \quad e_{j,k} = O(m^{-\frac{j-1}{2}})\ (j \notin 2\mathbb{Z}), \quad (m \to \infty).$$

Let $j = \frac{\mu+1}{2}$. Then the leading term of $e_{j,j}e_{j+1,j+1}$ is determined by

$$-r_{j-1}(e_{j-1,j+1})^2.$$

Thus $e_{j,j}e_{j+1,j+1} < 0$ for sufficiently large m.

In any case, for all $C > 0$, there exists j such that $r_{j+1} < 0$ as $m \to \infty$. We can also apply the same argument to the case of $\delta > \mu/2$ and obtain the same result. Therefore, we complete the proof of Theorem 3.1.1.

□

Proof of Theorem 3.1.2. Let $\phi(z)^m$ and $P_m(z)$ be the forms of (3.5) and (3.11) respectively. Let $h_{m,k} = \phi(D)^m z^k$. Then by Lemma 3.3.1,

$$\lim_{m\to\infty} \frac{h_{k,m}(z\sqrt{m})}{m^{k/2}} = \alpha_0 \left(-\frac{a_2}{2}\right)^{k/2} H_k\left(\frac{z}{\sqrt{-2a_2}}\right) \quad (k \geq 1).$$

Thus we can show that $m^{-d/2}P_m(z\sqrt{m})$ converges to

$$\alpha_0 \left(-\frac{a_2}{2}\right)^{\frac{d}{2}} H_d\left(\frac{z}{\sqrt{-2a_2}}\right) \tag{3.21}$$

uniformly on compact sets in the complex plane as $m \to \infty$. For $r > 0$, we denote the disc with center $\rho_j\sqrt{-2a_2m}$ and radius $r\sqrt{-2a_2m}$ by D_j. Then each $D_1, \ldots, D_d$ contains only one zero of $P_m(z)$ for sufficiently large m. Thus the proof is completed by Theorem 3.1.1. □

Remark. Let $P(z)$ be a real polynomial. Then there is no such a μ in Theorem 3.1.1. In the proof of Theorem 3.1.1, let C and δ be arbitrary positive numbers. Then we can see that (3.14)-(3.17) hold for all j and k. Hence, there exists $m_1 > 0$ such that $\forall r_j > 0$ for all $m \geq m_1$. Therefore, we can also obtain the Theorem 2.1.1 of Craven and Csordas . In fact, we can know the simplicity of zeros by the same method as in the proof of Theorem 3.1.2.

Analogously to $I_{\mathcal{A}_p}$, we can define $I^{\,i}_{\mathcal{A}_p}$. Let P be a polynomial of degree d with leading coefficient α. Consider the polynomial $\frac{1}{\alpha}P(z + \text{ Re } \mathcal{A}_p) = \sum_{k=0}^{d} \alpha_k z^{d-k}$, $\alpha_0 = 1$. Obviously, Re $\alpha_1 = 0$. If there is a μ such that, for $1 \leq k \leq \mu - 1$, Im $(i^k \alpha_k) = 0$ and Im $(i^\mu \alpha_\mu) \neq 0$, we set $I^{\,i}_{\mathcal{A}_p} = \mu$. Let $W_v(P) = \sup\{|\text{Re}(\mathcal{A}_p - z)| : P(z) = 0\}$. Then we state the analogue of Theorem 3.1.1 when $a_2 > 0$ as a corollary.

Corollary 3.3.2. *Let $\phi(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k}/(2k)!$ be an even real power series with $a_0 = 1$, $a_2 > 0$. Let $P(z) = \sum_{k=0}^{d} \alpha_k z^{d-k}$ be a polynomial of degree at least two and $I^{\,i}_{\mathcal{A}_p} = \mu$, $(2 \leq \mu \leq d)$. Then, for positive integer m,*

$$\overline{\lim_{m\to\infty}}\, m^{(\mu-1)/2} W_v([\phi(D)]^m P(z)) < \infty,$$

$$\overline{\lim_{m\to\infty}}\, m^{\delta} W_v([\phi(D)]^m P(z)) = \infty, \quad \forall \delta > \frac{\mu - 1}{2}\ .$$

From (3.21), we can rephrase Theorem 3.1.2 as follows.

