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Crossed products of Cuntz-Pimsner algebras by coactions of Hopf C^*-algebras

(호프 C^*-대수의 쌍대작용에 의한 쿤쯔-핌스너 대수의 교차곱)

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Abstract

Crossed products of Cuntz-Pimsner algebras by coactions of Hopf C^*-algebras

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Unifying two notions of an action and coaction of a locally compact group on a C^*-correspondence we introduce a coaction (σ, δ) of a Hopf C^*-algebra S on a C^*-correspondence (X, A). We show that this coaction naturally induces a coaction ζ of S on the associated Cuntz-Pimsner algebra $\mathcal{O}_X$ under the weak δ-invariancy for the ideal J_X. When the Hopf C^*-algebra S is a reduced Hopf C^*-algebra of a well-behaved multiplicative unitary, we construct from the coaction (σ, δ) a C^*-correspondence $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$, and show that it has a representation on the reduced crossed product $\mathcal{O}_X \rtimes_\zeta \widehat{S}$ by the induced coaction ζ. If this representation is covariant, particularly if either the ideal $J_{X \rtimes_\sigma \widehat{S}}$ of $A \rtimes_\delta \widehat{S}$ is generated by the canonical image of J_X in $M(A \rtimes_\delta \widehat{S})$ or the left action on X by A is injective, the C^*-algebra $\mathcal{O}_X \rtimes_\zeta \widehat{S}$ is shown to be isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}_{X \rtimes_\sigma \widehat{S}}$ associated to $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$. Under the covariance assumption, our results extend the isomorphism result known for actions of amenable groups to arbitrary locally compact groups. Also, the Cuntz-Pimsner covariance condition which was assumed for the same isomorphism result concerning group coactions is shown to be redundant.

Key words: C^*-correspondence, Cuntz-Pimsner algebra, multiplier correspondence, Hopf C^*-algebra, coaction, reduced crossed product
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Contents

Chapter 1

Introduction

In this dissertation, we introduce coactions of Hopf C^*-algebras on C^*-correspondences, and study the *induced* coactions on the associated Cuntz-Pimsner algebras and their crossed products.

A C^*-correspondence (X, A) is a (right) Hilbert A-module X equipped with a left action $\varphi_A : A \to \mathcal{L}(X)$. For each C^*-correspondence (X, A) with injective φ_A, a C^*-algebra $\mathcal{O}_X$ was constructed in [36] generalizing crossed product by $\mathbb{Z}$ and Cuntz-Krieger algebra [10]. The construction was extended in [25] to arbitrary C^*-correspondences (X, A) by considering an ideal J_X of A — the largest ideal that is mapped injectively into $\mathcal{K}(X)$ by φ_A — and requiring that a covariance-like relation should hold on J_X. The C^*-algebra $\mathcal{O}_X$, called the Cuntz-Pimsner algebra associated to (X, A), is generated by $k_X(X)$ and $k_A(A)$ for the universal covariant representation (k_X, k_A) of (X, A). The class of Cuntz-Pimsner algebras is known to be large enough and include in particular graph C^*-algebras. In addition, there have been significant results concerning Cuntz-Pimsner algebras such as gauge invariant uniqueness theorem, criteria on nuclearity or exactness, six-term exact sequence, and description of ideal structure ([36, 25, 26]). Thus the Cuntz-Pimsner algebras can be viewed as a well-understood class of C^*-algebras, and in view of this, it would be advantageous to know that a given C^*-algebra is a Cuntz-Pimsner algebra.

Our work was inspired by [18] and [23] in which group actions and coactions on C^*-correspondences are shown to induce actions and coactions of the same groups on the associated Cuntz-Pimsner algebras, and the crossed products by the induced actions or coactions are proved to be realized as Cuntz-Pimsner algebras. (We refer to [13] for the definition of actions and coactions of locally compact groups on C^*-correspondences.) More precisely, if (γ, α) is an action of a locally compact group G on a C^*-correspondence (X, A), one can form two constructions: an action β of G on $\mathcal{O}_X$ induced by (γ, α) [18, Lemma 2.6.(b)] on the one hand, and the crossed product correspondence $(X \rtimes_{\gamma,r} G, A \rtimes_{\alpha,r} G)$ of (X, A) by (γ, α) ([13, Proposition 3.2] or [18]) on the other. It was shown in [18] that if G is amenable, then the crossed product by the action β is isomorphic to the Cuntz-Pimsner algebra associated to $(X \rtimes_\gamma G, A \rtimes_\alpha G)$:

$$\mathcal{O}_X \rtimes_\beta G \cong \mathcal{O}_{X \rtimes_\gamma G}. \tag{1.1}$$

Similarly, it was shown in [23] that a nondegenerate coaction (σ, δ) of a locally compact group G on (X, A) satisfying an invariance condition induces a coaction ζ of G on $\mathcal{O}_X$ [23, Proposition 3.1], and under the hypothesis of Cuntz-Pimsner covariance, the crossed product by ζ is again a Cuntz-Pimsner algebra [23, Theorem 4.4]:

$$\mathcal{O}_X \rtimes_\zeta G \cong \mathcal{O}_{X \rtimes_\sigma G}, \tag{1.2}$$

where $X \rtimes_\sigma G$ is the C^*-correspondence over $A \rtimes_\delta G$ arising from the coaction (σ, δ) [13, Proposition 3.9].

The study in [3] proposed the framework of reduced Hopf C^*-algebras arising from multiplicative unitaries including both Kac algebras [14] and compact quantum groups [46, 48] (of course locally compact groups as well). The study also established the reduced crossed products of C^*-algebras by reduced Hopf C^*-algebra coactions, which are shown to be a natural generalization of crossed products by group actions and coactions. To each multiplicative unitary V, two reduced Hopf C^*-algebras S_V and $\widehat{S}_V$ are associated in [3] under the regularity condition which was modified later in [47, 38] with manageability or modularity; in particular, the multiplicative unitaries of locally compact quantum groups [31] are known to be manageable. Thus the reduced Hopf

C^*-algebras arising from multiplicative unitaries are a vast generalization of groups and their dual structures.

The goal of this dissertation is to show that essentially the same results can be obtained if group actions or coactions studied in [18, 23] are replaced by Hopf C^*-algebra coactions. To this end, we first need a concept of a coaction of a Hopf C^*-algebra on a C^*-correspondence. In [2], coaction of a Hopf C^*-algebra S on a Hilbert A-module X was introduced as a pair (σ, δ) of a linear map $\sigma : X \to M(X \otimes S)$ and a homomorphism $\delta : A \to M(A \otimes S)$ which are required to be, among other things, compatible with the Hilbert module structure of X. This notion was originally aimed to define equivariant KK-groups and generalize the Kasparov product in the setting of Hopf C^*-algebras. Since then, the notion of coactions on Hilbert modules has been extensively dealt with in various situations: for example [7, 20, 17, 41, 9, 11, 8, 39, 43]. In this dissertation, we propose a definition of coaction of a Hopf C^*-algebra S on a C^*-correspondence (X, A) as a coaction (σ, δ) of S on the Hilbert A-module X which is also compatible with the left action φ_A (see Definition 3.2.1 for the precise definition), and show that this definition unifies the separate notions of group actions and nondegenerate group coactions on (X, A) (Remark 3.2.3). We then proceed to show that the passage from a group action or coaction on (X, A) to an action or coaction on $\mathcal{O}_X$ can be generalized nicely in the Hopf C^*-algebra framework (Theorem 3.2.7). When the Hopf C^*-algebra under consideration is a reduced one defined by a well-behaved multiplicative unitary in the sense of [40], we construct the reduced crossed product correspondences (Theorem 4.2.1), and prove an isomorphism result analogous to (1.1) and (1.2) under a suitable condition (Theorem 5.2.4). Applying our results we improve and extend the main results of [18] and [23] (Remark 5.1.6 and Corollary 5.2.5).

There have been plenty of works concerning "natural" coactions of compact quantum groups on the Cuntz algebra $\mathcal{O}_n$ with the focus on their fixed point algebras: for example, see [16, 27, 33, 35] among others. These coactions are the ones induced by coactions on the finite dimensional C^*-correspondences $(\mathbb{C}^n, \mathbb{C})$, and actually, can be considered within a more general context of graph C^*-algebras [29, 28, 15]. In fact, we extend in Section 6 the notion of labeling of a graph E given in [21] to the setting of compact quantum groups (Definition 6.2.1), and show that this labeling gives rise to a coaction on the graph correspondence $(X(E), A)$ [24], which in turn induces a coaction on the graph C^*-algebra $C^*(E)$ (Corollary 6.2.4). We also give a definition of coaction of a compact quantum group on a finite graph (Definition 6.2.5) and show that this coaction gives rise to a coaction on the graph C^*-algebra (Theorem 6.2.11). Natural coactions on $\mathcal{O}_n$ then can be viewed as the ones arising from labelings of the graph consisting of one vertex and n edges, or alternatively, the ones arising from coactions on such a graph. Moreover, the crossed products by those natural coactions can be realized as Cuntz-Pimsner algebras (Example 6.2.16). In light of these facts, it is natural and desirable to extend the works of [18, 23] from the point of view of Hopf C^*-algebra coactions. It should be pointed out that a coaction of a compact quantum group on a finite graph E was considered in [5] under the aim of constructing the quantum automorphism group coacing on E along the principle of [45]. The definition however was given only for finite graphs with at most one edge from a vertex to another. Our concern lies in the coactions of compact quantum groups on the graph C^*-algebras $C^*(E)$ arising from coactions on E as well as the quantum automorphism groups for any finite graphs (Theorem 6.2.14).

This dissertation is organized as follows.

In Chapter 2, we review basic facts from [13, Chapter 1] and [12, Appendix A] on multiplier correspondences. We also collect from [25, 3] definitions and facts on Cuntz-Pimsner algebras and reduced crossed products by Hopf C^*-algebra coactions on C^*-algebras. Note that in [13], Hilbert A-B bimodules were considered while we are concerned only with Hilbert A-A bimodules, namely nondegenerate C^*-correspondences (X, A).

In Chapter 3, we define a coaction (σ, δ) of a Hopf C^*-algebra S on a C^*-correspondence (X, A) (Definition 3.2.1) generalizing both an action and nondegenerate coaction of a locally compact group on (X, A). We justify our definition in Theorem A.2.1 by showing that it agrees with the definition of an action of a locally compact group G on (X, A) when S is the commutative Hopf C^*-algebra $C_0(G)$. We prove in Theorem 3.2.7 that if (σ, δ) is a coaction of S on (X, A) such that the ideal J_X is weakly δ-invariant (Definition 3.2.5), then (σ, δ) induces a coaction ζ of S on the associated Cuntz-Pimsner algebra

$\mathcal{O}_X$.

Chapter 4 is devoted to constructing the reduced crossed product correspondence $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ from a coaction (σ, δ) of a reduced Hopf C^*-algebra S defined by a well-behaved multiplicative unitary. Our space $X \rtimes_\sigma \widehat{S}$ coincides with the one given in [7, Definition 1.2] as a Hilbert $(A \rtimes_\delta \widehat{S})$-module, but further can be said, that is, it is a C^*-correspondence over $A \rtimes_\delta \widehat{S}$. An important step of [7] toward its main result of imprimitivity was the Baaj-Skandalis type lemma [7, Proposition 1.3] in which the proof invokes a linking algebra technique to utilize the strict continuity of slice maps. We provide an alternative and intuitive proof for the lemma in our C^*-correspondence setting. We show that slice maps on the algebraic tensor product of a C^*-correspondence X and a C^*-algebra B can be extended strictly to the multiplier correspondence $M(X \otimes B)$ (see Proposition 4.1.2), which enables us to construct the C^*-correspondence $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ directly. We show that our construction of $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ reduces to the crossed product correspondence in the sense of [13] if the Hopf C^*-algebra coaction under consideration comes from a group action or nondegenerate group coaction on (X, A) (Corollary B.2.3 and Remark 4.2.4).

In Chapter 5, we prove an isomorphism analogous to (1.1) and (1.2) in the reduced Hopf C^*-algebra setting. Along the way we answer the question posed in [23, Remark 4.5]; specifically, we prove that Theorem 4.4 of [23] still holds without the hypothesis of the Cuntz-Pimsner covariance for the canonical embedding of (X, A) into the crossed product correspondence (see Remark 5.1.6). The C^*-correspondence $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ is shown to have a canonical representation $(k_X \rtimes_\sigma \mathrm{id}, k_A \rtimes_\delta \mathrm{id})$ on the reduced crossed product $\mathcal{O}_X \rtimes_\zeta \widehat{S}$ by the induced coaction ζ (Proposition 5.1.4). We then prove in Theorem 5.2.4 that

$$\mathcal{O}_X \rtimes_\zeta \widehat{S} \cong \mathcal{O}_{X \rtimes_\sigma \widehat{S}}$$

under the assumption that $(k_X \rtimes_\sigma \mathrm{id}, k_A \rtimes_\delta \mathrm{id})$ is covariant. By applying this to group actions, we extend Theorem 2.10 of [18] (see Corollary 5.2.5) to any locally compact groups.

It is however, not so easy to determine whether the representation $(k_X \rtimes_\sigma$

$\mathrm{id}, k_A \rtimes_\delta \mathrm{id})$ is covariant or not without understanding the ideal $J_{X \rtimes_\sigma \widehat{S}}$ of $A \rtimes_\delta \widehat{S}$. Actually $J_{X \rtimes_\sigma \widehat{S}}$ is not known even for the commutative case with some exceptions. For an action (γ, α) of a locally compact group G, it was shown that $J_{X \rtimes_\gamma G} = J_X \rtimes_\alpha G$ if G is amenable ([18, Proposition 2.7]), which was the most difficult part in proving the main result of [18] as was mentioned in the introductory section there. Recently, the same has been shown for a discrete group G if G is exact or if the action α has Exel's Approximation Property ([4, Theorem 5.5]). However, we only know in general that $J_{X \rtimes_\sigma \widehat{S}}$ contains the ideal of $A \rtimes_\delta \widehat{S}$ generated by the image $\delta_\iota(J_X)$ (Proposition 5.1.5). We bypass the difficulty regarding the ideal $J_{X \rtimes_\sigma \widehat{S}}$ by focusing our attention on the $(A \otimes \mathscr{K})$-multiplier correspondence $(M_{A \otimes \mathscr{K}}(X \otimes \mathscr{K}), M(A \otimes \mathscr{K}))$ in which the C^*-correspondence $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ lies. This leads us to two equivalent conditions that the representation $(k_X \rtimes_\sigma \mathrm{id}, k_A \rtimes_\delta \mathrm{id})$ is covariant (Theorem 5.2.1). From these equivalent conditions we see that $(k_X \rtimes_\sigma \mathrm{id}, k_A \rtimes_\delta \mathrm{id})$ is covariant if, in particular, $J_{X \rtimes_\sigma \widehat{S}}$ is generated by $\delta_\iota(J_X)$ or the left action φ_A is injective (Corollary 5.2.2), even though we do not know the ideal $J_{X \rtimes_\sigma \widehat{S}}$ explicitly.

Applying the results obtained in Chapter 3–5, we consider in Chapter 6 coactions on crossed products by $\mathbb{Z}$ and directed graph C^*-algebras which form a special class of Cuntz-Pimsner algebras.

Finally, we provide two appendices. In Appendix A, we generalize [1, Corollary 3.4] to our C^*-correspondence setting, and then show that there exists a one-to-one correspondence between actions of a locally compact group G and coactions of the commutative Hopf C^*-algebra $C_0(G)$ on a C^*-correspondence. In Appendix B, we prove a C^*-correspondence analogue to the well-known fact that $\mathcal{L}_A(A \otimes \mathcal{H}) = M(A \otimes \mathcal{K}(\mathcal{H}))$ for a C^*-algebra A and a Hilbert space $\mathcal{H}$. Using this, we recover from our construction of $(X \rtimes \widehat{S}, A \rtimes \widehat{S})$ the crossed product correspondences $(X \rtimes_r G, A \rtimes_r G)$ for actions of locally compact groups G given in [13].

Chapter 2

Preliminaries

In this chapter, we review some definitions and properties related to multiplier correspondences, Cuntz-Pimsner algebras, and reduced crossed products by Hopf C^*-algebra coactions. Our references include [3, 12, 13, 25, 32]. We also fix some notations.

2.1 C^*-correspondences

Throughout the dissertation A denotes a C^*-algebra. A (right) *inner product A-module* is a right A-module X which is at the same time a linear space, together with an A-valued inner product $\langle \cdot, \cdot \rangle_A : X \times X \to A$ that is sesquilinear, respects the right action, and is positive definite. More precisely, the scalar multiplication on X is consistent with the right action by A such that $(c\xi \cdot a) = c(\xi \cdot a) = \xi \cdot (ca)$ for $c \in \mathbb{C}$, $\xi \in X$, and $a \in A$, and $\langle \cdot, \cdot \rangle_A$ satisfies the following:

(i) $\langle \xi, c\eta + \eta' \rangle_A = c\langle \xi, \eta \rangle_A + \langle \xi, \eta' \rangle_A$;

(ii) $\langle \xi, \eta \cdot a \rangle_A = \langle \xi, \eta \rangle_A a$;

(iii) $\langle \xi, \eta \rangle_A^* = \langle \eta, \xi \rangle_A$;

(iv) $\langle \xi, \xi \rangle_A \geq 0$; $\langle \xi, \xi \rangle_A = 0 \Leftrightarrow \xi = 0$.

An inner product A-module X is called a *Hilbert A-module* if it is complete with respect to the norm $\|\xi\| := \|\langle\xi,\xi\rangle_A\|^{1/2}$ $(\xi \in X)$.

Hilbert spaces are Hilbert $\mathbb{C}$-modules with the convention that the inner product is conjugate linear in the first variable. Every C^*-algebra A is itself a Hilbert A-module with the inner product given by $\langle a,b\rangle_A := a^*b$. When we view a C^*-algebra as a Hilbert module, we always refer to this Hilbert module.

For two Hilbert A-modules X and Y, a map $T : X \to Y$ is said to be *adjointable* if there exists a map $T^* : Y \to X$ such that

$$\langle T\xi,\eta\rangle_A = \langle\xi,T^*\eta\rangle_A \quad (\xi,\eta \in X).$$

It is not hard to see that an adjointable map is bounded, linear, and A-linear. We denote by $\mathcal{L}(X,Y)$ (or $\mathcal{L}_A(X,Y)$) the set of all adjointable maps from X to Y. It is straightforward to see that $\mathcal{L}(X,Y)$ is a Banach space with the operator norm. We denote by $\mathcal{K}(X,Y) = \mathcal{K}_A(X,Y)$ the closed subspace of $\mathcal{L}(X,Y)$ generated by the operators $\theta_{\xi,\eta}$:

$$\theta_{\xi,\eta}(\eta') = \xi \cdot \langle\eta,\eta'\rangle_A \quad (\xi \in Y,\ \eta,\eta' \in X).$$

We simply write $\mathcal{L}(X)$ and $\mathcal{K}(X)$ for $\mathcal{L}(X,X)$ and $\mathcal{K}(X,X)$; in this case $\mathcal{L}(X)$ becomes a maximal unital C^*-algebra containing $\mathcal{K}(X)$ as an essential ideal. For a C^*-algebra A, we write $M(A) = \mathcal{L}(A)$ and call it the *multiplier algebra* of A. Note that the left multiplication gives an isomorphism of A onto $\mathcal{K}(A)$, and we always regard A as a C^*-subalgebra of $M(A)$ through this isomorphism. For a Hilbert $\mathbb{C}$-module X, $\mathcal{L}(X)$ and $\mathcal{K}(X)$ are, respectively, the usual C^*-algebras of bounded operators and compact operators on the Hilbert space X.

A *C^*-correspondence* over a C^*-algebra A is a Hilbert A-module X equipped with a homomorphism $\varphi_A : A \to \mathcal{L}(X)$, called the *left action*. We use the notation (X,A) of [23] to refer to a C^*-correspondence X over A. We say that (X,A) is *nondegenerate* if φ_A is nondegenerate, that is, $\overline{\varphi_A(A)X} = X$. Every C^*-algebra A has the natural structure of a nondegenerate C^*-correspondence over itself with the left action identifying A with $\mathcal{K}(A)$ through left multiplication, called the *identity correspondence* (p. 368 of

[25]). When we regard A as a C^*-correspondence, we always mean this C^*-correspondence

2.2 Multiplier correspondences

Throughout the dissertation, we restrict ourselves to nondegenerate C^*-correspondences, which in particular allows us to consider their multiplier correspondences that are a generalization of multiplier C^*-algebras.

Let (X, A) be a C^*-correspondence, and let $M(X) := \mathcal{L}(A, X)$. The *multiplier correspondence* of X is the C^*-correspondence $M(X)$ over the multiplier algebra $M(A)$ with the Hilbert $M(A)$-module operations

$$m \cdot a := ma, \quad \langle m, n\rangle_{M(A)} := m^*n \tag{2.1}$$

and the left action

$$\varphi_{M(A)}(a)m := \overline{\varphi_A}(a)m \tag{2.2}$$

for $m, n \in M(X)$ and $a \in M(A)$, where $\overline{\varphi_A}$ is the strict extension of the nondegenerate homomorphism φ_A and ma, m^*n, and $\varphi_{M(A)}(a)m$ mean the compositions $m \circ a$, $m^* \circ n$, and $\varphi_{M(A)}(a) \circ m$, respectively. The identification of X with $\mathcal{K}(A, X)$, in which each $\xi \in X$ is regarded as the operator $A \ni a \mapsto \xi \cdot a \in X$, gives an embedding of X into $M(X)$, and we will always regard X as a subspace of $M(X)$ through this embedding. Note that $\mathcal{K}(M(X)) \subseteq M(\mathcal{K}(X))$ nondegenerately; but $\mathcal{K}(M(X)) \neq M(\mathcal{K}(X))$ in general. For example, if $X = \mathcal{H}$ then $\mathcal{K}(M(X)) = \mathcal{K}(\mathcal{H})$ and $M(\mathcal{K}(X)) = \mathcal{L}(\mathcal{H})$.

The *strict topology* on $M(X)$ is the locally convex topology such that a net $\{m_i\}$ in $M(X)$ converges strictly to 0 if and only if for $T \in \mathcal{K}(X)$ and $a \in A$, the nets $\{Tm_i\}$ and $\{m_i \cdot a\}$ both converge in norm to 0. It can be shown that $M(X)$ is the strict completion of X.

Let (X, A) and (Y, B) be C^*-correspondences. A pair

$$(\psi, \pi) : (X, A) \to (M(Y), M(B))$$

of a linear map $\psi : X \to M(Y)$ and a homomorphism $\pi : A \to M(B)$ is called a *correspondence homomorphism* if

(i) $\psi(\varphi_A(a)\xi) = \varphi_{M(B)}(\pi(a))\,\psi(\xi)$ for $a \in A$ and $\xi \in X$;

(ii) $\pi(\langle \xi, \eta\rangle_A) = \langle \psi(\xi), \psi(\eta)\rangle_{M(B)}$ for $\xi, \eta \in X$.

It is automatic that $\psi(\xi \cdot a) = \psi(\xi) \cdot \pi(a)$ (see the comment below [26, Definition 2.3]). We say that (ψ, π) is *injective* if π is injective; if so ψ is isometric. We also say that (ψ, π) is *nondegenerate* if $\overline{\psi(X) \cdot B} = Y$ and $\overline{\pi(A)B} = B$. In this case, (ψ, π) extends uniquely to a strictly continuous correspondence homomorphism

$$(\overline{\psi}, \overline{\pi}) : (M(X), M(A)) \to (M(Y), M(B))$$

([13, Theorem 1.30]). Note that if (ψ, π) is injective, then so is $(\overline{\psi}, \overline{\pi})$.

A correspondence homomorphism $(\psi, \pi) : (X, A) \to (M(Y), M(B))$ determines a (unique) homomorphism $\psi^{(1)} : \mathcal{K}(X) \to \mathcal{K}(M(Y)) \subseteq M(\mathcal{K}(Y))$ such that

$$\psi^{(1)}(\theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^* \quad (\xi, \eta \in X)$$

(see for example [26, Definition 2.4] and the comment below it). If (ψ, π) is nondegenerate, then so is $\psi^{(1)}$; it is straightforward to verify that

$$\psi(T\xi) = \overline{\psi^{(1)}}(T)\psi(\xi), \quad \overline{\psi^{(1)}}(mn^*) = \overline{\psi}(m)\overline{\psi}(n)^* \tag{2.3}$$

for $T \in \mathcal{L}(X)$, $\xi \in X$, and $m, n \in M(X)$. Indeed, write $\xi = \theta_{\zeta,\zeta}(\zeta)$ [37, Proposition 2.31]. Since $\psi^{(1)}(\theta_{\eta_1,\eta_2})\psi(\eta_3) = \psi(\eta_1 \cdot \langle \eta_2, \eta_3\rangle_A)$, we have

$$\begin{aligned}\psi(T\xi) &= \psi(T\zeta \cdot \langle \zeta, \zeta\rangle_A) = \psi^{(1)}(\theta_{T\zeta,\zeta})\psi(\zeta)\\ &= \overline{\psi^{(1)}}(T)\psi^{(1)}(\theta_{\zeta,\zeta})\psi(\zeta) = \overline{\psi^{(1)}}(T)\psi(\xi),\end{aligned}$$

which verifies the first relation of (2.3). It then follows that

$$\overline{\psi^{(1)}}(mn^*)\psi(\xi) \cdot b = \psi(mn^*\xi) \cdot b = \overline{\psi}(m) \cdot \pi(\langle n, \xi\rangle_A)b = \overline{\psi}(m)\overline{\psi}(n)^*\psi(\xi) \cdot b.$$

This verifies the second relation of (2.3) since (ψ, π) is nondegenerate. We note that the first relation of (2.3) shows that $\overline{\psi^{(1)}}$ is injective whenever ψ is injective. We also note the following analogue to the fact that $M(M(A)) = M(A)$ for a C^*-algebra A although we do not need it in the sequel.

Remark 2.2.1. We have $M(M(X)) = M(X)$ for a C^*-correspondence (X, A), whose proof can be given as follows. The identity maps $\mathrm{id}_{M(X)}$ and $\mathrm{id}_{M(A)}$ form a correspondence homomorphism $(\mathrm{id}_{M(X)}, \mathrm{id}_{M(A)})$ which is clearly nondegenerate. Consider the strict extension

$$(\overline{\mathrm{id}_{M(X)}}, \overline{\mathrm{id}_{M(A)}}) : (M(M(X)), M(M(A))) \to (M(X), M(A)).$$

Obviously, $\overline{\mathrm{id}_{M(X)}}$ is surjective. It is injective as well since $\overline{\mathrm{id}_{M(A)}} = \mathrm{id}_{M(A)}$ is injective. Consequently, $M(M(X)) = M(X)$.

2.3 Tensor product correspondences

In this dissertation, the tensor product of two C^*-algebras always means the minimal tensor product. For two Hilbert A_i-modules X_i $(i = 1, 2)$, their tensor product $X_1 \otimes X_2$ denotes the exterior tensor product given in [32, pp. 34–35], which is a Hilbert $(A_1 \otimes A_2)$-module such that

$$(\xi_1 \otimes \xi_2) \cdot (a_1 \otimes a_2) = \xi_1 \cdot a_1 \otimes \xi_2 \cdot a_2,$$
$$\langle \xi_1 \otimes \xi_2, \xi_1' \otimes \xi_2' \rangle_{A_1 \otimes A_2} = \langle \xi_1, \xi_1' \rangle_{A_1} \otimes \langle \xi_2, \xi_2' \rangle_{A_2}$$

for $\xi_1, \xi_1' \in X_1$, $\xi_2, \xi_2' \in X_2$, $a_1 \in A_1$, and $a_2 \in A_2$.

Let (X_1, A_1) and (X_2, A_2) be C^*-correspondences. We will freely use the following identification

$$\mathcal{K}(X_1 \otimes X_2) = \mathcal{K}(X_1) \otimes \mathcal{K}(X_2)$$

via $\theta_{\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2} = \theta_{\xi_1, \eta_1} \otimes \theta_{\xi_2, \eta_2}$. Equipped with the left action

$$\varphi_{A_1 \otimes A_2} = \varphi_{A_1} \otimes \varphi_{A_2},$$

the tensor product $X_1 \otimes X_2$ then becomes a C^*-correspondence over $A_1 \otimes A_2$ which we call the *tensor product correspondence*.

Let $(\psi_i, \pi_i) : (X_i, A_i) \to (M(Y_i), M(B_i))$ $(i = 1, 2)$ be correspondence homomorphisms. Then there exists a unique correspondence homomorphism

$$(\psi_1 \otimes \psi_2, \pi_1 \otimes \pi_2) : (X_1 \otimes X_2, A_1 \otimes A_2) \to (M(Y_1 \otimes Y_2), M(B_1 \otimes B_2))$$

such that $(\psi_1 \otimes \psi_2)(\xi_1 \otimes \xi_2) = \psi_1(\xi_1) \otimes \psi_2(\xi_2)$. If both (ψ_i, π_i) are nondegenerate then $(\psi_1 \otimes \psi_2, \pi_1 \otimes \pi_2)$ is also nondegenerate ([13, Proposition 1.38]).

Remark 2.3.1. We can easily check that

$$(\psi_1 \otimes \psi_2)^{(1)} = \psi_1^{(1)} \otimes \psi_2^{(1)},$$

that is, we have the commutative diagram

$$\begin{tikzcd}
\mathcal{K}(X_1 \otimes X_2) \arrow[r, "(\psi_1 \otimes \psi_2)^{(1)}"] \arrow[d, equal] & M(\mathcal{K}(Y_1 \otimes Y_2)) \arrow[d, equal] \\
\mathcal{K}(X_1) \otimes \mathcal{K}(X_2) \arrow[r, "\psi_1^{(1)} \otimes \psi_2^{(1)}"'] & M(\mathcal{K}(Y_1) \otimes \mathcal{K}(Y_2))
\end{tikzcd}$$

for two correspondence homomorphisms (ψ_1, π_1) and (ψ_2, π_2). Indeed,

$$\begin{aligned}
(\psi_1 \otimes \psi_2)^{(1)}(\theta_{\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2}) &= \big((\psi_1 \otimes \psi_2)(\xi_1 \otimes \xi_2)\big)\big((\psi_1 \otimes \psi_2)(\eta_1 \otimes \eta_2)\big)^* \\
&= \psi_1(\xi_1)\psi_1(\eta_1)^* \otimes \psi_2(\xi_2)\psi_2(\eta_2)^* \\
&= \big(\psi_1^{(1)} \otimes \psi_2^{(1)}\big)(\theta_{\xi_1, \eta_1} \otimes \theta_{\xi_2, \eta_2})
\end{aligned}$$

for $\xi_1, \eta_1 \in X_1$ and $\xi_2, \eta_2 \in X_2$.

