이학박사 학위논문

$W^{1,p}$ estimates in homogenization of elliptic systems with measurable coefficients in nonsmooth domains

(부드럽지 않은 영역에서 측정가능한 계수를 가지는 타원형 연립 방정식의 균질화 문제에 대한 $W^{1,p}$ 가늠)

2015년 8월

서울대학교 대학원

수리과학부

장 윤 수

$W^{1,p}$ estimates in homogenization of elliptic systems with measurable coefficients in nonsmooth domains

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

Yunsoo Jang

Dissertation Director : Professor Sun-Sig Byun

Department of Mathematical Sciences
Seoul National University

August 2015

Abstract

In this study, we establish uniform $W^{1,p}$ estimates for weak solutions in homogenization of elliptic systems in divergence-form with measurable coefficients in nonsmooth domains. We consider first an interior regularity and then we study boundary value problems, a Dirichlet problem and a conormal derivative problem. Our main purpose is to find an answer for minimal requirements on the coefficients and the boundary condition of the domains to ensure that Calderón-Zygmund theory holds in a homogenization problem.

Key words: Regularity theory, Homogenization, Elliptic system, BMO space, Reifenberg domain

Student Number: 2009-20280

Contents

Chapter 1

Introduction

This thesis is based on the papers [5, 6, 7]. In this thesis we consider a divergence-form elliptic system in the homogenization problem :

$$D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta u_\epsilon^j(x) \right) = D_\alpha f_\alpha^i(x) \quad \text{in } \Omega, \tag{1.1}$$

under suitable boundary conditions, a Dirichlet problem

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta u_\epsilon^j(x) \right) &= D_\alpha f_\alpha^i(x) \quad \text{in} \quad \Omega \\ u_\epsilon^i(x) &= 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.2}$$

and a conormal derivative problem

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta u_\epsilon^j(x) \right) &= D_\alpha f_\alpha^i(x) \quad \text{in} \quad \Omega \\ \left(A_{ij}^{\alpha\beta,\epsilon} D_\beta u_\epsilon^j - f_\alpha^i \right) \nu_\alpha &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$, $A_{ij}^{\alpha\beta,\epsilon} : \mathbb{R}^n \to \mathbb{R}$ for $1 \leq \alpha, \beta \leq n$, $1 \leq i, j \leq m$ and $0 < \epsilon \leq 1$, the nonhomogeneous term $F = \{f_\alpha^i\}$ is a given $m \times n$ matrix valued function, and $\nu = (\nu_1, \cdots, \nu_n)$ is the outward pointing unit normal vector to the boundary $\partial\Omega$ of a bounded domain Ω which is not well-defined in the classical sense, but is well-defined with a weak formulation of (1.3) in Definition 1.0.1. The tensor coefficients $A^\epsilon = \{A_{ij}^{\alpha\beta,\epsilon}\}$ are defined from $A = \{A_{ij}^{\alpha\beta}\}$, $A_{ij}^{\alpha\beta} : \mathbb{R}^n \to \mathbb{R}$, to be

$$A_{ij}^{\alpha\beta}(x) = A_{ij}^{\alpha\beta,1}(x) \quad \text{and} \quad A_{ij}^{\alpha\beta,\epsilon}(x) = A_{ij}^{\alpha\beta}\left(\frac{x}{\epsilon}\right). \tag{1.4}$$

The coefficients are assumed to have uniform ellipticity and uniform boundedness. More precisely, there exist positive constants λ and Λ such that

$$\lambda|\xi|^2 \ \leq \ A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j \tag{1.5}$$

for every matrix $\xi \in \mathbb{R}^{mn}$ and for almost every $x \in \mathbb{R}^n$ and

$$\|A\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{mn\times mn})} \ \leq \ \Lambda, \tag{1.6}$$

Further, we assume the following periodicity condition on $\left\{A_{ij}^{\alpha\beta}(x)\right\}$:

$$A_{ij}^{\alpha\beta}(x+z) = A_{ij}^{\alpha\beta}(x) \quad (x \in \mathbb{R}^n, z \in \mathbb{Z}^n). \tag{1.7}$$

We state now definitions of weak solutions for (1.1)-(1.3).

Definition 1.0.1.

1. We say that $u_\epsilon \in H^1(\Omega,\mathbb{R}^m)$ is a weak solution (1.1) if

$$\int_\Omega A_{ij}^{\alpha\beta,\epsilon} D_\beta u_\epsilon^j D_\alpha \phi^i dx = \int_\Omega f_\alpha^i D_\alpha \phi^i dx, \qquad \forall \phi \in H_0^1(\Omega,\mathbb{R}^m). \tag{1.8}$$

2. We say that $u_\epsilon \in H_0^1(\Omega,\mathbb{R}^m)$ is a weak solution of (1.2) if

$$\int_\Omega A_{ij}^{\alpha\beta,\epsilon} D_\beta u_\epsilon^j D_\alpha \phi^i dx = \int_\Omega f_\alpha^i D_\alpha \phi^i dx, \qquad \forall \phi \in H_0^1(\Omega,\mathbb{R}^m). \tag{1.9}$$

3. We say that $u_\epsilon \in H^1(\Omega,\mathbb{R}^m)$ is a weak solution of (1.3) if

$$\int_\Omega A_{ij}^{\alpha\beta,\epsilon} D_\beta u_\epsilon^j D_\alpha \phi^i dx = \int_\Omega f_\alpha^i D_\alpha \phi^i dx, \qquad \forall \phi \in H^1(\Omega,\mathbb{R}^m). \tag{1.10}$$

We remark that in this thesis the summation convention, where repeated indices are automatically summed over, is employed. Also, throughout this paper we denote by c to mean any universal constants that can be computed in terms of known data such as $\lambda, \Lambda, m, n, p$, and the domain structure, and may change from line to line. If necessary, we use $c_1, c_2, \cdots$, to specify them.

According to Lax-Milgram lemma, if $F \in L^2(\Omega,\mathbb{R}^{mn})$, then the problem (1.2) and (1.3) has a unique (up to a constant for (1.3)) weak solution $u_\epsilon \in H_0^1(\Omega,\mathbb{R}^m)$ ($H^1(\Omega,\mathbb{R}^m)$ for (1.3)) with the estimate

$$\|Du_\epsilon\|_{L^2(\Omega)} \leq c\|F\|_{L^2(\Omega)}, \tag{1.11}$$

where the constant c does not depend on ϵ, F and u_ϵ. The goal of this thesis is to obtain an optimal $W^{1,p}$ regularity for weak solutions of the periodic homogenization problems (1.2) and (1.3). More precisely, we want to ask what is a minimal regularity requirement on $A_{ij}^{\alpha\beta}$ and the boundary of Ω under which we have the following relation :

$$F \in L^p \Rightarrow Du_\epsilon \in L^p \quad \text{for every } 1 < p < \infty. \tag{1.12}$$

In particular, we are interested in the uniform $W^{1,p}$ estimate like

$$\|Du_\epsilon\|_{L^p(\Omega)} \leq c\|F\|_{L^p(\Omega)}, \tag{1.13}$$

where c is independent of F and u_ϵ, especially of ϵ. In other words, we want to obtain a uniform estimate like (1.13) with respect to ϵ.

Homogenization is a mathematical analysis for studying partial differential equations which have rapidly oscillating coefficients. Homogenization issues arise in many parts of science such as mechanics, physics, chemistry, engineering, etc., where we deal with inhomogeneous materials (or composite materials), molecular structure, etc., see [3, 26, 28, 42]. Starting from a microscopic structure of a problem, we find a macroscopic, or effective, description. This process of making an asymptotic analysis and seeking an averaged formulation is called homogenization. In this theory, we are interested in homogeneous effective parameters from heterogeneous media. Homogenization is not restricted to the periodic case but in this thesis we focus on the periodic homogenization.

Here we record some basic facts about the periodic homogenization problem (1.1). The matrix of correctors $\chi = \left\{\chi_\alpha^{ij}\right\}$, $1 \leq i, j \leq m$, $1 \leq \alpha \leq n$, is the weak solution of the following cell problem:

$$\begin{cases} -D_\alpha\left(A_{ij}^{\alpha\beta}(x) D_\beta \chi_\gamma^{jk}(x)\right) = D_\alpha A_{ik}^{\alpha\gamma}(x) & \text{in} \quad \mathbb{R}^n, \\ \fint_{[0,1]^n} \chi_\gamma^{jk} = 0, & \\ \chi_\gamma^{jk} \quad \mathbb{Z}^n \text{ periodic.} & \end{cases} \tag{1.14}$$

Under our condition on the coefficients of this paper (Definition 2.2.1 and Definition 2.2.2), we have the L^∞ estimate with the estimate

$$\|\chi\|_{L^\infty(\mathbb{R}^n)} \leq c(\nu, L, m, n), \tag{1.15}$$

see [4, 9, 12]. Let

$$A_{ij}^{\alpha\beta,0} = \int_{[0,1]^n} \left(A_{ij}^{\alpha\beta} + A_{ik}^{\alpha\gamma} D_\gamma \chi_\beta^{kj} \right). \tag{1.16}$$

Then the following linear elliptic system

$$D_\alpha \left(A_{ij}^{\alpha\beta,0} D_\beta u_0^j(x) \right) = D_\alpha f_\alpha^i(x) \quad \text{in } \Omega \tag{1.17}$$

is the homogenized problem whose weak solutions u_0 of (1.17) is the weak limit of weak solutions u_ϵ in $H_0^1(\Omega, \mathbb{R}^m)$ for the case (1.2) and $H^1(\Omega, \mathbb{R}^m)$ for the case (1.3) with the same boundary condition as $\epsilon \to 0$, see [3].

To obtain a uniform $W^{1,p}$ regularity in the homogenization problem, $W^{1,p}$ regulariry for $\epsilon = 1$, meaning there is no homogenization issue, will play an important role. This is because from the results of $W^{1,p}$ theory without homogenization issue we can extract our main results in the homogenization problem. More precisely, $W^{1,p}$ theory, where there is no homogenization, will be used in the following when we use a blow-up argument. In this sense, we study $W^{1,p}$ regularity for homogenization problems under the situation that $W^{1,p}$ theory for $\epsilon = 1$ is established.

Much research has been devoted to the global $W^{1,p}$ regularity theory, when there is no homogenization, in various situations, [2, 4, 9, 10, 11, 12, 13, 16, 17, 18, 23, 33, 37] and the references therein for related results. However, since $W^{1,p}$ regularity for every $1 < p < \infty$ does not always hold even when there is no homogenization issue ($\epsilon = 1$), see [27, 36], we need some additional conditions both on the coefficients $A_{ij}^{\alpha\beta}$ and on the boundary of Ω.

Without homogenization, for (1.2), $W^{1,p}$ regularity was proved when $A_{ij}^{\alpha\beta}$ are in the class VMO (vanishing mean oscillation) and the boundary of Ω is $C^{1,1}$, see [18]. This result extended to the class of small BMO (bounded mean ascillation) functions in a δ-Reifenberg flat domain, see [10, 12]. In recent papers [9, 13], $A_{ij}^{\alpha\beta}$ were allowed to be merely measurable with respect to one variable but have small BMO semi-norms with respect to the other variables. For (1.3), when $\epsilon = 1$, $W^{1,p}$ estimate was obtained in [23] for $\frac{3}{2} - \delta_1 < p < 3 + \delta_1$ when $n \geq 3$, and $\frac{4}{3} - \delta_1 < p < 4 + \delta_1$ when $n = 2$, for some small $\delta_1 > 0$, regarding a similar Neumann problem to (1.1) under the assumptions that the coefficients are in the class of VMO functions and the domain is a general Lipschitz domain. In [4, 11], $W^{1,p}$ estimate was obtained for the full range of $p \in (1, \infty)$ with small BMO coefficients and in a δ-Reifenberg flat domain. A δ-Reifenberg flat domain is a natural generalization of Lipschitz domains with a small Lipschitz constant whose boundary might be fractal, see [38].

Until now $W^{1,p}$ regularity theory of the homogenization problem has been developed in various ways, as follows from [1, 16, 21, 22, 30, 39]. For the Dirichlet problems, in [1], a uniform $W^{1,p}$ regularity for (1.1) was proved when the coefficients are Hölder continuous and the boundary of the domain is $C^{1,\alpha}$. Following this, given continuous coefficients, the interior $W^{1,p}$ regularity for linear elliptic equations was established in [16]. Also, the estimate (1.7) of a linear elliptic equation for $1 < p < 3 + \delta_1$ when $n \geq 3$, and for $1 < p < 4 + \delta_1$ when $n = 2$ under the conditions that the coefficients are in the VMO class and the domain is a general Lipschitz domain was established [40]. For the conormal derivative problems, research on global $W^{1,p}$ regularity for the problem (1.1) has been limited to C^α coefficients and $C^{1,\alpha}$ domains, [30]. From these points of view, we look for optimal global $W^{1,p}$ regularity theory in both a Dirichlet problem and a conormal derivative problem for (1.1) under weaker conditions as in [4, 12] than those in [30, 40]. To be more precise, we want to extend the previous results of $W^{1,p}$ regularity in [4, 12] to the homogenization problem (1.1) with the same assumptions that Ω is a δ-Reifenberg domain and the coefficients $A_{ij}^{\alpha\beta}$ are in the BMO class with small BMO seminorms.

It should be noted that for $\epsilon = 1$, $W^{1,p}$ regularity for the Dirichlet problem was established under a weaker condition on the coefficients than a small BMO condition, see [9, 13]. However, in order to remain consistent with the conditions between the periodic coefficients in the homogenization and the domain, we should use a small BMO condition on the coefficients for the global regularity.

The rest of this thesis is organized as follows. In the next chapter, we introduce notations, definitions related to our main assumptions, and basic tools to obtain main results. In chapter 3, we prove interior $W^{1,p}$ regularity for (1.1) when $A_{ij}^{\alpha\beta}$ are allowed to be merely measurable with respect to one variable but have small BMO semi-norms with respect to the other variables. In chapter 4 and chapter 5, we obtain global $W^{1,p}$ estimates for (1.2) and (1.3), respectively, under the assumptions that the coefficients are in the class of BMO functions with small BMO seminorms and the domain is a δ-Reifenberg flat domain.

Chapter 2

Preliminaries

This chapter describes the main assumptions on the coefficients and the boundary of the domain and introduces some tools to obtain the main results of the present thesis. We start with some notations.

2.1 Notations

We start this chapter with some notations.

1. The open ball in $\mathbb{R}^n$ with center 0 and radius $r > 0$ is defined by
$$B_r = \{x \in \mathbb{R}^n : |x| < r\}.$$
2. $B_r(y) = B_r + y$: the open balls in $\mathbb{R}^n$ with center y and radius $r > 0$.
3. The elliptic cylinder in $\mathbb{R}^n$ with center 0 and size $r > 0$ is defined by
$$Q_r = \{(x', x_n) = (x_1, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n : |x'| < r \text{ and } |x_n| < r\}.$$
4. The integral average of $g \in L^1(U)$ over a bounded domain U in $\mathbb{R}^n$ is denoted by
$$\bar{g}_U = \fint_U g(x)dx = \frac{1}{|U|}\int_U g(x)dx.$$
5. For each $x_n \in \mathbb{R}$ and each bounded subset E' of $\mathbb{R}^{n-1}$ the integral average of $g(\cdot, x_n)$ over E' is denoted by
$$\bar{g}_{E'}(x_n) = \fint_{E'} g(x', x_n)dx' = \frac{1}{|E'|}\int_{E'} g(x', x_n)dx'.$$

6. $B_r^+ = B_r \cap \{x_n > 0\}$ and $B_r^+(y) = B_r^+ + y$.
7. $T_r = B_r \cap \{x_n = 0\}$ and $T_r(y) = T_r + y$.
8. $\Omega_r = B_r \cap \Omega$ and $\Omega_r(y) = B_r(y) \cap \Omega$.
9. $\partial_w \Omega_r = B_r \cap \partial\Omega$: the wiggled part of $\partial\Omega_r$.
10. $\partial_c \Omega_r = \partial\Omega_r \backslash \partial_w \Omega_r$: the curved part of $\partial\Omega_r$.

2.2 Main assumptions

Here, we introduce some definitions related to our main assumptions.

To obtain $W^{1,p}$ regularity, we need some kinds of smallness conditions on the coefficients. First, the regularity requirement on the coefficients is that they belong to BMO space with their BMO semi-norms sufficiently small. We introduce the following definition :

Definition 2.2.1. Let U be a bounded domain in $\mathbb{R}^n$. We say that $A_{ij}^{\alpha\beta}$ is (δ, R)-*vanishing* if

$$\sup_{0<r\leq R} \sup_{y\in\mathbb{R}^n} \fint_{B_r(y)} \left| A_{ij}^{\alpha\beta}(x) - \overline{A_{ij}^{\alpha\beta}}_{B_r(y)} \right|^2 dx \leq \delta^2. \tag{2.1}$$

For the interior case, we can give a weaker condition on the coefficients than the condition in Definition 2.2.1. This condition is that $A_{ij}^{\alpha\beta}$ are allowed to be merely measurable with respect to one variable but have small BMO semi-norms with respect to the other variables.

Definition 2.2.2. We say that $A_{ij}^{\alpha\beta}$ is (δ, R)-*vanishing of codimension 1* if for every point $x_0 \in \mathbb{R}^n$ and for every number $r \in (0, R]$, there exists a coordinate system depending on x_0 and r, whose variables we still denote by $x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n)$, so that in this new coordinate system, x_0 is the origin and

$$\fint_{Q_{\sqrt{2}r}} \left| A_{ij}^{\alpha\beta}(x', x_n) - \overline{A_{ij}^{\alpha\beta}}_{B'_{\sqrt{2}r}}(x_n) \right|^2 dx \leq \delta^2. \tag{2.2}$$

We assume that the boundary of the bounded domain can be locally trapped between two hyperplanes sufficiently close.

Definition 2.2.3. Let U be a bounded domain in $\mathbb{R}^n$. We say that U is (δ, R)-*Reifenberg flat* if for every $x \in \partial\Omega$ and every $r \in (0, R]$, there exists a coordinate system $\{y_1, \ldots, y_n\}$ dependent on r and x so that $x = 0$ in this coordinate system and

$$B_r \cap \{y_n > \delta r\} \subset B_r \cap \Omega \subset B_r \cap \{y_n > -\delta r\}. \tag{2.3}$$

Remark 2.2.4. Throughout this paper we assume that δ is a small positive number since the concept of Reifenberg flatness (2.3) is only meaningful when $0 < \delta < \frac{1}{8}$, see [43]. Because our primary problems (1.1)-(1.3) have a scaling invariance property, the constant R can be 1 or any other constant greater than 1 while the constant δ is still invariant under this scaling. δ requires a small oscillation of the coefficients from being their local integral averages. At the same time it only allows locally a small deviation of $\partial\Omega$ from being $(n-1)$-dimensional hyperplanes for each sufficiently small scale $r > 0$.

Remark 2.2.5. By a change of variables, we know from Definition 2.2.1 (respectively, Definition 2.2.2) that if $A_{ij}^{\alpha\beta}$ is (δ, R)-vanishing (respectively, (δ, R)-vanishing of codimension 1), then $\tilde{A}_{ij}^{\alpha\beta}(z) = A_{ij}^{\alpha\beta}(\rho z)$ is $(\delta, \frac{R}{\rho})$-vanishing (respectively, $(\delta, \frac{R}{\rho})$-vanishing of codimension 1). Similarly from Definition 2.2.3, if Ω is (δ, R)-Reifenberg flat, then $\tilde{\Omega} = \{\frac{1}{\rho}x : x \in \Omega\}$ is $(\delta, \frac{R}{\rho})$-Reifenberg flat.

2.3 Tools

In this section, we introduce analytic and geometric tools that will be used later in the proof of our main theorem. Our approach is based on the Hardy-Littlewood maximal function, classical measure theory, and a Vitali-type covering argument.

First, let us recall the Hardy-Littlewood maximal function and its basic properties. If we suppose g is a locally integrable function on $\mathbb{R}^n$, then the Hardy-Littlewood maximal function is given by

$$(\mathcal{M}g)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y)| dy. \tag{2.4}$$

If g is defined only on a bounded subset of $\mathbb{R}^n$, then we define

$$\mathcal{M}g = \mathcal{M}\bar{g},$$

where $\bar{g}$ is the zero extension of g from the bounded set to $\mathbb{R}^n$. This maximal function satisfies the *weak 1-1 estimate* and *strong p-p estimate* as follows (see [41]) :

For $g \in L^1(\mathbb{R}^n)$, there is a constant $c = c(n) > 0$ such that

$$|\{x \in \mathbb{R}^n : (\mathcal{M}g)(x) > t\}| \leq \frac{c}{t}\|g\|_{L^1(\mathbb{R}^n)}, \quad \forall t > 0. \tag{2.5}$$

Also, given $g \in L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$, $\mathcal{M}g \in L^p(\mathbb{R}^n)$ holds with the estimate

$$\frac{1}{c}\|g\|_{L^p(\mathbb{R}^n)} \leq \|\mathcal{M}g\|_{L^p(\mathbb{R}^n)} \leq c\|g\|_{L^p(\mathbb{R}^n)} \tag{2.6}$$

for some constant $c = c(n, p) > 0$.

In order to apply it later, we need to review some classical measure theory.

Lemma 2.3.1. *[15] Assume g is a nonnegative, measurable function defined on the bounded domain $\Omega \subset \mathbb{R}^n$, and let $\theta > 0$ and $\lambda > 1$ be constants. Then for $0 < q < \infty$, we have*

$$g \in L^q(\Omega) \quad \Longleftrightarrow \quad S = \sum_{k \geq 1} \mu^{qk} \left|\left\{x \in \Omega : g(x) > \theta\mu^k\right\}\right| < \infty \tag{2.7}$$

and

$$\frac{1}{c}S \leq \|g\|^q_{L^q(\Omega)} \leq c(|\Omega| + S), \tag{2.8}$$

where the positive constant c depends only on θ, μ, and q.

In addition, we will use the following version of the Vitalli-type covering lemma for the proof of our main results.