Corollary 3.3.3. *Let $\phi(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k}/(2k)!$ be an even real power series with $a_0 = 1$, $a_2 > 0$. Let $P(z) = \sum_{k=0}^{d} \alpha_k z^{d-k}$ be a polynomial of degree at least two and $I^{\,i}_{\mathcal{A}_p} = \mu$, $(2 \leq \mu \leq d)$. Let $\rho_1, \rho_2, \ldots, \rho_d$ be distinct zeros of $H_d(z)$ and $r := \min\{\frac{|\rho_i - \rho_j|}{2} : i \neq j\}$. Then for every $\epsilon > 0$, there is a positive integer m_0 such that each open square $\{z : |\text{Re } (z - \mathcal{A}_p)| < \epsilon, |\text{Im } z - \sqrt{2a_2 m}\rho_j| < r\sqrt{2a_2 m}\}$ contains only one zero of $\phi(D)^m P(z)$ for all $m > m_0$ and $j = 1, 2, \ldots, d$.*

Remark. In Corollary 3.3.2 and 3.3.3, if there is no such a μ, there exists a $m_1 > 0$ such that, for all $m \geq m_1$, the zeros of the polynomial $\phi(D)^m P(z)$ are simple and all lie on the Re $z =$ Re $\mathcal{A}_p$.

Example 3.3.4. Corollary 3.3.2 and above remark do not extend to arbitrary real power series $\phi(z)$. Let $\phi(z) = 1 + z^2 + z^3$ and $P(z) = z^3 + 1$. Then $\mathcal{A}_p = 0$, $I^{\,i}_{\mathcal{A}_p} = 3$. Let $P_m(z) = [\phi(D)]^m P(z)$. Consider $(-i)^3 P_m(iz - \frac{1}{2})$. By Wall-Frank Theorem, we can see that $P_m(z)$ has a zero in the Re $z < -1/2$ as $m \to \infty$.

If $Q(z) = z^3 + i$, then $\mathcal{A}_p = 0$ and $\nexists I^{\,i}_{\mathcal{A}_Q}$. We can also obtain the same result as $P(z)$.

Chapter 4

De Bruijn-Newman constant of the polynomial $(z+i)^n + (z-i)^n$

Let λ_n be the largest zero of $2n$-th Hermite polynomial. In this chapter, We prove that the de Bruijn-Newman constant of the polynomial $(z+i)^n+(z-i)^n$ is $-(2\lambda_n)^{-2}$.

4.1 Main Result

A function of growth $(2,0)$ is a real entire function which is at most order 2 and type 0, that is,

$$f(z) = O(\exp(\epsilon|z|^2)) \qquad (|z| \to \infty)$$

for every $\epsilon > 0$. If f is of growth $(2,0)$ then it is known that $f \in \text{dom } e^{\alpha D^2}$ and $e^{\alpha D^2} f$ is of growth $(2,0)$ for every $\alpha \in \mathbb{C}$ [3]. When f is a real entire function of growth $(2,0)$, we define $\lambda(f)$ by

$$\lambda(f) = \sup\{\alpha \in \mathrm{R} : e^{\alpha D^2} f \text{ has real zeros only}\}.$$

We extend the notion of the de Bruijn-Newman constant to arbitrary real entire functions of growth $(2,0)$ by calling $-\lambda(f)$ the de Bruijn-Newman constant of f.

For $n = 0, 1, 2, \dots$ let F_n be the real polynomial defined by

$$F_n(z) = \frac{1}{2}((z+i)^n + (z-i)^n) = (\cos D\ M^n)(z),$$

where M^n is the monic monomial of degree n, that is, $M^n(z) = z^n$. We will establish the following:

Theorem 4.1.1. *Let λ_n be the largest zero of $H_{2n}(z)$ where $H_{2n}(z)$ is the $2n$-th Hermite polynomial defined by $H_{2n}(z) = e^{z^2} D^{2n} e^{-z^2}$. Then $\lambda(F_{2n}) = \lambda(F_{2n+1}) = (2\lambda_n)^{-2}$.*

In fact, it is well known [31, (6.32.5)] that $H_{2n}(z)$ has only real and simple zeros, especially

$$\lambda_n = \sqrt{4n+1} - 6^{-1/3}(4n+1)^{-1/6}(i_1 + \epsilon),$$

where $\epsilon \to 0$, as $n \to \infty$ and $i_1 = 3.372134408\dots$.

We obtain the following corollary.

