이학박사 학위논문

Symplectic geometry of orbifolds and Diophantine equations

(오비다양체의 사교기하와 디오판투스 방정식)

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Symplectic geometry of orbifolds and Diophantine equations

A dissertation
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by

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Abstract

Symplectic geometry of orbifolds and Diophantine equations

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We study symplectic geometry of orbifolds, especially which are necessary to extend the Lagrangian intersection Floer theory to the one of orbifold setting. First, we give another definition of the Maslov indices of bundle pairs via curvature integral of L-orthogonal unitary connection. This definition naturally extends to the one of orbi-bundle pairs with interior singularities. Secondly, we investigate the notion of orbifold embedding. When the target orbifold is a global quotient of a smooth manifold by the action of a Lie group G, we show that orbifold embeddings are equivariant with G-equivariant immersions.

In the last part of the dissertation, we compute quantum cohomology of elliptic $\mathbb{P}^1$ orbifolds via classifying holomorphic orbi-spheres in those orbifolds. Interestingly, we find that these orbi-spheres have an one-to-one correspondence with the solutions of certain Diophantine equations depending on the lattice structures on the universal covers of elliptic $\mathbb{P}^1$ orbifolds constructed from the preimages of three singular points.

Key words: orbifold, symplectic geometry, Maslov index, orbifold embedding, orbifold Quantum cohomology, Diophantine equation
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Chapter 1

Introduction

Originated from the phase space in the Hamiltonian mechanics, symplectic geometry has developed through interactions with many branches of physics. Especially, string theory has stimulated a lot of discoveries of topological invariants and their interesting relations. There are two types of strings considered in the string theory – *closed* string and *open* string (with boundary on D-branes). In very rough terms, quantum cohomology ring (or more generally Gromov-Witten invariants) is related with closed strings in the string theory. The counterpart of open strings is the Lagrangian intersection Floer cohomology, which is introduced by A. Floer and fully developed by Fukaya, Oh, Ohta, and Ono([FOOO]).

Inspired by the role of orbifolds in the string theory([DHVW]), Chen and Ruan ([CR1], [CR2]) introduced de Rham and Hodge cohomologies and Gromov-Witten invariants in the orbifold setting. Following their work, "closed theory" in the orbifold setting has been developed radically, for example, orbifold K-theory, Fan-Jarvis-Ruan-Witten theory, etc. On the other hand, "open theory" in the orbifold setting is poorly explored yet. There has been some partial results on the development of Lagrangian intersection Floer cohomology in the orbifold setting by Cho-Hong[CH], Cho-Poddar[CP], and Woodward[W].

In this thesis, placing major emphasis on developing the Lagrangian intersection Floer cohomology in the orbifold setting, we investigate the geometry of orbifold. A detailed introduction and description of our main results are now in order.

Chapter 3. Maslov index via Chern-Weil theory and its orbifold analogues. In Lagrangian intersection Floer cohomology, Maslov index is related to the (relative) grading of the cohomology. Here, we only concern the Maslov index associated with a bundle pair $(E, L) \to (\Sigma, \partial\Sigma)$, where $E \to \Sigma$ is a symplectic vector bundle equipped with a Lagrangian subbundle data L over the boundary of the bordered Riemann surface Σ. To define the Maslov index in this setting, the bundle should be trivialized. However, "orbi-bundle" can not be trivialized in general, and hence the usual definition does not extend to the one in orbifold setting. To remedy this, we give another definition of Maslov index via curvature integral of *L-orthogonal unitary connection*.

Definition 1.0.1. *Let ∇ be a connection on E which restricts, on the boundary of Σ, to an L-orthogonal unitary connection on $(E|_{\partial\Sigma}, J)$. The Chern-Weil Maslov index of the bundle pair (E, L) is defined by*

$$\mu_{CW}(E, L) = \frac{\sqrt{-1}}{\pi} \int_{\Sigma} tr(F_{\nabla})$$

where $F_{\nabla} \in \Omega^2(\Sigma, End(E))$ is the curvature induced by ∇.

The key feature of L-orthogonal unitary connections is that parallel transformation preserves the Lagrangian bundle L. This new definition is equivalent to the usual Maslov index :

Theorem 1.0.1. *Given a bundle pair (E, L), topological Maslov index equals Chern-Weil Maslov index.*

Moreover, this Maslov index easily extends to the one of orbifold setting when the Lagrangian subbundle does not contain any orbi-singular points.

Chapter 4. On orbifold embeddings. In Lagrangian Floer theory, the most important subobjects are Lagrangian submanifolds. Similar to the smooth setting, we need to study the geometry of Lagrangian "suborbifolds". However, there is a problem in defining the notion of "suborbifold" as follows.

The notion of orbifold can be defined using groupoid language (see Section 2.2.2) and there is already well-defined notion of subobjects in the groupoid category. However, this notion is too restrictive to include many interesting objects as suborbifolds. For example, for a given groupoid $\mathcal{G}$, the diagonal $\mathcal{G} \times_{\mathcal{G}} \mathcal{G}$ is not a subgroupoid of $\mathcal{G} \times \mathcal{G}$.

In [ALR], Adem, Leida, and Ruan defined the notion of *orbifold embedding* and extended the category of suborbifolds, which is more flexible and includes many interesting objects as a suborbifold (for example, the diagonal groupoid $\Delta : \mathcal{G} \times_{\mathcal{G}} \mathcal{G} \to \mathcal{G} \times \mathcal{G}$). However, its complicated local description makes one hard to study the geometry of suborbifolds.

To remedy this, we study the relation between the notion of orbifold embedding and equivariant immersion :

Theorem 1.0.2. *Let $\phi : \mathcal{H} \to \mathcal{G}$ be an orbifold embedding, where $\mathcal{G}$ is Morita equivalent to a translation groupoid $[M/G]$. Then, there exist a manifold N on which the Lie group G acts locally freely such that*

(i) $\mathcal{H} \simeq [N/G]$ *and* $\mathcal{G} \simeq [M/G]$

(ii) there exists a G-equivariant immersion $\iota : N \to M$ which makes the diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\ \phi\ } & \mathcal{G} \\ {\scriptstyle Morita}\updownarrow & & \updownarrow{\scriptstyle Morita} \\ [N/G] & \xrightarrow[\ \iota\ ]{} & [M/G] \end{array}$$

commute.

To prove this, we use the notion of *Hilsum-Skandalis map* (See Section 2.2.3).

Chapter 5. Holomorphic orbi-spheres in elliptic $\mathbb{P}^1$ orbifolds and Diophantine equations. Elliptic $\mathbb{P}^1$ orbifolds are $\mathbb{P}^1$ orbifolds admitting an elliptic curve as an orbifold covering space (with holomorphic covering map). There are four such elliptic $\mathbb{P}^1$ orbifolds: $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{2,3,6}$, $\mathbb{P}^1_{2,4,4}$, and $\mathbb{P}^1_{2,2,2,2}$, where subindices indicate the orders of singularities at orbifold points. In this chapter, we compute (small) quantum cohomology rings of elliptic projective lines by counting "orbi-spheres".

In fact, the Gromov-Witten potentials of elliptic $\mathbb{P}^1$ orbifolds were already computed : Satake-Takahashi [ST] computed the full genus-0 Gromov-Witten potential for $\mathbb{P}^1_{3,3,3}$ and $\mathbb{P}^1_{2,2,2,2}$. Furthermore, Krawitz-Shen [KS] independently computed the potentials for $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{2,3,6}$, and $\mathbb{P}^1_{2,4,4}$ for all genera. Their arguments are based on the algebraic structure of Gromov-Witten theory, for example, WDVV equation, string equation, etc. Our purpose, on the other hand, is to reproduce the (small) quantum product terms of the potentials by directly counting holomorphic orbi-spheres. For the counting, we first need to classify all holomorphic orbi-spheres(=stable orbifold morphism from a Riemann orbi-sphere to $\mathbb{P}^1_{a,b,c}$) contributing to nontrivial quantum product terms.

The kew ingredient of our argument for the classification is *orbifold covering theory*. Roughly, we show that "almost all" holomorphic orbi-spheres contributing to quantum cohomology rings can be lifted to linear maps on $\mathbb{C}$. Here, $\mathbb{C}$ is the universal orbifold covering space of the appropriate elliptic $\mathbb{P}^1$ orbifolds. Using the lattice structure in $\mathbb{C}$ induced from the orbi-singular points, we find an interesting relation between holomorphic orbi-spheres and Diophantine equations. Main results are as follows.

First, the classification of holomorphic orbi-spheres for the quantum product on $\mathbb{P}^1_{3,3,3}$ is related to the Diophantine equation $Q_F(a,b) := a^2 - ab + b^2 = d$.

Theorem 1.0.3. *For $\mathbb{P}^1_{3,3,3}$, the only nontrivial 3-fold Gromov-Witten invariants are*

$$\langle \Delta_1^{1/3}, \Delta_2^{1/3}, \Delta_3^{1/3} \rangle_{0,3}^{\mathbb{P}^1_{3,3,3}} \tag{1.0.1}$$

$$\langle \Delta_i^{1/3}, \Delta_i^{1/3}, \Delta_i^{1/3} \rangle_{0,3}^{\mathbb{P}^1_{3,3,3}} \tag{1.0.2}$$

for $i = 1,2,3$, where subindices of Δ indicate the singular points z_i. If one denotes the (compactified) moduli space of degree-d holomorphic orbi-spheres contributing to (1.0.1) *by $\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{3,3,3}; \Delta_1^{1/3}, \Delta_1^{1/3}, \Delta_1^{1/3})$ and that for* (1.0.2) *by $\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{3,3,3}; \Delta_i^{1/3}, \Delta_i^{1/3}, \Delta_i^{1/3})$, then*

$$\#\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{3,3,3}; \Delta_1^{1/3}, \Delta_2^{1/3}, \Delta_3^{1/3}) = \frac{1}{6}\#\{(a,b) : Q_F(a,b) = d,\ d \equiv 1 \pmod 3\}$$

$$\#\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{3,3,3}; \Delta_1^{1/3}, \Delta_1^{1/3}, \Delta_1^{1/3}) = \frac{1}{3}\#\{(a,b) : Q_F(a,b) = d,\ d \equiv 0 \pmod 3\}$$

Similarly, solutions the Diophantine equation $Q_G(a,b) := a^2 + b^2 = d$ are assigned to holomorphic orbi-spheres in $\mathbb{P}^1_{2,4,4}$, and the rest of argument is parallel to that for $\mathbb{P}^1_{3,3,3}$.

Theorem 1.0.4. *For $\mathbb{P}^1_{2,4,4}$, the nontrivial contributions to 3-fold Gromov-Witten invariants come only from the moduli spaces*

$$\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{2,4,4};\Delta_1^{1/2},\Delta_j^{1/4},\Delta_k^{1/4}),\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{2,4,4};\Delta_j^{2/4},\Delta_j^{1/4},\Delta_j^{1/4}),\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{2,4,4};\Delta_j^{2/4},\Delta_k^{1/4},\Delta_k^{1/4})$$

for $j,k = 2,3$ *and*

$$\#\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{2,4,4};\Delta_1^{1/2},\Delta_j^{1/4},\Delta_k^{1/4}) = \frac{1}{4}\#\{(a,b) : Q_G(a,b) = d,\ d \equiv 1 \pmod 4\}$$

$$\#\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{2,4,4};\Delta_j^{2/4},\Delta_j^{1/4},\Delta_j^{1/4}) = \frac{1}{4}\#\{(a,b) : Q_G(a,b) = d,\ d \equiv 0 \pmod 4\}$$

$$\#\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{2,4,4};\Delta_j^{2/4},\Delta_k^{1/4},\Delta_k^{1/4}) = \frac{1}{4}\#\{(a,b) : Q_G(a,b) = d,\ d \equiv 2 \pmod 4\}$$

The quantum product on $\mathbb{P}^1_{2,3,6}$ is also related to the Diophantine equation $Q_F(a,b) := a^2 + b^2 = d$, but now we consider d modulo 6.

Proposition 1.0.5. *For $\mathbb{P}^1_{2,3,6}$, we also have a similar statement related to the number of solutions of $Q_F(a,b) = d$ considering $d \pmod 6$ and $d \pmod 3$. Nontrivial 3-fold Gromov-Witten invariants are listed as follows.*

1.

$$\#\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{2,3,6};\Delta_1^{1/2},\Delta_2^{1/3},\Delta_3^{1/6}) = \frac{1}{6}\#\{(a,b) : Q_F(a,b) = d,\ d \equiv 1 \pmod 6\},$$

$$\#\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{2,3,6};\Delta_3^{3/6},\Delta_2^{1/3},\Delta_3^{1/6}) = \frac{1}{6}\#\{(a,b) : Q_F(a,b) = d,\ d \equiv 4 \pmod 6\},$$

$$\#\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{2,3,6};\Delta_1^{1/2},\Delta_3^{2/6},\Delta_3^{1/6}) = \frac{1}{6}\#\{(a,b) : Q_F(a,b) = d,\ d \equiv 3 \pmod 6\},$$

$$\#\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{2,3,6};\Delta_3^{3/6},\Delta_3^{2/6},\Delta_3^{1/6}) = \frac{1}{6}\#\{(a,b) : Q_F(a,b) = d,\ d \equiv 0 \pmod 6\}.$$

2.

$$\#\overline{\mathcal{M}}_{0,3,2d}(\mathbb{P}^1_{2,3,6};\Delta_3^{2/6},\Delta_3^{2/6},\Delta_3^{2/6}) = \frac{1}{6}\#\{(a,b) : Q_F(a,b) = d,\ d \equiv 0 \pmod 3\},$$

$$\#\overline{\mathcal{M}}_{0,3,2d}(\mathbb{P}^1_{2,3,6};\Delta_2^{1/3},\Delta_2^{1/3},\Delta_3^{2/6}) = \frac{1}{6}\#\{(a,b) : Q_F(a,b) = d,\ d \equiv 1 \pmod 3\},$$

$$\#\overline{\mathcal{M}}_{0,3,2d}(\mathbb{P}^1_{2,3,6};\Delta_2^{1/3},\Delta_2^{1/3},\Delta_2^{1/3}) = \frac{1}{3}\#\{(a,b) : Q_F(a,b) = d,\ d \equiv 0 \pmod 3\}.$$

In the item (2), there are only even-degree holomorphic orbi-spheres.

In addition to these, there are two more 3-fold Gromov-Witten invariants $\langle \Delta_3^{1/6}, \Delta_3^{1/6}, \Delta_3^{4/6} \rangle_{0,3}^{\mathbb{P}^1_{2,3,6}}$ and $\langle \Delta_2^{2/3}, \Delta_3^{1/6}, \Delta_3^{1/6} \rangle_{0,3}^{\mathbb{P}^1_{2,3,6}}$, for which we only give a heuristic counting in Conjecture 5.4.3. We are not able to associate solutions of a Diophantine equation to these holomorphic orbi-spheres since their domain is $\mathbb{P}^1_{3,6,6}$ which does not admit an elliptic curve as a covering unlike the other cases.

Chapter 2

Preliminaries

2.1 Symplectic geometry

Symplectic geometry is the study of manifolds equipped with an extra structure, a closed nondegenerate 2-form ω, which is called a *symplectic form*. We call a pair (M, ω) of manifold M equipped with a symplectic form ω as a *symplectic manifold*.

Although our main purpose is to investigate the geometry of orbifolds in the aspect of extending the Lagrangian intersection Floer theory to orbifold setting, explaining the whole theory is more than necessary for this thesis. Hence, we only recall some linear algebraic part of the symplectic geometry which is required for this thesis.

Symplectic vector spaces Fibers of a tangent bundle TM of a symplectic manifold (M, ω) have some algebraic properties inherited from the restriction of the symplectic form ω to each fiber. A vector space with such algebraic properties is called a symplectic vector space :

Definition 2.1.1. *A* symplectic vector space V *is a* $2n$*-dimensional vector space over* $\mathbb{R}$ *equipped with a nondegenerate skew-symmetric bilinear form* $\Omega : V \times V \to \mathbb{R}$.

A typical example of a symplectic vector space is the Euclidean space $\mathbb{R}^{2n}$ with the skew-symmetric form $\Omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$, where $(x_1, \cdots, x_n, y_1, \cdots, y_n)$

is the standard coordinate system of $\mathbb{R}^{2n}$.

Using a skew-symmetric version of the Gram-Schmidt process, one can find a basis of V such that the (V, Ω) is isomorphic to $(\mathbb{R}^{2n}, \Omega_0)$:

Theorem 2.1.1 ([MS2] Theorem 2.3)**.** *Let (V, Ω) be a symplectic vector space of dimension 2n. Then there exists a basis $u_1, \cdots, u_n, v_1, \cdots, v_n$ such that*

$$\Omega(u_j, u_k) = \Omega(v_j, v_k) = 0, \quad \Omega(u_j, v_k) = \delta_{jk}.$$

Such a basis is called a symplectic basis. Moreover, there exists a vector space isomorphism $\Psi : \mathbb{R}^{2n} \to V$ such that

$$\Psi^*\Omega = \Omega_0.$$

Lagrangian subspaces A subspace S in a symplectic vector space (V, Ω) is called *isotropic*, if the restriction of Ω on S vanishes : $\Omega|_{S\times S} = 0$. Moreover, if an isotropic subspace S does not properly contain any other isotropic subspace, S is called a *Lagrangian* subspace. From the non-degeneracy of Ω, a Lagrangian subspace S is exactly an n-dimensional isotropic subspace. For example, the planes $x = 0$ and $y = 0$ are Lagrangian subspace of $(\mathbb{R}^{2n}, \Omega_0)$.

Lagrangian Grassmannian We denote by $\Lambda(V, \Omega)$ the set of all (non-oriented) Lagrangian subspaces of V, which is called Lagrangian Grassmannian of V. The Lagrangian Grassmannian $\Lambda(V, \Omega)$ can be identified with the homogeneous space $U(n)/O(n)$ ($n := \frac{1}{2}\dim V$) as follows :
First, we fix a complex structure J on V which is compatible with Ω, i.e., an endomorphism $J \in \mathrm{End}(V)$ such that $J^2 = -Id_V$, $\Omega(v, w) = \Omega(Jv, Jw)$ for any $v, w \in V$, and $g(\ ,\ ) := \Omega(\ , J\ )$ is a positive definite bilinear form. If we define complex multiplication on V as $(a + ib)v = av + bJ(v)$ for $a, b \in \mathbb{R}$ and $v \in V$, then (V, J) is a complex vector space. Define a Hermitian inner product $\langle\ ,\ \rangle$ on (V, J) as

$$\langle v, w\rangle := g(v, w) - i\Omega(v, w) \quad \text{for } v, w \in V.$$

Note that the Hermitian inner product vanishes on any Lagrangian subspace $L \leq V$. Hence, one can find a real orthonormal basis $\beta := \{e_1, \cdots, e_n\}$ of L using $\langle\ ,\ \rangle|_L = g(\ ,\ )$, which is also a orthonormal basis of the complex vector space

(V, J). Fixing a Lagrangian subspace L and its real orthonormal basis β, we identify (V, J) with $\mathbb{C}^n$. Then, any real orthonormal basis of other Lagrangian subspaces can be written as columns of a unitary matrix. Conversely, columns of any unitary group $A \in U(n)$ is a real basis of a Lagrangian subspace of (V, J). Noting that elements in $O(n) = U(n) \cap GL(n, \mathbb{R})$ is also an orthonormal basis of L, we get an isomorphism

$$\Lambda(n) := U(n)/O(n) \cong \Lambda(V, J),$$

which depends on the choice of J and L. Moreover, this isomorphism is a diffeomorphism, and hence $\Lambda(V, J)$ is a compact connected manifold.

Related to the geometry of Lagrangian Grassmannian $\Lambda(n)$, there is an invariant which is called Maslov index and there are some variants of it according to various settings and purposes. In Chapter 3, we briefly recall the notion of Maslov index associated with a bundle pair and find an equivalent definition of it.

2.2 Orbifolds

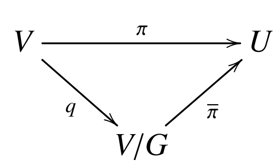
The notion of orbifolds had been firstly introduced by Satake [Sa] in the name of V-manifolds, and the theory of orbifolds was further developed by Thurston [Thu]. Here we follow the definition of orbifold as given in [CR1] and [ALR].

2.2.1 Orbifold via local charts

An orbifold is a topological space whose local model is $\mathbb{R}^n/G$ for a finite group G. More precisely, local chart of an orbifold is called uniformizing system which is defined as follows.

For a topological space U, an n-dimensional uniformizing system of U is a triple (V, G, π) :

- V is an open subset of $\mathbb{R}^n$ with an action of a finite group G.
- π is a continuous map inducing a homeomorphism $\overline{\pi}$ which can be factored as



where q is the quotient map.

If two uniformizing system (V_1, G_1, π_1) and (V_2, G_2, π_2) of U is called *isomorphic* if there is a diffeomorphism $\phi : V_1 \to V_2$ and an isomorphism $\lambda : G_1 \to G_2$ such that ϕ is λ-equivariant and make following diagram commutes.

$$\begin{array}{ccc} V_1 & \xrightarrow{\pi_1} & U \\ & {\scriptstyle \phi}\searrow \quad \nearrow{\scriptstyle \pi_2} & \\ & V_2 & \end{array}$$

For a connected open subset $\iota : U' \hookrightarrow U$, we say that a uniformizing system (V', G', π') of U' is induced from (V, G, π) if

- there is a monomorphism $\tau : G' \to G$ which is an isomorphism restricted to the kernels of the action of G' and G respectively, and
- there is a τ-equivariant open embedding $\psi : V' \to V$ such that $\iota \circ \pi' = \pi \circ \psi$.