Lemma 2.3.2. *[10, 44] Assume that C and D are measurable sets with $C \subset D \subset \Omega$ and Ω being $(\delta, 1)$-Reifenberg flat. Also assume there exists a small $\eta > 0$ such that*

$$|C| < \eta|B_1| \tag{2.9}$$

and that for each $x \in \Omega$ and $r \in (0, 1]$ with $|C \cap B_r(x)| > \eta|B_r(x)|$, we have

$$B_r(x) \cap \Omega \subset D. \tag{2.10}$$

Then

$$|C| \leq \left(\frac{10}{1-\delta}\right)^n \eta|D|. \tag{2.11}$$

Chapter 3

Interior estimates

3.1 Main result

In this chapter, we obtain uniform interior $W^{1,p}$ estimates for the problem (1.1). For this, we allow the coefficients to be merely measurable with respect to one variable but have small BMO semi-norms with respect to the other variables. This condition includes a small BMO condition which will be used in the next two chapters. Our main result in this chapter is the following :

Theorem 3.1.1. *For any constant $2 < p < \infty$, suppose $F \in L^p(\Omega, \mathbb{R}^{mn})$ and $B_7 \subset \Omega$. Then there exists a small positive constant $\delta = \delta(\lambda, \Lambda, m, n, p)$ such that if $A_{ij}^{\alpha\beta}$ is (δ, R)-vanishing of codimension 1, then for any weak solution $u_\epsilon \in H^1(\Omega, \mathbb{R}^m)$ of (1.1) we have*

$$Du_\epsilon \in L^p(B_1, \mathbb{R}^{mn}), \tag{3.1}$$

with estimate

$$\int_{B_1} |Du_\epsilon|^p \, dx \;\leq\; c \int_{B_5} |u_\epsilon|^p + |F|^p \, dx, \tag{3.2}$$

where the constant $c = c(\lambda, \Lambda, m, n, p)$ is independent of ϵ.

Remark 3.1.2. The case that $p = 2$ is a classical one. After the estimate (3.2) for $2 < p < \infty$ is obtained, the case $1 < p < 2$ follows from a duality argument.

3.2 Interior Hölder estimates

To obtain our main result in this chapter, we need boundedness of weak solutions of homogeneous systems. To do this, we first investigate interior Hölder regularity.

Theorem 3.2.1. *Let $\gamma \in (0,1)$. Suppose that $v_\epsilon \in H^1(B_r^+, \mathbb{R}^m)$ is a weak solution of*

$$D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon} D_\beta v_\epsilon^j \right) = 0 \quad \text{in } B_r. \tag{3.3}$$

Then there exists a small positive constant $\delta = \delta(\lambda, \Lambda, m, n)$ such that if $A_{ij}^{\alpha\beta}$ is (δ, R)-vanishing of codimension 1, then for any $x, y \in B_{\frac{r}{2}}$,

$$|v_\epsilon(x) - v_\epsilon(y)| \le c \left(\frac{|x-y|}{r} \right)^\gamma \left(\fint_{B_r} |v_\epsilon(z)|^2 dz \right)^{\frac{1}{2}}, \tag{3.4}$$

where $c > 0$ depends only on $\lambda, \Lambda, m, n,$ and γ.

The following two lemmas will be used for the proof of Theorem 3.2.1.

Lemma 3.2.2. *Let $\gamma \in (0,1)$. Then there exists $\epsilon_0 \in (0,1]$ and $\theta \in \left(0, \frac{1}{4}\right)$ depending only on $\lambda, \Lambda, m, n,$ and γ such that if for $0 < \epsilon < \epsilon_0$, v_ϵ is a weak solution of*

$$D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon} D_\beta v_\epsilon^j \right) = 0 \quad \text{in } B_1, \tag{3.5}$$

with

$$\fint_{B_1} |v_\epsilon - (\bar{v}_\epsilon)_{B_1}|^2 dx \le 1, \tag{3.6}$$

then

$$\fint_{B_\theta} |v_\epsilon - (\bar{v}_\epsilon)_{B_\theta}|^2 dx \le \theta^{2\gamma}. \tag{3.7}$$

Proof. We will prove this lemma by contradiction. If not, then there exists sequences ϵ_k, and v_{ϵ_k} such that $\epsilon_k \to 0$, v_{ϵ_k} is a weak solution of

$$D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon_k} D_\beta v_{\epsilon_k}^j \right) = 0 \quad \text{in } B_1, \tag{3.8}$$

with

$$\fint_{B_1} |v_{\epsilon_k} - (\bar{v}_{\epsilon_k})_{B_1}|^2 dx \le 1, \tag{3.9}$$

but for every $\theta \in (0, \frac{1}{4})$,

$$\fint_{B_\theta} |v_{\epsilon_k} - (\bar{v}_{\epsilon_k})_{B_\theta}|^2 dx > \theta^{2\gamma}. \tag{3.10}$$

By subtracting a constant, we assume that $(\bar{v}_{\epsilon_k})_{B_1} = 0$. Then from Caccioppoli inequality for (3.8) and (3.9), we have

$$\int_{B_{\frac{1}{2}}} |Dv_{\epsilon_k}|^2 dx \le c \int_{B_1} |v_{\epsilon_k}|^2 dx \le c. \tag{3.11}$$

Thus v_{ϵ_k} is uniformly bounded in $H^1(B_{\frac{1}{2}})$, and then by passing to a subsequence, we assume that $v_{\epsilon_k} \to v_0$ strongly in $L^2(B_{\frac{1}{2}})$ for some $v_\epsilon \in H^1(B_{\frac{1}{2}})$. Consequently we have that for any $\theta \in (0, \frac{1}{4})$,

$$\int_{B_\theta} |v_{\epsilon_k} - (\bar{v}_{\epsilon_k})_{B_r}|^2 dx \to \int_{B_\theta} |v_0 - (\bar{v}_0)_{B_\theta}|^2 dx, \tag{3.12}$$

and so from (3.10), we find that for every $\theta \in (0, \frac{1}{4})$,

$$\fint_{B_\theta} |v_0 - (\bar{v}_0)_{B_\theta}|^2 dx > \theta^{2\gamma}. \tag{3.13}$$

In addition, recalling (3.8) and existing homogenization theory as in [3], we see that v_0 solves

$$D_\alpha \left(A_{ij}^{\alpha\beta,0} D_\beta v_0^j \right) = 0 \quad \text{in } B_{\frac{1}{2}} \tag{3.14}$$

where $A_{ij}^{\alpha\beta,0}$ is the constant matrix defined as in (1.13). According to interior Hölder regularity for solutions of elliptic systems with constant coefficients, we discover that

$$\fint_{B_\theta} |v_0 - (\bar{v}_0)_{B_\theta}|^2 dx \le c_1 \theta^{1+\gamma}, \tag{3.15}$$

for some universal constant $c_1 = c_1(\lambda, \Lambda, m, n, \gamma)$.

We finally combine (3.13) and (3.15), to discover

$$\theta^{2\gamma} < \fint_{B_\theta} |v_0 - (\bar{v}_0)_{B_\theta^+}|^2 dx \le c_2 \theta^{1+\gamma} \tag{3.16}$$

for every $\gamma \in (0, 1)$ and every $\theta \in (0, \frac{1}{4})$. However, we take $\theta \in (0, \frac{1}{4})$ so small to deduce

$$\theta^{2\gamma} \ge c_1 \theta^{1+\gamma},$$

which contradicts (3.16). This finishes the proof. □

Lemma 3.2.3. *Fix $\gamma \in (0,1)$. Let ϵ_0 and θ be the constants as in Lemma 3.2.2 and let v_ϵ be a weak solution of (3.5). Then for all k such that $\epsilon < \theta^{k-1}\epsilon_0$, we have*

$$\fint_{B_{\theta^k}} |v_\epsilon - (\bar{v}_\epsilon)_{B_{\theta^k}}|^2 dx \le \theta^{2k\gamma} \fint_{B_1} |v_\epsilon - (\bar{v}_\epsilon)_{B_1}|^2 dx. \tag{3.17}$$

Proof. The proof is by induction on k. By Lemma 3.2.2, (3.17) holds for $k = 1$. Now we assume that (3.17) holds for some $k \ge 1$. Let

$$w(z) = \frac{v_\epsilon(\theta^k z)}{\left(\fint_{B_{\theta^k}} |v_\epsilon - (\bar{v}_\epsilon)_{B_{\theta^k}}|^2 dx\right)^{\frac{1}{2}}} \quad \text{for } z \in B_1 \tag{3.18}$$

(We divide $v_\epsilon(\theta^k z)$ into $\left(\fint_{B_{\theta^k}} |v_\epsilon - (\bar{v}_\epsilon)_{B_{\theta^k}}|^2 dx\right)^{\frac{1}{2}} + \sigma$ for any $\sigma > 0$ and then we let $\sigma \to 0^+$ if $\fint_{B_{\theta^k}} |v_\epsilon - (\bar{v}_\epsilon)_{B_{\theta^k}}|^2 dx = 0$). Then w satisfies

$$D_\alpha \left(A_{ij}^{\alpha\beta, \frac{\epsilon}{\theta^k}} D_\beta w^j \right) = 0 \quad \text{in } B_1 \tag{3.19}$$

with

$$\fint_{B_1} |w - \bar{w}_{B_1}|^2 dz \le 1. \tag{3.20}$$

Thus by applying Lemma 3.2.2 again to w, we obtain

$$\fint_{B_\theta} |w - \bar{w}_{B_\theta}|^2 dz \le \theta^{2\gamma}. \tag{3.21}$$

Then by the induction hypothesis we find that

$$\begin{aligned}\fint_{B_{\theta^{k+1}}} |v_\epsilon - (\bar{v}_\epsilon)_{B_{\theta^{k+1}}}|^2 dx &= \left(\fint_{B_\theta} |w - \bar{w}_{B_\theta}|^2 dz\right) \left(\fint_{B_{\theta^k}} |v_\epsilon - (\bar{v}_\epsilon)_{B_{\theta^k}}|^2 dx\right) \\ &\le \theta^{2\gamma} \fint_{B_{\theta^k}} |v_\epsilon - (\bar{v}_\epsilon)_{B_{\theta^k}}|^2 dx \\ &\le \theta^{2(k+1)\gamma} \fint_{B_1} |v_\epsilon - (\bar{v}_\epsilon)_{B_1}|^2 dx.\end{aligned}$$

This completes the proof. □

Remark 3.2.4. Before giving the proof of Theorem 3.2.1, we would like to point out that in the paper [9], $W^{1,p}$ regularity for a weak solution to (3.3) with $\epsilon = 1$ was established for all $1 < p < \infty$ where the coefficients $A_{ij}^{\alpha\beta}$ are assumed to be (δ, R)-vanishing of codimension 1. From this, we know that the equation (3.3) with $\epsilon = 1$ has $C^{0,\gamma}$ regularity for any fixed $\gamma \in (0,1)$ as a consequence of Morrey embedding for p large enough.

Proof of Theorem 3.2.1. Let ϵ_0 and θ be constants given in Lemma 3.2.2. By scaling, we may assume that $r = 1$. The case $\epsilon \geq \theta\epsilon_0$ follows from Remark 3.2.4 with an appropriate scaling.

We next consider $0 < \epsilon < \theta\epsilon_0$. We divide this into two cases, $\rho \geq \frac{\epsilon}{\epsilon_0}$ and $\rho < \frac{\epsilon}{\epsilon_0}$. For the first case, we can take $k \geq 0$ such that $\theta^{k+1} \leq \rho < \theta^k$. Since $\epsilon \leq \theta^k \epsilon_0$, we apply Lemma 3.2.3 to find that

$$
\begin{aligned}
\fint_{B_\rho} |v_\epsilon - (\bar{v}_\epsilon)_{B_r}|^2 dx &\leq c \fint_{B_{\theta^k}} |v_\epsilon - (\bar{v}_\epsilon)_{B_{\theta^k}}|^2 dx \\
&\leq c\theta^{2k\gamma} \fint_{B_1} |v_\epsilon - (\bar{v}_\epsilon)_{B_1}|^2 dx \\
&\leq c\rho^{2\gamma} \fint_{B_1} |v_\epsilon|^2 dx.
\end{aligned}
$$

For the second one, we use a blow-up argument by letting $w(z) = v_\epsilon(\epsilon z)$. Since $\frac{2}{\epsilon_0} < \frac{1}{\theta\epsilon_0} < \frac{1}{\epsilon}$, w satisfies

$$
D_\alpha \left(A_{ij}^{\alpha\beta,1} D_\beta w^j \right) = 0 \quad \text{in } B_{\frac{2}{\epsilon_0}}. \tag{3.22}
$$

By the $C^{0,\gamma}$ regularity for (3.22), we see that

$$
\fint_{B_{\frac{\rho}{\epsilon}}} |w - \bar{w}_{B_{\frac{\rho}{\epsilon}}}|^2 dz \leq c \left(\frac{\rho}{\epsilon} \right)^{2\gamma} \fint_{B_{\frac{1}{\epsilon_0}}} |w - \bar{w}_{B_{\frac{1}{\epsilon_0}}}|^2 dz \tag{3.23}
$$

for some constant $c = c(\gamma, \lambda, \Lambda, m, n)$. Since $\frac{\epsilon}{\epsilon_0} < \theta$, we apply Lemma 3.2.3

again to find that

$$
\begin{aligned}
\fint_{B_\rho} |v_\epsilon - (\bar{v}_\epsilon)_{B_\rho}|^2 dx &= \fint_{B_{\frac{\rho}{\epsilon}}} |w - \bar{w}_{B_{\frac{\rho}{\epsilon}}}|^2 dz \\
&\leq c\left(\frac{\rho}{\epsilon}\right)^{2\gamma} \fint_{B_{\frac{1}{\epsilon_0}}} |w - \bar{w}_{B_{\frac{1}{\epsilon_0}}}|^2 dz \\
&\leq c\left(\frac{\rho}{\epsilon}\right)^{2\gamma} \fint_{B_{\frac{\epsilon}{\epsilon_0}}} |v_\epsilon - (\bar{v}_\epsilon)_{B_{\frac{\epsilon}{\epsilon_0}}}|^2 dx \\
&\leq c\left(\frac{\rho}{\epsilon}\right)^{2\gamma}\left(\frac{\epsilon}{\epsilon_0}\right)^{2\gamma} \fint_{B_1} |v_\epsilon - (\bar{v}_\epsilon)_{B_1}|^2 dx \\
&\leq c\rho^{2\gamma} \fint_{B_1} |v_\epsilon|^2 dx.
\end{aligned} \tag{3.24}
$$

This completes the proof of Theorem 3.2.1. □

3.3 Uniform $W^{1,q}$ estimates for homogeneous systems

We first recall the local boundedness of weak solutions of

$$D_\alpha(A_{ij}^{\alpha\beta,\epsilon} D_\beta v_\epsilon^j) = 0 \quad \text{in } B_3 \tag{3.25}$$

with the estimate

$$\|v_\epsilon\|_{L^\infty(B_1)} \leq c\left(\fint_{B_2} |v_\epsilon|^2 dx\right)^{\frac{1}{2}} \tag{3.26}$$

for some constant c which is independent of ϵ from the result of the previous section. Also, by scaling the problem (1.14), we see from (1.1) that for the identity matrix I in $\mathbb{R}^m$ and each constant matrix $B \in \mathbb{R}^{mn}$

$$D_\alpha\left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta\left((I\otimes x)_\gamma^{jk} + \epsilon\chi_\gamma^{jk}\left(\frac{x}{\epsilon}\right)\right) B_\gamma^k\right) = 0. \tag{3.27}$$

For simplicity, in this section, we use the notation $\left(x + \epsilon\chi\left(\frac{x}{\epsilon}\right)\right)B$ instead of $\left((I\otimes x)_\gamma^{jk} + \epsilon\chi_\gamma^{jk}\left(\frac{x}{\epsilon}\right)\right)B_\gamma^k$.

To obtain our main result, we need to control the case that ϵ is sufficiently small. The following lemma gives us a criterion of sufficient smallness of ϵ, which was previously proved in [1, 34] by a compactness argument.

Lemma 3.3.1. *Let $v_\epsilon \in H^1(B_3, \mathbb{R}^m)$ be a weak solution of*

$$D_\alpha(A_{ij}^{\alpha\beta,\epsilon} D_\beta v_\epsilon^j) = 0 \quad \text{in } B_3. \tag{3.28}$$

Then there exist constants $\theta \in (0, \frac{1}{4})$ and $\epsilon_0 \in (0,1)$ both depending on λ, Λ, m, n such that for $0 < \epsilon < \epsilon_0$

$$\sup_{x\in B_\theta} \left| v_\epsilon(x) - v_\epsilon(0) - \left(x + \epsilon\chi\left(\frac{x}{\epsilon}\right)\right) \overline{Dv_\epsilon}_{B_\theta} \right| \leq \theta^{\frac{5}{4}} \|v_\epsilon\|_{L^\infty(B_1)}. \tag{3.29}$$

Proof. We will prove this lemma by contradiction. Without loss of generality, we assume that

$$\|v_\epsilon\|_{L^\infty(B_1)} \leq 1.$$

If not, then there exists sequences ϵ_k, and v_{ϵ_k} such that $\epsilon_k \to 0$, v_{ϵ_k} is a weak solution of

$$D_\alpha\left(A_{ij}^{\alpha\beta,\epsilon_k} D_\beta v_{\epsilon_k}^j\right) = 0 \quad \text{in } B_1, \tag{3.30}$$

with

$$\|v_{\epsilon_k}\|_{L^\infty(B_1)} \leq 1 \tag{3.31}$$

but

$$\sup_{x\in B_\theta} \left| v_{\epsilon_k}(x) - v_{\epsilon_k}(0) - \left(x + \epsilon_k\chi\left(\frac{x}{\epsilon_k}\right)\right) \overline{Dv_{\epsilon_k}}_{B_\theta} \right| > \theta^{\frac{5}{4}}. \tag{3.32}$$

As the proof of Lemma 3.2.2, v_{ϵ_k} is uniformly bounded in $H^1(B_{\frac{1}{2}})$, and then by passing to a subsequence, we assume that

$$\begin{cases} Dv_{\epsilon_k} \rightharpoonup Dv_0 & \text{weakly in} \quad L^2(B_{\frac{1}{2}}, \mathbb{R}^{mn}) \\ v_{\epsilon_k} \to v_0 & \text{strongly in} \quad L^2(B_{\frac{1}{2}}, \mathbb{R}^m) \end{cases} \tag{3.33}$$

as $k \to \infty$. Since χ is bounded in $L^\infty(\mathbb{R}^n)$, see (1.15), $\epsilon_k \to 0$ and for $\theta \in (0, \frac{1}{4})$

$$\overline{Dv_{\epsilon_k}}_{B_\theta} \to \overline{Dv_0}_{B_\theta}$$

as $k \to \infty$ by (3.33), we obtain

$$\sup_{x\in B_\theta} \left| v_0(x) - v_0(0) - x\overline{Dv_0}_{B_\theta} \right| > \theta^{\frac{5}{4}}. \tag{3.34}$$

In addition, recalling (3.30) and existing homogenization theory as in [3], we see that v_0 solves

$$D_\alpha\left(A_{ij}^{\alpha\beta,0} D_\beta v_0^j\right) = 0 \quad \text{in } B_{\frac{1}{2}} \tag{3.35}$$

where $A_{ij}^{\alpha\beta,0}$ is the constant matrix defined as in (1.16). According to the theory for elliptic systems with constant coefficients, we discover that

$$\sup_{x\in B_\theta}\left|v_0(x)-v_0(0)-x\overline{Dv_0}_{B_\theta}\right|\le c_2\theta^2\left(\fint_{B_{\frac{1}{2}}}|v_0|^2dx\right)^{\frac{1}{2}}, \tag{3.36}$$

for some universal constant $c_2=c_2(\lambda,\Lambda,m,n)$.

We finally combine (3.34) and (3.36) to have

$$\theta^{\frac{5}{4}}<\sup_{x\in B_\theta}\left|v_0(x)-v_0(0)-x\overline{Dv_0}_{B_\theta}\right|\le c_2\theta^2\left(\fint_{B_{\frac{1}{2}}}|v_0|^2dx\right)^{\frac{1}{2}}\le c_2\theta^2. \tag{3.37}$$

However, we take $\theta\in(0,\frac{1}{4})$ so small to deduce

$$\theta^{\frac{5}{4}}\ge c_2\theta^2, \tag{3.38}$$

which contradicts (3.37). This finishes the proof. □

Hereafter we fix the universal constants θ and ϵ_0 given in Lemma 3.3.1. Based on this lemma, we deal with (1.1) for $\epsilon\ge\theta\epsilon_0$ and $\epsilon<\theta\epsilon_0$ in two different ways.