2.4 Cuntz-Pimsner algebras

Let (X, A) be a C^*-correspondence, and let

$$J_X := \varphi_A^{-1}(\mathcal{K}(X)) \cap \{a \in A : ab = 0 \text{ for } b \in \ker \varphi_A\}.$$

Then J_X is characterized as the largest ideal of A which is mapped injectively into $\mathcal{K}(X)$ by φ_A. A correspondence homomorphism $(\psi, \pi) : (X, A) \to (B, B)$ into an identity correspondence (B, B) is called a *representation* of (X, A) on B and denoted simply by $(\psi, \pi) : (X, A) \to B$. We say that (ψ, π) is *covariant* if

$$\psi^{(1)}(\varphi_A(a)) = \pi(a) \quad (a \in J_X)$$

([25, Definition 3.4]). We denote by (k_X, k_A) the universal covariant representation of (X, A) which is known to be injective ([25, Proposition 4.9]). The *Cuntz-Pimsner algebra* $\mathcal{O}_X$ is the C^*-algebra generated by $k_X(X)$ and $k_A(A)$. Note that the embedding $k_A : A \hookrightarrow \mathcal{O}_X$ is nondegenerate by our standing assumption that (X, A) is nondegenerate. From the universality of (k_X, k_A), if (ψ, π) is a covariant representation of (X, A) on B, there exists a unique homomorphism $\psi \times \pi : \mathcal{O}_X \to B$ called the *integrated form* of (ψ, π) such that $\psi = (\psi \times \pi) \circ k_X$ and $\pi = (\psi \times \pi) \circ k_A$.

A representation (ψ, π) of (X, A) is said to *admit a gauge action* if there exists an action β of the unit circle $\mathbb{T}$ on the C^*-subalgebra generated by $\psi(X)$ and $\pi(A)$ such that $\beta_z(\psi(\xi)) = z\psi(\xi)$ and $\beta_z(\pi(a)) = \pi(a)$ for $z \in \mathbb{T}$, $\xi \in X$, and $a \in A$. The universal covariant representation (k_X, k_A) clearly admits a gauge action. The *gauge invariant uniqueness theorem* [25, Theorem 6.4] asserts that an injective covariant representation (ψ, π) admits a gauge action if and only if $\psi \times \pi$ is injective.

2.5 C-multiplier correspondences

Let (X, A) be a C^*-correspondence, C be a C^*-algebra, and $\kappa : C \to M(A)$ be a nondegenerate homomorphism. The *C-multiplier correspondence* $M_C(X)$ of X and the *C-multiplier algebra* $M_C(A)$ of A are defined by

$$M_C(X) := \{m \in M(X) : \varphi_{M(A)}(\kappa(C))m \cup m \cdot \kappa(C) \subseteq X\},$$

$$M_C(A) := \{a \in M(A) : \kappa(C)a \cup a\kappa(C) \subseteq A\}.$$

Under the restriction of the operations (2.1) and (2.2), $(M_C(X), M_C(A))$ becomes a C^*-correspondence ([12, Lemma A.9.(2)]).

Notations 2.5.1. We mean by $M_A(X)$ the A-multiplier correspondence

$$M_A(X) = \big\{m \in M(X) : \varphi_A(A)m \subseteq X\big\}$$

determined by $\kappa = \mathrm{id}_A$, and by $M_A(\mathcal{K}(X))$ the A-multiplier algebra

$$M_A(\mathcal{K}(X)) = \{m \in M(\mathcal{K}(X)) : \varphi_A(A)m \cup m\varphi_A(A) \subseteq \mathcal{K}(X)\}$$

determined by the left action φ_A.

Note that $\mathcal{K}(M_A(X)) \subseteq M_A(\mathcal{K}(X))$ ([12, Lemma A.9.(3)]).

The *C-strict topology* on $M_C(X)$ is the locally convex topology whose neighborhood system at 0 is generated by the family $\{m : \|\varphi_{M(A)}(\kappa(c))m\| \leq \epsilon\}$ and $\{m : \|m \cdot \kappa(c)\| \leq \epsilon\}$ $(c \in C,\ \epsilon > 0)$. The C-strict topology is stronger than the relative strict topology on $M_C(X)$, and $M_C(X)$ is the C-strict completion of X. Likewise, the *C-strict topology* on $M_C(A)$ is the locally convex topology defined by the family of seminorms $\|\kappa(c) \cdot \| + \| \cdot \kappa(c)\|$ $(c \in C)$.

Remark 2.5.2. Let (X, A) be a C^*-correspondence and $M_{C_i}(X)$ be the C_i-multiplier correspondence determined by a nondegenerate homomorphism $\kappa_i : C_i \to M(A)$ $(i = 1, 2)$. It is clear that if $\kappa_1(C_1)$ is nondegenerately contained in $M(\kappa_2(C_2))(\subseteq M(A))$, then $M_{C_1}(X) \subseteq M_{C_2}(X)$ and the C_1-strict topology on $M_{C_1}(X)$ is stronger than the relative C_2-strict topology. In particular, $M_C(X) \subseteq M_A(X)$ and the C-strict topology is stronger than the relative A-strict topology.

For a possibly degenerate correspondence homomorphism, we still have an extension by [12, Proposition A.11]. Let $(\psi, \pi) : (X, A) \to (M_D(Y), M_D(B))$ be a correspondence homomorphism, where $(M_D(Y), M_D(B))$ is a D-multiplier correspondence determined by a nondegenerate homomorphism $\kappa_D : D \to M(B)$. Assume that $\kappa_C : C \to M(A)$ and $\lambda : C \to M(\kappa_D(D))(\subseteq M(B))$ are nondegenerate homomorphisms such that

$$\pi(\kappa_C(c)a) = \lambda(c)\pi(a) \quad (c \in C,\ a \in A).$$

Then (ψ, π) extends uniquely to a C-strict to D-strictly continuous correspondence homomorphism

$$(\overline{\psi}, \overline{\pi}) : (M_C(X), M_C(A)) \to (M_D(Y), M_D(B)),$$

where $(M_C(X), M_C(A))$ is the C-multiplier correspondence determined by κ_C.

Remarks 2.5.3. (1) If (ψ, π) is nondegenerate, then every C-strict to D-strictly continuous extension of (ψ, π) coincides with the restriction of its usual strict extension.

(2) Suppose that $\overline{\psi}_i : M_{C_i}(X) \to M_{D_i}(Y)$ are C_i-strict to D_i-strictly continuous extensions ($i = 1, 2$). If $M_{C_1}(X) \subseteq M_{C_2}(X)$ and $M_{D_1}(Y) \subseteq M_{D_2}(Y)$ and if the C_1-strict and D_1-strict topologies are stronger than the relative C_2-strict and D_2-strict topologies, respectively, then $\overline{\psi}_1 = \overline{\psi}_2|_{M_{C_1}(X)}$.

We will frequently need the following special form of [12, Proposition A.11].

Theorem 2.5.4 ([12, Corollary A.14])**.** *Let $(\psi, \pi) : (X, A) \to B$ be a representation with π nondegenerate. Then*

(i) *(ψ, π) extends uniquely to an A-strictly continuous correspondence homomorphism*

$$(\overline{\psi}, \overline{\pi}) : (M_A(X), M(A)) \to M_A(B),$$

where $M_A(B)$ is the A-multiplier algebra determined by π.

(ii) *$\psi^{(1)} : \mathcal{K}(X) \to B$ extends uniquely to an A-strictly continuous homomorphism $\overline{\psi^{(1)}} : M_A(\mathcal{K}(X)) \to M_A(B)$; moreover,*

$$\overline{\psi^{(1)}} = \overline{\psi}^{(1)}$$

on $\mathcal{K}(M_A(X))$, that is, $\overline{\psi^{(1)}}(mn^*) = \overline{\psi}(m)\overline{\psi}(n)^*$ for $m, n \in M_A(X)$.

2.6 Reduced and dual reduced Hopf C^*-algebras

By a *Hopf C^*-algebra* we always mean a bisimplifiable Hopf C^*-algebra in the sense of [3], that is, a pair (S, Δ) of a C^*-algebra S and a nondegenerate homomorphism $\Delta : S \to M(S \otimes S)$ called the *comultiplication* of S satisfying

(i) $\overline{\Delta \otimes \mathrm{id}} \circ \Delta = \overline{\mathrm{id} \otimes \Delta} \circ \Delta$;

(ii) $\overline{\Delta(S)(1_{M(S)} \otimes S)} = S \otimes S = \overline{\Delta(S)(S \otimes 1_{M(S)})}$.

Let G be a locally compact group. Then $(C_0(G), \Delta_G)$ is a Hopf C^*-algebra with the comultiplication $\Delta_G(f)(r,s) = f(rs)$ for $f \in C_0(G)$ and $r, s \in G$. The full group C^*-algebra $C^*(G)$ equipped with the comultiplication given by $r \mapsto r \otimes r$ for $r \in G$ is also a Hopf C^*-algebra. The same is true for the reduced group C^*-algebra $C^*_r(G)$ such that the canonical surjection $\lambda : C^*(G) \to C^*_r(G)$ is a morphism in the sense of [3] (also see [40, Example 4.2.2]).

Let $\mathcal{H}$ be a Hilbert space. A unitary operator V acting on $\mathcal{H} \otimes \mathcal{H}$ is said to be *multiplicative* if it satisfies the pentagonal relation $V_{12}V_{13}V_{23} = V_{23}V_{12}$, where we use the leg-numbering notations V_{ij} such that $V_{12} \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$ denotes the unitary $V \otimes 1$ for example (see [3, p. 428]). For each functional $\omega \in \mathcal{L}(\mathcal{H})_*$, define the operators $L(\omega)$ and $\rho(\omega)$ in $\mathcal{L}(\mathcal{H})$ by

$$L(\omega) = \overline{\omega \otimes \mathrm{id}}(V), \quad \rho(\omega) = \overline{\mathrm{id} \otimes \omega}(V),$$

where the maps $\overline{\omega \otimes \mathrm{id}}$ and $\overline{\mathrm{id} \otimes \omega}$ denote the usual strict extension to the multiplier algebra $M(\mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\mathcal{H}))(= \mathcal{L}(\mathcal{H} \otimes \mathcal{H}))$. The *reduced algebra* S_V and the *dual reduced algebra* $\widehat{S}_V$ are defined as the following norm closed subspaces of $\mathcal{L}(\mathcal{H})$:

$$S_V = \overline{\{L(\omega) : \omega \in \mathcal{L}(\mathcal{H})_*\}}, \quad \widehat{S}_V = \overline{\{\rho(\omega) : \omega \in \mathcal{L}(\mathcal{H})_*\}}.$$

They are known to be nondegenerate subalgebras of $\mathcal{L}(\mathcal{H})$ ([3, Proposition 1.4]).

A multiplicative unitary V acting on $\mathcal{H} \otimes \mathcal{H}$ is said to be *well-behaved* if both S_V and $\widehat{S}_V$ are Hopf C^*-algebras with the comultiplications

$$\Delta_V(s) = V(s \otimes 1)V^*, \quad \widehat{\Delta}_V(x) = V^*(1 \otimes x)V \tag{2.4}$$

for $s \in S$ and $x \in \widehat{S}$, and $V \in M(\widehat{S} \otimes S)$ ([40, Definition 7.2.6.i)]).

Remark 2.6.1. When we consider a well-behaved multiplicative unitary V, we will not need the last property $V \in M(\widehat{S} \otimes S)$. In fact, we only need the property that V gives rise to two Hopf C^*-algebras S_V and $\widehat{S}_V$. We use the terminology of well-behavedness just because we do not want to define a new terminology. It should be stressed though that many important and significant Hopf C^*-algebras come from well-behaved multiplicative unitaries the class of which includes those with regularity [3], manageability [47] and modularity [38]. In particular, locally compact quantum groups [31] are the Hopf C^*-algebras arising from well-behaved multiplicative unitaries.

For a locally compact group G, let W_G and $\widehat{W}_G$ be the regular multiplicative unitaries acting on $L^2(G) \otimes L^2(G)$ by

$$(W_G\xi)(r,s) = \xi(r, r^{-1}s), \quad (\widehat{W}_G\xi)(r,s) = \xi(sr, s)$$

for $\xi \in C_c(G \times G)$ and $r, s \in G$. It can be shown that $S_{W_G} = C_r^*(G) = \widehat{S}_{\widehat{W}_G}$ as Hopf C^*-algebras. Let μ_G and $\check{\mu}_G$ be the nondegenerate embeddings $C_0(G) \hookrightarrow \mathcal{L}(L^2(G))$ given by

$$\big(\mu_G(f)h\big)(r) = f(r)h(r), \quad \big(\check{\mu}_G(f)h\big)(r) = f(r^{-1})h(r) \tag{2.5}$$

for $h \in C_c(G)$. Then μ_G and $\check{\mu}_G$ are isomorphisms from the Hopf C^*-algebra $C_0(G)$ onto $\widehat{S}_{W_G}$ and $S_{\widehat{W}_G}$, respectively (see for example [40, Example 9.3.11]).

2.7 Reduced crossed products $A \rtimes \widehat{S}$

By a *coaction* of a Hopf C^*-algebra (S, Δ) on a C^*-algebra A we always mean a nondegenerate homomorphism $\delta : A \to M(A \otimes S)$ such that

(i) δ satisfies the *coaction identity* $\overline{\delta \otimes \mathrm{id}} \circ \delta = \overline{\mathrm{id} \otimes \Delta} \circ \delta$;

(ii) δ satisfies the *coaction nondegeneracy* $\overline{\delta(A)(1_{M(A)} \otimes S)} = A \otimes S$.

Let V be a well-behaved multiplicative unitary acting on $\mathcal{H} \otimes \mathcal{H}$. Let δ be a coaction of the reduced Hopf C^*-algebra S_V on A and $\iota_{S_V} : S_V \hookrightarrow M(\mathcal{K}(\mathcal{H}))$ be the inclusion map. We denote by δ_ι the following composition

$$\delta_\iota := \overline{\mathrm{id}_A \otimes \iota_{S_V}} \circ \delta : A \to M(A \otimes \mathcal{K}(\mathcal{H})). \tag{2.6}$$

The *reduced crossed product* $A \rtimes_\delta \widehat{S}_V$ of A by the coaction δ of S_V is defined to be the following norm closed subspace of $M(A \otimes \mathcal{K}(\mathcal{H}))$

$$A \rtimes_\delta \widehat{S}_V = \overline{\delta_\iota(A)(1_{M(A)} \otimes \widehat{S}_V)},$$

where $1_{M(A)} \otimes \widehat{S}_V$ denotes the image of the canonical embedding $\widehat{S}_V \hookrightarrow M(A \otimes \mathcal{K}(\mathcal{H}))$. By [3, Lemma 7.2], $A \rtimes_\delta \widehat{S}_V$ is a C^*-algebra.

Remark 2.7.1. In the literature, the reduced crossed product $A \rtimes_\delta \widehat{S}_V$ is usually defined as a subalgebra of $\mathcal{L}_A(A \otimes \mathcal{H})$ which can be identified with $M(A \otimes \mathcal{K}(\mathcal{H}))$. For the arguments concerning multiplier correspondences and the relevant strict topologies, it seems to be more convenient to work with $M(A \otimes \mathcal{K}(\mathcal{H}))$ rather than $\mathcal{L}_A(A \otimes \mathcal{H})$. This leads us to regard $A \rtimes_\delta \widehat{S}_V$ as a subalgebra of $M(A \otimes \mathcal{K}(\mathcal{H}))$.

Let G be a locally compact group and A be a C^*-algebra. It is well-known that there exists a one-to-one correspondence between actions of G on A and coactions of $C_0(G)$ on A: to each action α there corresponds a coaction δ^α, and to a coaction δ there corresponds an action α^δ such that

$$\delta^\alpha(a)(r) = \alpha_r(a), \quad \alpha_r^\delta(a) = \delta(a)(r)$$

for $a \in A$ and $r \in G$. Moreover, if $\alpha : G \to Aut(A)$ is an action, then the reduced crossed product $A \rtimes_{\alpha,r} G$ coincides with the crossed product $A \rtimes_{\delta_G^\alpha} \widehat{S}_{\widehat{W}_G}$ by the coaction

$$\delta_G^\alpha = \overline{\mathrm{id}_A \otimes \check{\mu}_G} \circ \delta^\alpha : A \to M(A \otimes S_{\widehat{W}_G})$$

when viewed as subalgebras of $M(A \otimes \mathcal{K}(\mathcal{H}))$ (see for example [40, Chapter 9]). We will freely use these facts in the proof of Corollary 5.2.5, Theorem A.2.1, and Corollary B.2.3 with no further explanation.

A *nondegenerate coaction* of G on a C^*-algebra A is an injective coaction δ of the Hopf C^*-algebra $C^*(G)$ on A ([13, Definition A.21]). Let

$$\delta_\lambda := \overline{\mathrm{id}_A \otimes \lambda} \circ \delta : A \to M(A \otimes C_r^*(G)) = M(A \otimes S_{W_G}). \tag{2.7}$$

The crossed product $A \rtimes_\delta G$ by δ is defined to be the reduced crossed crossed product $A \rtimes_{\delta_\lambda} \widehat{S}_{W_G}$ by δ_λ ([13, Definition A.39]).

Chapter 3

Coactions of Hopf C^*-algebras on C^*-correspondences

In this chapter, we define a coaction of a Hopf C^*-algebra S on a C^*-correspondence (X, A) which unifies two notions of an action and nondegenerate coaction of a locally compact group on a C^*-correspondence. We prove that a coaction of S on (X, A) induces a coaction of S on the associated Cuntz-Pimsner algebra $\mathcal{O}_X$ under an invariance condition (Theorem 3.2.7). This generalizes both [18, Lemma 2.6.(b)] for group actions and [23, Proposition 3.1] for group coactions.

3.1 The extensions $(\overline{k_X \otimes \mathrm{id}}, \overline{k_A \otimes \mathrm{id}})$

In this section, we prove among others that for a C^*-correspondence (X, A) and a C^*-algebra C, the relation $(k_X \otimes \mathrm{id}_C)^{(1)} \circ \varphi_{A\otimes C} = k_A \otimes \mathrm{id}_C$ on $J_X \otimes C$ is still valid on the strict closure of $J_X \otimes C$ (Lemma 3.1.3).

Recall that the C^*-correspondences considered in this dissertation are always nondegenerate.

Notations 3.1.1. Let (X, A) be a C^*-correspondence and C be a C^*-algebra. Consider the representation $(k_X \otimes \mathrm{id}_C, k_A \otimes \mathrm{id}_C) : (X \otimes C, A \otimes C) \to \mathcal{O}_X \otimes C$. Since $k_A \otimes \mathrm{id}_C$ is nondegenerate, $k_X \otimes \mathrm{id}_C$ extends to the $(A \otimes C)$-strictly continuous map

$$\overline{k_X \otimes \mathrm{id}_C} : M_{A\otimes C}(X \otimes C) \to M_{A\otimes C}(\mathcal{O}_X \otimes C)$$

by Theorem 2.5.4.(i). Throughout the dissertation, we mean by $\overline{k_X \otimes \mathrm{id}_C}$ this extension, and by $M_{A\otimes C}(\mathcal{O}_X \otimes C)$ the $(A \otimes C)$-multiplier algebra determined by $k_A \otimes \mathrm{id}_C$. On the other hand, $(M_C(X \otimes C), M_C(A \otimes C))$ is the C-multiplier correspondence determined by the embedding $C \hookrightarrow M(A \otimes C)$ onto the last factor.

For an ideal I of a C^*-algebra B, let

$$M(B; I) := \{m \in M(B) : mB \cup Bm \subseteq I\}.$$

By [22, Lemma 2.4.(i)], $M(B; I)$ is the strict closure of I in $M(B)$.

Lemma 3.1.2. *Let (X, A) be a C^*-correspondence. Then the ideal $J_{M_A(X)}$ is contained in the strict closure of J_X, that is,*

$$J_{M_A(X)} \subseteq M(A; J_X).$$

Proof. We need to show that the ideal $AJ_{M_A(X)}$ is contained in J_X. By definition, we have

$$\varphi_A(AJ_{M_A(X)}) \subseteq \varphi_A(A)\mathcal{K}(M_A(X)) \subseteq \varphi_A(A)M_A(\mathcal{K}(X)) \subseteq \mathcal{K}(X).$$

We also have

$$AJ_{M_A(X)} \ker \varphi_A \subseteq J_{M_A(X)} \ker \varphi_{M(A)} = 0.$$

Consequently, $AJ_{M_A(X)} \subseteq J_X$. □

The next lemma, contained in the proof of [23, Lemma 2.5], will be useful in proving Theorem 3.2.7, Proposition 5.1.5, and Theorem 5.2.1.

Lemma 3.1.3. *Let (X, A) be a C^*-correspondence and C be a C^*-algebra. Then*

$$\overline{(k_X \otimes \mathrm{id}_C)^{(1)}} \circ \varphi_{M(A\otimes C)} = \overline{k_A \otimes \mathrm{id}_C} \tag{3.1}$$

holds on $M(A\otimes C; J_X\otimes C)$*, that is, the diagram*

$$\begin{tikzcd} M_{A\otimes C}(\mathcal{K}(X\otimes C)) \arrow[drr, "\overline{(k_X\otimes \mathrm{id}_C)^{(1)}}"] & & \\ M(A\otimes C; J_X\otimes C) \arrow[u, "\varphi_{M(A\otimes C)}"] \arrow[rr, "\overline{k_A\otimes \mathrm{id}_C}"'] & & M_{A\otimes C}(\mathcal{O}_X\otimes C) \end{tikzcd}$$

commutes.

Proof. By definition, the vertical map makes sense and is $(A\otimes C)$-strictly continuous. Also, Theorem 2.5.4.(ii) says that $(k_X\otimes \mathrm{id}_C)^{(1)}$ extends $(A\otimes C)$-strictly to the homomorphism $\overline{(k_X\otimes \mathrm{id}_C)^{(1)}}$ indicated by the lower right arrow. Hence the composition on the left side of (3.1) is well-defined on $M(A\otimes C; J_X\otimes C)$ and $(A\otimes C)$-strictly continuous. On the other hand, the horizontal map is the restriction of the usual strict extension $\overline{k_A\otimes \mathrm{id}_C}$ and $(A\otimes C)$-strictly continuous. Since (3.1) is valid on $J_X\otimes C$ by Remark 2.3.1, the conclusion now follows by $(A\otimes C)$-strict continuity and the fact that $J_X\otimes C$ is $(A\otimes C)$-strictly dense in $M(A\otimes C; J_X\otimes C)$. □

Remarks 3.1.4. (1) Let $(\psi,\pi) : (X,A)\to C$ be a covariant representation such that π is nondegenerate. Then the relation $\overline{\psi^{(1)}}\circ\varphi_{M(A)} = \overline{\pi}$ holds on $M(A;J_X)$ for the strict extension $(\overline{\psi},\overline{\pi}) : (M_A(X), M(A))\to M_A(C)$. In particular, $(\overline{\psi},\overline{\pi})$ is covariant. A proof of this can be given in the same way as the one of Lemma 3.1.3.

(2) Let $(\rho,\omega) : (Y,B)\to (M_A(X), M(A))$ be a nondegenerate correspondence homomorphism such that $\omega(J_X)\subseteq M(A;J_X)$, that is, (ρ,ω) is *Cuntz-Pimsner covariant* in the sense of [22, Definition 3.1] (see also [22, Lemma 3.2]). Let (ψ,π) be as above. Then the representation $(\overline{\psi}\circ\rho, \overline{\pi}\circ\omega) : (Y,B)\to M_A(C)$ is covariant. Indeed, $(\overline{\psi}\circ\rho)^{(1)} = \overline{\psi^{(1)}}\circ\rho^{(1)}$ on $\mathcal{K}(Y)$, and $\varphi_{M(A)}\circ\omega = \overline{\rho^{(1)}}\circ\varphi_B$ by [22, Lemma 3.3]. It then follows that

$$(\overline{\psi}\circ\rho)^{(1)}\circ\varphi_B = \overline{\psi^{(1)}}\circ\rho^{(1)}\circ\varphi_B = \overline{\psi^{(1)}}\circ\varphi_{M(A)}\circ\omega = \overline{\pi}\circ\omega$$

on J_Y, and consequently, $(\overline{\psi}\circ\rho, \overline{\pi}\circ\omega)$ is covariant.

Recall from [6, Definition 12.4.3] the following terminology. Let A and C be C^*-algebras and J be a closed subspace of A. The triple (J, A, C) is said to *satisfy the slice map property* if the space

$$F(J, A, C) = \{x \in A \otimes C : (\mathrm{id} \otimes \omega)(x) \in J \text{ for } \omega \in C^*\}$$

equals the norm closure $J \otimes C$ of the algebraic tensor product $J \odot C$ in $A \otimes C$.

Remarks 3.1.5. (1) If J is an ideal of A, then (J, A, C) satisfies the slice map property if and only if the sequence

$$0 \longrightarrow J \otimes C \longrightarrow A \otimes C \longrightarrow (A/J) \otimes C \longrightarrow 0$$

is exact; this is the case if A is locally reflexive or C is exact (see below [6, Definition 12.4.3]).

(2) Let $\mathcal{H}$ be a Hilbert space. If C is a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$, then $F(J, A, C)$ equals the norm closure of the following space

$$\{x \in A \otimes C : (\mathrm{id} \otimes \omega)(x) \in J \text{ for } \omega \in \mathcal{L}(\mathcal{H})_*\}.$$

Indeed, let E be the latter space. Obviously the closure $\overline{E}$ contains $F(J, A, C)$. Conversely let $x \in E$, $\omega \in C^*$ with $\|\omega\| = 1$, and $\epsilon > 0$. Take an $x_0 \in A \odot C$ with $\|x - x_0\| < \epsilon$. Since the unit ball of $\mathcal{L}(\mathcal{H})_*$ is weak-star dense in the unit ball of $\mathcal{L}(\mathcal{H})^*$, we can choose an $\omega_0 \in \mathcal{L}(\mathcal{H})_*$ such that $\|\omega_0\| = 1$ and $\|(\mathrm{id}_A \otimes \omega)(x_0) - (\mathrm{id}_A \otimes \omega_0)(x_0)\| < \epsilon$. The triangle inequality then verifies that

$$\begin{aligned}\|(\mathrm{id}_A \otimes \omega)(x) - (\mathrm{id}_A \otimes \omega_0)(x)\| \leq \|&(\mathrm{id}_A \otimes \omega)(x - x_0)\| \\ &+ \|(\mathrm{id}_A \otimes \omega)(x_0) - (\mathrm{id}_A \otimes \omega_0)(x_0)\| \\ &+ \|(\mathrm{id}_A \otimes \omega_0)(x_0 - x)\| < 3\epsilon.\end{aligned}$$

This prove that $(\mathrm{id}_A \otimes \omega)(x) \in \overline{E}$ since $(\mathrm{id}_A \otimes \omega_0)(x) \in E$.

Corollary 3.1.6. *Let (X, A) be a C^*-correspondence and C be a C^*-algebra.*

Suppose that (J_X, A, C) *satisfies the slice map property. Then*

$$J_{X \otimes C} = J_X \otimes C.$$

Furthermore,

$$J_{M_{A\otimes C}(X\otimes C)} \subseteq M(A \otimes C; J_X \otimes C)$$

and the injective representation

$$(\overline{k_X \otimes \mathrm{id}_C}, \overline{k_A \otimes \mathrm{id}_C}) : (M_{A\otimes C}(X \otimes C), M(A \otimes C)) \to M_{A\otimes C}(\mathcal{O}_X \otimes C)$$

is covariant.

Proof. We always have $J_{X\otimes C} \supseteq J_X \otimes C$ as shown in the first part of the proof of [23, Lemma 2.6]. We thus only need to show the converse $J_{X\otimes C} \subseteq F(J_X, A, C) = J_X \otimes C$. But, this can be done in the same way as the second part of the proof of [23, Lemma 2.6], and then the first assertion of the corollary follows. Lemma 3.1.2 then verifies the second assertion on the inclusion. Finally, since $\varphi_{M(A\otimes C)}$ maps $J_{M_{A\otimes C}(X\otimes C)}$ into $\mathcal{K}(M_{A\otimes C}(X \otimes C))$ on which

$$\overline{(k_X \otimes \mathrm{id})^{(1)}} = \overline{k_X \otimes \mathrm{id}_C}^{(1)}$$

by Theorem 2.5.4.(ii), the representation is covariant by Lemma 3.1.3. $\square$

Corollary 3.1.7. *Under the same hypothesis of Corollary 3.1.6, the injective representation*

$$(k_X \otimes \mathrm{id}_C, k_A \otimes \mathrm{id}_C) : (X \otimes C, A \otimes C) \to \mathcal{O}_X \otimes C$$

is covariant and the integrated form $(k_X \otimes \mathrm{id}_C) \times (k_A \otimes \mathrm{id}_C) : \mathcal{O}_{X\otimes C} \to \mathcal{O}_X \otimes C$ *is a surjective isomorphism.*

Proof. Generally we have $J_Y \subseteq J_{M_B(Y)}$ for a C^*-correspondence (Y, B) since J_Y is an ideal of $M(B)$ and is mapped injectively into $\mathcal{K}(Y) \subseteq \mathcal{K}(M_B(Y))$ by $\varphi_{M(B)}$. Hence $J_{X\otimes C} \subseteq J_{M_{A\otimes C}(X\otimes C)}$, and therefore $(k_X \otimes \mathrm{id}_C, k_A \otimes \mathrm{id}_C)$ is covariant by Corollary 3.1.6. The integrated form is clearly surjective. Since

(k_X, k_A) admits a gauge action, and hence so does $(k_X \otimes \mathrm{id}_C, k_A \otimes \mathrm{id}_C)$, the integrated form must be injective by [25, Theorem 6.4]. $\square$

3.2 Coactions on C^*-correspondences and their induced coactions

Definition 3.2.1. A *coaction* of a Hopf C^*-algebra (S, Δ) on a C^*-correspondence (X, A) is a nondegenerate correspondence homomorphism

$$(\sigma, \delta) : (X, A) \to (M(X \otimes S), M(A \otimes S))$$

such that

(i) δ is a coaction of S on the C^*-algebra A;

(ii) σ satisfies the *coaction identity* $\overline{\sigma \otimes \mathrm{id}_S} \circ \sigma = \overline{\mathrm{id}_X \otimes \Delta} \circ \sigma$;

(iii) σ satisfies the *coaction nondegeneracy*

$$\overline{\varphi_{M(A\otimes S)}(1_{M(A)} \otimes S)\, \sigma(X)} = X \otimes S.$$

Note that the strict extensions $\overline{\sigma \otimes \mathrm{id}_S}$ and $\overline{\mathrm{id}_X \otimes \Delta}$ in (ii) are well-defined because the tensor product of two nondegenerate correspondence homomorphisms is also nondegenerate ([13, Proposition 1.38]).

Remarks 3.2.2. (1) If we replace in Definition 3.2.1 the requirements on the left actions such as the compatibility $\sigma(\varphi_A(a)\xi) = \varphi_{M(A\otimes S)}(\delta(a))\, \sigma(\xi)$ and the coaction nondegeneracy by the corresponding requirements on the right actions, we get the notion of coaction of a Hopf C^*-algebra S on a Hilbert A-module X given in [3, Definition 2.2].

(2) It should be noted that

$$\overline{\sigma(X) \cdot (1_{M(A)} \otimes S)} = X \otimes S,$$

which follows by the same argument as [13, Remark 2.11.(1) and (2)]. We then have $\sigma(X) \subseteq M_S(X \otimes S) \subseteq M_{A\otimes S}(X \otimes S)$.

Remark 3.2.3. Let G be a locally compact group and (X, A) be a C^*-correspondence. We show in Theorem A.2.1 that every action of G on (X, A) in the sense of [13, Definition 2.5] determines a coaction of the Hopf C^*-algebra $C_0(G)$ on (X, A), and one can define in this way a one-to-one correspondence between actions of G on (X, A) and coactions of $C_0(G)$ on (X, A). On the other hand, a nondegenerate coaction of G [13, Definition 2.10] is by definition a coaction (σ, δ) of the Hopf C^*-algebra $C^*(G)$ on (X, A) such that δ is injective. Definition 3.2.1 thus unifies the notions of actions and nondegenerate coactions of locally compact groups on C^*-correspondences.