The pair (ψ, τ) is called an *injection*.

Consider two injections $(\psi_j, \tau_j) : (V'_j, G'_j, \pi'_j) \to (V, G, \pi)$ for $j = 1, 2$. These two injections are called *isomorphic* if

- there is an isomorphism (ϕ', λ') between (V'_1, G'_1, π'_1) and (V'_2, G'_2, π'_2), and
- there is an automorphism (ϕ, λ) of (V, G, π) such that

$$\begin{array}{ccc} (V'_1, G'_1) & \xrightarrow{(\psi_1,\tau_1)} & (V, G) \\ {\scriptstyle (\phi',\lambda')}\downarrow & & \downarrow{\scriptstyle (\phi,\lambda)} \\ (V'_2, G'_2) & \xrightarrow[(\psi_2,\tau_2)]{} & (V, G) \end{array} \tag{2.2.1}$$

commutes.

We also define the *germ* of uniformizing systems at x using the following equivalence relation: Let (V_1, G_1, π_1) and (V_2, G_2, π_2) be uniformizing systems of neighborhoods U_1 and U_2 of x. We say (V_1, G_1, π_1) and (V_2, G_2, π_2) are *equivalent* at x if they induce isomorphic uniformizing systems of a neighborhood $U_3 \subset U_1 \cap U_2$ of x.

Definition 2.2.1. *An n-dimensional orbifold X consists of a paracompact Hausdorff space $|X|$ with an orbifold structure defined by following data:*
For any point $x \in |X|$, there exists an open neighborhood $U_x \subset |X|$ and an n-dimensional uniformizing system (V_x, G_x, π_x) *of U_x such that for any point $y \in U_x$, (V_x, G_x, π_x) and (V_y, G_y, π_y) are equivalent at q. With a given germ of orbifold structures, X is called an orbifold. We call $|X|$ the underlying topological space of X.*

Example 2.2.1. *Consider the 2-dimensional sphere S^2 and choose two open disc neighborhood of north pole and south pole, U_0 and U_1, such that $U_0 \cup U_1 = S^2$ and $U_0 \cap U_1$ does not contain north pole and south pole.*

We give an orbifold structure on S^2 as follows : For north pole or south pole, open neighborhood U_j is uniformized by $(V_j, \mathbb{Z}_{m_j}, \pi_j)$ where $\mathbb{Z}_{m_j}$ acts on V_j by rotation for $j = 0, 1$. For other points, we take small open disc neighborhoods in $U_0 \cap U_1$ and uniformizing charts which are induced from either $(V_0, \mathbb{Z}_{m_0}, \pi_0)$ or $(V_1, \mathbb{Z}_{m_1}, \pi_1)$. Note that these induced uniformizing systems have trivial isotropy groups.

Now, for two orbifolds X and Y, the morphism between them can be defined as follows.

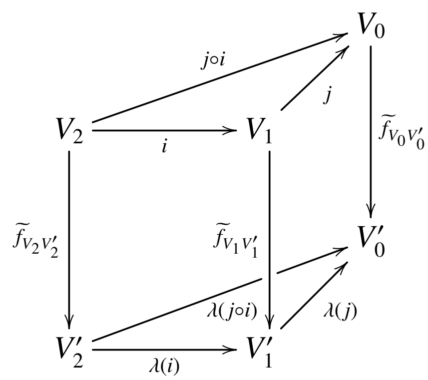
Definition 2.2.2.

1. *A smooth map f between X and Y is a continuous map $|f| : |X| \to |Y|$, which has the following local property. For each $x \in |X|$, there exist uniformizing chart (V_x, G_x) and $(V_{|f|(x)}, G_{|f|(x)})$ of x and $|f|(x)$ respectively and an injective group homomorphism $G_x \to G_{|f|(x)}$ such that $|f|$ admits a local smooth lifting $\widetilde{f}_{V_x V_{|f|(x)}} : V_x \to V_{|f|(x)}$ which is $(G_x, G_{|f|(x)})$-equivariant.*

2. *A smooth map f between X and Y is called good if it admits a collection of maps $\{\widetilde{f}_{UU'}, \lambda\}$ which is called a compatible system of f and defined as follows :*

   For each equivariant embedding $i : (V_2, G_2, \pi_2) \to (V_1, G_1, \pi_1)$ of local uniformizing charts of X, there is an equivariant embedding $\lambda(i) : (V_2', G_2', \pi_2') \to$

(V_1', G_1', π_1') *of charts of* Y *with the following commutative diagram,*



where each maps are equivariant maps. In the above diagram, we omit to write relevant groups and group homomorphisms for simplicity.

Denote by $|X|_{reg}$ the smooth part of $|X|$. In other words, for each $x \in |X|_{reg}$, there is a uniformizing chart (V_x, G_x, π_x) with a trivial group G_x.

Definition 2.2.3. *A smooth map* $f : X \to Y$ *is called* regular *if the underlying continuous map* $|f| : |X| \to |Y|$ *has the following property :* $|f|^{-1}(|Y|_{reg})$ *is an open dense and connected subset of* $|X|$*.*

Lemma 2.2.2 ([CR2],Lemma 4.4.11)**.** *If* f *is regular, then* f *is the unique germ of smooth liftings of* $|f|$*. Moreover* f *is good with a unique isomorphism class of compatible systems.*

There is another notion of maps between orbifolds introduced by Takeuchi [T]. We will compare two notions in Lemma 5.1.2.

2.2.2 Orbifold via groupoid

In the modern approach of orbifolds, one usually uses the language of groupoids in the definition of orbifolds. This generalizes the notion of classical orbifolds allowing noneffective orbifolds. Recall that a groupoid is a (small) category whose morphisms are all invertible. Giving a topological structure and smooth structure on groupoids, we get the notion of Lie groupoids.

Definition 2.2.4. *A topological groupoid $\mathcal{G}$ is a pair of topological spaces $G_0 := Obj(\mathcal{G})$ and $G_1 := Mor(\mathcal{G})$ together with continuous structure maps:*

1. *The source and target map $s, t : G_1 \rightrightarrows G_0$, which assigns to each arrow $g \in G_1$ its source object and target object, respectively.*
2. *The multiplication map $m : G_{1\,s} \times_t G_1 \to G_1$, which compose two arrows.*
3. *The unit map $u : G_0 \to G_1$, which is a two-sided unit for the multiplication.*
4. *The inverse $i : G_1 \to G_1$, which assigns to each arrow its inverse arrow. This map is well-defined since all morphisms are invertible.*

If all of the above maps are smooth and s (or t) is a surjective submersion(so that the domain $G_{1\,s} \times_t G_1$ of m is a smooth manifold), then $\mathcal{G}$ is called a Lie groupoid.

Definition 2.2.5. *Let $\mathcal{G}$ be a Lie groupoid.*

1. *$\mathcal{G}$ is proper if $(s, t) : G_1 \to G_0 \times G_0$ is a proper map.*
2. *$\mathcal{G}$ is called a foliation groupoid if each isotropy group G_x is discrete.*
3. *$\mathcal{G}$ is ètale if s and t are local diffeomorphisms.*

Note the every ètale groupoid is a foliation groupoid. It can be easily checked that a proper foliation groupoid $\mathcal{G}$ has only finite isotropy groups $G_x := (s, t)^{-1}(x, x)$ for each $x \in G_0$

Definition 2.2.6. *We define an orbifold groupoid to be a proper ètale Lie groupoid.*

Let us recall morphisms and Morita equivalence of orbifolds.

Definition 2.2.7. *Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids. A homomorphism $\phi : \mathcal{H} \to \mathcal{G}$ consists of two smooth maps $\phi_0 : H_0 \to G_0$ and $\phi_1 : H_1 \to G_1$, that together commute with all the structure maps for the two groupoids $\mathcal{G}$ and $\mathcal{H}$. It means that Lie groupoid morphisms are smooth functors between categories.*

The following notion of equivalence is restrictive (it does not define equivalence relation), and later we will recall Morita equivalence which is indeed the correct notion of equivalences between orbifold groupoids.

Definition 2.2.8. *A homomorphism between $\phi : \mathcal{H} \to \mathcal{G}$ between Lie groupoids is called equivalence if*

(i) (essentially surjective) the map

$$t\pi_1 : G_1 \, {}_s\times_{\phi_0} H_0 \to G_0$$

defined on the fibered product of manifolds

$$\{(g, y) \,|\, g \in G_1, y \in H_0, s(g) = \phi(y)\}$$

is a surjective submersion where $\pi_1 : G_1 \, {}_s\times_{\phi_0} H_0 \to G_1$ is the projections to the first factor;

(ii) the square

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi_1} & G_1 \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\ H_0 \times H_0 & \xrightarrow{\phi_0 \times \phi_0} & G_0 \times G_0 \end{array}$$

is a fibered product of manifolds.

An equivalence in the Definition 2.2.8 may not have an inverse. The notion of Morita equivalence is obtained by formally inverting equivalences in Definition 2.2.8.

Definition 2.2.9. *$\mathcal{G}$ and $\mathcal{G}'$ are said to be Morita equivalent if there exists a groupoid $\mathcal{H}$ and two equivalences*

$$\mathcal{G} \xleftarrow{\phi} \mathcal{H} \xrightarrow{\phi'} \mathcal{G}.$$

It is well known that "Morita equivalence" defines an equivalence relation. (See the discussion below Definition 1.43 in [ALR].) It is clear from the definition that equivalence is a special case of Morita equivalence. Lastly, we give an example of Morita equivalent groupoid which are not equivalent. For example, if $\mathcal{G}$ can be made by tearing off some part $\mathcal{G}'$ and adding arrows which contains the original gluing information, then we have an equivalence from $\mathcal{G}$ to $\mathcal{G}'$. However, since "tearing off" process is not continuous, there is no map in the opposite direction in general.

Example 2.2.3. *Consider two orbifold groupoids, $\mathcal{G}$ and $\mathcal{G}'$, which are equivalent to the closed interval. From the figure 2.1, it is clear that they are Morita equivalent, but there are no maps neither from $\mathcal{G}$ to $\mathcal{G}'$ nor from $\mathcal{G}'$ to $\mathcal{G}$.*

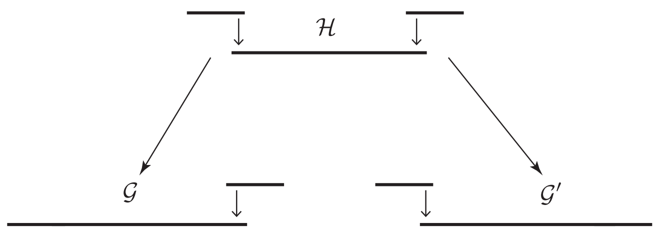


Figure 2.1: Morita equivalence

Using groupoid language, we define an orbifold as an equivalence class of orbifold groupoids. For a groupoid $\mathcal{G}$, we denote by $|\mathcal{G}|$ the quotient space of G_0 under the equivalence relation $x \sim y$ if and only if there is an element $g \in G_1$ such that $s(g) = x$ and $t(g) = y$.

Definition 2.2.10. *An orbifold structure on a paracompact Hausdorff space X consists of an orbifold groupoid $\mathcal{G}$ and a homeomorphism $f : |\mathcal{G}| \to X$. If $\phi : \mathcal{H} \to \mathcal{G}$ is an equivalence, then $|\phi| : |\mathcal{H}| \to |\mathcal{G}|$ is a homeomorphism, and we say the composition $f \circ |\phi| : |\mathcal{H}| \to X$ defines an equivalent orbifold structure on X.*

In groupoid language, the notion of morphism between orbifolds is defined as follows.

Definition 2.2.11. *Suppose that $\mathcal{H}$ and $\mathcal{G}$ are orbifold groupoids. An orbifold morphism from $\mathcal{H}$ to $\mathcal{G}$ is a pair of groupoid homomorphisms*

$$\mathcal{H} \stackrel{\epsilon:\simeq}{\leftarrow} \mathcal{H}' \stackrel{\phi}{\to} \mathcal{G}$$

such that the left arrow is an equivalence.

In the above definition, $\mathcal{H}'$ can be understood as a refinement of $\mathcal{H}$ in very rough terms. There is also the notion of equivalence of orbifold morphisms :

1. If there is a natural transformation between two homomorphisms $\phi, \phi' : \mathcal{H}' \to \mathcal{G}$, then $\mathcal{H} \overset{\epsilon:\simeq}{\leftarrow} \mathcal{H}' \overset{\phi}{\to} \mathcal{G}$ is equivalent to $\mathcal{H} \overset{\epsilon:\simeq}{\leftarrow} \mathcal{H}' \overset{\phi'}{\to} \mathcal{G}$.

2. If $\delta : \mathcal{H}'' \overset{\simeq}{\to} \mathcal{H}'$ is an equivalence morphism, $\mathcal{H} \overset{\epsilon:\simeq}{\leftarrow} \mathcal{H}' \overset{\phi}{\to} \mathcal{G}$ is equivalent to $\mathcal{H} \overset{\epsilon\circ\delta:\simeq}{\longleftarrow} \mathcal{H}'' \overset{\phi\circ\delta}{\longrightarrow} \mathcal{G}$.

Let $\mathcal{R}$ be the minimal equivalence relation among orbifold morphisms from $\mathcal{H}$ to $\mathcal{G}$ generated by the above two relations.

Definition 2.2.12. *Two orbifold morphisms are said to be equivalent if they belong to the same $\mathcal{R}$-equivalence class.*

The set of Morita equivalence classes of orbifold groupoids form a category with morphisms the equivalence classes of orbifold morphisms. This orbifold category defined via orbifold groupoids and orbifold morphisms is equivalent to the orbifold category defined via local charts and good maps (See [LU]).

Definition 2.2.13. *Let $\phi : \mathcal{H} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{G}$ be homomorphisms of Lie groupoids. The fibered product $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$ is the Lie groupoid that makes the following diagram a fibered product.*

$$\begin{array}{ccc} \mathcal{H} \times_{\mathcal{G}} \mathcal{K} & \xrightarrow{pr_2} & \mathcal{K} \\ \Big\downarrow{\scriptstyle pr_1} & & \Big\downarrow{\scriptstyle \psi} \\ \mathcal{H} & \xrightarrow{\phi} & \mathcal{G} \end{array}$$

which commutes up to a natural transformation. More explicitly,

$$(\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_0 \;:=\; H_0 \times_{\phi_0, G_0, s} G_1 \times_{t, G_0, \psi_0} K_0 \tag{2.2.2}$$
$$(\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_1 \;:=\; H_1 \times_{s\phi_1, G_0, s} G_1 \times_{t, G_0, s\psi_1} K_1 \tag{2.2.3}$$

with following source and target maps

$$s(h, g, k) = (s(h), g, s(k)),$$
$$t(h, g, k) = (t(h), \psi(k) g \phi(h)^{-1}, t(k)).$$

We will also write $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$ as $\phi^* \mathcal{K}$ occasionally.

To be more precise, an element of $(\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_0$ is a triple (x, g, z) such that

$$x \qquad \phi_0(x) \xrightarrow{g} \psi_0(z) \qquad z \tag{2.2.4}$$

and a morphism between (x, g, z) and (x', g', z') is a triple (h, g, k) which makes the following diagram commutative.

$$\begin{array}{cccccc} x & & \phi_0(x) & \xrightarrow{g} & \psi_0(z) & z \\ \downarrow{\scriptstyle h} & & \scriptstyle\phi_1(h)\downarrow & & \downarrow{\scriptstyle\psi_1(k)} & \downarrow{\scriptstyle k} \\ x' & & \phi_0(x') & \xrightarrow{g'} & \psi_0(z') & z' \end{array} \tag{2.2.5}$$

i.e. for $(h, g, k) \in (\mathcal{H} \times_{\mathcal{K}} \mathcal{G})_1$ which satisfies $s(g) = \phi_0(s(h))$ and $t(g) = \psi_0(s(k))$ by definition,

$$s(h, g, k) \quad = \quad s(h) \qquad \phi_0(s(h)) \xrightarrow{g} \psi_0(s(k)) \qquad s(k)$$

and

$$t(h, g, k) \quad = \quad t(h) \qquad \phi_0(t(h)) \xrightarrow{g'} \psi_0(t(k)) \qquad t(k)$$

where $g' = \psi_1(k) g \phi_1(h)^{-1}$.

Remark 2.2.4. *The fibered product $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$ may not be a Lie groupoid, since $(\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_0$ or $(\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_1$ may not be manifolds.*

The following lemma is well-known.

Lemma 2.2.5. *If $\psi : \mathcal{K} \to \mathcal{G}$ is an equivalence, then $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$ is a Lie groupoid and the projection $\mathcal{H} \times_{\mathcal{G}} \mathcal{K} \to \mathcal{H}$ is an equivalence*

$$\begin{array}{ccc} \mathcal{H} \times_{\mathcal{G}} \mathcal{K} & \xrightarrow{pr_2} & \mathcal{K} \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \psi:\cong} \\ \mathcal{H} & \xrightarrow[\phi]{} & \mathcal{G}. \end{array} \tag{2.2.6}$$

Proof. From (2.2.2), one can see that if $s \circ pr_1 : G_1 \times_{t,G_0,\psi_0} K_0 \to G_0$ is a submersion, then $(\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_0$ is a manifold. This happens when ψ is an equivalence. Since $s : K_1 \to K_0$ is a submersion, a similar argument shows that $(\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_1$ is a manifold for the equivalence ψ.

Recall that $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$ is a Lie groupoid whose set of objects and arrows are

$$\begin{aligned} (\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_0 &= H_0 \times_{\phi_0,G_0,s} G_1 \times_{t,G_0,\psi_0} K_0, \\ (\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_1 &= H_1 \times_{s\phi_1,G_0,s} G_1 \times_{t,G_0,s\psi_1} K_1 \end{aligned}$$

respectively. We first check the condition (i) of Definition 2.2.8. We have to show that the following map

$$t\pi_1 : H_1 \times_{s,H_0,pr_1} (H_0 \times_{\phi_0,G_0,s} G_1 \times_{t,G_0,\psi_0} K_0) \to H_0$$

is a surjective submersion where π_1 is the projection to the first factor H_1. Consider the following diagrams of fiber products.

$$\begin{array}{ccccc} H_1 \times_{s,H_0,pr_1} (\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_0 & \longrightarrow & (\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_0 & \longrightarrow & G_1 \times_{t,G_0,\psi_0} K_0 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow & & \downarrow \\ H_1 & \longrightarrow & H_0 & \longrightarrow & G_0 \end{array}$$

The rightmost vertical map $G_1 \times_{t,G_0,\psi_0} K_0 \to G_0$ is a surjective submersion, since $\psi : \mathcal{K} \to \mathcal{G}$ is an equivalence. Then, it follows from a general property of fiber product diagrams that the middle vertical map $H_0 \times_{\phi_0,G_0,s} G_1 \times_{t,G_0,\psi_0} K_0 \to H_0$ is also a surjective submersion, and hence so is π_1. Finally, $t\pi_1$ is a surjective submersion since it is given by a composition of two such kinds of maps.

To show the second condition of equivalence, we consider the following diagram

$$\begin{array}{ccc} (\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_1 & \xrightarrow{pr_1} & H_1 \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\ (\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_0 \times (\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_0 & \xrightarrow{pr_1 \times pr_1} & H_0 \times H_0 \end{array}$$

Since $(s,t) : H_1 \to H_0 \times H_0$ is a submersion, we only need to check that the above diagram is a fibered product of sets. Suppose $h \in H_1$, and denote $x = s(h)$ and

$x' = t(h)$. Since $pr_1 : H_0 \times_{\phi_0,G_0,s} G_1 \times_{t,G_0,\psi_0} K_0 \to H_0$ is surjective, there exists (x, g, y) and (x', g', y') in $H_0 \times_{\phi_0,G_0,s} G_1 \times_{t,G_0,\psi_0} K_0$. Since ψ is equivalence, there exists a unique $k \in K_1$ satisfying $\psi_1(k) = g'\phi_1(h)g^{-1}$. Since $h \in H_1$ determines a unique element (h, g, k) in the fiber over $((x, g, y), (x', g', y'))$, the above diagram is a fiber product as sets. □

2.2.3 Hilsum-Skandalis map

There is another notion of morphism between Lie groupoids, Hilsum-Skandalis map. In fact, the equivalence classes of representable orbifold morphisms from $\mathcal{H}$ to $\mathcal{G}$ are in one-to-one correspondence with isomorphic classes of Hilsum-Skandalis maps from $\mathcal{H}$ to $\mathcal{G}$. We give brief review on Hilsum-Skandalis map and we refer readers to [PS] and [L] for further details. This notion plays a crucial role in Section 4.4 and 4.5.

We first recall the definition of the action of an orbifold groupoid on manifolds.