We first consider the case $\epsilon\ge\theta\epsilon_0$. In this case, we define

$$\begin{cases}\frac{1}{\epsilon}\Omega &=& \{\frac{1}{\epsilon}x:x\in\Omega\}, & \\ \tilde{u}_\epsilon(\tilde{x}) &=& \frac{u_\epsilon(\epsilon\tilde{x})}{\epsilon} & (\tilde{x}\in\frac{1}{\epsilon}\Omega), \\ \tilde{f}^i_\alpha(\tilde{x}) &=& f^i_\alpha(\epsilon\tilde{x}) & (\tilde{x}\in\frac{1}{\epsilon}\Omega), \\ \tilde{A}_{ij}^{\alpha\beta,\epsilon}(\tilde{x}) &=& A_{ij}^{\alpha\beta,\epsilon}(\epsilon\tilde{x})=A_{ij}^{\alpha\beta}(\tilde{x}) & (\tilde{x}\in\mathbb{R}^n).\end{cases} \tag{3.39}$$

Then, $\tilde{u}_\epsilon\in H^1(\frac{1}{\epsilon}\Omega,\mathbb{R}^n)$ is a weak solution of

$$D_\alpha(A_{ij}^{\alpha\beta,1}(\tilde{x})D_\beta\tilde{u}^j_\epsilon(\tilde{x}))=D_\alpha\tilde{f}^i_\alpha(\tilde{x})\qquad\text{in}\quad\frac{1}{\epsilon}\Omega. \tag{3.40}$$

According to the previous known results in [8] and [9], there exists a small positive constant $\delta=\delta(\lambda,\Lambda,m,n,p)$ such that if $A_{ij}^{\alpha\beta}$ is $(\delta,5)$-vanishing of codimension 1, then for any weak solution $\tilde{u}_\epsilon\in H^1(\frac{1}{\epsilon}\Omega,\mathbb{R}^n)$ of (3.40) with $B_7\subset\frac{1}{\epsilon}\Omega$, we have

$$\fint_{B_1}|D\tilde{u}_\epsilon|^pd\tilde{x}\le c\fint_{B_5}|\tilde{u}_\epsilon|^p+|\tilde{F}|^pd\tilde{x}$$

for some constant $c = c(\lambda, \Lambda, m, n, p)$. Rescale back and use $\theta\epsilon_0 \le \epsilon \le 1$ to find that

$$\begin{aligned}
\fint_{B_{\theta\epsilon_0}} |Du_\epsilon|^p dx &\le c \fint_{B_\epsilon} |Du_\epsilon|^p dx \\
&\le c \frac{1}{|B_{5\epsilon}|} \int_{B_{5\epsilon}} \frac{1}{\epsilon} |u_\epsilon|^p + |F|^p dx \\
&\le c \frac{|B_5|}{|B_{5\theta\epsilon_0}|} \fint_{B_5} \frac{1}{\theta\epsilon_0} |u_\epsilon|^p + |F|^p dx \\
&\le c \fint_{B_5} |u_\epsilon|^p + |F|^p dx
\end{aligned}$$

for some constant $c = c(\lambda, \Lambda, m, n, p)$. Then by standard covering argument , we get the required estimate (3.1).

From now we only consider the case $\epsilon < \theta\epsilon_0$. The following lemma comes from Lemma 3.3.1 by an iteration argument.

Lemma 3.3.2. *Let v_ϵ be a weak solution of (3.25). Then for all k with $\epsilon < \theta^k \epsilon_0$, there exist constants $a_k^\epsilon \in \mathbb{R}^n$ and $B_k^\epsilon \in \mathbb{R}^{mn}$ such that*

$$|a_k^\epsilon| + |B_k^\epsilon| \le c \|v_\epsilon\|_{L^\infty(B_1)} \tag{3.41}$$

for some constant $c = c(\lambda, \Lambda, m, n)$ and

$$\sup_{x \in B_{\theta^k}} \left| v_\epsilon(x) - v_\epsilon(0) - \epsilon a_k^\epsilon - \left(x + \epsilon \chi\left(\frac{x}{\epsilon}\right)\right) B_k^\epsilon \right| \le \theta^{\frac{5}{4}k} \|v_\epsilon\|_{L^\infty(B_1)}. \tag{3.42}$$

Proof. The proof is by induction on k. By Lemma 3.3.1, for $k = 1$, $a_1^\epsilon = 0$ and $B_1^\epsilon = \overline{Dv_\epsilon}_{B_\theta}$. Then by Caccioppoli inequality we see that

$$|B_1^\epsilon| \le \left(\fint_{B_\theta} |Dv_\epsilon|^2 dx \right)^{\frac{1}{2}} \le \frac{c}{1-\theta} \left(\fint_{B_1} |v_\epsilon|^2 dx \right)^{\frac{1}{2}} \le c \|v_\epsilon\|_{L^\infty(B_1)} \tag{3.43}$$

for some constant $c = c(\lambda, \Lambda, m, n)$. Thus, this holds for $k = 1$.

Now, we assume that (3.42) holds for some $k \ge 1$. Let

$$w(z) = v_\epsilon(\theta^k z) - v_\epsilon(0) - \epsilon a_k^\epsilon - \left(\theta^k z + \epsilon \chi\left(\frac{\theta^k z}{\epsilon}\right)\right) B_k^\epsilon \text{ for } z \in B_1. \tag{3.44}$$

Then w satisfies

$$D_\alpha \left(A_{ij}^{\alpha\beta, \frac{\epsilon}{\theta^k}} D_\beta w^j \right) = 0 \quad \text{in } B_1. \tag{3.45}$$

Thus by applying Lemma 3.3.1 again to w, we obtain

$$\sup_{x\in B_\theta}\left|w(x)-w(0)-\left(x+\epsilon\chi\left(\frac{x}{\epsilon}\right)\right)\overline{Dw}_{B_\theta}\right|\le\theta^{\frac{5}{4}}\|w\|_{L^\infty(B_1)}. \tag{3.46}$$

In addition, by the induction hypothesis, we find that

$$\begin{aligned}\|w\|_{L^\infty(B_1)} &= \sup_{x\in B_{\theta^k}}\left|v_\epsilon(x)-v_\epsilon(0)-\epsilon a_k^\epsilon-\left(x+\epsilon\chi\left(\frac{x}{\epsilon}\right)\right)B_k^\epsilon\right| \\ &\le \theta^{\frac{5}{4}k}\|v_\epsilon\|_{L^\infty(B_1)}.\end{aligned} \tag{3.47}$$

Now, we combine (3.44), (3.46), and (3.47) to find that

$$\begin{aligned}\sup_{x\in B_{\theta^{k+1}}}\Big|v_\epsilon(x)-v_\epsilon(0)-\epsilon\chi(0)B_k^\epsilon \\ -\left(x+\epsilon\chi\left(\frac{x}{\epsilon}\right)\right)\left(B_k^\epsilon+\theta^{-k}\overline{Dw}_{B_\theta}\right)\Big| \\ \le\theta^{\frac{5}{4}(k+1)}\|v_\epsilon\|_{L^\infty(B_1)}.\end{aligned} \tag{3.48}$$

Here we use, for simplicity, the expression w in (3.44). Therefore, a_{k+1}^ϵ and B_{k+1}^ϵ are inductively defined by

$$a_{k+1}^\epsilon=\chi(0)B_k^\epsilon \tag{3.49}$$

and

$$B_{k+1}^\epsilon=B_k^\epsilon+\theta^{-k}\overline{Dw}_{B_\theta}, \tag{3.50}$$

respectively.

Finally, we need to chech that a_{k+1}^ϵ and B_{k+1}^ϵ satisfy the condition (3.41). For a_{k+1}^ϵ, since χ is bounded in $L^\infty(\mathbb{R}^n)$, see (1.15) and $|B_k^\epsilon|\le c\|v_\epsilon\|_{L^\infty(B_1)}$ by the induction hypothesis, we see that

$$|a_{k+1}^\epsilon|\le c\|v_\epsilon\|_{L^\infty(B_1)}. \tag{3.51}$$

To compute B_{k+1}^ϵ, we use Caccioppoli inequality as in (3.43) and (3.47) to find that

$$|\overline{Dw}_{B_\theta}|\le c\|w\|_{L^\infty(B_1)}\le c\theta^{\frac{5}{4}k}\|v_\epsilon\|_{L^\infty(B_1)}, \tag{3.52}$$

for some constant $c=c(\lambda,\Lambda,m,n)$. Therefore, by the induction hypothesis, we have

$$\begin{aligned}|B_{k+1}^\epsilon| &\le |B_k^\epsilon|+c\theta^{\frac{1}{4}k}\|v_\epsilon\|_{L^\infty(B_1)} \\ &\le c\left(1+\theta^{\frac{1}{4}}+\cdots+\theta^{\frac{1}{4}k}\right)\|v_\epsilon\|_{L^\infty(B_1)} \\ &\le c\|v_\epsilon\|_{L^\infty(B_1)}\end{aligned} \tag{3.53}$$

since $\theta \in (0, \frac{1}{4})$. This completes the proof.

$\square$

According to Lemma 3.3.2, one can derive if $A_{ij}^{\alpha\beta}$ is $(\delta, 5)$-vanishing of codimension 1, then

$$\sup_{x \in B_{\frac{\epsilon}{\epsilon_0}}} \frac{|v_\epsilon(x) - v_\epsilon(0)|}{\epsilon} \leq c\|v_\epsilon\|_{L^\infty(B_1)}. \tag{3.54}$$

Indeed, choose k such that $\theta^{k+1}\epsilon_0 \leq \epsilon < \theta^k \epsilon_0$, then

$$\begin{aligned}
&\sup_{x \in B_{\frac{\epsilon}{\epsilon_0}}} \left(\frac{|v_\epsilon(x) - v_\epsilon(0)|}{\epsilon} - \frac{\left|\epsilon a_k^\epsilon + \left(x + \epsilon\chi\left(\frac{x}{\epsilon}\right)\right) B_k^\epsilon\right|}{\epsilon} \right) \\
&\qquad \leq \sup_{x \in B_{\theta^k}} \frac{\left|v_\epsilon(x) - v_\epsilon(0) - \epsilon a_k^\epsilon - \left(x + \epsilon\chi\left(\frac{x}{\epsilon}\right)\right) B_k^\epsilon\right|}{\epsilon} \\
&\qquad \leq \frac{\theta^{\frac{5}{4}k}}{\epsilon} \|v_\epsilon\|_{L^\infty(B_1)} \leq \frac{1}{\theta\epsilon_0} \|w_\epsilon\|_{L^\infty(B_1)}
\end{aligned}$$

since $\frac{\theta^{\frac{5}{4}k}}{\epsilon} \leq \frac{\theta^{\frac{1}{4}k}}{\theta\epsilon_0} \leq \frac{1}{\theta\epsilon_0}$. Thus, for $x \in B_{\frac{\epsilon}{\epsilon_0}}$,

$$\begin{aligned}
\frac{\left|\epsilon a_k^\epsilon + \left(x + \epsilon\chi\left(\frac{x}{\epsilon}\right)\right) B_k^\epsilon\right|}{\epsilon} &\leq |a_k^\epsilon| + \left|\frac{x}{\epsilon}\right| |B_k^\epsilon| + \left|\chi\left(\frac{x}{\epsilon}\right)\right| |B_k^\epsilon| \\
&\leq |a_k^\epsilon| + \frac{1}{\epsilon_0} |B_k^\epsilon| + \left|\chi\left(\frac{x}{\epsilon}\right)\right| |B_k^\epsilon|.
\end{aligned}$$

We recall (1.15) and use (3.41) to find that

$$\frac{\left|\epsilon a_k^\epsilon + \left(x + \epsilon\chi\left(\frac{x}{\epsilon}\right)\right) B_k^\epsilon\right|}{\epsilon} \leq c\|v_\epsilon\|_{L^\infty(B_1)},$$

then (3.54) follows.

For the case $0 < \epsilon < \theta\epsilon_0$, we need the following uniform regularity estimate of (3.28). We here point out that it is important that the following lemma holds for any $2 < q < \infty$.

Lemma 3.3.3. *Given any ϵ with $0 < \epsilon < \theta\epsilon_0$, let v_ϵ be a weak solution of (3.25). Then for any $2 < q < \infty$, there exists $\delta = \delta(\lambda, \Lambda, m, n, q)$ such that if $A_{ij}^{\alpha\beta}$ is $(\delta, 5)$-vanishing of codimension 1, then we have*

$$|Dv_\epsilon| \in L^q(B_1, \mathbb{R}^m)$$

with the estimate

$$\left(\fint_{B_1} |Dv_\epsilon|^q dx\right)^{\frac{1}{q}} \le c \left(\fint_{B_3} |Dv_\epsilon|^2 dx\right)^{\frac{1}{2}} \tag{3.55}$$

for some positive constant $c = c(\lambda, \Lambda, m, n, q)$*, independent of* ϵ*.*

Proof. Fix any ϵ with $0 < \epsilon < \theta\epsilon_0$ and any q with $2 < q < \infty$. Without loss of generality, we assume that $\overline{v_\epsilon}_{B_3} = 0$. Define $v(x) = \frac{1}{\epsilon} v_\epsilon(\epsilon x)$, $x \in B_{\frac{3}{\epsilon}}$, then one can readily check that $v \in H^1(B_{\frac{1}{\epsilon}}, \mathbb{R}^m)$ is a weak solution of

$$D_\alpha(A_{ij}^{\alpha\beta,1} D_\beta v^j) = 0 \quad \text{in } B_{\frac{1}{\epsilon}}. \tag{3.56}$$

In particular, since $\epsilon < \theta\epsilon_0$, $\frac{1}{\epsilon} > \frac{1}{\theta\epsilon_0} > \frac{4}{\epsilon_0} > \frac{2}{\epsilon_0}$ and we have

$$D_\alpha(A_{ij}^{\alpha\beta,1} D_\beta v^j) \ = 0 \quad \text{in } B_{\frac{2}{\epsilon_0}}. \tag{3.57}$$

By interior $W^{1,q}$ estimate (see [8, 9]) for (3.57), there exists $\delta = \delta(\lambda, \Lambda, m, n, q)$ such that if $A_{ij}^{\alpha\beta}$ is $(\delta, 5)$-vanishing of codimension 1, then we have

$$\left(\fint_{B_{\frac{1}{4\epsilon_0}}} |Dv|^q dx\right)^{\frac{1}{q}} \le c \left(\fint_{B_{\frac{1}{2\epsilon_0}}} |Dv|^2 dx\right)^{\frac{1}{2}} \tag{3.58}$$

for some positive constant $c = c(\lambda, \Lambda, m, n, q)$. Let $\zeta \in C_0^1(Q_{\frac{1}{\epsilon_0}})$ be a cutoff function with $|D\zeta| \le 2c\epsilon_0$, then by the Caccioppoli inequality we have

$$\begin{aligned} \left(\fint_{B_{\frac{1}{2\epsilon_0}}} |Dv|^2 dx\right)^{\frac{1}{2}} &\le \left(\fint_{B_{\frac{1}{\epsilon_0}}} \zeta^2 |Dv|^2 dx\right)^{\frac{1}{2}} \\ &\le c \left(\fint_{B_{\frac{1}{\epsilon_0}}} |D\zeta|^2 |v(x) - v(0)|^2 dx\right)^{\frac{1}{2}} \\ &\le c \sup_{x \in B_{\frac{1}{\epsilon_0}}} |v(x) - v(0)| \end{aligned} \tag{3.59}$$

for some constant $c = c(\lambda, \Lambda, m, n)$. We then rescale back and use (3.54), (3.58), and (3.59) to find that

$$\left(\fint_{B_{\frac{\epsilon}{4\epsilon_0}}} |Dv_\epsilon|^q dx\right)^{\frac{1}{q}} \leq c \sup_{x\in B_{\frac{\epsilon}{\epsilon_0}}} \frac{|v_\epsilon(x) - v_\epsilon(0)|}{\epsilon} \tag{3.60}$$

$$\leq c\|v_\epsilon\|_{L^\infty(B_1)} \leq c\left(\fint_{B_2} |v_\epsilon|^2 dx\right)^{\frac{1}{2}}$$

for some constant $c = c(\lambda, \Lambda, m, n, q)$ where for the last inequality we use (3.26). Note that $\frac{\epsilon}{4\epsilon_0} < \frac{1}{16}$, we then apply (3.60) to each $y \in B_1$ to find the following estimate

$$\left(\fint_{B_{\frac{\epsilon}{4\epsilon_0}}(y)} |Dv_\epsilon|^q dx\right)^{\frac{1}{q}} \leq c\left(\fint_{B_2(y)} |v_\epsilon|^2 dx\right)^{\frac{1}{2}}.$$

By the standard covering argument and Poincaré inequality, we get the required estimate (3.55). That is, by choosing y_k for $k = 1, \cdots, l$ appropriately such that $\{B_{\frac{\epsilon}{4\epsilon_0}}(y_k)\}_{k=1}^l$ covers B_1 and $l = c(n)\left(\frac{4\epsilon_0}{\epsilon}\right)^n$, we have

$$\begin{aligned}
\fint_{B_1} |Dv_\epsilon|^q dx &\leq c\int_{\cup_{k=1}^l B_{\frac{\epsilon}{4\epsilon_0}}(y_k)} |Dv_\epsilon|^q dx \\
&\leq c\sum_{k=1}^l \int_{B_{\frac{\epsilon}{4\epsilon_0}}(y_k)} |Dv_\epsilon|^q dx \\
&\leq c\sum_{k=1}^l \left|B_{\frac{\epsilon}{4\epsilon_0}}\right| c\left(\fint_{B_3} |v_\epsilon|^2 dx\right)^{\frac{q}{2}} \\
&\leq c\left(\fint_{B_3} |v_\epsilon|^2 dx\right)^{\frac{q}{2}},
\end{aligned}$$

and hence

$$\left(\fint_{B_1} |Dv_\epsilon|^q dx\right)^{\frac{1}{q}} \leq c\left(\fint_{B_3} |v_\epsilon|^2 dx\right)^{\frac{1}{2}} \leq c\left(\fint_{B_3} |Dv_\epsilon|^2 dx\right)^{\frac{1}{2}} \tag{3.61}$$

by the Poincaré inequality. This completes the proof.

□

3.4 Proof of Theorem 3.1.1

Now, we ready to prove the following lemma which is a key ingredient in our approach.

Lemma 3.4.1. *Let $2 < p < \infty$. Suppose that $u_\epsilon \in H^1(\Omega, \mathbb{R}^m)$ is a weak solution of (1.1). Then there exists a universal comstant $\eta = \eta(\lambda, \Lambda, m, n, p)$ so that one can select a small $\delta = \delta(\lambda, \Lambda, m, n, p) > 0$ such that if $A_{ij}^{\alpha\beta}$ is $(\delta, 5)$-vanishing of codimension 1 and if for all $y \in \Omega$ and for every $0 < r \leq 1$ with $B_{7r}(y) \subset \Omega$, $B_r(y)$ satisfies*

$$\left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^2\right\} \cap B_r(y)\right| > \eta\,|B_r(y)|\,, \tag{3.62}$$

where

$$\left(\frac{80}{7}\right)^n N^p \eta = \frac{1}{2}, \tag{3.63}$$

then there holds

$$B_r(y) \subset \left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\} \cup \left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2\right\}. \tag{3.64}$$

Proof. Since the problem (1.1) is invariant under scaling and translation, it suffices to prove this lemma for B_1. We prove it by contradiction. Assume that (3.62) and (3.63) hold but (3.64) is false. Then there is a point $x_1 \in B_1$ such that

$$\frac{1}{|B_\rho(x_1)|}\int_{B_\rho(x_1)\cap\Omega} |Du_\epsilon|^2 dx \leq 1 \quad \text{and} \quad \frac{1}{|B_\rho(x_1)|}\int_{B_\rho(x_1)\cap\Omega} |F|^2 dx \leq \delta^2 \tag{3.65}$$

for all $\rho > 0$. Since $x_1 \in B_1$, we see that

$$B_5 \subset B_6(x_1) \subset B_7 \subset \Omega. \tag{3.66}$$

Then a direct computation and (3.65) yield

$$\fint_{B_5} |Du_\epsilon|^2 dx \leq \frac{|B_6(x_1)|}{|B_5|}\fint_{B_6(x_1)} |Du_\epsilon|^2 dx \leq c. \tag{3.67}$$

Similarly, we have

$$\fint_{B_5} |F|^2 dx \leq c\delta^2. \tag{3.68}$$

Let $v_\epsilon \in H^1(B_4, \mathbb{R}^m)$ be the weak solution of

$$\begin{cases} D_\alpha\left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta v_\epsilon^j(x)\right) = 0 & \text{in} \quad B_4, \\ v_\epsilon^i(x) = u_\epsilon^i(x) & \text{on} \quad \partial B_4. \end{cases} \tag{3.69}$$

Then $u_\epsilon - v_\epsilon \in H_0^1(B_4, \mathbb{R}^m)$ is the weak solution of

$$\begin{cases} D_\alpha\left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta (u_\epsilon^j(x) - v_\epsilon^j(x)\right) = D_\alpha f_\alpha^i(x) & \text{in} \quad B_4, \\ u_\epsilon^i(x) - v_\epsilon^i(x) = 0 & \text{on} \quad \partial B_4. \end{cases} \tag{3.70}$$

By the definition of weak solution v_ϵ of (3.69) with $\phi = v_\epsilon - u_\epsilon$ and (3.67), we see that

$$⨍_{B_4} |Dv_\epsilon|^2 dx \leq c. \tag{3.71}$$

By a standard L^2 estimate and (3.68), we also see that

$$⨍_{B_4} |Du_\epsilon - Dv_\epsilon|^2 dx \leq c\delta^2. \tag{3.72}$$

We now apply Lemma 3.3.3 to (3.69) with $q = p+1$, there exists a small $\delta = \delta(\lambda, \Lambda, m, n, p)$ such that if $A_{ij}^{\alpha\beta}$ is $(\delta, 5)$-vanishing of codimension 1, then we have

$$\left(⨍_{B_1} |Dv_\epsilon|^{p+1} dx\right)^{\frac{1}{p+1}} \leq c\left(⨍_{B_3} |Dv_\epsilon|^2 dx\right)^{\frac{1}{2}} \leq c \tag{3.73}$$

for some constant $c = c(\lambda, \Lambda, m, n, p)$ where we have used (3.67) for the last inequality.