By Proposition 2.27 (Proposition 2.30, respectively) of [13], an action (nondegenrate coaction, respectively) of a locally compact group G on (X, A) determines an action (coaction, respectively) of G on $\mathcal{K}(X)$, and the left action φ_A satisfies an equivariance condition. The next proposition generalizes this in the Hopf C^*-algebra setting. Recall that we identify $\mathcal{K}(X_1 \otimes X_2) = \mathcal{K}(X_1) \otimes \mathcal{K}(X_2)$ for two Hilbert modules X_1 and X_2. In particular, if (X, A) is a C^*-correspondence and C is a C^*-algebra then $\mathcal{K}(X \otimes C) = \mathcal{K}(X) \otimes C$ because $\mathcal{K}(C) = C$.

Proposition 3.2.4. *Let (σ, δ) be a coaction of a Hopf C^*-algebra S on a C^*-correspondence (X, A). Then the nondegenerate homomorphism*

$$\sigma^{(1)} : \mathcal{K}(X) \to M(\mathcal{K}(X \otimes S)) = M(\mathcal{K}(X) \otimes S)$$

is a coaction of S on $\mathcal{K}(X)$ and the left action φ_A is δ-$\sigma^{(1)}$ equivariant, that is, $\overline{\varphi_A \otimes \mathrm{id}_S} \circ \delta = \overline{\sigma^{(1)}} \circ \varphi_A$. If δ is injective then so is $\sigma^{(1)}$.

Proof. Let $S = (S, \Delta)$. For $\xi, \eta \in X$, we have

$$\begin{aligned}\overline{(\sigma \otimes \mathrm{id}_S)^{(1)}}(\sigma(\xi)\sigma(\eta)^*) &= \overline{\sigma \otimes \mathrm{id}_S}(\sigma(\xi))\big(\overline{\sigma \otimes \mathrm{id}_S}(\sigma(\eta))\big)^* \\ &= \overline{\mathrm{id}_X \otimes \Delta}(\sigma(\xi))\big(\overline{\mathrm{id}_X \otimes \Delta}(\sigma(\eta))\big)^* \\ &= \overline{(\mathrm{id}_X \otimes \Delta)^{(1)}}(\sigma(\xi)\sigma(\eta)^*)\end{aligned}$$

by the second relation of (2.3) and the coaction identity of σ. It then follows by Remark 2.3.1 that

$$\begin{aligned}\overline{\sigma^{(1)} \otimes \mathrm{id}_S} \circ \sigma^{(1)}(\theta_{\xi,\eta}) &= \overline{(\sigma \otimes \mathrm{id}_S)^{(1)}}\big(\sigma(\xi)\sigma(\eta)^*\big) \\ &= \overline{(\mathrm{id}_X \otimes \Delta)^{(1)}}\big(\sigma(\xi)\sigma(\eta)^*\big) \\ &= \overline{\mathrm{id}_{\mathcal{K}(X)} \otimes \Delta} \circ \sigma^{(1)}(\theta_{\xi,\eta})\end{aligned}$$

which verifies the coaction identity of $\sigma^{(1)}$. We also have

$$\begin{aligned}\overline{\sigma^{(1)}(\mathcal{K}(X))(1_{M(\mathcal{K}(X))} \otimes S)} &= \overline{\sigma(X)\sigma(X)^*\varphi_{M(A\otimes S)}(1_{M(A)} \otimes S)} \\ &= \overline{\sigma(X)\big(\varphi_{M(A\otimes S)}(1_{M(A)} \otimes S)\,\sigma(X)\big)^*} \\ &= \overline{\sigma(X)\big((\sigma(X)\cdot(1_{M(A)} \otimes S))\cdot(1_{M(A)} \otimes S)\big)^*} \\ &= \overline{\big(\sigma(X)\cdot(1_{M(A)} \otimes S)\big)\big(\sigma(X)\cdot(1_{M(A)} \otimes S)\big)^*} \\ &= \overline{(X \otimes S)(X \otimes S)^*} = \mathcal{K}(X \otimes S),\end{aligned} \tag{3.2}$$

in the third and fifth step of which we use the coaction nondegeneracy of σ. This shows that $\sigma^{(1)}$ satisfies the coaction nondegeneracy, and thus $\sigma^{(1)}$ is a coaction.

The first relation of (2.3) and the fact that (σ, δ) is a correspondence homomorphism yield

$$\overline{\sigma^{(1)}}(\varphi_A(a))\,\sigma(\xi) = \sigma(\varphi_A(a)\xi) = \varphi_{M(A\otimes S)}(\delta(a))\,\sigma(\xi).$$

for $a \in A$ and $\xi \in X$. Multiplying by $1_{M(A)} \otimes s$ on both end sides from the right gives

$$\overline{\sigma^{(1)}}(\varphi_A(a))\big(\sigma(\xi)\cdot(1_{M(A)} \otimes s)\big) = \varphi_{M(A\otimes S)}(\delta(a))\big(\sigma(\xi)\cdot(1_{M(A)} \otimes s)\big)$$

which leads to $\overline{\sigma^{(1)}}(\varphi_A(a)) = \varphi_{M(A\otimes S)}(\delta(a))$ by the coaction nondegeneracy of σ. But $\varphi_{M(A\otimes S)} = \overline{\varphi_A \otimes \mathrm{id}_S}$ by definition, and then the δ-$\sigma^{(1)}$ equivariancy of φ_A follows.

For the last assertion, see the comment below [25, Lemma 2.4]. $\square$

Definition 3.2.5. Let (σ, δ) be a coaction of a Hopf C^*-algebra S on a C^*-correspondence (X, A). We say that the ideal J_X is *weakly δ-invariant* if

$$\delta(J_X)(1_{M(A)} \otimes S) \subseteq J_X \otimes S.$$

Remark 3.2.6. The coaction nondegeneracy of δ implies that J_X is weakly δ-invariant if and only if $\delta(J_X)(A \otimes S) \subseteq J_X \otimes S$, namely

$$\delta(J_X) \subseteq M(A \otimes S; J_X \otimes S).$$

Under the assumption of the last inclusion in Remark 3.2.6 with $S = C^*(G)$, it was proved in [23, Proposition 3.1] that every coaction of a locally compact group G on (X, A) induces a coaction of G on the associated Cuntz-Pimsner algebra $\mathcal{O}_X$. Modifying the proof of [23, Proposition 3.1] we now prove the next theorem.

Theorem 3.2.7. *Let (σ, δ) be a coaction of a Hopf C^*-algebra S on a C^*-correspondence (X, A) such that the ideal J_X is weakly δ-invariant. Then the representation*

$$(\overline{k_X \otimes \mathrm{id}_S} \circ \sigma, \overline{k_A \otimes \mathrm{id}_S} \circ \delta) : (X, A) \to M_{A \otimes S}(\mathcal{O}_X \otimes S)$$

is covariant, and its integrated form $\zeta := (\overline{k_X \otimes \mathrm{id}_S} \circ \sigma) \times (\overline{k_A \otimes \mathrm{id}_S} \circ \delta)$ is a coaction of S on $\mathcal{O}_X$ such that the diagram

$$\begin{tikzcd}
(X, A) \arrow[r, "(\sigma,\delta)"] \arrow[d, "(k_X, k_A)"'] & (M_{A\otimes S}(X \otimes S), M(A \otimes S)) \arrow[d, "(\overline{k_X \otimes \mathrm{id}_S}, \overline{k_A \otimes \mathrm{id}_S})"] \\
\mathcal{O}_X \arrow[r, "\zeta"'] & M_{A\otimes S}(\mathcal{O}_X \otimes S)
\end{tikzcd} \tag{3.3}$$

commutes. If δ is injective then so is ζ.

Proof. Let us first prove that $(\overline{k_X \otimes \mathrm{id}_S} \circ \sigma, \overline{k_A \otimes \mathrm{id}_S} \circ \delta)$ is covariant, that is,

$$\left(\overline{k_X \otimes \mathrm{id}_S} \circ \sigma\right)^{(1)} \circ \varphi_A = \overline{k_A \otimes \mathrm{id}_S} \circ \delta$$

on J_X. Since $\sigma(X) \subseteq M_{A\otimes S}(X \otimes S)$ and thus $\sigma^{(1)}(\mathcal{K}(X)) \subseteq \mathcal{K}(M_{A\otimes S}(X\otimes S))$, we have

$$\big(\overline{k_X \otimes \mathrm{id}_S} \circ \sigma\big)^{(1)} = \overline{(k_X \otimes \mathrm{id}_S)^{(1)}} \circ \sigma^{(1)}$$

on $\mathcal{K}(X)$ by Theorem 2.5.4.(ii). We then have

$$\begin{aligned}\big(\overline{k_X \otimes \mathrm{id}_S} \circ \sigma\big)^{(1)} \circ \varphi_A &= \overline{(k_X \otimes \mathrm{id}_S)^{(1)}} \circ \sigma^{(1)} \circ \varphi_A \\ &= \overline{(k_X \otimes \mathrm{id}_S)^{(1)}} \circ \overline{\varphi_{A\otimes S}} \circ \delta\end{aligned}$$

on J_X since $\overline{\sigma^{(1)}} \circ \varphi_A = \overline{\varphi_{A\otimes S}} \circ \delta$ by [22, Lemma 3.3]. Hence, the requirement that $(\overline{k_X \otimes \mathrm{id}_S} \circ \sigma, \overline{k_A \otimes \mathrm{id}_S} \circ \delta)$ be covariant amounts to that

$$\overline{(k_X \otimes \mathrm{id}_S)^{(1)}} \circ \overline{\varphi_{A\otimes S}} \circ \delta = \overline{k_A \otimes \mathrm{id}_S} \circ \delta$$

holds on J_X. By Remark 3.2.6, this equality will follow if we show that

$$\overline{(k_X \otimes \mathrm{id}_S)^{(1)}} \circ \overline{\varphi_{A\otimes S}} = \overline{k_A \otimes \mathrm{id}_S}$$

on $M(A \otimes S; J_X \otimes S)$. But, this is the content of Lemma 3.1.3, and therefore the representation $(\overline{k_X \otimes \mathrm{id}_S} \circ \sigma, \overline{k_A \otimes \mathrm{id}_S} \circ \delta)$ is covariant.

We now show that ζ is a coaction of (S, Δ) on $\mathcal{O}_X$. Since

$$\begin{aligned}\overline{(1_{M(\mathcal{O}_X)} \otimes S)\zeta(k_X(X))} &= \overline{\overline{k_A \otimes \mathrm{id}_S}(1_{M(A)} \otimes S)\overline{k_X \otimes \mathrm{id}_S}(\sigma(X))} \\ &= \overline{\overline{k_X \otimes \mathrm{id}_S}\big(\varphi_{M(A\otimes S)}(1_{M(A)} \otimes S)\,\sigma(X)\big)} \\ &= \overline{k_X(X) \odot S},\end{aligned}$$

we have

$$\overline{\zeta(k_X(X)^*)(1_{M(\mathcal{O}_X)} \otimes S)} = \Big(\overline{(1_{M(\mathcal{O}_X)} \otimes S)\zeta(k_X(X))}\Big)^* = \overline{k_X(X)^* \odot S}.$$

We also have $\overline{\zeta(k_X(X))(1_{M(\mathcal{O}_X)} \otimes S)} = \overline{k_X(X) \odot S}$. From these and the coaction nondegeneracy of δ, we can deduce that ζ satisfies the coaction nondegeneracy.

The coaction nondegeneracy of ζ implies $\zeta(\mathcal{O}_X) \subseteq M_{A\otimes S}(\mathcal{O}_X \otimes S)$, and then we have the commutative diagram (3.3).

We can easily see that $\overline{\zeta \otimes \mathrm{id}_S} \circ \zeta \circ k_A = \overline{\mathrm{id}_{\mathcal{O}_X} \otimes \Delta} \circ \zeta \circ k_A$ by (3.3), strict continuity, and the coaction identity of δ. To prove the corresponding equality for k_X, we first note the followings. Let $x \in A \otimes S$ and $m \in M_{A\otimes S}(\mathcal{O}_X \otimes S)$. Then

$$(\zeta \otimes \mathrm{id}_S)\big((k_A \otimes \mathrm{id}_S)(x)\, m\big) = \overline{k_A \otimes \mathrm{id}_S \otimes \mathrm{id}_S}\big((\delta \otimes \mathrm{id}_S)(x)\big)\, \overline{\zeta \otimes \mathrm{id}_S}(m),$$
$$(\mathrm{id}_{\mathcal{O}_X} \otimes \Delta)\big((k_A \otimes \mathrm{id}_S)(x)\, m\big) = \overline{k_A \otimes \mathrm{id}_S \otimes \mathrm{id}_S}\big((\mathrm{id}_A \otimes \Delta)(x)\big)\, \overline{\mathrm{id}_{\mathcal{O}_X} \otimes \Delta}(m),$$

and similarly for $(\zeta \otimes \mathrm{id}_S)\big(m\,(k_A \otimes \mathrm{id}_S)(x)\big)$ and $(\mathrm{id}_{\mathcal{O}_X} \otimes \Delta)\big(m\,(k_A \otimes \mathrm{id}_S)(x)\big)$. From these relations and also the nondegeneracy of $\delta \otimes \mathrm{id}_S$ and $\mathrm{id}_A \otimes \Delta$, we deduce that the restrictions

$$\overline{\zeta \otimes \mathrm{id}_S},\ \overline{\mathrm{id}_{\mathcal{O}_X} \otimes \Delta} : M_{A\otimes S}(\mathcal{O}_X \otimes S) \to M_{A\otimes S\otimes S}(\mathcal{O}_X \otimes S \otimes S)$$

are $(A \otimes S)$-strict to $(A \otimes S \otimes S)$-strictly continuous (cf. [12, Lemma A.5]). Therefore the following compositions

$$\overline{\zeta \otimes \mathrm{id}_S} \circ \overline{k_X \otimes \mathrm{id}_S},\ \overline{\mathrm{id}_{\mathcal{O}_X} \otimes \Delta} \circ \overline{k_X \otimes \mathrm{id}_S} : M_{A\otimes S}(X \otimes S) \to M_{A\otimes S\otimes S}(\mathcal{O}_X \otimes S \otimes S) \tag{3.4}$$

are $(A \otimes S)$-strict to $(A \otimes S \otimes S)$-strictly continuous. Similarly, both maps

$$\overline{\sigma \otimes \mathrm{id}_S},\ \overline{\mathrm{id}_X \otimes \Delta} : M_{A\otimes S}(X \otimes S) \to M_{A\otimes S\otimes S}(X \otimes S \otimes S)$$

are $(A \otimes S)$-strict to $(A \otimes S \otimes S)$-strictly continuous, and hence so are the maps

$$\overline{k_X \otimes \mathrm{id}_S \otimes \mathrm{id}_S} \circ \overline{\sigma \otimes \mathrm{id}_S},\ \overline{k_X \otimes \mathrm{id}_S \otimes \mathrm{id}_S} \circ \overline{\mathrm{id}_X \otimes \Delta} : M_{A\otimes S}(X \otimes S) \to M_{A\otimes S\otimes S}(\mathcal{O}_X \otimes S \otimes S). \tag{3.5}$$

Since the equalities

$$\overline{\zeta \otimes \mathrm{id}_S} \circ \overline{k_X \otimes \mathrm{id}_S} = \overline{k_X \otimes \mathrm{id}_S \otimes \mathrm{id}_S} \circ \overline{\sigma \otimes \mathrm{id}_S},$$
$$\overline{k_X \otimes \mathrm{id}_S \otimes \mathrm{id}_S} \circ \overline{\mathrm{id}_X \otimes \Delta} = \overline{\mathrm{id}_{\mathcal{O}_X} \otimes \Delta} \circ \overline{k_X \otimes \mathrm{id}_S}$$

hold on $X \odot S$ which is $(A \otimes S)$-strictly dense in $M_{A \otimes S}(X \otimes S)$ and since $\sigma(X) \subseteq M_{A \otimes S}(X \otimes S)$, we now have

$$\begin{aligned}
\overline{\zeta \otimes \mathrm{id}_S} \circ \zeta \circ k_X &= \overline{\zeta \otimes \mathrm{id}_S} \circ \overline{k_X \otimes \mathrm{id}_S} \circ \sigma \\
&= \overline{k_X \otimes \mathrm{id}_S \otimes \mathrm{id}_S} \circ \overline{\sigma \otimes \mathrm{id}_S} \circ \sigma \\
&= \overline{k_X \otimes \mathrm{id}_S \otimes \mathrm{id}_S} \circ \overline{\mathrm{id}_X \otimes \Delta} \circ \sigma \\
&= \overline{\mathrm{id}_{\mathcal{O}_X} \otimes \Delta} \circ \overline{k_X \otimes \mathrm{id}_S} \circ \sigma = \overline{\mathrm{id}_{\mathcal{O}_X} \otimes \Delta} \circ \zeta \circ k_X
\end{aligned}$$

by the $(A \otimes S)$-strict to $(A \otimes S \otimes S)$-strict continuity of the maps of (3.4) and (3.5) and also by the coaction identity of σ. Thus ζ satisfies the coaction identity.

For the last assertion of the theorem, assume that δ is injective. We only need to show by [25, Theorem 6.4] that the injective covariant representation $(\overline{k_X \otimes \mathrm{id}_S} \circ \sigma, \overline{k_A \otimes \mathrm{id}_S} \circ \delta)$ admits a gauge action. Let $\beta : \mathbb{T} \to Aut(\mathcal{O}_X)$ be the gauge action. Note that for each $z \in \mathbb{T}$, the strict extension $\overline{\beta_z \otimes \mathrm{id}_S}$ on $M(\mathcal{O}_X \otimes S)$ maps $M_{A \otimes S}(\mathcal{O}_X \otimes S)$ onto itself. Then the composition

$$(\overline{\beta_z \otimes \mathrm{id}_S} \circ \overline{k_X \otimes \mathrm{id}_S},\ \overline{\beta_z \otimes \mathrm{id}_S} \circ \overline{k_A \otimes \mathrm{id}_S}) : \\ (M_{A \otimes S}(X \otimes S), M(A \otimes S)) \to M_{A \otimes S}(\mathcal{O}_X \otimes S) \quad (3.6)$$

gives a representation which is clearly $(A \otimes S)$-strictly continuous. Since the equalities

$$\overline{\beta_z \otimes \mathrm{id}_S} \circ \overline{k_X \otimes \mathrm{id}_S}(m) = z\, \overline{k_X \otimes \mathrm{id}_S}(m),$$
$$\overline{\beta_z \otimes \mathrm{id}_S} \circ \overline{k_A \otimes \mathrm{id}_S}(n) = \overline{k_A \otimes \mathrm{id}_S}(n)$$

are valid for $m \in X \odot S$ and $n \in A \odot S$, and the representation (3.6) is $(A \otimes S)$-strictly continuous, the above equalities still hold for $m \in M_{A \otimes S}(X \otimes S)$ and

$n \in M(A \otimes S)$. Since $\sigma(X) \subseteq M_{A\otimes S}(X \otimes S)$, it thus follows that

$$\overline{\beta_z \otimes \mathrm{id}_S} \circ \overline{k_X \otimes \mathrm{id}_S} \circ \sigma = z\, \overline{k_X \otimes \mathrm{id}_S} \circ \sigma,$$

and similarly that $\overline{\beta_z \otimes \mathrm{id}_S} \circ \overline{k_A \otimes \mathrm{id}_S} \circ \delta = \overline{k_A \otimes \mathrm{id}_S} \circ \delta$. This proves that the restrictions of $\overline{\beta_z \otimes \mathrm{id}_S}$ to $\zeta(\mathcal{O}_X)$ $(z \in \mathbb{T})$ define a gauge action of $\mathbb{T}$ on $\zeta(\mathcal{O}_X)$, which establishes the theorem. □

Definition 3.2.8. We call ζ in Theorem 3.2.7 the coaction *induced* by (σ, δ).

Remarks 3.2.9. (1) Let G be a locally compact group. If (σ, δ) is a coaction of $C_0(G)$ on (X, A), then $\overline{\delta(J_X)(1_{M(A)} \otimes S)} = J_X \otimes S$ by [18, Lemma 2.6.(a)] and Theorem A.2.1. Hence, J_X is automatically weakly δ-invariant in this case.

(2) Replacing in the diagram (3.3) the $(A \otimes S)$-multiplier correspondence and $(A \otimes S)$-multiplier algebra by $(M_S(X \otimes S), M_S(A \otimes S))$ and $M_S(\mathcal{O}_X \otimes S)$, respectively, we can regard $(\overline{k_X \otimes \mathrm{id}_S}, \overline{k_A \otimes \mathrm{id}_S})$ as the S-strict extension by Remarks 2.5.3.(2).

Chapter 4

Reduced crossed product correspondences

This chapter is dedicated to constructing the reduced crossed product correspondence $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ from a coaction (σ, δ) on (X, A) of a Hopf C^*-algebra S defined by a well-behaved multiplicative unitary. When the Hopf C^*-algebra coaction under consideration comes from a group action or nondegenerate group coaction, the C^*-correspondence $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ is shown to be equal to the crossed product correspondence in [13] (see Remark 4.2.4). The first section proves the Baaj-Skandalis type lemma for C^*-correspondences (Lemma 4.1.5), which serves as a technical tool for the construction of $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$. The last section constructs $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ and provides some consequences of the construction needed to prove our results to be obtained in Chapter 5.

4.1 Baaj-Skandalis type lemma for C^*-correspondences

Recall that the *Toeplitz algebra* $\mathcal{T}_X$ is the C^*-algebra generated by $i_X(X)$ and $i_A(A)$, where (i_X, i_A) is the universal representation of (X, A) ([25]). The following lemma is a Toeplitz algebra analogue of Corollary 3.1.7.

Lemma 4.1.1. *Let (X, A) be a C^*-correspondence and C be a C^*-algebra. Then the injective representation*

$$(i_X \otimes \mathrm{id}_C, i_A \otimes \mathrm{id}_C) : (X \otimes C, A \otimes C) \to \mathcal{T}_X \otimes C$$

gives rise to an isomorphism from $\mathcal{T}_{X \otimes C}$ onto $\mathcal{T}_X \otimes C$.

Proof. By the universality of the Toeplitz algebra $\mathcal{T}_{X\otimes C}$, there exists a homomorphism $\Psi : \mathcal{T}_{X\otimes C} \to \mathcal{T}_X \otimes C$ such that $\Psi(i_{X\otimes C}(\xi \otimes c)) = i_X(\xi) \otimes c$ and $\Psi(i_{A\otimes C}(a \otimes c)) = i_A(a) \otimes c$ for $\xi \in X$, $a \in A$, and $c \in C$. Clearly, Ψ is surjective.

To see that Ψ is injective, we first note that $(i_X \otimes \mathrm{id}_C, i_A \otimes \mathrm{id}_C)$ admits a gauge action. Thus, we only need to show by [25, Theorem 6.2] that the space

$$I' = \left\{x \in A \otimes C : (i_A \otimes \mathrm{id}_C)(x) \in (i_X \otimes \mathrm{id}_C)^{(1)}(\mathcal{K}(X \otimes C)) = i_X^{(1)}(\mathcal{K}(X)) \otimes C\right\}$$

is zero. But for $x \in I'$ and $\omega \in C^*$, applying the slice map $\mathrm{id}_{\mathcal{T}_X} \otimes \omega$ to $(i_A \otimes \mathrm{id}_C)(x)$ yields

$$(\mathrm{id}_{\mathcal{T}_X} \otimes \omega)\big((i_A \otimes \mathrm{id}_C)(x)\big) = i_A\big((\mathrm{id}_A \otimes \omega)(x)\big) \in i_X^{(1)}(\mathcal{K}(X)),$$

which implies by [25, Theorem 6.2] that $(\mathrm{id}_A \otimes \omega)(x) = 0$. Therefore, $x = 0$ as desired. □

In what follows, for $c \in C$ and $\omega \in C^*$, we denote by ωc and $c\omega$ the functionals on C given by

$$(\omega c)(b) = \omega(cb), \quad (c\omega)(b) = \omega(bc) \quad (b \in C).$$

Proposition 4.1.2. *Let (X, A) be a C^*-correspondence, C be a C^*-algebra, and $\omega \in C^*$. Then the slice map $\mathrm{id}_X \odot \omega : X \odot C \to X$ extends uniquely to a strictly continuous linear map*

$$\overline{\mathrm{id}_X \otimes \omega} : M(X \otimes C) \to M(X)$$

between the two multiplier correspondences.

Proof. Uniqueness assertion will follow immediately once we show the existence of a strict extension of $\mathrm{id}_X \odot \omega$ since $X \odot C$ is strictly dense in $M(X \otimes C)$.

By Lemma 4.1.1, $X \otimes C$ can be embedded isometrically into $\mathcal{T}_X \otimes C$. Restricting to $X \otimes C$ the slice map $\mathrm{id}_{\mathcal{T}_X} \otimes \omega$ on $\mathcal{T}_X \otimes C$, we thus obtain a norm continuous extension $\mathrm{id}_X \otimes \omega : X \otimes C \to X$ of $\mathrm{id}_X \odot \omega$.

We claim that the map $\mathrm{id}_X \otimes \omega$ just obtained is strictly continuous. Indeed, let $\{x_i\}$ be a net in $X \otimes C$ converging strictly to an $x \in X \otimes C$, $T \in \mathcal{K}(X)$, and $a \in A$. Factor ω into $\omega_1 c_1$ or $c_2 \omega_2$ for some $\omega_1, \omega_2 \in C^*$ and $c_1, c_2 \in C$. (The Hewitt-Cohen factorization theorem allows us to do this; see for example [37, Proposition 2.33].) By norm continuity, we have

$$T(\mathrm{id}_X \otimes (\omega_1 c_1))(y) = (\mathrm{id}_X \otimes \omega_1)((T \otimes c_1)y) \quad (y \in X \otimes C).$$

Hence the net $\{T(\mathrm{id}_X \otimes \omega_1 c_1)(x_i)\} = \{(\mathrm{id}_X \otimes \omega_1)((T \otimes c_1)x_i)\}$ in X converges to $(\mathrm{id}_X \otimes \omega_1)((T \otimes c_1)x) = T(\mathrm{id}_X \otimes \omega_1 c_1)(x)$ again by norm continuity. Similarly, $\{(\mathrm{id}_X \otimes c_2\omega_2)(x_i) \cdot a\}$ converges to $(\mathrm{id}_X \otimes c_2\omega_2)(x) \cdot a$, which proves our claim.

By standard argument on continuous extensions (for example, see [30, Proposition 7.2]), $\mathrm{id}_X \otimes \omega$ extends strictly to all of $M(X \otimes C)$. $\square$

Remarks 4.1.3. (1) We note that $\overline{X_r}^{\mathrm{str}} \supseteq M(X)_r \supseteq \overline{M(X)_r}^{\mathrm{str}}$, where X_r and $M(X)_r$ are, respectively, the r-balls in X and $M(X)$, and the closures are taken with respect to the strict topology. In particular, $\overline{X_r}^{\mathrm{str}} = M(X)_r$. One can see the first inclusion by considering an approximate identity for A with the norms bounded by 1. For the second, let $m \in \overline{M(X)_r}^{\mathrm{str}}$, $a \in A$ with $\|a\| = 1$, and $\epsilon > 0$. Take a net $\{m_i\}$ in $M(X)_r$ strictly converging to m. Then $\|m \cdot a - m_i \cdot a\| < \epsilon$ for some i so that $\|m \cdot a\| \leq \|m \cdot a - m_i \cdot a\| + \|m_i \cdot a\| < \epsilon + r$. This verifies that $\|m\| \leq r$ and then the second inclusion follows.

(2) The strict extension $\overline{\mathrm{id}_X \otimes \omega}$ on $M(X \otimes C)$ is norm bounded with $\|\overline{\mathrm{id}_X \otimes \omega}\| \leq \|\omega\|$. Indeed, let $x \in M(X \otimes C)$ with $\|x\| = 1$ and $\{u_i\}$ be an approximate identity of $A \otimes C$ with $\|u_i\| \leq 1$. Then $\overline{\mathrm{id}_X \otimes \omega}(x)$ is the strict limit of $(\mathrm{id}_X \otimes \omega)(x \cdot u_i)$ by Proposition 4.1.2, and the latter vectors have norms at most $\|\omega\|$ by the proof of Proposition 4.1.2. The conclusion then follows by the previous observation.

In the rest of this chapter and the next one, we restrict our attention to coactions of reduced Hopf C^*-algebras defined by well-behaved multiplicative unitaries.

Notations 4.1.4. Until the end of Chapter 6, we will denote by $\mathcal{H}$ the Hilbert space on the two-fold tensor product of which a well-behaved multiplicative unitary V acts. To simplify notation, we often write S and $\widehat{S}$ for the "reduced" and "dual reduced" Hopf C^*-algebras S_V and $\widehat{S}_V$ defined by V, respectively.

Let (σ, δ) be a coaction of S on (X, A) and $\iota_S : S \hookrightarrow M(\mathcal{K}(\mathcal{H}))$ be the inclusion map. As δ_ι in (2.6), we denote by σ_ι the composition

$$\sigma_\iota = \overline{\mathrm{id}_X \otimes \iota_S} \circ \sigma,$$

where $\overline{\mathrm{id}_X \otimes \iota_S}$ is the strict extension. Evidently, $(\sigma_\iota, \delta_\iota)$ is a nondegenerate correspondence homomorphism:

$$\begin{tikzcd}
(X, A) \arrow[rr, "(\sigma_\iota,\delta_\iota)"] \arrow[dr, "(\sigma,\delta)"'] & & (M(X \otimes \mathcal{K}(\mathcal{H})), M(A \otimes \mathcal{K}(\mathcal{H}))) \\
& (M(X \otimes S), M(A \otimes S)) \arrow[ur, "(\overline{\mathrm{id}_X\otimes\iota_S}, \overline{\mathrm{id}_A\otimes\iota_S})"'] &
\end{tikzcd}$$

If B is a C^*-algebra, the canonical embeddings of $x \in \widehat{S}$ and $s \in S$ in $M(B \otimes \mathcal{K}(\mathcal{H}))$ will be written as $1_{M(B)} \otimes x$ and $1_{M(B)} \otimes s$.

The next lemma generalizes [3, Lemma 7.2]. The proof is not significantly different, but we provide it here for the reader's convenience.