Corollary 4.1.2. $\lambda(F_{2n}) \sim \frac{1}{16n}$ *as $n \to \infty$.*

If $x \in \mathbb{R}$, then $F_n(x) = \mathrm{Re}\ (x+i)^n$; hence we have

$$\begin{aligned} F_{2n}(x) &= (-1)^n (1+x^2)^n \cos(2n \tan^{-1} x) \\ &= \prod_{k=1}^{n} \left(x^2 - \tan^2 \frac{(2k-1)\pi}{4n} \right), \end{aligned}$$

and

$$\begin{aligned} F_{2n+1}(x) &= (-1)^n \sqrt{1+x^2}(1+x^2)^n \sin((2n+1)\tan^{-1} x) \\ &= x \prod_{k=1}^{n} \left(x^2 - \tan^2 \frac{k\pi}{2n+1} \right). \end{aligned}$$

This factorization formula exhibits the location of zeros of F_n explicitly. In particular, all the zeros of F_n are real and simple. However, it will not be used in our proof of Theorem 4.1.1.

4.2 Preliminaries

We denote the function $z \mapsto z^n$ by M^n. A direct calculation shows that

$$F_n = \cos D\ M^n. \tag{4.1}$$

If $\langle \alpha_k \rangle = \langle \alpha_k \rangle_{k=0}^{\infty}$ is a sequence of numbers and if f is a polynomial, we define $\langle \alpha_k \rangle f$ by

$$\langle \alpha_k \rangle f = \sum_{k=0}^{\infty} \frac{\alpha_k f^{(k)}(0)}{k!} M^k.$$

In other words, if $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, then

$$\langle \alpha_k \rangle f(z) = \alpha_0 a_0 + \alpha_1 a_1 z + \cdots + \alpha_n a_n z^n.$$

A real entire function ϕ is said to be a Laguerre-Pólya function if there are real polynomials $f_1, f_2, \ldots$ such that $f_n \to \phi$ uniformly on compact sets in the complex plane and that all the zeros of $f_1, f_2, \ldots$ are real; if all the zeros of $f_1, f_2, \ldots$ are real and of the same sign, then ϕ is called a Laguerre-Pólya function of the first kind.

The Pólya-Schur Theorem. *If ϕ is a Laguerre-Pólya function of the first kind, and f is a real polynomial with real zeros only, then $\langle \phi^{(k)}(0) \rangle f$ has real zeros only.*

Proof. See [27]. □

For $\alpha \in \mathbb{R}$ define sg α by

$$\text{sg } \alpha = \begin{cases} 0 & (\alpha = 0), \\ |\alpha|/\alpha & (\alpha \neq 0). \end{cases} \tag{4.2}$$

Suppose $\langle s_k \rangle = \langle s_k \rangle_{k=0}^{n}$ is a finite sequence such that $s_k \in \{-1, 0, 1\}$ for every k and $s_n \neq 0$. For example, if f is a real polynomial of degree n and $a \in \mathbb{R}$, then $\langle \text{sg } f^{(k)}(a) \rangle$ is such a sequence. For $k = 0, 1, \ldots, n$ define s_k^+ and s_k^- as follows: If $s_k \neq 0$, $s_k^+ = s_k^- = s_k$; otherwise, $s_k^+ = s_{k+l}$ and $s_k^- = (-1)^l s_{k+l}$, where l is the smallest positive integer such that $s_{k+l} \neq 0$. Thus $s_k^+,\ s_k^- \neq 0$ for all k. We denote the new sequences $\langle s_k^+ \rangle$ and $\langle s_k^- \rangle$ by $\langle s_k \rangle^+$ and $\langle s_k \rangle^-$, respectively. If $s_k \neq 0$ for all k, we denote by $W\langle s_k \rangle$ the number of sign-changes in $\langle s_k \rangle$, that is,

$$W\langle s_k \rangle = \sum_{k=1}^{\infty} \frac{1 - s_{k-1}s_k}{2}.$$

If f is a non-constant real polynomial and $a \in \mathbb{R}$, then

$$\langle \text{sg } f^{(k)}(a) \rangle^+ = \langle \text{sg } f^{(k)}(a+\epsilon) \rangle \text{ and } \langle \text{sg } f^{(k)}(a) \rangle^- = \langle \text{sg } f^{(k)}(a-\epsilon) \rangle$$

for all sufficiently small $\epsilon > 0$. Thus we may state the Budan-Fourier-Hurwitz theorem in the following form:

The Budan-Fourier-Hurwitz Theorem. *If f is a non-constant real polynomial and $a < b$, then*

$$N(f;(a,b)) = W\langle \text{sg } f^{(k)}(a) \rangle^+ - W\langle \text{sg } f^{(k)}(b) \rangle^- - 2K(f;(a,b)).$$

Here, $N(f,I)$ denotes the number of zeros of f in the interval I, and $K(f,I)$ denotes the number of critical points of f in the interval I.