Definition 2.2.14. *Let $\mathcal{G}$ be an orbifold groupoid. A left $\mathcal{G}$-space is a manifold E equipped with an action by $\mathcal{G}$. Such an action is given by two maps:*

- *an anchor $\pi : E \to G_0$;*
- *an action $\mu : G_1 \times_{G_0} E \to E$.*

The latter map is defined on pairs (g, e) with $\pi(e) = s(g)$, and written $\mu(g, e) = g \cdot e$. It satisfies the usual identities for an action:

- *$\pi(g \cdot e) = t(g)$;*
- *$1_x \cdot e = e$;*
- *$g \cdot (h \cdot e) = (gh) \cdot e$.*

for $x \xrightarrow{h} y \xrightarrow{g} z$ in G_1 with $\pi(e) = x$.

A right $\mathcal{G}$-space is the same thing as a left $\mathcal{G}^{op}$-space, where $\mathcal{G}^{op}$ is the opposite groupoid obtained by exchanging the roles of the target and source maps.

Definition 2.2.15. *A left $\mathcal{G}$-bundle over a manifold M is a manifold R with smooth maps*

$$\begin{array}{ccc} R & \xrightarrow{\rho} & M \\ \downarrow{\scriptstyle r} & & \\ G_0 & & \end{array}$$

and a left $\mathcal{G}$-action μ on R, with anchor map $r : R \to G_0$, such that $\rho(gx) = \rho(x)$ for any $x \in R$ and any $g \in G_1$ with $r(x) = s(g)$.

Such a bundle R is principal if

1. *ρ is a surjective submersion,*

2. *the map $(\pi_1, \mu) : R \times_{r,G_0,s} G_1 \to R \times_M R$, sending (x, g) to (x, gx), is a diffeomorphism.*

Definition 2.2.16. *A Hilsum-Skandalis map $\mathcal{G} \to \mathcal{H}$ is represented by a principal left $\mathcal{H}$-bundle R over G_0*

$$\begin{array}{ccc} R & \xrightarrow{\rho} & G_0 \\ \downarrow{\scriptstyle r} & & \\ H_0 & & \end{array}$$

which is also a right $\mathcal{G}$-bundle (over H_0), and the right $\mathcal{G}$-action commutes with the $\mathcal{H}$-action. R is called the Hilsum-Skandalis bibundle.

Definition 2.2.17. *For two bibundles $R : \mathcal{G} \to \mathcal{H}$ and $Q : \mathcal{H} \to \mathcal{K}$, their composition is defined by the quotient of the fiber product $Q \times_{H_0} R$ by the action of $\mathcal{H}$:*

$$Q \circ R := (Q \times_{H_0} R)/H_1, \tag{2.2.7}$$

where the action of H_1 on $Q \times_{H_0} R$ is given by $h \cdot (q, r) := (qh, h^{-1}r)$. Since the left action of $\mathcal{H}$ on R is principal, the action of $\mathcal{H}$ on $Q \times_{H_0} R$ is free and proper; hence, the $Q \circ R$ is a smooth manifold. It also admits a principal $\mathcal{K}$-bundle structure with a right $\mathcal{G}$-action, because $\mathcal{H}$-action commutes with $\mathcal{G}$- and $\mathcal{K}$-actions on R and Q, respectively.

One can compose two Hilsum-Skandalis maps as follows:

Definition 2.2.18. *Two Hilsum-Skandalis maps $P, R : \mathcal{G} \to \mathcal{H}$ are isomorphic if they are diffeomorphic as left $\mathcal{H}$- and right $\mathcal{G}$-bundles: i.e, there is a diffeomorphism $\alpha : P \to R$ satisfying $\alpha(h \cdot p \cdot g) = h \cdot \alpha(p) \cdot g$ for all $(h, p, g) \in H_1 \times_{H_0} P \times_{G_0} G_1$.*

For example, any Lie groupoid homomorphism $\phi : \mathcal{G} \to \mathcal{H}$ defines a Hilsum-Skandalis map

$$\begin{array}{ccc} R_\phi := H_1 \times_{s,H_0,\phi_0} G_0 & \xrightarrow{\pi_2} & G_0 \\ \Big\downarrow{\scriptstyle t\circ\pi_1} & & \\ H_0 & & \end{array}$$

where π_1 and π_2 are the projection maps. One can easily check that π_2 is principal in this case. We will use this construction crucially in the next section to construct an equivariant immersion from an orbifold embedding.

Remark 2.2.6.

1. *Not every Hilsum-Skandalis map is induced from Lie groupoid homomorphisms. In fact, a Hilsum-Skandalis map $R : \mathcal{G} \to \mathcal{H}$ is isomorphic to some R_ϕ for some Lie groupoid homomorphism $\phi : \mathcal{G} \to \mathcal{H}$ if and only if the map $\rho : R \to G_0$ has a global section. See Lemma 3.36 in [L].*

2. *We use slightly different notion of the Hilsum-Skandalis map from [PS]. In [PS],*

$$R_\phi^{ps} = H_0 \times_{\phi_0, G_0, t} G_1$$

is used to construct a Hilsum-Skandalis map from ϕ. Here, we use R_ϕ^* to make it a left G-space. See the following diagrams.

$$\begin{array}{ccc} R_\phi & \xrightarrow{\pi_1} & G_1 \\ {\scriptstyle \pi_2}\Big\downarrow & & \Big\downarrow{\scriptstyle s} \\ H_0 & \xrightarrow{\phi_0} & G_0 \end{array} \qquad \begin{array}{ccc} R_\phi^{ps} & \longrightarrow & H_0 \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \phi_0} \\ G_1 & \xrightarrow{t} & G_0 \end{array}$$

Now, we want to translate the notion of equivalence in the category of orbifold groupoids into Hilsum-Skandalis maps. We first refer to the following two lemmas from [L].

Lemma 2.2.7. *([L], Lemma 3.34) A Lie groupoid homomorphism $\phi : \mathcal{G} \to \mathcal{H}$ is an equivalence of Lie groupoids if and only if the corresponding R_ϕ is $\mathcal{G}$-principal.*

Lemma 2.2.8. *([L], Lemma 3.37) Let $P : \mathcal{G} \to \mathcal{H}$ be a Hilsum-Skandalis map. Then, there is a cover $\phi : \mathcal{U} \to G_0$ and a groupoid homomorphism $f : \phi^*\mathcal{G} \to \mathcal{H}$ so that*

$$P \circ R_{\tilde{\phi}} \xrightarrow{\simeq} R_f,$$

where $\tilde{\phi} : \phi^*\mathcal{G} \to \mathcal{G}$ is the induced functor and "$\xrightarrow{\simeq}$" an isomorphism of Hilsum-Skandalis maps. Here, $\phi^*\mathcal{G}$ is the Lie groupoid with $(\phi^*\mathcal{G})_0 = \mathcal{U}$ and $(\phi^*\mathcal{G})_1$ given by

$$\begin{array}{ccc} (\phi^*\mathcal{G})_1 & \longrightarrow & G_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathcal{U} \times \mathcal{U} & \xrightarrow[(\phi,\phi)]{} & G_0 \times G_0 \end{array}$$

From the above two lemmas, we obtain the following characterization of Morita equivalence in terms of Hilsum-Skandalis language.

Lemma 2.2.9. *If a Hilsum-Skandalis map $P : \mathcal{G} \to \mathcal{H}$ is also right $\mathcal{G}$-principal, then $f : \phi^*\mathcal{G} \to \mathcal{H}$ obtained from the above lemma is an equivalence of groupoids. Note that $\tilde{\phi} : \phi^*\mathcal{G} \to \mathcal{G}$ is trivially an equivalence of groupoids.*

Proof. Note that $R_{\tilde{\phi}}$ is biprincipal, since $\tilde{\phi}$ is an equivalence of groupoids. The composition of two biprincipal bundle $P \circ R_{\tilde{\phi}}$ is also biprincipal, and hence the isomorphic bibundle R_f also biprincipal. Therefore f is an equivalence of groupoids $\phi^*\mathcal{G}$ and $\mathcal{H}$. □

The above lemma justifies the notion of the Morita equivalence in the Hilsum-Skandalis setting.

Definition 2.2.19. *A Hilsum-Skandalis map (R, ρ, r) is a Morita equivalence when it is both a principal $\mathcal{G}$-bundle and a principal $\mathcal{H}$-bundle.*

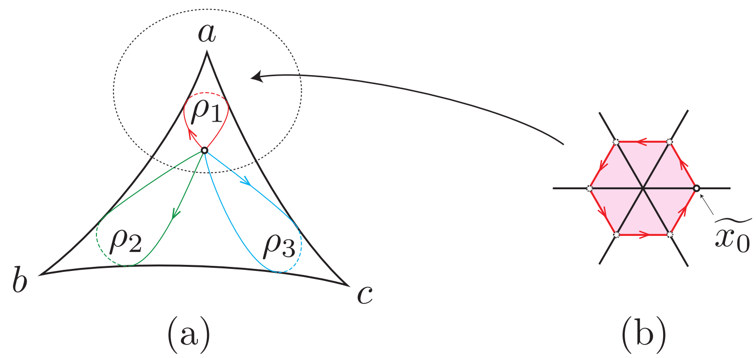


Figure 2.2: (a) generators of $\pi_1^{orb}(\mathbb{P}^1_{a,b,c})$ and (b) the relation $\rho_1^a = 1$

2.3 Orbifold fundamental group

In this section, we recall the notion of the orbifold fundamental group (introduced by Thurston), which is closely related to orbifold covering theory explained below. This is analogous to the connection between usual covering theory and fundamental groups. It is enough to consider global quotient orbifolds for our purpose. Consider a finite group action G on a manifold M which preserving orientation. (One may consider an infinite group G, if an orbifold is represented as a quotient of non-compact manifold by an locally free action of a discrete group.) Then generalized loops in M are defined as follows:

Definition 2.3.1. *A path $\gamma : [0,1] \to M$ is called a generalized loop based at $\widetilde{x_0} \in M$ if $\gamma(0) = \widetilde{x_0}$, and there exists $g_\gamma \in G$ such that $\gamma(1) = g \cdot \gamma(0)$.*

Choose a point $\widetilde{x_0} \in M$ with $G_{\widetilde{x_0}} = 1$, and let $\pi_1^{orb}([M/G])$ be the set of equivalence classes of generalized loops based at $\widetilde{x_0}$ where the equivalence relation is given as homotopies fixing end points. One can check that $\pi_1^{orb}([M/G])$ has a natural group structure by defining

$$[\gamma] \cdot [\delta] = [\gamma \# g_\gamma(\delta)]$$

for generalized loops γ, δ based at $\widetilde{x_0}$ where '#' denotes the concatenation of paths.

For example, $\pi_1^{orb}(\mathbb{P}^1_{a,b,c})$ has a presentation

$$\pi_1^{orb}(\mathbb{P}_{a,b,c}) =< \rho_1, \rho_2, \rho_3 \mid (\rho_1)^a = (\rho_2)^b = (\rho_3)^c = \rho_1\rho_2\rho_3 = 1 >$$

where ρ_1, ρ_2 and ρ_3 are generalized loops in the universal cover of $\mathbb{P}^1_{a,b,c}$ whose images in the underlying space of the orbifold look as in (a) of Figure 2.2. The relation $\rho_1^a = 1$ can be seen in the uniformizing chart around the singular point of order a (see (b) of Figure 2.2). One can observe the relation $\rho_1\rho_2\rho_3 = 1$ even more directly on the orbifold itself.

Remark 2.3.1. *See [T] for details on the link between orbifold fundamental groups and orbifold coverings which we shall explain below.*

2.4 Orbifold covering theory

There is an analogue of covering space for orbifolds whose local model is $\mathbb{R}^n/G' \to \mathbb{R}^n/G$ for some finite group G which acting on $\mathbb{R}^n$ with $G' \leq G$.

Definition 2.4.1. *[T, Section 1] An orbifold $\widetilde{X}$ is called a covering orbifold, if there is a continuous surjective map $p : |\widetilde{X}| \to |X|$ satisfying following condition :*

For each point $x \in |X|$, there is a local uniformizing chart $\widetilde{U}_x/G_x \cong U_x$ such that each point $\widetilde{x} \in p^{-1}(x)$ has a local uniformizing chart $\widetilde{U}_x/G_{x,i} \cong V_{x,i}$ for some $G_{x,i} \leq G_x$ such that the following diagram commutes :

$$\begin{array}{ccccc} & \nearrow & \widetilde{U}_x/G_{x,i} & \xrightarrow{\cong} & V_{x,i} \\ & & \downarrow q & & \downarrow p \\ \widetilde{U}_x & \longrightarrow & \widetilde{U}_x/G_x & \xrightarrow{\cong} & U_x \end{array}$$

where q is the natural projection.

An orbifold which admits a manifold covering is called a *good orbifold*. For example, the orbifold projective lines $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{2,4,4}$ and $\mathbb{P}^1_{2,3,6}$ are all good orbifolds, as they are given by quotients of a 2-torus. Throughout the section, we assume that all orbifolds are good.

Following [T] we introduce the notion of orbi-map.

Definition 2.4.2. *[T, Section 2] An orbi-map $f : X \to Y$ consists of a continuous map $h : |X| \to |Y|$ between underlying spaces and a fixed continuous map $\tilde{h} : \widetilde{X} \to \widetilde{Y}$ which satisfy*

1. $h \circ p = q \circ \tilde{h}$

2. *For each* $\sigma \in \mathrm{Aut}(\widetilde{X}, p)(\cong \pi_1^{orb}(X))$, *there exists* $\tau \in \mathrm{Aut}(\widetilde{Y}, q)$ *such that* $\tilde{h} \circ \sigma = \tau \circ \tilde{h}$

3. $h(|X|)$ *does not lie in the singular loci of* Y *entirely.*

Remark 2.4.1. *Indeed, covering theory in [T] only concerns about good orbifolds.*

For orbi-maps, we have usual lifting theorems in covering theory as well, whose proof is not very much different from the standard one.

Proposition 2.4.2. *[T, Proposition 2.7] Let* $f : (X, x) \to (Y, y)$ *be an orbi-map and* $p : (Y', y') \to (Y, y)$ *be a covering. Then* f *can be lifted to an orbi-map* $\widetilde{f} : X \to Y'$ *if and only if* $f_*\pi_1^{orb}(X, x) \subset p_*\pi_1^{orb}(Y', y')$.

2.5 Orbifold Gromov-Witten theory

In this section, we briefly review the quantum cohomology of orbifolds developed by Chen and Ruan. The key ingredients in defining the product on this cohomology are holomorphic orbi-spheres (or orbifold stable maps in general) in orbifolds with three marked points. If we consider such spheres with arbitrary number of markings as well, then we obtain the orbifold (genus-0) Gromov-Witten invariants [CR2] (See Subsection 2.5.1 and 2.5.3, also.)

2.5.1 Description of $\overline{\mathcal{M}}_{g,k,\beta}(X)$

Let (X, ω) be an compact effective symplectic orbifold with a compatible almost complex structure J. (See [CR2, Definition 2.1.1, 2.1.5].) We begin with the description of the compactified moduli space of orbifold stable maps into X. Details can be found in [CR2].

Definition 2.5.1 ([CR2], Definition 2.2.2)**.** *An orbi-Riemann surface of genus g is a triple* $(\Sigma, \boldsymbol{z}, \boldsymbol{m})$ *:*

- Σ *is a smooth Riemann surface of genus* g.

- $z = (z_1, \cdots, z_k)$ *is a set of orbi-singular points on* Σ *with isotropy group of order* $\boldsymbol{m} = (m_1, \cdots, m_k)$ *for some integer* $m_i (\geq 2)$*. The orbifold structure on* Σ *is given as follows: at each point* z_i*, a disc neighborhood of* z_i *is uniformized by the branched covering map* $z \to z^{m_i}$*.*

In order to compactify the moduli space, we should also include nodal Riemann surfaces as domains of holomorphic maps.

Definition 2.5.2 ([CR2], Definition 2.3.1)**.** *A nodal Riemann surface with k marked points is a pair* (Σ, z) *of a connected topological space* $\Sigma = \bigcup \pi_{\Sigma_\nu}(\Sigma_\nu)$ *and a set of k-distinct points* $z = (z_1, \cdots, z_k)$ *in* Σ *with the following properties.*

- Σ_ν *is a smooth Riemann surface of genus* g_ν*, and* $\pi_\nu : \Sigma_\nu \to \Sigma$ *is a continuous map. The number of* Σ_ν *is finite.*

- *For each* $z \in \Sigma_\nu$*, there is a neighborhood of it such that the restriction of* $\pi_\nu : \Sigma_\nu \to \Sigma$ *to this neighborhood is a homeomorphism to its image.*

- *For each* $z \in \Sigma$*, we have* $\sum_\nu \#\pi_\nu^{-1}(z) \leq 2$*, and the set of nodal points* $\left\{ z | \sum_\nu \#\pi_\nu^{-1}(z) = 2 \right\}$ *is finite.*

- z *does not contain any nodal point.*

We next allow cone singularities on nodal Riemann surfaces.

Definition 2.5.3 ([CR2], Definition 2.3.2)**.** *A nodal orbi-Riemann surface is a nodal marked Riemann surface* (Σ, z) *with an orbifold structure as follows:*

- *The set of orbi-singular points is contained in the set of marked points and nodal points* z*;*

- *A disk neighborhood of a marked point is uniformized by a branched covering map* $z \to z^{m_i}$*;*

- *A neighborhood of a nodal point is uniformized by the chart* $(\widetilde{U}, \mathbb{Z}_{n_j})$*, where* $\widetilde{U} = \left\{ (x, y) \in \mathbb{C}^2 \middle| xy = 0 \right\}$ *on which* $\mathbb{Z}_{n_j}$ *acts by* $e^{\frac{2\pi i}{n_j}} \cdot (x, y) = (e^{\frac{2\pi i}{n_j}} x, e^{-\frac{2\pi i}{n_j}} y)$*.*

Here m_i *and* n_j *are allowed to be* 1*. We denote the corresponding nodal orbi-Riemann surface by* $(\Sigma, z, \boldsymbol{m}, \boldsymbol{n})$*, and if there is no confusion, simply write it as* (Σ, z)*. (See Figure 2.3.)*

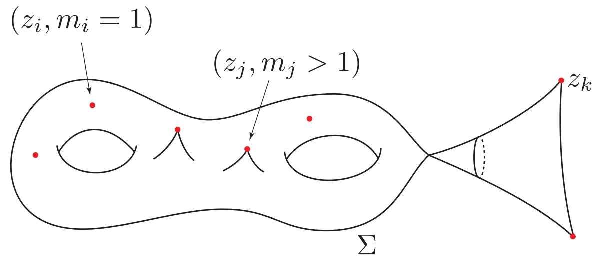


Figure 2.3: A nodal orbi-Riemann surface

Having a nodal orbi-Riemann surface as the domain, an orbifold stable map is defined as follows.

Definition 2.5.4 ([CR2], Definition 2.3.3)**.** *For given an almost complex orbifold (X, J), an orbifold stable map is a triple $(f, (\Sigma, \boldsymbol{z}, \boldsymbol{m}, \boldsymbol{n}), \xi)$:*

- *$f : \Sigma \to |X|$ is a continuous map from a nodal Riemann surface Σ such that $f_v = f \circ \pi_v$ is a J-holomorphic map.*
- *Consider the local lifting $\widetilde{f}_{V_x V_{f(x)}} : (V_x, G_x) \to (V_{f(x)}, G_{f(x)})$ of f. Then the homomorphism $G_x \to G_{f(x)}$ is injective.*
- *Let k_v be the number of points in Σ_v which is marked or nodal. If f_v is a constant map, then $k_v + 2g_v \geq 3$.*
- *ξ is an isomorphism class of compatible systems.*

For the definition of an isomorphism between compatible systems, see [CR2, Definition 4.4.4].

We are now ready to define the moduli space relevant to the orbifold Gromov-Witten invariants of X.

Definition 2.5.5. *1. Two stable maps $(f, (\Sigma, z), \xi)$ and $(f', (\Sigma', z'), \xi')$ are equivalent if there is an isomorphism $\theta : (\Sigma, \boldsymbol{z}, \boldsymbol{m}, \boldsymbol{n}) \to (\Sigma', \boldsymbol{z}', \boldsymbol{m}', \boldsymbol{n}')$ such that $f' \circ \theta = f$ and $\xi' \circ \theta = \xi$.*

2. Given a homology class $\beta \in H_2(|X|; \mathbb{Z})$, $\overline{\mathcal{M}}_{g,n,\beta}(X, J)$ is defined as the moduli space of equivalence classes of orbifold stable maps of genus g, with k marked points, and of homology class β.

2.5.2 Orbifold cohomology theory

Similar to the smooth case, orbifold Gromov-Witten invariants is defined as some pairing of some cohomology classes. The cohomology theory relevant to orbifold Gromov-Witten theory is the orbifold cohomology group $H^*_{orb}(X, \mathbb{Q})$ introduced by Chen and Ruan in [CR1].

In order to describe the orbifold cohomology group, we first need to recall the notion of inertia orbifolds. A precise definition of inertia orbifold in groupoid language is given in Section 4.2. However, one need to refine it according to the type of isotropy groups in order to define *twisted sectors*, which is the main ingredient for defining the orbifold cohomology group. In case of general orbifolds, this "refining" procedure is quite cumbersome to describe and is more that necessary for our purpose in Chapter 5. Hence, we only consider a global quotient orbifold $X = [M/G]$ for some finite group G in this section. For the general definition of inertia orbifolds and twisted sectors, see [CR1] or [ALR].