Lemma 4.1.5. *Let $(\sigma, \delta) : (X, A) \to (M(X \otimes S), M(A \otimes S))$ be a coaction of S on a C^*-correspondence (X, A). Then the norm closures in $M(X \otimes \mathcal{K}(\mathcal{H}))$ of the subspaces $\sigma_\iota(X) \cdot (1_{M(A)} \otimes \widehat{S})$ and $\varphi_{M(A \otimes \mathcal{K}(\mathcal{H}))}(1_{M(A)} \otimes \widehat{S})\, \sigma_\iota(X)$ coincide.*

Proof. Let us show that each of the subspaces is contained in the norm closure of the other. Let $S = S_V$ be the reduced Hopf C^*-algebra obtained from a well-behaved multiplicative unitary $V \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$. Set $\mathscr{K} = \mathcal{K}(\mathcal{H})$. For $\xi \in X$ and $\omega \in \mathcal{L}(\mathcal{H})_*$, let m be the following element of $\varphi_{M(A\otimes\mathscr{K})}(1_{M(A)} \otimes \widehat{S})\, \sigma_\iota(X)$:

$$m = \varphi_{M(A\otimes\mathscr{K})}\big(1_{M(A)} \otimes \rho(\omega)\big)\sigma_\iota(\xi) = \big(\mathrm{id}_X \otimes \overline{\mathrm{id}_{\mathscr{K}} \otimes \omega}(V)\big) \circ \sigma_\iota(\xi).$$

Write $V_{23} = 1_{M(A)} \otimes V \in M(A \otimes \mathscr{K} \otimes \mathscr{K})$. Similarly, write $\sigma(\xi)_{12} = \sigma(\xi) \otimes 1_{M(S)} \in M(X \otimes S \otimes S)$ and $\sigma_\iota(\xi)_{12} = \overline{\mathrm{id}_X \otimes \iota_S \otimes \iota_S}(\sigma(\xi)_{12}) \in M(X \otimes \mathscr{K} \otimes \mathscr{K})$. Consider a net $\{v_i\}_i$ in $X \odot S$ strictly converging to $\sigma(\xi)$. Since S is a nondegenerate subalgebra of $M(\mathscr{K})$, we can see that the net $\{(\mathrm{id}_X \otimes \iota_S)(v_i) \otimes 1_{M(\mathscr{K})}\}_i$ in $M(X \otimes \mathscr{K} \otimes \mathscr{K})$ converges strictly to $\sigma_\iota(\xi)_{12}$. Hence we deduce from Proposition 4.1.2 that

$$m = \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega}\big(\varphi_{M(A \otimes \mathscr{K} \otimes \mathscr{K})}(V_{23})\, \sigma_\iota(\xi)_{12}\big).$$

We then have

$$\begin{aligned} m &= \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega}\big(\big(\varphi_{M(A \otimes \mathscr{K} \otimes \mathscr{K})}(V_{23})\sigma_\iota(\xi)_{12} \cdot V_{23}^*\big) \cdot V_{23}\big) \\ &= \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega}\big(\overline{\mathrm{id}_X \otimes \iota_S \otimes \iota_S}\big(\overline{\mathrm{id}_X \otimes \Delta_V}(\sigma(\xi))\big) \cdot V_{23}\big) \\ &= \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega}\big(\overline{\mathrm{id}_X \otimes \iota_S \otimes \iota_S}\big(\overline{\sigma \otimes \mathrm{id}_S}(\sigma(\xi))\big) \cdot V_{23}\big) \\ &= \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega}\big(\overline{\sigma_\iota \otimes \iota_S}(\sigma(\xi)) \cdot V_{23}\big) \end{aligned}$$

again by Proposition 4.1.2 and also by the definition of Δ_V in (2.4) and the coaction identity of σ. Write $\omega = \omega' s$. Then

$$m = \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega'}\big(\varphi_{M(A \otimes \mathscr{K} \otimes \mathscr{K})}(1_{M(A)} \otimes 1_{M(\mathscr{K})} \otimes s)\big(\overline{\sigma_\iota \otimes \iota_S}(\sigma(\xi))\big) \cdot V_{23}\big).$$

Since $(\overline{\sigma_\iota \otimes \iota_S}, \overline{\delta_\iota \otimes \iota_S})$ is a correspondence homomorphism,

$$m = \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega'}\big(\overline{\sigma_\iota \otimes \iota_S}\big(\varphi_{M(A \otimes S)}(1_{M(A)} \otimes s)\sigma(\xi)\big) \cdot V_{23}\big).$$

The coaction nondegeneracy of σ then implies that m belongs to the space

$$M = \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega'}\big((\sigma_\iota \otimes \iota_S)(X \otimes S) \cdot V_{23}\big)$$

in which the elements $\overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega'}((\sigma_\iota(\xi') \otimes s') \cdot V_{23})$ for $\xi' \in X$ and $s' \in S$

are linearly dense by Remark 4.1.3.(2). But

$$\begin{aligned}\overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega'}((\sigma_\iota(\xi') \otimes s') \cdot V_{23}) &= \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega' s'}\big(\sigma_\iota(\xi')_{12} \cdot V_{23}\big) \\ &= \sigma_\iota(\xi') \cdot (1_{M(A)} \otimes \rho(\omega' s')) \\ &\in \sigma_\iota(X) \cdot (1_{M(A)} \otimes \widehat{S}),\end{aligned}$$

and therefore $m \in M \subseteq \overline{\sigma_\iota(X) \cdot (1_{M(A)} \otimes \widehat{S})}$.

For the converse, let $m' = \sigma_\iota(\xi) \cdot (1_{M(A)} \otimes \rho(\omega s))$. Then

$$\begin{aligned}m' &= \sigma_\iota(\xi) \cdot \big(1_{M(A)} \otimes \overline{\mathrm{id}_{\mathscr{K}} \otimes \omega s}(V)\big) \\ &= \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega}\big(\sigma_\iota(\xi)_{12} \cdot \big((1_{M(A)} \otimes 1_{M(\mathscr{K})} \otimes s)V_{23}\big)\big) \\ &= \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega}\big((\sigma_\iota \otimes \iota_S)(\xi \otimes s) \cdot V_{23}\big)\end{aligned}$$

so that m' is an element of the space

$$M' = \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega}\big((\sigma_\iota \otimes \iota_S)(X \otimes S) \cdot V_{23}\big).$$

By coaction nondegeneracy and the fact that $(\overline{\sigma_\iota \otimes \iota_S}, \overline{\delta_\iota \otimes \iota_S})$ is a correspondence homomorphism, we have

$$\begin{aligned}M' &= \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega}\big((\sigma_\iota \otimes \iota_S)(\overline{\varphi_{M(A\otimes S)}(1_{M(A)} \otimes S)\,\sigma(X)}) \cdot V_{23}\big) \\ &\subseteq \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega}\big(\overline{\varphi_{M(A\otimes \mathscr{K}\otimes \mathscr{K})}(1_{M(A)} \otimes 1_{M(\mathscr{K})} \otimes S)\,\overline{\sigma_\iota \otimes \iota_S}(\sigma(X)) \cdot V_{23}}\big).\end{aligned}$$

Since

$$\begin{aligned}\overline{\sigma_\iota \otimes \iota_S}\big(\sigma(X)\big) &= \overline{\mathrm{id}_X \otimes \iota_S \otimes \iota_S}\big(\overline{\sigma \otimes \mathrm{id}_S}(\sigma(X))\big) \\ &= \overline{\mathrm{id}_X \otimes \iota_S \otimes \iota_S}\big(\overline{\mathrm{id}_X \otimes \Delta}(\sigma(X))\big) \\ &= \varphi_{M(A\otimes \mathscr{K}\otimes \mathscr{K})}(V_{23})\,\sigma_\iota(X)_{12} \cdot V_{23}^*\end{aligned}$$

by coaction identity and strict continuity, we then have

$$\begin{aligned} M' &\subseteq \overline{\mathrm{id}_X \otimes \mathrm{id}_{\mathscr{K}} \otimes \omega}\big(\overline{\varphi_{M(A\otimes\mathscr{K}\otimes\mathscr{K})}\big((1_{M(A)} \otimes 1_{M(\mathscr{K})} \otimes S)V_{23}\big)\sigma_\iota(X)_{12}}\big) \\ &\subseteq \overline{\varphi_{M(A\otimes\mathscr{K})}(1_{M(A)} \otimes \widehat{S})\,\sigma_\iota(X)}. \end{aligned}$$

Consequently, $m' \in M' \subseteq \overline{\varphi_{M(A\otimes\mathscr{K})}(1_{M(A)} \otimes \widehat{S})\,\sigma_\iota(X)}$. □

4.2 Reduced crossed product correspondences $(X \rtimes \widehat{S}, A \rtimes \widehat{S})$

For a coaction (σ, δ) of S on (X, A), we denote by $X \rtimes_\sigma \widehat{S}$ the norm closure of the subspaces considered in Lemma 4.1.5:

$$X \rtimes_\sigma \widehat{S} := \overline{\sigma_\iota(X) \cdot (1_{M(A)} \otimes \widehat{S})} = \overline{\varphi_{M(A\otimes\mathcal{K}(\mathcal{H}))}(1_{M(A)} \otimes \widehat{S})\,\sigma_\iota(X)}.$$

The space $X \rtimes_\sigma \widehat{S}$ is a Hilbert $(A \rtimes_\delta \widehat{S})$-module as considered in [7], but more can be said:

Theorem 4.2.1. *Let (σ, δ) be a coaction of a reduced Hopf C^*-algebra S on a C^*-correspondence (X, A). Then $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ is a nondegenerate C^*-correspondence such that the inclusion*

$$(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S}) \hookrightarrow (M(X \otimes \mathcal{K}(\mathcal{H})), M(A \otimes \mathcal{K}(\mathcal{H})))$$

is a nondegenerate correspondence homomorphism. The left action $\varphi_{A\rtimes_\delta \widehat{S}}$ is injective if φ_A is injective. Also,

$$\mathcal{K}(X \rtimes_\sigma \widehat{S}) = \mathcal{K}(X) \rtimes_{\sigma^{(1)}} \widehat{S},$$

where $\sigma^{(1)}$ is the coaction in Proposition 3.2.4, and

$$\varphi_{A\rtimes_\delta \widehat{S}}\big(\delta_\iota(a)(1_{M(A)} \otimes x)\big) = \overline{\sigma_\iota^{(1)}}(\varphi_A(a))(1_{M(\mathcal{K}(X))} \otimes x)$$

for $a \in A$ *and* $x \in \widehat{S}$.

Proof. Set $\mathscr{K} = \mathcal{K}(\mathcal{H})$. The first assertion is clearly equivalent to saying that the following three conditions are satisfied:

(i) $X \rtimes_\sigma \widehat{S}$ is a Hilbert $(A \rtimes_\delta \widehat{S})$-module with respect to the operations on the Hilbert $M(A \otimes \mathscr{K})$-module $M(X \otimes \mathscr{K})$, namely

$$\big(\sigma_\iota(X) \cdot (1_{M(A)} \otimes \widehat{S})\big) \cdot \big(\delta_\iota(A)(1_{M(A)} \otimes \widehat{S})\big) \subseteq X \rtimes_\sigma \widehat{S},$$
$$\big\langle \sigma_\iota(X) \cdot (1_{M(A)} \otimes \widehat{S}), \sigma_\iota(X) \cdot (1_{M(A)} \otimes \widehat{S}) \big\rangle_{M(A \otimes \mathscr{K})} \subseteq A \rtimes_\delta \widehat{S};$$

(ii) the Hilbert $(A \rtimes_\delta \widehat{S})$-module $X \rtimes_\sigma \widehat{S}$ is a nondegenerate C^*-correspondence such that $\varphi_{A \rtimes_\delta \widehat{S}} = \varphi_{M(A \otimes \mathscr{K})}|_{A \rtimes_\delta \widehat{S}}$, namely

$$\overline{\varphi_{M(A \otimes \mathscr{K})}\big(\delta_\iota(A)(1_{M(A)} \otimes \widehat{S})\big)\, \sigma_\iota(X) \cdot (1_{M(A)} \otimes \widehat{S})} = X \rtimes_\sigma \widehat{S};$$

(iii) the inclusion $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S}) \hookrightarrow (M(X \otimes \mathscr{K}), M(A \otimes \mathscr{K}))$ is a nondegenerate correspondence homomorphism, namely

$$\overline{(X \rtimes_\sigma \widehat{S}) \cdot (A \otimes \mathscr{K})} = X \otimes \mathscr{K}, \quad \overline{(A \rtimes_\delta \widehat{S})(A \otimes \mathscr{K})} = A \otimes \mathscr{K}.$$

The condition (i) is clearly satisfied since $(\sigma_\iota, \delta_\iota)$ is a correspondence homomorphism and $(1_{M(A)} \otimes \widehat{S})\delta_\iota(A)(1_{M(A)} \otimes \widehat{S})$ is contained in $\overline{\delta_\iota(A)(1_{M(A)} \otimes \widehat{S})}$. Lemma 4.1.5 shows that

$$\overline{\varphi_{M(A \otimes \mathscr{K})}(1_{M(A)} \otimes \widehat{S})\, \sigma_\iota(X) \cdot (1_{M(A)} \otimes \widehat{S})} = \overline{\sigma_\iota(X) \cdot (1_{M(A)} \otimes \widehat{S})}.$$

Since φ_A is nondegenerate, this equality combined with the following

$$\varphi_{M(A \otimes \mathscr{K})}(\delta_\iota(A))\, \sigma_\iota(X) = \sigma_\iota(\varphi_A(A)X)$$

gives (ii). Since S and $\widehat{S}$ are both nondegenerate subalgebras of $M(\mathscr{K})$, we

have
$$\begin{aligned}\overline{(X \rtimes_\sigma \widehat{S}) \cdot (A \otimes \mathscr{K})} &= \overline{\sigma_\iota(X) \cdot (A \otimes \widehat{S}\mathscr{K})} \\ &= \overline{\sigma_\iota(X) \cdot (1_{M(A)} \otimes S) \cdot (A \otimes \widehat{S}\mathscr{K})} \\ &= \overline{(X \otimes S) \cdot (A \otimes \mathscr{K})} = X \otimes \mathscr{K}\end{aligned} \tag{4.1}$$

and similarly $\overline{(A \rtimes_\delta \widehat{S})(A \otimes \mathscr{K})} = A \otimes \mathscr{K}$. This verifies (iii), and the first assertion of the theorem is established. Since $\varphi_{A \rtimes_\delta \widehat{S}}$ is the restriction of $\overline{\varphi_A \otimes \mathrm{id}_{\mathscr{K}}}$ which is injective if φ_A is, the assertion on the injectivity of $\varphi_{A \rtimes_\delta \widehat{S}}$ follows.

As in the computation (3.2), but using Lemma 4.1.5 instead of coaction nondegeneracy, we can deduce the equality $\mathcal{K}(X \rtimes_\sigma \widehat{S}) = \mathcal{K}(X) \rtimes_{\sigma^{(1)}} \widehat{S}$. Finally,

$$\begin{aligned}\varphi_{A \rtimes_\delta \widehat{S}}\big(\delta_\iota(a)(1_{M(A)} \otimes x)\big) &= \overline{\varphi_A \otimes \mathrm{id}_{\mathscr{K}}} \circ \overline{\mathrm{id}_A \otimes \iota_S}(\delta(a))\,(1_{M(\mathcal{K}(X))} \otimes x) \\ &= \overline{\mathrm{id}_{\mathcal{K}(X)} \otimes \iota_S} \circ \overline{\varphi_A \otimes \mathrm{id}_S}(\delta(a))\,(1_{M(\mathcal{K}(X))} \otimes x) \\ &= \overline{\mathrm{id}_{\mathcal{K}(X)} \otimes \iota_S} \circ \overline{\sigma^{(1)}}(\varphi_A(a))\,(1_{M(\mathcal{K}(X))} \otimes x) \\ &= \overline{\sigma_\iota^{(1)}}(\varphi_A(a))(1_{M(\mathcal{K}(X))} \otimes x),\end{aligned}$$

in the third step of which we use the δ-$\sigma^{(1)}$ equivariancy of φ_A obtained in Proposition 3.2.4. This completes the proof. □

Definition 4.2.2. We call the C^*-correspondence $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ in Theorem 4.2.1 the *reduced crossed product correspondence* of (X, A) by the coaction (σ, δ) of S.

Remark 4.2.3. We require no universal property of the crossed product $A \rtimes_\delta \widehat{S}$ to define the left action $\varphi_{A \rtimes_\delta \widehat{S}} : A \rtimes_\delta \widehat{S} \to \mathcal{L}(X \rtimes_\sigma \widehat{S})$. It is just the restriction of $\varphi_{M(A \otimes \mathcal{K}(\mathcal{H}))}$.

Remark 4.2.4. For an action (γ, α) of a locally compact group G on (X, A), one can form the crossed product correspondence $(X \rtimes_{\gamma,r} G, A \rtimes_{\alpha,r} G)$ by [13, Proposition 3.2]. We will see in Corollary B.2.3 that it is isomorphic to the reduced crossed product correspondence $(X \rtimes_{\sigma_G^\gamma} \widehat{S}_{\widehat{W}_G}, A \rtimes_{\delta_G^\alpha} \widehat{S}_{\widehat{W}_G})$, where $(\sigma_G^\gamma, \delta_G^\alpha)$ is the coaction of the Hopf C^*-algebra $S_{\widehat{W}_G}$ given in (B.5). On the other hand, if (σ, δ) is a nondegenerate coaction of G on (X, A) ([13, Definition 2.10]) and if $\sigma_\lambda := \overline{\mathrm{id}_X \otimes \lambda} \circ \sigma$ as (2.7), then the crossed product correspondence by (σ, δ) in the sense of [13, Proposition 3.9] is just the reduced crossed product correspondence by the coaction $(\sigma_\lambda, \delta_\lambda)$ of the Hopf C^*-algebra S_{W_G}. Construction in Theorem 4.2.1 thus extends both of the crossed product correspondences by actions and nondegenerate coactions of locally compact groups on C^*-correspondences.

As in [23, Remark 2.7], we have the following corollary, the proof of which is routine.

Corollary 4.2.5. *Let (σ, δ) be a coaction of S on (X, A). Then the map*

$$(j_X^\sigma, j_A^\delta) : (X, A) \to (M(X \rtimes_\sigma \widehat{S}), M(A \rtimes_\delta \widehat{S}))$$

defined by

$$j_X^\sigma(\xi) \cdot c := \sigma_\iota(\xi) \cdot c, \quad j_A^\delta(a)c := \delta_\iota(a)c$$

for $\xi \in X$, $a \in A$, and $c \in A \rtimes_\delta \widehat{S}$ is a nondegenerate correspondence homomorphism such that $j_X^\sigma(X) \subseteq M_{A \rtimes_\delta \widehat{S}}(X \rtimes_\sigma \widehat{S})$.

Remarks 4.2.6. (1) It will be seen that $j_A^\delta(J_X) \subseteq M(A \rtimes_\delta \widehat{S}; J_{X \rtimes_\sigma \widehat{S}})$ (see Proposition 5.1.5). Hence (j_X^σ, j_A^δ) is Cuntz-Pimsner covariant in the sense of [22, Definition 3.1].

(2) Applying Remarks 3.1.4 for $(\psi, \pi) = (k_{X \rtimes_\sigma \widehat{S}}, k_{A \rtimes_\delta \widehat{S}})$ and $(\rho, \omega) = (j_X^\sigma, j_A^\delta)$, we see that the representation

$$(\overline{k_{X \rtimes_\sigma \widehat{S}}} \circ j_X^\sigma, \overline{k_{A \rtimes_\delta \widehat{S}}} \circ j_A^\delta) : (X, A) \to M_{A \rtimes_\delta \widehat{S}}(\mathcal{O}_{X \rtimes_\sigma \widehat{S}})$$

is covariant.

Chapter 5

Reduced crossed products

In this chapter, we first show that the C^*-correspondence $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ has a representation $(k_X \rtimes_\sigma \mathrm{id}, k_A \rtimes_\delta \mathrm{id})$ on the reduced crossed product $\mathcal{O}_X \rtimes_\zeta \widehat{S}$. We then provide a couple of equivalent conditions that this representation is covariant, which is readily seen to be the case if the ideal $J_{X \rtimes_\sigma \widehat{S}}$ of $A \rtimes_\delta \widehat{S}$ is generated by the image $\delta_\iota(J_X)$ or the left action φ_A is injective. Under this covariance condition, the integrated form of the representation $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ will give an isomorphism between the C^*-algebra $\mathcal{O}_X \rtimes_\zeta \widehat{S}$ and the Cuntz-Pimsner algebra $\mathcal{O}_{X \rtimes_\sigma \widehat{S}}$.

Throughout this chapter, we simply write $\mathscr{K} = \mathcal{K}(\mathcal{H})$ as before. The representation

$$(\overline{k_X \otimes \mathrm{id}_{\mathscr{K}}}, \overline{k_A \otimes \mathrm{id}_{\mathscr{K}}}) : (M_{A\otimes\mathscr{K}}(X \otimes \mathscr{K}), M(A \otimes \mathscr{K})) \to M_{A\otimes\mathscr{K}}(\mathcal{O}_X \otimes \mathscr{K})$$

will play an important role in our analysis.

5.1 Representations of $(X \rtimes \widehat{S}, A \rtimes \widehat{S})$ on $\mathcal{O}_X \rtimes \widehat{S}$

Recall that $\overline{k_X \otimes \mathrm{id}_C}$ denotes the $(A \otimes C)$-strict extension to $M_{A\otimes C}(X \otimes C)$.

Lemma 5.1.1. *Let (X, A) be a C^*-correspondence. Let S be a reduced Hopf C^*-algebra and $\iota_S : S \hookrightarrow M(\mathscr{K})$ be the inclusion. Then the following diagram* *commutes:*

$$\begin{tikzcd}
(M_{A\otimes S}(X\otimes S), M(A\otimes S)) \arrow[r, "(\overline{\mathrm{id}_X\otimes\iota_S}, \overline{\mathrm{id}_A\otimes\iota_S})"] \arrow[d, "(\overline{k_X\otimes\mathrm{id}_S}, \overline{k_A\otimes\mathrm{id}_S})"'] & (M_{A\otimes\mathscr{K}}(X\otimes\mathscr{K}), M(A\otimes\mathscr{K})) \arrow[d, "(\overline{k_X\otimes\mathrm{id}_{\mathscr{K}}}, \overline{k_A\otimes\mathrm{id}_{\mathscr{K}}})"] \\
M_{A\otimes S}(\mathcal{O}_X\otimes S) \arrow[r, "\overline{\mathrm{id}_{\mathcal{O}_X}\otimes\iota_S}"'] & M_{A\otimes\mathscr{K}}(\mathcal{O}_X\otimes\mathscr{K})
\end{tikzcd} \tag{5.1}$$

Proof. By [12, Proposition A.11], we see that the upper and lower horizontal maps are $(A\otimes S)$-strict to $(A\otimes\mathscr{K})$-strictly continuous. Hence the two compositions in (5.1) are $(A\otimes S)$-strict to $(A\otimes\mathscr{K})$-strictly continuous. Since the diagram commutes on $(X\odot S, A\odot S)$, the conclusion follows by strict continuity. $\square$

Corollary 5.1.2. *Let (σ,δ) be a coaction of S on (X,A) such that J_X is weakly δ-invariant. Then $\sigma_\iota(X)\subseteq M_{A\otimes\mathscr{K}}(X\otimes\mathscr{K})$ and*

$$X\rtimes_\sigma\widehat{S}\subseteq M_{A\otimes\mathscr{K}}(X\otimes\mathscr{K}).$$

Also,

$$\overline{k_X\otimes\mathrm{id}_{\mathscr{K}}}(\sigma_\iota(\xi))=\zeta_\iota(k_X(\xi)),\quad \overline{k_A\otimes\mathrm{id}_{\mathscr{K}}}(\delta_\iota(a))=\zeta_\iota(k_A(a)) \tag{5.2}$$

for $\xi\in X$ and $a\in A$.

Proof. By Theorem 3.2.7, we can consider the induced coaction ζ on $\mathcal{O}_X$ making the diagram (3.3) commute. Combining (3.3) and (5.1) we see that

$$\sigma_\iota(X)=\overline{\mathrm{id}_X\otimes\iota_S}(\sigma(X))\subseteq M_{A\otimes\mathscr{K}}(X\otimes\mathscr{K}),$$

and thus

$$\begin{aligned}X\rtimes_\sigma\widehat{S}&=\overline{\sigma_\iota(X)\cdot(1_{M(A)}\otimes\widehat{S})}\\&\subseteq M_{A\otimes\mathscr{K}}(X\otimes\mathscr{K})\cdot M(A\otimes\mathscr{K})=M_{A\otimes\mathscr{K}}(X\otimes\mathscr{K}).\end{aligned}$$

The equalities of (5.2) are also immediate from (3.3) and (5.1). $\square$

Remark 5.1.3. From Corollary 5.1.2 (and also from Theorem 4.2.1), we have an injective correspondence homomorphism

$$(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S}) \hookrightarrow (M_{A\otimes\mathscr{K}}(X \otimes \mathscr{K}), M(A \otimes \mathscr{K})).$$

We then have

$$\mathcal{K}(X \rtimes_\sigma \widehat{S}) \subseteq \mathcal{K}\big(M_{A\otimes\mathscr{K}}(X \otimes \mathscr{K})\big) \subseteq M_{A\otimes\mathscr{K}}\big(\mathcal{K}(X \otimes \mathscr{K})\big).$$

Proposition 5.1.4. *Let (σ, δ) be a coaction of S on (X, A) such that J_X is weakly δ-invariant. Then, the restriction of $(\overline{k_X \otimes \mathrm{id}_{\mathscr{K}}}, \overline{k_A \otimes \mathrm{id}_{\mathscr{K}}})$ to $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ defines an injective representation*

$$(k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}, k_A \rtimes_\delta \mathrm{id}_{\widehat{S}}) : (X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S}) \to \mathcal{O}_X \rtimes_\zeta \widehat{S}$$

such that

$$\begin{aligned} k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}\big(\sigma_\iota(\xi) \cdot (1_{M(A)} \otimes x)\big) &= \zeta_\iota(k_X(\xi))(1_{M(\mathcal{O}_X)} \otimes x), \\ k_A \rtimes_\delta \mathrm{id}_{\widehat{S}}\big(\delta_\iota(a)(1_{M(A)} \otimes x)\big) &= \zeta_\iota(k_A(a))(1_{M(\mathcal{O}_X)} \otimes x), \\ k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}\big(\varphi_{M(A\otimes\mathscr{K})}(1_{M(A)} \otimes x)\sigma_\iota(\xi)\big) &= (1_{M(\mathcal{O}_X)} \otimes x)\zeta_\iota(k_X(\xi)) \end{aligned} \tag{5.3}$$

for $\xi \in X$, $x \in \widehat{S}$, and $a \in A$.

Proof. Since $X \rtimes_\sigma \widehat{S} \subseteq M_{A\otimes\mathscr{K}}(X \otimes \mathscr{K})$, the restriction

$$(k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}, k_A \rtimes_\delta \mathrm{id}_{\widehat{S}}) := (\overline{k_X \otimes \mathrm{id}_{\mathscr{K}}}|_{X\rtimes_\sigma\widehat{S}}, \overline{k_A \otimes \mathrm{id}_{\mathscr{K}}}|_{A\rtimes_\delta\widehat{S}})$$

makes sense and is an injective representation of $(X\rtimes_\sigma\widehat{S}, A\rtimes_\delta\widehat{S})$ on $M_{A\otimes\mathscr{K}}(\mathcal{O}_X\otimes \mathscr{K})$. Using the equalities (5.2), we have

$$\begin{aligned} k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}\big(\sigma_\iota(\xi) \cdot (1_{M(A)} \otimes x)\big) &= \overline{k_X \otimes \mathrm{id}_{\mathscr{K}}}\big(\sigma_\iota(\xi) \cdot (1_{M(A)} \otimes x)\big) \\ &= \overline{k_X \otimes \mathrm{id}_{\mathscr{K}}}(\sigma_\iota(\xi))\, \overline{k_A \otimes \mathrm{id}_{\mathscr{K}}}(1_{M(A)} \otimes x) \\ &= \zeta_\iota(k_X(\xi))(1_{M(\mathcal{O}_X)} \otimes x) \end{aligned}$$

for $\xi \in X$ and $x \in \widehat{S}$, and similarly for $k_A \rtimes_\delta \mathrm{id}_{\widehat{S}}$. This proves the first two equalities of (5.3), and hence $(k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}, k_A \rtimes_\delta \mathrm{id}_{\widehat{S}})$ is a representation on $\mathcal{O}_X \rtimes_\zeta \widehat{S}$. The last of (5.3) can be seen similarly. □

For an action (γ, α) of a locally compact group group G on (X, A), the ideal $J_{X\rtimes_{\gamma,r}G}$ for the crossed product correspondence $(X \rtimes_{\gamma,r} G, A \rtimes_{\delta,r} G)$ is known to be equal to the crossed product $J_X \rtimes_{\alpha,r} G$ if G is amenable ([18, Proposition 2.7]) or if G is discrete such that it is exact or α has Exel's Approximation Property ([4, Theorem 5.5]). We now give a partial analogue of this fact in the Hopf C^*-algebra setting.

Proposition 5.1.5. *Let (σ, δ) be a coaction of S on (X, A) such that J_X is weakly δ-invariant. Then*

$$\delta_\iota(J_X)(1_{M(A)} \otimes \widehat{S}) \subseteq J_{X\rtimes_\sigma \widehat{S}}. \tag{5.4}$$

In particular, if $J_X = A$ then $J_{X\rtimes_\sigma \widehat{S}} = A \rtimes_\delta \widehat{S}$.

Proof. The last assertion of the proposition is an immediate consequence of the first. Hence we only need to prove (5.4), which will follow by [26, Proposition 3.3] if we show that

$$k_A \rtimes_\delta \mathrm{id}_{\widehat{S}}\big(\delta_\iota(J_X)(1_{M(A)} \otimes \widehat{S})\big) \subseteq (k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}})^{(1)}\big(\mathcal{K}(X \rtimes_\sigma \widehat{S})\big)$$

since the representation $(k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}, k_A \rtimes_\delta \mathrm{id}_{\widehat{S}})$ is injective. Let us first note the following. By Theorem 2.5.4.(ii), we have

$$\overline{(k_X \otimes \mathrm{id}_{\mathscr{K}})^{(1)}} = \overline{k_X \otimes \mathrm{id}_{\mathscr{K}}}^{(1)}$$

on $\mathcal{K}(M_{A\otimes\mathscr{K}}(X \otimes \mathscr{K}))$. Hence

$$\begin{aligned}\overline{(k_X \otimes \mathrm{id}_{\mathscr{K}})^{(1)}}\big(\mathcal{K}(X \rtimes_\sigma \widehat{S})\big) &= \overline{k_X \otimes \mathrm{id}_{\mathscr{K}}}^{(1)}\big(\mathcal{K}(X \rtimes_\sigma \widehat{S})\big) \\ &= (k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}})^{(1)}\big(\mathcal{K}(X \rtimes_\sigma \widehat{S})\big)\end{aligned}$$

by Remark 5.1.3 and Proposition 5.1.4.

In much the same way as the calculation (4.1) in the proof of Theorem 4.2.1, we see that

$$\delta_\iota(J_X)(1_{M(A)} \otimes \widehat{S}) \subseteq M(A \otimes \mathscr{K}; J_X \otimes \mathscr{K})$$

since J_X is weakly δ-invariant. It therefore follows by Proposition 5.1.4, Lemma 3.1.3, and the above equality that

$$\begin{aligned} k_A \rtimes_\delta \mathrm{id}_{\widehat{S}}\big(\delta_\iota(J_X)(1_{M(A)} \otimes \widehat{S})\big) & \\ &= \overline{k_A \otimes \mathrm{id}_{\mathscr{K}}}\big(\delta_\iota(J_X)(1_{M(A)} \otimes \widehat{S})\big) \\ &= \overline{(k_X \otimes \mathrm{id}_{\mathscr{K}})^{(1)}} \circ \varphi_{M(A\otimes\mathscr{K})}\big(\delta_\iota(J_X)(1_{M(A)} \otimes \widehat{S})\big) \\ &= \overline{(k_X \otimes \mathrm{id}_{\mathscr{K}})^{(1)}}\big(\sigma_\iota^{(1)}(\varphi_A(J_X))(1_{M(\mathcal{K}(X))} \otimes \widehat{S})\big) \\ &\subseteq \overline{(k_X \otimes \mathrm{id}_{\mathscr{K}})^{(1)}}\big(\mathcal{K}(X) \rtimes_{\sigma^{(1)}} \widehat{S}\big) \\ &= (k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}})^{(1)}\big(\mathcal{K}(X \rtimes_\sigma \widehat{S})\big), \end{aligned}$$

where the third and last step come from the δ-$\sigma^{(1)}$ equivariancy of φ_A and equality $\mathcal{K}(X) \rtimes_{\sigma^{(1)}} \widehat{S} = \mathcal{K}(X \rtimes_\sigma \widehat{S})$, respectively. This establishes the proposition. $\square$

Remark 5.1.6. Recall from Definition 3.1 and Lemma 3.2 of [22] that a nondegenerate correspondence homomorphism $(\psi, \pi) : (X, A) \to (M(Y), M(B))$ is *Cuntz-Pimsner covariant* if $\psi(X) \subseteq M_B(Y)$ and $\pi(J_X) \subseteq M(B; J_Y)$. Corollary 4.2.5 and Proposition 5.1.5 then assure us that the representation (j_X^σ, j_A^δ) is always Cuntz-Pimsner covariant since (5.4) is obviously equivalent to

$$j_A^\delta(J_X) \subseteq M(A \rtimes_\delta \widehat{S}; J_{X \rtimes_\sigma \widehat{S}})$$

which was a hypothesis of [23, Theorem 4.4] for $S = C^*(G)$. Therefore, Theorem 4.4 of [23] can be improved as follows: if (σ, δ) is a nondegenerate coaction of a locally compact group G on (X, A) such that $\delta(J_X) \subseteq M(A \otimes C^*(G); J_X \otimes C^*(G))$, then we always have $\mathcal{O}_X \rtimes_\zeta G \cong \mathcal{O}_{X \rtimes_\sigma G}$.

