이학박사 학위논문

Finite C*-algebras associated with labeled graphs

(유한 C^*-대수로서의 라벨 그래프 C^*-대수)

2015년 8월

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Finite C*-algebras associated with labeled graphs

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
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by

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Abstract

Finite C*-algebras associated with labeled graphs

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We study the properties of the C^*-algebras $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ associated to labeled spaces $(E, \mathcal{L}, \overline{\mathcal{E}})$. It is shown that if $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is AF, then the labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ has no loops. We also prove that some of the known equivalent conditions for usual graph C^*-algebras $C^*(E)$ to be AF are not necessarily equivalent for labeled graph C^*-algebras by providing examples. For this, we use generalized Morse sequences. These examples are also shown to be non-AF simple finite C^*-algebras, which contrasts with the fact that the usual simple graph C^*-algebras are either AF or purely infinite.

Besides, we find a sufficient condition for a labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ to give rise to an infinite C^*-algebra in the sense that every nonzero hereditary C^*-subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ contains an infinite projection.

Key words: graph C^*-algebras, labeled graph C^*-algebras, finite C^*-algebras, AF C^*-algebas, purely infinite C^*-algebras, generalized Morse sequences
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Contents

Chapter 1

Introduction

About forty years ago, Cuntz [7] introduced a class of C^*-algebras $\mathcal{O}_n$ called the Cuntz algebras which are generated by n isometries satisfying certain relations. In [9], Cuntz and Krieger constructed a generalized version of the Cuntz algebras associated to a finite $\{0,1\}$-matrix A. The Cuntz-Krieger algebra $\mathcal{O}_A$ which is defined to be the C^*-algebra generated by partial isometries satisfying relations determined by A played an important role for the study of the topological Markov chain associated with the matrix A. It is natural to try to generalize this sort of C^*-algebras of partial isometries satisfying some relations given by finite matrices to C^*-algebras of (infinitely many) partial isometries with relations given by objects like directed graphs, since matrices with positive integer entries are nothing but the adjacency matrices of some directed graphs. Actually in [30], a class of C^*-algebras $C^*(E)$ for directed graphs E (briefly, graph C^*-algebras or graph algebras) was introduced as groupoid C^*-algebras using the groupoid structure of the infinite path spaces when the graphs E are locally finite, that is, every vertex emits and receives only finitely many edges. Then in [29] for row-finite graphs E (every vertex emits only finitely many edges), the C^*-algebras $C^*(E)$ were shown to be defined without using groupoid machineries. It is also shown later in [11] that an arbitrary graph can be transformed into a row-finite graph with no sinks through the so-called desingularization and that the C^*-algebra of the original graph is isomorphic to a full conner of the C^*-algebra of the desingularized row-finite one. Thus every graph C^*-algebra is stably isomorphic to a C^*-algebra of a row-finite graph. This fact allows us to focus on row-finite graphs and thier C^*-algebras.

Besides the graph C^*-algebras, there have been various generalizations of Cuntz-Kreiger algebras. The Exel-Lace algebras [13], the ultra graph algebras [35], and the higher-rank graph algebras [28] are those generalizations which also include the C^*-algebras of row-finite graphs with no sinks. On the other hand, it is known [26] that the class of graph algebras, Exel-Laca algebras, and ultra graph algebras coincide up to Morita equivalence.

Working with graph algebras has attractive benefit because many complex properties and structures of graph algebras can be explained in terms of graph conditions. For example, a graph algebra $C^*(E)$ is an AF (approximately finite dimensional) algebra if and only if E has no loops [29], and similarly if a higher-rank graph C^*-algebra $C^*(\Lambda)$ is AF, then the higher-rank graph Λ has no loops [12].

A C^*-algebra is called *infinite* if it contains infinite projections and *finite* otherwise. AF algebras are C^*-algebras that are best understood among finite C^*-algebras, and Cuntz algebras (or, more generally purely infinite simple C^*-algebras) are infinite C^*-algebras. It is also known in [29] that $C^*(E)$ is purely infinite simple if and only if E satisfies Condition (L), namely every loop has an exit, and every vertex connects to a loop.

As a generalization of graph C^*-algebras, a class of C^*-algebras $C^*(E, \mathcal{L}, \mathcal{B})$ associated with labeled spaces $(E, \mathcal{L}, \mathcal{B})$ was introduced in [4] and has been studied in [1, 4, 5, 17, 18, 19, 20]. Briefly, these are the C^*-algebras generated by a family of partial isometies satisfying certain relations from the labeled spaces $(E, \mathcal{L}, \mathcal{B})$, where $\mathcal{L}$ is a labeling map assigning a label (or alphabet) to each of the edges of the graph E and $\mathcal{B}$, called an accommodating set, is a collection of vertex subsets which plays the role of vertices in graph algebras.

In this thesis we first investigate the question of when a labeled graph C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is AF, where $(E, \mathcal{L}, \overline{\mathcal{E}})$ is a labeled space with the smallest non-degenerate accommodating set $\overline{\mathcal{E}}$. We will define a notion of loop for a labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ and show that if $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is AF, the labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ has no loops. Unlike the graph algebra case, it turns out that the converse may not be true in general, namely there is a labeled space with no loops whose C^*-algebra is not AF. A sufficient condition for a labeled space to be associated to an AF labeled graph C^*-algebra will also be given.

It is also well-known [29] that there is a dichotomy for simple graph C^*-algebras: it is either AF or purely infinite. On the other hand, in [5, Proposition 7.2], Bates and Pask provide an example of a simple unital purely infinite labeled graph C^*-algebra which is not isomorphic to any unital graph C^*-algebra. We also know from [33] that there exist simple higher-rank graph C^*-algebras which are neither AF nor purely infinite, more specifically there exist such simple C^*-algebras which are stably isomorphic to irrational rotation algebras or Bunce-Deddens algebras. Since the property of being AF or pure infiniteness is preserved under stable isomorphism, the examples of higher-rank graph C^*-algebras constructed in [33] are not stably isomorphic to any graph C^*-algebras. This leads us to ask if there exists a simple unital labeled graph C^*-algebra which is neither AF nor purely infinite. We will show that there exist simple labeled graph C^*-algebras $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ associated to generalized Morse sequences ω that are not AF, but finite (with unique traces). To see that $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ is non AF, we show that $K_1(C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})) \neq 0$ by applying the K-theory formula obtained in [1]. The fact that $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ is finite comes from the existence of a unique trace which is the extension of the unique ergodic measure on the closed orbit space of ω. This result says that the dichotomy for simple graph C^*-algebras does not hold for simple labeled graph C^*-algebras.

We then turn our attention to the question of what conditions on a labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ guarantee that the labeled graph C^*-algebra $C^*(E, \mathcal{L}, \mathcal{B})$ contains sufficiently many infinite projections in the sense that every nonzero hereditary C^*-subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ contains infinite projections. This property is well-known to be equivalent to the pure infiniteness of [31] at least for simple C^*-algebras. As mentioned earlier, a simple graph algebra $C^*(E)$ is purely infinite exactly when the graph E satisfies Condition (L) and every vertex connects to a loop. To extend this fact to labeled graph C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$, we will make clear the meaning of connecting a vertex to a loop in $(E, \mathcal{L}, \overline{\mathcal{E}})$, and then show that every nonzero hereditary subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is infinite for a disagreeable $(E, \mathcal{L}, \overline{\mathcal{E}})$ with this property. Here the disagreeability of a labeled space is an extended notion of Condition (L) for a graph ([5]).

This thesis is organized as follows. We begin in Chapter 2 with reviewing necessary background on graph C^*-algebras, labeled graph C^*-algebras, and generalized Morse sequences.

In Chapter 3, we find conditions of a labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ which give rise to an AF C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$. Based on the fact that a graph C^*-algebra $C^*(E)$ is AF exactly when the graph E has no loops, we first consider several conditions on a directed graph E that are equivalent to the existence of a loop in E (Proposition 3.1.2), and then we will define a notion of loop for a labeled space (Definition 3.1.3) by extending one of these conditions. Each of the other equivalent conditions can also be restated in terms of labeled spaces or labeled graph C^*-algebras. We also discuss those equivalent conditions are not always equivalent in the class of labeled graph C^*-algebras.

In Chapter 4, we consider the question of whether the dichotomy for simple graph C^*-algebras (a simple graph C^*-algebra is either AF or purely infinite) would hold true for the class of simple labeled graph C^*-algebras. To answer this question we prove in Theorem 4.1.7 that there exists a simple unital finite, but non-AF labeled graph C^*-algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$. This is a C^*-algebra associated to a labeled space $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ which is labeled by a generalized Morse sequence ω.

Finally in Chapter 5, we investigate conditions of labeled spaces $(E, \mathcal{L}, \overline{\mathcal{E}})$ that generate infinite C^*-algebras. We shall define an analogue of connecting every vertex to a loop in the context of labeled spaces and show that if $(E, \mathcal{L}, \overline{\mathcal{E}})$ is a disagreeable labeled space in which every generalized vertex connects to a loop, then every nonzero hereditary C^*-subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ contains an infinite projection. It will be one of our future projects to explore whether the converse holds and when the (possibly non-simple) labeled graph C^*-algebra would be *purely infinite* in the sense of [31, 32].

Chapter 2

Preliminaries

In this chapter, we review basic definitions and properties of graph C^*-algebras, labeled graph C^*-algebras, and generalized Morse sequences and set up our notation that are frequently used throughout this thesis.

2.1 Directed graphs and their C^*-algebras

A *directed graph* $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and range and source maps $r, s : E^1 \to E^0$. The directed graph E is *row-finite* if each vertex emits only finitely many edges. A row-finite graph is *locally finite* if every vertex receives only finitely many edges. A vertex $v \in E^0$ which emits no edges is called a *sink* and a vertex $v \in E^0$ which does not receive any edges is called a *source*. By E^0_{sink} we denote the set of all sinks of E. A *path of length* n in a directed graph E is a sequence of edges $\lambda = \lambda_1 \cdots \lambda_n$ with $r(\lambda_i) = s(\lambda_{i+1})$ for $1 \le i < n$. We write $|\lambda| := n$ for the length of λ. Let E^n denote the set of all paths of length n. By convention E^0 is regarded as the set of paths of length 0. We let $E^* := \cup_{n\ge 0} E^n$ be the set of all finite paths and let $E^{\le n}$ and $E^{\ge n}$ be the sets $\cup_{i=1}^{n} E^i$ and $\cup_{i=n}^{\infty} E^i$, respectively. The maps r and s naturally extend to E^*, where $r(v) = s(v) = v$ for $v \in E^0$. If a sequence of edges $\lambda_i \in E^1 (i \ge 1)$ satisfies $r(\lambda_i) = s(\lambda_{i+1})$, one obtains an infinite path $\lambda_1\lambda_2\lambda_3 \cdots$ with the source $s(\lambda_1\lambda_2\lambda_3 \cdots) := s(\lambda_1)$ and

E^∞ will denote the set of all infinite paths.

If E is a directed graph, a *Cuntz-Kreiger E-family* consists of a set $\{p_v : v \in E^0\}$ of mutually orthogonal projections and a set $\{s_e : e \in E^1\}$ of partial isometries satisfying the following *Cuntz-Kreiger relations*:

(i) $s_e^* s_e = p_{r(e)}$ for $e \in E^1$,

(ii) $s_e s_e^* \le p_{s(e)}$ for $e \in E^1$,

(iii) $p_v = \sum_{s(e)=v} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$.

It is shown ([29, Theorem 1.2] and [14]) that there is a C^*-algebra $C^*(E)$ generated by a universal Cuntz-Krieger E-family $\{s_e, p_v : e \in E^1, v \in E^0\}$ (or, briefly $\{s_e, p_v\}$). More precisely, for every Cuntz-Krieger E-family $\{S_e, P_v\}$ of partial isometries on a Hilbert space $\mathcal{H}$, there is a representation $\pi := \pi_{S,P}$ of $C^*(E)$ on $\mathcal{H}$ such that $\pi(s_e) = S_e$ and $\pi(p_v) = P_v$ for all $e \in E^1, v \in E^0$. The C^*-algebra $C^*(E)$ is called the *graph C^*-algebra* of E. Since one can construct families $\{S_e, P_v\}$ in which all projections P_v are non-zero, we have that p_v is non-zero for all $v \in E^0$.

It is known in [11] that if E is an arbitrary graph, there is a row-finite graph E' with no sinks or sources such that $C^*(E)$ and $C^*(E')$ are stably isomorphic. Since we are mainly interested in properties of graph C^*-algebras that are preserved under stable isomorphism, we will restrict ourselves to graph C^*-algebras $C^*(E)$ of row-finite graphs E. So, from now on E will be a row-finite directed graph unless stated otherwise.

The Cuntz-Krieger relations imply that $s_e s_f \neq 0$ only if $r(e) = s(f)$ and that $s_e^* s_f = 0$ unless $e = f$. More generally, a product $s_\lambda := s_{\lambda_1} s_{\lambda_2} \cdots s_{\lambda_n}$ of partial isometries $s_{\lambda_1}, \cdots, s_{\lambda_n}$ is non-zero precisely when $\lambda = \lambda_1 \lambda_2 \cdots \lambda_n$ is a path in E^n and one has the following relations.

Lemma 2.1.1. ([29, Lemma 1.1]) *Let E be a row-finite graph, $\{s_e, p_v\}$ a Cuntz-*

Krieger E-family, and $\nu, \lambda \in E^*$. Then

$$s_\nu^* s_\lambda = \begin{cases} s_{\lambda'}, & \textit{if } \lambda = \nu\lambda' \\ s_{\nu'}^*, & \textit{if } \nu = \lambda\nu' \\ p_{r(\lambda)}, & \textit{if } \nu = \lambda \\ 0, & \textit{otherwise.} \end{cases}$$

Moreover, every non-zero finite product of s_e, p_v and s_f^* is a partial isometry of the form $s_\mu s_\nu^*$ for some $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$.

It follows by Lemma 2.1.1 that for a universal Cuntz-Krieger E-family $\{s_e, p_v\}$

$$C^*(E) = \overline{span}\{s_\mu s_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\},$$

where $s_v := p_v$ for $v \in E^0$.

A finite path λ with $|\lambda| > 0$ is called a *loop* based at $v \in E^0$ if $s(\lambda) = r(\lambda) = v$, that is, if it comes back to its source vertex. An *exit* of a loop $\lambda = \lambda_1\lambda_2 \cdots \lambda_n$ is an edge $f \in E^1$ which satisfies that $s(f) = s(\lambda_i)$ for some $i \in \{1, \cdots, n\}$, but $f \neq \lambda_i$. We say that a directed graph E satisfies *Condition* (L) if every loop has an exit.

Theorem 2.1.2. (The Cuntz-Krieger uniqueness theorem [3, Theorem 3.1]). *Let E be a row-finite graph which satisfies Condition (L) and let $\{T_e, Q_v\}$ be a Cuntz-Krieger E-family such that $Q_v \neq 0$ for every $v \in E^0$. Then there is an isomorphism π of $C^*(E)$ onto $C^*(T_e, Q_v)$ such that $\pi(s_e) = T_e$ and $\pi(p_v) = Q_v$ for all $e \in E^1$ and $v \in E^0$.*

For each $z \in \mathbb{T}$, the family $\{zs_e, p_v : e \in E^1, v \in E^0\}$ is a Cuntz-Kreiger E-family which generates $C^*(E) = C^*(s_e, p_v)$. Thus the universal property of $C^*(E)$ defines an automorphism $\gamma_z : C^*(E) \to C^*(E)$ such that

$$\gamma_z(s_e) = zs_e \text{ and } \gamma_z(p_v) = p_v$$

for all $e \in E^1$ and $v \in E^0$. Moreover, $\gamma : \mathbb{T} \to Aut(C^*(E))$ given by $\gamma(z) := \gamma_z$ is a strongly continuous action of $\mathbb{T}$ on the C^*-algebra $C^*(E)$ which is called the *gauge action*.

Theorem 2.1.3. (The Gauge Invariant Uniqueness Theorem [3, Theorem 2.1]). *Let E be a row-finite graph and $\{T_e, Q_v\}$ be a Cuntz-Krieger E-family in which Q_v is non-zero for all $v \in E^0$. Let $\pi := \pi_{S,P}$ be the representation of $C^*(E)$ such that $\pi(s_e) = T_e$ and $\pi(p_v) = Q_v$. If there is a strongly continuous action β of $\mathbb{T}$ on $C^*(T_e, Q_v)$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}$, then π is faithful.*

A directed graph E is said to be *cofinal* if for every vertex $v \in E^0$ and for every infinite path $\lambda = \lambda_1\lambda_2 \cdots \in E^\infty$, there exists $\mu \in E^*$ such that $s(\mu) = v$ and $r(\mu) = s(\lambda_i)$ for some $i \geq 1$. For the simplicity of graph C^*-algebras, the following is known.

Theorem 2.1.4. ([3, Proposition 5.1]) *Let E be a row-finite directed graph with no sinks. Then $C^*(E)$ is simple if and only if E is cofinal and satisfies Condition (L).*

For $v, w \in E^0$ we write $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set H is *saturated* if $v \in H$ whenever $\{r(e) : s(e) = v\} \subset H$. If H is a hereditary set, the *saturation* of H is the smallest saturated subset $\overline{H}$ of E^0 containing H. For each subset H of E^0, let I_H be the ideal of $C^*(E)$ generated by the projections $\{p_v : v \in H\}$.

Theorem 2.1.5. ([3, Theorem 4.1]) *Let E be a row-finite directed graph. Then we have the following.*

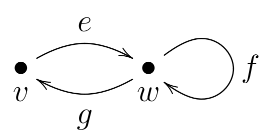
(i) *The map $H \mapsto I_H$ is an isomorphism from the lattice of saturated hereditary subsets of E^0 onto the lattice of gauge-invariant ideals of $C^*(E)$.*

(ii) *Let H be a saturated hereditary subset of E^0 and $E \setminus H := (E^0 \setminus H, E^1 \setminus$*

$r^{-1}(H), r, s)$ be the subgraph of E. Then $C^*(E)/I_H$ is canonically isomorphic to $C^*(E \setminus H)$.

A loop $\lambda = \lambda_1\lambda_2 \cdots \lambda_n$ at $v = s(\lambda)$ is called a *first-return path* if $s(\lambda_i) \neq v$ for all $i = 2, \cdots, n$. A directed graph E is said to satisfy *Condition* (K) if no vertex is the base of exactly one first-return path.

Example 2.1.6. The following directed graph E satisfies Condition (K). The vertex v is a base of two distinct first-return paths eg and efg. Also, f and ge are distint first-return paths at w. (fge is not a first-return path.)



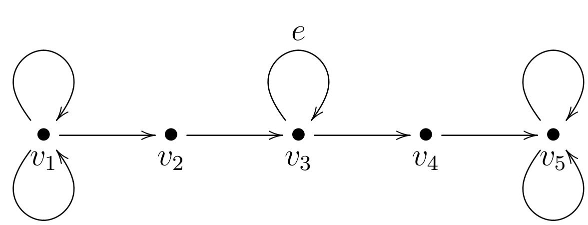
Theorem 2.1.7. ([3, Theorem 4.4]) *Let E be a row-finite directed graph. Then the following are equivalent.*

(i) *E satisfies Condition (K),*

(ii) *all ideals of $C^*(E)$ are gauge invariant,*

(iii) *the map $H \mapsto I_H$ is a lattice isomorphism from the saturated hereditary subsets of E onto the ideals of $C^*(E)$.*

Example 2.1.8. The following graph E satisfies Condition (L), but does not satisfy Condition (K) since v_3 is the base of only the first-return path e.

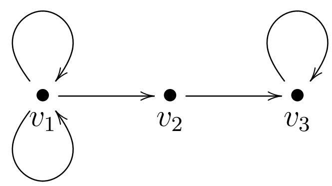


The hereditary saturated subsets of E^0 are

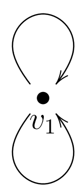
$$\emptyset,\ H_1 = \{v_4, v_5\},\ H_2 = \{v_2, v_3, v_4, v_5\},\ E^0.$$

Thus $C^*(E)$ has two non-trivial gauge invariant ideals, namely I_{H_1} and I_{H_2}. It is easy to see that the subgraphs $E \setminus H_1$ and $E \setminus H_2$ are as follows:

$E \setminus H_1$:



$E \setminus H_2$:



By Theorem 2.1.5 (ii), one sees that $C^*(E)/I_{H_2} \cong C^*(E \setminus H_2)$. Note that the hereditary subalgebra $p_{v_3}C^*(E)p_{v_3}$ is isomorphic to $C(\mathbb{T})$. Since $C(\mathbb{T})$ has many ideals that are not gauge invariant, it follows that $C^*(E)$ also has many ideals that are not gauge invariant.

If a graph E has a loop with an exit, the C^*-algebra $C^*(E)$ is infinite in the sense that $C^*(E)$ contains an infinite projection. In fact, if $\lambda = \lambda_1\lambda_2 \cdots \lambda_n$ is a loop at $v = s(\lambda_1)$ with an exit $f \in E^1$ at v, the projection p_v is Murray-von Neumann equivalent to its proper subprojection $s_\lambda s_\lambda^*$ in $C^*(E)$ since $p_v = s_\lambda^* s_\lambda \sim s_\lambda s_\lambda^* \leq s_{\lambda_1} s_{\lambda_1}^* < s_{\lambda_1} s_{\lambda_1}^* + s_f s_f^* \leq p_v$. Thus if $C^*(E)$ is finite, the graph E should not have any loops with exits. If E has a loop with no exits, it can be seen that $C^*(E)$ has a quotient C^*-algebra which is stably isomorphic to $C(\mathbb{T})$. Thus, if $C^*(E)$ is an AF algebra, the graph E can not have any loops. Moreover, the converse is also known to hold true.

Theorem 2.1.9. ([29, Theorem 2.4]) *A directed graph E has no loops if and only if $C^*(E)$ is an AF algebra.*

If a graph C^*-algebra $C^*(E)$ is AF, by Theorem 2.1.7 all ideals of $C^*(E)$ are gauge invariant because E (with no loops) trivially satisfies Condition (K). Since any quotients of AF algebras are also AF, the graph $E \setminus H$ must have no loops for any saturated hereditary subset H of E^0.

Recall ([7]) that a simple C^*-algebra A is said to be *purely infinite* if every non-zero hereditary C^*-subalgebra of A contains an infinite projection. In many works including [29], a (possibly non-simple) C^*-algebra A that has this property was called purely infinite. On the other hand, another notion of pure infiniteness for non-simple C^*-algebras was studied intensively by Kirchberg and Rørdam in [31, 32] and it was suggested to call a non-simple C^*-algebra A *purely infinite* if there are no characters on A and if for every pair of positive elements $a, b \in A$ such that $a \in \overline{AbA}$, there exists a sequence $(x_i)_{i=1}^{\infty}$ in A with $x_i^* b x_i \to a$ ([31, Definition 4.1]).

In this thesis, if a C^*-algebra A has the property that every non-zero hereditary C^*-subalgebra of A contains an infinite projection, we will say that A has the property (SP_∞) to distinguish this one from the pure infiniteness of [31]. The reason we have chosen (SP_∞) to refer the property is because in the literature a C^*-algebra is said to have the property (SP) if every nonzero hereditary C^*-subalgebra of A contains a nonzero projection. In gereral, the property (SP_∞) is neither weaker nor stronger than pure infiniteness ([31, Example 4.6]). But, both definitions are equivalent for simple C^*-algebras ([31, Proposition 4.6 and Proposition 5.4]).

Theorem 2.1.10. ([3, Proposition 5.3] or [29, Theorem 3.9]) *Let E be a row-finite directed graph with no sinks. Then $C^*(E)$ has the property (SP_∞) if and only if E satisfies Condition (L) and every vertex connects to a loop.*

If a directed graph E is cofinal and has a loop, every vertex automatically connects to every loop. Combining this fact together with Theorem 2.1.9, we have the following dichotomy for simple graph C^*-algebras.