For some large constant N, as selected below along with η according to

(3.63), we compute

$$
\begin{aligned}
&\frac{1}{|B_1|}|\{x \in B_1 : \mathcal{M}(|Du_\epsilon|^2 > N^2\}| \\
&\quad \leq \quad \frac{1}{|B_1|}|\{x \in B_1 : \mathcal{M}(2|Du_\epsilon - Dv_\epsilon|^2 + 2|Dv_\epsilon|^2) > N^2\}| \\
&\quad \leq \quad \frac{1}{|B_1|}\left|\left\{x \in B_1 : \mathcal{M}(|Du_\epsilon - Dv_\epsilon|^2) > \frac{N^2}{4}\right\}\right| \\
&\qquad\qquad\qquad + \frac{1}{|B_1|}\left|\left\{x \in B_1 : \mathcal{M}(|Dv_\epsilon|^2) > \frac{N^2}{4}\right\}\right| \\
&\quad \leq \quad c_3\left(\frac{4}{N^2}\right)\fint_{B_1}|Du_\epsilon - Dv_\epsilon|^2 dx + c_3\left(\frac{4}{N^2}\right)^{\frac{p+1}{2}}\fint_{B_1}|Dv_\epsilon|^{p+1}dx \\
&\quad \leq \quad \frac{c_3}{N^2}\fint_{B_4}|F|^2 dx + \frac{c_3}{N^{p+1}}\left(\fint_{B_4}|Dv_\epsilon|^2 dx\right)^{\frac{p+1}{2}} \\
&\quad \leq \quad \frac{c_3}{N^2}\delta^2 + \frac{c_3}{N^{p+1}} \text{ by } (3.72) - (3.73) \\
&\quad \leq \quad (c_3\eta^{\frac{p+1}{p}} + c_3\eta^{\frac{2}{p}}\delta^2) \text{ by } (3.63) \\
&\quad \leq \quad \eta\left[c_3\left(\eta^{\frac{1}{p}} + \eta^{\frac{2}{p}-1}\delta^2\right)\right]
\end{aligned}
$$

for some constant $c_3 = c_3(\lambda, \Lambda, m, n, p)$. Finally, we first take η so that

$$c_3\eta^{\frac{1}{p}} = \frac{1}{2},$$

and then select N from (3.63). We then select δ in order to have

$$c_3\eta^{\frac{2}{p}-1}\delta^2 \leq \frac{1}{2}.$$

Consequently, we conclude that for such N and η,

$$|\{x \in B_1 : \mathcal{M}(|Du_\epsilon|^2 > N^2\}| \leq \eta|B_1| \tag{3.74}$$

which contradicts to (3.62). This completes the proof. □

We now derive the required an interior $W^{1,p}$ estimate for the homogenization problem.

Proof of Theorem 3.1.1. Given any p with $2 < p < \infty$, assume $F \in L^p(\Omega, \mathbb{R}^{mn})$ and $A_{ij}^{\alpha\beta}$ is $(\delta, 5)$-vanishing of codimension 1. Also, let $u_\epsilon \in H^1(\Omega, \mathbb{R}^m)$ be a weak solution of (1.1). We now take η, N, and δ given by Lemma 3.4.1.

We can further suppose that

$$\|u_\epsilon\|_{L^p(B_5)} + \|F\|_{L^p(B_5)} \le \delta \tag{3.75}$$

by replacing u_ϵ and F by $\frac{u_\epsilon}{\frac{1}{\delta}\left(\|u_\epsilon\|_{L^p(B_5)}+\|F\|_{L^p(B_5)}\right)+\sigma}$ and $\frac{F}{\frac{1}{\delta}\left(\|u_\epsilon\|_{L^p(B_5)}+\|F\|_{L^p(B_5)}\right)+\sigma}$ for $\sigma > 0$, respectively. We want to show that

$$\|\mathcal{M}(|Du_\epsilon|^2)\|_{L^{\frac{p}{2}}(B_1)} \le c$$

for some universal constant $c > 0$, after letting $\sigma \to 0$.

To do this, we write

$$C = \left\{x \in B_1 : \mathcal{M}(|Du_\epsilon|^2) > N^2\right\}$$

and

$$D = \left\{x \in B_1 : \mathcal{M}(|Du_\epsilon|^2) > 1\right\} \cup \left\{x \in B_1 : \mathcal{M}(|F|^2) > \delta^2\right\}.$$

We use weak 1-1 estimates, the standard L^2 estimates, and Hölder's inequality, we see that

$$\begin{aligned} |C| &\le \tfrac{c}{N^2}\int_{B_1} |Du_\epsilon|^2 dx \\ &\le \tfrac{c}{N^2}\int_{B_5} |u_\epsilon|^2 + |F|^2 dx \\ &\le \tfrac{c}{N^2}\left(\|u_\epsilon\|^2_{L^p(B_5)} + \|F\|^2_{L^p(B_5)}\right) \\ &\le \tfrac{c\delta^2}{N^2} < \eta|B_1|, \end{aligned} \tag{3.76}$$

by further taking $\delta > 0$ satisfying the inequality (3.76). This asserts the first condition of Lemma 2.3.2. On the other hand, the second condition of Lemma 2.3.2 follows from Lemma 3.4.1. Then, we apply Lemma 2.3.2 to discover that

$$|C| < \eta_1|D|,$$

where

$$\eta_1 = \left(\frac{80}{7}\right)^n \eta, \tag{3.77}$$

by Remark 2.2.4.

Note that the problem (1.1) is invariant under normalization, we obtain the same results for $(\frac{u_\epsilon}{N}, \frac{F}{N})$, $(\frac{u_\epsilon}{N^2}, \frac{F}{N^2})$, $(\frac{u_\epsilon}{N^3}, \frac{F}{N^3}), \ldots$, inductively. Therefore, we obtain the following power decay estimates of $\mathcal{M}(|Du_\epsilon|^2)$:

$$\begin{aligned}\left|\left\{x \in B_1 : \mathcal{M}(|Du_\epsilon|^2) > N^{2k}\right\}\right| & \\ \leq \ \eta_1^k \left|\{x \in B_1 : \mathcal{M}(|Du_\epsilon|^2) > 1\}\right| & \\ + \sum_{i=1}^{k} \eta_1^i \left|\left\{x \in B_1 : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right|. &\end{aligned}$$

Applying Lemma 2.3.1 to

$$g = \mathcal{M}(|Du_\epsilon|^2), \quad \mu = N^2, \quad \theta = 1, \quad \text{and} \quad q = \frac{p}{2},$$

we compute as follows :

$$\begin{aligned}\|\mathcal{M}(|Du_\epsilon|^2)\|_{L^{\frac{p}{2}}(B_1)}^{\frac{p}{2}} & \\ \leq \ & c\left(|B_1| + \sum_{k\geq 1} N^{2k\frac{p}{2}} \left|\left\{x \in B_1 : \mathcal{M}(|Du_\epsilon|^2) > N^{2k}\right\}\right|\right) \\ \leq \ & c\left(1 + \sum_{k\geq 1} N^{kp}\eta_1^k \left|\{x \in B_1 : \mathcal{M}(|Du_\epsilon|^2) > 1\}\right| \right. \\ & \left. + \sum_{k\geq 1} N^{kp} \sum_{i=1}^{k} \eta_1^i \left|\left\{x \in B_1 : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right|\right) \\ =: \ & S_1 + S_2.\end{aligned}$$

$$\begin{aligned}S_1 \ \leq \ & c\left(1 + \sum_{k\geq 1} N^{kp}\eta_1^k \left|\{x \in B_1 : \mathcal{M}(|Du_\epsilon|^2) > 1\}\right|\right) \\ \leq \ & c\left(1 + |B_1| \sum_{k\geq 1} N^{kp}\eta_1^k\right).\end{aligned}$$

$$\begin{aligned}
S_2 &\le c\sum_{k\ge1} N^{kp}\sum_{i=1}^{k}\eta_1^i\left|\left\{x\in B_1 : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right| \\
&= c\sum_{i\ge1}\sum_{k\ge i} N^{kp}\eta_1^i\left|\left\{x\in B_1 : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right| \\
&= c\sum_{i\ge1}(N^p\eta_1)^i\sum_{k\ge i}(N^p)^{k-i}\left|\left\{x\in B_1 : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right| \\
&= c\sum_{i\ge1}(N^p\eta_1)^i\sum_{j\ge0}(N^p)^{j}\left|\left\{x\in B_1 : \mathcal{M}\left(\left|\frac{F}{\delta}\right|^2\right) > N^{2j}\right\}\right| \\
&\le c\sum_{i\ge1}(N^p\eta_1)^i\left\|\mathcal{M}\left(\left|\frac{F}{\delta}\right|^2\right)\right\|_{L^{\frac{p}{2}}(B_1)} \\
&\le c\sum_{i\ge1}(N^p\eta_1)^i\,\frac{\|F\|^2_{L^p(B_5)}}{\delta^2} \ \le\ c\sum_{i\ge1}(N^p\eta_1)^i \ \text{ by (3.75).}
\end{aligned}$$

Therefore, we have

$$\|\mathcal{M}(|Du_\epsilon|^2)\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(B_1)} \ \le\ c\left(1+\sum_{k\ge1}(N^p\eta_1)^k\right) \le c,$$

since $N^p\eta_1 = N^p\left(\frac{80}{7}\right)\eta = \frac{1}{2}$ from (3.63) and (3.77). Using the strong p-p estimate of maximal operator, we finally obtain

$$\|Du_\epsilon\|_{L^p(B_1)} \le c$$

which is the required one. This finishes the proof.

□

Chapter 4

Dirichlet problems

4.1 Main result

To start with boundary value problems in chapter 4 and chapter 5, we recall that, as we stated in the first chapter, we use (δ, R)-vanishing condition on the coefficients instead of (δ, R)-vanishing of codimension 1 condition for consistency with the conditions between the periodic coefficients in the homogenization and the domain. Also, as in chapter 3, by proving global $W^{1,p}$ estimates for $2 < p < \infty$, we will prove our main results for $2 < p < \infty$ since we can obtain the same results for every $1 < p < \infty$ by the classical estimate and a duality argument, see Remark 3.1.2. First, in this chapter, we consider the problem (1.2) which has the Dirichlet boundary condition. The following is our main result.

Theorem 4.1.1. *For any positive constant $2 < p < \infty$, suppose $F \in L^p(\Omega, \mathbb{R}^{mn})$. Then there exists a small positive constant $\delta = \delta(\lambda, \Lambda, m, n, p)$ such that if $A_{ij}^{\alpha\beta}$ is (δ, R)-vanishing and Ω is (δ, R)-Reifenberg flat, then for the weak solution $u_\epsilon \in H_0^1(\Omega, \mathbb{R}^m)$ of (1.2) we have*

$$Du_\epsilon \in L^p(\Omega, \mathbb{R}^{mn}) \tag{4.1}$$

with estimate

$$\|Du_\epsilon\|_{L^p(\Omega)} \leq c\|F\|_{L^p(\Omega)}, \tag{4.2}$$

where the positive constant $c = c(|\Omega|, \lambda, \Lambda, m, n, p)$ is independent of ϵ.

4.2 Boundary Hölder estimates and uniform $W^{1,q}$ estimates for homogeneous systems for the flat boundary

Similar to the interior case, for the global regularity, we need the following boundary Hölder estimates and $W^{1,q}$ estimates for homogeneous systems up to the flat boundary. In fact, the contents in this section can be proved in the same ways as in section 5.2 and section 5.3 except for using the result in [12] instead of [4]. For this reason, in this section, we state some results without proofs and we will give precise proofs later in chapter 5.

Boundary Hölder estimates is the following :

Theorem 4.2.1. *Let $\gamma \in (0,1)$. Suppose that $v_\epsilon \in H^1(B_r^+, \mathbb{R}^m)$ is a weak solution of*

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon} D_\beta v_\epsilon^j \right) &= 0 \quad in \quad B_r^+ \\ v_\epsilon^i &= 0 \quad on \quad T_r. \end{cases} \tag{4.3}$$

Then there exists a small positive constant $\delta = \delta(\lambda, \Lambda, m, n)$ such that if $A_{ij}^{\alpha\beta}$ is (δ, R)-vanishing, then for any $x, y \in B_{\frac{r}{2}}^+$,

$$|v_\epsilon(x) - v_\epsilon(y)| \leq c \left(\frac{|x-y|}{r} \right)^\gamma \left(\fint_{B_r^+} |v_\epsilon(z)|^2 dz \right)^{\frac{1}{2}}, \tag{4.4}$$

where $c > 0$ depends only on $\lambda, \Lambda, m, n,$ and γ.

In addition, $W^{1,q}$ regularity for homogeneous systems is given by the following lemma :

Lemma 4.2.2. *Let $v_\epsilon \in H^1(B_r^+, \mathbb{R}^m)$ be a weak solution of (4.3). Then for any $2 < q < \infty$, there exists $\delta = \delta(\lambda, \Lambda, m, n, q)$ such that if $A_{ij}^{\alpha\beta}$ is (δ, R)-vanishing, then we have*

$$\left(\fint_{B_{\frac{r}{2}}^+} |Dv_\epsilon|^q dx \right)^{\frac{1}{q}} \leq c \left(\fint_{B_r^+} |Dv_\epsilon|^2 dx \right)^{\frac{1}{2}} \tag{4.5}$$

for some positive constant $c = c(\lambda, \Lambda, m, n, q)$, independent of ϵ.

4.3 Approximation Lemmas

We next localize our problem near the flat boundary. We first assume that

$$B_5^+ \subset \Omega_5 \subset B_5 \cap \{x_n > -10\delta\}. \tag{4.6}$$

Let us suppose that $u_\epsilon \in H^1(\Omega_5, \mathbb{R}^m)$ is a weak solution of

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta u_\epsilon^j(x) \right) = D_\alpha f_\alpha^i(x) & \text{in} \quad \Omega_5 \\ u_\epsilon^i(x) = 0 & \text{on} \quad \partial_w \Omega_5, \end{cases} \tag{4.7}$$

which means

$$\int_{\Omega_5} A_{ij}^{\alpha\beta,\epsilon} D_\beta u_\epsilon^j D_\alpha \phi^i dx = \int_{\Omega_5} f_\alpha^i D_\alpha \phi^i dx \tag{4.8}$$

for all $\phi \in H_0^1(\Omega_5, \mathbb{R}^m)$ and the zero extension $\bar{u}_\epsilon$ of u_ϵ is in $H^1(B_5, \mathbb{R}^m)$. We further assume that

$$\frac{1}{|B_5|} \int_{\Omega_5} |Du_\epsilon|^2 dx \leq 1. \tag{4.9}$$

Then we consider the homogeneous problem :

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta w_\epsilon^j(x) \right) = 0 & \text{in} \quad \Omega_4 \\ w_\epsilon^i(x) = u_\epsilon^i(x) & \text{on} \quad \partial \Omega_4. \end{cases} \tag{4.10}$$

and the following homogeneous problem on the flat boundary :

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta v_\epsilon^j(x) \right) = 0 & \text{in} \quad B_3^+ \\ v_\epsilon^i(x) = 0 & \text{on} \quad T_3 \end{cases} \tag{4.11}$$

with the following definitions.

Definition 4.3.1. 1. $w_\epsilon \in H^1(\Omega_4, \mathbb{R}^m)$ is a weak solution of (4.10) if

$$\int_{\Omega_4} A_{ij}^{\alpha\beta,\epsilon} D_\beta w_\epsilon^j D_\alpha \phi^i dx = 0 \tag{4.12}$$

for all $\phi \in H_0^1(\Omega_4, \mathbb{R}^m)$ and the zero extension $\bar{w}_\epsilon$ of w_ϵ is in $H^1(B_4, \mathbb{R}^m)$.

2. $v_\epsilon \in H^1(B_3^+, \mathbb{R}^m)$ is a weak solution of (4.11) if

$$\int_{B_3^+} A_{ij}^{\alpha\beta,\epsilon} D_\beta v_\epsilon^j D_\alpha \phi^i dx = 0 \tag{4.13}$$

for all $\phi \in H_0^1(B_3^+, \mathbb{R}^m)$ and the zero extension $\bar{v}_\epsilon$ of v_ϵ is in $H^1(B_3, \mathbb{R}^m)$.

(4.7), (4.9), and (4.10) lead us to the following regularity result.

Lemma 4.3.2. *[31] Let $u_\epsilon \in H^1(\Omega_5, \mathbb{R}^m)$ be a weak solution of (4.7) satisfying (4.9) and let $w_\epsilon \in H^1(\Omega_4, \mathbb{R}^m)$ be the weak solution of (4.10). Then there exist small positive constants σ_1 and c, which depend only on λ, Λ, m, and n, such that*

$$\|Dw_\epsilon\|_{L^{2+\sigma_1}(\Omega_3)} \leq c. \tag{4.14}$$

In order to justify our argument in a Reifenberg domain, we need the following approximation lemma.

Lemma 4.3.3. *Let $u_\epsilon \in H^1(\Omega_5, \mathbb{R}^m)$ be a weak solution of (4.7) satisfying (4.9), and let $w_\epsilon \in H^1(\Omega_4, \mathbb{R}^m)$ be the weak solution of (4.10). Then for any fixed $\kappa > 0$, there exists a small $\delta = \delta(\kappa, \lambda, \Lambda, m, n) > 0$ such that if*

$$B_5^+ \subset \Omega_5 \subset B_5 \cap \{x_n > -10\delta\} \tag{4.15}$$

holds for δ, then there exists a weak solution $v_\epsilon \in H^1(B_3^+, \mathbb{R}^m)$ of (4.11) with

$$\fint_{B_3^+} |Dv_\epsilon|^2 dx \leq c \tag{4.16}$$

for some positive constant $c = c(\lambda, \Lambda, m, n)$ such that

$$\fint_{B_1^+} |D(w_\epsilon - v_\epsilon)|^2 dx \leq \kappa^2. \tag{4.17}$$

Proof. We argue this by contradiction. To do this, we assume that there exist $\kappa_0 > 0$, $\{u_{\epsilon,k}\}_{k=1}^\infty$, and $\left\{\Omega_5^k\right\}_{k=1}^\infty$ such that $u_{\epsilon,k}$ is a weak solution of

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta (u_{\epsilon,k})^j \right) = 0 & \text{in} \quad \Omega_5^k \\ (u_{\epsilon,k})^i = 0 & \text{on} \quad \partial_w \Omega_5^k \end{cases} \tag{4.18}$$

with

$$\fint_{\Omega_5^k} |Du_{\epsilon,k}|^2 dx \leq 1 \tag{4.19}$$

and

$$B_5^+ \subset \Omega_5^k \subset B_5 \cap \left\{ x_n > -\frac{10}{k} \right\}. \tag{4.20}$$

However,

$$\fint_{B_1^+} |D(w_{\epsilon,k} - v_\epsilon)|^2 dx > \kappa_0^2 \tag{4.21}$$

for any weak solution v_ϵ of

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta v_\epsilon^j \right) = 0 & \text{in} \quad B_3^+ \\ v_\epsilon^i = 0 & \text{on} \quad T_3, \end{cases} \tag{4.22}$$

where

$$\fint_{B_3^+} |Dv_\epsilon|^2 dx \le c \tag{4.23}$$

for the same positive constant c as in (4.16) and $w_{\epsilon,k}$ is the weak solution of

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta w_{\epsilon,k}^j(x) \right) = 0 & \text{in} \quad \Omega_4^k \\ w_{\epsilon,k}^i(x) = u_{\epsilon,k}^i(x) & \text{on} \quad \partial\Omega_4^k. \end{cases} \tag{4.24}$$

Applying (4.19) and the standard L^2-estimate for (4.24), we know that

$$\frac{1}{|B_4|} \int_{\Omega_4^k} |Dw_{\epsilon,k}|^2 dx \le c \frac{1}{|B_5|} \int_{\Omega_5^k} |Du_{\epsilon,k}|^2 dx \le c. \tag{4.25}$$

Also, using the fact that $w_{\epsilon,k} = 0$ on $\partial_w \Omega_3^k$ and (4.20), we apply Poincaré's inequality to find that

$$\begin{aligned} \frac{1}{|B_3|} \int_{B_3^+} |w_{\epsilon,k}|^2 dx &\le \frac{1}{|B_3|} \int_{\Omega_3^k} |w_{\epsilon,k}|^2 dx \le \frac{c}{|B_3|} \int_{\Omega_3^k} |Dw_{\epsilon,k}|^2 dx \\ &\le \frac{c}{|B_5|} \int_{\Omega_5^k} |Du_{\epsilon,k}|^2 dx \le c \end{aligned} \tag{4.26}$$

for some positive constant $c = c(\nu, L, m, n)$. If we apply the zero extension of $w_{\epsilon,k}$ from Ω_3^k to B_3, say, $\bar{w}_{\epsilon,k}$, then (4.25) and (4.26) imply that $\{\bar{w}_{\epsilon,k}\}_{k=1}^\infty$ is uniformly bounded in $H^1(B_3, \mathbb{R}^m)$. Thus, there exists a subsequence, which we will continue to denote as $\{\bar{w}_{\epsilon,k}\}$, and $\bar{w}_{\epsilon,0} \in H^1(B_3, \mathbb{R}^m)$ is such that

$$\begin{cases} D\bar{w}_{\epsilon,k} \rightharpoonup D\bar{w}_{\epsilon,0} & \text{weakly in} \quad L^2(B_3, \mathbb{R}^{mn}) \\ \bar{w}_{\epsilon,k} \to \bar{w}_{\epsilon,0} & \text{strongly in} \quad L^2(B_3, \mathbb{R}^m) \end{cases} \tag{4.27}$$

as $k \to \infty$. We define $w_{\epsilon,0}$ on $B_3^+ \cup T_3$ by $w_{\epsilon,0}(x) = \bar{w}_{\epsilon,0}(x)$ for all $x \in B_3^+ \cup T_3$. Hence, $w_{\epsilon,0}$ is a weak solution of

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta w_{\epsilon,0}^j \right) = 0 & \text{in} \quad B_3^+ \\ w_{\epsilon,0}^i = 0 & \text{on} \quad T_3. \end{cases} \tag{4.28}$$

From (4.25) and the lower semicontinuity with respect to weak convergence, we see that

$$\fint_{B_3^+} |Dw_{\epsilon,0}|^2 dx \le \liminf_{k\to\infty} \fint_{B_3^+} |Dw_{\epsilon,k}|^2 dx \le c. \tag{4.29}$$

Thus, we derive a contradiction by showing that

$$Dw_{\epsilon,k} \to Dw_{\epsilon,0} \text{ strongly in } L^2(B_1^+, \mathbb{R}^{mn}).$$

In order to do this, we begin with the cut-off function $\phi \in C_0^\infty(B_3)$ that satisfies

$$0 \le \phi \le 1, \quad \phi = 1 \text{ on } B_1, \quad \phi = 0 \text{ on } B_3 \backslash B_2, \quad \text{and } |D\phi| \le 2. \tag{4.30}$$

Then,

$$\begin{aligned}
\int_{B_1^+} |D(w_{\epsilon,k} - w_{\epsilon,0})|^2 dx \quad \le \quad & \int_{B_1} |D(\bar{w}_{\epsilon,k} - \bar{w}_{\epsilon,0})|^2 dx \\
&\le c \int_{B_1} A_{ij}^{\alpha\beta,\epsilon} D_\beta(\bar{w}_{\epsilon,k} - \bar{w}_{\epsilon,0})^j D_\alpha(\bar{w}_{\epsilon,k} - \bar{w}_{\epsilon,0})^i dx \\
&\le c \int_{B_3} A_{ij}^{\alpha\beta,\epsilon} D_\beta(\bar{w}_{\epsilon,k} - \bar{w}_{\epsilon,0})^j D_\alpha(\bar{w}_{\epsilon,k} - \bar{w}_{\epsilon,0})^i \phi^2 dx \\
&\le c \int_{B_3} A_{ij}^{\alpha\beta,\epsilon} D_\beta(\bar{w}_{\epsilon,k})^j D_\alpha(\bar{w}_{\epsilon,k} - \bar{w}_{\epsilon,0})^i \phi^2 dx \\
&\quad - c \int_{B_3} A_{ij}^{\alpha\beta,\epsilon} D_\beta(\bar{w}_{\epsilon,0})^j D_\alpha(\bar{w}_{\epsilon,k} - \bar{w}_{\epsilon,0})^i \phi^2 dx \\
&\le c \int_{B_3} A_{ij}^{\alpha\beta,\epsilon} D_\beta(\bar{w}_{\epsilon,k})^j D_\alpha(\phi^2(\bar{w}_{\epsilon,k} - \bar{w}_{\epsilon,0}))^i dx \\
&\quad - c \int_{B_3} A_{ij}^{\alpha\beta,\epsilon} D_\beta(\bar{w}_{\epsilon,k})^j (2\phi D_\alpha \phi)(\bar{w}_{\epsilon,k} - \bar{w}_{\epsilon,0})^i dx \\
&\quad - c \int_{B_3} A_{ij}^{\alpha\beta,\epsilon} D_\beta(\bar{w}_{\epsilon,0})^j D_\alpha(\bar{w}_{\epsilon,k} - \bar{w}_{\epsilon,0})^i \phi^2 dx \\
&\to 0
\end{aligned}$$

as $k \to \infty$ by applying (4.24) and (4.27). This completes the proof.