5.2 An isomorphism between $\mathcal{O}_X \rtimes \widehat{S}$ and $O_{X \rtimes \widehat{S}}$

In this section, we present our main results.

Theorem 5.2.1. *Let (σ, δ) be a coaction of a reduced Hopf C^*-algebra S on a C^*-correspondence (X, A) such that J_X is weakly δ-invariant. Then the following conditions are equivalent:*

(i) *The representation $(k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}, k_A \rtimes_\delta \mathrm{id}_{\widehat{S}}) : (X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S}) \to \mathcal{O}_X \rtimes_\zeta \widehat{S}$ is covariant.*

(ii) *The ideal $J_{X \rtimes_\sigma \widehat{S}}$ is contained in $M(A \otimes \mathscr{K}; J_X \otimes \mathscr{K})$.*

(iii) *The product $J_{X \rtimes_\sigma \widehat{S}}(\ker \varphi_A \otimes \mathscr{K})$ is zero.*

Proof. (i) $\Leftrightarrow$ (ii): Suppose (i). Since $(k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}, k_A \rtimes_\delta \mathrm{id}_{\widehat{S}})$ is injective, we have

$$J_{X \rtimes_\sigma \widehat{S}} = (k_A \rtimes_\sigma \mathrm{id}_{\widehat{S}})^{-1}\big((k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}})^{(1)}(\mathcal{K}(X \rtimes_\sigma \widehat{S}))\big)$$

by the comment below [26, Proposition 5.14]. The same reason shows

$$J_{M_{A\otimes\mathscr{K}}(X\otimes\mathscr{K})} = (\overline{k_A \otimes \mathrm{id}_{\mathscr{K}}})^{-1}\Big(\overline{k_X \otimes \mathrm{id}_{\mathscr{K}}}^{(1)}\big(\mathcal{K}(M_{A\otimes\mathscr{K}}(X \otimes \mathscr{K}))\big)\Big)$$

since $\mathscr{K}$ is nuclear and then $(\overline{k_X \otimes \mathrm{id}_{\mathscr{K}}}, \overline{k_A \otimes \mathrm{id}_{\mathscr{K}}})$ is covariant by Corollary 3.1.6. It thus follows that $J_{X \rtimes_\sigma \widehat{S}} \subseteq J_{M_{A\otimes\mathscr{K}}(X\otimes\mathscr{K})}$ by Remark 5.1.3 and Proposition 5.1.4. But, the latter is contained in $M(A \otimes \mathscr{K}; J_X \otimes \mathcal{K})$ again by Corollary 3.1.6. This proves (i) $\Rightarrow$ (ii). Conversely, suppose (ii). Restricting the equality (3.1) of Lemma 3.1.3 to the subalgebra $J_{X \rtimes_\sigma \widehat{S}}$, we can write

$$(k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}})^{(1)} \circ \varphi_{A \rtimes_\delta \widehat{S}} = k_A \rtimes_\delta \mathrm{id}_{\widehat{S}},$$

which verifies (ii) $\Rightarrow$ (i).

(ii) $\Leftrightarrow$ (iii): Assuming (ii) we have

$$\begin{aligned} J_{X \rtimes_\sigma \widehat{S}}(\ker \varphi_A \otimes \mathscr{K}) &= J_{X \rtimes_\sigma \widehat{S}}(A \otimes \mathscr{K})(\ker \varphi_A \otimes \mathscr{K}) \\ &\subseteq (J_X \otimes \mathscr{K})(\ker \varphi_A \otimes \mathscr{K}) = 0, \end{aligned}$$

and hence we get (iii). Finally, we always have

$$\varphi_{A\otimes\mathscr{K}}\big((A\otimes\mathscr{K})J_{X\rtimes_\sigma\widehat{S}}\big)\subseteq\varphi_{A\otimes\mathscr{K}}(A\otimes\mathscr{K})\mathcal{K}(X\rtimes_\sigma\widehat{S})\subseteq\mathcal{K}(X\otimes\mathscr{K})$$

by Remark 5.1.3. Since $\ker\varphi_{A\otimes\mathscr{K}}=\ker(\varphi_A\otimes\mathrm{id}_{\mathscr{K}})=\ker\varphi_A\otimes\mathscr{K}$ by the exactness of $\mathscr{K}$, (iii) implies

$$\big((A\otimes\mathscr{K})J_{X\rtimes_\sigma\widehat{S}}\big)\ker\varphi_{A\otimes\mathscr{K}}=(A\otimes\mathscr{K})\big(J_{X\rtimes_\sigma\widehat{S}}(\ker\varphi_A\otimes\mathscr{K})\big)=0.$$

Therefore $(A\otimes\mathscr{K})J_{X\rtimes_\sigma\widehat{S}}\subseteq J_{X\otimes\mathscr{K}}$. But $J_{X\otimes\mathscr{K}}=J_X\otimes\mathscr{K}$ by Corollary 3.1.6, which proves (iii) $\Rightarrow$ (ii). $\square$

Corollary 5.2.2. *Let (σ,δ) be a coaction of S on (X,A) such that J_X is weakly δ-invariant. Assume that either (i) the ideal $J_{X\rtimes_\sigma\widehat{S}}$ of $A\rtimes_\delta\widehat{S}$ is generated by $\delta_\iota(J_X)$ or (ii) φ_A is injective. Then $(k_X\rtimes_\sigma\mathrm{id}_{\widehat{S}},k_A\rtimes_\delta\mathrm{id}_{\widehat{S}})$ is covariant.*

Proof. Assume (i), that is, $J_{X\rtimes_\sigma\widehat{S}}=\overline{(1_{M(A)}\otimes\widehat{S})\delta_\iota(J_X)(1_{M(A)}\otimes\widehat{S})}$. The nondegeneracy of S and $\widehat{S}$ shows that

$$\begin{aligned}J_{X\rtimes_\sigma\widehat{S}}(A\otimes\mathscr{K})&=\overline{(1_{M(A)}\otimes\widehat{S})\delta_\iota(J_X)(1_{M(A)}\otimes\widehat{S})\,(A\otimes\mathscr{K})}\\&=\overline{(1_{M(A)}\otimes\widehat{S})\delta_\iota(J_X)\,(A\otimes\mathscr{K})}\\&=\overline{(1_{M(A)}\otimes\widehat{S})\delta_\iota(J_X)(1_{M(A)}\otimes S)\,(A\otimes\mathscr{K})}\\&\subseteq\overline{(1_{M(A)}\otimes\widehat{S})(J_X\otimes S)(A\otimes\mathscr{K})}=J_X\otimes\mathscr{K},\end{aligned}$$

in which the last inclusion follows from the weak δ-invariancy of J_X. Hence we get the equivalent condition (ii) in Theorem 5.2.1. On the other hand, assuming (ii) we have (iii) in Theorem 5.2.1. $\square$

Corollary 5.2.3. *Let (σ,δ) be a coaction of S on (X,A) such that δ is trivial, that is, $\delta(a)=a\otimes 1_{M(S)}$ for $a\in A$. If the triple $(J_X,A,\widehat{S})$ satisfies the slice map property, then $(k_X\rtimes_\sigma\mathrm{id}_{\widehat{S}},k_A\rtimes_\delta\mathrm{id}_{\widehat{S}})$ is covariant. Moreover, $J_{X\rtimes_\sigma\widehat{S}}=J_X\otimes\widehat{S}$.*

Proof. Since δ is trivial, J_X is evidently weakly δ-invariant. Hence the representation $(k_X\rtimes_\sigma\mathrm{id}_{\widehat{S}},k_A\rtimes_\delta\mathrm{id}_{\widehat{S}})$ on $\mathcal{O}_X\rtimes_\zeta\widehat{S}$ makes sense by Proposition 5.1.4.

To show that it is covariant, we check the equivalent condition (iii) in Theorem 5.2.1. First note that $\varphi_{A\rtimes_\delta \widehat{S}} = \varphi_{A\otimes\widehat{S}} = \varphi_A \otimes \mathrm{id}_{\widehat{S}}$. Then

$$\ker \varphi_{A\rtimes_\delta \widehat{S}} = \ker(\varphi_A \otimes \mathrm{id}_{\widehat{S}}) = \ker\varphi_A \otimes \widehat{S}$$

by Remarks 3.1.5.(1). Since $\widehat{S}$ is a nondegenerate subalgebra of $\mathcal{L}(\mathcal{H})$, it follows that

$$J_{X\rtimes_\sigma \widehat{S}}(\ker\varphi_A \otimes \mathscr{K}) = \big(J_{X\rtimes_\sigma \widehat{S}}(\ker\varphi_A \otimes \widehat{S})\big)(1_{M(A)} \otimes \mathscr{K}) = 0,$$

and therefore $(k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}, k_A \rtimes_\delta \mathrm{id}_{\widehat{S}})$ is covariant.

Let $\omega \in \mathcal{L}(\mathcal{H})_*$ and $T \in \mathscr{K}$. Applying the slice map $\mathrm{id}_A \otimes (\omega T)$ to $J_{X\rtimes_\sigma \widehat{S}}$ and then multiplying $a \in A$ yields

$$a(\mathrm{id}_A \otimes (\omega T))(J_{X\rtimes_\sigma \widehat{S}}) = (\mathrm{id}_A \otimes \omega)\big((a\otimes T)J_{X\rtimes_\sigma \widehat{S}}\big) \subseteq J_X,$$

in which the last inclusion is due to the equivalent condition (ii) of Theorem 5.2.1. We thus have $(\mathrm{id}_A \otimes \omega)(J_{X\rtimes_\sigma \widehat{S}}) \subseteq J_X$ for $\omega \in \mathcal{L}(\mathcal{H})_*$, and conclude by Remarks 3.1.5.(2) that $J_{X\rtimes_\sigma \widehat{S}} \subseteq F(J_X, A, \widehat{S}) = J_X \otimes \widehat{S}$. The converse follows from Proposition 5.1.5. □

We now state and prove our main theorem.

Theorem 5.2.4. *Let (σ, δ) be a coaction of a reduced Hopf C^*-algebra S on a C^*-correspondence (X, A) such that J_X is weakly δ-invariant. Suppose that the representation $(k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}, k_A \rtimes_\delta \mathrm{id}_{\widehat{S}})$ is covariant. Then the integrated form*

$$(k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}) \times (k_A \rtimes_\delta \mathrm{id}_{\widehat{S}}) : \mathcal{O}_{X\rtimes_\sigma \widehat{S}} \to \mathcal{O}_X \rtimes_\zeta \widehat{S}$$

is a surjective isomorphism.

Proof. Set $\Psi = (k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}) \times (k_A \rtimes_\delta \mathrm{id}_{\widehat{S}})$. Note that the embedding $k_A \rtimes_\delta \mathrm{id}_{\widehat{S}}$ is clearly nondegenerate, and hence Ψ is also nondegenerate.

We claim that $\Psi(\mathcal{O}_{X\rtimes_\sigma \widehat{S}})$ contains all the elements of the form

$$(1_{M(\mathcal{O}_X)} \otimes x)\big(\zeta_\iota\big(k_X(\xi_1)\cdots k_X(\xi_n)k_X(\eta_m)^*\cdots k_X(\eta_1)^*\big)\big)(1_{M(\mathcal{O}_X)} \otimes y)$$

for nonnegative integers m and n, vectors $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m \in X$, and $x, y \in \widehat{S}$. This will prove that Ψ is surjective ([25, Proposition 2.7]). Since

$$\zeta_\iota(k_A(A))(1_{M(\mathcal{O}_X)} \otimes \widehat{S}) \subseteq \Psi(\mathcal{O}_{X \rtimes_\sigma \widehat{S}})$$

by (5.3) of Proposition 5.1.4, we only show by considering adjoints that

$$(1_{M(\mathcal{O}_X)} \otimes x)\zeta_\iota(k_X(\xi_1) \cdots k_X(\xi_n)) \in \Psi(\mathcal{O}_{X \rtimes_\sigma \widehat{S}}) \tag{5.5}$$

for positive integers n, vectors $\xi_1, \dots, \xi_n \in X$, and $x \in \widehat{S}$. We now proceed by induction on n. For $n = 1$, (5.5) follows from the last equality of (5.3). Suppose that (5.5) is true for an n. Let $\xi, \xi_1, \dots, \xi_n$ be $n+1$ vectors in X and $x \in \widehat{S}$. Take an element $C \in \mathcal{O}_{X \rtimes_\sigma \widehat{S}}$ such that

$$\Psi(C) = (1_{M(\mathcal{O}_X)} \otimes x)\zeta_\iota\big(k_X(\xi_1)k_X(\xi_2) \cdots k_X(\xi_n)\big).$$

By Remarks 4.2.6.(2), we have

$$\overline{k_{X \rtimes_\sigma \widehat{S}}}(j_X^\sigma(\xi)) \in M_{A \rtimes_\delta \widehat{S}}(\mathcal{O}_{X \rtimes_\sigma \widehat{S}}).$$

We claim that

$$\overline{\Psi}\big(\overline{k_{X \rtimes_\sigma \widehat{S}}}(j_X^\sigma(\xi))\big) = j_{\mathcal{O}_X}^\zeta(k_X(\xi)), \tag{5.6}$$

where $j_{\mathcal{O}_X}^\zeta : \mathcal{O}_X \to M(\mathcal{O}_X \rtimes_\zeta \widehat{S})$ is the canonical homomorphism such that $j_{\mathcal{O}_X}^\zeta(c)D = \zeta_\iota(c)D$ for $c \in \mathcal{O}_X$ and $D \in \mathcal{O}_X \rtimes_\zeta \widehat{S}$. In fact, for

$$v = \Psi\big(k_{A \rtimes_\delta \widehat{S}}\big(\delta_\iota(a)(1_{M(A)} \otimes x)\big)\big) = \zeta_\iota(k_A(a))(1_{M(\mathcal{O}_X)} \otimes x),$$

we have

$$\begin{aligned}\overline{\Psi}\big(\overline{k_{X\rtimes_\sigma\widehat{S}}}(j_X^\sigma(\xi))\big)\,v &= \Psi\big(\overline{k_{X\rtimes_\sigma\widehat{S}}}(j_X^\sigma(\xi))\,k_{A\rtimes_\delta\widehat{S}}\big(\delta_\iota(a)(1_{M(A)}\otimes x)\big)\big)\\ &= \Psi\big(k_{X\rtimes_\sigma\widehat{S}}\big(j_X^\sigma(\xi)\cdot(\delta_\iota(a)(1_{M(A)}\otimes x))\big)\big)\\ &= \Psi\big(k_{X\rtimes_\sigma\widehat{S}}\big(\sigma_\iota(\xi\cdot a)\cdot(1_{M(A)}\otimes x)\big)\big)\\ &= \zeta_\iota\big(k_X(\xi\cdot a)\big)(1_{M(\mathcal{O}_X)}\otimes x)\\ &= j^\zeta_{\mathcal{O}_X}(k_X(\xi))\,\zeta_\iota(k_A(a))(1_{M(\mathcal{O}_X)}\otimes x) = j^\zeta_{\mathcal{O}_X}(k_X(\xi))\,v,\end{aligned}$$

which verifies the equality (5.6) since $k_A \rtimes_\delta \mathrm{id}_{\widehat{S}}$ is nondegenerate. It is now obvious that for the product $C\,\overline{k_{X\rtimes_\sigma\widehat{S}}}(j_X^\sigma(\xi)) \in \mathcal{O}_{X\rtimes_\sigma\widehat{S}}$ we have

$$\Psi\big(C\,\overline{k_{X\rtimes_\sigma\widehat{S}}}(j_X^\sigma(\xi))\big) = (1_{M(\mathcal{O}_X)}\otimes x)\zeta_\iota\big(k_X(\xi_1)\cdots k_X(\xi_n)k_X(\xi)\big).$$

Consequently, the statement (5.5) is shown to be true for all positive integer n, and hence Ψ is surjective.

Let $\beta : \mathbb{T} \to Aut(\mathcal{O}_X)$ be the gauge action. Then the strict extensions $\overline{\beta_z \otimes \mathrm{id}_{\mathscr{K}}}$ are automorphisms on $M(\mathcal{O}_X \otimes \mathscr{K})$. We have

$$\begin{aligned}\overline{\beta_z\otimes \mathrm{id}_{\mathscr{K}}}\big(\zeta_\iota(k_X(\xi))\big) &= z\zeta_\iota(k_X(\xi)),\\ \overline{\beta_z\otimes \mathrm{id}_{\mathscr{K}}}\big(\zeta_\iota(k_A(a))\big) &= \zeta_\iota(k_A(a))\end{aligned}$$

in the same way as the last part of the proof of Theorem 3.2.7. Therefore,

$$\begin{aligned}&\overline{\beta_z\otimes \mathrm{id}_{\mathscr{K}}}\big((k_X\rtimes_\sigma \mathrm{id}_{\widehat{S}})\big(\sigma_\iota(\xi)\cdot(1_{M(A)}\otimes x)\big)\big)\\ &\qquad= \overline{\beta_z\otimes \mathrm{id}_{\mathscr{K}}}\big(\zeta_\iota(k_X(\xi))(1_{M(\mathcal{O}_X)}\otimes x)\big)\\ &\qquad= z\,\zeta_\iota(k_X(\xi))(1_{M(\mathcal{O}_X)}\otimes x)\\ &\qquad= z\,(k_X\rtimes_\sigma \mathrm{id}_{\widehat{S}})\big(\sigma_\iota(\xi)\cdot(1_{M(A)}\otimes x)\big)\end{aligned}$$

and similarly

$$\overline{\beta_z\otimes \mathrm{id}_{\mathscr{K}}}\big((k_A\rtimes_\delta \mathrm{id}_{\widehat{S}})\big(\delta_\iota(a)(1_{M(A)}\otimes x)\big)\big) = (k_A\rtimes_\delta \mathrm{id}_{\widehat{S}})\big(\delta_\iota(a)(1_{M(A)}\otimes x)\big).$$

This proves that the restriction $\overline{\beta_z\otimes \mathrm{id}_{\mathscr{K}}}|_{\mathcal{O}_{X\rtimes_\zeta\widehat{S}}}$ defines an automorphism on

$\mathcal{O}_X \rtimes_\zeta \widehat{S} = \Psi(\mathcal{O}_{X \rtimes_\sigma \widehat{S}})$, and the injective covariant representation $(k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}, k_A \rtimes_\delta \mathrm{id}_{\widehat{S}})$ admits a gauge action. We thus conclude by [25, Theorem 6.4] that Ψ is injective as well, which completes the proof. □

Applying Theorem 5.2.4 to group actions we can extend Theorem 2.10 of [18] as Corollary 5.2.5 states below, the proof of which will be given in Appendix B. Let (γ, α) be an action of a locally compact group G on (X, A). By Theorem A.2.1, (γ, α) defines a coaction $(\sigma^\gamma, \delta^\alpha)$ of $C_0(G)$ on (X, A), which induces a coaction ζ of $C_0(G)$ on $\mathcal{O}_X$ by Theorem 3.2.7 and Remarks 3.2.9.(1). Let β^ζ be the action of G on $\mathcal{O}_X$ corresponding to the coaction ζ. In a similar way to [18, Corollary 2.9], we define a representation

$$(k_X \rtimes_\gamma G, k_A \rtimes_\alpha G) : (X \rtimes_{\gamma,r} G, A \rtimes_{\alpha,r} G) \to \mathcal{O}_X \rtimes_{\beta^\zeta,r} G$$

by

$$(k_X \rtimes_\gamma G)(f)(r) = k_X(f(r)), \quad (k_A \rtimes_\alpha G)(g)(r) = k_A(g(r))$$

for $f \in C_c(G, X)$, $g \in C_c(G, A)$, and $r \in G$.

Corollary 5.2.5. *Let (γ, α) be an action of a locally compact group G on (X, A). If the representation $(k_X \rtimes_\gamma G, k_A \rtimes_\alpha G)$ is covariant, then its integrated form $(k_X \rtimes_\gamma G) \times (k_A \rtimes_\alpha G) : \mathcal{O}_{X \rtimes_{\gamma,r} G} \to \mathcal{O}_X \rtimes_{\beta^\zeta,r} G$ is a surjective isomorphism.*

For the amenability in the next theorem, we refer to [3]. See also [34].

Theorem 5.2.6. *Let (σ, δ) be a coaction on (X, A) of a reduced Hopf C^*-algebra S defined by an amenable regular multiplicative unitary such that J_X is weakly δ-invariant. If A is nuclear (or exact, respectively), then the same is true for $\mathcal{O}_X \rtimes_\zeta \widehat{S}$.*

Proof. If A is nuclear (or exact, respectively), then so is $A \rtimes_\delta \widehat{S}$ by [34, Theorem 3,4] (or by [34, Theorem 3.13], respectively). Hence, the Toeplitz algebra $\mathcal{T}_{X \rtimes_\sigma \widehat{S}}$ is nuclear by [25, Corollary 7.2] (or exact by [25, Theorem 7.1], respectively). Since nuclearity or exactness passes to quotients, it suffices to show that the representation $(k_X \rtimes_\sigma \mathrm{id}_{\widehat{S}}, k_A \rtimes_\delta \mathrm{id}_{\widehat{S}})$ gives rise to a surjection from $\mathcal{T}_{X \rtimes_\sigma \widehat{S}}$ onto $\mathcal{O}_X \rtimes_\zeta \widehat{S}$. The proof of this then goes parallel to the one given in the proof of Theorem 5.2.4 using the embedding $(i_{X\rtimes_\sigma\widehat{S}}, i_{A\rtimes_\delta\widehat{S}})$ of $(X\rtimes_\sigma\widehat{S}, A\rtimes_\delta\widehat{S})$ into $\mathcal{T}_{X\rtimes_\sigma\widehat{S}}$ instead of $(k_{X\rtimes_\sigma\widehat{S}}, k_{A\rtimes_\delta\widehat{S}})$ used in there. $\square$

Chapter 6

Examples

Applying the previous results we consider in this chapter coactions on crossed products by $\mathbb{Z}$ and directed graph C^*-algebras which form an important example of Cuntz-Pimsner algebras.

6.1 Coactions on crossed products by $\mathbb{Z}$

Let φ be an automorphism on a C^*-algebra A. Equipped with the left action $\varphi_A(a)b = \varphi(a)b$ for $a, b \in A$, the Hilbert A-module A then becomes a C^*-correspondence ([36, Examples (3)]), which we call a *φ-correspondence* and denote by $A(\varphi)$. For a φ-correspondence $A(\varphi)$, it is clear that the multiplier correspondence $M(A(\varphi))$ coincides with the $\overline{\varphi}$-correspondence $M(A)(\overline{\varphi})$ and the strict topology on $M(A(\varphi))$ is the usual one on the multiplier algebra $M(A)$.

We want to consider a coaction of a Hopf C^*-algebra on a φ-correspondence $(A(\varphi), A)$ and its induced coaction on $\mathcal{O}_{A(\varphi)}$. Before that, let us observe the following.

Lemma 6.1.1. *Let φ and φ' be automorphisms on C^*-algebras A and B, respectively, and $\pi : A \to M(B)$ be a nondegenerate homomorphism. Let*

$v \in M(B)$ be a unitary such that

$$v\,\pi(\varphi(a)) = \overline{\varphi'}(\pi(a))\,v \quad (a \in A).$$

Define

$$\psi(a) := v\pi(a) \quad (a \in A).$$

Then $(\psi, \pi) : (A(\varphi), A) \to (M(B(\varphi')), M(B))$ is a nondegenerate correspondence homomorphism. Moreover, every nondegenerate correspondence homomorphism from $(A(\varphi), A)$ into $(M(B(\varphi')), M(B))$ is of this form.

Proof. For $a, a' \in A$,

$$\psi(\varphi(a)a') = v\pi(\varphi(a)a') = v\pi(\varphi(a))\,\pi(a') = \overline{\varphi'}(\pi(a))\,v\pi(a') = \overline{\varphi'}(\pi(a))\,\psi(a')$$

and $\langle \psi(a), \psi(a') \rangle_{M(B)} = \pi(a)^* v^* v \pi(a') = \pi(\langle a, a' \rangle_A)$. Hence (ψ, π) is a correspondence homomorphism, and obviously nondegenerate.

For the converse, let $(\psi, \pi) : (A(\varphi), A) \to (M(B(\varphi')), M(B))$ be a nondegenerate correspondence homomorphism, and consider its strict extension $(\overline{\psi}, \overline{\pi})$. Let $v = \overline{\psi}(1_{M(A)})$. Since $(\overline{\psi}, \overline{\pi})$ is a correspondence homomorphism, we have

$$v^* v = \langle v, v \rangle_{M(B)} = \overline{\pi}(\langle 1_{M(A)}, 1_{M(A)} \rangle_{M(A)}) = 1_{M(B)}.$$

We also have

$$vv^*(\psi(a)b) = \overline{\psi}(1_{M(A)}) \langle \overline{\psi}(1_{M(A)}), \psi(a)b \rangle_{M(B)} = \overline{\psi}(1_{M(A)})\pi(a)b = \overline{\psi}(a)b$$

for $a \in A$ and $b \in B$ so that $vv^* = 1_{M(B)}$. Hence v is a unitary in $M(B)$. Finally,

$$v\pi(\varphi(a)) = \psi(\varphi(a)) = \psi\big(\varphi(a)1_{M(A)}\big) = \overline{\varphi'}(\pi(a))v.$$

This completes the proof. □

Let δ be a coaction of a Hopf C^*-algebra (S, Δ) on a C^*-algebra A and $\varphi \in Aut(A)$. Let v be a cocycle for the coaction δ, that is, a unitary $v \in M(A \otimes S)$ satisfying

$$v_{12}\, \overline{\delta \otimes \mathrm{id}_S}(v) = \overline{\mathrm{id}_A \otimes \Delta}(v)$$

([3, Definition 0.4]), and suppose that

$$v\,\delta(\varphi(a)) = \overline{\varphi \otimes \mathrm{id}_S}(\delta(a))\, v \quad (a \in A). \tag{6.1}$$

Define $\sigma : A(\varphi) \to M(A \otimes S(\varphi \otimes \mathrm{id}_S)) = M(A(\varphi) \otimes S)$ by

$$\sigma(a) := v\delta(a) \quad (a \in A).$$

Then (σ, δ) is a coaction of S on the φ-correspondence $(A(\varphi), A)$. Indeed, it is a nondegenerate correspondence homomorphism by Lemma 6.1.1. Also, the computation

$$\begin{aligned}
\overline{\sigma \otimes \mathrm{id}_S}(\sigma(a)) &= v_{12}\,\overline{\delta \otimes \mathrm{id}_S}(v\delta(a)) \\
&= v_{12}\,\overline{\delta \otimes \mathrm{id}_S}(v)\,\overline{\delta \otimes \mathrm{id}_S}(\delta(a)) \\
&= \overline{\mathrm{id}_A \otimes \Delta}(v)\,\overline{\mathrm{id}_A \otimes \Delta}(\delta(a)) = \overline{\mathrm{id}_A \otimes \Delta}(\sigma(a))
\end{aligned}$$

verifies the coaction identity of σ. The coaction nondegeneracy of δ gives

$$\begin{aligned}
\overline{(1_{M(A)} \otimes S)\sigma(A)} &= \overline{(1_{M(A)} \otimes S)\, v\delta(A)} = \overline{(1_{M(A)} \otimes S)\, v\delta(\varphi(A))} \\
&= \overline{(1_{M(A)} \otimes S)\,\big(\overline{\varphi \otimes \mathrm{id}_S}\,\delta(A)\big)\, v} \\
&= \overline{\overline{\varphi \otimes \mathrm{id}_S}\big((1_{M(A)} \otimes S)\,\delta(A)\big)\, v} = (A \otimes S)v = A \otimes S
\end{aligned}$$

so that σ satisfies coaction nondegeneracy. Hence (σ, δ) is a coaction.

The Cuntz-Pimsner algebra $\mathcal{O}_{A(\varphi)}$ is isomorphic to the crossed product $A \rtimes_\varphi \mathbb{Z}$ and an isomorphism $\mathcal{O}_{A(\varphi)} \cong A \rtimes_\varphi \mathbb{Z}$ can be given as follows. Let (π, u) be the canonical covariant representation of the C^*-dynamical system $(A, \mathbb{Z}, \varphi)$ on $M(A \rtimes_\varphi \mathbb{Z})$. Define $\psi : A(\varphi) \to A \rtimes_\varphi \mathbb{Z}$ by

$$\psi(a) = u^*\pi(a) \quad (a \in A(\varphi)).$$

It can be easily checked that (ψ, π) is a covariant representation of $(A(\varphi), A)$ on $A \rtimes_\varphi \mathbb{Z}$. Furthermore, the integrated form $\psi \times \pi : \mathcal{O}_{A(\varphi)} \to A \rtimes_\varphi \mathbb{Z}$ gives a surjective isomorphism. We will identify in this way the universal covariant representations $(k_X, k_A) = (\psi, \pi)$ as well as the C^*-algebras $\mathcal{O}_{A(\varphi)} = A \rtimes_\varphi \mathbb{Z}$.

Since $J_{A(\varphi)} = A$ is evidently weakly δ-invariant, it follows by Theorem 3.2.7 that (σ, δ) induces a coaction ζ of S on $\mathcal{O}_{A(\varphi)} = A \rtimes_\varphi \mathbb{Z}$ which can be described explicitly on the canonical generators of $A \rtimes_\varphi \mathbb{Z}$ as follows. Theorem 3.2.7 says that

$$\zeta(\pi(a)) = \overline{\pi \otimes \mathrm{id}_S}(\delta(a)),$$

$$\zeta(u^*\pi(a)) = \zeta(\psi(a)) = \overline{\psi \otimes \mathrm{id}_S}(\sigma(a)) = (u^* \otimes 1_{M(S)})\,\overline{\pi \otimes \mathrm{id}_S}(v\delta(a))$$

for $a \in A$. Note that $\overline{\zeta}(u^*) = (u^* \otimes 1_{M(S)})\,\overline{\pi \otimes \mathrm{id}_S}(v)$. Hence,

$$\zeta(\pi(a)u^n) = \overline{\pi \otimes \mathrm{id}_S}(\delta(a))\big((u^* \otimes 1_{M(S)})\,\overline{\pi \otimes \mathrm{id}_S}(v)\big)^{-n}$$

for $a \in A$ and $n \in \mathbb{Z}$.