Inertia orbifolds and twisted sectors Let $X := [M/G]$ be an almost complex orbifold where M is an almost complex manifold with a holomorphic G-action for a finite group G. Denote by $X_{(g)}$ the global quotient orbifold $[M^g/C(g)]$ where $C(g)$ is the centralizer of $g \in G$. Then each $X_{(g)}$ naturally inherits an almost complex structure from the one on M. The inertia orbifold of X is defined as $IX = \coprod_{(g)} X_{(g)}$ where the disjoint union is over conjugate classes (g) in G. The group $H^*_{orb}(X, \mathbb{Q})$ is freely generated by the elements of the cohomology groups of twisted sectors of X :

$$H^*_{orb}(X, \mathbb{Q}) := H^*(IX, \mathbb{Q}) = \oplus_{(g)} H^*(X_{(g)}, \mathbb{Q})$$

as a *vector space*. Here, $H^*(X_{(g)}, \mathbb{Q})$ is isomorphic to $H^*(|X_{(g)}|, \mathbb{Q})$ and the degrees of elements in $H^*(X_{(g)}, \mathbb{Q})$ are shifted by $2\iota(g)$ where $\iota(g)$ is the "age" of the element g. To define the age of g, we first choose a generic point $x \in M^g$. Note that the cycle group $\langle g \rangle$ generated by g acts on the tangent space T_xM. Decomposing it into eigenspaces, we can find a representation of $\rho : \langle g \rangle \to T_xM \cong \mathbb{C}^n$ $(n = dim_{\mathbb{C}} M)$:

$$\rho(g) = diag\left(e^{\frac{2\pi\sqrt{-1}m_1}{m}}, \cdots, e^{\frac{2\pi\sqrt{-1}m_n}{m}}\right)$$

for some integers $0 \leq m_j < m := |\langle g \rangle|$ $(j = 1, \cdots, n)$. The age of $X_{(g)}$ is defined to be $\sum_j \frac{m_j}{m} \in \mathbb{Q}$.

$H^*_{orb}(X,\mathbb{Q})$ also admits a natural Poincarè pairing which is compatible with these shifted degrees:

$$\int_{IX}^{orb}(-)\cup_{orb}(-): H^*_{orb}(X,\mathbb{Q})\otimes H^*_{orb}(X,\mathbb{Q})\to\mathbb{Q} \tag{2.5.8}$$

defined by $\alpha\otimes\beta\mapsto\int_{IX}^{orb}\alpha\cup_{orb}I^*\beta$ for the natural involution map $I: X_{(g)}\to X_{(g^{-1})}$ induced from the identity map.

Desingularization of orbi-bundles In order to understand the degree shifting in orbifold cohomology, we briefly recall the desingularization of orbi-bundles and orbifold index theorem from [CR1]. Let $(\Sigma,\boldsymbol{z},\boldsymbol{m})$ be an orbi-Riemann surface and consider a complex orbi-bundle $E\to(\Sigma,\boldsymbol{z},\boldsymbol{m})$. The desingularization of E is defined as follows :

For each disc neighborhood D_i of orbi-singular points z_i in $\boldsymbol{z}$, E can be uniformized by $(D_i\times\mathbb{C}^n,\mathbb{Z}_{m_i},\pi)$ so that the action is linear and diagonal. Hence the action can be written as

$$e^{\frac{2\pi\sqrt{-1}}{m_i}}\cdot(z,f)=\left(e^{\frac{2\pi\sqrt{-1}}{m_i}}z,\operatorname{diag}\left(e^{\frac{2\pi\sqrt{-1}m_{i,1}}{m_i}},\cdots,e^{\frac{2\pi\sqrt{-1}m_{i,n}}{m_i}}\right)f\right) \tag{2.5.9}$$

for some integers $0\le m_{i,j}<m_i$ $(j=1,\cdots,n)$. Let $\Phi_i: D^*\times\mathbb{C}^n\to D^*\times\mathbb{C}^n$ be a $\mathbb{Z}_{m_i}$-equivariant map over the punctured disc $D^*=D_i-\{0\}$ defined by

$$\Phi_i(z,f_1,\cdots,f_n)=(z^{m_i},z^{-m_{i,1}}f_1,\cdots,z^{-m_{i,n}}f_n) \tag{2.5.10}$$

where the $\mathbb{Z}_{m_i}$ trivially acts on the right side. Consider the natural map $\phi:(\Sigma,\boldsymbol{z},\boldsymbol{m})\to|\Sigma|$ which can be written as $z\to z^{m_i}$ over each D_i. Then the local holomorphic chart on $|\Sigma|$ is $w=z^{m_i}$ over each $\phi(D_i)$. We construct a complex vector bundle $|E|$ over the underlying space $|\Sigma|$ of $(\Sigma,\boldsymbol{z},\boldsymbol{m})$ by extending the complex vector bundle over $|\Sigma|-\{z_1,\cdots z_k\}$ whose trivialization is given by the right hand side of (2.5.10).

Chen and Ruan observed that the first Chern number of an orbi-bundle is the sum of the first Chern number of its desingularization and the *age*s of representations which are induced from local trivialization of the orbi-bundle over orbi-singular points. More precisely, let E be an orbi-bundle over a closed orbi-Riemann surface $(\Sigma,\boldsymbol{z},\boldsymbol{m})$ and set $\boldsymbol{m}=(m_1,\cdots,m_k)$ for $m_i\in\mathbb{Z}_{\ge 0}$. Then for each orbi-singular points z_i, the induced representation $\rho_i : \mathbb{Z}_{m_i} \to \mathrm{End}(\mathbb{C}^n)$ can be written as

$$\rho_i\left(e^{\frac{2\pi\sqrt{-1}}{m_i}}\right) = \mathrm{diag}\left(e^{\frac{2\pi\sqrt{-1}m_{i,1}}{m_i}}, \cdots, e^{\frac{2\pi\sqrt{-1}m_{i,n}}{m_i}}\right)$$

for some integers $0 \le m_{i,j} < m_i$ ($j = 1, \cdots, n$). Using Chern-Weil definition of the Chern class, Chen and Ruan proved following formula:

Proposition 2.5.1. *The Chern number of orbi-bundle and that of its de-singularization satisfies (Proposition 4.2.1 [CR1])*

$$c_1(E)([\Sigma]) = c_1(|E|)([\Sigma]) + \sum_{i=1}^{k}\sum_{j=1}^{n}\frac{m_{i,j}}{m_i}. \tag{2.5.11}$$

For a holomorphic orbi-bundle $E \to (\Sigma, \boldsymbol{z}, \boldsymbol{m})$, let us denote sheaves of holomorphic sections of E and $|E|$ over $(\Sigma, \boldsymbol{z}, \boldsymbol{m})$ and Σ by $\mathcal{O}(E)$ and $\mathcal{O}(|E|)$, respectively. Then we have $\mathcal{O}(E) = \mathcal{O}(|E|)$ [CR1, Proposition 4.2.2] from the removability of isolated singularities of J-holomorphic maps. In detail, if $g : D_i \to \mathbb{C}^n$ is a local holomorphic section over $|E|\big|_{D_i} \to D_i$, then $g(w) = (g_1(w), \cdots, g_n(w))$ for some holomorphic maps $g_i : D_i \to \mathbb{C}$ with respect to the trivialization taken as above. If we pullback this section via Φ_i, then the corresponding section on $E|_{D_i} \to D_i$ is the holomorphic map $f : D_i \to \mathbb{C}^n$ whose components are $f_j(z) = z^{m_{i,j}} g_j(z^{m_i})$ for each $j = 1, \cdots, n$.

Conversely, let $f = (f_1, \cdots f_n) : D_i \to \mathbb{C}^n$ be a given local holomorphic section on orbi-bundle $E|_{D_i} \to D_i$, i.e., f is a $\mathbb{Z}_{m_i}$-equivariant holomorphic section. Define a map $g : D^* \to \mathbb{C}^n$ whose components are $g_j(w) := z^{-m_{i,j}} f_j(z)$ for $z = w^{\frac{1}{m_i}}$. Note that the $\mathbb{Z}_{m_i}$-equivariantness of f says that the section g is well-defined, although there is an ambiguity on the choice of brach cut for $z = w^{\frac{1}{m_i}}$. Moreover, if we expand the holomorphic function f_j as

$$f_j(z) = \sum_{n=0}^{\infty} a_{n,j} z^n,$$

then $a_{n,j} = 0$ unless $n \equiv m_{i,j}$ modulo m_i. Thus, the $\left|\frac{f_j(z)}{z^{m_{i,j}}}\right|$ is bounded on D_i, and we conclude that g can be extended to a holomorphic section over D_i using the Riemann extension theorem. One can easily check that this process is the inverse of the pullback via Φ_i, which gives $\mathcal{O}(E) = \mathcal{O}(|E|)$.

From this observation, we recall the Chen-Ruan's index formula:

Proposition 2.5.2. *Let E be a complex orbifold bundle of rank n over a complex orbi-Riemann surface $(\Sigma, \boldsymbol{z}, \boldsymbol{m})$ of genus g. Then*

$$\mathcal{X}(\mathcal{O}(E)) = \mathcal{X}(\mathcal{O}(|E|)) = c_1(|E|)([\Sigma]) + n(1-g)$$
$$= n(1-g) + c_1(E)([\Sigma]) + \sum_{i=1}^{k}\sum_{j=1}^{n}\frac{m_{i,j}}{m_i}.$$

2.5.3 Gromov-Witten invariants of orbifolds

We fix a $\mathbb{Q}$-basis $\{\gamma_i\}_{i=1,\cdots,N}$ of $H^*_{orb}(X,\mathbb{Q})$. Then the k-fold Gromov-Witten invariants is defined by the following equation:

$$\langle \gamma_1, \cdots, \gamma_k\rangle^X_{g,k,\beta} := \int_{[\overline{\mathcal{M}}_{0,k,\beta}(X)]^{vir}} ev_1^*\gamma_1 \wedge \cdots \wedge ev_k^*\gamma_k. \tag{2.5.12}$$

We also define $\langle \gamma_1, \cdots, \gamma_k\rangle^X_{g,k}$ to be the weighted sum $\sum_\beta \langle \gamma_1, \cdots, \gamma_k\rangle^X_{g,k,\beta} q^{\omega(\beta)}$.

Remark 2.5.3. *The compactified moduli space $\overline{\mathcal{M}}_{0,k,\beta}(X)$ admits a virtual fundamental class $[\overline{\mathcal{M}}_{0,k,\beta}(X)]^{vir}$ which can be defined with help of an abstract perturbation technique in general. (Readers are referred to [CR2] for more details.)*

For a tuple $\boldsymbol{x} = (X_{(g_1)}, X_{(g_2)}, \cdots, X_{(g_k)})$ of twisted sectors, we say that $(f, (\Sigma, \boldsymbol{z}), \xi)$ is of type $\boldsymbol{x}$, if orbi-insertions at the marked point z_i lies in $H^*(X_{(g_i)}, \mathbb{Q})$ for each i. Let $\overline{\mathcal{M}}_{g,k,\beta}(X, J, \beta, \boldsymbol{x})$ denote the moduli space of equivalence classes of orbifold stable maps of genus g with k marked points and of homology class β and type $\boldsymbol{x}$. Then $\overline{\mathcal{M}}_{0,k,\beta}(X)$ is the union of $\overline{\mathcal{M}}_{g,k,\beta}(X, J, \beta, \boldsymbol{x})$ over all types $\boldsymbol{x}$, and the integration in (2.5.12) is nonzero on components $\overline{\mathcal{M}}_{g,k,\beta}(X, J, \beta, \boldsymbol{x})$ with $\gamma_i \in H^*(X_{(g_i)}, \mathbb{Q})$.

For later purpose, we give the virtual dimension of $\overline{\mathcal{M}}_{g,k,\beta}(X, J, \beta, \boldsymbol{x})$ explicitly:

$$2c_1(TX)(\beta) + 2(n-3)(1-g) + 2k - 2\iota(\boldsymbol{x}), \tag{2.5.13}$$

where $2n = \dim_{\mathbb{R}} X$ and $\iota(\boldsymbol{x}) = \sum_{i=1}^k \iota(g_i)$. (See [CR2, Proposition 3.2.5].)

Remark 2.5.4. *If c_1 of X vanishes as in our elliptic examples, the virtual dimension of the moduli is independent of the homology class β. In particular, $n = 1$ and $g = 0$ in our main examples.*

If we set $\boldsymbol{t} := \sum t_i \gamma_i$, then the generating function for the Gromov-Witten invariants is defined as

$$F_0^X(\boldsymbol{t}) := \sum_{k,\beta} \frac{1}{k!} \langle \boldsymbol{t}, \cdots, \boldsymbol{t} \rangle_{0,k,\beta}^X q^{\omega(\beta)},$$

which we will call the genus-0 Gromov-Witten potential for X.

In particular when $k = 3$, the counting given in (2.5.12) defines a product $*$ on $H^*_{orb}(X, \Lambda)$ where Λ is the Novikov ring over $\mathbb{Q}$, which is called the quantum product. More precisely,

$$\int_X^{orb} (\gamma_i * \gamma_j) \cup_{orb} \gamma_l := \langle \gamma_i, \gamma_j, \gamma_l \rangle_{0,3}^X,$$

or equivalently,

$$\gamma_i * \gamma_j := \sum_{l=1}^{N} \sum_{\beta} \langle \gamma_i, \gamma_j, \gamma_l \rangle_{0,3,\beta}^X PD(\gamma_l) q^{\omega(\beta)}$$

where $PD(-)$ denotes the Poincarè dual with respect to the pairing (2.5.8). Therefore, (2.5.12) with $n = 3$ gives structure constants of this product. The associativity of $*$ is proved in [CR2]. We remark that what we have defined is the small quantum cohomology of X while the big quantum cohomology involves the full Gromov-Witten invariants.

2.5.4 Elliptic orbifolds $\mathbb{P}^1_{a,b,c}$ and review on Satake-Takahashi's work

We now focus on elliptic orbifolds with three cone points $\mathbb{P}^1_{a,b,c}$ and its Gromov-Witten potential. $\mathbb{P}^1_{a,b,c}$ is elliptic if and only if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, and hence there are three elliptic orbifold projective lines $\mathbb{P}^1_{a,b,c}$ where (a, b, c) are $(3, 3, 3)$, $(2, 3, 6)$ and $(2, 4, 4)$. They are called elliptic since these orbifolds can be described as a global quotient of an elliptic curve E by a finite cyclic group G.

We first fix the notation for generators of their orbifold cohomology rings in the following way: Let w_1, w_2, and w_3 be the three cone points $\mathbb{P}^1_{a,b,c}$ with isotropy groups $\mathbb{Z}_a$, $\mathbb{Z}_b$, and $\mathbb{Z}_c$, respectively. We fix a choice of a $\mathbb{Q}$-basis of $H^*_{orb}(\mathbb{P}^1_{a,b,c}, \mathbb{Q})$, which is fairly standard. The $\mathbb{Q}$-basis

$$\left\{1, \Delta_1^{1/a}, \cdots, \Delta_1^{(a-1)/a}, \Delta_2^{1/b}, \cdots, \Delta_2^{(b-1)/b}, \Delta_3^{1/c}, \cdots, \Delta_3^{(c-1)/c}, [\mathrm{pt}]\right\} \tag{2.5.14}$$

of $H^*_{orb}(\mathbb{P}^1_{a,b,c}, \mathbb{Q})$ is defined by the following conditions. The basis of smooth sector are

$$H^0_{orb}(\mathbb{P}^1_{a,b,c}, \mathbb{Q}) = \mathbb{Q}\langle 1\rangle, \qquad H^2_{orb}(\mathbb{P}^1_{a,b,c}, \mathbb{Q}) = \mathbb{Q}\langle [\mathrm{pt}]\rangle.$$

For twist sectors, let $\Delta_1^{j/a} \in H^{\frac{2j}{a}}_{orb}(\mathbb{P}^1_{a,b,c}, \mathbb{Q})$, $\Delta_2^{j/b} \in H^{\frac{2j}{b}}_{orb}(\mathbb{P}^1_{a,b,c}, \mathbb{Q})$, and $\Delta_3^{j/c} \in H^{\frac{2j}{c}}_{orb}(\mathbb{P}^1_{a,b,c}, \mathbb{Q})$ which are supported at singular points w_1, w_2, and w_3, respectively. For $\boldsymbol{t} := \sum t_{j,i}\Delta^i_j$,

Orbifold cup products with respect to these basis are given as follows.

$$\int_X^{orb} \Delta_1^{j/a} \cup \Delta_1^{k/a} = \frac{1}{a}\delta_{j+k-a,0} \qquad \int_X^{orb} \Delta_1^{j/b} \cup \Delta_1^{k/b} = \frac{1}{b}\delta_{j+k-b,0},$$

$$\int_X^{orb} \Delta_1^{j/c} \cup \Delta_1^{k/c} = \frac{1}{a}\delta_{j+k-c,0} \qquad \int_X^{orb} 1 \cup [pt] = 1$$

where $\delta_{i,j}$ is 1 if $i = j$ and zero otherwise. The last cup product does not have any fraction since both 1 and $[pt]$ live in smooth(untwisted) sector of IX.

Remark 2.5.5. *Readers are hereby warned that the Poincarè dual $PD(\Delta_1^{j/a})$ of $\Delta_1^{j/a}$ is not $\Delta_1^{(a-j)/a}$, but $a \times \Delta_1^{(a-j)/a}$, and the same happens for b and c. However,* 1 *and* $[pt]$ *are still Poincarè dual to each other.*

In the remaining part, we briefly recall the work of Satake and Takahashi [ST] on $\mathbb{P}^1_{3,3,3}$. We first give a description of $\mathbb{P}^1_{3,3,3}$ as a quotient of an elliptic curve. Let E be the elliptic curve associated with the lattice $\mathbb{Z}\langle 1, \tau\rangle$ in $\mathbb{C}$ where $\tau = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$. Then the $\mathbb{Z}_3$-action on $\mathbb{C}$ generated by $2\pi/3$-rotation descends to E since this action preserves the lattice $\mathbb{Z}\langle 1, \tau\rangle$. By taking quotients of E via the induced $\mathbb{Z}_3$-action, we finally obtain the global quotient orbifold $\mathbb{P}^1_{3,3,3} = [E/\mathbb{Z}_3]$. (The shaded region in (a) of Figure 2.4 represents a fundamental domain of the $\mathbb{Z}_3$-action on E.) Since each fixed point in E has the isotropy group isomorphic to $\mathbb{Z}_3$, $\mathbb{P}^1_{3,3,3}$ has three cone points each of which has $\mathbb{Z}_3$-singularity. We denote these singular points by w_1, w_2 and w_3, respectively. Therefore, the inertia orbifold $I\mathbb{P}^1_{3,3,3}$ consists of the trivial sector together with three $B\mathbb{Z}_3$(equivalent to $[\mathbb{Z}_3 \ltimes \{*\}]$)'s which are associated with the point w_i's.

Remark 2.5.6. *Consider the universal covering $\mathbb{C} \to E$ of the elliptic curve. The composition $p : \mathbb{C} \to E \to \mathbb{P}^1_{3,3,3}$ as well as the quotient map $E \to \mathbb{P}^1_{3,3,3}$ is*

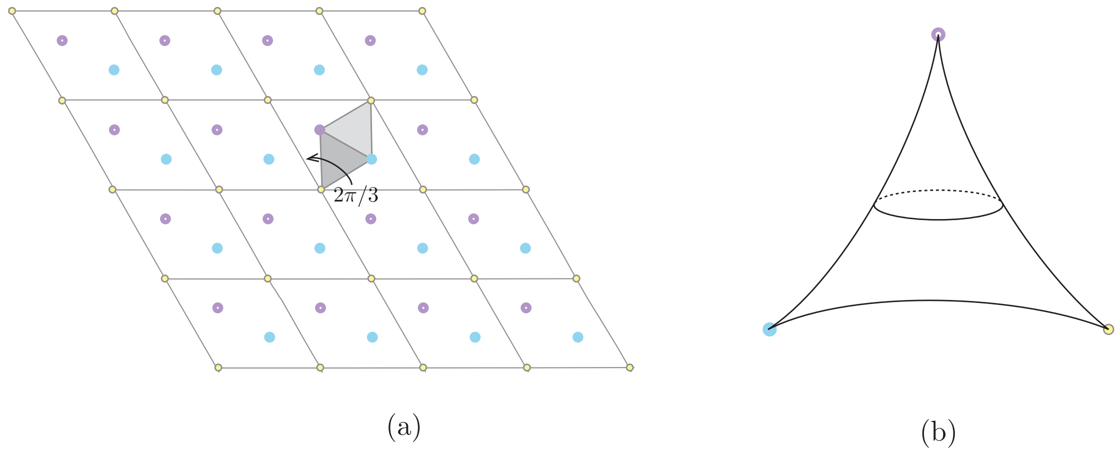


Figure 2.4: (a) The $\mathbb{Z}_3$-action on E and (b) its quotient

a holomorphic orbifold covering map in the sense of [T] (see Definition 2.4.1, also). Indeed, p is the orbifold universal cover. We will use this fact crucially to classify holomorphic orbi-spheres in $\mathbb{P}^1_{3,3,3}$.