□

4.4 Proof of Theorem 4.1.1

Now we are ready to prove the following lemma, which is a key ingredient in our argument.

Lemma 4.4.1. *Let $2 < p < \infty$. Suppose that $u_\epsilon \in H^1_0(\Omega, \mathbb{R}^m)$ is the weak solution of (1.2). Then there exists a universal constant $\eta = \eta(\lambda, \Lambda, m, n, p)$ so that one can select a small $\delta = \delta(\lambda, \Lambda, m, n, p) > 0$ such that if $A^{\alpha\beta}_{ij}$ is $(\delta, 70)$-vanishing, if Ω is $(\delta, 70)$-Reifenberg flat, and if ,for all $y \in \Omega$ and every $0 < r \leq 1$, $B_r(y)$ satisfies*

$$\left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^2\right\} \cap B_r(y)\right| > \eta\, |B_r(y)|\,, \tag{4.31}$$

where

$$\left(\frac{80}{7}\right)^n N^p \eta = \frac{1}{2}, \tag{4.32}$$

then there holds

$$\Omega \cap B_r(y) \subset \left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\} \cup \left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2\right\}. \tag{4.33}$$

Proof. We prove this by contradiction. Using a scaling argument, it suffices to prove this lemma for $r = 1$. We assume (4.31) holds, but (4.33) is false. Then there is a point $x_1 \in \Omega \cap B_1(y)$ such that

$$\frac{1}{|B_\rho(x_1)|} \int_{\Omega_\rho(x_1)} |Du_\epsilon|^2 dx \leq 1 \quad \text{and} \quad \frac{1}{|B_\rho(x_1)|} \int_{\Omega_\rho(x_1)} |F|^2 dx \leq \delta^2 \tag{4.34}$$

for all $\rho > 0$.

We divide this into the two cases : an interior case when $B_7(y) \subset \Omega$ and a boundary case where $B_7(y) \not\subset \Omega$. Here, we only consider the boundary case as we already proved the interior case in Lemma 3.4.1. Because Ω is $(\delta, 70)$-Reifenberg flat, there exists an appropriate coordinate system such that

$$B_7(y) \cap \Omega \subset B_{14} \cap \Omega \tag{4.35}$$

and

$$B^+_{70} \subset \Omega_{70} \subset B_{70} \cap \{x_n > -140\delta\}. \tag{4.36}$$

It directly follows from (4.34) that

$$\frac{1}{|B_{70}|} \int_{\Omega_{70}} |Du_\epsilon|^2 dx \leq \frac{|B_{140}(x_1)|}{|B_{70}|} \frac{1}{|B_{140}|} \int_{\Omega_{140}(x_1)} |Du_\epsilon|^2 dx \leq 2^n \tag{4.37}$$

since $B_{70} \subset B_{140}(x_1)$. Similarly, we have

$$\frac{1}{|B_{70}|}\int_{\Omega_{70}} |F|^2 dx \leq 2^n\delta^2. \tag{4.38}$$

We consider the following rescaled maps :

$$\tilde{u}_\epsilon(z) = \frac{u_\epsilon(14z)}{14\sqrt{2^n}}, \quad \tilde{F}(z) = \frac{F(14z)}{\sqrt{2^n}}, \quad \tilde{A}_{ij}^{\alpha\beta,\epsilon}(z) = A_{ij}^{\alpha\beta,\epsilon}(14z) \quad (z \in \tilde{\Omega}_5) \tag{4.39}$$

where $\tilde{\Omega}_5 = \frac{1}{14}\Omega_{70}$ satisfying

$$B_5^+ \subset \tilde{\Omega}_5 \subset B_5 \cap \{z_n > -10\delta\}. \tag{4.40}$$

Therefore, $\tilde{u}_\epsilon \in H^1(\tilde{\Omega}_5, \mathbb{R}^m)$ is a weak solution of

$$\begin{cases} D_\alpha\left(\tilde{A}_{ij}^{\alpha\beta,\epsilon}(z) D_\beta \tilde{u}_\epsilon^j(z)\right) = D_\alpha \tilde{f}_\alpha^i(z) & \text{in} \quad \tilde{\Omega}_5 \\ \tilde{u}_\epsilon^i(z) = 0 & \text{on} \quad \partial_w\tilde{\Omega}_5 \end{cases} \tag{4.41}$$

with

$$\frac{1}{|B_5|}\int_{\tilde{\Omega}_5} |D\tilde{u}_\epsilon|^2 dz \leq 1 \quad \text{and} \quad \frac{1}{|B_5|}\int_{\tilde{\Omega}_5} |\tilde{F}|^2 dz \leq \delta^2. \tag{4.42}$$

Let $\tilde{w}_\epsilon \in H^1(\tilde{\Omega}_4, \mathbb{R}^m)$ be the weak solution of

$$\begin{cases} D_\alpha\left(\tilde{A}_{ij}^{\alpha\beta,\epsilon}(z) D_\beta \tilde{w}_\epsilon^j(z)\right) = 0 & \text{in} \quad \tilde{\Omega}_4 \\ \tilde{w}_\epsilon^i(z) = \tilde{u}_\epsilon^i(z) & \text{on} \quad \partial\tilde{\Omega}_4. \end{cases} \tag{4.43}$$

Then $\tilde{u}_\epsilon - \tilde{w}_\epsilon \in H_0^1(\tilde{\Omega}_4, \mathbb{R}^m)$ is the weak solution of

$$\begin{cases} D_\alpha\left(\tilde{A}_{ij}^{\alpha\beta,\epsilon}(z) D_\beta (\tilde{u}_\epsilon^j(z) - \tilde{w}_\epsilon^j(z)\right) = D_\alpha \tilde{f}_\alpha^i(z) & \text{in} \quad \tilde{\Omega}_4, \\ \tilde{u}_\epsilon^i(z) - \tilde{w}_\epsilon^i(z) = 0 & \text{on} \quad \partial\tilde{\Omega}_4. \end{cases} \tag{4.44}$$

Applying a standard L^2 estimate to (4.44) and (4.42), we obtain

$$\frac{1}{|B_4|}\int_{\tilde{\Omega}_4} |D\tilde{u}_\epsilon - D\tilde{w}_\epsilon|^2 dz \leq \frac{c}{|B_4|}\int_{\tilde{\Omega}_4} |\tilde{F}|^2 dz \leq c\delta^2 \tag{4.45}$$

for some positive constant $c = c(\lambda, \Lambda, m, n)$.

In addition, if we apply Lemma 4.3.3, then for any fixed $\kappa > 0$, there exists a small $\delta = \delta(\kappa, \lambda, \Lambda, m, n) > 0$ such that a weak solution $\tilde{v}_\epsilon \in H^1(B_3^+, \mathbb{R}^m)$ exists for

$$\begin{cases} D_\alpha \left(\tilde{A}_{ij}^{\alpha\beta,\epsilon}(z) D_\beta \tilde{v}_\epsilon^j \right) &= 0 \quad \text{in} \quad B_3^+ \\ \tilde{v}_\epsilon^i &= 0 \quad \text{on} \quad T_3, \end{cases} \tag{4.46}$$

with

$$\fint_{B_3^+} |D\tilde{v}_\epsilon|^2 dz \le c \tag{4.47}$$

for some constant $c = c(\lambda, \Lambda, m, n)$ such that

$$\fint_{B_1^+} |D(\tilde{w}_\epsilon - \tilde{v}_\epsilon)|^2 dz \le \kappa^2. \tag{4.48}$$

Applying Lemma 4.2.2 to (4.46) with $q = p + 1$, we know there is a small $\delta = \delta(\lambda, \Lambda, m, n, p)$ so that

$$\left(\fint_{B_1^+} |D\tilde{v}_\epsilon|^{p+1} dz \right)^{\frac{1}{p+1}} \le c \left(\fint_{B_3^+} |D\tilde{v}_\epsilon|^2 dz \right)^{\frac{1}{2}} \tag{4.49}$$

for some constant $c = c(\lambda, \Lambda, m, n, p)$. Therefore, for the zero extension $\bar{\tilde{v}}_\epsilon$ of $\tilde{v}_\epsilon$ from B_3^+ to B_3 we have

$$\begin{aligned}
&\frac{1}{|B_1|} |\{x \in \Omega : \mathcal{M}(|Du_\epsilon|)^2 > N^2\} \cap B_1(y)| \\
\le\ & \frac{c}{|B_1|} |\{z \in \tilde{\Omega}_1 : \mathcal{M}(3|D\tilde{u}_\epsilon - D\tilde{w}_\epsilon|^2 + 3|D\tilde{w}_\epsilon - D\bar{\tilde{v}}_\epsilon|^2 + 3|D\bar{\tilde{v}}_\epsilon|^2) > N^2\}| \\
\le\ & \frac{c}{|B_1|} \left| \left\{ z \in \tilde{\Omega}_1 : \mathcal{M}(|D\tilde{u}_\epsilon - D\tilde{w}_\epsilon|^2) > \frac{N^2}{9} \right\} \right| \\
&\quad + \frac{c}{|B_1|} \left| \left\{ z \in \tilde{\Omega}_1 : \mathcal{M}(|D\tilde{w}_\epsilon - D\bar{\tilde{v}}_\epsilon|^2) > \frac{N^2}{9} \right\} \right| \\
&\quad + \frac{c}{|B_1|} \left| \left\{ z \in \tilde{\Omega}_1 : \mathcal{M}(|D\bar{\tilde{v}}_\epsilon|^2) > \frac{N^2}{9} \right\} \right| \\
\le\ & c \left(\frac{9}{N^2} \right) \frac{1}{|B_1|} \int_{\tilde{\Omega}_1} |D\tilde{u}_\epsilon - D\tilde{w}_\epsilon|^2 dz + c \left(\frac{9}{N^2} \right) \frac{1}{|B_1|} \int_{\tilde{\Omega}_1} |D\tilde{w}_\epsilon - D\bar{\tilde{v}}_\epsilon|^2 dz \\
&\quad + c \left(\frac{9}{N^2} \right)^{\frac{p+1}{2}} \fint_{B_1^+} |D\tilde{v}_\epsilon|^{p+1} dz \\
=:\ & I_1 + I_2 + I_3.
\end{aligned}$$

Estimate of I_1 : the inequality (4.45) gives us

$$I_1 \leq \frac{c}{N^2}\delta^2 \tag{4.50}$$

for some positive constant $c = c(\lambda, \Lambda, m, n)$.

Estimate of I_2 : applying Lemma 4.3.2, Hölder's inequality, and (4.48), we see that

$$\begin{aligned} I_2 &\leq \frac{c}{N^2}\left(\frac{1}{|B_1|}\int_{B_1^+}|D\tilde{w}_\epsilon - D\tilde{v}_\epsilon|^2 dz + \frac{1}{|B_1|}\int_{\Omega_1\setminus B_1^+}|D\tilde{w}_\epsilon|^2 dz\right) \quad (4.51)\\ &\leq \frac{c}{N^2}\left(\kappa^2 + \left(\int_{\Omega_3}|D\tilde{w}_\epsilon|^{2+\sigma_1}dz\right)^{\frac{2}{2+\sigma_1}}\left(\int_{\Omega_1\setminus B_1^+}dz\right)^{\frac{\sigma_1}{2+\sigma_1}}\right)\\ &\leq \frac{c}{N^2}\left(\kappa^2 + \delta^{\frac{\sigma_1}{2+\sigma_1}}\right) \end{aligned}$$

for some positive constant $c = c(\lambda, \Lambda, m, n)$.

Estimate of I_3 : from (4.47) and (4.49) we can conclude

$$I_3 \leq \frac{c}{N^{p+1}}\left(\fint_{B_3^+}|D\tilde{v}_\epsilon|^2 dz\right)^{\frac{p+1}{2}} \leq \frac{c}{N^{p+1}} \tag{4.52}$$

for some positive constant $c = c(\lambda, \Lambda, m, n, p)$.

Therefore, if we combine (4.50), (4.51), and (4.52), we see that

$$\begin{aligned} &\frac{1}{|B_1|}|\{x \in \Omega : \mathcal{M}(|Du_\epsilon|)^2 > N^2\} \cap B_1(y)|\\ &\quad\leq I_1 + I_2 + I_3\\ &\quad\leq \frac{c_4}{N^{p+1}} + \frac{c_4}{N^2}\left(\kappa^2 + \delta^2 + \delta^{\frac{\sigma_1}{2+\sigma_1}}\right)\\ &\quad\leq c_4\eta^{\frac{p+1}{p}} + c_4\eta^{\frac{2}{p}}\left(\kappa^2 + \delta^2 + \delta^{\frac{\sigma_1}{2+\sigma_1}}\right) \text{ by (4.32)}\\ &\quad= \eta\left[c_4\left(\eta^{\frac{1}{p}} + \eta^{\frac{2}{p}-1}\left(\kappa^2 + \delta^2 + \delta^{\frac{\sigma_1}{2+\sigma_1}}\right)\right)\right] \end{aligned}$$

for some constant $c_4 = c_4(\lambda, \Lambda, m, n, p)$. Finally, we first take η so that

$$0 < c_4\eta^{\frac{1}{p}} \leq \frac{1}{3}$$

and then select N from (4.32). Secondly, we select *kappa* in order to have

$$0 < c_4\eta^{\frac{2}{p}-1}\kappa^2 \leq \frac{1}{3}. \tag{4.53}$$

Finally, one can find the corresponding $\delta = \delta(\lambda, \Lambda, m, n, p)$ satisfying (4.53) and

$$0 < c_4 \eta^{\frac{2}{p}-1} \left(\delta^2 + \delta^{\frac{\sigma_1}{2+\sigma_1}}\right) \leq \frac{1}{3}$$

such that this η and δ we can conclude that

$$|\{x \in \Omega : \mathcal{M}(|Du_\epsilon|)^2 > N^2\} \cap B_1(y)| \leq \eta |B_1|. \tag{4.54}$$

This contradicts (4.31) and completes the proof. □

Now, we are all ready to prove our main result in this chapter.

Proof of Theorem 4.1.1. Given any p with $2 < p < \infty$, assume that $F \in L^p(\Omega, \mathbb{R}^{mn})$, $A_{ij}^{\alpha\beta}$ is $(\delta, 70)$-vanishing and Ω is $(\delta, 70)$-Reifenberg flat. Also let $u_\epsilon \in H^1(\Omega, \mathbb{R}^m)$ be the weak solution of (1.2). We now take η, N, and δ given by Lemma 4.4.1.

We can further suppose that

$$\|F\|_{L^p(\Omega)} \leq \delta \tag{4.55}$$

by replacing u_ϵ and F with $\frac{u_\epsilon}{\frac{1}{\delta}\|F\|_{L^p(\Omega)}+\sigma}$ and $\frac{F}{\frac{1}{\delta}\|F\|_{L^p(\Omega)}+\sigma}$ for $\sigma > 0$, respectively. We want to show that

$$\|\mathcal{M}(|Du_\epsilon|^2)\|_{L^{\frac{p}{2}}(Q_1)} \leq c$$

for some universal constant $c > 0$ when $\sigma \to 0$.

To do this, we write

$$C = \left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^2\right\}$$

and

$$D = \left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\} \cup \left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2\right\}.$$

Using the weak 1-1 estimate, the standard L^2 estimate, and Hölder's inequality, we see that

$$\begin{aligned} |C| &\leq \frac{c}{N^2}\int_\Omega |Du_\epsilon|^2 dx \leq \frac{c}{N^2}\int_\Omega |F|^2 dx \\ &\leq \frac{c}{N^2}|\Omega|^{\frac{p-2}{p}}\|F\|^2_{L^p(\Omega)} \leq \frac{c\delta^2}{N^2} < \eta|B_1|, \end{aligned} \tag{4.56}$$

by further taking δ satisfying the inequality (4.56). This asserts the first condition of Lemma 2.3.2. On the other hand, the second condition of Lemma 2.3.2 follows from Lemma 4.4.1. Then we apply Lemma 2.3.2 to discover that

$$|C| < \eta_1 |D|$$

where

$$\eta_1 = \left(\frac{10}{1-\delta}\right)^n \eta \le \left(\frac{80}{7}\right)^n \eta, \tag{4.57}$$

by Remark 2.2.4.

Note that the problem (1.2) is invariant under normalization, we obtain the same results for $(\frac{u_\epsilon}{N}, \frac{F}{N})$, $(\frac{u_\epsilon}{N^2}, \frac{F}{N^2})$, $(\frac{u_\epsilon}{N^3}, \frac{F}{N^3}), \dots$ inductively. Therefore, we obtain the following power decay estimates of $\mathcal{M}(|Du_\epsilon|^2)$:

$$\begin{aligned} &\left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^{2k}\right\}\right| \\ &\qquad \le \eta_1^k \left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\}\right| \\ &\qquad\qquad + \sum_{i=1}^{k} \eta_1^i \left|\left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right|. \end{aligned}$$

Applying Lemma 2.3.1 to

$$g = \mathcal{M}(|Du_\epsilon|^2), \quad \mu = N^2, \quad \theta = 1, \quad \text{and} \quad q = \frac{p}{2},$$

we compute as follows :

$$\begin{aligned} &\|\mathcal{M}(|Du_\epsilon|^2)\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{p}{2}} \\ &\quad \le \quad c\left(|\Omega| + \sum_{k\ge1} N^{2k\frac{p}{2}} \left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^{2k}\right\}\right|\right) \\ &\quad \le \quad c\left(1 + \sum_{k\ge1} N^{kp}\eta_1^k \left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\}\right|\right. \\ &\qquad\qquad \left. + \sum_{k\ge1} N^{kp} \sum_{i=1}^{k} \eta_1^i \left|\left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right|\right) \\ &\quad =: \quad S_1 \ + \ S_2. \end{aligned}$$

$$\begin{aligned}
S_1 &\leq c\left(1+\sum_{k\geq 1} N^{kp}\eta_1^k \left|\{x\in\Omega : \mathcal{M}(|Du_\epsilon|^2)>1\}\right|\right)\\
&\leq c\left(1+|\Omega|\sum_{k\geq 1} N^{kp}\eta_1^k\right).
\end{aligned}$$

$$\begin{aligned}
S_2 &\leq c\sum_{k\geq 1} N^{kp}\sum_{i=1}^{k}\eta_1^i\left|\left\{x\in\Omega : \mathcal{M}(|F|^2)>\delta^2 N^{2(k-i)}\right\}\right|\\
&= c\sum_{i\geq 1}\sum_{k\geq i} N^{kp}\eta_1^i\left|\left\{x\in\Omega : \mathcal{M}(|F|^2)>\delta^2 N^{2(k-i)}\right\}\right|\\
&= c\sum_{i\geq 1}(N^p\eta_1)^i\sum_{k\geq i}(N^p)^{k-i}\left|\left\{x\in\Omega : \mathcal{M}(|F|^2)>\delta^2 N^{2(k-i)}\right\}\right|\\
&= c\sum_{i\geq 1}(N^p\eta_1)^i\sum_{j\geq 0}(N^p)^{j}\left|\left\{x\in\Omega : \mathcal{M}\left(\left|\frac{F}{\delta}\right|^2\right)>N^{2j}\right\}\right|\\
&\leq c\sum_{i\geq 1}(N^p\eta_1)^i\left\|\mathcal{M}\left(\left|\frac{F}{\delta}\right|^2\right)\right\|_{L^{\frac{p}{2}}(\Omega)}\\
&\leq c\sum_{i\geq 1}(N^p\eta_1)^i\,\frac{\|F\|^2_{L^p(\Omega)}}{\delta^2}\ \leq\ c\sum_{i\geq 1}(N^p\eta_1)^i \text{ by (4.55).}
\end{aligned}$$

Therefore, we have

$$\|\mathcal{M}(|Du_\epsilon|^2)\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\Omega)} \leq c\left(1+\sum_{k\geq 1}(N^p\eta_1)^k\right)\leq c,$$

since $N^p\eta_1 = N^p\left(\frac{10}{1-\delta}\right)^n\eta \leq N^p\left(\frac{80}{7}\right)^n\eta = \frac{1}{2}$ from (4.32) and (4.57). Using the strong p-p estimate of the maximal operator, we finally obtain

$$\|Du_\epsilon\|_{L^p(\Omega)}\leq c,$$

which is the required one. This completes the proof.