Corollary 2.1.11. ([29, Corollary 3.10]) *Let E be a row-finite directed graph. If $C^*(E)$ is simple, then either*

(i) *$C^*(E)$ is an AF-algebra if E has no loops; or*

(ii) *$C^*(E)$ is purely infinite if E has a loop.*

For a path $\mu \in E^* \cup E^\infty$, μ^0 will denote the following subset of E^0

$$\mu^0 = \{v \in E^0 : v = s(e) \text{ or } v = r(e) \text{ for some edge } e \text{ appearing in } \mu\},$$

namely μ^0 is the set of all vertices that μ is passing through. A path $\nu \in E^*$ is called a *detour* of μ if $s(\nu) \in \mu^0$ and $r(\nu) \in \mu^0$. Obviously, subpaths of μ are detours of μ. In [16], Hjelmborg shows among others that:

Theorem 2.1.12. ([16, Theorem 3.1]) *Let E be a locally finite directed graph with no sinks. The following are equivalent.*

(i) *$C^*(E)$ is purely infinite,*

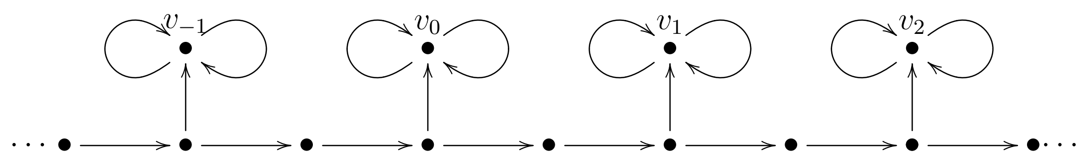
(ii) *$C^*(E)$ has no quotient that contains a two-sided ideal that is an AF-algebra or contains a corner that is $*$-isomorphic to $M_n(C(\mathbb{T}))$ for some $n \in \mathbb{N}$,*

(iii) *every infinite path in E admits a detour β such that there are two or more loops based at some vertex of β^0,*

(iv) *the subgraph $E \setminus H := (E^0 \setminus H, E^1 \setminus r^{-1}(H), r, s)$ has the property that every vertex connects to a loop with an exit for every hereditary saturated subset H of E^0.*

Theorem 2.1.12 (iv) says that a purely infinite graph C^*-algebra $C^*(E)$ has the property (SP_∞). The converse may not be true as we see from the following example.

Example 2.1.13. Consider a directed graph E as follows:



It is easy to see that E satisfies Condition (L) and E is not cofinal, but every vertex in E^0 connects to a loop. Thus, $C^*(E)$ is a non-simple C^*-algebra by Theorem 2.1.4 and has the property (SP_∞) by Theorem 2.1.10. Whereas the hereditary saturated subset $H := \{v_i : i \in \mathbb{Z}\}$ of E^0 gives rise to the non-trivial ideal I_H in $C^*(E)$ generated by the projections $\{p_v : v \in H\}$ and $C^*(E)/I_H$ is isomorphic to $C^*(E \setminus H)$, where the directed subgraph $E \setminus H = (E^0 \setminus H, E^1 \setminus r^{-1}(H), r, s)$ is as below:

The subgraph $E \setminus H$ has no loops at all, so $C^*(E \setminus H)$ is AF by Theorem 2.1.9, which means that $C^*(E)$ contains an AF quotient. Thus $C^*(E)$ is not purely infinite. This also can be seen from Theorem 2.1.12 (iii) since E has an infinite path in which every vertex admits no loops at all.

2.2 Labeled spaces and their C^*-algebras

We use notational conventions of [1, 5] for labeled spaces and their C^*-algebras. A *labeled graph* $(E, \mathcal{L})$ over a countable alphabet $\mathcal{A}$ consists of a directed graph E and a *labeling map* $\mathcal{L} : E^1 \to \mathcal{A}$. We assume that the map $\mathcal{L}$ is onto. Let $\mathcal{A}^*$ be the set of all finite sequences of length greater than or equal to 1 in the symbols of $\mathcal{A}$. Then the map $\mathcal{L}$ extends naturally to the map $\mathcal{L} : E^* \to \mathcal{A}^*$ given by $\mathcal{L}(\lambda) := \mathcal{L}(\lambda_1) \cdots \mathcal{L}(\lambda_n) \in \mathcal{A}^*$ for $\lambda = \lambda_1 \cdots \lambda_n \in E^n$. Similarly, $\mathcal{A}^\infty$ denotes the set of all infinite sequences in $\mathcal{A}$ and the map $\mathcal{L}$ extends to E^∞ via $\mathcal{L}(\delta) := \mathcal{L}(\delta_1)\mathcal{L}(\delta_2) \cdots \in \mathcal{L}(E^\infty) \subset \mathcal{A}^\infty$ for $\delta = \delta_1\delta_2 \cdots \in E^\infty$. We use notation $\mathcal{L}^*(E) := \mathcal{L}(E^{\geq 1})$. For $\alpha = \alpha_1\alpha_2\cdots\alpha_{|\alpha|} \in \mathcal{L}^*(E)$, we denote the subsegment $\alpha_i\cdots\alpha_j$ of α by $\alpha_{[i,j]}$ for $1 \leq i \leq j \leq |\alpha|$. A subsegment of the form $\alpha_{[1,j]}$ is called an *initial path* of α. The *range* $r(\alpha)$ and *source* $s(\alpha)$ of a labeled path $\alpha \in \mathcal{L}^*(E)$ are subsets of E^0 defined by

$$\begin{aligned} r(\alpha) &= \{r(\lambda) : \lambda \in E^{\geq 1},\, \mathcal{L}(\lambda) = \alpha\}, \\ s(\alpha) &= \{s(\lambda) : \lambda \in E^{\geq 1},\, \mathcal{L}(\lambda) = \alpha\}. \end{aligned}$$

The *relative range of* $\alpha \in \mathcal{L}^*(E)$ *with respect to* $A \subset E^0$ is defined to be

$$r(A, \alpha) = \{r(\lambda) : \lambda \in E^{\geq 1},\ \mathcal{L}(\lambda) = \alpha,\ s(\lambda) \in A\}.$$

A collection $\mathcal{B} \subset 2^{E^0}$ of subsets of E^0 is said to be *closed under relative ranges* for $(E, \mathcal{L})$ if $r(A, \alpha) \in \mathcal{B}$ whenever $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$. We call $\mathcal{B}$ an *accommodating set* for $(E, \mathcal{L})$ if it contains $r(\alpha)$ for all $\alpha \in \mathcal{L}^*(E)$ and it is closed under relative ranges, finite intersections and unions. Moreover, if an accommodation set $\mathcal{B}$ is closed under relative complements, then it is said to be *non-degenerate* ([1]).

Definition 2.2.1. Let $(E, \mathcal{L})$ be a labeled graph. A *labeled space* consists of a triple $(E, \mathcal{L}, \mathcal{B})$ where $\mathcal{B}$ is an accommodating set for $(E, \mathcal{L})$. If in addition $\mathcal{B}$ is non-degenerate, then the labeled space $(E, \mathcal{L}, \mathcal{B})$ is called *normal* ([1]).

A labeled space $(E, \mathcal{L}, \mathcal{B})$ is *weakly left-resolving* if

$$r(A, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha)$$

holds for all $A, B \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$. A labeled graph $(E, \mathcal{L})$ is *left-resolving* if the map $\mathcal{L} : r^{-1}(v) \to \mathcal{A}$ is injective for each $v \in E^0$ and *label-finite* if $|\mathcal{L}^{-1}(a)| < \infty$ for each $a \in \mathcal{A}$. If $(E, \mathcal{L})$ is left-resolving, then it is label-finite if and only if $r(a)$ is finite for all $a \in \mathcal{A}$. A set $A \in \mathcal{B}$ is called *minimal* (in $\mathcal{B}$) if A does not have any proper subset in $\mathcal{B}$.

We denote by $\mathcal{E}$ the smallest subset of 2^{E^0} which is an accommodating set for $(E, \mathcal{L})$ and by $\mathcal{E}^{0,-}$ the smallest accommodating set containing

$$\{r(\alpha) : \alpha \in \mathcal{L}^*(E)\} \cup \{\{v\} : v \text{ is a sink or a source}\}.$$

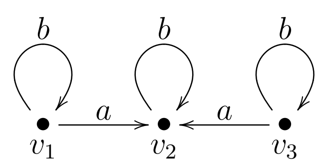
If E has no sinks or sources, $\mathcal{E}^{0,-} = \mathcal{E}$ and if $(E, \mathcal{L}, \mathcal{E})$ is weakly left-resolving,

$$\mathcal{E} = \{\cup_{k=1}^{m} \cap_{i=1}^{n} r(\beta_{i,k}) : \ \beta_{i,k} \in \mathcal{L}^*(E)\}$$

from [5, Remarks 2.1(i)]. For a vertex subset $A \subset E^0$, A_{sink} denotes the sinks $A \cap E^0_{\text{sink}}$ in A, and for $\mathcal{B} \subset 2^{E_0}$ we simply denote the set $\{A_{\text{sink}} : \ A \in \mathcal{B}\}$ by $\mathcal{B}_{\text{sink}}$. For a labeled space $(E, \mathcal{L}, \mathcal{B})$, we denote by $\overline{\mathcal{B}}$ the smallest non-degenerate accommodating set that contains $\mathcal{B} \cup \mathcal{B}_{\text{sink}}$. The existence of $\overline{\mathcal{B}}$ clearly follows from considering the intersection of all those accommodating sets. $\overline{\mathcal{E}}$ will thus denote the smallest non-degenerate accommodating set containing $\mathcal{E}_{\text{sink}} = \{A_{\text{sink}} : A \in \overline{\mathcal{E}}\}$ (We wrote $\overline{\mathcal{E}^0}$ for $\overline{\mathcal{E}}$ in the paper [17]). Also with abuse of notation, for $\mathcal{B} \subset 2^{E_0}$ and $A \subset E_0$, we write

$$\mathcal{B} \cap A := \{B \in \mathcal{B} : \ B \subset A\}.$$

Example 2.2.2. For the following labeled graph $(E, \mathcal{L})$ (see [1])



,

the smallest accommodating set is $\mathcal{E} = \{\{v_2\}, \{v_1, v_2, v_3\}\}$, while the smallest non-degenerate accommodating set is $\overline{\mathcal{E}} = \{\emptyset, \{v_2\}, \{v_1, v_3\}, \{v_1, v_2, v_3\}\}$. Thus, $\mathcal{E} \subsetneq \overline{\mathcal{E}}$. The set $\mathcal{B} = \{\emptyset, \{v_2\}, \{v_3\}, \{v_2, v_3\}, \{v_1, v_3\}, \{v_1, v_2, v_3\}\}$ is also an accommodating set for $(E, \mathcal{L})$, which is not closed under relative complements: $\{v_1, v_3\} \setminus \{v_3\} = \{v_1\} \notin \mathcal{B}$. The labeled space $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving. Of course, 2^{E^0} is the largest accommodating set for $(E, \mathcal{L})$. But, $(E, \mathcal{L}, 2^{E^0})$ is not weakly left-resolving because $r(\{v_1\}, a) \cap r(\{v_2, v_3\}, a) = \{v_2\}$, but $r(\{v_1\} \cap \{v_2, v_3\}, a) = \emptyset$.

Remark 2.2.3. As we have seen in Example 2.2.2, the smallest accommodating set $\mathcal{E}$ is not necessarily closed under relative complements. On the other hand, in the construction of the C^*-algebra $C^*(E, \mathcal{L}, \mathcal{B})$ ([4, 5]) of a labeled space $(E, \mathcal{L}, \mathcal{B})$, to each nonempty set $A \in \mathcal{B}$ there is associated a nonzero projection p_A in $C^*(E, \mathcal{L}, \mathcal{B})$ in such a manner that $p_A \leq p_B$ whenever $A \subset B$. Hence $p_B - p_A$ belongs to $C^*(E, \mathcal{L}, \mathcal{B})$ and it seems reasonable to write $p_{B \setminus A}$ for $p_B - p_A$, which leads us to consider accommodating sets that are closed under relative complements and the results in [19] was obtained under this assumption. But then quite recently in [1], a labeled space $(E, \mathcal{L}, \mathcal{B})$ with an accommodating set $\mathcal{B}$ which is closed under relative complements is newly termed as *normal* and discussed that the original definition of $C^*(E, \mathcal{L}, \mathcal{B})$ given in [4, 5] is correct to establish the so-called Gauge Invariant Uniqueness Theorem only when $(E, \mathcal{L}, \mathcal{B})$ is normal. For a general case, a correct definition of $C^*(E, \mathcal{L}, \mathcal{B})$ is also given in Appendix A of [1].

For $A, B \in 2^{E^0}$ and $n \geq 1$, let

$$AE^n = \{\lambda \in E^n : s(\lambda) \in A\}, \quad E^nB = \{\lambda \in E^n : r(\lambda) \in B\},$$

and $AE^nB = AE^n \cap E^nB$. We write E^nv for $E^n\{v\}$ and vE^n for $\{v\}E^n$. Then the sets $AE^{\geq k}$ and vE^∞ must have their obvious meaning. We also take conventions like $AE^0 = A$ and $\mathcal{L}(A) = A$ for $A \in \mathcal{B}$. A labeled space $(E, \mathcal{L}, \mathcal{B})$ is said to be *set-finite* if the set $\mathcal{L}(AE^l)$ is finite for every $A \in \mathcal{B}$ and $l \geq 1$ and it is said to be *receiver set-finite* if $\mathcal{L}(E^lA)$ is finite for all $A \in \mathcal{B}$ and $l \geq 1$.

Assumptions. We assume that a labeled space $(E, \mathcal{L}, \mathcal{B})$ considered in this thesis always satisfies the following:

(i) $(E, \mathcal{L}, \mathcal{B})$ is normal.

(ii) $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving.

(iii) $(E, \mathcal{L}, \mathcal{B})$ is set-finite and receiver set-finite.

By $\Omega_0(E)$ we mean the set of all vertices of E that are not sources. For

$v, w \in \Omega_0(E) \subset E^0$, we write $v \sim_l w$ if $\mathcal{L}(E^{\le l}v) = \mathcal{L}(E^{\le l}w)$ as in [5]. Then $\sim_l$ is an equivalence relation on $\Omega_0(E)$. The equivalence class $[v]_l$ of v is called a *generalized vertex*. Let $\Omega_l(E) := \Omega_0(E)/{\sim_l}$ for $l \ge 1$. If $k > l$, $[v]_k \subset [v]_l$ is obvious and $[v]_l = \cup_{i=1}^m [v_i]_{l+1}$ for some vertices $v_1, \ldots, v_m \in [v]_l$ ([5, Proposition 2.4]).

Note that every set in $\mathcal{E}$ can be expressed as a finite union of generalized vertices ([5, Remark 2.1 and Proposition 2.4.(ii)], where $\mathcal{E}^{0,-}$ denotes our $\mathcal{E}$):

$$\mathcal{E} \subseteq \{\cup_{i=1}^n [v_i]_l : \ v_i \in \Omega_0(E), \ n, l \ge 1\}.$$

Generalized vertices $[v]_l$ are not always members of the accommodating set $\mathcal{E}$ but always the relative complements of sets in $\mathcal{E}$, namely $[v]_l = X_l(v) \backslash r(Y_l(v))$, where $X_l(v)$, $Y_l(v)$ are given by

$$X_l(v) := \cap_{\alpha \in \mathcal{L}(E^{\le l}v)} r(\alpha) \ \text{ and } \ Y_l(v) := \cup_{w \in X_l(v)} \mathcal{L}(E^{\le l}w) \setminus \mathcal{L}(E^{\le l}v)$$

so that $X_l(v)$, $r(Y_l(v)) \in \mathcal{E}$ ([5, Proposition 2.4]). One can easily check that the expression $[v]_l = X_l(v) \backslash r(Y_l(v))$ is valid even for a sink v and $[v]_l \cap r(Y_l(v)) = \emptyset$.

Notice also that the smallest non-degenerate accommodating set $\overline{\mathcal{E}}$ contains all generalized vertices, and hence

$$\{\cup_{i=1}^n [v_i]_l : \ v_i \in \Omega_0(E), \ n, l \ge 1\} \subset \overline{\mathcal{E}}.$$

More precisely, we see the following.

Proposition 2.2.4. *Let $(E, \mathcal{L})$ be a labeled graph and $A \in \overline{\mathcal{E}}$. Then A is of the form*

$$A = \big(\cup_{i=1}^{n_1} [v_i]_l \big) \cup \big(\cup_{j=1}^{n_2} ([u_j]_l)_{\text{sink}} \big) \cup \big(\cup_{k=1}^{n_3} [w_k]_l \setminus ([w_k]_l)_{\text{sink}} \big) \tag{2.1}$$

for some $v_i, u_j, w_k \in \Omega_0(E)$ and $l \ge 1$, $n_1, n_2, n_3 \ge 0$.

Proof. Let $\mathcal{B}$ be the set of all vertex subsets that are expressed as in the right hand side of (2.1). Then $\mathcal{B} \subset \overline{\mathcal{E}}$ is obvious since $\overline{\mathcal{E}}$ contains all generalized vertices. Now it suffices to show that $\mathcal{B}$ is an accommodating set that is closed under relative complements. By the proof of [5, Proposition 2.4], $r(\alpha) \in \mathcal{B}$ for all labeled paths $\alpha \in \mathcal{L}^*(E)$. It is easy to see that $\mathcal{B}$ is closed under finite unions, finite intersections and relative complements.

In order to show that $\mathcal{B}$ is closed under relative ranges, it suffices to see that $r([v]_l, \alpha) \in \mathcal{B}$ for $v \in \Omega_0(E)$ and $\alpha \in \mathcal{L}^*(E)$. Since $r([v]_l, \alpha) \cap r(r(Y_l(v)), \alpha) = r([v]_l \cap r(Y_l(v)), \alpha) = r(\emptyset, \alpha) = \emptyset$, we have

$$r([v]_l, \alpha) = r\big(X_l(v) \setminus r(Y_l(v)), \alpha\big) = r(X_l(v), \alpha) \setminus r\big(r(Y_l(v)), \alpha\big)$$

which belongs to $\mathcal{B}$ since $r(X_l(v), \alpha)$, $r(r(Y_l(v)), \alpha) \in \mathcal{E} \subset \mathcal{B}$ and $\mathcal{B}$ is closed under relative complements. $\square$

Notation 2.2.5. Let $(E, \mathcal{L})$ be a labeled graph.

(a) As in [1], $\mathcal{L}^{\#}(E)$ will denote the union of all labeled paths $\mathcal{L}^*(E)$ and empty word ϵ, where ϵ is a symbol such that $r(\epsilon) = E^0$, $r(A, \epsilon) = A$ for all $A \subset E^0$.

(b) If $\mathcal{L}$ is the identity map $id : E^1 \to E^1$, it is called the *trivial labeling* and will be denoted by $\mathcal{L}_{id}$. For a labeled graph $(E, \mathcal{L}_{id})$, the accommodating set $\overline{\mathcal{E}}$ is equal to the collection of all finite subsets of E^0.

Recall [5] that $\alpha \in \mathcal{L}^*(E)$ with $s(\alpha) \cap [v]_l \neq \emptyset$ is said to be *agreeable* for $[v]_l$ if $\alpha = \beta\alpha' = \alpha'\gamma$ for some $\alpha', \beta, \gamma \in \mathcal{L}^*(E)$ with $|\beta| = |\gamma| \leq l$. Otherwise α is said to be *disagreeable*.

Definition 2.2.6. ([4, Definition 5.1]) Let E be a graph with no sinks or sources and $(E, \mathcal{L}, \mathcal{B})$ be a labeled space.

(i) $[v]_l \in \mathcal{B}$ is called *disagreeable* if there is an $N > 0$ such that for all $n > N$ there is an $\alpha \in \mathcal{L}(E^{\geq n})$ that is disagreeable for $[v]_l$.

(ii) $(E, \mathcal{L}, \mathcal{B})$ is called *disagreeable* if $[v]_l$ is disagreeable for all $v \in E^0$ and $l \geq 1$.

Note [20, Proposition 3.9] that a generalized vertex $[v]_l$ is *not disagreeable* if and only if there is an $N > 0$ such that every path $\alpha \in \mathcal{L}([v]_l E^{\geq N})$ is *agreeable*, namely is of the form $\alpha = \beta^k \beta'$ for some $k \geq 0$ and some paths $\beta, \beta' \in \mathcal{L}(E^{\leq l})$, where β' is an initial path of β. In case of trivial labeling, $(E, \mathcal{L}_{id}, \overline{\mathcal{E}})$ is disagreeable exactly when the graph E satisfies condition (L) [4, Lemma 5.3].

We now define a representation of a labeled space $(E, \mathcal{L}, \mathcal{B})$ such that $\overline{\mathcal{E}} \subset \mathcal{B}$, where $\overline{\mathcal{E}}$ is the smallest non-degenerate accommodating set containing $\mathcal{E}_{\text{sink}} = \{A_{\text{sink}} : A \in \overline{\mathcal{E}}\}$.

Definition 2.2.7. Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space such that $\overline{\mathcal{E}} \subset \mathcal{B}$. A *representation* of $(E, \mathcal{L}, \mathcal{B})$ consists of projections $\{p_A : A \in \mathcal{B}\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ such that for $A, B \in \mathcal{B}$ and $a, b \in \mathcal{A}$,

(i) $p_\emptyset = 0$, $p_A p_B = p_{A\cap B}$, and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$,

(ii) $p_A s_a = s_a p_{r(A,a)}$,

(iii) $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$,

(iv) for each $A \in \mathcal{B}$,

$$p_A = \sum_{a\in\mathcal{L}(AE^1)} s_a p_{r(A,a)} s_a^* + p_{A_{\text{sink}}}.$$

Remark 2.2.8. For a weakly left-resolving normal labeled space $(E, \mathcal{L}, \mathcal{B})$ such that $\mathcal{E}_{\text{sink}} \not\subset \mathcal{B}$, a definition of a representation of $(E, \mathcal{L}, \mathcal{B})$ is given in [1, Definition 2.1]. As pointed out in [1, Remark 2.3], if $A \in \mathcal{B}$ and $A \cap E^0_{\text{sink}} \in \mathcal{B}$, $p_A = p_{A\cap E^0_{\text{sink}}} + \sum_{a\in\mathcal{L}(AE^1)} s_a p_{r(A,a)} s_a^*$, which agrees with our definition of the representation of $(E, \mathcal{L}, \mathcal{B})$.

Theorem 2.2.9. *Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space such that $\overline{\mathcal{E}} \subset \mathcal{B}$. Then there exists a C^*-algebra B generated by a universal representation $\{s_a, p_A\}$ of $(E, \mathcal{L}, \mathcal{B})$.*

Proof. The assertion can be obtained by a slight modification of the proof of [4, Theorem 4.5], namely we should mod out the $*$-algebra $k_{(E,\mathcal{L},\mathcal{B})}$ by the ideal J generated by the elements $q_{A\cup B}-q_A-q_B+q_{A\cap B}$ and $q_A-\sum_{a\in\mathcal{L}(AE^1)} s_a q_{r(A,a)} s_a^*-q_{A_{\text{sink}}}$ for $A,\ B\in\mathcal{B}$. □

Remark 2.2.10. Let $(E,\mathcal{L},\mathcal{B})$ be a labeled space such that $\overline{\mathcal{E}}\subset\mathcal{B}$.