Assume now that the Hopf C^*-algebra S is reduced. Then we can form the reduced crossed product correspondence $(A(\varphi) \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ by Theorem 4.2.1. Let $v_\iota = \overline{\mathrm{id}_A \otimes \iota_S}(v)$. Since the multiplication by v_ι from the left gives a Hilbert module isomorphism from $A \rtimes_\delta \widehat{S}$ onto $A(\varphi) \rtimes_\sigma \widehat{S}$, we may — and do — regard the C^*-correspondence $A(\varphi) \rtimes_\sigma \widehat{S}$ as the Hilbert module $A \rtimes_\delta \widehat{S}$ with the left action

$$\varphi_{A \rtimes_\delta \widehat{S}}(c)\, d = v_\iota^* \overline{\varphi \otimes \mathrm{id}_{\mathcal{K}(\mathcal{H})}}(c) v_\iota\, d \tag{6.2}$$

for an element c in the C^*-algebra $A \rtimes_\delta \widehat{S}$ and a vector d in the Hilbert module $A \rtimes_\delta \widehat{S}$. Note that $\varphi_{A \rtimes_\delta \widehat{S}}$ is injective. Since φ_A is injective, $\mathcal{O}_{A(\varphi)} \rtimes_\zeta \widehat{S}$ is the Cuntz-Pimsner algebra $\mathcal{O}_{A(\varphi) \rtimes_\sigma \widehat{S}}$ by Corollary 5.2.2 and Theorem 5.2.4.

We can summerize what we have seen so far as follows.

Proposition 6.1.2. *Let φ be an automorphism on a C^*-algebra A and δ be a coaction of a Hopf C^*-algebra S on A. Let v be a cocyle for δ satisfying* (6.1)*. Define $\sigma : A(\varphi) \to M(A(\varphi) \otimes S)$ by $\sigma(a) = v\delta(a)$. Then the following hold.*

(i) *(σ, δ) is a coaction of S on the φ-correspondence $(A(\varphi), A)$.*

(ii) *Let (π, u) be the canonical covariant representation of $(A, \mathbb{Z}, \varphi)$ on $M(A \rtimes_\varphi \mathbb{Z})$. Then, the homomorphism*

$$\overline{\pi \otimes \mathrm{id}_S} \circ \delta : A \to M((A \rtimes_\varphi \mathbb{Z}) \otimes S)$$

and the unitary $\overline{\pi \otimes \mathrm{id}_S}(v^*)(u \otimes 1_{M(S)}) \in M((A \rtimes_\varphi \mathbb{Z}) \otimes S)$ *form a covariant representation of* $(A, \mathbb{Z}, \varphi)$ *on* $M((A \rtimes_\varphi \mathbb{Z}) \otimes S)$ *such that the integrated form gives a coaction* ζ *of* S *on* $A \rtimes_\varphi \mathbb{Z}$ *and coincides with the coaction induced by* (σ, δ)*.*

(iii) *If* S *is reduced then* $A(\varphi) \rtimes_\sigma \widehat{S} = A \rtimes_\delta \widehat{S}$ *as Hilbert* $(A \rtimes_\delta \widehat{S})$*-modules and the left action is given by* (6.2)*. The reduced crossed product* $(A \rtimes_\varphi \mathbb{Z}) \rtimes_\zeta \widehat{S} = \mathcal{O}_{A(\varphi)} \rtimes_\zeta \widehat{S}$ *is the Cuntz-Pimsner algebra* $\mathcal{O}_{A(\varphi) \rtimes_\sigma \widehat{S}}$*.*

We can say further if we take the cocycle v in Proposition 6.1.2 to be the identity. Let $v = 1_{M(A \otimes S)}$. Then (6.1) reduces to

$$\delta \circ \varphi = \overline{\varphi \otimes \mathrm{id}_S} \circ \delta,$$

and then $\varphi_{A \rtimes_\delta \widehat{S}}$ maps $A \rtimes_\delta \widehat{S}$ onto itself:

$$\varphi_{A \rtimes_\delta \widehat{S}}\big(\delta_\iota(a)(1_{M(A)} \otimes x)\big) = \overline{\varphi \otimes \mathrm{id}_{\mathcal{K}(\mathcal{H})}}\big(\delta_\iota(a)(1_{M(A)} \otimes x)\big) = \delta_\iota(\varphi(a))(1_{M(A)} \otimes x)$$

for $a \in A$ and $x \in \widehat{S}$. Hence $\varphi_{A \rtimes_\delta \widehat{S}}$ defines an automorphism $\varphi \rtimes \mathrm{id}$ on $A \rtimes_\delta \widehat{S}$ such that

$$(\varphi \rtimes \mathrm{id})\big(\delta_\iota(a)(1_{M(A)} \otimes x)\big) = \delta_\iota(\varphi(a))(1_{M(A)} \otimes x)$$

for $a \in A$ and $x \in \widehat{S}$. We thus see that $A(\varphi) \rtimes_\sigma \widehat{S}$ is the $(\varphi \rtimes \mathrm{id})$-correspondence $A \rtimes_\delta \widehat{S}(\varphi \rtimes \mathrm{id})$. We have the equality

$$\mathcal{O}_{A(\varphi) \rtimes_\sigma \widehat{S}} = \mathcal{O}_{A \rtimes_\delta \widehat{S}(\varphi \rtimes \mathrm{id})} = (A \rtimes_\delta \widehat{S}) \rtimes_{\varphi \rtimes \mathrm{id}} \mathbb{Z} \tag{6.3}$$

as well as

$$\mathcal{O}_{A(\varphi)} \rtimes_\zeta \widehat{S} = (A \rtimes_\varphi \mathbb{Z}) \rtimes_\zeta \widehat{S}, \tag{6.4}$$

and then have a surjective isomorphism

$$\Psi : (A \rtimes_\delta \widehat{S}) \rtimes_{\varphi \rtimes \mathrm{id}} \mathbb{Z} = \mathcal{O}_{A(\varphi) \rtimes_\sigma \widehat{S}} \to \mathcal{O}_{A(\varphi)} \rtimes_\zeta \widehat{S} = (A \rtimes_\varphi \mathbb{Z}) \rtimes_\zeta \widehat{S}.$$

Let us describe Ψ on the canonical generators of the iterated crossed products $(A \rtimes_\delta \widehat{S}) \rtimes_{\varphi \rtimes \mathrm{id}} \mathbb{Z}$ and $(A \rtimes_\varphi \mathbb{Z}) \rtimes_\zeta \widehat{S}$. As (π, u) in Proposition 6.1.2, let $(\widetilde{\pi}, \widetilde{u})$ be the canonical covariant representation of the C^*-dynamical system $(A \rtimes_\delta$

$\widehat{S}$, $\mathbb{Z}$, $\varphi \rtimes \mathrm{id}$) on $M((A \rtimes_\delta \widehat{S}) \rtimes_{\varphi \rtimes \mathrm{id}} \mathbb{Z})$. Let

$$\begin{aligned}
d_1 &= k_{A(\varphi) \rtimes_\delta \widehat{S}}\big(\delta_\iota(a) \cdot (1_{M(A)} \otimes x)\big) \in \mathcal{O}_{A(\varphi) \rtimes_\sigma \widehat{S}}, \\
d_2 &= \widetilde{u}^* \widetilde{\pi}\big(\delta_\iota(a)(1_{M(A)} \otimes x)\big) \in (A \rtimes_\delta \widehat{S}) \rtimes_{\varphi \rtimes \mathrm{id}} \mathbb{Z}, \\
d_3 &= \zeta_\iota(k_{A(\varphi)}(a))(1_{M(\mathcal{O}_{A(\varphi)})} \otimes x) \in \mathcal{O}_{A(\varphi)} \rtimes_\zeta \widehat{S}, \\
d_4 &= \zeta_\iota(u^* \pi(a))(1_{M(A \rtimes_\varphi \mathbb{Z})} \otimes x) \in (A \rtimes_\varphi \mathbb{Z}) \rtimes_\zeta \widehat{S}.
\end{aligned}$$

We then have $d_1 = d_2$ in (6.3), and $d_3 = d_4$ in (6.4). Since $\Psi(d_1) = d_3$, we may write $\Psi(d_2) = d_4$. Note that $\overline{\Psi}(\widetilde{u}) = \overline{\zeta_\iota}(u)$. Therefore

$$\Psi\big(\widetilde{u}^n\, \widetilde{\pi}\big(\delta_\iota(a)(1_{M(A)} \otimes x)\big)\big) = \zeta_\iota(u^n \pi(a))(1_{M(A \rtimes_\varphi \mathbb{Z})} \otimes x),$$

or equivalently, by the fact that (π, u) and $(\widetilde{\pi}, \widetilde{u})$ are covariant representations,

$$\Psi\big(\widetilde{\pi}\big(\delta_\iota(a)(1_{M(A)} \otimes x)\big)\, \widetilde{u}^n\big) = \zeta_\iota(\pi(a) u^n)(1_{M(A \rtimes_\varphi \mathbb{Z})} \otimes x) \tag{6.5}$$

for $a \in A$, $x \in \widehat{S}$, and $n \in \mathbb{Z}$. We summarize this in the next corollary.

Corollary 6.1.3. *Under the hypothesis and notation in Proposition 6.1.2 with v replaced by $1_{M(A \otimes S)}$, the formula*

$$\zeta(\pi(a)u^n) = \overline{\pi \otimes \mathrm{id}_S}(\delta(a))(u^n \otimes 1_{M(S)}) \quad (a \in A,\ n \in \mathbb{Z})$$

defines a coaction ζ of S on $A \rtimes_\varphi \mathbb{Z}$. Moreover, if S is reduced then there exists a surjective isomorphism

$$\Psi : (A \rtimes_\delta \widehat{S}) \rtimes_{\varphi \rtimes \mathrm{id}} \mathbb{Z} \to (A \rtimes_\varphi \mathbb{Z}) \rtimes_\zeta \widehat{S}$$

between the iterated crossed products such that (6.5) *holds.*

6.2 Coactions on directed graph C^*-algebras

In this section, we consider coactions of compact quantum groups on (directed) graph C^*-algebras arising from *labelings* or *coactions on finite graphs.* We begin by recalling some of basic facts about graph C^*-algebras and compact quamtum groups.

A *directed graph* $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, a range map $r : E^1 \to E^0$ and source map $s : E^1 \to E^0$ describing the terminal and initial vertices of the edges, respectively. The *graph C^*-algebra* $C^*(E)$ of E is the universal C^*-algebra generated by mutually orthogonal projections p_v ($v \in E^0$) and partial isometries s_e ($e \in E^1$) with mutually orthogonal ranges such that $s_e^* s_e = p_{r(e)}$ and $p_{s(e)} s_e s_e^* = s_e s_e^*$ for $e \in E^1$, and $p_v = \sum_{e \in s^{-1}(v)} s_e s_e^*$ for v with $0 < |s^{-1}(v)| < \infty$ ([29, 28, 15]). Throughout we consider graphs with nonempty edge sets.

As in [24], one can associate to a directed graph E a C^*-correspondence. Denote by χ_e and χ_v the functions $\chi_e(f) = \delta_{e,f}$ and $\chi_v(w) = \delta_{v,w}$, where the symbols $\delta_{e,f}$ and $\delta_{v,w}$ are the Kronecker deltas. Let A be the commutative C^*-algebra of functions on E^0 vanishing at infinity. The *graph correspondence* $(X(E), A)$ associated to E is the completion of the $C_c(E^0)$-bimodule $C_c(E^1)$ such that

$$\chi_e \cdot \chi_v = \delta_{r(e),v}\, \chi_e, \quad \langle \chi_e, \chi_f \rangle_A = \delta_{e,f}\, \chi_{r(e)}, \quad \varphi_A(\chi_v)\chi_e = \delta_{s(e),v}\, \chi_e$$

for $e, f \in E^1$ and $v \in E^0$. It can be easily seen that $J_{X(E)} = \{\chi_v \in A : 0 < |s^{-1}(v)| < \infty\}$ and the map $(\psi, \pi) : (X(E), A) \to C^*(E)$ given by $\psi(\chi_e) = s_e$ and $\pi(\chi_v) = p_v$ is covariant. The integrated form $\psi \times \pi$ gives an isomorphism from the Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$ onto the graph C^*-algebra $C^*(E)$. We will identify $\mathcal{O}_{X(E)} = C^*(E)$ through this isomorphism so that $k_{X(E)}(\chi_e) = s_e$ and $k_A(\chi_v) = p_v$.

A unital Hopf C^*-algebra (S, Δ) is called a *compact quantum group*. A *finite dimensional unitary corepresentation* of S is a unitary $U = (u_{ij})$ in a matrix algebra $M_n(S)$ such that $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$ for $i, j = 1, \ldots, n$ ([46, 48]). We write d_U for the dimension n of the unitary corepresentation $U = (u_{ij}) \in M_n(S)$. We say that U is *fundamental* [46] if the C^*-algebra is generated by the matrix elements of U.

Let $(\mathcal{H}, \Lambda)$ be the GNS-representation of a compact quantum group S associated to its Haar state which is known to exist uniquely ([48, 42]). Then there corresponds to S a regular multiplicative unitary $V_S \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ ([3, Proposition 3.4.4.b)]) such that Λ gives a compact quantum group morphism (in the sense of [3, Definition 0.5]) from S onto the reduced Hopf C^*-algebra S_{V_S}.

6.2.1 Labelings and coactions on graph C^*-algebras

Definition 6.2.1. Let $E = (E^0, E^1, r, s)$ be a directed graph. Let S be a compact quantum group and $\mathscr{R}_S$ be a set of finite dimensional unitary corepresentations of S. We call a function $c : E^1 \to \mathscr{R}_S$ a *labeling* if the edge set E^1 admits a partition $\{(v, w; U)_n\}_{(v,w),U,n}$ such that

(i) $(v, w; U)_n \subseteq s^{-1}(v) \cap r^{-1}(w) \cap c^{-1}(U)$,

(ii) $|(v, w; U)_n| = d_U$,

where the indices (v, w) and U range over the sets $\{(s(e), r(e)) : e \in E^1\}$ and $c(E^1)$, respectively.

Remark 6.2.2. The terminology of labeling comes from [21] in which for a directed graph E and a countable discrete group G, a labeling is defined to be a function $c : E^1 \to G$. It was shown in [21] that a labeling gives rise to a coaction ζ of G on the graph C^*-algebra $C^*(E)$ and the crossed product $C^*(E) \rtimes_\zeta G$ can be realized as a graph C^*-algebra. Corollary 6.2.4 below generalizes this fact to compact quantum groups. It is clear that if we let $\mathscr{R}_{C^*(G)} = G$, the set of one dimensional unitary corepresentations of the compact quantum group $C^*(G)$, then our definition of labeling coincides with the definition in [21]; in this case E^1 admits a partition consisting of singleton sets.

Let $c : E^1 \to \mathscr{R}_S$ be a labeling and $\{(v, w; U)_n\}$ be a partition of E^1. For each $(v, w; U)_n$, fix an order of the elements of $(v, w; U)_n$ so that $1, 2, \ldots, d_U$ represents the elements of the set. The reason why we consider $(v, w; U)_n$ will be clear from the proof of the next proposition.

Proposition 6.2.3. *Let E be a directed graph and S be a compact quantum group. Let $c : E^1 \to \mathscr{R}_S$ be a labeling. Then there exists a coaction (σ, δ) :*

$(X(E), A) \to (X(E) \otimes S, A \otimes S)$ *of* S *on the graph correspondence* $(X(E), A)$ *such that*

$$\sigma(\chi_j) = \sum_{i \in (v,w;U)_n} \chi_i \otimes u_{ij} \quad (j \in (v,w;U)_n)$$

and $\delta(a) = a \otimes 1_S$ *for* $a \in A$.

Proof. Let $j \in (v,w;U)_n$ and $l \in (v',w';U')_{n'}$. Then

$$\begin{aligned}\langle \sigma(\chi_j), \sigma(\chi_l)\rangle_{A\otimes S} &= \Big\langle \sum_i \chi_i \otimes u_{ij}, \sum_k \chi_k \otimes u'_{kl}\Big\rangle_{A\otimes S} = \sum_{i,k} \langle \chi_i, \chi_k\rangle_A \otimes {u_{ij}}^* u'_{kl} \\ &= \delta_{v,v'}\delta_{w,w'}\delta_{n,n'} \sum_i \chi_w \otimes {u_{ij}}^* u_{il} \\ &= \delta_{v,v'}\delta_{w,w'}\delta_{n,n'}\delta_{j,l}(\chi_w \otimes 1_S) = \langle \chi_j, \chi_l\rangle_A \otimes 1_S.\end{aligned}$$

It then follows that $\langle \sigma(\xi), \sigma(\eta)\rangle_{A\otimes S} = \langle \xi, \eta\rangle_A \otimes 1_S$ for $\xi, \eta \in C_c(E^1)$. In particular, σ is isometric on $C_c(E^1)$, and hence extends to all of $X(E)$. Also,

$$\begin{aligned}\sigma(\varphi_A(\chi_v)\chi_j) = \delta_{v,s(j)}\sigma(\chi_j) &= (\varphi_A(\chi_v) \otimes 1_S) \sum_{i \in (v,w;U)_n} \chi_i \otimes u_{ij} \\ &= \varphi_{A\otimes S}(\delta(\chi_v))\sigma(\chi_j).\end{aligned}$$

Therefore, (σ, δ) is a correspondence homomorphism. It is readily seen that σ satisfies the coaction identity. For the coaction nondegeneracy, consider the antipode κ of the dense Hopf $*$-algebra generated by the matrix elements of all finite dimensional unitary corepresentations of S [48]. Then

$$\kappa\Big(\sum_j \kappa^{-1}(u_{jk})u_{ij}\Big) = \sum_j \kappa(u_{ij})u_{jk} = \delta_{i,k}1_S,$$

which gives $\sum_j \kappa^{-1}(u_{jk})u_{ij} = \delta_{i,k}1_S$. Therefore,

$$\sum_j \varphi_{A\otimes S}(1_A \otimes \kappa^{-1}(u_{jk}))\, \sigma(\chi_j) = \sum_{ij} \chi_i \otimes \kappa^{-1}(u_{jk})u_{ij} = \chi_k \otimes 1_S. \tag{6.6}$$

This proves that $\overline{\varphi_{A\otimes S}(1_A \otimes S)\, \sigma(X(E))}$ contains $\chi_e \otimes 1_S$ for $e \in E^1$, and consequently, coincides with $X(E) \otimes S$. $\square$

Applying Theorem 3.2.7, Corollary 5.2.3, and Theorem 5.2.4 to the coaction (σ, δ) in Proposition 6.2.3 we now obtain the following corollary in which $\sigma_\Lambda = (\mathrm{id}_{C^*(E)} \otimes \Lambda) \circ \sigma$ and similarly for δ_Λ and ζ_Λ.

Corollary 6.2.4. *Let E be a directed graph and S be a compact quantum group. Let $c : E^1 \to \mathscr{R}_S$ be a labeling. Then there exists a coaction $\zeta : C^*(E) \to C^*(E) \otimes S$ of S on the graph C^*-algebra $C^*(E)$ such that*

$$\zeta(s_j) = \sum_{i \in (v,w;U)_n} s_i \otimes u_{ij} \quad (j \in (v,w;U)_n)$$

and $\zeta(p_v) = p_v \otimes 1_S$ for $v \in E^0$. Moreover, the crossed product $C^*(E) \rtimes_{\zeta_\Lambda} \widehat{S}_{V_S}$ is the Cuntz-Pimsner algebra associated to $(X(E) \rtimes_{\sigma_\Lambda} \widehat{S}_{V_S}, A \rtimes_{\delta_\Lambda} \widehat{S}_{V_S})$.

6.2.2 Coactions on finite graphs

Definition 6.2.5. Let $E = (E^0, E^1, r, s)$ be a finite directed graph and S be a compact quantum group. A *coaction of S on E* is a pair (σ, δ) such that

(i) σ and δ are coactions of S on the commutative C^*-algebras $C(E^1)$ and $C(E^0)$, respectively,

(ii) the diagram

$$\begin{tikzcd} C(E^0) \arrow[r, "\delta"] \arrow[d, "r_* \ (s_*,\ \text{resp.})"'] & C(E^0) \otimes S \arrow[d, "r_* \otimes \mathrm{id} \ (s_* \otimes \mathrm{id},\ \text{resp.})"] \\ C(E^1) \arrow[r, "\sigma"'] & C(E^1) \otimes S \end{tikzcd} \tag{6.7}$$

commutes, where r_* is the homomorphism given by $r_*(\chi_v) = \chi_v \circ r$ and similarly for s_*.

Remark 6.2.6. The notion of coaction of a compact quantum group on a finite graph was considered in [5] under the aim of constructing the quantum automorphism group coacting on the graph, and given only for a finite graph with at most one edge from a vertex to another ([5, Definition 3.1]). Our definition allows for finitely many edges between two vertices.

Notations 6.2.7. For a coaction (σ, δ) of S on E, we denote by $a_{f\,e}$ and $b_{w\,v}$ the elements of S such that

$$\sigma(\chi_e) = \sum_{f \in E^1} \chi_f \otimes a_{f\,e}, \quad \delta(\chi_v) = \sum_{w \in E^0} \chi_w \otimes b_{w\,v}.$$

It is well-known by [45, Theorem 3.1] that the elements $a_{f\,e}$ are projections such that $\sum_e a_{f\,e} = 1_S = \sum_f a_{f\,e}$, and similarly for $b_{f\,e}$.

Lemma 6.2.8. *Let E be a finite graph and S be a compact quantum group. Let σ and δ be coactions of S on $C(E^1)$ and $C(E^0)$, respectively. Then the diagram (6.7) commutes if and only if for $f \in E^1$ the following are satisfied:*

(i) *if $r^{-1}(v) = \emptyset$ then $b_{r(f)\,v} = 0$,*

(ii) *if $r^{-1}(v) \neq \emptyset$ then $b_{r(f)\,v} = \sum_{e \in r^{-1}(v)} a_{f\,e}$,*

(iii) *if $s^{-1}(v) = \emptyset$ then $b_{s(f)\,v} = 0$,*

(iv) *if $s^{-1}(v) \neq \emptyset$ then $b_{s(f)\,v} = \sum_{e \in s^{-1}(v)} a_{f\,e}$.*

Proof. Assume that (6.7) commutes. We only prove (i) and (ii). The others are treated in the same way. Note that

$$\begin{aligned} (r_* \otimes \mathrm{id}_S) \circ \delta(\chi_v) &= \sum_{w \in E^0} r_*(\chi_w) \otimes b_{w\,v} \\ &= \sum_{r^{-1}(w) \neq \emptyset} \sum_{f \in r^{-1}(w)} \chi_f \otimes b_{w\,v} = \sum_{f \in E^1} \chi_f \otimes b_{r(f)\,v}. \end{aligned} \tag{6.8}$$

If $r^{-1}(v) = \emptyset$ then $r_*(\chi_v) = 0$ so that $\sigma \circ r_*(\chi_v) = 0$. Since $\sigma \circ r_* = (r_* \otimes \mathrm{id}_S) \circ \delta$, we then have $b_{r(f)\,v} = 0$. If $r^{-1}(v) \neq \emptyset$ then

$$\sigma \circ r_*(\chi_v) = \sigma\Bigg(\sum_{e \in r^{-1}(v)} \chi_e \Bigg) = \sum_{e \in r^{-1}(v)} \sum_{f \in E^1} \chi_f \otimes a_{f\,e}. \tag{6.9}$$

Comparing (6.8) and (6.9) we have (ii). Conversely, assume (i)-(iv). From (6.8) and (6.9) we readily see that the diagram (6.7) commutes. □

Applying the antipode of S to each of (i)-(iv) in Lemma 6.2.8, we have the following.

Corollary 6.2.9. *For $f \in E^1$ the following hold:*

(i) *if $r^{-1}(v) = \emptyset$ then $b_{v\,r(f)} = 0$,*

(ii) *if $r^{-1}(v) \neq \emptyset$ then $b_{v\,r(f)} = \sum_{e \in r^{-1}(v)} a_{e\,f}$,*

(iii) *if $s^{-1}(v) = \emptyset$ then $b_{v\,s(f)} = 0$,*

(iv) *if $s^{-1}(v) \neq \emptyset$ then $b_{v\,s(f)} = \sum_{e \in s^{-1}(v)} a_{e\,f}$.*

The next corollary corresponds to [5, Theorem 3.2].

Corollary 6.2.10. *Let (σ, δ) be a coaction of a compact quantum group S on a finite graph E. Then*

(i) *the products $b_{s(f)\,v} b_{r(f)\,w}$, $b_{r(f)\,w} b_{s(f)\,v}$, $b_{w\,r(f)} b_{v\,s(f)}$, and $b_{v\,s(f)} b_{w\,r(f)}$ are all zero whenever $s^{-1}(v) \cap r^{-1}(w) = \emptyset$,*

(ii) *the projections $b_{s(f)\,s(e)}$ and $b_{r(f)\,r(e)}$ commute,*

(iii) *the sum $\sum_{f \in E^1} b_{s(f)\,v} b_{r(f)\,w}$ is equal to $|s^{-1}(v) \cap r^{-1}(w)|$.*

Proof. (i) We only show $b_{s(f)\,v} b_{r(f)\,w} = 0$. The others are followed from this by considering the adjoint or the antipode. We may assume that both $b_{s(f)\,v}$ and $b_{r(f)\,w}$ are nonzeoro. By Lemma 6.2.8 we have $s^{-1}(v) \neq \emptyset$ and $r^{-1}(w) \neq \emptyset$ so that

$$b_{s(f)\,v} b_{r(f)\,w} = \sum_{e \in s^{-1}(v)} a_{f\,e} \sum_{e \in r^{-1}(v)} a_{f\,e}$$

which must be zero since $s^{-1}(v) \cap r^{-1}(w) = \emptyset$.

(ii) Let $v = s(e)$ and $w = r(e)$. By Lemma 6.2.8,

$$b_{s(f)\,v} b_{r(f)\,w} = \sum_{e \in s^{-1}(v)} a_{f\,e} \sum_{g \in r^{-1}(w)} a_{f\,g} = \sum_{e \in s^{-1}(v) \cap r^{-1}(w)} a_{f\,e}. \tag{6.10}$$

Hence the product $b_{s(f)\,s(e)}b_{r(f)\,r(e)}$ is a projection, which verifies (ii).

(iii) We may assume that $s^{-1}(v) \neq \emptyset$ and $r^{-1}(w) \neq \emptyset$. As (6.10) we have

$$\sum_{f\in E^1} b_{s(f)\,v}b_{r(f)\,w} = \sum_{f\in E^1}\sum_{e\in s^{-1}(v)} a_{f\,e}\sum_{g\in r^{-1}(w)} a_{f\,g},$$

which gives (iii). $\square$

We now identify the space $C(E^1)$ and the graph correspondence $X(E)$ (algebraically) for a finite graph E.

Theorem 6.2.11. *Let (σ, δ) be a coaction of a compact quantum group S on a finite directed graph E. Write*

$$\sigma(\chi_e) = \sum_{f\in E^1} \chi_f \otimes a_{f\,e}, \quad \delta(\chi_v) = \sum_{w\in E^0} \chi_w \otimes b_{w\,v}.$$

for $e \in E^1$ and $v \in E^0$. Then

(i) *(σ, δ) is a coaction on $(X(E), A)$ such that $J_{X(E)}$ is weakly δ-invariant.*

(ii) *(σ, δ) induces a coaction ζ of S on the graph C^*-algebra $C^*(E)$ such that*

$$\zeta(s_e) = \sum_{f\in E^1} s_f \otimes a_{fe}, \quad \zeta(p_v) = \sum_{w\in E^0} p_w \otimes b_{wv}.$$

Moreover, $C^*(E) \rtimes_{\zeta_\Lambda} \widehat{S}_{V_S} \cong \mathcal{O}_{X(E)\rtimes_{\sigma_\Lambda}\widehat{S}_{V_S}}$.

Proof. We have

$$\begin{aligned}\varphi_{A\otimes S}(\delta(\chi_v))\,\sigma(\chi_e) &= \Big(\sum_w \varphi_A(\chi_w)\otimes b_{w\,v}\Big)\Big(\sum_f \chi_f \otimes a_{f\,e}\Big)\\ &= \sum_f \chi_f \otimes b_{s(f)\,v}\,a_{f\,e}\\ &= \delta_{v,\,s(e)}\sum_f \chi_f\otimes a_{f\,e} = \sigma(\varphi_A(\chi_v)\chi_e)\end{aligned}$$

since $b_{s(f)\,v} = \sum_{g \in s^{-1}(v)} a_{f\,g}$ if $s^{-1}(v) \neq \emptyset$ and zero otherwise by Lemma 6.2.8. For $e, g \in E^1$ we also have

$$\begin{aligned}\delta(\langle \chi_e, \chi_g \rangle_A) &= \delta_{e,g}\delta(\chi_{r(e)}) = \delta_{e,g} \sum_w \chi_w \otimes b_{w\,r(e)} \\ &= \delta_{e,g} \sum_{r^{-1}(w) \neq \emptyset} \chi_w \otimes b_{w\,r(e)}\end{aligned}$$

since $b_{w\,r(e)} = 0$ if $r^{-1}(w) = \emptyset$ by Corollary 6.2.9, and then

$$= \delta_{e,g} \sum_{r^{-1}(w) \neq \emptyset} \chi_w \otimes \sum_{f \in r^{-1}(w)} a_{f\,e}$$

again by Corollary 6.2.9, and hence

$$\begin{aligned}&= \delta_{e,g} \sum_f \chi_{r(f)} \otimes a_{f\,e} \\ &= \sum_f \chi_{r(f)} \otimes a_{f\,e} a_{f\,g} = \langle \sigma(\chi_e), \sigma(\chi_g) \rangle_{A \otimes S}.\end{aligned}$$

Hence (σ, δ) is a correspondence homomorphism. By definition, σ satisfies the coaction identity. Computation like (6.6) shows that σ satisfies the coaction nondegeneracy, and hence (σ, δ) is a coaction of S on $(X(E), A)$. Let $\chi_v \in J_{X(E)}$, that is, $s^{-1}(v) \neq \emptyset$. Take an $e \in s^{-1}(v)$. Then

$$\delta(\chi_v) = \sum_w \chi_w \otimes b_{w\,s(e)} = \sum_{s^{-1}(w) \neq \emptyset} \chi_w \otimes b_{w\,s(e)}$$

by Corollary 6.2.9.(iii). This proves that $J_{X(E)}$ is weakly δ-invariant. The assertion on the induced coaction ζ follows from Theorem 3.2.7. Finally, since A is finite dimensional, the space $\delta_\iota(A)(1_A \otimes \widehat{S})$ is already closed and equal to $A \rtimes_\delta \widehat{S}$. Hence, if $s^{-1}(v) = \emptyset$ then $\delta_\iota(\chi_v)(1_A \otimes \widehat{S}) \nsubseteq J_{X \rtimes_\sigma \widehat{S}}$, and consequently, $J_{X \rtimes_\sigma \widehat{S}}$ coincides with $(1_A \otimes \widehat{S})\delta_\iota(J_X)(1_A \otimes \widehat{S})$. The last assertion on the isomorphism now follows from Corollary 5.2.2. $\square$

It is easily seen that $\delta(J_{X(E)})(1_A \otimes S) = J_{X(E)} \otimes S$. Hence, the restriction $\delta|_{J_{X(E)}}$ gives a coaction of S on $J_{X(E)}$.