□

Chapter 5

Conormal derivative problems

5.1 Main result

In this chapter, we consider the conormal derivative problem (1.3). The following is our desired global $W^{1,p}$ regularity.

Theorem 5.1.1. *For any positive constant* $2 < p < \infty$*, suppose* $F \in L^p(\Omega, \mathbb{R}^{mn})$*. Then there exists a small positive constant* $\delta = \delta(\lambda, \Lambda, m, n, p)$ *such that if* $A_{ij}^{\alpha\beta}$ *is* (δ, R)*-vanishing and* Ω *is* (δ, R)*-Reifenberg flat, then for any weak solution* $u_\epsilon \in H^1(\Omega, \mathbb{R}^m)$ *of (1.3) we have*

$$Du_\epsilon \in L^p(\Omega, \mathbb{R}^{mn}) \tag{5.1}$$

with estimate

$$\|Du_\epsilon\|_{L^p(\Omega)} \leq c\|F\|_{L^p(\Omega)}, \tag{5.2}$$

where the constant $c = c(|\Omega|, \lambda, \Lambda, m, n, p)$ *is independent of* ϵ*.*

5.2 Boundary Hölder estimates

We begin this section with boundary Hölder regularity for homogeneous systems. This will be crucially used in the next section.

Theorem 5.2.1. *Let $\gamma \in (0,1)$. Suppose that $v_\epsilon \in H^1(B_r^+, \mathbb{R}^m)$ is a weak solution of*

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon} D_\beta v_\epsilon^j \right) = 0 & \text{in} \quad B_r^+ \\ A_{ij}^{\alpha\beta,\epsilon} D_\beta v_\epsilon^j \nu_\alpha = 0 & \text{on} \quad T_r. \end{cases} \tag{5.3}$$

Then there exists a small positive constant $\delta = \delta(\lambda, \Lambda, m, n)$ such that if $A_{ij}^{\alpha\beta}$ is (δ, R)-vanishing, then for any $x, y \in B_{\frac{r}{2}}^+$,

$$|v_\epsilon(x) - v_\epsilon(y)| \le c \left(\frac{|x-y|}{r} \right)^\gamma \left(\fint_{B_r^+} |v_\epsilon(z)|^2 dz \right)^{\frac{1}{2}}, \tag{5.4}$$

where $c > 0$ depends only on $\lambda, \Lambda, m, n,$ and γ.

The following two lemmas are needed for the proof of Theorem 5.2.1.

Lemma 5.2.2. *Let $\gamma \in (0,1)$. Then there exists $\epsilon_0 \in (0,1]$ and $\theta \in \left(0, \frac{1}{4}\right)$ depending only on $\lambda, \Lambda, m, n,$ and γ such that if for $0 < \epsilon < \epsilon_0$, v_ϵ is a weak solution of*

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon} D_\beta v_\epsilon^j \right) = 0 & \text{in} \quad B_1^+ \\ A_{ij}^{\alpha\beta,\epsilon} D_\beta v_\epsilon^j \nu_\alpha = 0 & \text{on} \quad T_1, \end{cases} \tag{5.5}$$

with

$$\fint_{B_1^+} |v_\epsilon - (\bar{v}_\epsilon)_{B_1^+}|^2 dx \le 1, \tag{5.6}$$

then

$$\fint_{B_\theta^+} |v_\epsilon - (\bar{v}_\epsilon)_{B_\theta^+}|^2 dx \le \theta^{2\gamma}. \tag{5.7}$$

Proof. We will prove this lemma by contradiction. If not, then there exists sequences ϵ_k, and v_{ϵ_k} such that $\epsilon_k \to 0$, v_{ϵ_k} is a weak solution of

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon_k} D_\beta v_{\epsilon_k}^j \right) = 0 & \text{in} \quad B_1^+ \\ A_{ij}^{\alpha\beta,\epsilon_k} D_\beta v_{\epsilon_k}^j \nu_\alpha = 0 & \text{on} \quad T_1, \end{cases} \tag{5.8}$$

with

$$\fint_{B_1^+} |v_{\epsilon_k} - (\bar{v}_{\epsilon_k})_{B_1^+}|^2 dx \le 1, \tag{5.9}$$

but for every $\theta \in (0, \frac{1}{4})$,

$$\fint_{B_\theta^+} |v_{\epsilon_k} - (\bar{v}_{\epsilon_k})_{B_\theta^+}|^2 dx > \theta^{2\gamma}. \tag{5.10}$$

By subtracting a constant, we assume that $(\bar{v}_{\epsilon_k})_{B_1^+} = 0$. Then from Caccioppoli inequality for (5.8) and (5.9), we have

$$\int_{B_{\frac{1}{2}}^+} |Dv_{\epsilon_k}|^2 dx \le c \int_{B_1^+} |v_{\epsilon_k}|^2 dx \le c. \tag{5.11}$$

Thus v_{ϵ_k} is uniformly bounded in $H^1(B_{\frac{1}{2}}^+)$, and then by passing to a subsequence, we assume that $v_{\epsilon_k} \to v_0$ strongly in $L^2(B_{\frac{1}{2}}^+)$ for some $v_\epsilon \in H^1(B_{\frac{1}{2}}^+)$. Consequently we have that for any $\theta \in (0, \frac{1}{4})$,

$$\int_{B_\theta^+} |v_{\epsilon_k} - (\bar{v}_{\epsilon_k})_{B_r^+}|^2 dx \to \int_{B_\theta^+} |v_0 - (\bar{v}_0)_{B_\theta^+}|^2 dx, \tag{5.12}$$

and so from (5.10), we find that for every $\theta \in (0, \frac{1}{4})$,

$$\fint_{B_\theta^+} |v_0 - (\bar{v}_0)_{B_\theta^+}|^2 dx > \theta^{2\gamma}. \tag{5.13}$$

In addition, recalling (5.8) and existing homogenization theory as in [3, 30], we see that v_0 solves

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,0} D_\beta v_0^j \right) = 0 & \text{in} \quad B_{\frac{1}{2}}^+ \\ A_{ij}^{\alpha\beta,0} D_\beta v_0^j \nu_\alpha = 0 & \text{on} \quad T_{\frac{1}{2}}, \end{cases} \tag{5.14}$$

where $A_{ij}^{\alpha\beta,0}$ is the constant matrix defined as in (1.16). According to boundary Hölder regularity for solutions of elliptic systems with constant coefficients on the flat boundaries, we discover that

$$\fint_{B_\theta^+} |v_0 - (\bar{v}_0)_{B_\theta^+}|^2 dx \le c_5 \theta^{1+\gamma}, \tag{5.15}$$

for some universal constant $c_5 = c_5(\lambda, \Lambda, m, n, \gamma)$.

We finally combine (5.13) and (5.15), to discover

$$\theta^{2\gamma} < \fint_{B_\theta^+} |v_0 - (\bar{v}_0)_{B_\theta^+}|^2 dx \le c_5 \theta^{1+\gamma} \tag{5.16}$$

for every $\gamma \in (0,1)$ and every $\theta \in (0, \frac{1}{4})$. However, we take $\theta \in (0, \frac{1}{4})$ so small to deduce

$$\theta^{2\gamma} \ge c_5 \theta^{1+\gamma}, \tag{5.17}$$

which contradicts (5.16). This finishes the proof. □

Lemma 5.2.3. *Fix $\gamma \in (0,1)$. Let ϵ_0 and θ be the constants as in Lemma 5.2.2 and let v_ϵ be a weak solution of (5.5). Then for all k such that $\epsilon < \theta^{k-1}\epsilon_0$, we have*

$$\fint_{B^+_{\theta^k}} |v_\epsilon - (\bar{v}_\epsilon)_{B^+_{\theta^k}}|^2 dx \leq \theta^{2k\gamma} \fint_{B^+_1} |v_\epsilon - (\bar{v}_\epsilon)_{B^+_1}|^2 dx. \tag{5.18}$$

Proof. The proof is by induction on k. By Lemma 5.2.2, (5.18) holds for $k = 1$. Now we assume that (5.18) holds for some $k \geq 1$. Let

$$w(z) = \frac{v_\epsilon(\theta^k z)}{\left(\fint_{B^+_{\theta^k}} \left| v_\epsilon - (\bar{v}_\epsilon)_{B^+_{\theta^k}} \right|^2 dx \right)^{\frac{1}{2}}} \quad \text{for } z \in B^+_1 \tag{5.19}$$

(We divide $v_\epsilon(\theta^k z)$ by $\left(\fint_{B^+_{\theta^k}} |v_\epsilon - (\bar{v}_\epsilon)_{B^+_{\theta^k}}|^2 dx \right)^{\frac{1}{2}} + \sigma$ for any $\sigma > 0$ and then we let $\sigma \to 0^+$ if $\fint_{B^+_{\theta^k}} |v_\epsilon - (\bar{v}_\epsilon)_{B^+_{\theta^k}}|^2 dx = 0$). Then w satisfies

$$\begin{cases} D_\alpha \left(A^{\alpha\beta, \frac{\epsilon}{\theta^k}}_{ij} D_\beta w^j \right) = 0 & \text{in } B^+_1 \\ A^{\alpha\beta, \frac{\epsilon}{\theta^k}}_{ij} D_\beta w^j \nu_\alpha = 0 & \text{on } T_1 \end{cases} \tag{5.20}$$

with

$$\fint_{B^+_1} |w - \bar{w}_{B^+_1}|^2 dz \leq 1. \tag{5.21}$$

Thus by applying Lemma 5.2.2 again to w, we obtain

$$\fint_{B^+_\theta} |w - \bar{w}_{B^+_\theta}|^2 dz \leq \theta^{2\gamma}. \tag{5.22}$$

Then by the induction hypothesis, we find that

$$\begin{aligned} \fint_{B^+_{\theta^{k+1}}} |v_\epsilon - (\bar{v}_\epsilon)_{B^+_{\theta^{k+1}}}|^2 dx &= \left(\fint_{B^+_\theta} |w - \bar{w}_{B^+_\theta}|^2 dz \right) \left(\fint_{B^+_{\theta^k}} |v_\epsilon - (\bar{v}_\epsilon)_{B^+_{\theta^k}}|^2 dx \right) \\ &\leq \theta^{2\gamma} \fint_{B^+_{\theta^k}} |v_\epsilon - (\bar{v}_\epsilon)_{B^+_{\theta^k}}|^2 dx \\ &\leq \theta^{2(k+1)\gamma} \fint_{B^+_1} |v_\epsilon - (\bar{v}_\epsilon)_{B^+_1}|^2 dx. \end{aligned}$$

This completes the proof. □

Remark 5.2.4. Before giving the proof of Theorem 5.2.1, we would like to point out that in the paper [4], $W^{1,p}$ regularity for a weak solution to (5.3) with $\epsilon = 1$ was established for all $1 < p < \infty$ where the coefficients $A_{ij}^{\alpha\beta}$ are assumed to be (δ, R)-vanishing. From this, we know that the equation (5.3) with $\epsilon = 1$ has $C^{0,\gamma}$ regularity for any fixed $\gamma \in (0,1)$ as a consequence of Morrey embedding for p large enough.

Proof of Theorem 5.2.1. Let ϵ_0 and θ be constants given in Lemma 5.2.2. By scaling, we may assume that $r = 1$. The case $\epsilon \geq \theta\epsilon_0$ follows from Remark 5.2.4 with an appropriate scaling.

We next consider $0 < \epsilon < \theta\epsilon_0$. We divide this into two cases, $\rho \geq \frac{\epsilon}{\epsilon_0}$ and $\rho < \frac{\epsilon}{\epsilon_0}$. For the first case, we can take $k \geq 0$ such that $\theta^{k+1} \leq \rho < \theta^k$. Since $\epsilon \leq \theta^k \epsilon_0$, we apply Lemma 5.2.3 to find that

$$\begin{aligned} \fint_{B_\rho^+} |v_\epsilon - (\bar{v}_\epsilon)_{B_r^+}|^2 dx &\leq c \fint_{B_{\theta^k}^+} |v_\epsilon - (\bar{v}_\epsilon)_{B_{\theta^k}^+}|^2 dx \\ &\leq c\theta^{2k\gamma} \fint_{B_1^+} |v_\epsilon - (\bar{v}_\epsilon)_{B_1^+}|^2 dx \\ &\leq c\rho^{2\gamma} \fint_{B_1^+} |v_\epsilon|^2 dx. \end{aligned}$$

For the second one, we use a blow-up argument by letting $w(z) = v_\epsilon(\epsilon z)$. Since $\frac{2}{\epsilon_0} < \frac{1}{\theta\epsilon_0} < \frac{1}{\epsilon}$, w satisfies

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,1} D_\beta w^j \right) = 0 & \text{in} \quad B_{\frac{2}{\epsilon_0}}^+ \\ A_{ij}^{\alpha\beta,1} D_\beta w^j \nu_\alpha = 0 & \text{on} \quad T_{\frac{2}{\epsilon_0}}. \end{cases} \tag{5.23}$$

By the $C^{0,\gamma}$ regularity for (5.23), we see that

$$\fint_{B_{\frac{\rho}{\epsilon}}^+} |w - \bar{w}_{B_{\frac{\rho}{\epsilon}}^+}|^2 dz \leq c \left(\frac{\rho}{\epsilon}\right)^{2\gamma} \fint_{B_{\frac{1}{\epsilon_0}}^+} |w - \bar{w}_{B_{\frac{1}{\epsilon_0}}^+}|^2 dz \tag{5.24}$$

for some constant $c = c(\gamma, \lambda, \Lambda, m, n)$. Since $\frac{\epsilon}{\epsilon_0} < \theta$, we apply Lemma 5.2.3

again to find that

$$\begin{aligned}\fint_{B_\rho^+}|v_\epsilon-(\bar v_\epsilon)_{B_\rho^+}|^2dx &= \fint_{B^+_{\frac{\rho}{\epsilon}}}|w-\bar w_{B^+_{\frac{\rho}{\epsilon}}}|^2dz \\ &\le c\left(\frac{\rho}{\epsilon}\right)^{2\gamma}\fint_{B^+_{\frac{1}{\epsilon_0}}}|w-\bar w_{B^+_{\frac{1}{\epsilon_0}}}|^2dz \\ &\le c\left(\frac{\rho}{\epsilon}\right)^{2\gamma}\fint_{B^+_{\frac{\epsilon}{\epsilon_0}}}|v_\epsilon-(\bar v_\epsilon)_{B^+_{\frac{\epsilon}{\epsilon_0}}}|^2dx \\ &\le c\left(\frac{\rho}{\epsilon}\right)^{2\gamma}\left(\frac{\epsilon}{\epsilon_0}\right)^{2\gamma}\fint_{B_1^+}|v_\epsilon-(\bar v_\epsilon)_{B_1^+}|^2dx \\ &\le c\rho^{2\gamma}\fint_{B_1^+}|v_\epsilon|^2dx.\end{aligned} \tag{5.25}$$

This completes the proof of Theorem 5.2.1. □

5.3 Uniform $W^{1,q}$ estimates for homogeneous systems for the flat boundary

Now, we are now ready to derive uniform $W^{1,q}$ regularity.

Lemma 5.3.1. *Let $v_\epsilon \in H^1(B_r^+,\mathbb{R}^m)$ be a weak solution of (5.3). Then for any $2 < q < \infty$, there exists $\delta = \delta(\lambda,\Lambda,m,n,q)$ such that if $A_{ij}^{\alpha\beta}$ is (δ,R)-vanishing, then we have*

$$\left(\fint_{B^+_{\frac{r}{2}}}|Dv_\epsilon|^qdx\right)^{\frac{1}{q}} \le c\left(\fint_{B_r^+}|Dv_\epsilon|^2dx\right)^{\frac{1}{2}} \tag{5.26}$$

for some positive constant $c = c(\lambda,\Lambda,m,n,q)$, independent of ϵ.

Proof. By dilation, we assume that $r = 1$ and it suffices to show that

$$\int_{B^+_{\frac{1}{4}}}|Dv_\epsilon(y)|^qdy \le c\left(\int_{B_1^+}|Dv_\epsilon|^2dz\right)^{\frac{q}{2}} \tag{5.27}$$

for some constant $c = c(\lambda,\Lambda,m,n,q)$ since we can obtain (5.26) by using Lemma 3.3.3, (5.27), and standard covering argument.

For any $x = (x_1, \cdots, x_n) \in B^+_{\frac{1}{4}}$, by the interior $W^{1,q}$ regularity, see Theorem 3.1.1, we find that

$$\left(\fint_{B_{\frac{1}{4}x_n(x)}} |Dv_\epsilon|^q dy \right)^{\frac{1}{q}} \leq \frac{c}{x_n} \left(\fint_{B_{\frac{1}{2}x_n(x)}} |v_\epsilon(y) - v_\epsilon(x)|^q dy \right)^{\frac{1}{q}} \tag{5.28}$$

for some constant $c = c(\lambda, \Lambda, m, n, q)$ which is independent of ϵ.

Here, we observe that if $y = (y_1, \cdots, y_n) \in B_{tx_n}(x)$ for some $t \in (0, 1)$, then

$$|x_n - y_n| \leq |x - y| \leq tx_n. \tag{5.29}$$

This implies that

$$(1 - t)x_n \leq y_n \leq (1 + t)x_n. \tag{5.30}$$

Now, we apply (5.30), boundary Hölder estimates and Poincaré inequality, then for any $\gamma \in (0, 1)$, (5.28) becomes

$$\frac{1}{(x_n)^n} \int_{B_{\frac{1}{4}x_n(x)}} |Dv_\epsilon|^q dy \leq \frac{c}{(x_n)^n} \left(\int_{B_{\frac{1}{2}x_n(x)}} \frac{(x_n)^{\gamma q}}{(y_n)^q} dy \right) \left(\fint_{B_1^+} |Dv_\epsilon|^2 dz \right)^{\frac{q}{2}} \tag{5.31}$$

for some constant $c = c(\lambda, \Lambda, m, n, q, \gamma)$. Now, we integrate (5.31) over $B^+_{\frac{1}{4}}$. Then we apply (5.30) to the left hand side of (5.31) to see that

$$\begin{aligned} &\int_{B^+_{\frac{1}{4}}} \int_{B_{\frac{1}{4}x_n(x)}} \frac{|Dv_\epsilon(y)|^q}{(x_n)^n} dydx \\ &= \int_{B^+_{\frac{1}{4}}} |Dv_\epsilon(y)|^q \int_{x \in B^+_{\frac{1}{4}}, |x-y| \leq \frac{1}{4}x_n} \frac{1}{(x_n)^n} dxdy \\ &\geq c \int_{B^+_{\frac{1}{4}}} |Dv_\epsilon(y)|^q \int_{x \in B^+_{\frac{1}{4}}, |x-y| \leq \frac{1}{5}y_n} \frac{1}{(y_n)^n} dxdy \\ &\geq c \int_{B^+_{\frac{1}{4}}} |Dv_\epsilon(y)|^q dy \end{aligned} \tag{5.32}$$

for some constant $c = c(n)$. Similarly, we apply (5.30) again to the right hand side of (5.31) to find that

$$\begin{aligned}
&\int_{B^+_{\frac{1}{4}}}\int_{B_{\frac{1}{2}x_n(x)}} \frac{(x_n)^{\gamma q}}{(x_n)^n(y_n)^q}dydx \qquad (5.33)\\
&\quad\le c\int_{B^+_{\frac{1}{2}}}\int_{B_{\frac{1}{2}x_n(x)}} \frac{(y_n)^{\gamma q}}{(x_n)^n(y_n)^q}dydx\\
&\quad= c\int_{B^+_{\frac{1}{2}}}\frac{1}{(y_n)^{q(1-\gamma)}}\int_{x\in B^+_{\frac{1}{2}},|x-y|\le\frac{1}{2}x_n}\frac{1}{(x_n)^n}dxdy\\
&\quad\le c\int_{B^+_{\frac{1}{2}}}\frac{1}{(y_n)^{q(1-\gamma)}}\int_{x\in B^+_{\frac{1}{2}},|x-y|\le y_n}\frac{1}{(y_n)^n}dxdy\\
&\quad\le c\int_{B^+_{\frac{1}{2}}}\frac{1}{(y_n)^{q(1-\gamma)}}dy
\end{aligned}$$

for some constant $c = c(n)$.

Now, we choose $\gamma \in (0,1)$ so that $q(1-\gamma) < 1$ for $q > 2$, and then we insert (5.32) and (5.33) into (5.31) to discover that

$$\begin{aligned}
\int_{B^+_{\frac{1}{4}}}|Dv_\epsilon(y)|^q dy &\le \int_{B^+_{\frac{1}{4}}}\int_{B_{\frac{1}{4}x_n(x)}}\frac{|Dv_\epsilon(y)|^q}{(x_n)^n}dydx \qquad (5.34)\\
&\le \frac{c}{(x_n)^n}\left(\int_{B_{\frac{1}{2}x_n(x)}}\frac{(x_n)^{\gamma q}}{(y_n)^q}dy\right)\left(⨍_{B^+_1}|Dv_\epsilon|^2dz\right)^{\frac{q}{2}}\\
&\le c\left(\int_{B^+_{\frac{1}{2}}}\frac{1}{(y_n)^{q(1-\gamma)}}dy\right)\left(⨍_{B^+_1}|Dv_\epsilon|^2dz\right)^{\frac{q}{2}}\\
&\le c\left(\int_{B^+_1}|Dv_\epsilon|^2dz\right)^{\frac{q}{2}}
\end{aligned}$$

for some constant $c = c(\lambda, \Lambda, m, n, q)$. This completes the proof. □

5.4 Approximation lemmas

We next localize our problem near the flat boundary. We first assume that

$$B^+_5 \subset \Omega_5 \subset B_5 \cap \{x_n > -10\delta\}. \qquad (5.35)$$

Then we let $u_\epsilon \in H^1(\Omega_5, \mathbb{R}^m)$ be a weak solution of

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta u_\epsilon^j(x) \right) = D_\alpha f_\alpha^i(x) & \text{in} \quad \Omega_5 \\ \left(A_{ij}^{\alpha\beta,\epsilon} D_\beta u_\epsilon^j - f_\alpha^i \right) \nu_\alpha = 0 & \text{on} \quad \partial_w \Omega_5 \end{cases} \tag{5.36}$$

and $v_\epsilon \in H^1(B_4^+, \mathbb{R}^m)$ be a weak solution of

$$\begin{cases} D_\alpha \left(A_{ij}^{\alpha\beta,\epsilon}(x) D_\beta v_\epsilon^j(x) \right) = 0 & \text{in} \quad B_4^+ \\ A_{ij}^{\alpha\beta,\epsilon} D_\beta v_\epsilon^j \nu_\alpha = 0 & \text{on} \quad T_4 \end{cases} \tag{5.37}$$

with the following definitions.