(i) If $\{s_a,p_A\}$ is a universal representation of $(E,\mathcal{L},\mathcal{B})$, we simply write $C^*(E,\mathcal{L},\mathcal{B})=C^*(s_a,p_A)$ and call $C^*(E,\mathcal{L},\mathcal{B})$ the *labeled graph C^*-algebra* of a labeled space $(E,\mathcal{L},\mathcal{B})$. Note that $s_a\neq 0$ and $p_A\neq 0$ for $a\in\mathcal{A}$ and $A\in\mathcal{B}$, $A\neq\emptyset$, and that $s_\alpha p_A s_\beta^*\neq 0$ if and only if $A\cap r(\alpha)\cap r(\beta)\neq\emptyset$. By Definition 2.2.7(iv) and [4, Lemma 4.4] saying that with $s_\alpha:=p_\alpha$ for $\alpha\in\mathcal{B}$,

$$(s_\alpha p_A s_\beta^*)(s_\gamma p_B s_\delta^*)=\begin{cases} s_{\alpha\gamma'}p_{r(A,\gamma')\cap B}s_\delta^*, & \text{if } \gamma=\beta\gamma' \\ s_\alpha p_{A\cap r(B,\beta')}s_{\delta\beta'}^*, & \text{if } \beta=\gamma\beta' \\ s_\alpha p_{A\cap B}s_\delta^*, & \text{if } \beta=\gamma \\ 0, & \text{otherwise,}\end{cases}$$

for $\alpha,\beta,\gamma,\delta\in\mathcal{L}^\#(E)$ and $A,B\in\mathcal{B}$, it follows that

$$C^*(E,\mathcal{L},\mathcal{B})=\overline{\operatorname{span}}\{s_\alpha p_A s_\beta^*\ :\ \alpha,\ \beta\in\mathcal{L}^\#(E),\ A\in\mathcal{B}\},\tag{2.2}$$

where s_ϵ denotes the unit of the multiplier algebra of $C^*(E,\mathcal{L},\mathcal{B})$ [1]. It is observed in [20] that if E has no sinks nor sources, then

$$C^*(E,\mathcal{L},\mathcal{E}^{0,-})\cong C^*(E,\mathcal{L},\overline{\mathcal{E}}).$$

(ii) From Definition 2.2.7(iv), we have for each $n\geq 1$,

$$p_A=\sum_{\alpha\in\mathcal{L}(AE^n)} s_\alpha p_{r(A,\alpha)}s_\alpha^*+\sum_{\gamma\in\mathcal{L}(AE^{\leq n-1})} s_\gamma p_{r(A,\gamma)_{\text{sink}}}s_\gamma^*,$$

where $\sum_{\gamma\in\mathcal{L}(AE^0)} s_\gamma p_{r(A,\gamma)_{\text{sink}}} s_\gamma^* := p_{A_{\text{sink}}}$. In fact,

$$\begin{aligned}
p_A &= \sum_{a\in\mathcal{L}(AE^1)} s_a p_{r(A,a)} s_a^* + p_{A_{\text{sink}}} \\
&= \sum_{a\in\mathcal{L}(AE^1)} s_a \Big(\sum_{b\in\mathcal{L}(r(A,a)E^1)} s_b p_{r(A,ab)} s_b^* + p_{r(A,a)_{\text{sink}}} \Big) s_a^* + p_{A_{\text{sink}}} \\
&= \sum_{\gamma\in\mathcal{L}(AE^2)} s_\gamma p_{r(A,\gamma)} s_\gamma^* + \sum_{a\in\mathcal{L}(AE^1)} s_a p_{r(A,a)_{\text{sink}}} s_a^* + p_{A_{\text{sink}}} \\
&= \sum_{\gamma\in\mathcal{L}(AE^2)} s_\gamma \Big(\sum_c s_c p_{r(A,\gamma c)} s_c^* + p_{r(A,\gamma)_{\text{sink}}} \Big) s_\gamma^* \\
&\qquad + \sum_{a\in\mathcal{L}(AE^1)} s_a p_{r(A,a)_{\text{sink}}} s_a^* + p_{A_{\text{sink}}} \\
&= \sum_{\alpha\in\mathcal{L}(AE^3)} s_\alpha p_{r(A,\alpha)} s_\alpha^* + \sum_{\gamma\in\mathcal{L}(AE^2)} s_\gamma p_{r(A,\gamma)_{\text{sink}}} s_\gamma^* \\
&\qquad + \sum_{a\in\mathcal{L}(AE^1)} s_a p_{r(A,a)_{\text{sink}}} s_a^* + p_{A_{\text{sink}}} \\
&= \cdots \\
&= \sum_{\alpha\in\mathcal{L}(AE^n)} s_\alpha p_{r(A,\alpha)} s_\alpha^* + \sum_{\gamma\in\mathcal{L}(AE^{\le n-1})} s_\gamma p_{r(A,\gamma)_{\text{sink}}} s_\gamma^*.
\end{aligned}$$

(iii) Universal property of $C^*(E,\mathcal{L},\mathcal{B}) = C^*(s_a,p_A)$ defines a strongly continuous action $\gamma : \mathbb{T} \to Aut(C^*(E,\mathcal{L},\mathcal{B}))$, called the *gauge action*, such that

$$\gamma_z(s_a) = zs_a \quad\text{and}\quad \gamma_z(p_A) = p_A$$

for $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{B}$.

(iv) For $B \in \overline{\mathcal{E}}$, one can easily show that the ideal I_B of $C^*(E,\mathcal{L},\overline{\mathcal{E}}) = C^*(s_a,p_A)$ generated by the projection p_B is equal to

$$I_B = \overline{\text{span}}\{s_\alpha p_A s_\beta^* \,:\, \alpha,\, \beta \in \mathcal{L}^\#(E),\ A \in \overline{\mathcal{E}} \cap r(\mathcal{L}(BE^{\ge 0}))\,\}, \qquad (2.3)$$

where $r(\mathcal{L}(BE^0)) := B$ and $\overline{\mathcal{E}} \cap A = \{B \in \overline{\mathcal{E}} : B \subset A\}$ for $A \in \overline{\mathcal{E}}$.

As we have seen in the previous section, Condition (L) meaning that every loop has an exit was introduced as an essential condition to obtain the Cuntz-Krieger uniqueness theorem for graph C^*-algebras. More generally, in [4, Theorem 5.5] it is known that if $(E, \mathcal{L}, \mathcal{B})$ is disagreeable, Cuntz-Krieger uniqueness Theorem holds:

Theorem 2.2.11. (The Cuntz-Krieger Uniqueness Theorem [4, Theorem 5.5]). *Let $(E, \mathcal{L}, \mathcal{B})$ be a disagreeable labeled space. If $\{S_a, P_A\}$ is a representation of a labeled space $(E, \mathcal{L}, \mathcal{B})$ such that $S_a \neq 0$ and $P_A \neq 0$ for all $a \in \mathcal{A}$ and $A \in \mathcal{B}$, then there is an isomorphism π of $C^*(E, \mathcal{L}, \mathcal{B}) = C^*(s_a, p_A)$ onto $C^*(S_a, P_A)$ such that $\pi(s_a) = S_a$ and $\pi(p_A) = P_A$ for all $a \in \mathcal{A}$ and $A \in \mathcal{B}$.*

Remark 2.2.12. As it is pointed out in [1], if $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resoling labeled space that is not normal, then $C^*(E, \mathcal{L}, \mathcal{B})$ may not satisfy the gauge invariant uniqueness theorem under our definitions given in Definition 2.2.7 and Definition 2.2.9. To treat the general case, the definition of representations of non-normal labeled spaces $(E, \mathcal{L}, \mathcal{B})$ has to be modified. See [1, Appendis A] for this.

Theorem 2.2.13. (The Gauge Invariant Uniqueness Theorem [1, Corollary 3.9]). *Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving normal labeled space, $\{S_a, P_A\}$ a representation of $(E, \mathcal{L}, \mathcal{B})$ on a Hilbert space, and $\pi := \pi_{S,P}$ the representation of $C^*(E, \mathcal{L}, \mathcal{B})$ satisfying $\pi(s_a) = S_a$ and $\pi(p_A) = P_A$. Suppose that $P_A \neq 0$ for all $\emptyset \neq A \in \mathcal{B}$ and that there is a strongly continuous action β of $\mathbb{T}$ on $C^*(S_a, P_A)$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}$. Then π is faithful.*

K-theory of labeled graph C^*-algebras. In [1], labeled graph C^*-algebras $C^*(E, \mathcal{L}, \mathcal{B})$ are shown to be realized as Cuntz-Pimsner algebras on the purpose of computing the K-theory of $C^*(E, \mathcal{L}, \mathcal{B})$ by applying the results on the K-theory of Cuntz-Pimsner algebras obtained in [25]. Let

$$\mathcal{B}_J := \{A \in \mathcal{B} : \mathcal{L}(AE^1) \text{ is finite and } A \cap B = \emptyset \text{ for all } B \in \mathcal{B} \text{ with } B \subseteq E^0_{\text{sink}}\}.$$

Note that the following theorem can be obtained without the assumption that $(E, \mathcal{L}, \mathcal{B})$ is set-finite.

Theorem 2.2.14. ([1, Theorem 4.4]) *Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving normal labeled space, then the linear map $(1 - \Phi) : \mathrm{span}_{\mathbb{Z}}\{\chi_A : A \in \mathcal{B}_J\} \to \mathrm{span}_{\mathbb{Z}}\{\chi_A : A \in \mathcal{B}\}$ given by*

$$(1 - \Phi)(\chi_A) = \chi_A - \sum_{a \in \mathcal{L}(AE^1)} \chi_{r(A,a)}, \quad A \in \mathcal{B}_J \tag{2.4}$$

determines $K_*(C^*(E, \mathcal{L}, \mathcal{B}))$ as follows:

$$K_0(C^*(E, \mathcal{L}, \mathcal{B})) \cong \mathrm{span}_{\mathbb{Z}}\{\chi_A : A \in \mathcal{B}\}/\mathrm{Im}(1 - \Phi) \tag{2.5}$$

$$K_1(C^*(E, \mathcal{L}, \mathcal{B})) \cong \ker(1 - \Phi). \tag{2.6}$$

In (2.5), the isomorphism is given by $[p_A]_0 \mapsto \chi_A + \mathrm{Im}(1 - \Phi)$ for $A \in \mathcal{B}$.

2.3 Generalized Morse sequences

We review from [22] definitions and basic properties of generalized Morse sequences. Let Ω be the space

$$\Omega := \{\, \omega = \cdots \omega_{-2}\omega_{-1}\omega_0\omega_1 \cdots \,:\, \omega_i \in \{0, 1\},\ i \in \mathbb{Z} \,\}$$

of all two-sided sequences of zeros and ones, and let

$$\Omega_+ := \{x = x_0 x_1 \cdots \,:\, x_i \in \{0, 1\},\ i \geq 0\}$$

the space of one-sided sequences. By $\mathfrak{B}$, we denote the set of all finite blocks of zeros and ones. We write $|b| := n+1$ for the *length* of a block $b = b_0 \cdots b_n \in \mathfrak{B}$. For $\omega \in \Omega$ ($x \in \Omega_+$, respectively), the set of all finite blocks appearing in ω (x, respectively) will be denoted by $\mathfrak{B}_\omega$ ($\mathfrak{B}_x$, respectively). For $\omega \in \Omega$, we write $\omega_{[t_1,t_2]}$ for the block $\omega_{t_1} \cdots \omega_{t_2} \in \mathfrak{B}_\omega$ at the position t_1 ($t_1 \leq t_2$) of ω. Similarly,

$\omega_{[t_1,\infty)}$ and $\omega_{(-\infty,t_2]}$ mean the infinite sequences $\omega_{t_1}\omega_{t_1+1}\cdots$ and $\cdots\omega_{t_2-1}\omega_{t_2}$, respectively.

The space Ω (and similarly Ω_+) endowed with the product topology becomes a totally disconnected compact Hausdorff space such that the clopen *cylinder sets*

$$_t[b] := \{\omega \in \Omega : \omega_{[t,t+n]} = b\},$$

$t \in \mathbb{Z}$, $b \in \mathfrak{B}$, $|b| = n+1 \geq 1$, form a base for the topology. Thus every characteristic function $\chi_{_t[b]}$ is continuous on Ω. For convenience, we use the following notation:

$$[.b] := {}_0[b], \quad [b.] := {}_{-|b|}[b], \quad [b.c] := {}_{-|b|}[bc]$$

for $b, c \in \mathfrak{B}$. Note that on the right side of the dot is the zeroth position.

The *shift map*

$$T : \Omega \to \Omega \;\text{ given by }\; (T\omega)_i = \omega_{i+1},$$

$\omega \in \Omega$, $i \in \mathbb{Z}$, is easily seen to be a homeomorphism; if we consider T on the one-sided compact space Ω_+, it is just a continuous (not invertible) map. The orbit $\mathscr{O}(\omega)$ of a point $\omega \in \Omega$ is given by

$$\mathscr{O}(\omega) := \{T^i(\omega) : i \in \mathbb{Z}\}$$

and its orbit closure is denoted by $\mathscr{O}_\omega := \overline{\mathscr{O}(\omega)}$.

A subset Ω_0 of Ω is said to be *invariant* if $T(\Omega_0) \subseteq \Omega_0$. A non-empty closed invariant subset Ω_0 of Ω is called *minimal* if it contains no proper closed invariant subsets. A subset Ω_0 of Ω is minimal if and only if the orbit of each point of Ω_0 is dense in Ω_0. It is known ([22]) that $\mathscr{O}_\omega = \overline{\mathscr{O}(\omega)}$ is minimal if and only if ω is almost periodic. The meaning of almost periodicity is as follows.

Definition 2.3.1. A point x of Ω_+ is *almost periodic* if for any cylinder set $[.b]$, $b \in \mathfrak{B}_x$, there exists $d \geq 1$ such that for any $n \geq 0$, $T^{n+j}x \in [.b]$ for some $0 \leq j \leq d$.

For each block $b = b_0 \cdots b_n \in \mathfrak{B}$ the mirror image of b, denoted by $\tilde{b}$, is defined by $\tilde{b}_i = b_i + 1 \pmod 2$ for $i = 0, \cdots, n$, that is, $\tilde{b}$ is obtained from b by changing zeros into ones and vice-versa. Given a fixed block $c = c_0 \cdots c_n \in \mathfrak{B}$, the product $b \times c$ of b and c denotes the block formed by putting $n+1$ copies of either b or $\tilde{b}$ next to each other according to the rule of choosing the ith copy as b if $c_i = 0$ and $\tilde{b}$ if $c_i = 1$. For example, if $b = 01$ and $c = 011$, then the product block $b \times c$ is equal to $b\tilde{b}\tilde{b} = 011010$.

For each $i \geq 0$, let $b^i = b^i_0 \cdots b^i_{|b^i|-1} \in \mathfrak{B}$ be a block such that $|b^i| \geq 2$ and $b^i_0 = 0$ for all $i \geq 0$. (Here the superscript i of b^i should not be confused with a repetition of b.) Since the product operation $\times$ is associative, one can consider a sequence of the form

$$x = b^0 \times b^1 \times b^2 \times \cdots \in \Omega_+$$

which is called a *one-sided recurrent sequence* (see [22, Definition 7]). For $x \in \Omega_+$, the set of all two-sided sequences ω such that $\mathfrak{B}_\omega \subset \mathfrak{B}_x$ is denoted by $\mathscr{O}_x$, namely

$$\mathscr{O}_x := \{\omega \in \Omega : \mathfrak{B}_\omega \subset \mathfrak{B}_x\}.$$

For $c \in \mathfrak{B}$, the quantity

$$r_b(c) := \frac{1}{|c|} \sum_{t=0}^{|c|-1} \chi_{t[b]}(c)$$

indicates the *relative frequency of occurrence* of b in c. In particular, $r_0(c)$ and $r_1(c)$ are the relative frequencies of zeros and ones in c respectively.

Definition 2.3.2. ([22, Definition 8]) A one-sided recurrent sequence $x = b^0 \times b^1 \times b^2 \times \cdots \in \Omega_+$ is called a *one-sided Morse sequence* if it is non-periodic and

$$\sum_{i=0}^{\infty} \min(r_0(b^i), r_1(b^i)) = \infty.$$

The poinst of $\mathscr{O}_x$ are called *two-sided Morse sequences.*

Definition 2.3.3. By a *generalized Morse sequence*, we mean a two-sided sequence $\omega \in \Omega$ such that $x := \omega_{[0,\infty)}$ is a one-sided Morse sequence and $\mathfrak{B}_\omega = \mathfrak{B}_x$.

A probability measure m on Ω is *T-invariant* if $m(A) = m(TA)$ for every borel subset A of Ω. A T-invariant measure is called *ergodic* if every invariant set has measure 0 or 1. A compact invariant non-empty subset Ω_0 of Ω is *uniquely ergodic* if there is only one T-invariant measure carried by Ω_0. We record the following known facts for later use:

Theorem 2.3.4. ([22, Lemma 2, Lemma 4, Theorem 3]) *Let $x \in \Omega_+$ be a non-periodic recurrent sequence. Then we have the following:*

(i) *x is almost periodic,*

(ii) *there exists $\omega \in \mathscr{O}_x$ with $x = \omega_{[0,\infty)}$. Moreover, x is a one-sided Morse sequence if and only if $\mathscr{O}_\omega$ is minimal and uniquely ergodic. We denote the T-invariant probability measure on $\mathscr{O}_\omega$ by m_ω.*

Remark 2.3.5. For a generalized Morse sequence ω, the unital commutative AF algebra $C(\mathscr{O}_\omega)$ of all continuous functions on $\mathscr{O}_\omega$ admits a (tracial) state

$$f \mapsto \int_{\mathscr{O}_\omega} f \mathrm{d}m_\omega : C(\mathscr{O}_\omega) \to \mathbb{C} \tag{2.7}$$

which we also write m_ω. Since m_ω is T-invariant, it easily follows that $m_\omega(\chi_{t[b]}) = m_\omega(\chi_{t[b]} \circ T) = m_\omega(\chi_{t+1[b]})$, and hence

$$m_\omega(\chi_{t[b]}) = m_\omega(\chi_{[.b]}) \tag{2.8}$$

holds for all $t \in \mathbb{Z}$ and $b \in \mathfrak{B}_\omega$.

Example 2.3.6. (**Thue-Morse sequence**) Let $b^i := b = 01 \in \mathfrak{B}$ for all $i \geq 0$. Then the recurrent sequence

$$x := b \times b \times b \times \cdots \ = \ 01 \times b \times \cdots = 0110 \times b \times \cdots = 01101001 \times b \times \cdots$$

is a one-sided Morse sequence and

$$\omega := x^{-1}.x = \cdots 10010110.011010011001 \cdots \in \mathscr{O}_x$$

is a two-sided Morse sequence which we call the Thue-Morse sequence, where $x^{-1} := \cdots x_2 x_1 x_0$ is the sequence obtained by writing $x = x_0 x_1 \cdots$ in reverse order. In fact, ω is the sequence constructed from the proof of Theorem 2.3.4(ii) (see [22, Lemma 4]), and it is well known [15] that ω has no blocks of the form bbb_0 for any block $b = b_0 \cdots b_{|b|-1} \in \mathfrak{B}_\omega$.

Notation 2.3.7. Throughout this thesis, $E_{\mathbb{Z}}$ will denote the following graph:

$$\cdots \underset{v_{-4}}{\bullet} \xrightarrow{e_{-4}} \underset{v_{-3}}{\bullet} \xrightarrow{e_{-3}} \underset{v_{-2}}{\bullet} \xrightarrow{e_{-2}} \underset{v_{-1}}{\bullet} \xrightarrow{e_{-1}} \underset{v_{0}}{\bullet} \xrightarrow{e_{0}} \underset{v_{1}}{\bullet} \xrightarrow{e_{1}} \underset{v_{2}}{\bullet} \xrightarrow{e_{2}} \underset{v_{3}}{\bullet} \xrightarrow{e_{3}} \underset{v_{4}}{\bullet} \cdots .$$

Given a two-sided sequence $\omega = \cdots \omega_{-1}\omega_0\omega_1 \cdots \in \Omega$ of zeros and ones, we obtain a labeled graph $(E_{\mathbb{Z}}, \mathcal{L}_\omega)$ shown below

$$(E_{\mathbb{Z}}, \mathcal{L}_\omega) \cdots \underset{v_{-4}}{\bullet} \xrightarrow{\omega_{-4}} \underset{v_{-3}}{\bullet} \xrightarrow{\omega_{-3}} \underset{v_{-2}}{\bullet} \xrightarrow{\omega_{-2}} \underset{v_{-1}}{\bullet} \xrightarrow{\omega_{-1}} \underset{v_{0}}{\bullet} \xrightarrow{\omega_{0}} \underset{v_{1}}{\bullet} \xrightarrow{\omega_{1}} \underset{v_{2}}{\bullet} \xrightarrow{\omega_{2}} \underset{v_{3}}{\bullet} \xrightarrow{\omega_{3}} \underset{v_{4}}{\bullet} \cdots ,$$

where the labeling map $\mathcal{L}_\omega : E^1_{\mathbb{Z}} \to \{0, 1\}$ is given by $\mathcal{L}_\omega(e_n) = \omega_n$ for $e_n \in E^1_{\mathbb{Z}}$. Then we also have a labeled space $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ with the smallest accommodating set $\overline{\mathcal{E}}_{\mathbb{Z}}$ which is closed under relative complements.

Chapter 3

AF labeled graph C^*-algebras

In this chapter, we find conditions of a labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ under which the C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ becomes an AF algebra. Since a graph C^*-algebra $C^*(E)$ is AF exactly when the graph E has no loops (Theorem 2.1.9), we need to consider labeled spaces with no loops. Hence it should be our first task to define a notion of loop in labeled spaces.

3.1 Loops in labeled spaces

Recall that a path $x \in E^{\geq 1}$ in a directed graph E is called a *loop* if $s(x) = r(x)$. Considering $(E, \mathcal{L}_{id})$ with the trivial labeling $\mathcal{L}_{id}$, it is rather obvious that the following are equivalent for a path $x = x_1 \cdots x_m \in E^{\geq 1} (= \mathcal{L}_{id}^*(E))$:

(i) x is a loop in E,

(ii) $\{r(x)\} = r(\{r(x)\}, x)$,

(iii) x is *repeatable*, that is, $x^n \in E^{\geq 1}$ for all $n \geq 1$,

(iv) $(A_1 x_1 A_2 x_2 \cdots A_m x_m)^n (A_1 x_1 A_2 x_2 \cdots A_i x_i) \in \mathcal{L}_{id}^*(E)$ for all $n \geq 1$ and $1 \leq i \leq m$, where $A_i = \{s(x_i)\} \in \overline{\mathcal{E}}$. (See the following Notation 3.1.1 for the meaning of $A_1 x_1 A_2 x_2 \cdots A_m x_m$.)

Notation 3.1.1. For $A_i \in \mathcal{B}$, $1 \leq i \leq n$, and $K \geq 1$, we adopt the notation

$$A_1 E^{\leq K} A_2 \cdots E^{\leq K} A_{n+1} := \{x_1 \cdots x_n \in E^{\geq 1} : x_i \in A_i E^{\leq K} A_{i+1},\ 1 \leq i \leq n\}$$

for the set of paths $x = x_1 \cdots x_n \in E^*$ consisting of sub-paths x_i passing through from A_i to A_{i+1} with length $|x_i| \leq K$ for $1 \leq i \leq n$. To stress the fact that a path $x = x_1 \cdots x_n$ belongs to $A_1 E^{\leq K} A_2 \cdots E^{\leq K} A_{n+1}$, we may write $A_1 x_1 A_2 \cdots x_n A_{n+1}$ for x.

From the equivalent conditions given above for a labeled space $(E, \mathcal{L}_{id}, \overline{\mathcal{E}})$, we can obtain several equivalent conditions for a graph C^*-algebra $C^*(E)$ to be AF as follows.

Proposition 3.1.2. *Let $(E, \mathcal{L}_{id}, \overline{\mathcal{E}})$ be a labeled space with the trivial labeling $\mathcal{L}_{id}$. Then the following are equivalent for $C^*(E, \mathcal{L}_{id}, \overline{\mathcal{E}}) \cong C^*(E)$.*

(i) *$C^*(E, \mathcal{L}_{id}, \overline{\mathcal{E}})$ is AF,*

(ii) *E has no loops,*

(iii) *there are no repeatable paths in $\mathcal{L}_{id}^*(E)$,*

(iv) *$A \not\subset r(A, x)$ for all $A \in \overline{\mathcal{E}}$ and $x \in \mathcal{L}_{id}^*(E)$,*

(v) *if $\{A_1, \ldots, A_m\}$ is a finite collection of sets from $\overline{\mathcal{E}}$ and $K \geq 1$, there is an $m_0 \geq 1$ such that $A_{i_1} E^{\leq K} A_{i_2} \cdots E^{\leq K} A_{i_{n+1}} = \emptyset$ for all $n > m_0$.*

Proof. We only need to prove that (ii) and (v) are equivalent since the equivalence of (i) and (ii) is well known (see Theorem 2.1.9) and the other implications are rather obvious. Suppose x is a loop in E, then with $A = \{s(x)\} \in \overline{\mathcal{E}}$ and $K := |x| \geq 1$, it is immediate that $(AxA)^n \neq \emptyset$ for all $n \geq 1$. For the converse, suppose that (v) dose not hold and so there are finitely many sets $A_1, \ldots, A_m$ in $\overline{\mathcal{E}}$ and $K \geq 1$ such that $A_{i_1} E^{\leq K} A_{i_2} \cdots E^{\leq K} A_{i_{n+1}} \neq \emptyset$ for all $n \geq 1$. Since every set in $\overline{\mathcal{E}}$ is finite, the number of vertices in $\cup_{i=1}^m A_i$ is also finite. Choose an integer N with $|\cup_{i=1}^m A_i| < N$. Then for any path in $A_{i_1}E^{\le K}A_{i_2}\cdots E^{\le K}A_{i_{N+1}}(\neq \emptyset)$, there is a vertex in $\cup_{i=1}^m A_i$ the path passes through at least two times, which proves the existence of a loop (at that vertex) in E. $\square$

Motivated by the fact in Proposition 3.1.2 that there is a set $A \in \overline{\mathcal{E}}$ satisfying $A \subset r(A,x)$ for a path x (in fact, $A = \{s(x)\} = \{r(x)\}$) is equivalent to the existence of a loop in E, we extend the notion of a loop to a labeled space as follows.

Definition 3.1.3. Let $(E,\mathcal{L},\mathcal{B})$ be a labeled space and $\alpha \in \mathcal{L}^*(E)$ be a labeled path.

(a) α is called a *generalized loop* at $A \in \mathcal{B}$ if $\alpha \in \mathcal{L}(AE^{\ge 1}A)$.

(b) α is called a *loop* at $A \in \mathcal{B}$ if it is a generalized loop such that $A \subset r(A,\alpha)$.