Following the notation in (2.5.14), the $\mathbb{Q}$-basis of $H^*_{orb}(\mathbb{P}^1_{3,3,3},\mathbb{Q})$ is given by

$$H^0_{orb}(\mathbb{P}^1_{3,3,3},\mathbb{Q}) = \mathbb{Q}\langle 1\rangle, \qquad H^2_{orb}(\mathbb{P}^1_{3,3,3},\mathbb{Q}) = \mathbb{Q}\langle [\mathrm{pt}]\rangle,$$
$$H^{\frac{2}{3}}_{orb}(\mathbb{P}^1_{3,3,3},\mathbb{Q}) = \mathbb{Q}\langle \Delta_1^{1/3},\Delta_2^{1/3},\Delta_3^{1/3}\rangle, \qquad H^{\frac{4}{3}}_{orb}(\mathbb{P}^1_{3,3,3},\mathbb{Q}) = \mathbb{Q}\langle \Delta_1^{2/3},\Delta_2^{2/3},\Delta_3^{2/3}\rangle,$$

and the Poincarè pairing for Δ^i_j's is determined by

$$\int^{orb}_{\mathbb{P}^1_{3,3,3}} \Delta^i_j \cup_{orb} \Delta^k_l = \begin{cases} \frac{1}{3}, & \text{if } i+k=1 \text{ and } j=l, \\ 0, & \text{otherwise.} \end{cases}$$

Satake and Takahashi [ST, Theorem 3.1] calculated the genus 0-Gromov-Witten potential of $\mathbb{P}^1_{3,3,3}$ and the quantum product term can be written as

$$\begin{aligned} f_0(q) &:= \textstyle\sum_{d\geq 0}\langle \Delta_1^{1/3},\Delta_2^{1/3},\Delta_3^{1/3}\rangle_{0,3,d}\, q^d = \frac{\eta(q^9)^3}{\eta(q^3)}, \\ f_1(q) &:= \textstyle\sum_{d\geq 0}\langle \Delta_1^{1/3},\Delta_1^{1/3},\Delta_1^{1/3}\rangle_{0,3,d}\, q^d = \left(1+\frac{1}{3}\left(\frac{\eta(q)}{\eta(q^9)}\right)^3\right) f_0(q). \end{aligned} \tag{2.5.15}$$

Here, $\eta(q)$ is the Dedekind's eta function

$$\eta(\tau) := \exp\left(\frac{\pi\sqrt{-1}\tau}{2}\right)\prod_{n=1}^{\infty}(1-q^n), \quad q = \exp(2\pi\sqrt{-1}\tau)$$

for $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$.

Write $f_0(q) = \sum_{N\geq 1} a_N q^N$. The Fourier coefficients a_N of f_0 depends on the prime factorization of N (or more precisely the quadratic reciprocity of N), and is given by

$$a_N = \begin{cases} 0 & n > 0, \text{ or one of } m_j \text{ is odd} \\ (n_1+1)\cdots(n_k+1) & \text{otherwise} \end{cases}$$

for $N = 3^n p_1^{n_1} \cdots p_k^{n_k} q_1^{m_1} \cdots q_l^{m_l}$ where p_i is a prime number with $p_i \equiv 1 (\mod 3)$, and q_i is a prime number with $q_i \equiv 2 \pmod 3$. (See [S].) The Fourier coefficients of f_1 also has a similar description which we will give in Section 5.5.

We also provide first few terms of f_0 and f_1 for readers to get an impression:

$$f_0 = q + q^4 + 2q^7 + 2q^{13} + q^{16} + 2q^{19} + O(q^{24}),$$

$$f_1 = \frac{1}{3} + 2q^3 + 2q^9 + 2q^{12} + 4q^{21} + O(q^{24}).$$

Chapter 3

Maslov index via Chern-Weil theory and its orbifold analogue[1]

In this chapter, we give Chern-Weil definitions of the Maslov indices of bundle pairs over a Riemann surface Σ with boundary, which consists of symplectic vector bundle on Σ and a Lagrangian subbundle on $\partial\Sigma$ as well as its generalization for transversely intersecting Lagrangian boundary conditions. We discuss their properties and relations to the known topological definitions. As a main application, we extend Maslov index to the case with orbifold interior singularites, via curvature integral, and find also an analogous topological definition in these cases.

3.1 Maslov index via orthogonal connection

In this section, we define an L-orthogonal unitatry connection(c.f. [V]) of a bundle pair to give a Chern-Weil definition of its Maslov index.

We recall the well-known definition of $c_1(E)$ of a complex line bundle E via curvature integral.

3.1.1 Chern-Weil definition of the first Chern class

Let ∇ be a connection of a complex line bundle E over a closed surface Σ, and denote by F_∇ its curvature. The following theorem is well-known

[1]This chapter is based on [CS].

Theorem 3.1.1. *The curvature F_∇ satisfies the following:*

1. $dF_\nabla = 0$
2. *If ∇ and ∇' are two connections of E, then $\nabla = \nabla' + \eta$ for a 1-form η and $F_\nabla = F_{\nabla'} + d\eta$*
3. *The first Chern number $c_1(E)([\Sigma])$ is given by*

$$c_1(E)([\Sigma]) = \frac{\sqrt{-1}}{2\pi} \int_\Sigma F_\nabla, \tag{3.1.1}$$

and it is independent of the choice of a connection.

In fact, η is a $End(E)$-valued 1-form. However, since $End(E)$ is a trivial line bundle, we can consider the η as a 1-form on Σ after fixing a trivialization of $End(E)$. Note that the difference of curvature integrals for two connections ∇ and ∇' is

$$\int_\Sigma F_\nabla - \int_\Sigma F_{\nabla'} = \int_\Sigma d\eta = \int_{\partial\Sigma} \eta = 0.$$

But for the case with Riemann surfaces with boundary, and the invariance of the curvature integrals does not hold for arbitrary connections since $\partial\Sigma \neq 0$.

To obtain an invariant curvature integrals for the case with boundaries, we introduce the notion of an orthogonal connection.

3.1.2 Orthogonal connection for a bundle pair

We first recall a definition of bundle pair. Let Σ be a Riemann surface with boundary $\partial\Sigma$.

Definition 3.1.1. *We denote by a bundle pair $(E, L) \to (\Sigma, \partial\Sigma)$, a symplectic vector bundle (E, ω_E) over Σ, a Lagrangian subbundle L over $\partial\Sigma$.*

We also consider a compatible complex structures J of (E, L) which makes E a complex vector bundle with an induced inner product $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$. Denote by $g_\mathbb{C} = g + \sqrt{-1}\omega$ the induced hermitian inner product of E. A connection ∇ is said to be *unitary* if ∇ is compatible with the metric $g_\mathbb{C}$, or equivalently, the holonomy of ∇ lies in $U(n)$.

Definition 3.1.2. *Let ∇ be a unitary connection on $E \to \Sigma$. Then, ∇ is called L-orthogonal if the parallel transport along $\partial\Sigma$ via ∇ preserves Lagrangian subbundle $L \to \partial\Sigma$.*

To construct such an L-orthogonal connection of a bundle pair $(E, L) \to (\Sigma, \partial\Sigma)$, we can proceed as follows. Given a g-orthogonal metric connection ∇ for L, by defining $\nabla Je = J\nabla e$ for any local section e of L, we can extend the connection to $E \to \partial\Sigma$, and by trivially extending to the neighborhood of $\partial\Sigma$ and using the partition of unity, we can extend it to a unitary connection to $E \to \Sigma$.

Remark 3.1.2. *The name, orthogonal connection is given following the work of Vaisman, who considered orthogonal unitary connection in [V]. He considered principal $Sp(2n)$-bundles and related principle $U(n)$ subbundle (by choosing an complex structure), and a unitary connection preserving certain principal $O(n)$-subbundle (which is defined from a real unitary frame of Lagrangian subbundle) was called orthogonal unitary connection. He considered the case that the Lagrangian subbundle is defined everywhere(not just on the boundary) and hence, the Maslov index vanishes in these cases. He used orthogonal unitary connections and Chern-Weil theory to study secondary invariants.*

Note that we only require orthogonality along $\partial\Sigma$. Another natural (and more restrictive) assumption would be to take a tubular neighborhood of $\partial\Sigma$ and require the connection to be a product form, i.e. it is a pullback of the orthogonal connection on $\partial\Sigma$ along normal direction. But we remark that the resulting curvature integral will be the same, which can be proved as in the proof of the proposition 3.3.1

Example 3.1.3. *Consider a Lagrangian submanifold $L = S^1 \subset \mathbb{C}$, where $\mathbb{C}$ is equipped with standard symplectic structure $\omega_0 = dx \wedge dy$. Consider inclusion of a unit disc $u : D^2 \subset \mathbb{C}$ (so that $u(\partial D^2) = L$).*

Consider the pull-back bundle $u^*T\mathbb{C} \cong D^2 \times \mathbb{C}$ and its Lagrangian subbundle $u|_{\partial D^2}^* TL$. Consider the trivialization of $u^*T\mathbb{C}$ as above and denote by (r, θ) the polar coordinate of D^2. For the complex frame $\{\epsilon(r, \theta) := \frac{\partial}{\partial x}\big|_{(r,\theta)}\}$ of $u^*T\mathbb{C}$, we define a connection

$$\nabla := d - \sqrt{-1}rd\theta,$$

In this trivialization, $\sqrt{-1}e^{\sqrt{-1}\theta}\epsilon(1,\theta)$ *defines a real frame of* L *and one can check that*

$$\nabla_{\frac{\partial}{\partial\theta}}\sqrt{-1}e^{\sqrt{-1}\theta}\epsilon(1,\theta) = ((d-\sqrt{-1}rd\theta)\sqrt{-1}e^{\sqrt{-1}\theta})(\frac{\partial}{\partial\theta})\epsilon(1,\theta) = 0.$$

Therefore the connection ∇ *is an* L*-orthogonal unitary connection.*

3.1.3 Maslov index

Recall the definition of a Maslov index for a bundle pair $(E, L) \to \Sigma$, where E is a symplectic vector bundle over Σ, and L is a Lagrangian subbundle over $\partial\Sigma$. Let J be a compatible complex structure on E, and consider E as a complex vector bundle.

Recall the following well-known lemma.

Lemma 3.1.4 ([Oh])**.** *Consider the subset*

$$\tilde{\mathcal{U}}(n) = \{A \in U(n,\mathbb{C}) \mid A = A^t\}.$$

Then the map

$$B : \Lambda(n) = U(n)/O(n) \to \tilde{\mathcal{U}}(n); A \mapsto A\bar{A}^{-1}$$

is a diffeomorphism.

The Maslov index $\mu(\gamma)$ of an oriented loop $\gamma : S^1 \to \Lambda(n)$ is defined to be the winding number of

$$\det \circ B \circ \gamma : S^1 \to \mathbb{C}\backslash\{0\}.$$

Now given a bundle pair $(E, L) \to \Sigma$, if $\partial\Sigma \neq \emptyset$, then vector bundle $E \to \Sigma$ can be trivialized. We fix a symplectic trivialization $\Phi : E \cong \Sigma \times \mathbb{C}^n$, and let $R_1, ..., R_h$ be the connected components of $\partial\Sigma$, with orientation induced by the orientation Σ. Then $\Phi(L|_{R_i})$ gives a loop $\gamma_i : S^1 \to \Lambda(n)$. Let us denote $\mu(\Phi, R_i) := \mu(\gamma_i)$.

Definition 3.1.3. *The Maslov index of the bundle pair* (E, L) *is defined by*

$$\mu(E, L) = \sum_{i=1}^{h} \mu(\Phi, R_i)$$

where $\Phi : E \to \Sigma \times \mathbb{C}^n$ *is any trivialization.*

The Maslov index is independent of the choice of trivialization Φ, and the choice of an complex structure J. (see [KL] for example.)

3.1.4 Chern-Weil Maslov index

The main objective of this chapter is to give another definition of the Maslov index μ_{CW} for the bundle pair $(E, L) \to \Sigma$ in terms of curvature integral:

Definition 3.1.4. *Let ∇ be a connection on E which restricts, on the boundary of Σ, to an L-orthogonal unitary connection on $(E|_{\partial\Sigma}, J)$. The Maslov index of the bundle pair (E, L) is defined by*

$$\mu_{CW}(E, L) = \frac{\sqrt{-1}}{\pi} \int_{\Sigma} tr(F_{\nabla})$$

where $F_{\nabla} \in \Omega^2(\Sigma, End(E))$ is the curvature induced by ∇.

Remark 3.1.5. *Note that the denominator of* (3.1.1) *is 2π.*

We consider the example 4.1.1.

Example 3.1.6. *For the connection ∇ defined in example 4.1.1, we have*

$$F_{\nabla} = d(-\sqrt{-1} r d\theta) = -\sqrt{-1} dr \wedge d\theta$$

Hence,

$$\frac{\sqrt{-1}}{\pi} \int_{D^2} tr(F_{\nabla}) = 2.$$

This shows that $\mu_{CW} = 2$ and it is equal to the topological Maslov index.

In the following section, we prove that $\mu_{CW}(E, L)$ is independent of the choice of the orthogonal connection and equal the topological Maslov index $\mu(E, L)$.

3.2 Equivalence of two Maslov indices

We will give two proofs of equivalence of two Maslov indices.

Theorem 3.2.1. *Given a bundle pair (E, L), topological Maslov index equals Chern-Weil Maslov index:*

$$\mu(E, L) = \mu_{CW}(E, L).$$

The first proof in subsection 3.1 is easier, but the second proof in subsection 3.2 using doubling construction can be extended to the case of orbifolds, and will be used in a later section.

3.2.1 First proof of $\mu = \mu_{CW}$

Proof. Consider a bundle pair (E, L) with orthogonal connection ∇. We fix an complex structure J of E and regard E as a complex vector bundle. Consider $\Lambda^n E$ the top exterior bundle of E, with an induced connection $\widetilde{\nabla}$. We have a trivialization $\Phi : E \to \Sigma \times \mathbb{C}^n$ as a complex vector bundle since $\partial\Sigma \neq 0$. With respect to the standard frame $\{\epsilon_1, \cdots, \epsilon_n\}$ of $\Sigma \times \mathbb{C}^n$, we can write $\nabla = d + A$ for some $n \times n$-matrix-valued one form $A = (a_{ij})$. Then it is easy to see that $\widetilde{\nabla} = d + tr(A)$ with respect to the frame $\{\epsilon := \epsilon_1 \wedge \cdots \wedge \epsilon_n\}$.

Recall that the curvature of ∇ and $\widetilde{\nabla}$ is given as

$$F_\nabla = dA + A \wedge A, \;\; F_{\widetilde{\nabla}} = d(tr(A)).$$

Lemma 3.2.2.

$$\int_\Sigma tr(F_\nabla) = \int_\Sigma tr(F_{\widetilde{\nabla}}).$$

Proof.

$$tr(A \wedge A) = \sum_{i,j} a_{ij} \wedge a_{ji} = \sum_i a_{ii} \wedge a_{ii} = 0.$$

Second equality follows from cancelation of $a_{ij} \wedge a_{ji}$ with $a_{ji} \wedge a_{ij} = -a_{ij} \wedge a_{ji}$ for $i \neq j$. □

Now, we recall the standard relation between holonomy and curvature integral for line bundles. Given a complex line bundle $\mathcal{L}$ over a manifold M and a connection ∇' on $\mathcal{L}$, the holonomy along a contractible loop γ (which is bounded by the 2-dimensional contractible domain $D \subset M$) is given by

$$Hol_\gamma(\mathcal{L}, \nabla') = exp(-\int_D F_{\nabla'}).$$

Note that if $\nabla' = d + A$ is a unitary connection on D, then A satisfies $A = -\overline{A}^t$. Therefore for a complex line bundle, A is the purely imaginary connection 1-form. Since $\xi(t) = e^{-\int_0^t A(\dot{\gamma}(s))ds}\xi(0)$ is a parallel transformation of $\xi(0)$, the integral $-\frac{1}{\sqrt{-1}}\int_D F_{\nabla'} = \sqrt{-1}\int_D F_{\nabla'}$ gives the rotation angle of a parallel section along γ. Note that $\int_\gamma A = \int_D F_{\nabla'}$ by Stokes' theorem in this case.

In general, the above relation for D extends to the case of Riemann surface Σ with boundary $\partial\Sigma$. Namely, the integral $\sqrt{-1}\int_\Sigma F_{\nabla'}$ gives the sum of rotation angles of parallel sections along boundaries $\partial\Sigma$ with the induced orientations from Σ.

Now we apply this to $\Lambda^n E$ and $\widetilde{\nabla}$. Note that, by the definition of L-orthogonal unitary connection, parallel transformation in E preserves frame vectors of L. More precisely, if $\{e_1(t), \cdots, e_n(t)\}$ is a horizontal sections of E along $\gamma(t) \subset \partial\Sigma$ and $\{e_1(t_0), \cdots, e_n(t_0)\}$ is a real orthogonal frame of L_{t_0} at some moment t_0, then $\{e_1(t), \cdots, e_n(t)\}$ would be a real orthogonal frame of L_t for all t.

Define a matrix $u(t) := (e_1(t), \cdots, e_n(t)) \in U(n)$ using the frame as column vectors. Then, we have (for the standard frame $\{\epsilon_1, \cdots, \epsilon_n\}$)

$$e_1(t) \wedge \cdots \wedge e_n(t) = det(u(t))\epsilon_1 \wedge \cdots \wedge \epsilon_n.$$

So $det(u(t))$ is a frame of the Lagrangian subbundle $\Lambda^n L \subset \Lambda^n E$. In the trivialization $det(\Phi) : \Lambda^n E \to \Sigma \times \mathbb{C}$, we have $det(u(t)) \in U(1)$.

Observe that the $det(u(t))$ gives a horizontal section. Hence the $\sqrt{-1}\int_\Sigma F_{\widetilde{\nabla}}$ measures the rotating angle of $det(u(t))$ in $U(1)$.

As topological Maslov index μ corresponds to the rotation number of $det^2(u(t))$ in $U(1)$, hence it is equal to

$$\frac{\sqrt{-1}}{\pi}\int_\Sigma F_{\widetilde{\nabla}}.$$

□

3.2.2 Second proof of $\mu = \mu_{CW}$

We will use the doubling construction (and the equivalence between the topological and Chern-Weil definition of the first Chern class).

To explain the doubling construction, we recall the following well-known theorem (see [AG]).

Theorem 3.2.3. *Let Σ be a bordered Riemann surface. There exists a double cover $\pi : \Sigma_{\mathbb{C}} \to \Sigma$ of Σ by a compact Riemann surface $\Sigma_{\mathbb{C}}$ and an antiholomorphic involution $\sigma : \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ such that $\pi \circ \sigma = \pi$. There is a holomorphic embedding $i : \Sigma \to \Sigma_{\mathbb{C}}$ such that $\pi \circ i$ is the identity map. The triple $(\Sigma_{\mathbb{C}}, \pi, \sigma)$ is unique up to isomorphism.*

Definition 3.2.1. *We call the triple* $(\Sigma_{\mathbb{C}}, \pi, \sigma)$ *in Theorem 3.2.3 the* complex double *of* Σ*, and* $\overline{\Sigma} = \sigma(i(\Sigma))$ *the* complex conjugate *of* Σ.

We recall the following theorem 3.3.8 of [KL] and its proof for reader's convenience.

Theorem 3.2.4. *Let* (E, L) *be a bundle pair over a bordered Riemann surface* Σ*. Then there is a complex vector bundle* $E_{\mathbb{C}}$ *on* $\Sigma_{\mathbb{C}}$ *together with a conjugate linear involution* $\tilde{\sigma} : E_{\mathbb{C}} \to E_{\mathbb{C}}$ *covering the antiholomorphic involution* $\sigma : \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ *such that* $E_{\mathbb{C}}|_{\Sigma} = E$ *(where* Σ *is identified with its image under* i *in* $\Sigma_{\mathbb{C}}$*) and the fixed locus of* $\tilde{\sigma}$ *is* $L \to \partial\Sigma$*. Moreover, we have*

$$\deg E_{\mathbb{C}} = \mu(E, L).$$

Proof. Let $R_1, \ldots, R_h$ be the connected components of $\partial\Sigma$, and let $N_i \cong R_i \times [0, 1)$ be a neighborhood of R_i in Σ such that $N_1, \ldots, N_h$ are disjoint. Then $(N_i)_{\mathbb{C}} = N_i \cup \overline{N_i}$ is a tubular neighborhood of R_i in $\Sigma_{\mathbb{C}}$, and $N \equiv \cup_{i=1}^{h} (N_i)_{\mathbb{C}}$ is a tubular neighborhood of $\partial\Sigma$ in $\Sigma_{\mathbb{C}}$. Let $U_1 = \Sigma \cup N$, $U_2 = \overline{\Sigma} \cup N$, so that $U_1 \cup U_2 = \Sigma_{\mathbb{C}}$ and $U_1 \cap U_2 = N$.