Definition 5.4.1. 1. $u_\epsilon \in H^1(\Omega_5, \mathbb{R}^m)$ is a weak solution of (5.36) if

$$\int_{\Omega_5} A_{ij}^{\alpha\beta,\epsilon} D_\beta u_\epsilon^j D_\alpha \phi^i dx = \int_{\Omega_5} f_\alpha^i D_\alpha \phi^i dx \tag{5.38}$$

for all $\phi \in H^1(\Omega_5, \mathbb{R}^m)$ with $\phi = 0$ on $\partial_c \Omega_5$.

2. $v_\epsilon \in H^1(B_4^+, \mathbb{R}^m)$ is a weak solution of (5.37) if

$$\int_{B_4^+} A_{ij}^{\alpha\beta,\epsilon} D_\beta v_\epsilon^j D_\alpha \phi^i dx = 0 \tag{5.39}$$

for all $\phi \in H^1(B_4^+, \mathbb{R}^m)$ with $\phi = 0$ on $\partial_c B_4^+$.

We need the following approximation lemma.

Lemma 5.4.2. *Let $u_\epsilon \in H^1(\Omega_5, \mathbb{R}^m)$ be a weak solution of (5.36) satisfying*

$$\frac{1}{|B_5|} \int_{\Omega_5} |Du_\epsilon|^2 dx \le 1. \tag{5.40}$$

Then for any $0 < \tau < 1$ fixed, there exists a small $\delta = \delta(\tau, \lambda, \Lambda, m, n) > 0$ such that if

$$B_5^+ \subset \Omega_5 \subset B_5 \cap \{x_n > -10\delta\}, \tag{5.41}$$

and

$$\frac{1}{|B_5|} \int_{\Omega_5} |F|^2 dx \le \delta^2 \tag{5.42}$$

for such δ, then there exists a weak solution $v_\epsilon \in H^1(B_4^+, \mathbb{R}^m)$ of (5.37) with

$$\frac{1}{|B_4|} \int_{B_4^+} |Dv_\epsilon|^2 dx \le c \tag{5.43}$$

for some constant $c = c(m, n)$ such that

$$\frac{1}{|B_4|}\int_{B_4^+}|u_\epsilon - v_\epsilon|^2 dx \le \tau^2. \tag{5.44}$$

Proof. We argue this by contradiction. To do this, we assume that there exist $\tau_0 > 0$, $\{u_{\epsilon,k}\}_{k=1}^\infty$, $\{F_k\}_{k=1}^\infty$ and $\left\{\Omega_5^k\right\}_{k=1}^\infty$ such that $u_{\epsilon,k}$ is a weak solution of

$$\begin{cases} D_\alpha\left(A_{ij}^{\alpha\beta,\epsilon}(x)D_\beta u_{\epsilon,k}^j(x)\right) = D_\alpha f_{k,\alpha}^i(x) & \text{in} \quad \Omega_5^k \\ \left(A_{ij}^{\alpha\beta,\epsilon}D_\beta u_{\epsilon,k}^j - f_{k,\alpha}^i\right)\nu_\alpha = 0 & \text{on} \quad \partial_w\Omega_5^k \end{cases} \tag{5.45}$$

with

$$\frac{1}{|B_5|}\int_{\Omega_5^k}|Du_{\epsilon,k}|^2 dx \le 1, \tag{5.46}$$

$$B_5^+ \subset \Omega_5^k \subset B_5 \cap \left\{x_n > -\frac{10}{k}\right\}, \tag{5.47}$$

and

$$\frac{1}{|B_5|}\int_{\Omega_5^k}|F_k|^2 dx \le \left(\frac{1}{k}\right)^2. \tag{5.48}$$

However,

$$\frac{1}{|B_4|}\int_{B_4^+}|u_{\epsilon,k} - v_\epsilon|^2 dx > \tau_0^2 \tag{5.49}$$

for any weak solution v_ϵ of

$$\begin{cases} D_\alpha\left(A_{ij}^{\alpha\beta,\epsilon}(x)D_\beta v_\epsilon^j(x)\right) = 0 & \text{in} \quad B_4^+ \\ A_{ij}^{\alpha\beta,\epsilon}D_\beta v_\epsilon^j\nu_\alpha = 0 & \text{on} \quad T_4 \end{cases} \tag{5.50}$$

with

$$\frac{1}{|B_4|}\int_{B_4^+}|Dv_\epsilon|^2 dx \le c. \tag{5.51}$$

In view of (5.46), the Poincaré inequality, and the property of average which minimizes variance, we have

$$\begin{aligned} \frac{1}{|B_5|}\int_{B_5^+}|u_{\epsilon,k} - \bar{u}_{\epsilon,k_{B_5^+}}|^2 dx &\le \frac{1}{|B_5|}\int_{\Omega_5^k}|u_{\epsilon,k} - \bar{u}_{\epsilon,k_{\Omega_5^k}}|^2 dx \\ &\le c\frac{1}{|B_5|}\int_{\Omega_5^k}|Du_{\epsilon,k}|^2 dx \\ &\le c \end{aligned}$$

for some constant $c = c(m,n)$. This implies that $\{u_{\epsilon,k} - \bar{u}_{\epsilon,k_{B_5^+}}\}_{k=1}^{\infty}$ is bounded in $H^1(B_5^+)$. Therefore, there exists a subsequence, which we still denote by $\{u_{\epsilon,k} - \bar{u}_{\epsilon,k_{B_5^+}}\}_{k=1}^{\infty}$, and $u_{\epsilon,0} \in H^1(B_4^+)$ such that

$$\begin{cases} u_{\epsilon,k} - \bar{u}_{\epsilon,k_{B_5^+}} \rightharpoonup u_{\epsilon,0} & \text{weakly in} \quad H^1(B_4^+) \\ u_{\epsilon,k} - \bar{u}_{\epsilon,k_{B_5^+}} \to u_{\epsilon,0} & \text{strongly in} \quad L^2(B_4^+) \end{cases} \tag{5.52}$$

as $k \to \infty$. Using (5.47), (5.48) and (5.52) and letting $k \to \infty$ in (5.45), we discover that $u_{\epsilon,0}$ is a weak solution of (5.50). On the other hand, by using weakly lower semicontinuity for weak convergence,

$$\begin{aligned} \frac{1}{|B_4|}\int_{B_4^+} |Du_{\epsilon,0}|^2 dx &\leq \liminf_{k\to\infty} \frac{1}{|B_4|}\int_{B_4^+} |Du_{\epsilon,k}|^2 dx \\ &\leq c \liminf_{k\to\infty} \frac{1}{|B_5|}\int_{\Omega_5^k} |Du_{\epsilon,k}|^2 dx \leq c \end{aligned} \tag{5.53}$$

for some constant $c = c(m,n)$. Then $u_{\epsilon,0}$ is a weak solution of (5.50) satisfying (5.51) by (5.53), but (5.49) can not hold from (5.52). Hence we reach a contradiction. This finishes the proof. □

Lemma 5.4.3. *Let $2 < q < \infty$. Let $u_\epsilon \in H^1(\Omega_5, \mathbb{R}^m)$ be a weak solution of (5.36) satisfying*

$$\frac{1}{|B_5|}\int_{\Omega_5} |Du_\epsilon|^2 dx \leq 1. \tag{5.54}$$

Then for any $0 < \kappa < 1$ fixed, there exists a small $\delta = \delta(\kappa, \lambda, \Lambda, m, n, q) > 0$ such that if $A_{ij}^{\alpha\beta}$ is $(\delta, 5)$-vanishing,

$$B_5^+ \subset \Omega_5 \subset B_5 \cap \{x_n > -10\delta\}, \tag{5.55}$$

and

$$\frac{1}{|B_5|}\int_{\Omega_5} |F|^2 dx \leq \delta^2 \tag{5.56}$$

for such δ, then there exists a weak solution $v_\epsilon \in H^1(B_4^+, \mathbb{R}^m)$ of (5.37) such that

$$\frac{1}{|B_1|}\int_{\Omega_1} |D(u_\epsilon - \bar{v}_\epsilon)|^2 dx \leq \kappa^2. \tag{5.57}$$

where $\bar{v}_\epsilon$ is an $W^{1,q}$ extension of v_ϵ from B_4^+ to B_4.

Proof. According to Lemma 5.4.2, for each $0 < \tau < 1$, with the assumptions (5.54),(5.55),and (5.56), there exists a small δ such that there exists a weak solution $v_\epsilon \in H^1(B_4^+, \mathbb{R}^m)$ of (5.37) with

$$\frac{1}{|B_4|}\int_{B_4^+} |Dv_\epsilon|^2 dx \leq c \tag{5.58}$$

satisfying

$$\frac{1}{|B_4|}\int_{B_4^+} |u_\epsilon - v_\epsilon|^2 dx \leq \tau^2. \tag{5.59}$$

By a standard $W^{1,q}$ extension of v_ϵ from B_4^+ to B_4, there exists $\bar{v}_\epsilon \in H^1(B_4)$ such that $\bar{v}_\epsilon = v_\epsilon$ in B_4^+ and

$$\|D\bar{v}_\epsilon\|_{L^q(B_4)} \leq c\|Dv_\epsilon\|_{L^q(B_4^+)}, \tag{5.60}$$

where $c = c(m, n, q)$ is independent of v_ϵ.

Now we choose a standard cut-off function $\phi \in C_0^\infty(B_2)$ that satisfies

$$0 \leq \phi \leq 1, \quad \phi = 1 \text{ on } B_1, \quad \phi = 0 \text{ on } B_2 \backslash B_{\frac{3}{2}}, \quad \text{and } |D\phi| \leq 4. \tag{5.61}$$

Since u_ϵ is a weak solution of (5.36), we take $\phi^2(u_\epsilon - \bar{v}_\epsilon)$ as a test function in the definition of a weak solution (5.38) for Ω_5 to discover that

$$\frac{1}{|B_2|}\int_{\Omega_2} A_{ij}^{\alpha\beta,\epsilon} D_\beta u_\epsilon^j D_\alpha(\phi^2(u_\epsilon - \bar{v}_\epsilon))^i dx = \frac{1}{|B_2|}\int_{\Omega_2} f_\alpha^i D_\alpha(\phi^2(u_\epsilon - \bar{v}_\epsilon))^i dx. \tag{5.62}$$

We compute the left hand side of (5.62) as follows :

$$\begin{aligned}
&\frac{1}{|B_2|}\int_{\Omega_2} A_{ij}^{\alpha\beta,\epsilon} D_\beta u_\epsilon^j D_\alpha(\phi^2(u_\epsilon - \bar{v}_\epsilon))^i dx \\
&\quad = \frac{1}{|B_2|}\int_{\Omega_2} A_{ij}^{\alpha\beta,\epsilon} D_\beta (u_\epsilon - \bar{v}_\epsilon)^j D_\alpha(\phi^2(u_\epsilon - \bar{v}_\epsilon))^i dx \\
&\quad\quad + \frac{1}{|B_2|}\int_{\Omega_2} A_{ij}^{\alpha\beta,\epsilon} D_\beta \bar{v}_\epsilon^j D_\alpha(\phi^2(u_\epsilon - \bar{v}_\epsilon))^i dx \\
&\quad = \frac{1}{|B_2|}\int_{\Omega_2} \phi^2 A_{ij}^{\alpha\beta,\epsilon} D_\beta (u_\epsilon - \bar{v}_\epsilon)^j D_\alpha(u_\epsilon - \bar{v}_\epsilon)^i dx \\
&\quad\quad + \frac{1}{|B_2|}\int_{\Omega_2} 2\phi A_{ij}^{\alpha\beta,\epsilon} D_\beta (u_\epsilon - \bar{v}_\epsilon)^j D_\alpha\phi(u_\epsilon - \bar{v}_\epsilon)^i dx \\
&\quad\quad + \frac{1}{|B_2|}\int_{B_2^+} A_{ij}^{\alpha\beta,\epsilon} D_\beta \bar{v}_\epsilon^j D_\alpha(\phi^2(u_\epsilon - \bar{v}_\epsilon))^i dx \\
&\quad\quad + \frac{1}{|B_2|}\int_{\Omega_2\setminus B_2^+} A_{ij}^{\alpha\beta,\epsilon} D_\beta \bar{v}_\epsilon^j D_\alpha(\phi^2(u_\epsilon - \bar{v}_\epsilon))^i dx \\
&\quad =: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

The uniform ellipticity condition (1.5) implies

$$I_1 \geq \lambda \frac{1}{|B_2|}\int_{\Omega_2} \phi^2 |D(u_\epsilon - \bar{v}_\epsilon)|^2 dx. \tag{5.63}$$

Cauchy's inequality with s, (1.6), and (5.61) imply that

$$|I_2| \leq \frac{s}{|B_2|}\int_{\Omega_2} \phi^2 |D(u_\epsilon - \bar{v}_\epsilon)|^2 dx + \frac{c(s,\Lambda)}{|B_2|}\int_{\Omega_2} |u_\epsilon - \bar{v}_\epsilon|^2 dx. \tag{5.64}$$

In order to estimate the second term on the right hand side of (5.64), we use Sobolev inequality, (5.54), (5.55), (5.58), (5.59), and (5.60) to see that

$$\begin{aligned}
&\frac{1}{|B_2|}\int_{\Omega_2} |u_\epsilon - \bar{v}_\epsilon|^2 dx \qquad (5.65) \\
&\quad = \frac{1}{|B_2|}\int_{B_2^+} |u_\epsilon - \bar{v}_\epsilon|^2 dx + \frac{1}{|B_2|}\int_{\Omega_2\setminus B_2^+} |u_\epsilon - \bar{v}_\epsilon|^2 dx \\
&\quad \leq c\left(\tau^2 + \frac{1}{|B_2|}\left(\int_{\Omega_2\setminus B_2^+} |u_\epsilon - \bar{v}_\epsilon|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}} \left(\int_{\Omega_2\setminus B_2^+} dx\right)^{\frac{2}{n}}\right) \\
&\quad \leq c\left(\tau^2 + \frac{1}{|B_2|}\left(\int_{\Omega_2} |D(u_\epsilon - \bar{v}_\epsilon)|^2 dx\right)\delta^{\frac{2}{n}}\right) \\
&\quad \leq c\left(\tau^2 + \delta^{\frac{2}{n}}\right)
\end{aligned}$$

for some constant $c = c(\lambda, \Lambda, m, n)$ (This is valid for $n \geq 3$ but we can justify for $n = 2$ by using any $p_1 > 2$ instead of $2^* = \frac{2n}{n-2}$ and then applying Hölder's inequality to the exponents $\frac{1}{p_1}$ and $\frac{p_1-1}{p_1}$). From (5.64) with $s = \frac{\lambda}{2}$ and (5.65) we have

$$|I_2| \leq \frac{\lambda}{2|B_2|} \int_{\Omega_2} \phi^2 |D(u_\epsilon - \bar{v}_\epsilon)|^2 dx + c(\tau^2 + \delta^{\frac{2}{n}}) \tag{5.66}$$

for some constant $c = c(\lambda, \Lambda, m, n)$.

As $\bar{v}_\epsilon = v_\epsilon$ in B_4^+ and v_ϵ is a weak solution of (5.37), we find that

$$I_3 = 0 \tag{5.67}$$

We next estimate I_4 as follows : we recall (1.6) and apply Hölder's inequality with $\frac{1}{2} + \frac{1}{q} + \frac{q-2}{2q} = 1$ to discover that

$$|I_4| \leq \frac{c}{|B_2|} \left(\int_{\Omega_2} |D\bar{v}_\epsilon|^q dx \right)^{\frac{1}{q}} \left(\int_{\Omega_2 \setminus B_2^+} |D(\phi^2(u_\epsilon - \bar{v}_\epsilon))|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_2 \setminus B_2^+} dx \right)^{\frac{q-2}{2q}} \tag{5.68}$$

for some constant $c = c(\Lambda)$. Since $A_{ij}^{\alpha\beta}$ is $(\delta, 5)$-vanishing, we obtain by Lemma 5.3.1, (5.58), and (5.60)

$$\begin{aligned} \frac{1}{|B_2|} \int_{\Omega_2} |D\bar{v}_\epsilon|^q dx &\leq \frac{1}{|B_2|} \int_{B_2} |D\bar{v}_\epsilon|^q dx \\ &\leq c \left(\frac{1}{|B_2|} \int_{B_2^+} |Dv_\epsilon|^q dx \right) \\ &\leq c \left(\frac{1}{|B_2|} \int_{B_4^+} |Dv_\epsilon|^2 dx \right)^{\frac{2}{q}} \\ &\leq c \end{aligned} \tag{5.69}$$

for some constant $c = c(\lambda, \Lambda, m, n, q)$ and by (5.54), (5.55), (5.58), (5.60), (5.61), and Sobolev inequality

$$\begin{aligned}
&\frac{1}{|B_2|}\int_{\Omega_2\setminus B_2^+}|D(\phi^2(u_\epsilon-\bar{v}_\epsilon))|^2dx \qquad (5.70)\\
&\le \frac{c}{|B_2|}\left(\int_{\Omega_2\setminus B_2^+}\phi^2|D\phi|^2|u_\epsilon-\bar{v}_\epsilon|^2dx+\int_{\Omega_2}\phi^4|D(u_\epsilon-\bar{v}_\epsilon)|^2dx\right)\\
&\le \frac{c}{|B_2|}\left(\int_{\Omega_2\setminus B_2^+}|u_\epsilon-\bar{v}_\epsilon|^2dx+\int_{\Omega_2}|D(u_\epsilon-\bar{v}_\epsilon)|^2dx\right)\\
&\le c(1+\delta^{\frac{2}{n}})
\end{aligned}$$

for some constant $c=c(\lambda,\Lambda,m,n)$ with the same computation as in (5.65). Thus, we have from (5.68), (5.69), and (5.70)

$$|I_4|\le c(1+\delta^{\frac{2}{n}})^{\frac{1}{2}}\delta^{\frac{q-2}{2q}}\le c\delta^{\frac{q-2}{2q}} \qquad (5.71)$$

for some constant $c=c(\lambda,\Lambda,m,n,q)$ with δ small.

Using (5.56) and Hólder's inequality we compute the right hand side of (5.62) with the same computations as in (5.65) and (5.70) to see that

$$\begin{aligned}
&\frac{1}{|B_2|}\int_{\Omega_2}f_\alpha^iD_\alpha(\phi^2(u_\epsilon-\bar{v}_\epsilon))^idx \qquad (5.72)\\
&\le c\left(\frac{1}{|B_2|}\int_{\Omega_2}|F|^2dx\right)^{\frac{1}{2}}\left(\frac{1}{|B_2|}\int_{\Omega_2}|D(\phi^2(u_\epsilon-\bar{v}_\epsilon))|^2dx\right)^{\frac{1}{2}}\\
&\le c\delta(1+\tau^2+\delta^{\frac{2}{n}})^{\frac{1}{2}}\le c\delta
\end{aligned}$$

for some constant $c=c(\lambda,\Lambda,m,n)$, as τ is small and δ is small.

Now, we insert the estimates (5.63), (5.66), (5.67), (5.71), and (5.72) into (5.62) to discover that

$$\frac{1}{|B_1|}\int_{\Omega_1}|D(u-\bar{v}_\epsilon)|^2dx\le\frac{c}{|B_2|}\int_{\Omega_2}\phi^2|D(u-\bar{v}_\epsilon)|^2dx\le c_6\left(\tau^2+\delta+\delta^{\frac{q-2}{2q}}+\delta^{\frac{2}{n}}\right) \qquad (5.73)$$

for some constant $c_6=c_6(\lambda,\Lambda,m,n,q)$. Thus, we can select τ satisfying

$$c_6\tau^2\le\frac{1}{2}\kappa^2$$

and then we can take δ which depends on the choice of τ according to Lemma 5.4.2 and satisfies

$$c_6\left(\delta+\delta^{\frac{q-2}{2q}}+\delta^{\frac{2}{n}}\right)\le\frac{1}{2}\kappa^2$$

to obtain

$$c_6\left(\tau^2+\delta+\delta^{\frac{q-2}{2q}}+\delta^{\frac{2}{n}}\right)\leq\kappa^2. \tag{5.74}$$

This finishes the proof.

□

5.5 Proof of Theorem 5.1.1

Now we are ready to prove the following key lemma of our argument.

Lemma 5.5.1. *Let $2<p<\infty$. Suppose that $u_\epsilon\in H^1(\Omega,\mathbb{R}^m)$ is a weak solution of (1.3). Then there exists a universal constant $\eta=\eta(\lambda,\Lambda,m,n,p)$ so that one can select a small $\delta=\delta(\eta,\lambda,\Lambda,m,n,p)>0$ such that if $A_{ij}^{\alpha\beta}$ is $(\delta,70)$-vanishing, if Ω is $(\delta,70)$-Reifenberg flat, and if, for all $y\in\Omega$ and every $0<r\leq 1$, $B_r(y)$ satisfies*

$$\left|\left\{x\in\Omega:\mathcal{M}(|Du_\epsilon|^2)>N^2\right\}\cap B_r(y)\right|>\eta\left|B_r(y)\right|, \tag{5.75}$$

where

$$\left(\frac{80}{7}\right)^n N^p\eta=\frac{1}{2}, \tag{5.76}$$

then there holds

$$\Omega\cap B_r(y)\subset\left\{x\in\Omega:\mathcal{M}(|Du_\epsilon|^2)>1\right\}\cup\left\{x\in\Omega:\mathcal{M}(|F|^2)>\delta^2\right\}. \tag{5.77}$$

Proof. We prove this by contradiction. Using a scaling argument, it suffices to prove this lemma for $r=1$. We assume (5.75) holds, but (5.77) is false. Then there is a point $x_1\in\Omega\cap B_1(y)$ such that

$$\frac{1}{|B_\rho(x_1)|}\int_{\Omega_\rho(x_1)}|Du_\epsilon|^2dx\leq 1 \quad\text{and}\quad \frac{1}{|B_\rho(x_1)|}\int_{\Omega_\rho(x_1)}|F|^2dx\leq\delta^2 \tag{5.78}$$

for all $\rho>0$.