(c) A loop α at $A \in \mathcal{B}$ has an *exit* if one of the following holds:

  (i) $\{\alpha_{[1,k]} : 1 \le k \le |\alpha|\} \subsetneq \mathcal{L}(AE^{\le|\alpha|})$,

  (ii) $r(A,\alpha_{[1,i]})_{\text{sink}} \neq \emptyset$ for some $i = 1,\dots,|\alpha|$,

  (iii) $A \subsetneq r(A,\alpha)$.

If α is a loop at $A \in \mathcal{B}$, we also say that A admits a loop α. Note that every loop α is *repeatable*, that is, $\alpha^n \in \mathcal{L}^*(E)$ for all $n \ge 1$ ([5, Definition 6.6]), and every repeatable path is a generalized loop at its range. Not every repeatable path is a loop as we can see in Example 3.2.3(iii).

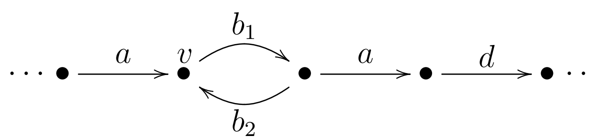
Remark 3.1.4. Let $(E,\mathcal{L},\mathcal{B})$ be a labeled space and $A \in \mathcal{B}$.

(i) A generalized loop α at a minimal set $A \in \mathcal{B}$ is always a loop because $A \subset r(A,\alpha)$ follows from the minimality of A since $\emptyset \neq A \cap r(A,\alpha) \subset A$. A labeled graph $(E,\mathcal{L})$ might have a loop α even though the underlying graph E itself has no loops at all as we will see in Example 3.2.3(i) and (ii).

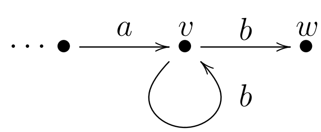
(ii) If $A \in \mathcal{B}$ admits a loop α and $\{s_a, p_A\}$ is a representation of $(E, \mathcal{L}, \mathcal{B})$, then evidently $p_A \le p_{r(A,\alpha)}$.

Example 3.1.5. We give three examples of labeled spaces with a loop each of which has an exit of different type from other two.

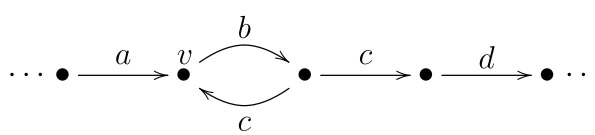
(i) The loop $\alpha := b_1 b_2$ at $A := r(b_2) = \{v\} \in \overline{\mathcal{E}}$ has an exit of type (i) of Definition 3.1.3(c) because $\{\alpha_{[1,k]} : 1 \le k \le 2\} = \{b_1, b_1 b_2\}$ while $\mathcal{L}(AE^{\le|\alpha|}) = \{b_1, b_1 b_2, b_1 a\}$.



(ii) Let $A := r(b) = \{v, w\} \in \overline{\mathcal{E}}$. Since $A = r(A, b)$, b is a loop at A with an exit of type (ii) of Definition 3.1.3(c); $r(A, b)_{\text{sink}} = \{w\} \neq \emptyset$.



(iii) The loop $\alpha := bc$ at $A := \{v\} \in \overline{\mathcal{E}}$ has an exit of type (iii) of Definition 3.1.3(c) because $A \subsetneq r(A, \alpha)$.



The following proposition is an extended version of the fact that if a directed graph E has a loop with an exit, its graph C^*-algebra has an infinite projection.

Proposition 3.1.6. *Let $(E, \mathcal{L})$ be a labeled graph and $A \in \overline{\mathcal{E}}$ admit a loop α with an exit. Then p_A is an infinite projection in $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$.*

Proof. If $A \subsetneq r(A,\alpha)$, the projection $p_{r(A,\alpha)}$ is infinite because

$$p_{r(A,\alpha)} > p_A \geq s_\alpha p_{r(A,\alpha)} s_\alpha^* \sim p_{r(A,\alpha)}.$$

If either $\mathcal{L}(AE^{\leq|\alpha|}) \supsetneq \{\alpha_{[1,k]} : 1 \leq k \leq |\alpha|\}$ or $r(A,\alpha_{[1,i]})_{\text{sink}} \neq \emptyset$ for some i, $1 \leq i \lneq |\alpha|$, by Remark 2.2.10(iii) we have

$$p_A = \sum_{\beta \in \mathcal{L}(AE^{|\alpha|})} s_\beta p_{r(A,\beta)} s_\beta^* + \sum_{1 \leq |\gamma| \leq |\alpha|-1} s_\gamma p_{r(A,\gamma)_{\text{sink}}} s_\gamma^* \gneq s_\alpha p_{r(A,\alpha)} s_\alpha^*.$$

Thus $p_{r(A,\alpha)} \geq p_A > s_\alpha p_{r(A,\alpha)} s_\alpha^* \sim p_{r(A,\alpha)}$ and we see that the projection $p_{r(A,\alpha)}$ (hence p_A) is infinite. Now it remains to prove the assertion in case $r(A,\alpha)_{\text{sink}} \neq \emptyset$ and $A = r(A,\alpha)$. The set $A_0 := A \setminus A_{\text{sink}}$ $(\neq \emptyset)$ then satisfies $A_0 \subsetneq A = r(A,\alpha) = r(A_0,\alpha)$, and by the first argument of the proof p_{A_0} is infinite. Hence $p_A (\gneq p_{A_0})$ is infinite. □

Remark 3.1.7. Proposition 3.1.6 can be slightly generalized as follows: Let $(E,\mathcal{L})$ be a labeled graph and $\alpha_1,\ldots,\alpha_n$ be distinct labeled paths with the same length, say $l \geq 1$, such that $A \subseteq \cup_{i=1}^n r(A,\alpha_i)$. Then p_A is an infinite projection in $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ if one of the following holds:

(i) $\cup_{i=1}^n \{\alpha_i' : \alpha_i' \text{ is an initial path of } \alpha_i\} \subsetneq \mathcal{L}(AE^{\leq l})$

(ii) $r(A,\alpha_i')_{\text{sink}} \neq \emptyset$ for some i and an initial path α_i' of α_i

(iii) $A \subsetneq \cup_{i=1}^n r(A,\alpha_i)$.

To prove this, first assume the case (iii) and set $A_1 := r(A,\alpha_1)$ and $A_i := r(A,\alpha_i) \setminus \cup_{j=1}^{i-1} r(A,\alpha_j)$, $i = 2,\ldots,n$, so that $\cup_{i=1}^n r(A,\alpha_i) = \cup_{i=1}^n A_i$ is the union of disjoint sets A_i's. Then we have

$$p_A \geq \sum_{i=1}^n s_{\alpha_i} p_{r(A,\alpha_i)} s_{\alpha_i}^* \geq \sum_{i=1}^n s_{\alpha_i} p_{A_i} s_{\alpha_i}^* \sim \sum_{i=1}^n p_{A_i} = p_{\cup A_i} = p_{\cup_{i=1}^n r(A,\alpha_i)} \gneq p_A$$

and so the projection p_A is infinite, where the equivalence is given by the partial isometry $u := \sum_{i=1}^n s_{\alpha_i} p_{A_i}$. It is not hard to see that the same argument in the proof of Proposition 3.1.6 shows the assertion for the rest cases.

Proposition 3.1.8. *Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a labeled space such that $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ has no infinite projections. Let $A \in \overline{\mathcal{E}}$ admit a loop. Then there exists a loop α at A such that $A = r(A, \alpha)$ and*

$$\mathcal{L}(AE^{\geq 1}) = \{\alpha^k \alpha' : k \geq 0,\ \alpha' \text{ is an initial path of } \alpha\}.$$

Proof. Choose a loop α at A with the smallest length; $|\alpha| \leq |\gamma|$ for all loops γ at A. Since $C^*(E, \mathcal{L}, \overline{\mathcal{E}^0})$ has no infinite projections, α does not have an exit by Proposition 3.1.6, hence $A = r(A, \alpha)$ and

$$\mathcal{L}(AE^{\leq |\alpha|}) = \{\alpha_{[1,k]} :\ 1 \leq k \leq |\alpha|\}. \tag{3.1}$$

Now let $\beta \in \mathcal{L}(AE^{\geq 1})$ be a path with $|\beta| > |\alpha|$. Then by (3.1), $\mathcal{L}(AE^{|\alpha|}) = \{\alpha\}$ and so we can write $\beta = \alpha\beta'$ for a path β'. But then from $A = r(A, \alpha)$, β' must be either an initial path of α or of the form $\alpha\beta''$ for some path β''. Applying the argument repeatedly, we finally end up with $\beta = \alpha^k \alpha'$ for some $k \geq 1$ and an initial path α' of α. □

3.2 Labeled spaces associated with AF algebras

In the previous section, we studied several equivalent conditions on $(E, \mathcal{L}_{id}, \overline{\mathcal{E}})$ to give rise to an AF C^*-algebra $C^*(E, \mathcal{L}_{id}, \overline{\mathcal{E}}) \cong C^*(E)$ and defined a notion of loop in a labeled space $(E, \mathcal{L}, \mathcal{B})$ based on one of the equivalent conditions given in the first paragraph of the previous section. Here, we will show that if a labeled graph C^*-algebra is AF, the labeled space has no loops and that other equivalent conditions for a graph C^*-algebra to be AF in the setting of labeled spaces are not always equivalent by invoking various examples.

Remark 3.2.1. We will consider the following properties (a)-(d) of a labeled space $(E,\mathcal{L},\overline{\mathcal{E}})$ and its C^*-algebra $C^*(E,\mathcal{L},\overline{\mathcal{E}})$. These properties are equivalent if $\mathcal{L}$ is the trivial labeling $\mathcal{L}_{id}$ as we have seen in Proposition 3.1.2.

(a) For every finite set $\{A_1,\dots,A_N\}$ of $\overline{\mathcal{E}}$ and every $K\geq 1$, there exists an $m_0\geq 1$ such that $A_{i_1}E^{\leq K}A_{i_2}\cdots E^{\leq K}A_{i_n}=\emptyset$ for all $n>m_0$ and $A_{i_j}\in\{A_1,\dots,A_N\}$.

(b) $(E,\mathcal{L},\overline{\mathcal{E}})$ has no repeatable paths.

(c) $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is an AF algebra.

(d) $(E,\mathcal{L},\overline{\mathcal{E}})$ has no loops (in the sense of Definition 3.1.3).

Note that (a) $\Rightarrow$ (b) follows from a simple observation that if α is a repeatable path, then with $A:=r(\alpha)$ one has $A_{i_1}E^{|\alpha|}A_{i_2}\cdots E^{|\alpha|}A_{i_n}\neq\emptyset$ for all $n\geq 1$, where $A_{i_j}=A$, $j=1,\dots,n$. The implication (b) $\Rightarrow$ (d) is obvious.

For the other implications, we shall see (b) $\not\Rightarrow$ (a) and (b) $\not\Rightarrow$ (c), in general throughout Example 3.2.10. Consequently (d) $\not\Rightarrow$ (c) follows although it can also be seen from Example 3.2.3(iii). We will show that (c) $\Rightarrow$ (d) and (a) $\Rightarrow$ (c) hold true in Theorem 3.2.2 and Theorem 3.2.8, respectively.

It would be interesting to know whether the remaining implication (c) $\Rightarrow$ (b) is true, that is, whether $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ will never be AF whenever $(E,\mathcal{L},\overline{\mathcal{E}})$ contains a repeatable path. In Theorem 3.2.12, we will show that this is the case under some additional conditions.

Theorem 3.2.2. *Let $(E,\mathcal{L})$ be a labeled graph. If $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is an AF algebra, the labeled space $(E,\mathcal{L},\overline{\mathcal{E}})$ has no loops.*

Proof. Suppose, for contradiction, that $(E,\mathcal{L},\overline{\mathcal{E}})$ has a loop α at $A\in\overline{\mathcal{E}}$. By Proposition 3.1.6, $A=r(A,\alpha)$ and so $p_As_\alpha=s_\alpha p_{r(A,\alpha)}=s_\alpha p_A$. Then $U:=s_\alpha p_A$ satisfies

$$p_A=U^*U\sim UU^*=s_\alpha p_A s_\alpha^*=s_\alpha p_{r(A,\alpha)}s_\alpha^*\leq p_A.$$

Since p_A is a finite projection, it follows that U is a unitary of the unital hereditary subalgebra $p_AC^*(E,\mathcal{L},\overline{\mathcal{E}})p_A$. Since $\gamma_z(p_A)=p_A$ for any $z\in\mathbb{T}$, the algebra $p_A C^*(E,\mathcal{L},\overline{\mathcal{E}})p_A$ admits an action of $\mathbb{T}$ which is the restriction of the gauge action γ on $C^*(E,\mathcal{L},\overline{\mathcal{E}})$. Then the fact that $\gamma_z(U) = \gamma_z(s_\alpha)p_A = z^{|\alpha|}U$ shows that U is not in the unitary path connected component of the unit p_A ([12, Proposition 3.9]), which is a contradiction to the assumption that $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ (hence any nonzero hereditary subalgebra) is an AF algebra. $\square$

In Example 3.2.3(iii) below, we see that the converse of Theorem 3.2.2 may not be true, in general.

Example 3.2.3. (i) For the following labeled graph $(E,\mathcal{L})$

$$\cdots \quad \underset{v_{-2}}{\bullet} \xrightarrow{a} \underset{v_{-1}}{\bullet} \xrightarrow{a} \underset{v_0}{\bullet} \xrightarrow{a} \underset{v_1}{\bullet} \xrightarrow{a} \underset{v_2}{\bullet} \quad \cdots,$$

we have $\overline{\mathcal{E}} = \{r(a)\} = \{E^0\}$ and the path a is a loop at $r(a)$. By Theorem 3.2.2, $C^*(E,\mathcal{L},\overline{\mathcal{E}}) := C^*(s_a, p_A)$ is not AF. Actually $C^*(E,\mathcal{L},\overline{\mathcal{E}}) \cong C(\mathbb{T})$ is the universal C^*-algebra generated by the unitary s_a.

(ii) $\overline{\mathcal{E}}$ of the following labeled graph consists of three sets $r(a) = E^0$, $r(a)_{\rm sink} = \{v_0\}$, and $A := r(a) \setminus r(a)_{\rm sink} = \{v_{-1}, v_{-2}, \dots\}$.

$$\cdots \quad \underset{v_{-4}}{\bullet} \xrightarrow{a} \underset{v_{-3}}{\bullet} \xrightarrow{a} \underset{v_{-2}}{\bullet} \xrightarrow{a} \underset{v_{-1}}{\bullet} \xrightarrow{a} \underset{v_0}{\bullet}$$

Since $A \subsetneq r(A,a)$, the labeled path a is a loop at A, hence $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is not AF by the above theorem. In fact, since the loop a at A has an exit, $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ contains an infinite projection by Proposition 3.1.6.

(iii) If $(E,\mathcal{L})$ is as follows

$$\underset{v_0}{\bullet} \xrightarrow{a} \underset{v_1}{\bullet} \xrightarrow{a} \underset{v_2}{\bullet} \xrightarrow{a} \underset{v_3}{\bullet} \xrightarrow{a} \underset{v_4}{\bullet} \quad \cdots,$$

it is not hard to see that $\overline{\mathcal{E}}$ consists of all finite sets F with $v_0 \notin F$ and all sets of the form $F \cup \{v_k, v_{k+1}, \dots\}$ for some $k \geq 1$. It is also easy to see that every $A \in \overline{\mathcal{E}}$ containing at least two vertices always admits a generalized loop. But there does not exist a loop at any $A \in \overline{\mathcal{E}}$. Nevertheless we shall show that $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ contains an infinite projection and so the C^*-algebra is not AF. Let $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) := C^*(p_A, s_a)$. Then

$$p_{r(a)} = s_a p_{r(r(a),a)} s_a^* = s_a p_{r(a^2)} s_a^* \sim p_{r(a^2)} < p_{r(a)}$$

since $r(a^2) \subsetneq r(a)$, which proves that $p_{r(a)}$ is an infinite projection.

The C^*-algebra is unital with the unit $s_a s_a^*$; $(s_a s_a^*) p_A = s_a p_{r(A,a)} s_a^* = p_A$, $p_A(s_a s_a^*) = s_a p_{r(A,a)} s_a^* = p_A$ for all $A \in \overline{\mathcal{E}}$ and $(s_a s_a^*) s_a = s_a = s_a p_{r(a)} = s_a p_{r(a)}(s_a s_a^*) = s_a(s_a s_a^*)$. Also we have $s_a s_a^* \gneq p_{r(a)} = s_a^* s_a$ since $s_a s_a^* \geq s_a p_{\{v_1\}} s_a^* (\neq 0)$ and $(s_a p_{\{v_1\}} s_a^*) p_A = s_a p_{\{v_1\}} p_{r(A,a)} s_a^* = s_a p_{\{v_1\} \cap r(A,a)} s_a^* = 0$ because $\{v_1\} \cap r(A, a) = \emptyset$ for all $A \in \overline{\mathcal{E}}$. Moreover every projection p_A belongs to the $*$-algebra generated by s_a. Therefore $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is the universal C^*-algebra generated by a proper coisometry s_a, and thus $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is the Toeplitz algebra. The ideal $I_{\{v_1\}}$ generated by the projection $p_{\{v_1\}}$ is in fact isomorphic to the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space as $I_{\{v_1\}} = \overline{\operatorname{span}}\{s_a^m p_{\{v_i\}} (s_a^*)^n : m, n \geq 0 \text{ and } i \geq 1\}$ (see (2.3)). The quotient algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})/I_{\{v_1\}}$ is therefore isomorphic to $C(\mathbb{T})$.

For a labeled graph $(E, \mathcal{L}_E)$, $v \sim w$ if and only if $v \sim_l w$ for all $l \geq 1$ defines an equivalence relation on E^0. We denote the equivalence class of $v \in E^0$ by $[v]_\infty$. If $(E, \mathcal{L}_E)$ has no sinks or sources, there exists a labeled graph $(F, \mathcal{L}_F)$ called the *merged labeled graph* of $(E, \mathcal{L}_E)$ with vertices $F^0 := \{[v]_\infty : v \in E^0\}$ and edges $F^1 := \{e_\lambda : \lambda \in E^1\}$, where e_λ is a path with $s_F(e_\lambda) = [s(\lambda)]_\infty$, $r_F(e_\lambda) = [r(\lambda)]_\infty$, and $\mathcal{L}_F(e_\lambda) = \mathcal{L}_E(\lambda)$. The range of $a \in \mathcal{L}_F(F^1)$ is defined by $r_F(a) = \{r_F(e_\lambda) : \mathcal{L}_F(e_\lambda) = a\}$. Here we use notation r_F to denote both the range map of paths in F^* and of labeled paths in $\mathcal{L}_F^*(F)$. It is known in [20, Theorem 6.10] that if $[v]_\infty \in \overline{\mathcal{E}}$ for all $v \in E^0$, then $\{[v]_\infty\} \in \overline{\mathcal{F}}$ for all $[v]_\infty \in F^0$ and moreover $C^*(E, \mathcal{L}_E, \overline{\mathcal{E}}) \cong C^*(F, \mathcal{L}_F, \overline{\mathcal{F}})$. Even when $(E, \mathcal{L}_E)$ has sinks or sources, we can obtain $C^*(E, \mathcal{L}_E, \overline{\mathcal{E}}) \cong C^*(F, \mathcal{L}_F, \overline{\mathcal{F}})$ whenever $[v]_\infty \in \overline{\mathcal{E}}$ for all $v \in E^0$ without significant modification of the proof of [20, Theorem 6.10].

The following proposition is a slightly generalized version of the result well known for graph C^*-algebras. Actually in case $\mathcal{L}$ is the trivial labeling, $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is isomorphic to $C^*(E)$ and the minimal sets in $\overline{\mathcal{E}}$ are the single vertex sets $\{v\}$, $v\in E^0$.

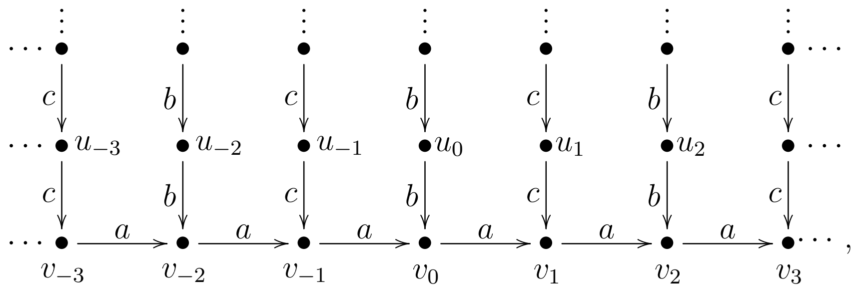
Proposition 3.2.4. *Let $(E,\mathcal{L})$ be a row-finite labeled graph with no sinks or sources such that every generalized vertex is a finite union of minimal sets in $\overline{\mathcal{E}}$. Then $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is AF if and only if no minimal set of $\overline{\mathcal{E}}$ admits a loop.*

Proof. Let $(F,\mathcal{L}_F)$ be the merged labeled graph of $(E,\mathcal{L})$. We first show that $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is isomorphic to the graph C^*-algebra $C^*(F)$.

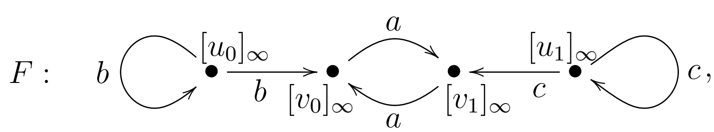
Our assumption implies $[v]_\infty\in\overline{\mathcal{E}}$ for all $v\in E^0$, so $\{[v]_\infty\}\in\overline{\mathcal{F}}$ for all $v\in E^0$ and $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is isomorphic to $C^*(F,\mathcal{L}_F,\overline{\mathcal{F}})$ ([20, Theorem 6.10]). For each $a\in\mathcal{L}(E^1)$, its range $r(a)$ can be written as the union $r(a)=\cup_{i=1}^n[w_i]_{l_i}$ of finitely many minimal sets $[w_i]_{l_i}$ by the assumption, but minimality of each $[w_i]_{l_i}$ implies that $[w_i]_{l_i}=[w_i]_\infty$ for $1\le i\le n$. Hence $r_F(a)=[r(a)]_\infty:=\{[w]_\infty : w\in r(a)\}=\{[w_1]_\infty,\dots,[w_n]_\infty\}$ is finite for each $a\in\mathcal{A}$. But from the construction ([20, Definition 6.1]), the merged labeled graph $(F,\mathcal{L}_F)$ is left-resolving. Thus the finiteness of each range set $r_F(a)$ implies that $(F,\mathcal{L}_F)$ is label-finite. Then by [4, Theorem 6.6], we have $C^*(F,\mathcal{L}_F,\overline{\mathcal{F}})\cong C^*(F)$.

Suppose that there is no loop at any minimal set $[v]_\infty$ in $\overline{\mathcal{E}}$. Since $\mathcal{L}_E([v]_\infty E^k v')=\mathcal{L}_F([v]_\infty F^k[v]_\infty)$ for all $v'\in[v]_\infty$ and $k\ge 1$ ([20, Lemma 6.7]), if F has a loop α at a vertex $[v]_\infty\in F^0$, $\alpha\in\mathcal{L}_E([v]_\infty E^k v')$ for all $v'\in[v]_\infty$. This means that $[v]_\infty(\in\overline{\mathcal{E}})$ satisfies $[v]_\infty\subset r([v]_\infty,\alpha)$, a contradiction. Hence F has no loops and the C^*-algebra $C^*(F)$ is AF. The converse was proved in Theorem 3.2.2. $\square$

Example 3.2.5. In the following labeled graph $(E,\mathcal{L})$

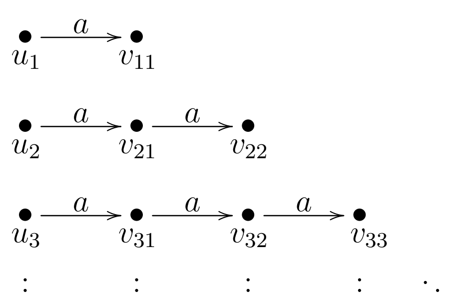


the path $\alpha := a^2$ is a loop at $\{v_{2k} : k \in \mathbb{Z}\}$ and also at $\{v_{2k+1} : k \in \mathbb{Z}\}$. By Theorem 3.2.2, the C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is not AF. In fact, $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is isomorphic to the graph algebra $C^*(F)$, where F is the underlying graph of the merged labeled graph $(F, \mathcal{L}_F)$ of $(E, \mathcal{L})$



and $C^*(F)$ has infinite projection because F has loops with exits.

Example 3.2.6. The following labeled graph $(E, \mathcal{L})$ does not have any infinite paths, but it has a repeatable path a.