Fix a trivialization $\Phi : E \cong \Sigma \times \mathbb{C}^n$, where n is the rank of E. Then $\Phi(L|_{R_i})$ gives rise to a loop $B_i : R_i \to \widetilde{\mathcal{R}}_n \subset GL(n, \mathbb{C})$. To construct $E_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$, we glue trivial bundles $U_1 \times \mathbb{C}^n \to U_1$ and $U_2 \times \mathbb{C}^n \to U_2$ along N by identifying $(x, u) \in (N_i)_{\mathbb{C}} \times \mathbb{C}^n \subset U_1 \times \mathbb{C}^n$ with $(x, B_i^{-1} \circ p_i(x) u) \in (N_i)_{\mathbb{C}} \times \mathbb{C}^n \subset U_2 \times \mathbb{C}^n$, where $p_i : (N_i)_{\mathbb{C}} \cong R_i \times (-1, 1) \to R_i$ is the projection to the first factor and $B_i^{-1} : R_i \to \widetilde{\mathcal{R}}_n$ denotes the map $B_i^{-1}(x) = (B_i(x))^{-1}$. There is a conjugate linear involution $\tilde{\sigma} : E_{\mathbb{C}} \to E_{\mathbb{C}}$ given by $(x, u) \in U_1 \times \mathbb{C}^n \mapsto (\sigma(x), \bar{u}) \in U_2 \times \mathbb{C}^n$ and $(y, v) \in U_2 \times \mathbb{C}^n \mapsto (\sigma(y), \bar{v}) \in U_1 \times \mathbb{C}^n$. It is clear from the above construction that $\tilde{\sigma} : E_{\mathbb{C}} \to E_{\mathbb{C}}$ covers the antiholomorphic involution $\sigma : \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$, and the fixed locus of $\tilde{\sigma}$ is $L \to \partial\Sigma$.

Proof of [MS2] Theorem 2.69 shows that $\deg(E_{\mathbb{C}})$, which is $(c_1(E_{\mathbb{C}}) \cap \Sigma_{\mathbb{C}})$ can be defined by winding number (degree) of the overlap map from trivializations of $E_{\mathbb{C}}$ over $\overline{\Sigma}$ to that of Σ. But in our setting, this map is given by B_i whose winding number defines the Maslov index $\mu(E, L)$. This proves the desired identity.

□

Now, given an orthogonal connection ∇ on a bundle pair (E, L), let us assume that it has a product form near the boundary. More precisely, on the normal neighborhood $N := \partial\Sigma \times [0, 1)$, we have $\nabla|_{\partial\Sigma \times [0,\epsilon)} = \pi^*(\nabla|_{\partial\Sigma})$ where $\pi : \partial\Sigma \times [0, \epsilon) \to \partial\Sigma$

is the projection map. It is easy to see that such an orthogonal connection always exists.

We construct a connection $\nabla_{\mathbb{C}}$ on the complex double $E_{\mathbb{C}}$ from the orthogonal connection ∇ on a bundle pair (E, L). Fix a trivialization $\Phi : E \cong \Sigma \times \mathbb{C}^n$ and let $\{\epsilon_1, \cdots, \epsilon_n\}$ be the frame of E, where $\epsilon_j = \frac{\partial}{\partial x_j}$, with $x_j + i \cdot y_j$ the j-th coordinate of $\mathbb{C}^n$. By deforming the frame near the boundaries, we may assume that $\epsilon_j|_{\partial\Sigma\times[0,\epsilon)} = \pi^*(\epsilon_j|_{\partial\Sigma})$. Then $\nabla = d + A$ where A is an $End(E)$-valued 1-form on Σ, which is defined by

$$\nabla\epsilon_i(z) = \sum_i (A)_{ji}(z) \cdot \epsilon_j(z) \tag{3.2.2}$$

for the standard hermitian metric h. Define a connection on $\overline{\Sigma} \times \mathbb{C}^n \to \overline{\Sigma}$ by $\overline{\nabla} := d + \overline{A}$, i.e.,

$$(\overline{A})_{ij}(z) := \overline{(A)_{ij}(\sigma(z))}$$

where $\sigma : \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ is the involution map.

Proposition - Definition 3.2.5. *We define a connection $\nabla_{\mathbb{C}}$ on $E_{\mathbb{C}}$ which restrict to $\nabla_{\mathbb{C}}|_{\Sigma} \equiv \nabla$ and $\nabla_{\mathbb{C}}|_{\overline{\Sigma}} \equiv \overline{\nabla}$. Namely, A is compatible to $\overline{A}$ on the tubular neighborhood N of $\partial\Sigma$.*

Proof. On $\partial\Sigma$, we fix a starting point $z \in R_i \subset \partial\Sigma$ and parameterize R_i by $\gamma : [0, 1] \to R_i$ with $\gamma(0) = \gamma(1) = z$. Image of Lagrangian subbundle $L_{\gamma(t)}$ under the map Φ can be written as $u(t)\cdot\mathbb{R}^n \subset \mathbb{C}^n$. In fact, we choose $u(t)$ as follows: Consider $u(0)$ with its column vectors $(e_1(0), \cdots, e_n(0))$. Using ∇ on (E, L), denote parallel transport of $e_j(0)$ at $\gamma(t)$ by $e_j(t)$. We have $\nabla e_j(t) = 0$ on R_i. And $u(t)$ with its column vectors $(e_1(t), \cdots, e_n(t))$, is a unitary matrix with $\Phi(L_{\gamma(t)}) = u(t) \cdot \mathbb{R}^n$. Denote entries of $u(t)$ as $(e_{ij}(t))$. Since $u(t)u(t)^* = I$,

$$\epsilon_i(t) = \overline{e_{i1}(t)}e_1(t) + \cdots + \overline{e_{in}(t)}e_n(t).$$

$$\begin{aligned}\nabla\epsilon_i(t) &= \nabla(\sum_{j=1}^{n} \overline{e_{ij}(t)}e_j(t)) \\ &= \sum_j d\overline{e_{ij}} \otimes e_j + \overline{e_{ij}}\nabla e_j \\ &= \sum_j d\overline{e_{ij}} \otimes e_j\end{aligned}$$

From the equation (3.2.2) on Σ, if we represent A with respect to the frame $\{\epsilon_1, \cdots, \epsilon_n\}$, the i-th column of $A(t)$ is equal to $\nabla\epsilon_i(t)$ and it is a linear combination of e_j's with coefficients $(\overline{de_{i1}}, \cdots, \overline{de_{in}})$ by last equation. Since this coefficient vector is i-th row of $\overline{du(t)}$, we have

$$A(t) = u(t) \cdot \frac{\partial \overline{u(t)}^T}{\partial t} dt, \text{ on } \gamma(t) \in \partial\Sigma.$$

Note that since ∇ is a product form near the boundary, $A(t,r) = A(t)$ on N, where r is the normal coordinate, i.e., $(t,r) \in \partial\Sigma \times [0,\epsilon) = N$. Recall that, in the construction of $E_{\mathbb{C}}$, the transition map was given by the inverse of B_i. Note that $B_i(\gamma(t)) = u(t) \cdot u(t)^T$. Hence, under the transition map, the connection 1-form A is transformed to

$$\begin{aligned}
A(\tilde{t}, r) &:= B(t)^{-1} \cdot dB(t) + B(t)^{-1} \cdot A(t) \cdot B(t) \\
&= \overline{u(t)} \cdot \overline{u(t)}^T d(u(t) \cdot u(t)^T) + \overline{u(t)} \cdot \overline{u(t)}^T \cdot A(t) \cdot u(t) \cdot u(t)^T \\
&= [\overline{u(t)} \cdot \overline{u(t)}^T \frac{\partial u(t)}{\partial t} u(t)^T + \overline{u(t)} \cdot \frac{\partial u(t)^T}{\partial t} + \overline{u(t)} \frac{\partial \overline{u(t)}^T}{\partial t} u(t) u(t)^T] dt \\
&= [\overline{u(t)} \frac{\partial}{\partial t} \{\overline{u(t)}^T u(t)\} u(t)^T + \overline{u(t)} \cdot \frac{\partial u(t)^T}{\partial t}] dt \\
&= \overline{u(t)} \cdot \frac{\partial u(t)^T}{\partial t} dt \\
&= \overline{A(t,r)}
\end{aligned}$$

Hence the connection ∇ and $\overline{\nabla}$ can be pasted near $\partial\Sigma$. ◻

Now we start the second proof of Theorem 3.2.1.

Proof. We first consider the case that orthogonal connection ∇ is of product form near the boundary. Later, we prove that μ_{CW} is independent of the choice of orthogonal connection.

Note that $F_{\overline{\nabla}} \circ \sigma = d\overline{A} + \overline{A} \wedge \overline{A} = \overline{dA + A \wedge A} = \overline{F_\nabla}$. Since ∇ is unitary connection, $tr(F_\nabla)$ is purely imaginary. Hence

$$tr(F_\nabla) + tr(F_{\overline{\nabla}} \circ \sigma) = tr(F_\nabla) + tr(\overline{F_\nabla}) = 0.$$

Recall that the holomorphic structure of $\overline{\Sigma}$ is given by antiholomorphic structure of Σ, hence the orientation of $\overline{\Sigma}$ is reversed to the one of Σ.

$$\int_\Sigma tr(F_\nabla) = -\int_\Sigma tr(F_{\overline{\nabla}} \circ \sigma) = \int_{\overline{\Sigma}} tr(F_{\overline{\nabla}}).$$

Since the first Chern number of doubling $E_\mathbb{C}$ gives the Maslov index of (E, L),

$$\frac{\sqrt{-1}}{\pi}\int_\Sigma tr(F_\nabla) = \frac{\sqrt{-1}}{2\pi}\int_{\Sigma_\mathbb{C}} tr(F_{\nabla_\mathbb{C}}) = c_1(E_\mathbb{C})([\Sigma_\mathbb{C}]) = \mu(E, L).$$

Therefore we can conclude that

$$\mu_{CW}(E, L) = \frac{\sqrt{-1}}{\pi}\int_\Sigma tr(F_\nabla) = \mu(E, L),$$

for a connection which is a product form near the boundary. This proves the theorem 3.2.1 together with the following lemma proposition which claims that $\mu_{CW}(E, L)$ is the same for any L-orthogonal unitary connection. □

3.3 Properties of Chern-Weil Maslov index

In this section, we prove several properties of μ_{CW}. We prove that μ_{CW} is independent of the choices of orthogonal connection and of compatible complex structures. Although this follows from the equivalence which is proved in the previous section, but the proofs given here will naturally extend to the case of orbifolds.

We also give a couple of examples to demonstrate that it is important to have unitary condition in the definition of orthogonal connection at the end of the section.

3.3.1 Independence of μ_{CW}

Proposition 3.3.1. *$\int_\Sigma tr(F_\nabla)$ is independent of the choice of an L-orthogonal unitary connection ∇.*

Proof. Let ∇^1 and ∇^2 are L-orthogonal unitary connections. Then $\nabla^1 - \nabla^2 = A$ for some $A \in \Omega^1(\Sigma) \otimes End(E)$, and we have

$$tr(F_{\nabla^1}) - tr(F_{\nabla^2}) = d(tr(A)).$$

To prove the lemma, it is enough to show that $\int_\Sigma d(tr(A)) = \int_{\partial\Sigma} tr(A) = 0$. After fixing a compatible complex structure J and a trivialization $\Phi : E \to \Sigma \times \mathbb{C}^n$, let $\{e_1, \cdots, e_n\}$ be an real orthonormal frame of L. Then

$$(\nabla^1 - \nabla^2)_{\frac{\partial}{\partial t}} e_i = \sum_j A_{ji}(\frac{\partial}{\partial t}) e_j$$

where t is a local coordinate of $\partial\Sigma$. Note that $A_{ji}(\frac{\partial}{\partial t})$s are real-valued functions over $\partial\Sigma$ since $\nabla^k (k = 1, 2)$ preserves L. Hence $\int_{\partial\Sigma} tr(A)$ is real-valued and it vanishes since $tr(F_{\nabla^k})$ are imaginary. □

One can define the notion of isomorphism between two bundle pairs over Σ, and it is easy to show that isomorphic bundle pairs have the same Maslov index. If the bundle pair is defined from a smooth map $u : (\Sigma, \partial\Sigma) \to (M, L)$ to symplectic manifolds via pulling back tangent bundles, then homotopic maps define isomorphic bundle pairs, hence has the same Maslov index.

The following corollary also follows from the equivalence $\mu = \mu_{CW}$, but we give a direct proof.

Corollary 3.3.2. *The Maslov index $\mu_{CW}(E, L)$ of a bundle pair (E, L) does not depend on a compatible complex structure J.*

Proof. Suppose we have a symplectic vector bundle $E \to \Sigma$ and Lagrangian subbundle $L \to \partial\Sigma$. Let J_0, J_1 be two compatible complex structures of E. It is well-known that the set of compatible complex structures are path connected, and take J_t connecting J_0 and J_1.

Now, consider a symplectic bundle $\widetilde{E} = E \times I \to \Sigma \times I$ and Lagrangian subbundle $L \times I \to \partial\Sigma \times I$.

We can choose an $(L \times I)$-orthogonal unitary connection $\widetilde{\nabla}$ on the complex vector bundle $\widetilde{E}$ with complex structure $\{J_t\}$. Here, $(L\times I)$-orthogonal unitary connection means a unitary connection which preserves $(L\times I)$-subbundle along $\partial\Sigma\times I$. Then $\widetilde{\nabla}$ restricts to an $(L \times \{t\})$-orthogonal unitary connection $\widetilde{\nabla}_t$ on

$$\widetilde{E}_t = E \times \{t\} \to \Sigma \times \{t\}.$$

It is enough to show that

$$\int_\Sigma tr(F_{\widetilde{\nabla}_0}) = \int_\Sigma tr(F_{\widetilde{\nabla}_1}).$$

As we have

$$0 = \int_{\Sigma\times I} tr(dF_{\widetilde{\nabla}}) = \int_{\Sigma\times\{1\}} tr(F_{\widetilde{\nabla}_0}) - \int_{\Sigma\times\{0\}} tr(F_{\widetilde{\nabla}_1}) + \int_{\partial\Sigma\times I} tr(F_{\widetilde{\nabla}})$$

it is enough to show that imaginary part of $\int_{\partial\Sigma\times I} tr(F_{\widetilde{\nabla}})$ vanishes.

Note that over $\partial\Sigma\times I$, the bundle $E\times I$ with $\{J_t\}$ is isomorphic to the complexification of real bundle $L\times I$. Using a similar argument in the Proposition 3.3.1, it is easy to show that $tr(F_{\widetilde{\nabla}})|_{\partial\Sigma\times I}$ is indeed real-valued. This proves the corollary. □

3.3.2 On the unitary condition.

Note that the property of the L-orthogonal unitary ∇ which preserving the hermitian product along the $\partial\Sigma$ is important in the above proof. The hermitian property guarantees that the induced connection $\widetilde{\nabla}$ on the determinant line bundle $det(E)|_{\partial\Sigma}$ can be identified with one of $U(1)$-principal line bundle over $\partial\Sigma$. If we drop the hermitian condition and choose a connection which preserves only the Lagrangian subbundle data over $\partial\Sigma$, the Chern-Weil definition of Maslov index fails. It is because the curvature integral captures not only the rotations of horizontal sections, but also the change of norm of them. See the following example.

Example 3.3.3. *Consider a bundle pair $E := D^2\times\mathbb{C}\to D^2$ with a trivial Lagrangian subbundle $L := \partial D^2\times\mathbb{R}$. Define a connection $\nabla := d + rd\theta$ with respect to the standard complex frame $\{\epsilon\}$ of E. Then ∇ preserves the Lagrangian structure, since*

$$\nabla_{\frac{\partial}{\partial\theta}} f\epsilon = 0 \Leftrightarrow f(\theta) = f(\theta_0)e^{\theta_0}e^{-\theta}.$$

Note that

$$\begin{aligned}\int_{D^2} F_\nabla &= \int_{D^2} dr\wedge d\theta \\ &= 2\pi\end{aligned}$$

Note that the 2π measures the ratio of the change of the norm of parallel sections along D^2.

3.4 The case of transversely intersecting Lagrangian submanifolds

Recall that to define Fukaya category, J-holomorphic maps from holomorphic polygons with boundary on several Lagrangian submanifolds (which intersects transversely) are used. There exist a Maslov index attached to such a map, which determines the virtual dimension of the moduli spaces of such maps. In general, one can consider maps from bordered Riemann surfaces with boundary condition on transversely intersecting Lagrangian submanifolds. In this section, we give a Chern-Weil definition of the Maslov index of a bundle pair arising from such maps, and find a relation with the virtual dimension of the related moduli spaces.

For simplicity of exposition, we assume that Riemann surface Σ has a boundary $\partial\Sigma$ which is connected. (In the general case, the same thing holds by taking the Maslov index in the sense of Definition 3.1.3.) We consider marked points (or punctures) $v_0, \cdots, v_k \in \partial\Sigma$ placed in a cyclic order for the induced orientation of $\partial\Sigma$. Holomorphic polygons are genus 0 cases.

We discuss orthogonal connection on a bundle pair in transversal case. Let Σ be a Riemann surface with boundary with vertices labeled as $\{v_0, \cdots, v_k\}$ and with $k+1$ edges labeled as $\{l_0, l_1, \cdots, l_k\}$ such that $v_i = l_i \cap l_{i+1}$ for $i = 0, \cdots, k$ modulo $k+1$. For each i, we fix a small closed neighborhood U_i of v_i and a conformal isomorphism

$$U_i \backslash \{v_i\} \to (-\infty, 0] \times [0, 1].$$

Let (M, ω) be a symplectic manifold with a compatible almost complex structure J and $L_0, \cdots, L_k$ be Lagrangian submanifolds intersecting transversely in M. Suppose $p_i \in L_i \cap L_{i+1}$ for $i = 0, \cdots, k$ modulo $k+1$.

Let $u : \Sigma \to M$ be a J-holomorphic map with boundary condition $u(l_j) \subset L_j$ and asymptotic condition $lim_{z \to v_i} u(z) = p_i$. By pulling back via u the tangent bundles, we obtain the following notion of bundle pair.

Definition 3.4.1. *We denote by a* bundle pair $(E, \mathbf{L}) \to \Sigma$*, a symplectic vector bundle E over Σ with Lagrangian subbundles* $\mathbf{L} := \{L_0, L_1, \cdots, L_k\}$ *over the edges* $\{l_0, l_1, \cdots, l_k\}$ *of Σ with a compatible complex structure J of E. At v_i, Lagrangian subbundles $L_i|_{v_i}$ and $L_{i+1}|_{v_i}$ are assumed to intersect transversely.*

We give a definition of **L**-orthogonal unitary connection for the bundle pair $(E, \mathbf{L})$ over Σ.

Definition 3.4.2. *Let $(E, \boldsymbol{L})$ be a bundle pair over Σ as above with J. A unitary connection ∇ on E is called $\boldsymbol{L}$-orthogonal unitary connection of $(E, \boldsymbol{L})$ if the connection ∇_{l_i} on $E|_{l_i}$, which is obtained by restriction, is L_i-orthogonal on l_i, for each $i = 0, 1, \cdots, k$.*

Lemma 3.4.1. *Given a bundle pair $(E, \boldsymbol{L})$ over Σ, an orthogonal connection exists.*

Proof. It is enough to show that such connection exists in a neighborhood of v_i. Since then, one can obtain the global one via partition of unities.

For convenience, we identify a neighborhood of $v_i \in \Sigma$ with

$$Z^o := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1, x \geq 0, y \geq 0\}$$

where v_i corresponds to 0.

Recall from [MS2] that we can take unitary trivialization of E over Z^o, $\Phi : E \to Z^o \times \mathbb{C}^n$. Here $\mathbb{C}^n$ is equipped with the standard complex structure J_0 and the standard symplectic form ω_0, and $\Phi^*\omega_0 = \omega$ and $\Phi^* J_0 = J$. On the real (resp. imaginary) axis $R \subset Z^o$ (resp. $I \subset Z^o$), we have Lagrangian subbundle L_{i+1} (resp. L_i) of E. By modifying the trivialization Φ (by multiplying elements of $U(n)$, in the neighborhood of R and I), we may assume that the image of L_i and L_{i+1} under Φ is constant along R and I in $\mathbb{C}^n$.

Now choose a trivial connection on $Z^o \times \mathbb{C}^n$ and pull back via Φ to obtain an orthogonal connection of E on Z^o. □

Now, we associate Chern-Weil Maslov index to the above bundle pair as before.

Definition 3.4.3. *Let ∇ be an $\boldsymbol{L}$-orthogonal unitary connection of $(E, \boldsymbol{L})$. The Maslov index of the bundle pair $(E, \boldsymbol{L})$ is defined by*

$$\mu_{CW}(E, \boldsymbol{L}) := \frac{\sqrt{-1}}{\pi} \int_{\Sigma} tr(F_{\nabla})$$

As in the previous case, we have

Proposition 3.4.2. *$\mu_{CW}(E, \boldsymbol{L})$ is independent of the choice of $\boldsymbol{L}$-orthogonal unitary connection ∇. It is also independent of the choice of an almost complex structure.*

Proof. The proof is similar to that of Proposition 3.3.1. □

Note that we can choose a compatible complex structure J satisfying

$$J \cdot L_i|_{v_i} = L_{i+1}|_{v_i}$$

at the marked point $v_i = l_i \cap l_{i+1}$ for each $i = 0, 1, \cdots, k$. We will use such a J in the following discussions.

We recall the usual topological Maslov index associated to the bundle pair $(E, \mathbf{L})$. First, given two Lagrangian subspaces L_0 and L_1 which intersects transversely in V, there exist a path from L_0 to L_1 that moves in the positive definite direction (which is unique up to fixed end points). If $L_1 = J \cdot L_0$, then such a path can be taken to be $t \mapsto e^{\pi J t/2} L_0$. For example, a loop in Lagrangian Grassmannian obtained by joining positive definite paths from L_0 to L_1 and from L_1 to L_0 has Maslov index $n = dim(L_0)$.