We divide this into the two cases : an interior case when $B_7(y)\subset\Omega$ and a boundary case where $B_7(y)\not\subset\Omega$. Here, we only consider the boundary case as we have already proved the interior case in Lemma 3.4.1. Because Ω is $(\delta,70)$-Reifenberg flat, there exists an appropriate coordinate system such that

$$B_7(y)\cap\Omega\subset B_{14}\cap\Omega \tag{5.79}$$

and

$$B_{70}^{+} \subset \Omega_{70} \subset B_{70} \cap \{x_n > -140\delta\}. \tag{5.80}$$

It directly follows from (5.78) that

$$\frac{1}{|B_{70}|}\int_{\Omega_{70}} |Du_\epsilon|^2 dx \le \frac{|B_{140}(x_1)|}{|B_{70}|}\frac{1}{|B_{140}|}\int_{\Omega_{140}(x_1)} |Du_\epsilon|^2 dx \le 2^n \tag{5.81}$$

since $B_{70} \subset B_{140}(x_1)$. Similarly, we have

$$\frac{1}{|B_{70}|}\int_{\Omega_{70}} |F|^2 dx \le 2^n\delta^2. \tag{5.82}$$

We consider the following rescaled maps :

$$\tilde{u}_\epsilon(z) = \frac{u_\epsilon(14z)}{14\sqrt{2^n}}, \quad \tilde{F}(z) = \frac{F(14z)}{\sqrt{2^n}}, \quad \tilde{A}_{ij}^{\alpha\beta,\epsilon}(z) = A_{ij}^{\alpha\beta,\epsilon}(14z) \quad (z \in \tilde{\Omega}_5) \tag{5.83}$$

where $\tilde{\Omega}_5 = \frac{1}{14}\Omega_{70}$.

$$B_5^{+} \subset \tilde{\Omega}_5 \subset B_5 \cap \{z_n > -10\delta\}. \tag{5.84}$$

Then $\tilde{u}_\epsilon \in H^1(\tilde{\Omega}_5, \mathbb{R}^m)$ is a weak solution of

$$\left\{ \begin{array}{rcll} D_\alpha\left(\tilde{A}_{ij}^{\alpha\beta,\epsilon}(z) D_\beta \tilde{u}_\epsilon^j(z)\right) & = & D_\alpha \tilde{f}_\alpha^i(z) & \text{in} \quad \tilde{\Omega}_5 \\ \left(\tilde{A}_{ij}^{\alpha\beta,\epsilon} D_\beta \tilde{u}_\epsilon^j - \tilde{f}_\alpha^i\right)\nu_\alpha & = & 0 & \text{on} \quad \partial_w \tilde{\Omega}_5 \end{array} \right. \tag{5.85}$$

satisfying that

$$\frac{1}{|B_5|}\int_{\tilde{\Omega}_5} |D\tilde{u}_\epsilon|^2 dz \le 1, \tag{5.86}$$

$$\tilde{A}_{ij}^{\alpha\beta} \text{ is } (\delta, 5)\text{-vanishing}, \tag{5.87}$$

$$B_5^{+} \subset \tilde{\Omega}_5 \subset B_5 \cap \{z_n > -10\delta\}, \tag{5.88}$$

and

$$\frac{1}{|B_5|}\int_{\tilde{\Omega}_5} |\tilde{F}|^2 dz \le \delta^2. \tag{5.89}$$

We now apply Lemma 5.4.3 to find that for any fixed $\kappa > 0$, there exists a small $\delta = \delta(\kappa, \lambda, \Lambda, m, n) > 0$ such that there exists a weak solution $\tilde{v}_\epsilon \in H^1(B_4^{+}, \mathbb{R}^m)$ of

$$\left\{ \begin{array}{rcll} D_\alpha\left(\tilde{A}_{ij}^{\alpha\beta,\epsilon}(z) D_\beta \tilde{v}_\epsilon^j\right) & = & 0 & \text{in} \quad B_4^{+} \\ \tilde{A}_{ij}^{\alpha\beta,\epsilon} D_\beta \tilde{v}_\epsilon^j \nu_\alpha & = & 0 & \text{on} \quad T_4, \end{array} \right. \tag{5.90}$$

with

$$\frac{1}{|B_4|}\int_{B_4^+}|D\tilde{v}_\epsilon|^2 dz \leq c \tag{5.91}$$

such that

$$\frac{1}{|B_1|}\int_{\tilde{\Omega}_1}|D(\tilde{u}_\epsilon - \bar{\tilde{v}}_\epsilon)|^2 dz \leq \kappa^2 \tag{5.92}$$

where $\bar{\tilde{v}}_\epsilon$ is a standard $W^{1,p+1}$ extension of $\tilde{v}_\epsilon$ from B_4^+ to B_4. Applying Lemma 5.3.1 to $q = p+1$, we see that

$$\left(\frac{1}{|B_2|}\int_{B_2^+}|D\bar{\tilde{v}}_\epsilon|^{p+1} dz\right)^{\frac{1}{p+1}} \leq c\left(\frac{1}{|B_4|}\int_{B_4^+}|D\tilde{v}_\epsilon|^2 dz\right)^{\frac{1}{2}} \leq c \tag{5.93}$$

for some constant $c = c(\lambda, \Lambda, m, n, p)$.

Consequently, we have

$$\begin{aligned}
&\frac{1}{|B_1|}|\{x \in \Omega : \mathcal{M}(|Du_\epsilon|)^2 > N^2\} \cap B_1(y)| \\
&\leq \frac{c_7}{|B_1|}|\{z \in \tilde{\Omega}_1 : \mathcal{M}(2|D\tilde{u}_\epsilon - D\bar{\tilde{v}}_\epsilon|^2 + 2|D\bar{\tilde{v}}_\epsilon|^2) > N^2\}| \\
&\leq \frac{c_7}{|B_1|}\left|\left\{z \in \tilde{\Omega}_1 : \mathcal{M}(|D\tilde{u}_\epsilon - D\bar{\tilde{v}}_\epsilon|^2) > \frac{N^2}{4}\right\}\right| \\
&\qquad + \frac{c_7}{|B_1|}\left|\left\{z \in \tilde{\Omega}_1 : \mathcal{M}(|D\bar{\tilde{v}}_\epsilon|^2) > \frac{N^2}{4}\right\}\right| \\
&\leq c_7\left(\frac{4}{N^2}\right)\frac{1}{|B_1|}\int_{\tilde{\Omega}_1}|D\tilde{u}_\epsilon - D\bar{\tilde{v}}_\epsilon|^2 dz \\
&\qquad + c_7\left(\frac{4}{N^2}\right)^{\frac{p+1}{2}}\frac{1}{|B_1|}\int_{B_1^+}|D\bar{\tilde{v}}_\epsilon|^{p+1} dz \\
&\leq \frac{c_7}{N^2}\kappa^2 + \frac{c_7}{N^{p+1}} \text{ by (5.92)} - \text{(5.93)} \\
&= c_7\eta^{\frac{2}{p}}\kappa^2 + c_7\eta^{\frac{p+1}{p}} \text{ by (5.76)} \\
&= \eta\left[c_7\left(\eta^{\frac{2}{p}-1}\kappa^2 + \eta^{\frac{1}{p}}\right)\right]
\end{aligned}$$

for some constant $c_7 = c_7(\lambda, \Lambda, m, n, p)$. Finally, we first take η so that

$$0 < c_7\eta^{\frac{1}{p}} \leq \frac{1}{2},$$

and then select N from (5.76). We then select κ in order to have

$$0 < c_7\eta^{\frac{2}{p}-1}\kappa^2 \leq \frac{1}{2}.$$

From this choice of κ, one can find the corresponding small $\delta = \delta(\lambda, \Lambda, m, n, p)$ such that this η and δ we can conclude that

$$|\{x \in \Omega : \mathcal{M}(|Du_\epsilon|)^2 > N^2\} \cap B_1(y)| \leq \eta |B_1|. \tag{5.94}$$

This contradicts (5.75) and completes the proof. □

We are all set to prove Theorem 5.1.1.

Proof of Theorem 5.1.1. Given any p with $2 < p < \infty$, assume that $F \in L^p(\Omega, \mathbb{R}^{mn})$, $A_{ij}^{\alpha\beta}$ is $(\delta, 70)$-vanishing and Ω is $(\delta, 70)$-Reifenberg flat. Also let $u_\epsilon \in H^1(\Omega, \mathbb{R}^m)$ be a weak solution of (1.3). We now take η, N, and δ given by Lemma 5.5.1.

We can further suppose that

$$\|F\|_{L^p(\Omega)} \leq \delta \tag{5.95}$$

by replacing u_ϵ and F with $\frac{u_\epsilon}{\frac{1}{\delta}\|F\|_{L^p(\Omega)} + \sigma}$ and $\frac{F}{\frac{1}{\delta}\|F\|_{L^p(\Omega)} + \sigma}$ for $\sigma > 0$, respectively. We want to show that

$$\|\mathcal{M}(|Du_\epsilon|^2)\|_{L^{\frac{p}{2}}(Q_1)} \leq c$$

for some universal constant $c > 0$ when $\sigma \to 0$.

To do this, we write

$$C = \left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^2\right\}$$

and

$$D = \left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\} \cup \left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2\right\}.$$

Using the weak 1-1 estimate, the standard L^2 estimate, and Hölder's inequality, we see that

$$\begin{aligned} |C| &\leq \frac{c}{N^2}\int_\Omega |Du_\epsilon|^2 dx \leq \frac{c}{N^2}\int_\Omega |F|^2 dx \\ &\leq \frac{c}{N^2}|\Omega|^{\frac{p-2}{p}}\|F\|_{L^p(\Omega)}^2 \leq \frac{c\delta^2}{N^2} < \eta|B_1|, \end{aligned} \tag{5.96}$$

by further taking δ satisfying the inequality (5.96). This asserts the first condition of Lemma 2.3.2. On the other hand, the second condition of Lemma

2.3.2 follows from Lemma 5.5.1. Then we apply Lemma 2.3.2 to discover that

$$|C| < \eta_1 |D|$$

where

$$\eta_1 = \left(\frac{10}{1-\delta}\right)^n \eta \leq \left(\frac{80}{7}\right)^n \eta, \tag{5.97}$$

by Remark 2.2.4.

Note that the problem (1.3) is invariant under normalization, we obtain the same results for $(\frac{u_\epsilon}{N}, \frac{F}{N})$, $(\frac{u_\epsilon}{N^2}, \frac{F}{N^2})$, $(\frac{u_\epsilon}{N^3}, \frac{F}{N^3}), \dots$ inductively. Therefore, we obtain the following power decay estimates of $\mathcal{M}(|Du_\epsilon|^2)$:

$$\begin{aligned} &\left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^{2k}\right\}\right| \\ &\quad \leq \eta_1^k \left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\}\right| \\ &\qquad + \sum_{i=1}^{k} \eta_1^i \left|\left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right|. \end{aligned}$$

Applying Lemma 2.3.1 to

$$g = \mathcal{M}(|Du_\epsilon|^2), \quad \mu = N^2, \quad \theta = 1, \quad \text{and} \quad q = \frac{p}{2},$$

we compute as follows :

$$\begin{aligned} &\|\mathcal{M}(|Du_\epsilon|^2)\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{p}{2}} \\ &\quad \leq \ c\left(|\Omega| + \sum_{k\geq 1} N^{2k\frac{p}{2}} \left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^{2k}\right\}\right|\right) \\ &\quad \leq \ c\left(1 + \sum_{k\geq 1} N^{kp}\eta_1^k \left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\}\right|\right. \\ &\qquad\qquad \left. + \sum_{k\geq 1} N^{kp} \sum_{i=1}^{k} \eta_1^i \left|\left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right|\right) \\ &\quad =: \ S_1 \ + \ S_2. \end{aligned}$$

$$
\begin{aligned}
S_1 &\leq c\left(1+\sum_{k\geq 1} N^{kp}\eta_1^k \left|\left\{x\in\Omega : \mathcal{M}(|Du_\epsilon|^2)>1\right\}\right|\right)\\
&\leq c\left(1+|\Omega|\sum_{k\geq 1} N^{kp}\eta_1^k\right).
\end{aligned}
$$

$$
\begin{aligned}
S_2 &\leq c\sum_{k\geq 1} N^{kp}\sum_{i=1}^{k}\eta_1^i\left|\left\{x\in\Omega : \mathcal{M}(|F|^2)>\delta^2 N^{2(k-i)}\right\}\right|\\
&= c\sum_{i\geq 1}\sum_{k\geq i} N^{kp}\eta_1^i\left|\left\{x\in\Omega : \mathcal{M}(|F|^2)>\delta^2 N^{2(k-i)}\right\}\right|\\
&= c\sum_{i\geq 1}(N^p\eta_1)^i\sum_{k\geq i}(N^p)^{k-i}\left|\left\{x\in\Omega : \mathcal{M}(|F|^2)>\delta^2 N^{2(k-i)}\right\}\right|\\
&= c\sum_{i\geq 1}(N^p\eta_1)^i\sum_{j\geq 0}(N^p)^{j}\left|\left\{x\in\Omega : \mathcal{M}\left(\left|\frac{F}{\delta}\right|^2\right)>N^{2j}\right\}\right|\\
&\leq c\sum_{i\geq 1}(N^p\eta_1)^i\left\|\mathcal{M}\left(\left|\frac{F}{\delta}\right|^2\right)\right\|_{L^{\frac{p}{2}}(\Omega)}\\
&\leq c\sum_{i\geq 1}(N^p\eta_1)^i\frac{\|F\|^2_{L^p(\Omega)}}{\delta^2} \leq c\sum_{i\geq 1}(N^p\eta_1)^i \text{ by (5.95).}
\end{aligned}
$$

Therefore, we have

$$
\|\mathcal{M}(|Du_\epsilon|^2)\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\Omega)} \leq c\left(1+\sum_{k\geq 1}(N^p\eta_1)^k\right)\leq c,
$$

since $N^p\eta_1 = N^p\left(\frac{10}{1-\delta}\right)^n\eta \leq N^p\left(\frac{80}{7}\right)^n\eta = \frac{1}{2}$ from (5.76) and (5.97). Using the strong p-p estimate of the maximal operator, we finally obtain

$$
\|Du_\epsilon\|_{L^p(\Omega)}\leq c,
$$

which is the required one. This completes the proof.

□

Bibliography

[1] M. Avellaneda and F. Lin, *Compactness methods in the theory of homogenization*, Comm. Pure Appl. Math., **40** (1987), 803–847.

[2] P. Auscher and M. Qafsaoui, *Observation on $W^{1,p}$ estimates for divergence elliptic equations with VMO coefficients*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat., no. 2, **5** (8) (2002), 487–509.

[3] A. Bensoussan, J. L. Lions, and G. C. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, AMS Chelsea Publishing, (2011).

[4] S. Byun, H. Chen, M. Kim and L. Wang, *L^p regularity theory for linear elliptic systems*, Discrete Contin. Dyn. Syst. **18** (1) (2007), 121–134.

[5] S. Byun and Y. Jang, *$W^{1,p}$ estimates in homogenization of elliptic systems with measurable coefficients*, Preprint.

[6] S. Byun and Y. Jang, *Global $W^{1,p}$ estimates for elliptic systems in homogenization problems in Reifenberg domains*, Preprint.

[7] S. Byun and Y. Jang, *Homogenization of the conormal derivative problem for elliptic systems in Reifenberg domains*, Preprint.

[8] S. Byun and D. Palagachev, *Weighted L^p-estimates for elliptic equations with measurable coefficients in nonsmooth domains*, Potential Anal., **41** (1) (2014), 51-79.

[9] S. Byun, S. Ryu, and L. Wang, *Gradient estimates for elliptic systems with measurable coefficients in nonsmooth domains*, Manuscripta Math., **133** (2010), 225–245.

[10] S. Byun and L. Wang, *Elliptic equations with BMO coefficients in Reifenberg domains*, Comm. Pure Appl. Math., **57** (10) (2004), 1283–1310.

[11] S. Byun and L. Wang, *The conormal derivative problem for elliptic equations with BMO coefficients on Reifenberg flat domains*, Proc. London Math. Soc. (3) **90** (1) (2005), 245–272.

[12] S. Byun and L. Wang, *Gradient estimates for elliptic systems in non-smooth domains*, Math. Ann., **341** (3) (2008), 629–650.

[13] S. Byun and L. Wang, *Elliptic equations with measurable coefficients in Reifenberg domains*, Adv. in Math., **225** (5) (2010), 2648-2673.

[14] S. Byun and L. Wang, *L^p-regularity for fourth order parabolic systems with measurable coefficients*, Math. Z., **272**, (1-2) (2012), 515-530.

[15] X. Cabré and L. A. Caffarelli, *Fully nonlinear elliptic equations*, Amer. Math. Soc. Colloq. Publ., vol. 43, Amer. Math. Soc., Providence, RI (1995).

[16] L. A. Caffarelli and I. Peral, *On $W^{1,p}$ estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math., **51** (1) (1998), 1–21.

[17] A. P. Calderon and A. Zygmund, *On the existence of certain singular integrals*, Acta Math., **88** (1952), 85–139.

[18] G. Di Fazio, *L^p estimates for divergence form elliptic equations with discontinuous coefficients*, Boll. Unione Mat. Ital. A (7) **10** (2) (1996), 409–420.

[19] H. Dong and D. Kim, *Parabolic and elliptic systems in divergence form with variably partially BMO coefficients*, SIAM J. Math. Anal., **43** (3) (2011), 1075–1098.

[20] L. C. Evans, *Partial differential equations. Second edition*, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, (2010). xxii+749 pp. ISBN: 978-0-8218-4974-3.

[21] N. Fusco and G. Moscariello, *Further results on the homogenization of quasilinear operators*, Ricerche Mat. **35** (2) (1986), 231–246.

[22] N. Fusco and G. Moscariello, *On the homogenization of quasilinear divergence structure operators*, Ann. Mat. Pura Appl. **146** (4) (1987), 1–13.

[23] J. Geng, $W^{1,p}$ *estimates for elliptic problems with Neumann boundary conditions in Lipschitz domains*, Adv. Math., **229** (4) (2012), 2427–2448.

[24] J. Geng and Z. Shen, *Uniform Regularity Estimates in Parabolic Homogenization*, arXiv:1308.5726v1.

[25] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Mathematics Studies, **105**, Princeton University Press, Princeton, NJ, (1983).

[26] U. Hornung, *Homogenization and Porous Media*, Springer, New York, 1997.

[27] D. Jerison and C. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal., **130** (1) (1995), 161–219.

[28] V. Jikov, S. Kozlov and O. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, Heidelberg, 1994.

[29] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), 415-426.

[30] C. Kenig, F. Lin and Z. Shen, *Homogenization of elliptic systems with Neumann boundary conditions*, J. Amer. Math. Soc., **26** (4) (2013), 901–937.

[31] T. Kilpeläinen and P. Koskela, *Global integrability of the gradients of solutions to partial differential equations*, Nonlinear Anal., **23** (7) (1994), 899–909.

[32] N. V. Krylov, *Parabolic and elliptic equations with VMO coefficients*, Comm. Partial Differential Equations, **32** (1-3) (2007), 453–475.

[33] N. V. Krylov, *Second-order elliptic equations with variably partially VMO coefficients*, J. Funct. Anal. **257** (6) (2009), 1695–1712.

[34] Y. Li and L. Nirenberg, *Estimates for elliptic systems from composite material*, Comm. Pure Appl Math., **56** (7) (2003), 892-925.

[35] G. M. Lieberman, *The conormal derivative problem for equations of variational type in nonsmooth domains*, Trans. Amer. Math. Soc., **330** (1992), 41–67.

[36] N. Meyers, *An L^p estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa, **17** (3) (1963), 189–206.

[37] C. B. Morrey, *Multiple integrals in the calculus of variations*, Grundlehren Math. Wiss., vol. 130, Springer, New York (1966).

[38] E. Reifenberg, *Solutions of the plateau problem for m-dimensional surfaces of varying topological type*, Acta Math., (1960), 1–92.

[39] Z. Shen, *The L^p boundary value problems on Lipschitz domains*, Adv. Math., **216** (2007), 212-254.

[40] Z. Shen, *$W^{1,p}$ estimates for elliptic homogenization problems in nonsmooth domains*, Indiana Univ. Math. J., **57** (2008), 2283–2298.

[41] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Math. Series **43**, Princeton Univ. Press, Princeton, NJ, (1993).

[42] L. Tartar, *The General Theory of Homogenization*, A Personalized Introduction, Lect. Notes Unione Mat. Ital., vol. 7, Springer-Verlag, UMI, Berlin, Bologna, 2009.

[43] T. Toro, *Doubling and flatness: geometry of measures*, Notices Amer. Math. Soc., (1997), 1087–1094.

[44] L. Wang, *A geometric approach to the Calderón-Zygmund estimates*, Acta Math. Sin. (Engl. Ser.), **19** (2) (2003), 381–396.

국문초록

이 논문에서 우리는 부드럽지 않은 영역에서 측정 가능한 계수를 가지는 발산 함수 형태의 타원형 연립 편미분 방정식의 균질화 문제의 약해에 대한 고른 $W^{1,p}$ 가늠에 대해서 연구한다. 우리는 먼저 내부에서의 정칙성에 대해서 고려할 것이며, 이어서 경계값 문제인 디리클레 문제와 쌍대 정규 도함수 문제에 대해서 살펴볼 것이다. 우리의 주요 목적은 균질화 문제에서 칼데론-지그문트 이론이 성립하는 계수와 주어진 영역의 경계의 최소 조건을 찾는 데 있다.

주요어휘: 정칙성 이론, 균질화 문제, 타원형 연립 방정식, BMO 공간, 라이펜버그 영역
학번: 2009-20280