We now consider the quantum automorphism group of a finite graph E, whose definition was given in [5, Definition 3.1] for E with at most one edge from a vertex to another.

Definition 6.2.12. Let E be a finite directed graph. Let $\mathcal{C}_E$ be the category such that

(i) an object is a pair $(S, (\sigma, \delta))$ of a compact quantum group S and a coaction (σ, δ) of S on E,

(ii) a morphism from an object $(S, (\sigma, \delta))$ to another $(S', (\sigma', \delta'))$ is a compact quantum group morphism $\phi : S \to S'$ satisfying

$$\sigma' \circ \phi = (\mathrm{id}_{C(E^1)} \otimes \phi) \circ \sigma, \quad \delta' \circ \phi = (\mathrm{id}_{C(E^0)} \otimes \phi) \circ \delta.$$

The *quantum automorphism group of* E is an initial object $(S, (\sigma, \delta))$ in $\mathcal{C}_E$, that is, for an object $(S', (\sigma', \delta'))$ in $\mathcal{C}_E$ there exists a morphism from $(S, (\sigma, \delta))$ to $(S', (\sigma', \delta'))$.

Recall from [44, Definition 2.9] that a *Wononowicz ideal* of a compact quantum group (S, Δ) is an ideal I of the C^*-algebra S such that $(\pi \otimes \pi) \circ \Delta(I) = 0$, where $\pi : S \to S/I$ is the quotient map. In this case, there exists a unique compact quantum group structure on S/I such that π is a compact quantum group morphism [44, Theorem 3.4].

Let $E = (E^0, E^1, r, s)$ be a finite graph. Let $(A_{aut}(E^i), \alpha_i)$ be the quantum permutation group of E^i $(i = 0, 1)$ [45, Theorem 3.1]. Denote by $(\tilde{a}_{f\,e})_{E^1 \times E^1}$ and $(\tilde{b}_{w\,v})_{E^0 \times E^0}$ the fundamental unitaries of $A_{aut}(E^1)$ and $A_{aut}(E^0)$, respectively, such that

$$\alpha_1(\chi_e) = \sum_f \chi_f \otimes \tilde{a}_{f\,e}, \quad \alpha_0(\chi_v) = \sum_w \chi_w \otimes \tilde{b}_{w\,v}.$$

Notations 6.2.13. We denote by F_E the amalgamated free product $A_{aut}(E^0) * A_{aut}(E^1)$ over $\mathbb{C}$. By [44, Theorem 3.4], F_E has a unique compact quantum group structure such that the canonical embedding

$$\iota_i : A_{aut}(E^i) \hookrightarrow F_E \quad (i = 0, 1)$$

is a compact quantum group morphism. We denote by I_E the ideal of the C^*-algebra F_E generated by the relations (i)-(iv) in Lemma 6.2.8.

Theorem 6.2.14. *Let $E = (E^0, E^1, r, s)$ be a finite graph, and $(A_{aut}(E^i), \alpha_i)$ be the quantum permutation group on E^i $(i = 0, 1)$. Define*

$$\sigma := \big(\mathrm{id}_{C(E^1)} \otimes (\pi \circ \iota_1)\big) \circ \alpha_1, \quad \delta := \big(\mathrm{id}_{C(E^0)} \otimes (\pi \circ \iota_0)\big) \circ \alpha_0,$$

where $\pi : F_E \to F_E/I_E$ is the quotient map. Then

(i) *I_E is a Woronowicz ideal of the compact quantum group F_E,*

(ii) *$(F_E/I_E, (\sigma, \delta))$ is a quantum automorphism group of E. More precisely, if $(S', (\sigma', \delta'))$ is an object in $\mathcal{C}_E$ such that $\sigma'(\chi_e) = \sum_f \chi_f \otimes a'_{f\,e}$ and $\delta'(\chi_v) = \sum_w \chi_w \otimes b'_{w\,v}$, then the formulas*

$$\phi(\pi(\tilde{a}_{f\,e})) := a'_{f\,e}, \quad \phi(\pi(\tilde{b}_{w\,v})) = b'_{w\,v}$$

define a morphism $\phi : (F_E/I_E, (\sigma, \delta)) \to (S', (\sigma', \delta'))$. The spectrum of F_E/I_E is the usual automorphism group of E.

Proof. To simplify the notations, we identify $A_{aut}(E^i)$ with its image in F_E. By definition, I_E is generated by the elements of the following type:

(i) $\tilde{b}_{r(f)\,v}$ for $f \in E^1$ and $v \in E^0$ with $r^{-1}(v) = \emptyset$,

(ii) $\tilde{b}_{r(f)\,v} - \sum_{e \in r^{-1}(v)} \tilde{a}_{f\,e}$ for $f \in E^1$ and $v \in E^0$ with $r^{-1}(v) \neq \emptyset$,

(iii) $\tilde{b}_{s(f)\,v}$ for $f \in E^1$ and $v \in E^0$ with $s^{-1}(v) = \emptyset$,

(iv) $\tilde{b}_{s(f)\,v} - \sum_{e \in s^{-1}(v)} \tilde{a}_{f\,e}$ for $f \in E^1$ and $v \in E^0$ with $s^{-1}(v) \neq \emptyset$.

Let Δ be the comultiplication of F_E. If $r^{-1}(v) = \emptyset$ then

$$\begin{aligned}(\pi \otimes \pi) \circ \Delta(\tilde{b}_{r(f)\,v}) &= \sum_w \pi(\tilde{b}_{r(f)\,w}) \otimes \pi(\tilde{b}_{w\,v}) \\ &= \sum_{r^{-1}(w)\neq\emptyset} \pi(\tilde{b}_{r(f)\,w}) \otimes \pi(\tilde{b}_{w\,v}) = 0\end{aligned}$$

since the elements $\tilde{b}_{r(f)\,w}$ for $r^{-1}(w) = \emptyset$ are of the type (i), and so are $\tilde{b}_{w\,v}$ for $r^{-1}(w) \neq \emptyset$. If $r^{-1}(v) \neq \emptyset$ then

$$\begin{aligned}(\pi \otimes \pi) \circ \Delta(\tilde{b}_{r(f)\,v}) &= \sum_w \pi(\tilde{b}_{r(f)\,w}) \otimes \pi(\tilde{b}_{w\,v}) \\ &= \sum_{r^{-1}(w)\neq\emptyset} \pi(\tilde{b}_{r(f)\,w}) \otimes \pi(\tilde{b}_{w\,v}) \\ &= \sum_{r^{-1}(w)\neq\emptyset} \sum_{g\in r^{-1}(w)} \pi(\tilde{a}_{f\,g}) \otimes \pi(\tilde{b}_{r(g)\,v}) \\ &= \sum_{r^{-1}(w)\neq\emptyset} \sum_{g\in r^{-1}(w)} \sum_{e\in r^{-1}(v)} \pi(\tilde{a}_{f\,g}) \otimes \pi(\tilde{a}_{g\,e}) \\ &= \sum_{e\in r^{-1}(v)} \sum_g \pi(\tilde{a}_{f\,g}) \otimes \pi(\tilde{a}_{g\,e}) = (\pi \otimes \pi) \circ \Delta\Bigg(\sum_{e\in r^{-1}(v)} \tilde{a}_{f\,e}\Bigg).\end{aligned}$$

This proves that $(\pi \otimes \pi) \circ \Delta$ maps the elements of the type (ii) to 0. Similarly, the same is true for the elements of the type (iii) and (iv), which proves the assertion (i) of the theorem.

It is clear that σ and δ are coactions of the compact quantum group F_E/I_E on $C(E^1)$ and $C(E^0)$, respectively. The elements $\pi(\tilde{a}_{f\,e})$ and $\pi(\tilde{b}_{w\,v})$ satisfy (i)-(iv) in Lemma 6.2.8 by definition, and hence (σ, δ) is a coaction of F_E/I_E on E, namely $(F_E/I_E, (\sigma, \delta))$ is an object of the category $\mathcal{C}_E$. Let $(S', (\sigma', \delta'))$ be an object in $\mathcal{C}_E$ such that $\sigma'(\chi_e) = \sum_f \chi_f \otimes a'_{f\,e}$ and $\delta'(\chi_v) = \sum_w \chi_w \otimes b'_{w\,v}$. Combining [45, Theorem 3.1] and [44, Theorem 3.4] we see that there exists a compact quantum group morphism $\phi_0 : F_E \to S'$ such that $\phi_0(\tilde{a}_{f\,e}) = a'_{f\,e}$ and $\phi_0(\tilde{b}_{w\,v}) = b'_{w\,v}$. By Lemma 6.2.8, ϕ_0 factors through F_E/I_E, that is, there exists a morphism $\phi : F_E/I_E \to S'$ such that $\phi(\pi(\tilde{a}_{f\,e})) = a'_{f\,e}$ and $\phi(\pi(\tilde{b}_{w\,v})) = b'_{w\,v}$.

This establishes the first part of the assertion (ii) of the theorem. The last part is clear. □

Remark 6.2.15. Theorem 6.2.14 together with Corollary 6.2.10 extends [5, Theorem 3.2] to any finite graphs, and shows that our notion of quantum automorphism group on E coincides with the one given in [5] for E with at most one edge from a vertex to another.

Example 6.2.16. For a positive integer $n \geq 2$, let E be the directed graph consisting of only one vertex and n edges $\{1, \dots, n\}$. Its graph C^*-algebra $C^*(E)$ is the Cuntz algebra $\mathcal{O}_n$. Let S be a compact quantum group and $\mathscr{R}_S = \{U\}$ for an n dimensional unitary corepresentation $U = (u_{ij})$ of S. Define a labeling c by the constant function $c(j) = u$ for $j \in E^1$. By Corollary 6.2.4, the formula $\zeta(s_j) = \sum_{i=1}^n s_i \otimes u_{ij}$ then determines a coaction ζ of S on $\mathcal{O}_n$ and the crossed product $\mathcal{O}_n \rtimes_{\zeta_\Lambda} \widehat{S}_{V_S}$ is a Cuntz-Pimsner algebra. Alternatively, the formulas $\sigma(\chi_j) = \sum_i \chi_i \otimes u_{ij}$ and $\delta(1_{\mathbb{C}}) = 1_{\mathbb{C}} \otimes 1_S$ define a coaction (σ, δ) of S on E. By Theorem 6.2.11, (σ, δ) induces the coaction ζ, and the crossed product is a Cuntz-Pimsner algebra.

Example 6.2.17. Let E be the following two copies of the directed cycle of length 2:

$$\begin{tikzcd} v_1 \bullet \arrow[r, bend left, "e_1"] & \bullet v_2 \arrow[l, bend left, "e_2"] & v_3 \bullet \arrow[r, bend left, "e_3"] & \bullet v_4 \arrow[l, bend left, "e_4"] \end{tikzcd}$$

One can readily check that $C^*(E) = C(\mathbb{T}, M_2 \oplus M_2)$, the C^*-algebra of continuous functions from the unit circle $\mathbb{T}$ to the direct sum of the 2×2 matrix algebras M_2. Hence $C^*(E)$ is generated by the four elements e_{12}, ze_{21}, e_{34}, and ze_{43}, where e_{ij} is the matrix units and z is the identity function on $\mathbb{T}$.

Now consider the quantum automorphism group F_E/I_E of E. We write $a_{i\,j}$ and $b_{i\,j}$, respectively, for the generators $\pi(\tilde{a}_{e_i\,e_j})$ and $\pi(\tilde{b}_{v_i\,v_j})$ of F_E/I_E $(i, j = 1, \dots, 4)$. By Lemma 6.2.8 we have

$$\begin{aligned} a_{i\,j} &= a_{i+1\,j+1} = b_{i\,j} = b_{i+1\,j+1} \quad ((i,j) = (1,1), (1,3), (3,1), (3,3)), \\ a_{i\,j} &= a_{i+1\,j-1} = b_{i\,j} = b_{i+1\,j-1} \quad ((i,j) = (1,2), (1,4), (3,2), (3,4)). \end{aligned}$$

Thus, F_E/I_E is the universal C^*-algebra generated by 8 projections $a_{1\,j}$ and $a_{3,\,j}$ $(j = 1, \ldots, 4)$ subject to the relations

$$\sum_j a_{1\,j} = 1 = \sum_j a_{3\,j}, \quad a_{1\,3} + a_{1\,4} = a_{3\,5} + a_{3\,6},$$

and a compact quantum group with the fundamental unitary

$$\begin{pmatrix} a_{1\,1} & a_{1\,2} & a_{1\,3} & a_{1\,4} \\ a_{1\,2} & a_{1\,1} & a_{1\,4} & a_{1\,3} \\ a_{3\,1} & a_{3\,2} & a_{3\,3} & a_{3\,4} \\ a_{3\,2} & a_{3\,1} & a_{3\,4} & a_{3\,3} \end{pmatrix}.$$

This is another description for the quantum automorphism group $A_{aut}(E)$ given in [5, Proposition 3.3] (see also its proof). By Theorem 6.2.11 and Theorem 6.2.14, we thus obtain a coaction ζ of F_E/I_E on $C(\mathbb{T}, M_2 \oplus M_2)$ such that

$$\begin{aligned} \zeta(e_{12}) &= e_{12} \otimes a_{1\,1} + ze_{21} \otimes a_{1\,2} + e_{34} \otimes a_{3\,1} + ze_{43} \otimes a_{3\,2}, \\ \zeta(ze_{21}) &= e_{12} \otimes a_{1\,2} + ze_{21} \otimes a_{1\,1} + e_{34} \otimes a_{3\,2} + ze_{43} \otimes a_{3\,1}, \\ \zeta(e_{34}) &= e_{12} \otimes a_{1\,3} + ze_{21} \otimes a_{1\,4} + e_{34} \otimes a_{3\,3} + ze_{43} \otimes a_{3\,4}, \\ \zeta(ze_{43}) &= e_{12} \otimes a_{1\,4} + ze_{21} \otimes a_{1\,3} + e_{34} \otimes a_{3\,4} + ze_{43} \otimes a_{3\,3}. \end{aligned}$$

The crossed product is a Cuntz-Pimsner algebra.

Appendix A

Coactions of $C_0(G)$ on C^*-correspondences

The goal of this chapter is to prove that there exists a one-to-one correspondence between actions of a locally compact group G on (X, A) in the sense of [13] and coactions of the commutative Hopf C^*-algebra $C_0(G)$ on (X, A) (Theorem A.2.1). For this, we first prove an Akemann-Pedersen-Tomiyama type theorem for C^*-correspondence (Theorem A.1.4), and using this we prove the bijective correspondence.

A.1 Akemann-Pedersen-Tomiyama type theorem for C^*-correspondences

Let us fix some notations. Let (X, A) be a nondegenerate C^*-correspondence as before, and G be a locally compact Hausdorff space. By $M(X)_s$ we mean the multiplier correspondence $M(X)$ endowed with the strict topology. We denote by $C_b(G, M(X)_s)$ the Banach space of all bounded continuous functions from G to $M(X)$ with the sup-norm, and by $C_b(G, X)$ the closed subspace of $C_b(G, M(X)_s)$ consisting of functions with values in X which are also continuous with respect to the norm topology on X. We denote by $C_c(G, X)$ the subspace of $C_b(G, X)$ of all compactly supported functions; $C_0(G, X)$ is the norm closure of $C_c(G, X)$.

For an identity correspondence $(X, A) = (A, A)$, the Banach space $C_b(G, M(X)_s)$ becomes a C^*-algebra under the usual point-wise operations. In this case,

$$C_b(G, M(X)_s) = M(X \otimes C_0(G))$$

([1, Corollary 3.4]). We first generalize this in Theorem A.1.4 to nondegenerate C^*-correspondences, which will enable us to prove the bijective correspondence between G-actions and $C_0(G)$-coactions on (X, A).

Proposition A.1.1. *Let (X, A) be a C^*-correspondence and G be a locally compact Hausdorff space. Then $(C_b(G, M(X)_s), C_b(G, M(A)_s)$ is a C^*-correspondence with respect to the following point-wise operations*

$$\begin{aligned} (m \cdot l)(r) &= m(r) \cdot l(r), \\ \langle m, n\rangle_{C_b(G,M(A)_s)}(r) &= \langle m(r), n(r)\rangle_{M(A)}, \\ \big(\varphi_{C_b(G,M(A)_s)}(l)\, m\big)(r) &= \varphi_{M(A)}(l(r))\, m(r) \end{aligned} \tag{A.1}$$

for $m, n \in C_b(G, M(X)_s)$, $l \in C_b(G, M(A)_s)$, and $r \in G$.

Proof. Write $\varphi = \varphi_{C_b(G,M(A)_s)}$. The only part requiring proof is that the functions on (A.1) are strictly continuous. We prove this only for the function $\varphi(l)\, m$. The others can be handled in the same way. Let $\{r_i\}$ be a net in G converging to an $r \in G$, $a \in A$, and $T \in \mathcal{K}(X)$. Evidently, the difference $(\varphi(l)m)(r_i) \cdot a - (\varphi(l)m)(r) \cdot a$ converges to 0. Factor $T = T'\varphi_A(a')$ for some $T' \in \mathcal{K}(X)$ and $a' \in A$, which is possible by the Hewitt-Cohen factorization theorem (see for example [37, Proposition 2.33]) since the left action φ_A is nondegenerate. Then the difference

$$\begin{aligned} T(\varphi(l)m)(r_i) - T(\varphi(l)m)(r) = &\big(T'\varphi_A(a'l(r_i))\, m(r_i) - T'\varphi_A(a'l(r))\, m(r_i)\big) \\ &+ \big(T\varphi_{M(A)}(l(r))\, m(r_i) - T\varphi_{M(A)}(l(r))\, m(r)\big) \end{aligned}$$

converges to 0 by the strict continuity of both l and m and also by the boundedness of m. Hence $\varphi(l)m$ is strictly continuous. □

It is clear that $(C_0(G,X), C_0(G,A))$ is also a C^*-correspondence with respect to the restriction of operations (A.1).

We call a correspondence homomorphism $(\psi, \pi) : (X, A) \to (Y, B)$ an *isomorphism* if both ψ and π are bijective. In this case, (X, A) and (Y, B) are said to be *isomorphic*. The next corollary is an easy consequence of Corollary 3.1.7.

Corollary A.1.2. *The C^*-correspondence $(C_0(G,X), C_0(G,A))$ and the tensor product correspondence $(X \otimes C_0(G), A \otimes C_0(G))$ are isomorphic.*

Lemma A.1.3. *With respect to the operations (A.1), the following hold.*

(i) $\varphi_{C_b(G,M(A)_s)}\big(C_b(G, M(A)_s)\big)\, C_0(G,X) = C_0(G,X),$

(ii) $C_0(G,X) \cdot C_b(G, M(A)_s) = C_0(G,X),$

(iii) $C_b(G, M(X)_s) \cdot C_0(G,A) = C_0(G,X).$

Proof. On each of (i) and (ii), the space in the right-hand side is evidently contained in the left-hand space. The same is true for (iii) by the Hewitt-Cohen factorization theorem since $(C_0(G,X), C_0(G,A))$ is isomorphic to the nondegenerate C^*-correspondence $(X \otimes C_0(G), A \otimes C_0(G))$. For the inclusion $\subseteq$ in (i), let $l \in C_b(G, M(A)_s)$ and $x \in C_0(G,X)$, and write $x = \varphi_{C_0(G,A)}(f)y$ for some $f \in C_0(G,A)$ and $y \in C_0(G,X)$. Then

$$\varphi_{C_b(G,M(A)_s)}(l)\, x = \varphi_{C_0(G,A)}(lf)\, y \in C_0(G,X),$$

which proves (i). Similarly write $x = z \cdot g$ for $z \in C_0(G,X)$ and $g \in C_0(G,A)$. Then $x \cdot l = y \cdot (gl) \in C_0(G,X)$, which verifies the inclusion $\subseteq$ in (ii). Finally, the triangle inequality verifies that the functions in the left-hand side space of (iii) are continuous, which gives $\subseteq$ in (iii). $\square$

Henceforth, we identify $C_0(G,X) = X \otimes C_0(G)$ as well as $C_b(G, M(A)_s) = M(A \otimes C_0(G))$. The next theorem generalizes [1, Corollary 3.4].

Theorem A.1.4. *The map*

$$(\psi, \mathrm{id}) : (C_b(G, M(X)_s), C_b(G, M(A)_s)) \to (M(X \otimes C_0(G)), M(A \otimes C_0(G)))$$

given by

$$\psi(m) \cdot f = m \cdot f$$

for $m \in C_b(G, M(X)_s)$ and $f \in A \otimes C_0(G)$ is an isomorphism.

Proof. By Lemma A.1.3, we can apply [13, Proposition 1.28] to see that (ψ, id) is an injective correspondence homomorphism. It thus remains to show that ψ is surjective. Let $n \in M(X \otimes C_0(G))$. For each $r \in G$, define $m_n(r) : A \to X$ and $m_n^*(r) : X \to A$ by

$$m_n(r)(a) := \big(n \cdot (a \otimes \phi_r)\big)(r), \quad m_n^*(r)(\xi) := \big(n^*(\xi \otimes \phi_r)\big)(r),$$

where $\phi_r \in C_c(G)$ such that $\phi_r \equiv 1$ on a neighborhood of r. It is immaterial which ϕ_r we take to define $m_n(r)$ and $m_n^*(r)$ as long as $\phi_r \equiv 1$ near r. Since

$$\langle n \cdot (a \otimes \phi_r), \xi \otimes \phi_r \rangle_{A \otimes C_0(G)} = \langle a \otimes \phi_r, n^*(\xi \otimes \phi_r) \rangle_{A \otimes C_0(G)},$$

we have $\langle m_n(r) \cdot a, \xi \rangle_A = \langle a, m_n^*(r)\xi \rangle_A$ by evaluating at r, and thus obtain a function $m_n : G \to M(X)$ with $m_n(r)^* = m_n^*(r)$. By definition, $\|m_n(r)\| \leq \|n\|$ for $r \in G$, and hence m_n is bounded. To see that m_n is strictly continuous, let $\{r_i\}$ be a net in G converging to an $r \in G$, $a \in A$, and $\xi, \eta \in X$. Evidently, $\{m_n(r_i) \cdot a\}$ converges to $m_n(r) \cdot a$. The same is true for the net $\{m_n(r_i)^*\xi\}$, and hence $\{\theta_{\eta,\xi} m_n(r_i)\} = \{\theta_{\eta, m_n(r_i)^*\xi}\}$ converges to $\theta_{\eta, m_n(r)^*\xi} = \theta_{\eta,\xi} m_n(r)$, and consequently $\{T m_n(r_i)\}$ converges to $T m_n(r)$ for $T \in \mathcal{K}(X)$. Therefore $m_n \in C_b(G, M(X)_s)$. Finally, we have

$$\big(n \cdot (a \otimes \phi_r)\big)(r) = m_n(r) \cdot a = \big(m_n \cdot (a \otimes \phi_r)\big)(r)$$

for $a \in A$ and $r \in G$, which shows $\psi(m_n) = n$. □

In what follows, we identify $C_b(G, M(X)_s) = M(X \otimes C_0(G))$.

Corollary A.1.5. *The $C_0(G)$-multiplier correspondence $M_{C_0(G)}(X \otimes C_0(G))$ coincides with $C_b(G, X)$.*

Proof. Evidently, $M_{C_0(G)}(X \otimes C_0(G)) \supseteq C_b(G, X)$. For the converse, let $m \in M_{C_0(G)}(X \otimes C_0(G))$. Let $r \in G$, and take a $\phi_r \in C_c(G)$ such that $\phi_r \equiv 1$ on a neighborhood U of r. Since the function $\varphi_{M(A\otimes C_0(G))}(1_{M(A)} \otimes \phi_r)m$ belongs to $X \otimes C_0(G)$ and agrees with m on U, we see that m is continuous at r with $m(r) \in X$. This proves the converse. $\square$

A.2 One-to-one correspondence between G-actions and $C_0(G)$-coactions

Let $Aut(X, A)$ be the group of isomorphisms from (X, A) onto itself. Recall from [13, Definition 2.5] that an *action* of a locally compact group G on (X, A) is a homomorphism $(\gamma, \alpha) : G \to Aut(X, A)$ such that for each $\xi \in X$ and $a \in A$, the maps

$$G \ni r \mapsto \gamma_r(\xi) \in X, \quad G \ni r \mapsto \alpha_r(a) \in A$$

are both continuous.

Theorem A.2.1. *If (γ, α) is an action of a locally compact group G on (X, A), then there exists a coaction $(\sigma^\gamma, \delta^\alpha)$ of $C_0(G)$ on (X, A) such that*

$$\sigma^\gamma(\xi)(r) = \gamma_r(\xi), \quad \delta^\alpha(a)(r) = \alpha_r(a) \tag{A.2}$$

for $\xi \in X$, $a \in A$, and $r \in G$. Moreover, the formulas in (A.2) *define a one-to-one correspondence between actions of G on (X, A) and coactions of $C_0(G)$ on (X, A).*

Proof. It is well-known that δ^α is a coaction of $C_0(G)$ on A (see for example [40, Chapter 9]). By Corollary A.1.5, the first formula in (A.2) defines a map

$$\sigma^\gamma : X \to M_{C_0(G)}(X \otimes C_0(G)) \subseteq M(X \otimes C_0(G)).$$

By definition, (γ_r, α_r) is a correspondence homomorphism for $r \in G$, that is,

(i) $\gamma_r(\varphi_A(a)\xi) = \varphi_A(\alpha_r(a))\gamma_r(\xi)$;

(ii) $\langle \gamma_r(\xi), \gamma_r(\eta)\rangle_A = \alpha_r(\langle \xi, \eta\rangle_A)$

for $r \in G$, which is equivalent to

(i) $\sigma^\gamma(\varphi_A(a)\xi) = \varphi_{M(A\otimes C_0(G))}(\delta^\alpha(a))\,\sigma^\gamma(\xi)$;

(ii) $\langle \sigma^\gamma(\xi), \sigma^\gamma(\eta)\rangle_{M(A\otimes C_0(G))} = \delta^\alpha(\langle \xi, \eta\rangle_A)$.

Hence $(\sigma^\gamma, \delta^\alpha)$ is a correspondence homomorphism. Let $\xi \in X$, $\phi \in C_c(G)$, and $\epsilon > 0$. Take a neighborhood U of the neutral element of G such that $\|\gamma_r(\xi) - \xi\| < \epsilon$ for $r \in U$. Choose a finite subcover $\{Ur_i\}$ of the support of ϕ, and a partition of unity $\{\phi_i\}$ subordinate to $\{Ur_i\}$. One can easily check that

$$\Big\| \xi \otimes \phi - \sum_i \varphi_{M(A\otimes C_0(G))}(1_{M(A)} \otimes \phi_i\phi)\,\sigma^\gamma(\gamma_{r_i}^{-1}(\xi)) \Big\| < \epsilon,$$

which proves that $\overline{\varphi_{M(A\otimes C_0(G))}\big(1_{M(A)} \otimes C_0(G)\big)\,\sigma^\gamma(X)} \supseteq X \otimes C_0(G)$. The opposite inclusion is obvious, and hence σ^γ satisfies the coaction nondegeneracy. For the coaction identity of σ^γ, let $\mathrm{ev}_r : C_0(G) \to \mathbb{C}$ be the evaluation at $r \in G$. It then suffices to show that

$$\overline{\mathrm{id}_X \otimes \mathrm{ev}_r \otimes \mathrm{ev}_s} \circ \overline{\sigma^\gamma \otimes \mathrm{id}_{C_0(G)}} \circ \sigma^\gamma = \overline{\mathrm{id}_X \otimes \mathrm{ev}_r \otimes \mathrm{ev}_s} \circ \overline{\mathrm{id}_X \otimes \Delta_G} \circ \sigma^\gamma$$

for $r, s \in G$ since the strict extensions $\overline{\mathrm{id}_X \otimes \mathrm{ev}_r \otimes \mathrm{ev}_s}$ on $M(X \otimes C_0(G) \otimes C_0(G))$ correspond to the evaluations $m(r, s)$ for $m \in C_b(G \times G, M(X)_s)$, and hence separate the points of $M(X \otimes C_0(G) \otimes C_0(G))$. Note that on $X \otimes C_0(G)$

$$\begin{aligned} \overline{\mathrm{id}_X \otimes \mathrm{ev}_r \otimes \mathrm{ev}_s} \circ (\sigma^\gamma \otimes \mathrm{id}_{C_0(G)}) &= (\overline{\mathrm{id}_X \otimes \mathrm{ev}_r} \circ \sigma^\gamma) \otimes \mathrm{ev}_s \\ &= \overline{\mathrm{id}_X \otimes \mathrm{ev}_r} \circ \sigma^\gamma \circ (\mathrm{id}_X \otimes \mathrm{ev}_s) \end{aligned}$$

and

$$\overline{\mathrm{id}_X \otimes \mathrm{ev}_r \otimes \mathrm{ev}_s} \circ (\mathrm{id}_X \otimes \Delta_G) = \mathrm{id}_X \otimes (\overline{\mathrm{ev}_r \otimes \mathrm{ev}_s} \circ \Delta_G) = \mathrm{id}_X \otimes \mathrm{ev}_{rs}.$$

Also note that if $(\psi_i, \pi_i) : (X_i, A_i) \to (M(X_{i+1}), M(A_{i+1}))$ are nondegenerate correspondence homomorphism ($i = 1, 2$), then $\overline{\psi_2} \circ \overline{\psi_1} = \overline{\overline{\psi_2} \circ \psi_1}$. We thus have

$$\begin{aligned}
\overline{\mathrm{id}_X \otimes \mathrm{ev}_r \otimes \mathrm{ev}_s} \circ \overline{\sigma^\gamma \otimes \mathrm{id}_{C_0(G)}} \circ \sigma^\gamma &= \overline{\overline{\mathrm{id}_X \otimes \mathrm{ev}_r \otimes \mathrm{ev}_s} \circ (\sigma^\gamma \otimes \mathrm{id}_{C_0(G)})} \circ \sigma^\gamma \\
&= \overline{\overline{\mathrm{id}_X \otimes \mathrm{ev}_r} \circ \sigma^\gamma \circ (\mathrm{id}_X \otimes \mathrm{ev}_s)} \circ \sigma^\gamma \\
&= \overline{\mathrm{id}_X \otimes \mathrm{ev}_r} \circ \sigma^\gamma \circ \overline{\mathrm{id}_X \otimes \mathrm{ev}_s} \circ \sigma^\gamma \\
&= \gamma_r \circ \gamma_s = \gamma_{rs} \\
&= \overline{\mathrm{id}_X \otimes \mathrm{ev}_{rs}} \circ \sigma^\gamma \\
&= \overline{\overline{\mathrm{id}_X \otimes \mathrm{ev}_r \otimes \mathrm{ev}_s} \circ (\mathrm{id}_X \otimes \Delta_G)} \circ \sigma^\gamma \\
&= \overline{\mathrm{id}_X \otimes \mathrm{ev}_r \otimes \mathrm{ev}_s} \circ \overline{\mathrm{id}_X \otimes \Delta_G} \circ \sigma^\gamma.
\end{aligned}$$

This establishes the first part of the theorem.