The topological Maslov index of the bundle pair $(E, \mathbf{L})$ can be defined by first taking a trivialization of E and taking a loop of Lagrangian subspaces along the boundary, by gluing the Lagrangian subbundle data of the edges at each marked point v_i via positive definite direction path from L_i to L_{i+1}. Denote this path by L_{loop}. The winding number of L_{loop} defines Maslov index $\mu_{top}(E, \mathbf{L})$.

We also recall how the Fredholm index arises in this setting. For a fixed $p > 2$, consider a Banach manifold $\mathcal{P}$ of $W^{1,p}$ maps $\Sigma \to M$ with boundary condition $u(l_j) \subset L_j$ and asymptotic condition $lim_{z \to v_i} u(z) = p_i$. Then the moduli space of J-holomorphic maps from Σ to M can be identified with a zero set of a smooth section $\overline{\partial}_J : \mathcal{P} \to \mathcal{E}$ for a Banach bundle $\mathcal{E}$. More precisely, the fiber of $\mathcal{E}$ at u is the space $L^p(\mathcal{A}^{0,1} \bigotimes u^*TM)$, and the section $\overline{\partial}_J$ is the antiholomorphic part of du with respect to J. If we linearize the $\overline{\partial}_J$ at $u \in \overline{\partial}_J^{-1}(0)$ and composite it with the projection map from $T_u\mathcal{E}$ to the fiber $\mathcal{E}_u$, we have a Fredholm operator $D_u\overline{\partial}_J : W^{1,p}(\Sigma, u^*TM) \to L^p(\mathcal{A}^{0,1}(\Sigma) \bigotimes u^*TM)$. The virtual dimension of the component of $\overline{\partial}_J^{-1}(0)$ containing u is defined by the Fredholm index of $D_u\overline{\partial}_J$. We denote the linearized Fredholm operator as $\overline{\partial}_{E,\mathbf{L}}$.

Now, we plan to compare the Fredholm index of $\overline{\partial}_{E,\mathbf{L}}$ with $\mu_{CW}(E, \mathbf{L}), \mu_{top}(E, \mathbf{L})$.

Proposition 3.4.3. *We have*

$$Ind(\bar{\partial}_{E,\boldsymbol{L}}) + (k+1)\frac{n}{2} = \mu_{CW}(E,\boldsymbol{L}) + n\chi(\Sigma) \tag{3.4.3}$$
$$Ind(\bar{\partial}_{E,\boldsymbol{L}}) + (k+1)n = \mu_{top}(E,\boldsymbol{L}) + n\chi(\Sigma). \tag{3.4.4}$$

Proof. From [KL] Theorem 3.4.2, we have

$$Ind(\bar{\partial}_{E,L}) = \mu(E,L) + n\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

The case of boundary condition on transversally intersecting Lagrangian submanifolds (with $k+1$ marked points), can be seen from the gluing principle of indices: At each marked point v_i, consider

$$Z_- = \{z \in \mathbb{C} \mid |z| \leq 1\} \cup \{z \in \mathbb{C} \mid Re\, z \geq 0, |Im\, z| \leq 1\}$$

and a trivial complex vector bundle $Z_- \times E|_{v_i}$ with Lagrangian boundary condition: $L_{i+1}|_{v_i}$ over $\partial Z_- \cap \{Im\, z = -1\}$, $L_i|_{v_i}$ over $\partial Z_- \cap \{Im\, z = +1\}$ and a path in the Lagrangian Grassmannian $\Lambda(E|_{v_i})$ of positive definite direction from $L_i|_{v_i}$ to $L_{i+1}|_{v_i}$ over the arc (left side of the unit circle). Denote by λ the above Lagrangian bundle data. Recall from Section 3.7 of [FOOO] that the index of $\bar{\partial}_{\lambda,Z_-}$ of weighted Cauchy-Riemann operator is n. By gluing Z_- at each marked point, we obtain the equation (3.4.4). (We refer readers to Section 8.8 of [FOOO] for more details on this argument).

To prove the first identity, we find a relation between $\mu_{top}(E,\mathbf{L})$ and $\mu_{CW}(E,\mathbf{L})$ by studying the index of a basic piece. Instead of Z_-, we consider the following domain Z to compute μ_{CW} in an easier way.

$$Z := \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$$

We consider the following bundle pair on Z. Consider $Z \times \mathbb{C}^n \to Z$ equipped with the standard symplectic structure, and we describe Lagrangian subbundle on ∂Z. Note that the boundary ∂Z consists of three parts, R, I and A which are real axis, imaginary axis and arc, respectively.

By identifying $E_{v_i} \cong \mathbb{C}^n$ (so that ω, J becomes ω_0, J_0), denote by Λ_i (resp. Λ_i') the Lagrangian subspace of $\mathbb{C}^n$ corresponding to $L_i|_{v_i}$ (resp. $L_{i+1}|_{v_i}$). Note that $J_0\Lambda_i = \Lambda_i'$.

We choose a Lagrangian subbundle over ∂Z as follows: the trivial bundle $R \times \Lambda_i \subset R \times \mathbb{C}^n$ over $R \subset \partial Z$, the trivial bundle $I \times \Lambda_i' \subset I \times \mathbb{C}^n$ over $I \subset \partial Z$ and a positive definite direction path from Λ_i to Λ_i' in the Lagrangian Grassmannian $\Lambda(n)$ over the arc $A \subset \partial Z$ in the counterclockwise direction. We assume that the path is constant near end points of the arc A. We may denote this Lagrangian subbundle on ∂Z by Λ.

Now, we compute $\mu_{CW}(Z\times\mathbb{C}^n, \Lambda)$. Now we can take a Λ-orthogonal connection as follows. We choose a map $\gamma : [0,1] \to U(n)$ whose column vectors form a unitary frame of Λ on the arc $A \subset \partial Z$ such that

$$\gamma(1) = e^{\pi\sqrt{-1}/2} \cdot \gamma(0),$$

and constant in $U(n)$ near end points 0 and 1. We may take a connection ∇ on the bundle satisfying $\nabla(\gamma \cdot \epsilon_j) \equiv 0$ (for $j = 1, \cdots, n$) near the arc, and $\nabla \equiv d$ near the real axis and imaginary axis. This defines a Λ-orthogonal connection and we define

$$\mu_{CW}(Z \times \mathbb{C}^n, \Lambda) := \frac{\sqrt{-1}}{\pi} \int_Z tr(F_\nabla).$$

Lemma 3.4.4. *$\mu_{CW}(Z\times\mathbb{C}^n, \Lambda)$ is equal to $\frac{n}{2}$ which is the half of topological Maslov index of loop $\gamma * (e^{\pi\sqrt{-1}/2} \cdot \gamma)$ in $\Lambda(n)$, where $\gamma * (e^{\pi\sqrt{-1}/2} \cdot \gamma)$ is a smooth function from [0,2] to $U(n)$ defined by*

$$\gamma * (e^{\pi\sqrt{-1}/2} \cdot \gamma)(t) := \begin{cases} \gamma(t) & \text{if } t \in [0,1] \\ e^{\pi\sqrt{-1}/2} \cdot \gamma(t-1) & \text{if } t \in [1,2] \end{cases} \tag{3.4.5}$$

Proof. Consider a (trivial) complex vector bundle on D^2. Consider a Lagrangian subbundle L_Γ over ∂D^2 by concatenating four paths $\Gamma := \gamma * (e^{\pi\sqrt{-1}/2} \cdot \gamma) * (e^{\pi\sqrt{-1}} \cdot \gamma) * (e^{3\pi\sqrt{-1}/2} \cdot \gamma)$ as in (3.4.5). Note that, since $e^{3\pi\sqrt{-1}/2} \cdot \gamma(1) = \gamma(0)$, this path Γ is a loop in $U(n)$ and hence loop in $\Lambda(n)$. Thus if ∇ is an orthogonal connection, $\frac{\sqrt{-1}}{\pi} \int_D tr(F_\nabla)$ gives the topological Maslov index of $(D^2 \times \mathbb{C}^n, L_\Gamma)$ from Theorem 3.2.1.

Now, we construct an orthogonal connection ∇ on D^2 from that of Z. Note that the ∇ constructed on Z induces a connection on D^2 by pullback of the map $m_k : D^2 \to D^2$, $z \mapsto e^{\frac{k\pi\sqrt{-1}}{2}} \cdot z$ for $k = 0,1,2,3$. Then, since $\nabla \equiv d$ near the real axis and imaginary axis, the pullback connection on each $e^{\frac{k\pi\sqrt{-1}}{2}} \cdot Z$ ($k = 0,1,2,3$)

can be glued to give a connection on D. It is easy to see that this connection is indeed an orthogonal connection for the pair $(D^2 \times \mathbb{C}^n, L_\Gamma)$. Note that the curvature integral on each quadrant of D^2 are the same:

$$\int_Z tr(F_\nabla) = \int_{m_k^{-1}(Z)} tr(m_k^* F_\nabla) = \int_{m_k^{-1}(Z)} tr(F_{m_k^*\nabla}).$$

Hence we have

$$\begin{aligned}\frac{\sqrt{-1}}{\pi}\int_Z tr(F_\nabla) &= \frac{1}{4}\frac{\sqrt{-1}}{\pi}\int_{D^2} tr(F_\nabla)\\ &= \frac{1}{4}(\text{topological Maslov index of } \Gamma \text{ in } \Lambda(n))\\ &= \frac{1}{2}(\text{topological Maslov index of } \gamma * (\sqrt{-1}\cdot\gamma) \text{ in } \Lambda(n))\\ &= \frac{n}{2}\end{aligned}$$

The last identity follows from the fact that since each γ is chosen to be positive definite, $\gamma * (e^{\pi\sqrt{-1}/2}\cdot\gamma)$ is a positive definite loop whose Maslov index is equal to n, which is the dimension of Lagrangian subspace. □

Now, as before, we attach bundle pair over Z to that over Σ at each marked points. After attaching $(k+1)$ copies of bundle data on Z, the resulting Lagrangian subbundle along the boundary is obtained by connecting L_i and L_{i+1} by positive definite direction paths and it becomes L_{loop} which defines μ_{top}. Thus we have the following identity.

$$\mu_{top}(E, \mathbf{L}) = \mu_{CW}(E, \mathbf{L}) + \sum_{i=0}^{k}\mu_{CW}(Z\times\mathbb{C}^n, \Lambda) = \mu_{CW}(E, \mathbf{L}) + \frac{(k+1)n}{2}$$

Combining with the equation (3.4.4), we obtain the equation (3.4.3). This finishes the proof of the proposition. □

When $k = 0$, and Σ is a bi-gon, $Ind(\bar\partial_{E,\mathbf{L}})$ which equals $\mu_{top}(E, \mathbf{L}) - n$, is called the Maslov-Viterbo index. Thus we obtain the following corollary.

Corollary 3.4.5.

$$\textit{Maslov-Viterbo index} = Ind(\bar\partial_{E,L}) = \mu_{CW}(E, \{L_0, L_1\}).$$

3.5 Orbifold Maslov Index

In this section, we extend the definition of Maslov index to the case of orbifolds. Namely, consider an orbi-bundle over a bordered orbifold Riemann surface with interior singularities, and a Lagrangian subbundle along the boundary. Note that orbi-bundles in these cases are not trivial bundles, and hence topological definition of Maslov index is not directly extended to these cases.

But Chern-Weil definition extends naturally by requiring the connection to be invariant under local group actions near orbifold singularities. Using the Chern-Weil definition, we show that there is a well-defined topological definition of orbifold Maslov index.

At the end of this section, we show a relation of orbifold Maslov index and desingularized Maslov index introduced by Cho and Poddar [CP], using the desingularization procedure introduced by Chen and Ruan [CR1].

3.5.1 Orbifold Chern-Weil Maslov index

We first recall the definition of bordered orbifold Riemann surface and J-holomorphic maps to almost complex orbifolds. We denote $\boldsymbol{z} = (z_1, \cdots, z_k), \boldsymbol{m} = (m_1, \cdots, m_k)$ in the following.

Definition 3.5.1. *Let Σ be a bordered Riemann surface with complex structure j. $(\Sigma, \boldsymbol{z}, \boldsymbol{m})$ is called a bordered orbifold Riemann surface with interior singularities if $\boldsymbol{z}$ are distinct interior of Σ, and if a disc neighborhood of each z_i is uniformized by a branched covering map $z \mapsto z^{m_i}$.*

Thus the disc neighborhood U_i of z_i is understood as a quotient space of D^2 by the standard rotation action of the local group $\mathbb{Z}_{m_i}$. We denote by $\boldsymbol{\Sigma} = (\Sigma, \boldsymbol{z}, \boldsymbol{m})$ the orbifold bordered Riemann surface.

In our case, we can also consider a smooth Riemann surface $\widetilde{\Sigma}$ with a branched covering map $\pi : \widetilde{\Sigma} \to \Sigma$ such that the orbifold $\boldsymbol{\Sigma}$ is obtained as the quotient of $\widetilde{\Sigma}$ by the action of deck transformation group G of π (i.e. $\boldsymbol{\Sigma}$ is good orbifold).

Such $\widetilde{\Sigma}$ can be obtained as follows. Consider two copies of $\boldsymbol{\Sigma}$ labelled as $\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2$, and glue $\boldsymbol{\Sigma}_1$ with $\overline{\boldsymbol{\Sigma}_2}$(opposite orientation) to obtain $\boldsymbol{\Sigma}_{double}$, which becomes a good orbifold. Hence it has a smooth Riemann surface $\widetilde{\Sigma}_{double}$ with a branched covering

$\pi : \widetilde{\Sigma}_{double} \to \mathbf{\Sigma}_{double}$. By considering only $\widetilde{\Sigma} := \pi^{-1}(\mathbf{\Sigma}_1)$, we obtain the desired Riemann surface with boundary $\widetilde{\Sigma}$.

Now, consider an orbifold vector bundle $E \to \mathbf{\Sigma}$. (see for example [CR1]). On the neighborhood U_i of z_i above, orbifold vector bundle $E|_{U_i} \to U_i$ has a uniformizing chart $D^2 \times \mathbb{R}^n \to D^2$ together with $\mathbb{Z}_{m_i}$-action compatible with the orbifold structure of U_i. This may be understood as a genuine vector bundle $\widetilde{E} \to \widetilde{\Sigma}$ with an action of deck transformation group G, which is compatible with that of $\widetilde{\Sigma}$.

Also recall that a connection ∇ on orbifold vector bundle $E \to \mathbf{\Sigma}$ is defined to be invariant under local group action.

We define a bundle pair over $(\mathbf{\Sigma}, \partial\mathbf{\Sigma})$.

Definition 3.5.2. *We denote by a* bundle pair $(E, L) \to (\mathbf{\Sigma}, \partial\mathbf{\Sigma})$*, an orbifold symplectic vector bundle E over $\mathbf{\Sigma}$ and a Lagrangian subbundle L over $\partial\Sigma$.*

We choose a compatible complex structure J of E. The bundle data in the orbifold case arises is obtained by a good map from $(\Sigma, \boldsymbol{z}, \boldsymbol{m})$ to a symplectic orbifold with Lagrangian boundary condition. The notion of a good map was introduced by Chen and Ruan, which enables one to pull-back bundles. Given a J-holomorphic map which is a good map, we obtain a bundle pair by pull-back tangent bundles.

We define L-orthogonal connection of a bundle pair as follows

Definition 3.5.3. *Let (E, L) be a bundle pair over $(\mathbf{\Sigma}, \partial\mathbf{\Sigma})$. A unitary connection ∇ on E is called orthogonal connection if the parallel transport along $\partial\Sigma$ via ∇ preserves Lagrangian subbundle $L \to \partial\Sigma$.*

Now, we give a definition of the Maslov index μ_{CW} for $(E, L) \to (\Sigma, \boldsymbol{z}, \boldsymbol{m})$ in terms of curvature integral:

Definition 3.5.4. *Let ∇ be an orthogonal connection of a bundle pair $(E, L) \to (\mathbf{\Sigma}, \partial\mathbf{\Sigma})$. We define the Maslov index of the bundle pair (E, L) as*

$$\mu_{CW}(E, L) = \frac{i}{\pi} \int_{\Sigma}^{orb} tr(F_\nabla)$$

where $F_\nabla \in \Omega^2(\Sigma, End(E))$ is the curvature induced by ∇.

As in the previous cases, we have

Proposition 3.5.1. *$\mu_{CW}(E, L)$ in Definition 3.5.4 is independent of the choice of L-orthogonal connection ∇. It is also independent of the choice of a complex structure J.*

Proof. The proof is similar to that of Proposition 3.3.1, using Stoke's theorem in the orbifold setting. □

3.5.2 Topological definition of orbifold Maslov index

One possible approach to define Maslov index topologically in the orbifold case is as follows.

Definition 3.5.5. *Consider a bundle pair $(E, L) \to (\mathbf{\Sigma}, \partial\mathbf{\Sigma})$. Take a branched covering $\pi : \widetilde{\Sigma} \to \mathbf{\Sigma}$ by a smooth Riemann surface $\widetilde{\Sigma}$ with boundary, and consider pull-back bundles (π^*E, π^*L) which becomes a smooth bundle pair on $(\widetilde{\Sigma}, \partial\widetilde{\Sigma})$.*

We define

$$\mu_\pi(E, L) = \frac{1}{|G|}\mu(\pi^*E, \pi^*L)$$

where $|G|$ is the degree of the branched covering map π.

A priori, it is not clear if $\mu_\pi(E, L)$ is independent of the choice of the branched covering map π. But we use Chern-Weil definition of Maslov index to prove that such a topological index is independent of the choice of a branched covering map, and prove that it is the same as Chern-Weil Maslov index. This should be useful in actual computations of Maslov indices.

Proposition 3.5.2. *We have*

$$\mu_\pi(E, L) = \mu_{CW}(E, L).$$

In particular, $\mu_\pi(E, L)$ is independent of the choice π of the branched covering map.

Proof. Let ∇ be an L-orthogonal connection on an orbifold vector bundle pair $(E, L) \to (\mathbf{\Sigma}, \partial\mathbf{\Sigma})$. Let $\pi^*\nabla$ be a pull-back connection on π^*E, which becomes a L-orthogonal connection of bundle pair (π^*E, π^*L)

By theorem 3.2.1, we have $\mu(\pi^*E, \pi^*L) = \mu_{CW}(\pi^*E, \pi^*L)$. Therefore, we have

$$\mu(\pi^*E, \pi^*L) = \mu_{CW}(\pi^*E, \pi^*L) = \int_{\widetilde{\Sigma}} F_{\pi^*\nabla} = |G| \int_{\Sigma} F_{\nabla} = |G|\mu_{CW}(E, L).$$

□

In fact, one may observe that the above argument works for branched coverings between two smooth bundle pairs also.

Consider a branched covering $\phi : \Sigma_1 \to \Sigma_2$ of degree m for bordered Riemann surfaces Σ_1, Σ_2. (Here we assume that the branching locus lies in the interior of Σ_2.) Then, given a smooth map $u : (\Sigma_2, \partial\Sigma_2) \to (M, L)$ for a symplectic manifold M and Lagrangian submanifold L, we obtain by composition another map $u \circ \phi : (\Sigma_1, \partial\Sigma_1) \to (M, L)$.

Define the Maslov index of u to be $\mu(u^*TM, u|^*_{\partial\Sigma_2}TL)$, and similarly for $u \circ \phi$. Then, the same argument as in the above proposition proves that we have

$$\mu(u \circ \phi) = m \cdot \mu(u).$$

3.5.3 Relation to desingularization

We first recall a definition of the desingularized Maslov index, which determines the virtual dimension of the moduli space of J-holomorphic orbi-discs from [CP]. Let X be a symplectic orbifold and N be a Lagrangian submanifold (which do not contain any orbifold singularity). Let $\mathbf{\Sigma}$ be an orbi-disc with interior orbifold singularity $(z_1, \cdots, z_k)$. Let $u : (\mathbf{\Sigma}, \partial\mathbf{\Sigma}) \to (X, N)$ be an orbifold J-holomorphic disc with Lagrangian boundary condition. Then, $E := u^*TX$ is a complex orbi-bundle over $\mathbf{\Sigma}$, with Lagrangian subbundle $L := u|^*_{\partial\mathbf{\Sigma}}TN$ at $\partial\mathbf{\Sigma}$.

Definition 3.5.6. *Let $|E|$ be the desingularized bundle over $\mathbf{\Sigma}$(or Σ), which still have the Lagrangian subbundle at the boundary from L. The Maslov index of the bundle pair $(|E|, L)$ over $(\Sigma, \partial\Sigma)$ is called the* desingularized Maslov index *of (E, L), and denoted as $\mu^{de}(E, L)$.*

We find a relation of the desingularized Maslov index of [CP] and the Maslov index in Definition 3.5.4.

Proposition 3.5.3. *We have*

$$\mu_{CW}(E,L) = \mu^{de}(E,L) + 2\sum_{i=1}^{k}\sum_{j=1}^{n}\frac{m_{i,j}}{m_i}.$$

Proof. We first consider the double $E_{\mathbb{C}}$ of the bundle pair (E,L) for bordered Riemann surface with interior orbifold singularities. Then we have from Chen-Ruan's formula that

$$c_1(E_{\mathbb{C}})([\Sigma_{\mathbb{C}}]) = deg(E_{\mathbb{C}}) = c_1(|E_{\mathbb{C}}|)([\Sigma_{\mathbb{C}}]) + 2\sum_{i=1}^{k}\sum_{j=1}^{n}\frac{m_{i,j}}{m_i}.$$

Note that from [KL], we have $\mu^{de}(E,L) = c_1(|E|_{\mathbb{C}})([\Sigma_{\mathbb{C}}])$, and as $|E_{\mathbb{C}}| = |E|_{\mathbb{C}}$ holds, we have $\mu^{de}(E,L) = c_1(|E_{\mathbb{C}}|)([\Sigma_{\mathbb{C}}])$.