Note that each finite path a^n is not a loop at any $A \in \overline{\mathcal{E}}$ but it is a generalized loop at $r(a^k)$ for all $k \geq 1$. $(E, \mathcal{L}, \overline{\mathcal{E}})$ has the generalized vertices as follows:

$$[v_{ij}]_k = \begin{cases} r(a^k), & \text{if } 1 \leq k \leq j \\ r(a^j) \setminus r(a^{j+1}), & \text{if } 1 \leq j < k \end{cases}$$
$$([v_{ij}]_k)_{\text{sink}} = \begin{cases} \{v_{mm} : m \geq k\}, & \text{if } 1 \leq k \leq j \\ \{v_{jj}\}, & \text{if } 1 \leq j < k \end{cases}$$
$$[v_{ij}]_k \setminus ([v_{ij}]_k)_{\text{sink}} = \begin{cases} \{v_{mn} : m > n \geq k\}, & \text{if } 1 \leq k \leq j \\ \{v_{mj} : m \geq j\}, & \text{if } 1 \leq j < k, \end{cases}$$

and every $A \in \overline{\mathcal{E}}$ is a finite union of these sets.

Let J be the ideal of $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(p_A, s_a)$ generated by the projection $p_{[v_{11}]_2}$. Then (2.3) shows that

$$J = \overline{\operatorname{span}}\{s_a^m p_B s_a^{*n} : \; B \in [v_{kk}]_{k+1} \cap \overline{\mathcal{E}}, \; m, n \geq 0, \; k \geq 1\}.$$

From $p_{r(a)} - p_{r(a^2)} = p_{r(a) \setminus r(a^2)} = p_{[v_{11}]_2} \in J$, we have

$$s_a + J = s_a p_{r(a)} + J = p_{r(a)} s_a + J.$$

Thus $s_a p_{r(a)} + J$ is a unitary of the unital hereditary subalgebra (with unit $p_{r(a)} + J$) of the quotient algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})/J$. The ideal J is obviously invariant under the gauge action $\gamma : \mathbb{T} \to \operatorname{Aut}(C^*(E, \mathcal{L}, \overline{\mathcal{E}}))$. Hence there exists an induced action $\gamma : \mathbb{T} \to \operatorname{Aut}(C^*(E, \mathcal{L}, \overline{\mathcal{E}})/J)$ such that $\gamma_z(s_a p_{r(a)} + J) = z(s_a p_{r(a)} + J)$ for $z \in \mathbb{T}$. Thus the unitary $s_a p_{r(a)} + K$ does not belong to the unitary path connected component of the unit of the hereditary subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})/J$, which implies as in the proof of Theorem 3.2.2 that $C^*(E, \mathcal{L}, \overline{\mathcal{E}})/J$ and hence $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is not AF.

Notation 3.2.7. If $x_i \in A_i E^{\leq K} A_{i+1}$ is a path with $\alpha_i = \mathcal{L}(x_i)$ for $i = 1, \ldots, n$ such that $x_1 \cdots x_n \in A_1 E^{\leq K} \cdots E^{\leq K} A_{n+1}$, then we set

$$\begin{aligned} \bar{r}(A_1 \alpha_1 A_2) &:= r(A_1, \alpha_1) \cap A_2 \\ \bar{r}(A_1 \alpha_1 A_2 \alpha_2 A_3) &:= r(\bar{r}(A_1 \alpha_1 A_2), \alpha_2) \cap A_3 = r(r(A_1, \alpha_1) \cap A_2, \alpha_2) \cap A_3, \end{aligned}$$

and so on, thus for $3 \leq i \leq n+1$,

$$\bar{r}(A_1\alpha_1A_2\cdots\alpha_{i-1}A_i) := r(\bar{r}(A_1\alpha_1A_2\cdots A_{i-1}), \alpha_{i-1}) \cap A_i.$$

Note that $\bar{r}(A_1\alpha_1A_2\cdots\alpha_{i-1}A_i)$ belongs to $\overline{\mathcal{E}}$ whenever $A_j \in \overline{\mathcal{E}}$ for $1 \leq j \leq i$. The notation $\bar{r}(A_1E^{\leq K}A_2\cdots E^{\leq K}A_{n+1})$ will then be used for the collection of all sets $\bar{r}(A_1\alpha_1A_2\cdots\alpha_{n-1}A_{n+1})$ for $\alpha_1\cdots\alpha_n \in \mathcal{L}(A_1E^{\leq K}A_2\cdots E^{\leq K}A_{n+1})$.

Theorem 3.2.8. *Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a labeled space such that for every finite subset $\{A_1, \ldots, A_N\}$ of $\overline{\mathcal{E}}$ and every $K \geq 1$, there exists an $m_0 \geq 1$ for which*

$$A_{i_1}E^{\leq K}A_{i_2}E^{\leq K}A_{i_3}\cdots E^{\leq K}A_{i_n} = \emptyset$$

for all $n > m_0$ and $1 \leq i_j \leq N$. Then $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is an AF algebra.

Proof. Let $F := \{s_{\alpha_i}p_{A_i}s^*_{\beta_i} : A_i \subset r(\alpha_i) \cap r(\beta_i),\ i = 1, \ldots, N\}$ be a finite set in the C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(s_a, p_A)$ with $F = F^*$. We shall show that F generates a finite dimensional C^*-algebra. Set $K := \max\{|\alpha_i|, |\beta_i| : i = 1, \ldots N\}$. By Remark 2.2.10(i), we have

$$(s_{\alpha_i}p_{A_i}s^*_{\beta_i})(s_{\alpha_j}p_{A_j}s^*_{\beta_j}) = \begin{cases} s_{\alpha_i\gamma'}p_{r(A_i,\gamma')\cap A_j}s^*_{\beta_j}, & \text{if } \alpha_j = \beta_i\gamma' \\ s_{\alpha_i}p_{A_i\cap r(A_j,\beta')}s^*_{\beta_j\beta'}, & \text{if } \beta_i = \alpha_j\beta' \\ s_{\alpha_i}p_{A_i\cap A_j}s^*_{\beta_j}, & \text{if } \beta_i = \alpha_j \\ 0, & \text{otherwise,} \end{cases}$$

and so if, for example, $\alpha_j = \beta_i\gamma'$ and $\alpha_k = \beta_j\gamma''$, we get

$$\begin{aligned}(s_{\alpha_i}p_{A_i}s^*_{\beta_i})(s_{\alpha_j}p_{A_j}s^*_{\beta_j})(s_{\alpha_k}p_{A_k}s^*_{\beta_k}) &= (s_{\alpha_i\gamma'}p_{r(A_i,\gamma')\cap A_j}s^*_{\beta_j})(s_{\alpha_k}p_{A_k}s^*_{\beta_k}) \\ &= s_{\alpha_i\gamma'\gamma''}p_{r(r(A_i,\gamma')\cap A_j,\gamma'')\cap A_k}s^*_{\beta_k}.\end{aligned}$$

Here note that $\gamma'\gamma''$ belongs to $\mathcal{L}(A_iE^{|\gamma'|}A_jE^{|\gamma''|}A_k)$ and the set $r(r(A_i, \gamma') \cap A_j, \gamma'') \cap A_k$ is equal to $\bar{r}(A_i\gamma'A_j\gamma''A_k)$. Continuing a similar computation once more, for example with $\beta_k = \alpha_l\beta'$, we have

$$\begin{aligned}
&(s_{\alpha_i}p_{A_i}s^*_{\beta_i})(s_{\alpha_j}p_{A_j}s^*_{\beta_j})(s_{\alpha_k}p_{A_k}s^*_{\beta_k})(s_{\alpha_l}p_{A_l}s^*_{\beta_l})\\
&= (s_{\alpha_i\gamma'\gamma''}p_{r(r(A_i,\gamma')\cap A_j,\gamma'')\cap A_k}s^*_{\beta_k})(s_{\alpha_l}p_{A_l}s^*_{\beta_l})\\
&= s_{\alpha_i\gamma'\gamma''}p_{r(r(A_i,\gamma')\cap A_j,\gamma'')\cap A_k\cap r(A_l,\beta')}s^*_{\beta_l\beta'}
\end{aligned}$$

which is nonzero only when $\gamma'\gamma'' \in \mathcal{L}(A_iE^{|\gamma'|}A_jE^{|\gamma''|}A_k)$ and $\beta' \in \mathcal{L}(A_lE^{|\beta'|}A_k)$. If this is the case, we have

$$s_{\alpha_i\gamma'\gamma''}p_{r(r(A_i,\gamma')\cap A_j,\gamma'')\cap A_k\cap r(A_l,\beta')}s^*_{\beta_l\beta'} = s_{\alpha_i\gamma'\gamma''}p_{\bar{r}(A_i\gamma'A_j\gamma''A_k)\cap\bar{r}(A_l\beta'A_k)}s^*_{\beta_l\beta'}$$

as before. Repeating the process of multiplying any finite elements from the set F actually produces an element of the form $s_{\alpha_i\mu}p_As^*_{\beta_j\nu}$, where A is a finite intersection of sets in

$$A(F) := \cup_{\substack{n\geq 1\\ 1\leq i_j\leq N}} \bar{r}\big(A_{i_1}E^{\leq K}A_{i_2}\cdots E^{\leq K}A_{i_n}\big)$$

and μ and ν are paths in

$$\mathcal{L}(F) := \cup_{\substack{n\geq 1\\ 1\leq i_j\leq N}} \mathcal{L}(A_{i_1}E^{\leq K}A_{i_2}\cdots E^{\leq K}A_{i_n}).$$

By our assumption, we find an $m_0 \geq 1$ such that $\mathcal{L}(A_{i_1}E^{\leq K}A_{i_2}\cdots E^{\leq K}A_{i_n}) = \emptyset$ for all $n > m_0$, so that $\mathcal{L}(F)$ turns out to be a finite set since our labeled space is always assumed receiver set-finite. Then the finiteness of the set $A(F)$ is immediate, and so we conclude that F generates the finite dimensional $*$-algebra;

$$\overline{\operatorname{span}}\big\{s_{\alpha_i\mu}p_As^*_{\beta_j\nu} : A = \cap B_k,\ B_k \in A(F),\ \mu,\nu \in \mathcal{L}(F),\ 1 \leq i,j \leq N\big\}.$$

$\square$

Example 3.2.9. In the following labeled graph $(E, \mathcal{L})$

$$
\begin{array}{cccccc}
\underset{u_1}{\bullet} \xrightarrow{1} \underset{v_{11}}{\bullet} & & & & & \\
\underset{u_2}{\bullet} \xrightarrow{1} \underset{v_{21}}{\bullet} \xrightarrow{2} \underset{v_{22}}{\bullet} & & & & & \\
\underset{u_3}{\bullet} \xrightarrow{1} \underset{v_{31}}{\bullet} \xrightarrow{2} \underset{v_{32}}{\bullet} \xrightarrow{3} \underset{v_{33}}{\bullet} & & & & & \\
\underset{u_4}{\bullet} \xrightarrow{1} \underset{v_{41}}{\bullet} \xrightarrow{2} \underset{v_{42}}{\bullet} \xrightarrow{3} \underset{v_{43}}{\bullet} \xrightarrow{4} \underset{v_{44}}{\bullet} & & & & & \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots, & & & & &
\end{array}
$$

one can show that the labeled graph C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is an AF algebra using Theorem 3.2.8. In fact, it is enough to see that for any finite subset $\{r(n_1), r(n_2), \cdots, r(n_N)\}$ of $\overline{\mathcal{E}}$ with $n_1 < n_2 < \cdots < n_N$, $n_i \in \mathbb{N}$ and every $K \geq 1$, actually only the $K := \max\{n_{i+1} - n_i : i = 1, \cdots, N-1\}$ is a matter of concern, we have

$$r(n_1)E^{\leq K}r(n_2)E^{\leq K}r(n_3)\cdots E^{\leq K}r(n_N)E^{\leq K}r(n_i) = \emptyset$$

for any $n_1 \leq n_i \leq n_N$.

In the following example, we see that the condition that $(E, \mathcal{L}, \overline{\mathcal{E}})$ has no repeatable paths is not a sufficient condition for $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ to be AF.

Example 3.2.10. Consider the following labeled graph $(E_{\mathbb{Z}}, \mathcal{L}_\omega)$:

$$\cdots \underset{v_{-4}}{\bullet} \xrightarrow{0} \underset{v_{-3}}{\bullet} \xrightarrow{1} \underset{v_{-2}}{\bullet} \xrightarrow{1} \underset{v_{-1}}{\bullet} \xrightarrow{0} \underset{v_0}{\bullet} \xrightarrow{x_0 = 0} \underset{v_1}{\bullet} \xrightarrow{x_1 = 1} \underset{v_2}{\bullet} \xrightarrow{1} \underset{v_3}{\bullet} \xrightarrow{0} \underset{v_4}{\bullet} \cdots,$$

where the $\{0, 1\}$ sequence ω is the *Thue-Morse sequence* (see Example 2.3.6). Recall that ω contains no block (no finite subsequence) of the form $\beta\beta\beta_1$ for $\beta = \beta_1 \cdots \beta_{|\beta|} \in \mathcal{L}^*(E)$. Thus $(E, \mathcal{L}, \overline{\mathcal{E}})$ has no repeatable paths satisfying (b) in Remark 3.2.1. But, the set $A := r(0)$, with $K := 3$, satisfies

$$A_{i_1}E^{\leq 3}A_{i_2}\cdots E^{\leq 3}A_{i_n} \neq \emptyset$$

for all $n \geq 1$, where $A_{i_j} = A$, $j \geq 1$. This is because the block 111 does not appear in the sequence ω. Thus $(E, \mathcal{L}, \overline{\mathcal{E}})$ does not meet the condition (a) in Remark 3.2.1. To see that $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ (equivalently, $M_2 \otimes C^*(E, \mathcal{L}, \overline{\mathcal{E}})$) is not AF, it is enough to show that $M_2 \otimes C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ contains a unitary U such that $(id_{M_2} \otimes \gamma)_z(U) = zU$ for all $z \in \mathbb{T}$, where γ is the gauge action of $\mathbb{T}$ on $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(s_i, p_A)$ ([12, Proposition 3.9]). Actually one can easily check that the unitary $U = (u_{ij})$, with entries $u_{ij} = \delta_{ij}\, s_0 + (1 - \delta_{ij})s_1$, is a desired one.

Now we turn to the implication (c) $\Rightarrow$ (b) of Remark 3.2.1. For a C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(s_a, p_A)$ and a set $A \in \overline{\mathcal{E}}$, we denote by I_A the ideal of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ generated by the projection p_A as before.

Lemma 3.2.11. *Let $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(s_a, p_A)$ be the C^*-algebra of a labeled graph $(E, \mathcal{L})$ with no sinks or sources. For $A, B \in \overline{\mathcal{E}}$, we have $p_A \in I_B$ if and only if there exist an $N \geq 1$ and finitely many paths $\{\mu_i\}_{i=1}^n$ in $\mathcal{L}(BE^{\geq 0})$ such that*

$$\cup_{|\beta|=N}\, r(A, \beta) \subset \cup_{i=1}^n r(B, \mu_i).$$

Proof. If $p_A \in I_B$, we can approximate p_A, within a small enough $\varepsilon > 0$, by an element $\sum_{i=1}^n c_i s_{\beta_i} p_{B_i \cap r(B,\mu_i)} s_{\gamma_i}^*$ of I_B, where $c_i \in \mathbb{C}$, $\beta_i, \gamma_i \in \mathcal{L}(AE^{\geq 0})$, $B_i \in \overline{\mathcal{E}}$, and $\mu_i \in \mathcal{L}(BE^{\geq 0})$ for $1 \leq i \leq n$ (see (2.3)). We assume $(\beta_i, \mu_i, \gamma_i) \neq (\beta_j, \mu_j, \gamma_j)$ if $i \neq j$. Considering the image of $X := p_A - \sum_{i=1}^n c_i s_{\beta_i} p_{B_i \cap r(B,\mu_i)} s_{\gamma_i}^*$ under the conditional expectation onto the AF core (the fixed point algebra of the gauge action), we may assume that $|\beta_i| = |\gamma_i|$, $1 \leq i \leq n$, since p_A is in the core. Moreover, since $(E, \mathcal{L})$ has no sinks, we can also assume that $|\beta_i| = |\beta_1|$ for all i. Put $N := |\beta_i|$, $1 \leq i \leq n$. From $p_A = \sum_{|\beta|=N} s_\beta p_{r(A,\beta)} s_\beta^*$, we have

$$\|X\| = \Big\| \sum_{|\beta|=N} s_\beta p_{r(A,\beta)} s_\beta^* - \sum_{i=1}^n c_i s_{\beta_i} p_{B_i \cap r(B,\mu_i)} s_{\gamma_i}^* \Big\| < \varepsilon.$$

If $r(A, \beta) \not\subset \cup_{i=1}^n r(B, \mu_i)$ for some $\beta \in \mathcal{L}(AE^N)$, that is, $A' := r(A, \beta) \setminus \cup_{i=1}^n r(B, \mu_i) \neq \emptyset$, one obtains a contradiction, $\varepsilon > \|p_{A'}(s_\beta^* X s_\beta) p_{A'}\| = \|p_{A'}\| = 1$.

For the reverse inclusion, it is enough to note that $p_{\cup_{i=1}^n r(B,\mu_i)} \in I_B$ (see [20, Lemma 3.5]). $\square$

If α is a repeatable path in a directed graph E, then α is a loop with the range $r(\alpha)$ consisting of a single vertex and every repetition α^n also has the same range as α, $r(\alpha^m) = r(\alpha)$, $m \geq 1$. The projection $p_{r(\alpha)\setminus r(\alpha^m)}$ is then equal to 0 in the C^*-algebra $C^*(E, \mathcal{L}_{id}, \overline{\mathcal{E}})$, and so the (zero) ideal generated by the projection $p_{r(\alpha)\setminus r(\alpha^m)}$ can not have the nonzero projection $p_{r(\alpha)}$. In this case, we already know that $C^*(E) = C^*(E, \mathcal{L}_{id}, \overline{\mathcal{E}})$ is not AF. But for a general labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ with a repeatable path α, this is no longer true, namely $r(\alpha^m) \subsetneq r(\alpha)$ can happen for some $m \geq 2$. Moreover, we have the following.

Theorem 3.2.12. *Let $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(s_a, p_A)$ be the C^*-algebra of a labeled graph $(E, \mathcal{L})$ with no sinks or sources. Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ have a repeatable path $\alpha \in \mathcal{L}^*(E)$. If $p_{r(\alpha^m)}$ does not belong to the ideal generated by a projection $p_{r(\alpha^m)\setminus r(\alpha^{m+1})}$ for some $m \geq 1$, $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is not AF.*

Proof. Let $A_m := r(\alpha^m) \setminus r(\alpha^{m+1})$ for $m \geq 1$. Then $\{I_{A_m}\}_{m=1}^{\infty}$ is a decreasing sequence of ideals because the generator $p_{r(\alpha^{m+1})\setminus r(\alpha^{m+2})}$ of $I_{A_{m+1}}$ belongs to I_{A_m};

$$p_{r(\alpha^{m+1})\setminus r(\alpha^{m+2})} = s_\alpha^* s_\alpha \, p_{r(r(\alpha^m)\setminus r(\alpha^{m+1}),\alpha)} = s_\alpha^* p_{r(\alpha^m)\setminus r(\alpha^{m+1})} s_\alpha \in I_{A_m}.$$

We first show the following claim.

Claim: If $p_{r(\alpha)}$ does not belong to the ideal generated by $p_{r(\alpha)\setminus r(\alpha^2)}$, then the C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is not AF.

To prove the claim, it is enough to show that the quotient algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})/I_{A_1}$ is not AF. Note that $p_{r(\alpha)} + I_{A_1} = p_{r(\alpha^2)} + I_{A_1}$ is a nonzero projection in the quotient algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})/I_{A_1}$ and that

$$I_{A_1} = \overline{\operatorname{span}}\{ s_\beta p_B s_\gamma^* : \beta, \gamma \in \mathcal{L}(E^{\geq 0}) \text{ and } B \in r(\mathcal{L}(A_1 E^{\geq 0})) \cap \overline{\mathcal{E}} \}$$

by (2.3). If $s_\alpha^* s_\alpha + I_{A_1} = s_\alpha s_\alpha^* + I_{A_1}$, the hereditary subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})/I_{A_1}$

with the unit projection $p_{r(\alpha)}+I_{A_1}$ is not AF since it contains a unitary $s_\alpha+I_{A_1}$ satisfying $\gamma_z(s_\alpha+I_{A_1}) = z^{|\alpha|}(s_\alpha+I_{A_1})$ for each $z\in\mathbb{C}$. Thus the hereditary subalgebra (hence $C^*(E,\mathcal{L},\overline{\mathcal{E}})$) is not an AF algebra. (The fact that $s_\alpha+I_{A_1}$ belongs to the hereditary subalgebra follows from $p_{r(\alpha)}s_\alpha+I_{A_1} = s_\alpha p_{r(\alpha^2)}+I_{A_1} = s_\alpha p_{r(\alpha)}+I_{A_1} = s_\alpha+I_{A_1}$.) If $s_\alpha^*s_\alpha+I_{A_1}\neq s_\alpha s_\alpha^*+I_{A_1}$, then $s_\alpha^*s_\alpha+I_{A_1} = p_{r(\alpha)}+I_{A_1}\geq s_\alpha p_{r(\alpha^2)}s_\alpha^*+I_{A_1} = s_\alpha p_{r(\alpha)}s_\alpha^*+I_{A_1} = s_\alpha s_\alpha^*+I_{A_1}$ and this shows that $s_\alpha^*s_\alpha+I_{A_1}\gneq s_\alpha s_\alpha^*+I_{A_1}$. Thus the projection $s_\alpha^*s_\alpha+I_{A_1}$ is infinite, and the quotient algebra is not AF as claimed.

Now suppose that $p_{r(\alpha^m)}\notin I_{A_m}$ for some $m\geq 2$. Since $\delta:=\alpha^m$ is a repeatable path, by the above claim, we only need to show that $p_{r(\delta)}$ does not belong to the ideal, say J, generated by the projection $p_{r(\delta)\setminus r(\delta^2)} = p_{r(\alpha^m)\setminus r(\alpha^{2m})}$. For this, assuming $p_{r(\delta)}\in J$ we have from Lemma 3.2.11 that there exist an $N\geq 1$ and paths $\{\mu_j\}_{j=1}^n$ such that

$$r(r(\delta),\beta)\subset \cup_{i=1}^n r\big(r(\alpha^m)\setminus r(\alpha^{2m}),\mu_i\big)$$

for all $\beta\in\mathcal{L}(r(\delta)E^N)$. Since each set $r(r(\alpha^m)\setminus r(\alpha^{2m}),\mu_i)$ coincides with

$$\cup_{j=0}^{m-1} r\big(r(\alpha^{m+j})\setminus r(\alpha^{m+j+1}),\,\mu_i\big) = \cup_{j=0}^{m-1} r\big(r(\alpha^m)\setminus r(\alpha^{m+1}),\alpha^j\mu_i\big),$$

we can write the set $\cup_{i=1}^n r\big(r(\alpha^m)\setminus r(\alpha^{2m}),\,\mu_i\big)$ as $\cup_{j=1}^{n'} r\big(r(\alpha^m)\setminus r(\alpha^{m+1}),\mu_j'\big)$ for some finitely many paths μ_j' which is of the form $\alpha^l\mu_i$. This means that $p_{r(\alpha^m)} = p_{r(\delta)}\in I_{A_m}$ again by Lemma 3.2.11, which is a contradiction. □

As pointed out in [5], a disagreeable labeled space $(E,\mathcal{L},\overline{\mathcal{E}})$ contains lots of aperiodic paths and in fact, $(E,\mathcal{L},\overline{\mathcal{E}})$ is disagreeable whenever it has no repeatable paths as it can be seen in the following proposition.

Proposition 3.2.13. *Let E be a directed graph with no sinks or sources. If the labeled space $(E,\mathcal{L},\overline{\mathcal{E}})$ has no repeatable paths, it is always disagreeable.*

Proof. Assuming that $(E,\mathcal{L},\overline{\mathcal{E}})$ is not disagreeable, one can pick a generalized vertex $[v]_l$ that is not disagreeable. Then there is an $N>0$ such that every α in $\mathcal{L}([v]_lE^{\geq N})$ is agreeable and of the form $\alpha=\beta^k\beta'$ for some $\beta\in\mathcal{L}([v]_lE^{\leq l})$

and its initial path β'. On the other hand, there are only finitely many labeled paths in $\mathcal{L}([v]_l E^{\leq l})$ while $\mathcal{L}([v]_l E^{\geq N})$ has infinitely many labeled paths. This shows that there should exist a path β in $\mathcal{L}([v]_l E^{\leq l})$ such that its repetitions β^n appear in $\mathcal{L}([v]_l E^{\geq N})$ for all sufficiently large n. $\square$

One might expect that a labeled space would be disagreeable if it has no loops, but this is not true in general: See the following example.