Proof. See [12]. □

4.3 Proof of the main result

If ϕ is analytic at 0, f is an entire function and the series

$$\sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{n!} f^{(n)}$$

converges absolutely and uniformly on compact sets in the complex plane, then we denote the resulting entire function by $\phi(D)f$ and say that $\phi(D)f$ is well defined. If f is a polynomial, it is obvious that $\phi(D)f$ is well defined and is a polynomial.

If ϕ is analytic at 0, then

$$\phi(D)M^n(z) = \sum_{l=0}^{n} \frac{\phi^{(l)}(0)}{l!}\frac{n!}{(n-l)!}z^{n-l} = n!\sum_{k=0}^{n}\frac{\phi^{(n-k)}(0)}{(n-k)!}\frac{z^k}{k!},$$

$$\begin{aligned}\phi(D^2)M^{2n}(z) &= \sum_{l=0}^{n}\frac{\phi^{l}(0)}{l!}\frac{(2n)!}{(2n-2l)!}z^{2n-2l}\\ &= (2n)!\sum_{k=0}^{n}\frac{\phi^{(n-k)}(0)}{(n-k)!}\frac{(z^2)^k}{(2k)!},\end{aligned}$$

and

$$\begin{aligned}\phi(D^2)M^{2n+1}(z) &= \sum_{l=0}^{n}\frac{\phi^{(l)}(0)}{l!}\frac{(2n+1)!}{(2n+1-2l)!}z^{2n+1-2l}\\ &= (2n+1)!z\sum_{k=0}^{n}\frac{\phi^{(n-k)}(0)}{(n-k)!}\frac{(z^2)^k}{(2k+1)!};\end{aligned}$$

hence

$$\frac{n!}{(2n)!}\phi(D^2)M^{2n}(z) = \left\langle \frac{k!}{(2k)!}\right\rangle \phi(D)M^n(z^2), \tag{4.3}$$

and

$$\frac{n!}{(2n+1)!}\phi(D^2)M^{2n+1}(z) = z\left\langle \frac{k!}{(2k+1)!}\right\rangle \phi(D)M^n(z^2). \tag{4.4}$$

For $\lambda \in \mathbb{R}$ let Ψ_λ be the real entire function defined by

$$\Psi_\lambda(z) = e^{\lambda z}\cos\sqrt{z}.$$

It is clear that Ψ_λ is a Laguerre-Pólya function. Since

$$e^{\lambda D}f(z) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}f^{(n)}(z) = f(z+\lambda)$$

for every entire function f, we have

$$\Psi_\lambda(D)f(z) = \Psi_0 f(z+\lambda),$$

whenever $\Psi_0(D)f$ is well defined. In fact, it is known that $\Psi_0(D)f$ is well defined for every entire function f. For a proof, see [3].

From (4.1), we have $e^{\lambda D^2} F_n = \Psi_\lambda(D^2)M^n$; hence (4.3) and (4.4) imply

$$\frac{n!}{(2n)!} e^{\lambda D^2} F_{2n}(z) = \left\langle \frac{k!}{(2k)!} \right\rangle \Psi_\lambda(D) M^n(z^2)$$

and

$$\frac{n!}{(2n+1)!} e^{\lambda D^2} F_{2n+1}(z) = z \left\langle \frac{k!}{(2k+1)!} \right\rangle \Psi_\lambda(D) M^n(z^2).$$

For simplicity, put

$$f = \Psi_\lambda(D)M^n, \quad g = \left\langle \frac{k!}{(2k)!} \right\rangle f \quad \text{and} \quad h = \left\langle \frac{k!}{(2k+1)!} \right\rangle f, \tag{4.5}$$

so that

$$e^{\lambda D^2} F_{2n}(z) = \frac{(2n)!}{n!} g(z^2) \quad \text{and} \quad e^{\lambda D^2} F_{2n+1}(z) = \frac{(2n+1)!}{n!} z h(z^2).$$