To prove the remaining part, let (σ, δ) be a coaction of $C_0(G)$ on (X, A). Note that $\sigma(\xi) \in M_{C_0(G)}(X \otimes C_0(G)) = C_b(G, X)$. Hence by setting

$$\gamma_r^\sigma(\xi) := \sigma(\xi)(r), \quad \alpha_r^\delta(a) := \delta(a)(r) \quad (\xi \in X,\ a \in A)$$

for each $r \in G$, we have a map $(\gamma_r^\sigma, \alpha_r^\delta) : (X, A) \to (X, A)$. Since α_r^δ is injective and $(\gamma_r^\sigma, \alpha_r^\delta)$ is the composition $(\overline{\mathrm{id}_X \otimes \mathrm{ev}_r} \circ \sigma, \overline{\mathrm{id}_A \otimes \mathrm{ev}_r} \circ \delta)$ of two correspondence homomorphisms, $(\gamma_r^\sigma, \alpha_r^\delta)$ is an injective correspondence homomorphism. Reversing the order of the above computation leading to the coaction identity of σ^γ shows that $\gamma_r^\sigma \circ \gamma_s^\sigma = \gamma_{rs}^\sigma$ for $r, s \in G$, which also proves that γ_r^σ is surjective. Consequently, $(\gamma^\sigma, \alpha^\delta)$ is an action of G on (X, A). It is now obvious that (A.2) gives a one-to-one correspondence between actions and coactions. This completes the proof. $\square$

Appendix B

C^*-correspondences $(X \rtimes \widehat{S}_{\widehat{W_G}}, A \rtimes \widehat{S}_{\widehat{W_G}})$

It is well-known that $\mathcal{L}_A(A \otimes \mathcal{H}) = M(A \otimes \mathcal{K}(\mathcal{H}))$ for a C^*-algebra A and a Hilbert space $\mathcal{H}$. We generalize this in Proposition B.1.3 to a nondegenerate C^*-correspondence:

$$\mathcal{L}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H}) = M(X \otimes \mathcal{K}(\mathcal{H})).$$

Using this, we show in Corollary B.2.3 that the construction of Theorem 4.2.1 reduces to the crossed product correspondence $(X \rtimes_{\gamma,r} G, A \rtimes_{\alpha,r} G)$ in the sense of [13] when the coaction under consideration comes from an action (γ, α) of G on (X, A). Since we always want to use the left Haar measure, our reduced crossed product correspondence in this commutative case must be regarded as the one by a coaction of the Hopf C^*-algebra $S_{\widehat{W_G}}$ defined by the multiplicative unitary $\widehat{W_G}$.

B.1 C^*-correspondences $(\mathcal{L}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H}), \mathcal{L}_A(A \otimes \mathcal{H}))$

We first clarify the C^*-correspondence $(\mathcal{L}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H}), \mathcal{L}_A(A \otimes \mathcal{H}))$ in the next lemma whose proof is trivial, and so we omit it.

Lemma B.1.1. *Let (X, A) be a C^*-correspondence and $\mathcal{H}$ be a Hilbert space. Then $(\mathcal{L}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H}), \mathcal{L}_A(A \otimes \mathcal{H}))$ is a C^*-correspondence with respect to the following operations*

$$m \cdot l = m \circ l, \quad \langle m, n \rangle_{\mathcal{L}_A(A \otimes \mathcal{H})} = m^* \circ n, \quad \varphi_{\mathcal{L}_A(A \otimes \mathcal{H})} = \varphi_{M(A \otimes \mathcal{K}(\mathcal{H}))} \tag{B.1}$$

for $m, n \in \mathcal{L}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H})$ and $l \in \mathcal{L}_A(A \otimes \mathcal{H})$.

Note that $(\mathcal{K}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H}), \mathcal{K}_A(A \otimes \mathcal{H}))$ is also a C^*-correspondence with the restriction of the operations given in (B.1).

Lemma B.1.2. *There exists an isomorphism*

$$(\psi_0, \mathrm{id}) : (\mathcal{K}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H}), \mathcal{K}_A(A \otimes \mathcal{H})) \to (X \otimes \mathcal{K}(\mathcal{H}), A \otimes \mathcal{K}(\mathcal{H}))$$

such that $\psi_0(\theta_{\xi \otimes h, a \otimes k}) = \xi \cdot a^* \otimes \theta_{h,k}$ for $\xi \in X$, $a \in A$, and $h, k \in \mathcal{H}$.

Proof. Let $\xi_i \in X$, $a_i \in A$, and $h_i, k_i \in \mathcal{H}$ for $i = 1, \ldots, n$. We claim that the norm of the operator $\sum_{i=1}^n \theta_{\xi_i \otimes h_i, a_i \otimes k_i}$ agrees with that of $\sum_{i=1}^n \xi_i \cdot a_i^* \otimes \theta_{h_i, k_i}$, which will proves that ψ_0 is well-defined and isometric. For this, we may assume that the vectors h_i are mutually orthonormal and similarly for k_i. Then

$$\begin{aligned}
\Big\| \sum_{i=1}^n \theta_{\xi_i \otimes h_i, a_i \otimes k_i} \Big\|^2 &= \Big\| \sum_{i,j=1}^n \theta_{a_i \otimes k_i, \xi_i \otimes h_i} \theta_{\xi_j \otimes h_j, a_j \otimes k_j} \Big\| \\
&= \Big\| \sum_{i,j=1}^n \theta_{(a_i \otimes k_i) \cdot \langle \xi_i \otimes h_i, \xi_j \otimes h_j \rangle_A, a_j \otimes k_j} \Big\| \\
&= \Big\| \sum_{i=1}^n \theta_{a_i \langle \xi_i, \xi_i \rangle_A \otimes k_i, a_i \otimes k_i} \Big\|.
\end{aligned}$$

By [19, Lemma 2.1], the last of the above equalities coincides with the norm of the following product of two positive $n \times n$ matrices

$$\Big(\big\langle a_i\langle \xi_i, \xi_i\rangle_A \otimes k_i,\, a_j\langle \xi_j, \xi_j\rangle_A \otimes k_j\big\rangle_A\Big)^{1/2}\Big(\langle a_i \otimes k_i, a_j \otimes k_j\rangle_A\Big)^{1/2}$$

which is diagonal by orthogonality. Let

$$b_i = \langle \xi_i \cdot a_i^*,\, \xi_i\rangle_A \quad (i = 1, \dots, n).$$

Then

$$\begin{aligned}\Big\|\sum_{i=1}^n \theta_{\xi_i\otimes h_i,\, a_i\otimes k_i}\Big\|^2 &= \max_{i=1,\dots,n} \Big\|\big(\langle \xi_i \cdot a_i^*,\, \xi_i\rangle_A^* \langle \xi_i \cdot a_i^*,\, \xi_i\rangle_A\big)^{1/2}(a_i^* a_i)^{1/2}\Big\| \\ &= \max_{i=1,\dots,n} \|(b_i^* b_i)^{1/2}(a_i^* a_i)^{1/2}\|.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Big\|\sum_{i=1}^n \xi_i \cdot a_i^* \otimes \theta_{h_i,k_i}\Big\|^2 &= \Big\|\sum_{i,j=1}^n \big\langle \xi_i \cdot a_i^* \otimes \theta_{h_i,k_i},\, \xi_j \cdot a_j^* \otimes \theta_{h_j,k_j}\big\rangle_{A\otimes\mathcal{K}(\mathcal{H})}\Big\| \\ &= \Big\|\sum_{i,j=1}^n \langle \xi_i \cdot a_i^*,\, \xi_j \cdot a_j^*\rangle_A \otimes \theta_{k_i\langle h_i,h_j\rangle,\, k_j}\Big\| \\ &= \max_{i=1,\dots,n} \|\langle \xi_i \cdot a_i^*,\, \xi_i\rangle_A\, a_i^*\| = \max_{i=1,\dots,n} \|b_i a_i^*\|\end{aligned}$$

again by orthonormality. Our claim then follows since

$$\begin{aligned}\|(b_i^* b_i)^{1/2}(a_i^* a_i)^{1/2}\|^2 &= \|(a_i^* a_i)^{1/2} b_i^* b_i (a_i^* a_i)^{1/2}\| \\ &= \|b_i (a_i^* a_i)^{1/2}\|^2 = \|b_i a_i^* a_i b_i^*\| = \|b_i a_i^*\|^2.\end{aligned}$$

What is left is now to show that (ψ_0, id) is a correspondence homomorphism. Let $a, b, b' \in A$, $h, k, k' \in \mathcal{H}$, $\xi, \xi' \in X$, and $T \in \mathcal{K}(\mathcal{H})$. Then,

$$\begin{aligned}
\psi_0\big(\varphi_{\mathcal{K}_A(A\otimes\mathcal{H})}(a \otimes T)\,\theta_{\xi\otimes h,\, b\otimes k}\big) &= \psi_0\big(\theta_{\varphi_A(a)\xi\otimes Th,\, b\otimes h}\big) \\
&= \varphi_A(a)\xi \cdot b^* \otimes \theta_{Th,k} \\
&= \varphi_{A\otimes\mathcal{K}(\mathcal{H})}(a \otimes T)(\xi \cdot b^* \otimes \theta_{h,k}) \\
&= \varphi_{A\otimes\mathcal{K}(\mathcal{H})}(a \otimes T)\,\psi_0(\theta_{\xi\otimes h,\, b\otimes k})
\end{aligned}$$

and

$$\begin{aligned}
\big\langle \psi_0(\theta_{\xi\otimes h,\, b\otimes k}),\, \psi_0(\theta_{\xi'\otimes h',\, b'\otimes k'})\big\rangle_{A\otimes\mathcal{K}(\mathcal{H})} &= \big\langle \xi \cdot b^* \otimes \theta_{h,k},\, \xi' \cdot b'^* \otimes \theta_{h',k'}\big\rangle_{A\otimes\mathcal{K}(\mathcal{H})} \\
&= \langle \xi \cdot b^*,\, \xi' \cdot b'^*\rangle_A \otimes \langle \theta_{h,k}, \theta_{h',k'}\rangle_{\mathcal{K}(\mathcal{H})} \\
&= \theta_{b\langle\xi,\xi'\rangle_A,\, b'} \otimes \theta_{k\langle h,h'\rangle,\, k'} \\
&= \theta_{b\langle\xi,\xi'\rangle_A\otimes k\langle h,h'\rangle,\, b'\otimes k'} \\
&= \theta_{b\otimes k,\, \xi\otimes h}\,\theta_{\xi'\otimes h',\, b'\otimes k'} \\
&= \big\langle \theta_{\xi\otimes h,\, b\otimes k},\, \theta_{\xi'\otimes h',\, b'\otimes k'}\big\rangle_{\mathcal{K}_A(A\otimes\mathcal{H})}.
\end{aligned}$$

This proves that (ψ_0, id) is a correspondence homomorphism. □

In the next proposition, we identify $\mathcal{K}_A(A\otimes\mathcal{H}, X\otimes\mathcal{H}) = X\otimes\mathcal{K}(\mathcal{H})$. Note that for $m \in \mathcal{L}_A(A\otimes\mathcal{H}, X\otimes\mathcal{H})$ and $f \in A\otimes\mathcal{K}(\mathcal{H}) = \mathcal{K}_A(A\otimes\mathcal{H})$, the right action $m \cdot f$ defines an element of $X\otimes\mathcal{K}(\mathcal{H})$.

Proposition B.1.3. *There exists an isomorphism*

$$(\psi, \mathrm{id}) : (\mathcal{L}_A(A\otimes\mathcal{H}, X\otimes\mathcal{H}), \mathcal{L}_A(A\otimes\mathcal{H})) \to (M(X\otimes\mathcal{K}(\mathcal{H})), M(A\otimes\mathcal{K}(\mathcal{H})))$$

such that

$$\psi(m) \cdot f = m \cdot f$$

for $m \in \mathcal{L}_A(A\otimes\mathcal{H}, X\otimes\mathcal{H})$ and $f \in A\otimes\mathcal{K}(\mathcal{H})$.

Proof. With respect to the operations on (B.1), the following can be easily seen to hold:

(i) $\mathcal{K}_A(A\otimes\mathcal{H}, X\otimes\mathcal{H}) \cdot \mathcal{L}_A(A\otimes\mathcal{H}) = \mathcal{K}_A(A\otimes\mathcal{H}, X\otimes\mathcal{H})$;

(ii) $\varphi_{\mathcal{L}_A(A\otimes\mathcal{H})}\big(\mathcal{L}_A(A\otimes\mathcal{H})\big)\,\mathcal{K}_A(A\otimes\mathcal{H}, X\otimes\mathcal{H}) = \mathcal{K}_A(A\otimes\mathcal{H}, X\otimes\mathcal{H})$;

(iii) $\mathcal{L}_A(A\otimes\mathcal{H}, X\otimes\mathcal{H})\cdot\mathcal{K}_A(A\otimes\mathcal{H}) = \mathcal{K}_A(A\otimes\mathcal{H}, X\otimes\mathcal{H})$.

Thus (ψ, id) is an injective correspondence homomorphism by [13, Proposition 1.28]. To see that ψ is surjective, let $n \in M(X\otimes\mathcal{K}(\mathcal{H}))$. Take a net $\{x_i\}$ in $X\otimes\mathcal{K}(\mathcal{H})$ strictly converging to n. Then the limits $\lim_i x_i h$ and $\lim_i x_i^* k$ clearly exist for $h \in A\otimes\mathcal{H}$ and $k \in X\otimes\mathcal{H}$. Define $m_n : A\otimes\mathcal{H} \to X\otimes\mathcal{H}$ and $m_n^* : X\otimes\mathcal{H} \to A\otimes\mathcal{H}$ by

$$m_n h = \lim_i x_i h, \quad m_n^* k = \lim_i x_i^* k.$$

We see from

$$\langle m_n h, k\rangle_A = \lim_i \langle x_i h, k\rangle_A = \lim_i \langle h, x_i^* k\rangle_A = \langle h, m_n^* k\rangle_A$$

that $m_n \in \mathcal{L}_A(A\otimes\mathcal{H}, X\otimes\mathcal{H})$ with the adjoint m_n^*. It is now obvious that $\psi(m_n) = n$, which completes the proof. $\square$

From now on, we identify $\mathcal{L}_A(A\otimes\mathcal{H}, X\otimes\mathcal{H}) = M(X\otimes\mathcal{K}(\mathcal{H}))$.

Remark B.1.4. Let $\mu_G : C_0(G) \hookrightarrow \mathcal{L}(L^2(G))$ be the embedding in (2.5). The strict extension $\overline{\mathrm{id}_X\otimes\mu_G}$ then embeds $M(X\otimes C_0(G))$ into $M(X\otimes\mathcal{K}(L^2(G)))$ such that if $m \in C_b(G, M(X)_s)$ and $h \in C_c(G, A) \subseteq A\otimes L^2(G)$, then

$$(\overline{\mathrm{id}\otimes\mu_G}(m)h)(r) = m(r)\cdot h(r) \quad (r \in G)$$

by strict continuity.

B.2 Crossed product correspondences $(X \rtimes_r G, A \rtimes_r G)$

Let (γ, α) be an action of a locally compact group G on (X, A). The *crossed product correspondence* $(X \rtimes_{\gamma,r} G, A \rtimes_{\alpha,r} G)$ is the completion of the $C_c(G, A)$-

bimodule $C_c(G, X)$ such that

$$(x \cdot f)(r) = \int_G x(s) \cdot \alpha_s(f(s^{-1}r))\, ds,$$
$$\langle x, y\rangle_{A \rtimes_{\alpha,r} G}(r) = \int_G \alpha_s^{-1}(\langle x(s), y(sr)\rangle_A)\, ds,$$
$$\big(\varphi_{A \rtimes_{\alpha,r} G}(f)\, x\big)(r) = \int_G \varphi_A(f(s))\, \gamma_s(x(s^{-1}r))\, ds$$

for $x, y \in C_c(G, X)$, $f \in C_c(G, A)$, and $r \in G$ ([13, Proposition 3.2]).

Remark B.2.1. The algebraic tensor product $X \odot C_c(G)$ is dense in $X \rtimes_{\gamma,r} G$. This is because $X \odot C_c(G)$ is L^1-norm dense in $C_c(G, X)$ and the crossed product norm on $C_c(G, A)$ is dominated by its L^1-norm.

Theorem B.2.2. *Let (γ, α) be an action of a locally compact group G on a C^*-correspondence (X, A). Then, there exists an injective correspondence homomorphism*

$$(\psi_\gamma, \pi_\alpha) : (X \rtimes_{\gamma,r} G, A \rtimes_{\alpha,r} G) \to (\mathcal{L}_A(A \otimes L^2(G), X \otimes L^2(G)), \mathcal{L}_A(A \otimes L^2(G)))$$

such that

$$\big(\psi_\gamma(x)h\big)(r) = \int_G \gamma_r^{-1}(x(s)) \cdot h(s^{-1}r)\, ds,$$
$$\big(\pi_\alpha(f)h\big)(r) = \int_G \alpha_r^{-1}(f(s))\, h(s^{-1}r)\, ds$$

for $x \in C_c(G, X)$, $f \in C_c(G, A)$, $h \in C_c(G, A)$, and $r \in G$.

Proof. It is well-known that π_α gives a nondegenerate embedding.

For each $x \in C_c(G, X) \subseteq X \rtimes_{\gamma,r} G$, define $\rho_\gamma(x) : C_c(G, X) \to C_c(G, A)$ by

$$(\rho_\gamma(x)k)(r) = \int_G \Delta(r^{-1})\langle \gamma_s^{-1}(x(sr^{-1})), k(s)\rangle_A\, ds$$

for $k \in C_c(G, X)$ and $r \in G$, where Δ is the modular function of G. We claim that

$$\langle \psi_\gamma(x)h, k\rangle_A = \langle h, \rho_\gamma(x)k\rangle_A, \quad \rho_\gamma(x)\psi_\gamma(y) = \pi_\alpha(\langle x, y\rangle_{A\rtimes_{\alpha,r}G}) \tag{B.2}$$

for $h \in C_c(G, A) \subseteq A \otimes L^2(G)$, $k \in C_c(G, X) \subseteq X \otimes L^2(G)$, and $x, y \in C_c(G, X)$. Indeed, we have

$$\begin{aligned}\langle \psi_\gamma(x)h, k\rangle_A &= \int_G \langle (\psi_\gamma(x)h)(r), k(r)\rangle_A \, dr \\ &= \int_G\int_G \langle \gamma_r^{-1}(x(s)) \cdot h(s^{-1}r), k(r)\rangle_A \, ds\, dr \\ &= \int_G\int_G h(s^{-1}r)^*\langle \gamma_r^{-1}(x(s)), k(r)\rangle_A \, ds\, dr\end{aligned}$$

by definition. Replacing s by rs and then s by s^{-1} we get

$$\langle \psi_\gamma(x)h, k\rangle_A = \int_G\int_G h(s)^*\Delta(s^{-1})\langle \gamma_r^{-1}(x(rs^{-1})), k(r)\rangle_A \, dr\, ds = \langle h, \rho_\gamma(x)k\rangle_A,$$

which verifies the first equality in (B.2). Also,

$$\begin{aligned}\big(\rho_\gamma(x)\psi_\gamma(y)h\big)(r) &= \int_G \Delta(r^{-1})\langle \gamma_s^{-1}(x(sr^{-1})), (\psi_\gamma(y)h)(s)\rangle_A \, ds \\ &= \int_G\int_G \Delta(r^{-1})\big\langle \gamma_s^{-1}(x(sr^{-1})), \gamma_s^{-1}(y(t)) \cdot h(t^{-1}s)\big\rangle_A \, dt\, ds \\ &= \int_G\int_G \big\langle \gamma_r^{-1}\gamma_s^{-1}(x(s)), \gamma_r^{-1}\gamma_s^{-1}\big\rangle_A h(t^{-1}sr)\, dt\, ds \\ &= \int_G\int_G \alpha_r^{-1}\alpha_s^{-1}\big(\langle x(s), y(st)\rangle_A\big)h(t^{-1}r)\, ds\, dt,\end{aligned}$$

the last of which is by definition equal to

$$\int_G \alpha_r^{-1}\big(\langle x, y\rangle_{A\rtimes_{\alpha,r}G}(t)\big)h(t^{-1}r)\, dt = \big(\pi_\alpha(\langle x, y\rangle_{A\rtimes_{\alpha,r}G})\, h\big)(r)$$

Thus we get the second equality in (B.2), and then the claim follows.

Now we can see from (B.2) that ψ_γ and ρ_γ both extend continuously to all of $X \rtimes_{\gamma,r} G$ and $\psi_\gamma(x) \in \mathcal{L}_A(A \otimes L^2(G), X \otimes L^2(G))$ for $x \in X \rtimes_{\gamma,r} G$ with the adjoint $\psi_\gamma(x)^* = \rho_\gamma(x)$.

The second relation in (B.2) also gives one of the condition that $(\psi_\gamma, \pi_\alpha)$ is a correspondence homomorphism, namely

$$\langle \psi_\gamma(x), \psi_\gamma(y) \rangle_{\mathcal{L}_A(A \otimes L^2(G))} = \pi_\alpha(\langle x, y \rangle_{A \rtimes_{\alpha,r} G})$$

for $x, y \in X \rtimes_{\gamma,r} G$. To see that the other one is also satisfied, it suffices to show that

$$\psi_\gamma\big(\varphi_{A \rtimes_{\alpha,r} G}(a \otimes \phi)\, x\big) = \varphi_{\mathcal{L}_A(A \otimes L^2(G))}\big(\pi_\alpha(a \otimes \phi)\big)\psi_\gamma(x)$$

for $a \in A$, $\phi \in C_c(G) \subseteq C_r^*(G)$, and $x \in C_c(G, X) \subseteq X \rtimes_{\gamma,r} G$, or

$$\big(\psi_\gamma\big(\varphi_{A \rtimes_{\alpha,r} G}(a \otimes \phi)x\big)h\big)(r) = \big(\varphi_{\mathcal{L}_A(A \otimes L^2(G))}\big(\pi_\alpha(a \otimes \phi)\big)\psi_\gamma(x)h\big)(r)$$

for $h \in C_c(G, A) \subseteq A \otimes L^2(G)$ and $r \in G$. For this, let us first note the following. Since the strict extension $\overline{\pi_\alpha}$ embeds A into $\mathcal{L}_A(A \otimes L^2(G))$ such that $\big(\overline{\pi_\alpha}(a)h\big)(r) = \alpha_r^{-1}(a)h(r)$, we can deduce that

$$\big(\varphi_{\mathcal{L}_A(A \otimes L^2(G))}(\overline{\pi_\alpha}(a))\, k\big)(r) = \varphi_A(\alpha_r^{-1}(a))k(r) \tag{B.3}$$

for $k \in C_c(G, X)$ and $r \in G$. Similarly,

$$\big(\varphi_{\mathcal{L}_A(A \otimes L^2(G))}\big(\overline{\pi_\alpha}(\phi)\big)k\big)(r) = \int_G \phi(s)k(s^{-1}r)\, ds. \tag{B.4}$$

We now have

$$
\begin{aligned}
&\big(\psi_\gamma\big(\varphi_{A\rtimes_{\alpha,r}G}(a\otimes\phi)x\big)h\big)(r)\\
&\qquad\qquad = \int_G \gamma_r^{-1}\big((\varphi_{A\rtimes_{\alpha,r}}(a\otimes\phi)x)(s)\big)\cdot h(s^{-1}r)\,ds\\
&\qquad\qquad = \int_G\int_G \gamma_r^{-1}\big(\varphi_A(a\phi(t))\gamma_t(x(t^{-1}s))\big)\cdot h(s^{-1}r)\,dt\,ds\\
&\qquad\qquad = \int_G\int_G \varphi_A(\alpha_r^{-1}(a))\phi(t)\,\gamma_r^{-1}\gamma_t(x(s))\cdot h(s^{-1}t^{-1}r)\,ds\,dt\\
&\qquad\qquad = \varphi_A(\alpha_r^{-1}(a))\int_G \phi(t)\,(\psi_\gamma(x)h)(t^{-1}r)\,dt\\
&\qquad\qquad = \Big(\varphi_{\mathcal{L}_A(A\otimes L^2(G))}(\overline{\pi_\alpha}(a))\big(\varphi_{\mathcal{L}_A(A\otimes L^2(G))}(\overline{\pi_\alpha}(\phi))\psi_\gamma(x)h\big)\Big)(r)\\
&\qquad\qquad = \big(\varphi_{\mathcal{L}_A(A\otimes L^2(G))}(\pi_\alpha(a\otimes\phi))\psi_\gamma(x)h\big)(r),
\end{aligned}
$$

in the fifth step of which we use (B.3) and (B.4). Since π_α is injective, $(\psi_\gamma, \pi_\alpha)$ is therefore an injective correspondence homomorphism. $\square$

Let (γ, α) be an action of G on (X, A), and $(\sigma^\gamma, \delta^\alpha)$ be the corresponding coaction. Define

$$\sigma_G^\gamma = \overline{\mathrm{id}_X \otimes \check{\mu}_G} \circ \sigma^\gamma, \quad \delta_G^\alpha = \overline{\mathrm{id}_X \otimes \check{\mu}_G} \circ \delta^\alpha, \tag{B.5}$$

where $\check{\mu}_G : C_0(G) \to S_{\widehat{W}_G}$ is the Hopf C^*-algebra isomorphism given in (2.5). Then $(\sigma_G^\gamma, \delta_G^\alpha)$ is a coaction of $S_{\widehat{W}_G}$ on (X, A). In the next corollary, we regard $\sigma_{G\iota}^\gamma(X) = \overline{\mathrm{id}_X \otimes \iota_{S_{\widehat{W}_G}}}(\sigma_G^\gamma(X))$ as a subspace of $\mathcal{L}_A(A \otimes L^2(G), X \otimes L^2(G))$.

Corollary B.2.3. *Let (γ, α) be an action of a locally compact group G on (X, A). Then $(\psi_\gamma, \pi_\alpha)$ in Theorem B.2.2 gives an isomorphism from $(X \rtimes_{\gamma,r} G, A \rtimes_{\alpha,r} G)$ onto $(X \rtimes_{\sigma_G^\gamma} \widehat{S}_{\widehat{W}_G}, A \rtimes_{\delta_G^\alpha} \widehat{S}_{\widehat{W}_G})$ such that*

$$\psi_\gamma(\xi \otimes \phi) = \sigma_{G\iota}^\gamma(\xi) \cdot (1_{M(A)} \otimes \phi), \quad \pi_\alpha(a \otimes \phi) = \delta_{G\iota}^\alpha(a)(1_{M(A)} \otimes \phi) \tag{B.6}$$

for $\xi \in X$, $a \in A$, and $\phi \in C_c(G)$.

Proof. We only need to prove that ψ_γ satisfies the first equality in (B.6) and gives a surjection onto $X \rtimes_{\sigma_G^\gamma} \widehat{S}_{\widehat{W}_G}$. Let $\xi \in X$ and $\phi \in C_c(G)$. We see from Remark B.1.4 that

$$(\sigma^\gamma_{G\iota}(\xi)\, h)(r) = \gamma_r^{-1}(\xi) \cdot h(r)$$

for $h \in C_c(G, A)$ and $r \in G$. Hence

$$\big(\psi_\gamma(\xi \otimes \phi)h\big)(r) = \gamma_r^{-1}(\xi) \cdot \int_G \phi(s)h(s^{-1}r)\, ds = \big(\sigma^\gamma_{G\iota}(\xi)\big((1_{M(A)} \otimes \phi)h\big)\big)(r),$$

which shows the first equality in (B.6). Since $X \odot C_c(G)$ is dense in $X \rtimes_{\gamma,r} G$ by Remark B.2.1 and ψ_γ is isometric, we must have $\psi_\gamma(X \rtimes_{\gamma,r} G) = X \rtimes_{\sigma_G^\gamma} \widehat{S}_{\widehat{W}_G}$. □

We now provide a proof of Corollary 5.2.5.

Proof of Corollary 5.2.5. Let

$$\zeta_G = \overline{\mathrm{id}_{\mathcal{O}_X} \otimes \check{\mu}_G} \circ \zeta.$$

Clearly, ζ_G is the coaction of $S_{\widehat{W}_G}$ on $\mathcal{O}_X$ induced by $(\sigma_G^\gamma, \delta_G^\alpha)$. Define a representation

$$(k_X \rtimes_\gamma G, k_A \rtimes_\gamma G) : (X \rtimes_{\gamma,r} G, A \rtimes_{\alpha,r} G) \to \mathcal{O}_X \rtimes_{\beta^\zeta,r} G$$

to be the composition as indicated in the following diagram:

$$\begin{tikzcd}
(X \rtimes_{\gamma,r} G, A \rtimes_{\alpha,r} G) \arrow[r, "(\psi_\gamma,\pi_\alpha)"] \arrow[d, dashed, "(k_X \rtimes_\gamma G, k_A \rtimes_\alpha G)"'] & (X \rtimes_{\sigma_G^\gamma} \widehat{S}_{\widehat{W}_G}, A \rtimes_{\delta_G^\alpha} \widehat{S}_{\widehat{W}_G}) \arrow[d, "(k_X \rtimes \mathrm{id}_{\widehat{S}_{\widehat{W}_G}}, k_A \rtimes \mathrm{id}_{\widehat{S}_{\widehat{W}_G}})"] \\
\mathcal{O}_X \rtimes_{\beta^\zeta,r} G \arrow[r, equal] & \mathcal{O}_X \rtimes_{\zeta_G} \widehat{S}_{\widehat{W}_G}.
\end{tikzcd}$$

By definition ((5.3) and (B.6)), we have

$$(k_X \rtimes_\gamma G)(f)(r) = k_X(f(r)), \quad (k_A \rtimes_\alpha G)(g)(r) = k_A(g(r))$$

for $f \in C_c(G, X)$, $g \in C_c(G, A)$, and $r \in G$. The conclusion then follows by Theorem 5.2.4. $\square$

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국문초록

본 연구에서는 C^*-대응 상에서 정의되는 군 작용 및 군 쌍대작용을 하나로 통합하여 C^*-대응 (X, A) 상에서 호프 C^*-대수 S 의 쌍대작용을 정의하고, 이로부터 쿤쯔-핌스너 대수 $\mathcal{O}_X$ 상에서의 S 의 쌍대작용이 유도됨을 보인다. 또한, S 가 곱 유니터리로 부터 정의되는 축소 호프 C^*-대수일 경우, C^*-대응 $(X \rtimes \widehat{S}, A \rtimes \widehat{S})$ 을 구성하고 이 C^*-대응이 축소 교차곱 $\mathcal{O}_X \rtimes \widehat{S}$ 상의 표현을 가짐을 보인다. 이 표현이 공변이라는 가정하에 C^*-대수 $\mathcal{O}_X \rtimes \widehat{S}$ 는 쿤쯔-핌스너 대수 $\mathcal{O}_{X \rtimes \widehat{S}}$ 와 동형임을 증명하고, 특히 $A \rtimes \widehat{S}$ 의 아이디얼 $J_{X \rtimes \widehat{S}}$ 가 $M(A \rtimes \widehat{S})$ 안에서의 J_X 의 상에 의해 생성되거나 혹은 A 의 왼쪽 작용 φ_A 가 단사라면 공변인 가정이 충족됨을 또한 증명한다. 본 연구결과를 하오와 엥의 군 작용 및 칼리스제우스키, 퀵, 그리고 로버트슨의 군 쌍대작용 연구에 적용한다면 기존의 고전적 결과를 크게 향상시킬 수 있다. 즉, 군 작용의 경우 공변의 가정이 충족된다면 평균가능 군에서 성립하는 동형의 결과를 임의의 국소콤펙트 군으로 확장할 수 있다. 또한, 군 쌍대작용에 관한 결과의 핵심 가정인 쿤쯔-핌스너 공변조건이 실은 필요없는 가정이었음을 증명한다.

주요어휘: C^*-대응, 쿤쯔-핌스너 대수, 곱 대응, 호프 C^*-대수, 쌍대작용, 축소 교차곱
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