Example 3.2.14. Consider the following labeled graph $(E, \mathcal{L})$

$$\begin{tikzcd} \cdots & \underset{v_{-3}}{\bullet} \arrow[r, "-3"] & \underset{v_{-2}}{\bullet} \arrow[r, "-2"] & \underset{v_{-1}}{\bullet} \arrow[r, "-1"] & \underset{v_0}{\bullet} \arrow[r, "0"] & \underset{v_1}{\bullet} \arrow[r, "0"] & \underset{v_2}{\bullet} \arrow[r, "0"] & \underset{v_3}{\bullet} & \cdots . \end{tikzcd}$$

Then $\overline{\mathcal{E}}$ is the collection of all finite sets F of E^0 and sets of the form $F \cup \{v_n, v_{n+1}, \dots\}$, $n \geq 1$. For the generalized vertex $\{v_0\} = [v_0]_1$, every path $\alpha \in \mathcal{L}(v_0 E^{\geq N})$ is agreeable since it must be equal to a^m for some $m \geq N$, so the labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ is not disagreeable, whereas it is obvious that$(E, \mathcal{L}, \overline{\mathcal{E}})$ has no loops.

Chapter 4

Non-AF finite simple labeled graph C^*-algebras

A simple graph C^*-algebra $C^*(E)$ is either AF or purely infinite ([29, Corollary 3.10]). In this chapter, we consider the question of whether this dichotomy for simple graph C^*-algebras would hold true for the simple labeled graph C^*-algebras.

4.1 Simple finite labeled graph C^*-algebras of generalized Morse sequences

We will provide a family of simple labeled graph C^*-algebras $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ associated to generalized Morse sequences ω and show that these C^*-algebras are equipped with unique traces, hence finite, and are not AF with non-zero K_1-groups.

Fixed point algebras $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ of gauge action. Let ω be a generalized Morse sequence and let $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}}) = C^*(s_a, p_A)$ be the labeled graph C^*-algebra associated with the labeled space $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ of a generalized Morse sequence ω (see Notaion 2.3.7). Then the fixed point algebra

$$C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma = \overline{\operatorname{span}}\{s_\alpha p_A s_\alpha^* : A \in \overline{\mathcal{E}}_{\mathbb{Z}},\ A \subset r(\alpha)\}$$

of the gauge action γ is easily seen to be a commutative C^*-algebra. For each $k \geq 1$, let

$$F_k := \operatorname{span}\{s_\alpha p_{r(\alpha'\alpha)} s_\alpha^* : \alpha, \alpha' \in \mathcal{L}_\omega(E_{\mathbb{Z}}^k)\}.$$

The set $\mathcal{L}_\omega(E_{\mathbb{Z}}^k)$ is finite and the elements $s_\alpha p_{r(\alpha'\alpha)} s_\alpha^*$ in F_k are easily seen to be mutually orthogonal. Hence F_k is a finite dimensional subalgebra of $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$. Moreover F_k is a subalgebra of F_{k+1} because

$$s_\alpha p_{r(\alpha'\alpha)} s_\alpha^* = \sum_{b \in \{0,1\}} s_{\alpha b} p_{r(\alpha'\alpha b)} s_{\alpha b}^* = \sum_{a,b \in \{0,1\}} s_{\alpha b} p_{r(a\alpha'\alpha b)} s_{\alpha b}^*.$$

This gives rise to an inductive sequence $F_1 \xrightarrow{\iota_1} F_2 \xrightarrow{\iota_2} \cdots$ of finite dimensional C^*algebras, where the connecting maps $\iota_k : F_k \to F_{k+1}$ are inclusions for all $k \geq 1$, from which we obtain an AF algebra $\varinjlim F_k$.

Proposition 4.1.1. *Let $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ be the labeled space of a generalized Morse sequence ω. Then*

$$C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}})^\gamma = \varinjlim F_k,$$

where $F_k := \operatorname{span}\{s_\alpha p_{r(\alpha'\alpha)} s_\alpha^* : \alpha, \alpha' \in \mathcal{L}(E^k)\}$ is a finite dimensional subalgebra of $C^*(E_{\mathbb{Z}}, \mathcal{L}, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ for $k \geq 1$.

Proof. Since $F_k \subset C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ for all $k \geq 1$ and $\overline{\cup_k F_k} = \varinjlim F_k$, it is clear that $\varinjlim F_k \subset C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$. Thus it suffices to know that $\cup_k F_k$ is dense in $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ and we only need to show that for $y := s_\alpha p_{r(\beta\alpha)} s_\alpha^*$, there is $k \geq 1$ with $y \in F_k$ as the span of the elements $s_\alpha p_{r(\beta\alpha)} s_\alpha^*$ ($|\alpha|, |\beta| \geq 0$) is dense in $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$. If $|\beta\alpha| = 2|\alpha|$, then $y \in F_k$ with $k = |\alpha|$. If $|\beta\alpha| > 2|\alpha|$, then $y = s_\alpha p_{r(\beta\alpha)} s_\alpha^* = \sum_{\sigma \in \mathcal{L}(E^{|\beta|-|\alpha|})} s_{\alpha\sigma} p_{r(\beta\alpha\sigma)} s_{\alpha\sigma}^* \in F_k$ with $k = |\beta|$. Finally if $|\beta\alpha| < 2|\alpha|$, we have $y = s_\alpha p_{r(\beta\alpha)} s_\alpha^* = \sum_{\sigma \in \mathcal{L}(E^{|\alpha|-|\beta|})} s_\alpha p_{r(\sigma\beta\alpha)} s_\alpha^* \in F_k$ with $k = |\alpha|$. ∎

Proposition 4.1.2. *Let $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ be the labeled space of a generalized Morse sequence ω. Then there is a surjective isomorphism*

$$\rho : C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma \to C(\mathscr{O}_\omega) \tag{4.1}$$

such that $\rho(s_\alpha p_{r(\alpha'\alpha)} s_\alpha^*) = \chi_{[\alpha'.\alpha]}$ for $s_\alpha p_{r(\alpha'\alpha)} s_\alpha^* \in F_k$, $k \geq 1$.

Proof. Note that for each $k \geq 1$, the map $\rho_k : F_k \to C(\mathscr{O}_\omega)$ given by

$$\rho_k(s_\alpha p_{r(\alpha'\alpha)} s_\alpha^*) = \chi_{[\alpha'.\alpha]}$$

is a $*$-homomorphism (we omit the proof) such that $\rho_k(y) = \rho_{k+1}(\iota_k(y))$ for $y = s_\alpha p_{r(\alpha'\alpha)} s_\alpha^* \in F_k$, where $\iota_k : F_k \to F_{k+1}$ is the inclusion map. In fact, $\iota_k(y) = \sum_{a,b\in\{0,1\}} s_{\alpha b} p_{r(a\alpha'\alpha b)} s_{\alpha b}^*$ and so

$$\rho_{k+1}(\iota_k(y)) = \rho_{k+1}\big(\sum_{a,b\in\{0,1\}} s_{\alpha b} p_{r(a\alpha'\alpha b)} s_{\alpha b}^*\big) = \sum_{a,b\in\{0,1\}} \chi_{[a\alpha'.\alpha b]}.$$

But $\sum_{a,b\in\{0,1\}} \chi_{[a\alpha'.\alpha b]} = \chi_{[\alpha'.\alpha]}$ is obvious from $\cup_{a,b\in\{0,1\}}[a\alpha'.\alpha b] = [\alpha'.\alpha]$. Thus there exists a $*$-homomorphism $\rho : \varinjlim F_k \to C(\mathscr{O}_\omega)$ satisfying $\rho(y) = \rho_k(y)$ for all $y \in F_k$, $k \geq 1$. Since each ρ_k is injective, so is ρ, and so we now show that ρ is surjective to complete the proof. Let $\chi_{t[\beta]} \in C(\mathscr{O}_\omega)$ for $t \in \mathbb{Z}$ and $\beta \in \mathcal{L}_\omega^*(E_{\mathbb{Z}})$. Assuming $t > 0$, we can write $\chi_{t[\beta]} = \sum_{\alpha,\sigma} \chi_{[\alpha.\sigma\beta]}$, where the sum is taken over all α, σ with $|\sigma| = t$ and $|\alpha| = |\sigma\beta|$. Thus we see that for $k := |\beta| + t$

$$\chi_{t[\beta]} = \rho_k\big(\sum_{\alpha,\sigma} s_\alpha p_{r(\alpha\sigma\beta)} s_\alpha^*\big) \in \rho(F_k).$$

In case where $t \leq 0$, a similar argument shows that $\chi_{t[\beta]} \in \rho(F_k)$ for some k. Thus ρ is surjective since the space $\operatorname{span}\{\chi_{t[\beta]} : t \in \mathbb{Z},\ \beta \in \mathcal{L}_\omega^*(E_{\mathbb{Z}})\}$ is a dense subalgebra of $C(\mathscr{O}_\omega)$. $\square$

Lemma 4.1.3. *Let $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ be the labeled space of a generalized Morse sequence ω and let $\rho : C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma \to C(\mathscr{O}_\omega)$ be the isomorphism in* (4.1).

Then the unique T-invariant ergodic measure $m_\omega : C(\mathscr{O}_\omega) \to \mathbb{C}$ defines a tracial state

$$\tau_0 := m_\omega \circ \rho : C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma \to \mathbb{C}$$

on the fixed point algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ such that for $\alpha, \beta \in \mathcal{L}^*_\omega(E_{\mathbb{Z}})$,

$$\tau_0(s_\alpha p_{r(\beta\alpha)} s^*_\alpha) = \tau_0(p_{r(\beta\alpha)}).$$

Proof. Note that $p_{r(\beta\alpha)} = \sum_\sigma s_\sigma p_{r(\beta\alpha\sigma)} s^*_\sigma$, where the sum is taken over the paths σ with $|\sigma| = |\beta\alpha|$ from which we have $\rho(p_{r(\beta\alpha)}) = \rho(\sum_{|\sigma|=|\beta\alpha|} s_\sigma p_{r(\beta\alpha\sigma)} s^*_\sigma) = \sum_{|\sigma|=|\beta\alpha|} \chi_{[\beta\alpha.\sigma]} = \chi_{\cup_\sigma [\beta\alpha.\sigma]} = \chi_{[\beta\alpha.]}$. Thus

$$\tau_0(p_{r(\beta\alpha)}) = m_\omega(\chi_{[\beta\alpha.]}).$$

On the other hand, if $|\beta\alpha| > 2|\alpha|$, $s_\alpha p_{r(\beta\alpha)} s^*_\alpha = \sum_{\sigma \in \mathcal{L}(E^{|\beta|-|\alpha|})} s_{\alpha\sigma} p_{r(\beta\alpha\sigma)} s^*_{\alpha\sigma}$ so that

$$\tau_0(s_\alpha p_{r(\beta\alpha)} s^*_\alpha) = m_\omega(\sum_{|\sigma|=|\beta|-|\alpha|} \chi_{[\beta.\alpha\sigma]}) = m_\omega(\chi_{[\beta.\alpha]}).$$

But the equality $m_\omega(\chi_{[\beta\alpha.]}) = m_\omega(\chi_{[\beta.\alpha]})$ follows from the fact that m_ω is T-invariant. The case where $|\beta\alpha| \le 2|\alpha|$ can be done in a similar way. $\square$

We also use the following notation

$$[\alpha) := \{\alpha\beta : \beta \in \mathcal{L}^\sharp_\omega(E_{\mathbb{Z}})\} \quad \text{and} \quad (\alpha] := \{\beta\alpha : \beta \in \mathcal{L}^\sharp_\omega(E_{\mathbb{Z}})\}.$$

Lemma 4.1.4. *Let $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ be the labeled space of a generalized Morse sequence ω. Then*

$$\tau_0 \circ \Psi : C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}}) \to \mathbb{C}$$

is a tracial state.

Proof. To see that $\tau_0 \circ \Psi$ is a trace, we claim

$$\tau_0(\Psi(XY)) = \tau_0(\Psi(YX)) \tag{4.2}$$

for $X, Y \in \operatorname{span}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}^*(E_\mathbb{Z}),\ A \in \overline{\mathcal{E}}_\mathbb{Z},\ A \subset r(\alpha) \cap r(\beta)\}$. We first show (4.2) with $X = s_\alpha p_A s_\beta^*$ and $Y = s_\mu p_B s_\nu^*$ by considering all possible cases as follows:

(a) $[\beta) \cap [\mu) = \emptyset$ and $[\alpha) \cap [\nu) = \emptyset$: Obviously $XY = YX = 0$.

(b) $[\beta) \cap [\mu) = \emptyset$ and $[\alpha) \cap [\nu) \neq \emptyset$: $XY = 0$ is clear, and we have either $\alpha \in [\nu)$ or $\nu \in [\alpha)$. If $\alpha \in [\nu)$ with $\alpha = \nu\alpha'$ for some $\alpha' \in \mathcal{L}_\omega^\sharp(E_\mathbb{Z})$, then

$$YX = s_\mu p_B s_\nu^* s_\alpha p_A s_\beta^* = s_{\mu\alpha'} p_{r(B,\alpha')\cap A} s_\beta^*$$

and $\Psi(YX) = 0$ if $|\mu\alpha'| \neq |\beta|$. If not, that is $|\mu\alpha'| = |\beta|$, then YX can possibly be nonzero only when $\mu\alpha' = \beta$, but this contradicts to the assumption $[\beta) \cap [\mu) = \emptyset$. If $\nu \in [\alpha)$ with $\nu = \alpha\nu'$ for some $\nu' \in \mathcal{L}_\omega^\sharp(E_\mathbb{Z})$, the same argument as above proves $\Psi(YX) = 0$.

(c) $[\beta) \cap [\mu) \neq \emptyset$ and $[\alpha) \cap [\nu) = \emptyset$: $YX = 0$ is obvious, and $\Psi(XY) = 0$ follows from the same argument as in (b) by exchanging the roles of X and Y.

(d) $[\beta) \cap [\mu) \neq \emptyset$ and $[\alpha) \cap [\nu) \neq \emptyset$:

(i) $|\beta| = |\mu|$ and $|\alpha| = |\nu|$: Then $\alpha = \nu$ and $\beta = \mu$, and so

$$XY = s_\alpha p_{A\cap B} s_\alpha^* \text{ and } YX = s_\beta p_{B\cap A} s_\beta^*.$$

Thus $\tau_0(\Psi(XY)) = \tau_0(XY) = \tau_0(s_\alpha p_{A\cap B} s_\alpha^*) = \tau_0(p_{A\cap B})$ and similarly $\tau_0(\Psi(YX)) = \tau_0(p_{A\cap B})$, so that (4.2) holds.

(ii) $|\beta| = |\mu|$ and $|\alpha| \neq |\nu|$: If $|\alpha| > |\nu|$ with $\alpha = \nu\alpha'$ for some $\alpha' \in \mathcal{L}_\omega^*(E_\mathbb{Z})$. Then $\Psi(XY) = \Psi(s_\alpha p_{A\cap B} s_\nu^*) = 0$. Also $\Psi(YX) = \Psi(s_{\mu\alpha'} p_{r(B,\alpha')\cap A} s_\beta^*) = 0$ because $|\mu\alpha'| > |\beta|$. If $|\nu| > |\alpha|$, a similar argument can be applied to have $\Psi(XY) = 0 = \Psi(YX)$.

(iii) $|\beta| \neq |\mu|$ and $|\alpha| = |\nu|$: We can exchange the roles of X and Y again to see that (4.2) holds in this case by (ii).

(iv) $|\beta| \neq |\mu|$ and $|\alpha| \neq |\nu|$: First suppose $|\beta| > |\mu|$ and $|\alpha| > |\nu|$ with $\beta = \mu\beta'$ and $\alpha = \nu\alpha'$ for some $\beta', \alpha' \in \mathcal{L}^*_\omega(E_\mathbb{Z})$, so that

$$XY = s_\alpha p_{A\cap r(B,\beta')} s^*_{\nu\beta'} \text{ and } YX = s_{\mu\alpha'} p_{r(B,\alpha')\cap A} s^*_\beta.$$

It is easily checked that $|\alpha| = |\nu\beta'|$ if and only if $|\mu\alpha'| = |\beta|$, and moreover if this is the case, we may assume $\alpha' = \beta'$ (because $\alpha' \neq \beta'$ implies $XY = YX = 0$) and so $\tau_0(\Psi(XY)) = \tau_0(p_{r(B,\alpha')\cap A}) = \tau_0(\Psi(XY))$. Otherwise (that is, $|\alpha| \neq |\nu\beta'|$ and $|\mu\alpha'| \neq |\beta|$), $\Psi(XY) = \Psi(YX) = 0$ is clear. This argument also proves the case when $|\beta| < |\mu|$ and $|\alpha| < |\nu|$. Now suppose $|\beta| > |\mu|$ and $|\alpha| < |\nu|$ with $\beta = \mu\beta'$ and $\nu = \alpha\nu'$ for some $\beta', \nu' \in \mathcal{L}^*_\omega(E_\mathbb{Z})$. Then

$$XY = s_\alpha p_{A\cap r(B,\beta')} s^*_{\nu\beta'} \text{ and } YX = s_\mu p_{B\cap r(A,\nu')} s^*_{\beta\nu'}.$$

Since $|\alpha| < |\nu\beta'|$ and $|\mu| < |\beta\nu'|$, we have $\Psi(XY) = \Psi(YX) = 0$. This, of course, proves the assertion when $|\beta| < |\mu|$ and $|\alpha| > |\nu|$.

In general, for $X = \sum_{i=1}^n c_i X_i$ and $Y = \sum_{j=1}^n c'_j Y_j$ with $X_i, Y_j \in \{s_\alpha p_A s^*_\beta : \alpha, \beta \in \mathcal{L}^*_\omega(E_\mathbb{Z}), A \in \overline{\mathcal{E}}_\mathbb{Z}\}$, $c_i, c'_j \in \mathbb{C}$, the above computations show that

$$\begin{aligned}\tau_0(\Psi(XY)) &= \sum_{i,j} c_i c'_j \tau_0(\Psi(X_i Y_j)) = \sum_{i,j} c_i c'_j \tau_0(\Psi(Y_j X_i)) \\ &= \tau_0(\Psi(\sum_{i,j} c'_j c_i Y_j X_i)) = \tau_0(\Psi(YX)).\end{aligned}$$

The positive linear functional $\tau_0 \circ \Psi$ is a state since

$$(\tau_0 \circ \Psi)(1) = \tau_0\big(\sum_{a,b\in\{0,1\}} s_b p_{r(ab)} s^*_b\big) = m_\omega(\sum_{a,b\in\{0,1\}} \chi_{[a.b]}) = m_\omega(\chi_{\Omega_\omega}) = 1.$$

□

We need several lemmas to show that the C^*-algebra $C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$ of a generalized Morse sequence ω is simple.

Lemma 4.1.5. *Let ω be a point of Ω. If $\beta \in \mathcal{L}^*_\omega(E_\mathbb{Z})$ is a path such that $\beta = \beta'\beta'' = \beta''\beta'$ for $\beta', \beta'' \in \mathcal{L}^\sharp_\omega(E_\mathbb{Z})$, then β' and β'' (hence β itself) are repetitions of an $\alpha \in \mathcal{L}^*_\omega(E_\mathbb{Z})$.*

Proof. If $|\beta'| = |\beta''|$, then $\beta = (\beta')^2$. So we assume that $|\beta''| < |\beta'|$. Let $\beta^{(1)} := \beta'$ and $\beta^{(2)} := \beta''$. Then the assumption $\beta^{(1)}\beta^{(2)} = \beta^{(2)}\beta^{(1)}$ implies that $\beta^{(1)} = (\beta^{(2)})^{n_1}\beta^{(3)}$ for some $n_1 \geq 1$ and $\beta^{(3)}$ with $|\beta^{(3)}| < |\beta^{(2)}|$. If $|\beta^{(3)}| = 0$, $\beta = (\beta^{(2)})^{n_1+1}$ as desired. If $|\beta^{(3)}| \neq 0$, again from the assumption $\beta^{(1)}\beta^{(2)} = \beta^{(2)}\beta^{(1)}$ we have $(\beta^{(2)})^{n_1}\beta^{(3)}\beta^{(2)} = \beta^{(2)}(\beta^{(2)})^{n_1}\beta^{(3)}$ which reduces to an equation $\beta^{(3)}\beta^{(2)} = \beta^{(2)}\beta^{(3)}$, $|\beta^{(3)}| < |\beta^{(2)}|$. Thus we have $\beta^{(2)} = (\beta^{(3)})^{n_2}\beta^{(4)}$ with $|\beta^{(4)}| < |\beta^{(3)}|$. In this way, we obtain a sequence $(\beta^{(m)})$ of subpaths of β such that $|\beta^{(m+1)}| < |\beta^{(m)}|$,

$$\beta^{(m-1)} = (\beta^{(m)})^{n_{m-1}}\beta^{(m+1)} \text{ and } \beta^{(m)}\beta^{(m+1)} = \beta^{(m+1)}\beta^{(m)}.$$

Since $|\beta| < \infty$, this process of obtaining $\beta^{(m)}$ should stop at some point where we must have $|\beta^{(m+1)}| = 0$ and so have $\beta^{(m-1)} = (\beta^{(m)})^{n_{m-1}}$. Then it is clear that every path $\beta^{(j)}$, $1 \leq j \leq m-1$, is equal to some repetition of $\beta^{(m)}$, and we complete the proof with $\alpha = \beta^{(m)}$. □

Lemma 4.1.6. *Let x be a one-sided recurrent sequence and let $\omega \in \mathscr{O}_x$ be a two-sided sequence with $x = \omega_{[0,\infty)}$. Then $(E_\mathbb{Z}, \mathcal{L}_\omega)$ has a repeatable path if and only if the sequence x is periodic. In particular, if ω is a generalized Morse sequence, the labeled graph $(E_\mathbb{Z}, \mathcal{L}_\omega)$ has no repeatable paths (hence $(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$ is disagreeable).*

Proof. If x is periodic, obviously $(E_\mathbb{Z}, \mathcal{L}_\omega)$ has repeatable paths.

For the converse, let $x = b^0 \times b^1 \times b^2 \times \cdots$ be a non-periodic sequence such that $(E_\mathbb{Z}, \mathcal{L}_\omega)$ has a repeatable path. We may assume that $l := |b^0| \geq 2$. Since any repetition of a repeatable path is repeatable, we can choose a repeatable path β with $|\beta| \geq 2$. Note also that if β is written as $\beta = \beta'\beta''$ for some $\beta', \beta'' \in \mathcal{L}^\#_\omega(E_\mathbb{Z})$, then $\beta''\beta'$ is repeatable. From this observation and the fact that for each $k \geq 1$, $x_{[kl,(k+1)l-1]}$ is equal to either b^0 or $\tilde{b}^0$, we may assume that there are a repeatable path β and $k, n, n' \geq 2$ such that

$$x_{[kl,n']} = \beta^n$$

and $x_{[kl,(k+2)l-1]} = b^0 b^0$ (this is possible because x is non-periodic) and $|\beta^n|$ is much larger than $l = |b^0|$. Taking $b^0 \times \cdots \times b^m$ instead of b^0 for a large m, we may also assume that $|\beta| < |b^0|$. Then $b^0 = \beta^d \beta'$ for some $d \geq 1$ and β' with $\beta = \beta'\beta''$, which gives $b^0 b^0 = \beta^d \beta' \beta^d \beta'$. On the other hand, $b^0 b^0 = x_{[kl,(k+2)l-1]}$, as an initial path of $x_{[kl,n']}$, starts with a repetition β^{d+2} of β and ends with some initial path of β. Thus $b^0 b^0 = \beta^d \beta' \beta^d \beta' = \beta^{d+2}\delta$ should hold (for some δ), which then implies $\beta = \beta'\beta'' = \beta''\beta'$. By Lemma 4.1.5, we see that $b^0 = \beta^k \beta'$ is a repetition of a path, so that x is periodic, a contradiction.

It is not hard to see that $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ is disagreeable if ω is a generalized Morse sequence (see [17, Proposition 4.12]). $\square$

Now we prove our main theorem of this chapter.

Theorem 4.1.7. *Let ω be a generalized Morse sequence of zeros and ones. Then the C^*-algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ is*

(i) *simple unital,*

(ii) *non AF,*

(iii) *finite with a unique tracial state τ which satisfies*

$$\tau(s_\alpha p_{r(\sigma\alpha)} s_\beta^*) = \tau(\Psi(s_\alpha p_{r(\sigma\alpha)} s_\beta^*)) = \delta_{\alpha,\beta}\tau(p_{r(\sigma\alpha)})$$

for $\alpha, \beta, \sigma \in \mathcal{L}_\omega^*(E_{\mathbb{Z}})$.

In particular, $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ is not stably isomorphic to any graph C^*-algebra.