Since

$$f(z) = \Psi_\lambda(D)M^n(z) = \Psi_0(D)M^n(z+\lambda) \tag{4.6}$$

and

$$\Psi_0(D)M^n(z) = n! \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(2n-2k)!} \frac{z^k}{k!} = (-1)^n \frac{n!}{(2n)!} z^n H_{2n}\left(\frac{1}{2\sqrt{z}}\right) \tag{4.7}$$

where $H_{2n}(z)$ is the $2n$-th Hermite polynomial (cf.[31, (5.5.4)]), f has real zeros only. Since the functions

$$z \mapsto \sum_{k=0}^{\infty} \frac{k!}{(2k)!} \frac{z^k}{k!} = \cosh\sqrt{z} \quad \text{and} \quad z \mapsto \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!} \frac{z^k}{k!} = \frac{\sinh\sqrt{z}}{\sqrt{z}}$$

are Laguerre-Pólya functions of the first kind, the Pólya-Schur theorem implies that g and h also have real zeros only. Thus $e^{\lambda D^2} F_{2n}$ and $e^{\lambda D^2} F_{2n+1}$ have exactly $2N(g; (-\infty, 0))$ and $2N(h; (-\infty, 0))$ non-real zeros, respectively; and all the non-real zeros are purely imaginary.

It is obvious that f, g and h are polynomials of the same degree n, and that

$$\langle \text{sg } f^k(0)\rangle = \langle \text{sg } g^k(0)\rangle = \langle \text{sg } h^k(0)\rangle.$$

Since f, g and h haver real zeros only, they have no critical points. Hence, by the Budan-Fourier-Hurwitz theorem, they have the same number of negative real zeros. In particular, the following are equivalent: (i) $e^{\lambda D^2}F_{2n}$ has non-real zeros, (ii) $e^{\lambda D^2}F_{2n+1}$ has non-real zeros, and (iii) f has a negative (real) zero.

From (4.6), f has a negative zero if and only if λ is greater than the smallest zeros of (4.7). Thus if λ_n is the largest zero of $2n$-th Hermite polynomial,

$$\lambda(F_{2n}) = \lambda(F_{2n+1}) = \frac{1}{(2\lambda_n)^2}.$$

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국문초록

본 논문에서는 정함수의 영점분포에 관해 연구하였다. 우선, 미탁-레플러 함수의 영점에 대해 연구하였다. 만약 α와 β가 복소수이고 Re $\alpha > 0$ 일때, 미탁-레플러 함수 $E_{\alpha,\beta}$는 다음과 같이 주어진다.

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}.$$

미탁-레플러 함수의 근에 관한 가장 최근의 결과는 $\alpha > 2$이고 $0 < \beta \leq 2\alpha - 1$이거나 $\alpha > 4$이고 $0 < \beta \leq 2\alpha$이면 $E_{\alpha,\beta}(z)$는 오직 실근만을 갖는다는 것이다. 이 결과를 개선하여 $\alpha \geq 4.07$이고 $0 < \beta \leq 3\alpha$이면 $E_{\alpha,\beta}(z)$가 오직 실근만을 갖는다는 것을 보였다.

다음으로 미분 연산자의 폴랴-위만 성질에 관해 연구하였다. $\phi(x) = \sum \alpha_n x^n$는 실계수를 갖는 멱급수이고 D는 미분연산자를 의미한다. 모든 실계수 다항식 f에 대해서 양의정수 m_0가 존재하여 $m \geq m_0$인 모든 정수 m에 대해 새로운 다항식 $\phi(D)^m f$이 오직 실근만을 갖기위한 필요충분 조건은 $\alpha_0 = 0$ 또는 $2\alpha_0\alpha_2 - \alpha_1^2 < 0$ 임을 보였다. 또한, ϕ가 라귀에르-폴랴 함수가 아닐때, 종수가 0인 라귀에르-폴랴 함수 f가 존재하여 모든 양의 정수 m에 대해 $\phi(D)^m f$가 무한히 많은 허근을 갖는 정함수가 된다는 것을 보였다.

λ_n은 다음과 같이 정의된 에르미트 다항식 H_{2n}

$$H_{2n}(z) = (2n)! \sum_{k=0}^{n} \frac{(-1)^k}{k!(2n-2k)!}(2z)^{2n-2k},$$

의 가장 큰 근이고 $M^n(z) = z^n$일때, 다음 등식이 성립함을 보였다.

$$\sup\{\alpha \in \mathbb{R} : e^{\alpha D^2} \cos D\ M^n \text{이 오직 실근만을 갖는다.}\} = 4{\lambda_n}^{-2}.$$

주요어휘: 미탁-레플러 함수, 폴랴-위만 정리, 다항식과 정함수의 근, 선형 미분 연산자, 라귀에르-폴랴 클래스, 에르미트 다항식, 드브루인-뉴먼 상수
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