Proof. By definition of a generalized Morse sequence, $x := \omega_{[0,\infty)}$ is a one-sided Morse sequence.

(i) For the simplicity of $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$, we show that any nonzero homomorphism $\pi : C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}}) \to C^*(q_A, t_i)$ onto a C^*-algebra generated by $q_A := \pi(p_A),\ t_i := \pi(s_i)$ for $A \in \overline{\mathcal{E}}$, $i = 0, 1$, is faithful. Since the labeled space $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ is disagreeable by Lemma 4.1.6, we see from [5, Theorem 5.5] that π is faithful whenever $\pi(p_{[v]_l}) \neq 0$ for all $v \in E^0$ and $l \geq 1$. Suppose on the contrary that

$$q_{[v]_m} = \pi(p_{[v]_m}) = 0$$

for some $[v]_m = r(\alpha)$ with $|\alpha| = m$. Since $\alpha \in \mathfrak{B}_x$ and x is almost periodic by Theorem 2.3.4(i), one finds a $d \geq 1$ such that for all $s \geq 0$,

$$T^{s+j} x \in [.\alpha],$$

for some $0 \leq j \leq d$. This means that if $\beta \in \mathfrak{B}_x$ is a block with length $|\beta| \geq d$, it must have α as its subpath. Thus β must be of the form $\beta = \beta'\alpha\beta''$ for some $\beta', \beta'' \in \mathcal{L}^\sharp_\omega(E)$. For these β's with $|\beta| \geq d$ we have $q_{r(\beta)} = 0$. In fact,

$$\begin{aligned} q_{r(\beta)} &= q_{r(\beta'\alpha\beta'')} = q_{r(r(\beta'\alpha),\beta'')} \\ &= q_{r(r(\beta'\alpha),\beta'')} t^*_{\beta''} t_{\beta''} q_{r(r(\beta'\alpha),\beta'')} \\ &\sim t_{\beta''} q_{r(r(\beta'\alpha),\beta'')} t^*_{\beta''} \\ &\leq q_{r(\beta'\alpha)} \leq q_{r(\alpha)} \\ &= q_{[v]_m}. \end{aligned}$$

On the other hand, since π is a nonzero homomorphism, there exists a $\delta \in \mathcal{L}^*_\omega(E_{\mathbb{Z}})$ with $q_{r(\delta)} = \pi(p_{r(\delta)}) \neq 0$. But then, with an $n > \max\{|\delta|, d\}$, we have

$$q_{r(\delta)} = \pi(p_{r(\delta)}) = \pi\big(\sum_{|\delta\mu_i|=n} s_{\mu_i} p_{r(\delta\mu_i)} s^*_{\mu_i} \big) = \sum_{|\delta\mu_i|=n} t_{\mu_i} q_{r(\delta\mu_i)} t^*_{\mu_i} = 0,$$

a contradiction.

(ii) With $\overline{\mathcal{E}}_{\mathbb{Z}}$ in place of $\mathcal{B}$ in (2.6) it is rather obvious that $\mathcal{N} = \emptyset$ and $\hat{\mathcal{B}} = \hat{\mathcal{B}}_J = \overline{\mathcal{E}}_{\mathbb{Z}}$. Since $\chi_A \in \ker(1 - \Phi)$ if and only if $\chi_A = \chi_{r(A,0)} + \chi_{r(A,1)}$ (see (2.4), and the vertex set $E^0_{\mathbb{Z}}$ is the disjoint union of two sets $r(E^0_{\mathbb{Z}}, 0)$ and

$r(E^0_{\mathbb{Z}},1)$ in $\overline{\mathcal{E}}_{\mathbb{Z}}$, we have $\chi_{E^0_{\mathbb{Z}}} \in \ker(1-\Phi)$. Thus $K_1(C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})) \neq 0$ and $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$ is not AF.

(iii) The tracial state $\tau := \tau_0 \circ \Psi : C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}}) \to \mathbb{C}$ of Lemma 4.1.4 satisfies

$$\tau(s_\alpha p_{r(\sigma\alpha)} s_\beta^*) = \delta_{\alpha,\beta}\tau(p_{r(\sigma\alpha)}) \tag{4.3}$$

for $s_\alpha p_{r(\sigma\alpha)} s_\beta^* \in C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$ by Lemma 4.1.3.

To show that τ is the unique tracial state on $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$, we claim that if τ' is a tracial state on $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$, then $\tau' \circ \Psi = \tau'$ holds, and that the state $\tau' \circ \rho^{-1}$ on $C(\mathscr{O}_\omega)$ is T-invariant. For the first claim, suppose $\tau' \circ \Psi \neq \tau'$. Then there exists an element $s_\alpha p_{r(\alpha)} s_\beta^*$ ($|\beta| < |\alpha|$) such that $\tau'(s_\alpha p_{r(\alpha)} s_\beta^*) \neq 0$. Since τ' is tracial, we have $0 \neq \tau'(s_\alpha p_{r(\alpha)} s_\beta^*) = \tau'(s_\beta^* s_\alpha p_{r(\alpha)})$. Thus α must be of the form $\alpha = \beta\alpha'$ for some path α', and then $0 \neq \tau'(s_\beta^* s_\alpha p_{r(\alpha)}) = \tau'(s_{\alpha'} p_{r(\alpha)})$. Again the tracial property of τ' gives

$$0 \neq \tau'(s_{\alpha'} p_{r(\alpha)}) = \tau'(p_{r(\alpha)} s_{\alpha'}) = \tau'(s_{\alpha'} p_{r(\alpha\alpha')}) = \cdots = \tau'(s_{\alpha'} p_{(r(\alpha),(\alpha')^n)})$$

for all $n \geq 1$. But this means that the generalized vertex $[v]_l := r(\alpha)$, $l = |\alpha|$, is not disagreeable emitting only agreeable paths, which is a contradiction to Lemma 4.1.6. To see that $\tau' \circ \rho^{-1} : C(\mathscr{O}_\omega) \to \mathbb{C}$ is T-invariant, let $\chi_{t[\beta]} \in C(\mathscr{O}_\omega)$. We assume $t > 0$. Since

$$\rho^{-1}(\chi_{t[\beta]}) = \rho^{-1}\Big(\sum_{\substack{\alpha,\beta \\ |\alpha|=|\sigma\beta|=t+|\beta|}} \chi_{[\alpha.\sigma\beta]}\Big) = \sum_{\substack{\alpha,\beta \\ |\alpha|=|\sigma\beta|=t+|\beta|}} s_{\sigma\beta} p_{r(\alpha\sigma\beta)} s_{\sigma\beta}^*,$$

we have $\tau'(\rho^{-1}(\chi_{t[\beta]})) = \tau'\Big(\sum_{\substack{\alpha,\beta \\ |\alpha|=|\sigma\beta|=t+|\beta|}} p_{r(\alpha\sigma\beta)}\Big) = \tau'(p_{r(\beta)})$. This implies that

$$\tau' \circ \rho^{-1}(\chi_{t[\beta]}) = \tau' \circ \rho^{-1}(\chi_{t+1[\beta]}) = \tau' \circ \rho^{-1}(\chi_{t[\beta]} \circ T),$$

which can also be shown for $t \leq 0$ in a similar way. Thus $\tau' \circ \rho^{-1}$ is T-invariant because the span of functions $\chi_{t[\beta]}$ is dense in $C(\mathscr{O}_\omega)$.

The last assertion follows from the fact that a simple graph C^*-algebra is either AF or purely infinite . □

Remark 4.1.8. Without using the result on the existence of a unique ergodic probability measure on $\mathscr{O}_\omega$ for a generalized Morse sequence ω, one can directly show that the simple unital C^*-algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ of the Thue-Morse sequence ω admits only one tracial state. Moreover, its values on typical elements of the form $s_\alpha p_A s_\beta^*$ can be obtained concretely, which is done in [23].

Chapter 5

Labeled graph C^*-algebras that are not finite

In previous chapters, we studied finite C^*-algebras of labeled spaces. Now we consider conditions of labeled spaces which give rise to infinite C^*-algebras. Throughout this chapter, we assume that a directed graph E has no sinks.

5.1 Labeled graph C^*-algebras whose nonzero hereditary subalgebras are all infinite

In a directed graph E satisfying Condition (L), if we further require every vertex to connect to a loop, any of nonzero hereditary C^*-subalgebras of the C^*-algebra $C^*(E)$ is well known to be infinite. Dealing with the labeled paths and the generalized vertices in a labeled space, we first need to define when a (generalized) vertex should be said to connects to a loop.

Definition 5.1.1. Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space. We say that *every vertex connects to a loop* in $(E, \mathcal{L}, \mathcal{B})$ if for every $[v]_m$, there exist an $A \in \mathcal{B}$ and labeled paths α, $\delta \in \mathcal{L}^*(E)$ such that

(i) $A \subseteq r([v]_m, \delta)$,

(ii) $A \subseteq r(A, \alpha)$.

Remarks 5.1.2. Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space.

(a) The condition (i) of Definition 5.1.1 can be replaced by

$$\text{(i')} \quad A \subseteq r([v]_m, \delta\alpha).$$

In fact, if a vertex $[v]_m$ connects to a loop so that (i), (ii) of Definition 5.1.1 hold for $A \in \mathcal{B}$ and $\alpha, \delta \in \mathcal{L}^*(E)$, then $A \subseteq r(A, \alpha) \subseteq r([v]_m, \delta\alpha)$ follows immediately. Conversely, for a vertex $[v]_m$ if there exist an $A \in \mathcal{B}$ and α, δ satisfying (i') and (ii), then from

$$A \subseteq r(A, \alpha) \cap r([v]_m, \delta\alpha) = r(A \cap r([v]_m, \delta), \alpha)$$

we see that the nonempty set $A' := A \cap r([v]_m, \delta) (\subseteq A)$ satisfies (i)($A' \subseteq r([v]_m, \delta)$) and (ii)($A' \subseteq r(A', \alpha)$).

(b) In [5, Definition 6.6], a property of $(E, \mathcal{L}, \mathcal{B})$ requiring every (generalized) vertex to connect to a loop which is based at a descending sequence $([w]_l)_l$ of generalized vertices was phrased as *every vertex connects to a repeatable path*. More precisely, this means that for every $[v]_m$ there exist a $w \in E^0, L(w) \geq 1$, and labeled paths $\alpha, \delta \in \mathcal{L}^*(E)$ such that

$$w \in r([v]_m, \delta\alpha) \ \text{ and } \ [w]_l \subseteq r([w]_l, \alpha) \text{ for all } l \geq L(w).$$

If we take $L(w)$ large enough, $[w]_l \subseteq r([v]_m, \delta\alpha)$ for all $l \geq L(w)$ and so $[w]_l \subseteq r([w]_l \cap r([v]_m, \delta), \alpha)$. Then $[w]_l \cap r([v]_m, \delta) \neq \emptyset$ for all $l \geq L(w)$, which implies that $[w]_l \subseteq r([v]_m, \delta)$ again for all sufficiently large l. Thus this property is equivalent to that for every $[v]_m$, there exist a $w \in E^0, L(w) \geq 1$, and labeled paths $\alpha, \delta \in \mathcal{L}^*(E)$ such that for all $l \geq L(w)$,

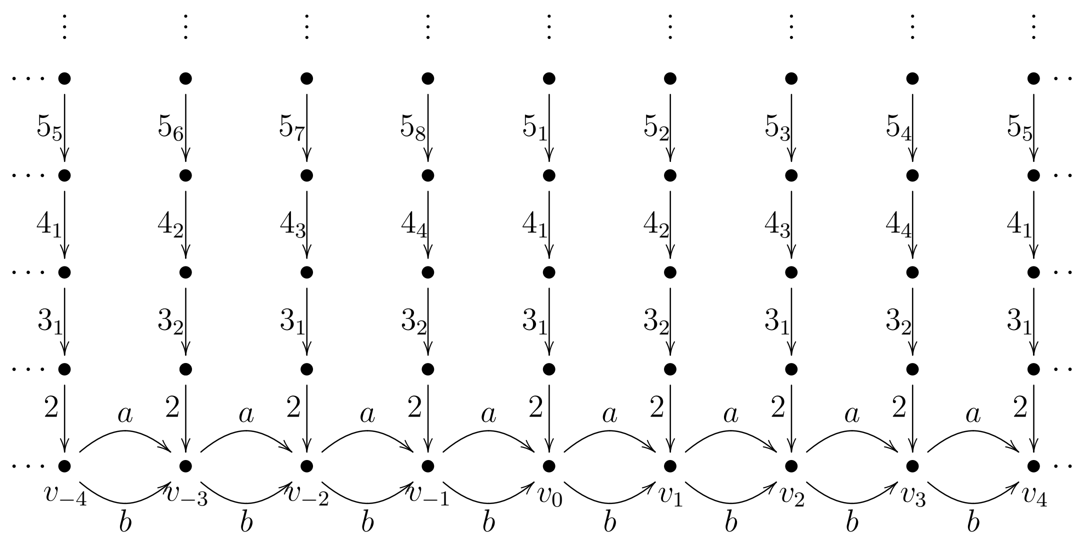
(i) $[w]_l \subseteq r([v]_m, \delta)$

(ii) $[w]_l \subseteq r([w]_l, \alpha)$ (that is, α is a loop at $[w]_l$).

So obviously this property is stronger than the one introduced in Definition 5.1.1 while the converse is not true because a loop at $[w]_{L(w)}$ may not be a loop at $[w]_l$ for $l > L(w)$ as Example 5.1.3 given below shows. Actually, Example 5.1.3 suggests that the notion of connecting every vertex to a repeatable path ([5]) might be said *every vertex connects to a loop at a nested sequence of generalized vertices.*

(c) It is known in [5, Theorem 6.9] that if a labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ is disagreeable and strongly cofinal and every vertex connects to a repeatable path, then the C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is simple and purely infinite.

Example 5.1.3. Consider the following labeled graph $(E, \mathcal{L})$.



Then one easily sees that for each $n \geq 1$,

$$[v_0]_n = \{v_{k\cdot 2^{n-1}} : k \in \mathbb{Z}\} = \{\cdots, v_{-2\cdot 2^{n-1}}, v_{-2^{n-1}}, v_0, v_{2^{n-1}}, v_{2\cdot 2^{n-1}}, \cdots\}$$

admits loops $a^{2^{k-1}}$ for all $k \geq n$. Specifically, for example, the path a is a loop at $[v_0]_1$, but not a loop at any $[v_0]_n$ for $n \geq 2$. Actually, $(E, \mathcal{L}, \overline{\mathcal{E}})$ does not have a path $\alpha \in \mathcal{L}^*(E)$ for which there are $w \in E^0$ and $L(w) \geq 1$

with $[w]_l \subseteq r([w]_l, \alpha)$ for all $l \geq L(w)$, while one can check that every vertex connects to a loop in this example. Since it is rather obvious that $(E, \mathcal{L}, \overline{\mathcal{E}})$ is disagreeable, we see that $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ has the property (SP_∞) by Theorem 5.1.6 below.

Lemma 5.1.4. *Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a disagreeable labeled space. Then every loop has an exit. Moreover, the projection p_A is infinite in $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(s_a, p_A)$ whenever $A \in \overline{\mathcal{E}}$ admits a loop.*

Proof. Let α be a loop at $A \in \overline{\mathcal{E}}$. Choose $w \in A$ and $l \geq 1$ so that $[w]_l \subseteq A$. Since $[w]_l$ is disagreeable, we may choose a labeled path $\beta \in \mathcal{L}^*(E)$ with $[w]_l \cap s(\beta) \neq \emptyset$ so that $|\beta| = |\alpha^i|$ and $\beta \neq \alpha^i$ for some $i > 1$, which means that the loop $\mu := \alpha^i$ at $A \in \overline{\mathcal{E}}$ has an exit β. Thus, p_A is an infinite projection by Proposition 3.1.6. □

The similar arguments as in [5, Theorem 6.9] and [3, Proposition 5.3] yield the following proposition. But it has to be modified to fit in our setting. Thus for convenience we provide a proof with details. In the following proposition, the notation that $p \preceq q$ where p, q are projections in a C^*-algebra will mean that p is Murray-von Neumann equivalent to a subprojection of q.

Proposition 5.1.5. *Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a disagreeable labeled space. Then every nonzero hereditary C^*-subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ contains a nonzero projection p such that $s_\mu p_A s_\mu^* \preceq p$ for some $\mu \in \mathcal{L}^*(E)$ and $A \in \overline{\mathcal{E}}$.*

Proof. Let B be a nonzero hereditary C^*-subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ and fix a positive element $a \in B$ with $\|\Phi(a)\| = 1$. Choose a positive element $b \in \operatorname{span}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}^*(E) \text{ and } A \subseteq r(\alpha) \cap r(\beta)\}$ so that $\|a - b\| < \frac{1}{4}$. From [5, Proposition 2.4 (ii) and (iii)], we may write $b = \sum_{(\alpha, [w]_l, \beta) \in F} c_{(\alpha, [w]_l, \beta)} s_\alpha p_{[w]_l} s_\beta^*$, where F is a finite subset of $\mathcal{L}^*(E) \times \Omega_l \times \mathcal{L}^*(E)$ for some $l \geq 1$. Let $b_0 = \Phi(b) \geq 0$. Since Φ is norm-decreasing, we have

$$|1 - \|b_0\|| = |\|\Phi(a)\| - \|\Phi(b)\|| \leq \|\Phi(a - b)\| \leq \|a - b\| < \frac{1}{4},$$

and hence $\|b_0\| \geq \frac{3}{4}$. Let $k = \max\{|\alpha|, |\beta| : (\alpha, [w]_l, \beta) \in F\}$. Applying the Definition 2.2.7(iv) and changing F (if necessary), we can choose a $k \in \mathbb{N}$ so that $\min\{|\alpha|, |\beta|\} = k$ for every $(\alpha, [w]_l, \beta) \in F$. Let $M = \max\{|\alpha|, |\beta| : (\alpha, [w]_l, \beta) \in F\}$. Applying [5, Proposition 2.4.(iii)] again, we may choose $m \geq M$ large enough so that

$$b_0 \in \oplus_{\{w:(\alpha,[w]_l,\beta)\in F\}}\mathcal{F}^k([w]_m).$$

Now, $\|b_0\|$ must be attained in some summand $\mathcal{F}^k([v]_m)$. Let b_1 be the component of b_0 in $\mathcal{F}^k([v]_m)$ so that $\|b_0\| = \|b_1\|$ and note that $b_1 \geq 0$. Then we can choose a projection $r \in C^*(b_1) \subseteq \mathcal{F}^k([v]_m)$ such that $rb_1r = \|b_1\|r$. Since b_1 is a finite sum of $s_\alpha p_{[v]_m} s_\beta^*$, we can write r as a sum $\sum c_{\alpha\beta} s_\alpha p_{[v]_m} s_\beta^*$ over all pairs of paths in

$$G = \{\alpha \in \mathcal{L}(E^k) : \text{ either } (\alpha, [v]_m, \beta) \in F \text{ or } (\beta, [v]_m, \alpha) \in F\}.$$

Note that $rb_0r = rb_1r$ and the $G \times G$-matrix $(c_{\alpha\beta})$ is also a projection in a finite dimensional matrix algebra $\mathcal{F}^k([v]_m) = \operatorname{span}\{s_\alpha p_{[v]_m} s_\beta^* : \alpha, \beta \in G\}$.

Since $[v]_m$ is disagreeable, we may choose a path $\lambda \in \mathcal{L}^*(E)$ with $|\lambda| > M$ so that λ has no factorization $\lambda = \lambda'\lambda'' = \lambda''\delta$ for some $|\lambda'|, |\delta| \leq m$. Then because $\operatorname{span}\{s_{\alpha\lambda} p_{r([v]_m,\lambda)} s_{\beta\lambda}^* : \alpha, \beta \in G\}$ is also a finite dimensional matrix algebra generated by the family of non-zero matrix units $\{s_{\alpha\lambda} p_{r([v]_m,\lambda)} s_{\beta\lambda}^* : \alpha, \beta \in G\}$,

$$Q = \sum_{\alpha,\beta \in G} c_{\alpha\beta} s_{\alpha\lambda} p_{r([v]_m,\lambda)} s_{\beta\lambda}^*$$

is a projection satisfying

$$r = \sum c_{\alpha\beta} s_\alpha p_{[v]_m} s_\beta^* = \sum c_{\alpha\beta} s_\alpha (s_\lambda p_{r([v]_m,\lambda)} s_\lambda^* + (p_{[v]_m} - s_\lambda p_{r([v]_m,\lambda)} s_\lambda^*)) s_\beta^* \geq Q.$$

We claim that for $(\mu, [v]_l, \nu) \in F$,

$$Q s_\mu p_{[v]_m} s_\nu^* Q = 0 \text{ unless } |\mu| = |\nu| = k \text{ and } [v]_m \subseteq r(\mu) \cap r(\nu).$$

Suppose that $(\mu, [v]_m, \nu) \in F$ with $|\mu| \neq |\nu|$. We may assume $|\mu| = k$ because either μ or ν has length k. Since $s^*_{\beta\lambda}s_\mu \neq 0$ if and only if $\beta = \mu$, we have

$$\begin{aligned}
Q(s_\mu p_{[v]_m} s^*_\nu)Q &= (\sum_{\alpha',\beta'\in G} c_{\alpha'\beta'} s_{\alpha'\lambda} p_{r([v]_m,\lambda)} s^*_{\beta'\lambda})(s_\mu p_{[v]_m} s^*_\nu)(\sum_{\alpha,\beta\in G} c_{\alpha\beta} s_{\alpha\lambda} p_{r([v]_m,\lambda)} s^*_{\beta\lambda}) \\
&= (\sum_{\alpha'\in G} c_{\alpha'\mu} s_{\alpha'\lambda} p_{r([v]_m,\lambda)} s^*_{\mu\lambda} s_\mu p_{[v]_m} s^*_\nu)(\sum_{\alpha,\beta\in G} c_{\alpha\beta} s_{\alpha\lambda} p_{r([v]_m,\lambda)} s^*_{\beta\lambda}) \\
&= \sum_{\alpha,\beta\in G} c_{\alpha\beta}(\sum_{\alpha'\in G} c_{\alpha'\mu} s_{\alpha'\lambda} p_{r([v]_m,\lambda)} s^*_{\nu\lambda}) s_{\alpha\lambda} p_{r([v]_m,\lambda)} s^*_{\beta\lambda}.
\end{aligned}$$

To be $s^*_{\nu\lambda}s_{\alpha\lambda} \neq 0$, it must be true that $\nu\lambda = \alpha\lambda\delta$ for some $\delta \in \mathcal{L}^*(E)$. Since $|\nu| > |\alpha| = k$, we may say that $\nu = \alpha\lambda'$ where $\lambda = \lambda'\lambda''$ for some $\lambda', \lambda'' \in \mathcal{L}^*(E)$. As $\nu\lambda = \alpha\lambda\delta = \alpha\lambda'\lambda''\delta = \alpha\lambda'\lambda$, we have

$$\lambda = \lambda'\lambda'' = \lambda''\delta$$

with $|\lambda'| = |\delta|$. Because $|\nu| = |\alpha\lambda'| \leq M$ with $|\alpha| = k$, we know $|\lambda'| \leq M - k \leq m$, which contradicts to the fact that λ is disagreeable for $[v]_m$.

Thus, we see that

$$QbQ = Qb_1Q = Qrb_1rQ = \|b_1\|rQ = \|b_0\|Q \geq \frac{3}{4}Q.$$

Since $\|a - b\| < \frac{1}{4}$, we have $QaQ \geq QbQ - \frac{1}{4}Q \geq \frac{1}{2}Q$. This implies that QaQ is invertible in $QC^*(E,\mathcal{L},\overline{\mathcal{E}})Q$. Let c be the inverse of QaQ in $QC^*(E,\mathcal{L},\overline{\mathcal{E}})Q$ and put $v = c^{\frac{1}{2}}Qa^{\frac{1}{2}}$. Then $v^*v = a^{\frac{1}{2}}QcQa^{\frac{1}{2}} \leq \|c\|a$, and hence $v^*v \in B$. Since

$$v^*v \sim vv^* = c^{\frac{1}{2}}QaQc^{\frac{1}{2}} = Q,$$

the hereditary C^*-subalgebra B contains a non-zero projection equivalent to Q. Note that Q belongs to the finite dimensional subalgebra $C := \operatorname{span}\{s_{\alpha\lambda} p_{r([v]_m,\lambda)} s^*_{\beta\lambda} : \alpha, \beta \in G\}$ for which the elements $\{s_{\alpha\lambda} p_{r([v]_m,\lambda)} s^*_{\beta\lambda}\}$ forms a matrix unit. This means that Q dominates a minimal projection in C. Since every minimal projecton in C is equivalent to a minial projection of the form $s_{\alpha\lambda} p_{r([v]_m,\lambda)} s^*_{\alpha\lambda}$, the hereditary subalgebra B also contains a projection equivalent to the desired form. □

Theorem 5.1.6. *Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a disagreeable labeled space in which every vertex connects to a loop. Then $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ has the property (SP_∞). Moreover every nonzero hereditary C^*-subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ contains an infinite projection equivalent to a projection p_A for some $A \in \overline{\mathcal{E}}$.*

Proof. We first show that $p_{[v]_l}$ is infinite for any generalized vertex $[v]_l$. By our assumption, the generalized vertex $[v]_l$ should connect to a loop, say α based at $A \in \overline{\mathcal{E}}$. Thus there is a $\delta \in \mathcal{L}^*(E)$ such that $A \subset r([v]_l, \delta)$. Then clearly $p_{r([v]_l,\delta)} \geq p_A$. Since p_A is an infinite projection by Lemma 5.1.4, the projeciton $p_{r([v]_l,\delta)}$ should also be infinite. From $p_{[v]_l} = \sum_{|\mu|=|\delta|} s_\mu p_{r([v]_l,\mu)} s_\mu^* \geq s_\delta p_{r([v]_l,\delta)} s_\delta^* \sim p_{r([v]_l,\delta)}$, we conclude that the projection $p_{[v]_l}$ is infinite.

Now let B be a nonzero hereditary subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$. By Proposition 5.1.5, B then contains a nonzero projection p such that $s_\mu p_A s_\mu^* \preceq p$ for some $\mu \in \mathcal{L}^*(E)$ and $A \in \overline{\mathcal{E}}$. But the projeciton $s_\mu p_A s_\mu^*$ is equivalent to $p_{A\cap r(\mu)}$ which is infinite by the first assertion. Thus p is infinite. $\square$

Remark 5.1.7. Let a disagreeable labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ satisfy the following property which is slightly weaker than the one assumed in Theorem 5.1.6: for every generalized vertex $[v]_l$, there exists a loop α based at $A \in \overline{\mathcal{E}}$ and a finite number of labeled paths $\delta_1, \ldots, \delta_m \in \mathcal{L}^k(E)$ with the same length k such that

$$A \subseteq \cup_{i=1}^m r([v]_l, \delta_i).$$

Then the conclusion in Theorem 5.1.6 still holds true. In fact, we can pick a path $\gamma \in \mathcal{L}^l(E)$ with $[v]_l \subset r(\gamma)$, then with $A_i := A \cap r([v]_l, \delta_i)$ and $B_i :=$

$A_i - \cup_{j=1}^{j-1} A_j$ ($B_1 := A_1$), $i = 1, \ldots, m$, we have

$$\begin{aligned}
p_A &= p_{B_1} + p_{B_2} + \cdots + p_{B_n} \\
&\le p_{B_1} s^*_{\gamma\delta_1} s_{\gamma\delta_1} p_{B_1} + \cdots + p_{B_n} s^*_{\gamma\delta_n} s_{\gamma\delta_n} p_{B_n} \\
&\sim s_{\gamma\delta_1} p_{B_1} s^*_{\gamma\delta_1} + \cdots + s_{\gamma\delta_n} p_{B_n} s^*_{\gamma\delta_n} \\
&= s_\gamma \big(s_{\delta_1} p_{B_1} s^*_{\delta_1} + \cdots + s_{\delta_n} p_{B_n} s^*_{\delta_n} \big) s^*_\gamma \\
&\le s_\gamma \big(s_{\delta_1} p_{r([v]_l, \delta_1)} s^*_{\delta_1} + \cdots + s_{\delta_n} p_{r([v]_l, \delta_n)} s^*_{\delta_n} \big) s^*_\gamma \\
&\le s_\gamma p_{[v]_l} s^*_\gamma \sim p_{[v]_l}.
\end{aligned}$$

But p_A is infinite and we see that $p_{[v]_l}$ is also infinite. The second part of the proof of Theorem 5.1.6 shows even in this case that every nonzero hereditary subalgebra of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ contains an infinite projection equivalent to p_A for some $A \in \overline{\mathcal{E}}$.

The set $\mathcal{L}^\infty(E) := \{\alpha = \alpha_1\alpha_2 \cdots \in \mathcal{A}^\infty : \ \alpha_1 \cdots \alpha_n \in \mathcal{L}^*(E) \text{ for all } n \ge 1\}$ that includes the infinite paths $\mathcal{L}(E^\infty)$ is considered in [23] to define a new version of strong cofinality of labeled spaces.

Definition 5.1.8. ([23]) A labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ is said to be *strongly cofinal* if for any generalized vertex $[v]_l \in \overline{\mathcal{E}}$ and any $x \in \mathcal{L}^\infty(E)$, there exist an $N \ge 1$ and finitely many labeled paths $\lambda_1, \ldots, \lambda_m \in \mathcal{L}^*(E)$ such that

$$r(x_{[1,N]}) \subseteq \cup_{i=1}^m r([v]_l, \lambda_i). \tag{5.1}$$

It is shown ([5, 19, 23]) that if $(E, \mathcal{L}, \overline{\mathcal{E}})$ is disagreeable and strongly cofinal, then $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is simple.

Corollary 5.1.9. *Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a disagreeable and strongly cofinal labeled space. If there is a vertex $w \in E^0$ such that $[w]_{l_i}$ admits a loop for a sequence $l_1 < l_2 < \cdots$, then $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is simple and purely infinite.*

Proof. By [19, Theorem 3.16], $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is simple. To see that $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is purely infinite, it is enough to show that $p_{[v]_l}$ is infinite for any $[v]_l \in \overline{\mathcal{E}}$. Let

α be a loop at $[w]_{l_1}$. Since $(E,\mathcal{L},\overline{\mathcal{E}})$ is strongly cofinal and $\alpha^n \in \mathcal{L}^*(E)$ for $n \geq 1$ (that is, $\alpha^\infty \in \mathcal{L}^\infty(E)$), $[v]_l$ connects to $\alpha^\infty \in \mathcal{L}^\infty(E)$. Thus there exist an $N \in \mathbb{N}$ and labeled paths $\delta_1, \cdots, \delta_m \in \mathcal{L}^*(E)$ such that

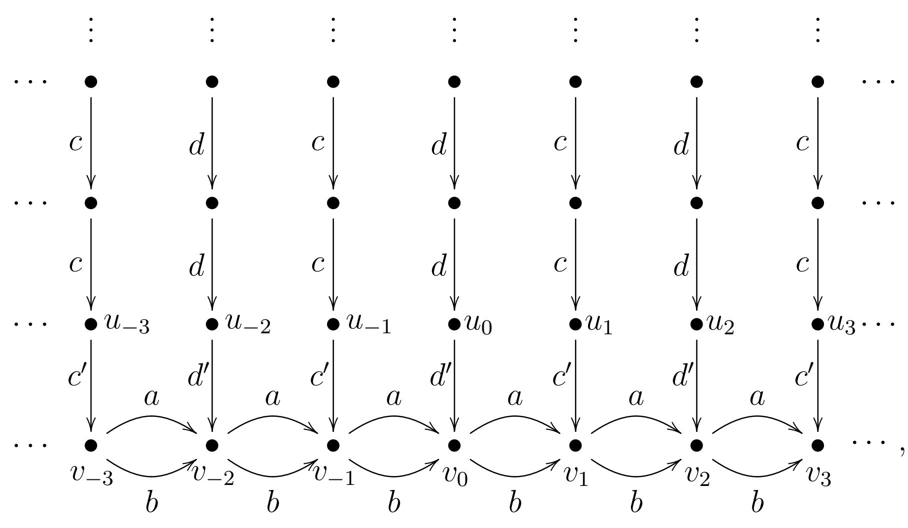
$$r(\alpha^k\alpha') = r(\alpha^\infty_{[1,N]}) \subset \cup_{i=1}^m r([v]_l, \delta_i),$$

where α' is an initial path of $\alpha(=\alpha'\alpha'')$ and some $k \geq 1$. Then

$$[w]_{l_1} \subset r([w]_{l_1}, \alpha^{k+1}) \subset r(\alpha^{k+1}) \subset \cup_{i=1}^m r([v]_l, \delta_i\alpha'').$$

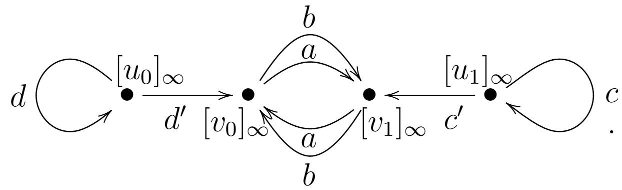
Setting $A_1 := [w]_{l_1} \cap r([v]_l, \delta_1\alpha'')$ for convenience, we choose $l > l_1$ large enough so that $[w]_l \subseteq A_1$. Then by assumption $[w]_l$ admits a loop, and hence the projection $p_{[w]_l}$ is infinite by Lemma 5.1.4. On the other hand, one sees that $p_{[v]_l} \geq s_{\delta_1\alpha''} p_{r([v]_l,\delta_1\alpha'')} s^*_{\delta_1\alpha''} \sim p_{r([v]_l,\delta_1\alpha'')} \geq p_{[w]_l}$, which implies that $p_{[v]_l}$ is also infinite. Then the second part of the proof of Theorem 5.1.6 completes the proof. $\square$

Example 5.1.10. For the following labeled graph $(E, \mathcal{L})$



Theorem 6.10 in [20] shows that $C^*(E,\mathcal{L},\overline{\mathcal{E}}) \simeq C^*(F,\mathcal{L},\overline{\mathcal{F}})$, where $(F,\mathcal{L},\overline{\mathcal{F}})$

is the associated merged labeled space of $(E, \mathcal{L}, \overline{\mathcal{E}})$ which is shown as below



Then it is clear that $(F, \mathcal{L}, \overline{\mathcal{F}})$ is disagreeable and every vertex connects to a loop. Thus by Theorem 5.1.6, $C^*(F, \mathcal{L}, \overline{\mathcal{F}})$ has the property (SP_∞), and hence $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ also has the property (SP_∞).

Extending the relation $\geq$ for vertices of directed graphs, we write $A \geq B$ for $A, B \in \overline{\mathcal{E}}$ if there exists a labeled path $\alpha \in \mathcal{L}^{\#}(E)$ such that $B \subseteq r(A, \alpha)$. The relation $\geq$ is reflexive and transitive. For $[v]_l \in \overline{\mathcal{E}}$, set

$$R_{[v]_l} := \{A \in \overline{\mathcal{E}} : [v]_l \geq A\} \text{ and}$$

$$\mathcal{F}_{[v]_l} := \{B \in \overline{\mathcal{E}} : B = \cup_{i=1}^{k} A_i \text{ for some } A_i \in R_{[v]_l} \text{ and } k \geq 1\}. \tag{5.2}$$

Lemma 5.1.11. *The set $\mathcal{F}_{[v]_l}$ defined above is a hereditary subset of $\overline{\mathcal{E}}$.*

Proof. Choose $B \in H := \mathcal{F}_{[v]_l}$. Then $B = \cup_{i=1}^{k} A_i$ where $A_i \in R_{[v]_l}$ for all $i = 1, \cdots, k$, that is, $A_i \subseteq r([v]_l, \alpha_i)$ for some $\alpha_i \in \mathcal{L}^{\#}(E)$. So, $r(B, \beta) = \cup_{i=1}^{k} r(A_i, \beta) \in H$ for all $\beta \in \mathcal{L}^{\#}(E)$ because $r(A_i, \beta) \subseteq r([v]_l, \alpha_i\beta)$ for all i. Also, if $C \in \overline{\mathcal{E}}$ with $C \subseteq B$, then $C = \cup_{i=1}^{k}(C \cap A_i)$ with $C \cap A_i \subseteq A_i \subseteq r([v]_l, \alpha_i)$ for all i, which implies $C \in H$. It is rather abvious that if $B_1, B_2 \in H$, then $B_1 \cup B_2 \in H$. $\square$

If H is any hereditary subset of $\overline{\mathcal{E}}$, we write $E_H := (E_H^0, E_H^1)$ for the subgraph of E whose vertices and edges are defined as follows:

$$E_H^0 := \{w \in E^0 : w \in B \text{ for some } B \in H\},$$

$$E_H^1 := \{f \in E^1 : s(f) \in E_H^0\}.$$

Note that the subgraph E_H can have a source and

$$E^0_{\mathcal{F}_{[v]_l}} = E^0_{R_{[v]_l}} := \{w \in E^0 : w \in A \text{ for some } A \in R_{[v]_l}\}.$$

Example 5.1.12. For the following labeled graph $(E, \mathcal{L})$

$$\begin{tikzcd}
\bullet_{v_0} \arrow[loop above, "\mathcal{L}(e_1) = b"] \arrow[loop below, "\mathcal{L}(e_2) = c"'] \arrow[r, "\mathcal{L}(f_1) = a"] & \bullet_{v_1} \arrow[r, "\mathcal{L}(f_2) = a"] & \bullet_{v_2} \arrow[r, "\mathcal{L}(f_3) = a"] & \bullet_{v_3} \arrow[r, "\mathcal{L}(f_4) = a"] & \bullet_{v_4} & \cdots ,
\end{tikzcd}$$

it is easy to see that $[v_2]_l = \{v_2\}$ for $l \geq 3$. Then $R_{\{v_2\}} = \{\{v_k\} : k \geq 2\}$, $H := \mathcal{F}_{\{v_2\}} = \{B \subset E^0 \setminus \{v_0, v_1\} : B \text{ is finite }\}$ and $E_H^0 = \{v_k : k \geq 2\}$. Now we have the directed subgraph $E_H = (E_H^0, E_H^1)$ of E as follows

$$\begin{tikzcd}
E_H : & \bullet_{v_2} \arrow[r, "f_3"] & \bullet_{v_3} \arrow[r, "f_4"] & \bullet_{v_4} \arrow[r, "f_5"] & \bullet_{v_5} \arrow[r, "f_6"] & \bullet_{v_6} \cdots .
\end{tikzcd}$$

Considering the restriction map $\mathcal{L}|_{E_H^1} : E_H^1 \to \mathcal{A}$, one can regard the directed subgraph E_H as a labeled graph $(E_H, \mathcal{L}|_{E_H^1})$ with a source as below

$$\begin{tikzcd}
\bullet_{v_2} \arrow[r, "a"] & \bullet_{v_3} \arrow[r, "a"] & \bullet_{v_4} \arrow[r, "a"] & \bullet_{v_5} \arrow[r, "a"] & \bullet_{v_6} \cdots .
\end{tikzcd}$$

For a hereditary subset H of $\overline{\mathcal{E}}$, we denote by I_H the ideal of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$

generated by the projections $\{p_A : A \in H\}$. It is easy to see ([20]) that

$$\begin{aligned} I_H &= \overline{span}\{s_\mu p_A s_\nu^* : \mu, \nu \in \mathcal{L}^*(E),\ A \in H\} \\ &= \overline{span}\{s_\mu p_A s_\nu^* : \mu, \nu \in \mathcal{L}^*(E),\ A \in \overline{H}\}. \end{aligned}$$

The following proposition is known for graph C^*-algebras (see [29, Proposition 2.1]).

Proposition 5.1.13. *For the hereditary set $H := \mathcal{F}_{[v]_l}$ given in (5.2), the ideal $I_H = I_{\mathcal{F}_{[v]_l}}$ is Morita equivalent to the hereditary C^*-subalgebra*

$$\mathcal{B}_H := \overline{span}\{s_\mu p_B s_\nu^* : \mu, \nu \in \mathcal{L}^*(E_H) \text{ and } B \in \mathcal{F}_{[v]_l}\}.$$

Proof. The relations

$$(s_\mu p_B s_\nu^*)(s_\alpha p_A s_\beta^*) = \begin{cases} s_\mu p_{B\cap r(A,\nu')} s_{\beta\nu'}^*, & \text{if } \nu = \alpha\nu' \\ s_{\mu\alpha'} p_{r(B,\alpha')\cap A} s_\beta^*, & \text{if } \alpha = \nu\alpha' \\ s_\mu p_{B\cap A} s_\beta^*, & \text{if } \nu = \alpha \\ 0, & \text{otherwise,} \end{cases}$$

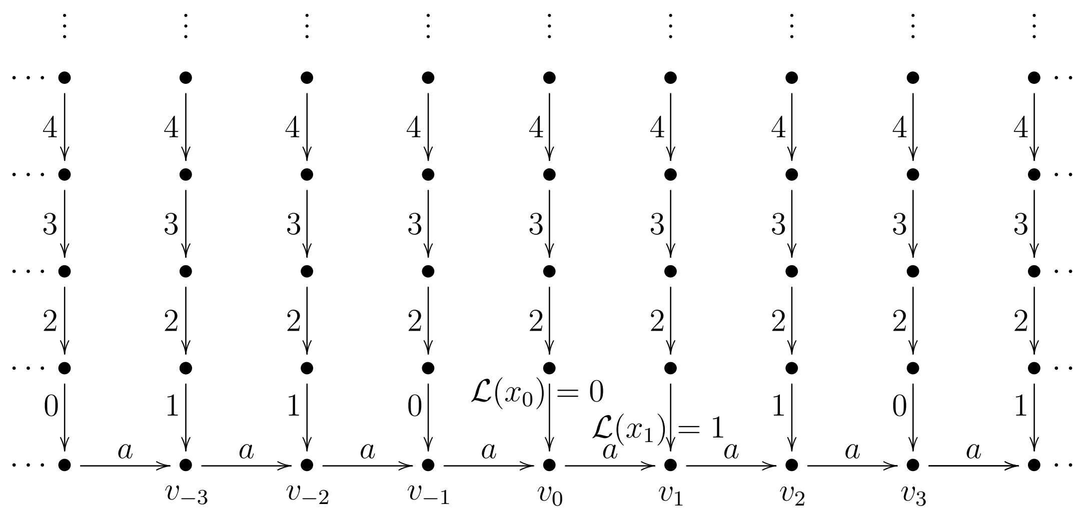
where $\alpha, \beta, \nu \in \mathcal{L}^*(E)$, $A \in \overline{\mathcal{E}}$ and $\mu \in \mathcal{L}^*(E_H)$, $B \in \mathcal{F}_{[v]_l}$, show that

$$X := \overline{span}\{s_\mu p_B s_\nu^* : \mu \in \mathcal{L}^*(E_H), \nu \in \mathcal{L}^*(E) \text{ and } B \in \mathcal{F}_{[v]_l}\}$$

is a right ideal of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ which satisfies $I_H = X^*X$ and $XX^* = \mathcal{B}_H$.

□

Remark 5.1.14. If a (generalized) vertex $[v]_l \in \overline{\mathcal{E}}$ does not connect to any loop, then obviously each set A in $R_{[v]_l}$ does not admit any loops. But some set $B \in \mathcal{F}_{[v]_l}$ of their union can be bases of loops. Consider the following labeled graph $(E, \mathcal{L})$:



By x_i we denote an edge with $s(x_i) \in r(2)$ and $r(x_i) = v_i$ for all $i \in \mathbb{Z}$. Then we give labels with the *Thue-Morse sequence* $\{0, 1\}$ starting from $\mathcal{L}(x_0) = 0$, $\mathcal{L}(x_1) = 1$. One can see that both of $r(0)$ and $r(0a^n)$ do not admit any loops for all $n \geq 1$. This proves that $r(0)$ can not connect to any loops in the sense of Definition 5.1.1. We see that

$$R_{r(0)} = \{A \in \overline{\mathcal{E}} : r(0) \geq A\} = \{r(0a^n) \in \overline{\mathcal{E}} : n \geq 0\},$$

$$\mathcal{F}_{r(0)} = \{\cup_{i=1}^{k} r(0a^{n_i}) \in \overline{\mathcal{E}} : n_i \geq 0 \text{ and } k \geq 1\},$$

where $r(0a^0) := r(0)$. Observe that $r(a) \subseteq r(0a) \cup r(0a^2) \cup r(0a^3)$. So, $r(a) \in \mathcal{F}_{r(0)}$ and $r(a)$ admits loops.

Corollary 5.1.15. *Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a labeled space and $[v]_l \in \overline{\mathcal{E}}$. If for every finite subset $\{A_1, \cdots, A_N\}$ of $H := \mathcal{F}_{[v]_l}$ and every $K \geq 1$, there exists an $m_0 \geq 1$ for which*

$$A_{i_1} E_{\overline{H}}^{\leq K} A_{i_2} E_{\overline{H}}^{\leq K} A_{i_3} \cdots E_{\overline{H}}^{\leq K} A_{i_n} = \emptyset$$

for all $n > m_0$ and $1 \leq i_j \leq N$, then the ideal $I_{\mathcal{F}_{[v]_l}}$ is an AF algebra.

Proof. Choose a generalized vertex $[v]_l \in \overline{\mathcal{E}}$. By Proposition 5.1.13, the ideal $I_{\mathcal{F}_{[v]_l}}$ is Morita equivalent to

$$\mathcal{B}_{\mathcal{F}_{[v]_l}} := \overline{span}\{s_\mu p_B s_\nu^* : \mu, \nu \in \mathcal{L}^*(E_H) \text{ and } B \in \mathcal{F}_{[v]_l}\}.$$

The assumtion asserts that $\mathcal{B}_{\mathcal{F}_{[v]_l}}$ is an AF algebra (see [17, Theorem 4.8]). $\square$

Example 5.1.16. Let us revisit the labeled graph in Example 5.1.12:

$$\begin{tikzcd}
\underset{v_0}{\bullet} \arrow[loop above, "\mathcal{L}(e_1)=b"] \arrow[loop below, "\mathcal{L}(e_2)=c"'] \arrow[r, "\mathcal{L}(f_1)=a"] & \underset{v_1}{\bullet} \arrow[r, "\mathcal{L}(f_2)=a"] & \underset{v_2}{\bullet} \arrow[r, "\mathcal{L}(f_3)=a"] & \underset{v_3}{\bullet} \arrow[r, "\mathcal{L}(f_4)=a"] & \underset{v_4}{\bullet} & \cdots ,
\end{tikzcd}$$

$$(E_H, \mathcal{L}|_{E_H^1}) : \underset{v_2}{\bullet} \xrightarrow{a} \underset{v_3}{\bullet} \xrightarrow{a} \underset{v_4}{\bullet} \xrightarrow{a} \underset{v_5}{\bullet} \xrightarrow{a} \underset{v_6}{\bullet} \cdots .$$

We first see that

$$I_{\{v_2\}} = \overline{span}\{s_\mu p_B s_\nu^* : \mu, \nu \in \mathcal{L}^*(E) \text{ and } B \in \overline{\mathcal{F}_{\{v_2\}}}\},$$

where $\overline{\mathcal{F}_{\{v_2\}}} = \{B \subset E^0 \setminus \{v_0\} : B \text{ is finite }\}$. By Proposition 5.1.13, the ideal $I_{\{v_2\}}$ is Morita equivalent to

$$\begin{aligned}
\mathcal{B}_{\{v_2\}} &= \overline{span}\{s_\mu p_B s_\nu^* : \mu, \nu \in \mathcal{L}^*(E_H) \text{ and } B \in \mathcal{F}_{\{v_2\}}\} \\
&= \overline{span}\{s_\mu p_B s_\nu^* : \mu = a^n, \nu = a^m, B \subset E^0 \setminus \{v_0, v_1\} \text{ and } B \text{ is finite }\}.
\end{aligned}$$

Corollary 5.1.15 says that $\mathcal{B}_{\{v_2\}}$ is an AF-algebra, which implies that $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ contains an AF hereditary C^*-subalgebra, namely $I_{\{v_2\}}$. Thus, $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ does not have the property (SP_∞).

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국문초록

본 학위논문에서는 라벨 그래프로 생성된 C^*-대수의 구조를 연구하였다. 라벨 그래프 C^*-대수 $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$가 AF-대수이면, 라벨 공간 $(E, \mathcal{L}, \overline{\mathcal{E}})$에 loop이 없음을 증명하였다. 또한 그래프 C^*-대수 $C^*(E)$가 AF-대수가 될 몇 가지 필요충분조건들이 라벨 그래프 C^*-대수에서는 더 이상 동치가 아님을 보였다. 이를 보이는 과정에서 일반화된 모스 수열을 이용하였다. 더 나아가 일반화된 모스 수열로 라벨을 준 그래프로 생성된 C^*-대수가 단순 유한 C^*-대수이면서 AF-대수는 아님을 증명하였다. 이는 단순 그래프 C^*-대수는 AF-대수이거나 순수 무한 C^*-대수라는 사실에 대비된다.

이 외에 라벨 그래프 C^*-대수의 모든 영이 아닌 유전적 부분 C^*-대수가 무한 C^*-대수가 될 충분조건을 제시하였다.

주요어휘: 그래프 C^*-대수, 라벨 그래프 C^*-대수, 유한 C^*-대수, AF C^*-대수, 순수 무한 C^*-대수, 일반화된 모스 수열
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