Note that given an orthogonal connection ∇ on (E,L) over $(\Sigma, \boldsymbol{z}, \boldsymbol{m})$, we can find a connection $\nabla_{\mathbb{C}}$ on $E_{\mathbb{C}}$ as in the section 3.2. From the Chern-Weil definition of Maslov index $\mu_{CW}(E,L)$ over $(\Sigma, \boldsymbol{z}, \boldsymbol{m})$, we find that

$$\mu_{CW}(E,L) = \frac{\sqrt{-1}}{\pi}\int_{\Sigma}^{orb} tr(F_{\nabla}) = \frac{\sqrt{-1}}{2\pi}\int_{\Sigma_{\mathbb{C}}}^{orb} tr(F_{\nabla_{\mathbb{C}}}) = c_1(E_{\mathbb{C}})([\Sigma_{\mathbb{C}}]).$$

Hence, we obtain the proposition. □

Chapter 4

On orbifold embeddings[1]

In this chapter, we develop several properties of orbifold embeddings. When the orbifold is equivariant to a translation groupoid, we show that such a notion is equivalent to a strong equivariant immersion.

4.1 Orbifold embeddings

In this section we recall the definition of an orbifold embedding, and explore its properties. The following notion is a slight modification from the one defined by Adem, Leida, and Ruan in their book [ALR].

Definition 4.1.1. *A homomorphism of orbifold groupoids $\phi : \mathcal{H} \to \mathcal{G}$ is an embedding if the following conditions are satisfied:*

1. *$\phi_0 : H_0 \to G_0$ is an immersion*

2. *Let $x \in im(\phi_0) \subset G_0$ and let U_x be a neighborhood such that $\mathcal{G}|_{U_x} \cong G_x \ltimes U_x$. Then, the $\mathcal{H}$-action on $\phi_0^{-1}(x)$ is transitive, and there exists an open neighborhood $V_y \subset H_0$ for each $y \in \phi_0^{-1}(x)$ such that $\mathcal{H}|_{V_y} \cong H_y \ltimes V_y$ and*

$$\mathcal{H}|_{\phi_0^{-1}(U_x)} \cong G_x \ltimes (G_x \times_{H_y} V_y) \tag{4.1.1}$$

3. *$|\phi| : |\mathcal{H}| \to |\mathcal{G}|$ is proper and injective.*

$\mathcal{H}$ together with ϕ is called an orbifold embedding of G.

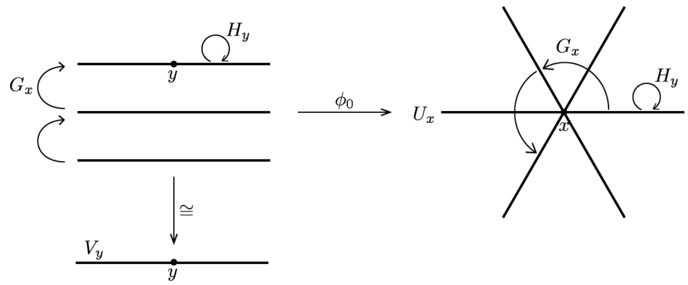


Figure 4.1: Local shape of an orbifold embedding

In (2) of Definition 4.1.1 the action of G_x is defined by

$$G_x \times (G_x \times_{H_y} V_y) \to G_x \times_{H_y} V_y, \quad (g, [k, z]) \mapsto [gk, z]$$

where $(k\phi_1(h), z) \sim (k, h \cdot z)$ is the equivalence relation defined by the action of H_y and $[k, z]$ denotes a class in the quotient $G_x \times_{H_y} V_y$.

There are two modifications in the definition from that of Adem, Leida, and Ruan (Definition 2.3 in [ALR]).

1. We use the local model $G_x \times_{H_y} V_y$ instead of $G_x/H_y \times V_y$.

2. We require that $|\phi| : |\mathcal{H}| \to |\mathcal{G}|$ is injective (which was not present in [ALR]).

Let us explain why we have made such modifications.

Firstly, in [ALR] $G_x/\phi_1(H_y) \times V_y$ was used instead of $G_x \times_{H_y} V_y$. But $\phi_1(H_y)$ may not be a normal subgroup of G_x (See the example 4.1.1). Also, it is not easy to find a natural G_x action on $G_x/\phi_1(H_y) \times V_y$ which reflects the H_y action on V_y. The only plausible action of G_x that may exist on $G_x/\phi_1(H_y) \times V_y$ is by the left multiplication on the first component. Now, any reasonable definition of an embedding should include the identity map, and therefore in this case we would have that $G_x \ltimes U_x \cong G_x \ltimes (G_x/G_x \times V_y)$ where $x = y$ and $U_x = V_y$ but, on U_x the group G_x acts and on $G_x/G_x \times V_y$ the action is trivial. Hence, $G_x \times_{H_y} V_y$ in (4.1.1) should be the correct local model.

[1]This chapter is based on [CHS].

Example 4.1.1. *Let S_3 act on $\mathbb{C}^3$ as permutations on three coordinates where S_3 is the permutation group on* 3 *letters. Consider $V := \mathbb{C} \times \mathbb{C} \times \{0\} \subset \mathbb{C}^3$ and the subgroup H of S_3 generated by the transposition* $(1,2)$. *Then, H acts on V and the natural map*

$$S_3 \times_H V \to \mathbb{C}^3$$

induces an orbifold embedding $S_3 \ltimes (S_3 \times_H V) \to S_3 \ltimes \mathbb{C}^3$. Note that H is not a normal subgroup of S_3.

Secondly, in [ALR], an orbifold embedding $\phi : \mathcal{H} \to \mathcal{G}$ does not necessarily induce an injective map $|\phi| : |\mathcal{H}| \to |\mathcal{G}|$. We first provide an example where $|\phi|$ is not injective but satisfies the other conditions of embedding. We will call a morphism ϕ of Lie groupoids *essentially injective* if $|\phi|$ is injective.

Example 4.1.2. *Let $\mathcal{G}$ be given by $G_0 = \mathbb{R} \amalg \mathbb{R}$ and $\mathbb{Z}/2\mathbb{Z}$ identifying two copies of $\mathbb{R}$. Suppose $\mathcal{H}$ is the disjoint union of two copies of $\mathbb{R}$ with only trivial arrows.*

Immerse (embed) H_0 to G_0 by $id_{\mathbb{R}} \amalg id_{\mathbb{R}}$. One can easily check that ϕ satisfies the other axioms of orbifold embedding, but $|\phi|$ not injective. The induced map between quotient space is rather a covering map from trivial double cover of $\mathbb{R}$ to $\mathbb{R}$.

Remark 4.1.3. *A morphism of groupoids $\phi : \mathcal{H} \to \mathcal{G}$ is essentially injective if the $\mathcal{H}$-action on $\phi_0^{-1}(t(s^{-1}(y)))$ (ϕ_0 inverse image of H_1-orbit) is transitive for every $y \in G_0$, i.e. if there exists an arrow in G_1 from $\phi_0(x)$ to $\phi_0(x')$, then one can find an arrow in H_1 from x to x'.*

Compare it with the notion of essential surjectivity: $\phi : \mathcal{H} \to \mathcal{G}$ is called essentially surjective if for any point x in G_0, there is an arrow $g : \phi(y) \to x$ from a point in the image of ϕ to x.

Remark 4.1.4. *The essential injectivity is a property which is Morita-invariant since it is a property of the induced map between quotient spaces.*

Let us mention why such a notion of orbifold embedding is needed. First, for orbifolds, one can define suborbifolds as sub Lie groupoids which are orbifold groupoids. But important objects, such as the diagonal, do *not* become suborbifolds. Hence we need a proper notion to consider such objects, or we need to enlarge the definition of suborbifolds to include orbifold embeddings. See example

4.1.2 to note that the diagonal homomorphism is indeed an orbifold embedding. We believe that orbifold embeddings are an important class of subobjects for an orbifold, but unfortunately, these notions has not been so far used nor developed further.

From now on, we develop the properties of orbifold embeddings.

Lemma 4.1.5. *If $\phi : \mathcal{H} \to \mathcal{G}$ is an orbifold embedding, then the restriction of ϕ_1 on local isotropy groups is injective.*

Proof. Note that the point y corresponds to $[e, y]$ in this model, where e is the identity element in G_x. Since equivalence between orbifolds preserves local isotropy groups, the local group $\phi_1(H_y)$ at $[e, y]$ of $G_x \ltimes (G_x \times_{H_y} V_y)$ has to be isomorphic to H_y, and it proves the lemma. □

Remark 4.1.6. *For the case of an effective orbifold $\mathcal{H}$, Lemma 4.1.5 follows directly from the 0-level immersion ϕ_0. Assume that there is a nontrivial element $h \in Ker(\phi_1|_{H_y})$. Fix a tangent vector $v \in T_yV_y$. Since the action of $\mathcal{H}$ is effective, the difference of two vectors $v - h_*v$ is not a zero vector. By the assumption on h,*

$$(\phi_0)_*(v - h_*v) = 0,$$

and it contradicts that ϕ_0 is an immersion.

Remark 4.1.7. *In the above Lemma 4.1.5, ϕ_1 may not be globally injective.*

We remark that the orbifold embedding is not Morita invariant. Indeed, the following two examples illustrate this phenomenon.

Example 4.1.8. *Let $\mathcal{H}$ be a circle with the trivial orbifold groupoid structure and $\mathcal{G}$ be a teardrop whose local group at the unique singular point x is $\mathbb{Z}/3$ as in Figure 4.2. The orbifold morphism $\phi : \mathcal{H} \to \mathcal{G}$ is not an orbifold embedding since it does not satisfy the second condition at $x \in \mathcal{G}$.*

However, we can change the orbifold structure of $\mathcal{H}$ as follows. Let $\phi(y) = x$ and U be a open neighborhood of y as in the figure. We add two more copies of U to get new objects U' and add additional arrows identifying three copies of U. Denote by $\mathcal{H}'$ the resulting orbifold. Note that there is an equivalence from $\mathcal{H}'$ to $\mathcal{H}$. The obvious modification $\phi' : \mathcal{H}' \to \mathcal{G}$ of ϕ is now an orbifold embedding.

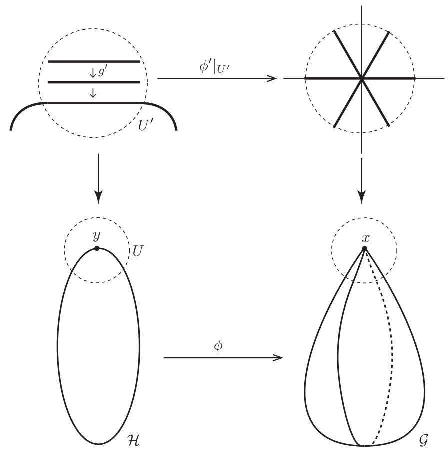


Figure 4.2: An orbifold embedding and equivalence 1

Example 4.1.9. *Let $\mathcal{H}$ be the disjoint union of three copies of real lines and $\mathcal{G}$ be $\mathbb{R}^2$ equipped with a $\mathbb{Z}/3$ action generated by $2\pi/3$-rotation. Consider an orbifold embedding $\phi : \mathcal{H} \to \mathcal{G}$ shown in Figure 4.3. We similarly change the orbifold structure of $\mathcal{H}$ by adding three more copies of $\mathbb{R}$ to $\mathcal{H}$ to get a new orbifold groupoid $\mathcal{H}'$, i.e. $H_0' = \mathbb{R} \times \mathbb{Z}_6$ and $((h, g), k) \in H_1' = H_0' \times \mathbb{Z}_6$ sends $(h, g) \to (h, kg)$. It is clear from Figure 4.3 that there is an equivalence $\mathcal{H}' \to \mathcal{H}$, which is induced by the projection $\mathbb{Z}_6 \to \mathbb{Z}_3$. The morphism $\phi' : \mathcal{H}' \to \mathcal{G}$ is defined by the composition of ϕ and this equivalence. Then, we see that ϕ' is no longer an orbifold morphism because there is no transitive G_x action on $\phi'^{-1}(x)$ where x is the unique singular point in $\mathcal{G}$.*

Example 4.1.10 (Orbifold diagonal). *As an example of an orbifold embedding, we introduce a diagonal suborbifold of product orbifolds.*

Definition 4.1.2. *The diagonal suborbifold Δ is defined as $\mathcal{G} \times_{\mathcal{G}} \mathcal{G}$*

Lemma 4.1.11. *The natural map $\Delta = \mathcal{G} \times_{\mathcal{G}} \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ is an orbifold embedding.*

Proof. We verify that Δ above is a subgroupoid in the sense of Definition 4.1.1. It suffices to prove this when $\mathcal{G}$ is a global quotient orbifold $G \ltimes M$. In this case, Δ is given by $(G \times G) \ltimes (\sqcup_g \Delta_g)$, where $\Delta_g = \{(x, gx) : x \in M\}$ and $(h, k) \in G \times G$ takes (x, g, gx) to (hx, kgh^{-1}, kgx). (The second terms in the triples are used to

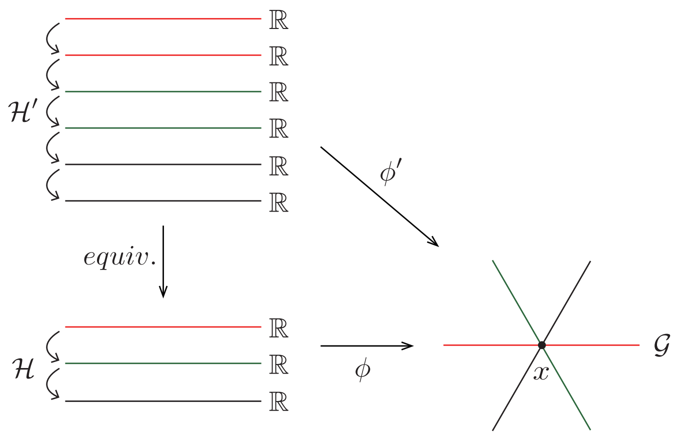


Figure 4.3: An orbifold embedding and equivalence 2

distinguish (x, gx) from (x, ghx) for $h \in G_x$.) The most natural choice of orbifold morphisms will be $\phi_0 : (x, g, gx) \mapsto (x, gx) \in M \times M$ and $\phi_1 : (h, k) \mapsto (h, k) \in G \times G$. ϕ_0 is clearly an immersion. Note that ϕ_1 is injective.

Choose a point $p = (x, y)$ in im(ϕ_0). Let U be a small connected neighborhood of p in $M \times M$, which is preserved under the $(G \times G)_p$-action. Since (x, y) is in the image of ϕ_0, there is some g in G satisfying $y = gx$. So, in particular $\phi_0^{-1}(x, y) = \{(x, g', g'x) \,|\, g'x = gx = y\}$. $(G \times G)_p$ acts on $\phi_0^{-1}(p)$ transitively since $(g^{-1}g', e)$ sends $(x, g', g'x) \in \phi_0^{-1}(p)$ to (x, g, gx) and $g^{-1}g' \in G_x$ when $g'x = gx$.

Let q denote $(x, g, gx) \in \phi_0^{-1}(p)$. Note that $(G \times G)_q = \{(h, k) : h \in G_x, k \in G_{gx}, kgh^{-1} = g\}$. Let V_g be the connected component of $\phi_0^{-1}(U)$ which contains q (V_g is given by $\Delta_g \cap \phi_0^{-1}(U)$). We define a smooth map ψ from $(G \times G)_p \times V_g$ to $\phi_0^{-1}(U)$ by

$$\psi : ((h, k), (x', g, gx')) \mapsto (hx', kgh^{-1}, kgx'). \tag{4.1.2}$$

Then, ψ is $(G \times G)_p$-equivariant by the definition of the $G \times G$-action on Δ. Since U is preserved under the $(G \times G)_p$ and V_g is a connected component, ψ should be surjective.

Suppose two different points

$$q_1 = ((h_1, k_1), (x_1, g, gx_1)) \quad \text{and} \quad q_2 = ((h_2, k_2), (x_2, g, gx_2))$$

in $(G \times G)_p \times V_g$ are mapped to the same point in $\phi_0^{-1}(U)$ by ψ. This happens pre-

cisely when $(h_2^{-1}h_1, k_2^{-1}k_1)$ *sends* (x_1, g, gx_1) *to* (x_2, g, gx_2). *In particular, we have* $(h_2^{-1}h_1, k_2^{-1}k_1) \in (G \times G)_q$. *Therefore,* ψ *descends to a map*

$$\bar{\psi} : (G \times G)_p \times_{(G\times G)_q} V_g \to \phi^{-1}(U)$$

which is bijective. (Here, $(a, b) \in (G \times G)_q$ *acts on the first factor of* $(G \times G)_p \times V_g$ *by* $(h, k) \mapsto (ha^{-1}, kb^{-1})$*.) Since the* $(G \times G)_q$*-action and the* $(G \times G)_p$*-action on* $(G \times G)_p \times V_g$ *commute, the* $(G \times G)_p$*-equivariance of* ψ *implies that of* $\bar{\psi}$*.* □

4.2 Inertia orbifolds and orbifold embeddings

In this section we show that given an orbifold embedding, there is an induced orbifold embedding between their inertia orbifolds under abelian assumption.

First, let us recall inertia orbifolds. The following diagram defines a smooth manifold $\mathcal{S}_{\mathcal{G}}$, which can be interpreted intuitively as a set of loops (i.e. elements of local groups) in $\mathcal{G}$:

$$\begin{array}{ccc} \mathcal{S}_{\mathcal{G}} & \longrightarrow & G_1 \\ \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle (s,t)} \\ G_0 & \xrightarrow{diag} & G_0 \times G_0. \end{array} \tag{4.2.3}$$

Then, the inertia orbifold $\Lambda\mathcal{G}$ will be an action groupoid $\mathcal{G} \ltimes \mathcal{S}_{\mathcal{G}}$. i.e.

$$(\Lambda\mathcal{G})_0 = \mathcal{S}_{\mathcal{G}},$$

$$(\Lambda\mathcal{G})_1 = G_1 \times_{G_0} \mathcal{S}_{\mathcal{G}}$$

where for $h \in G_1$ the induced map $h : \beta^{-1}(s(h)) \to \beta^{-1}(t(h))$ is given by the conjugation. More precisely, for any $g \in \beta^{-1}(s(h))$, set $h(g) = hgh^{-1}$. This gives a target map from $(\Lambda\mathcal{G})_1$ to $(\Lambda\mathcal{G})_0$ whereas the source map is simply the projection to the second factor of $(\Lambda\mathcal{G})_1$. Note that $\beta^{-1}(s(h))$ and $\beta^{-1}(t(h))$ are the sets of loops in $\mathcal{G}$ based at $s(h)$ and $t(h)$, respectively. Similarly, one can define $\mathcal{S}_{\mathcal{H}}$ and $\Lambda\mathcal{H}$ for a suborbifold $\mathcal{H}$ of $\mathcal{G}$.

Now, let us see how $\phi : \mathcal{H} \to \mathcal{G}$ induces a morphism $\Lambda\phi$ between inertia orbifolds. $\Lambda\phi_0$ should be a map from $(\Lambda\mathcal{H})_0 = \mathcal{S}_{\mathcal{H}}$ to $(\Lambda\mathcal{G})_0 = \mathcal{S}_{\mathcal{G}}$. Suppose (h, y), $y = \beta(h) \in H_0$ is a loop $h : y \to y$ in $\mathcal{H}$. Then, the image of this loop is $(\phi_1(h), \phi_0(y))$ or, $\phi_1(h) : \phi_0(y) \to \phi_0(y)$, i.e.

$$\Lambda\phi_0 : (h, y) \mapsto (\phi_1(h), \phi_0(y)).$$

$\Lambda\phi_1$ maps $(h', h) \in (\Lambda\mathcal{H})_1 = H_1 \times_{H_0} \mathcal{S}_{\mathcal{H}}$ as follows:

$$\Lambda\phi_1 : (h', h) \mapsto (\phi_1(h'), \phi_1(h)).$$

If $h : y \to y$, then $\phi_1(h) : \phi_0(y) \to \phi_0(y)$.

Lemma 4.2.1. *If $\mathcal{G}$ is abelian, i.e. G_x is an abelian group for each $x \in \mathcal{G}_0$, then $\Lambda\mathcal{H}$-action on $\Lambda\phi_0^{-1}(g, x)$ is transitive.*

Proof. To observe the local behavior of $\Lambda\phi$, we use the local model of embeddings. Near $y \in H_0$, the local model and the morphism, again denoted by ϕ, is given as follows:

$$\phi : G_x \ltimes (G_x \times_{H_y} V_y) \to G_x \ltimes U_x,$$

where V_y and U_x are suitable neighborhoods of y and x, respectively and $x = \phi_0(y)$. Note that $\phi_0 : G_x \times_{H_y} V_y \to U_x$ is given as $\phi_0[g, y'] = g \cdot \phi_0(y')$ and $\phi_1 = (id, \phi_0) : G_x \times (G_x \times_{H_y} V_y) \to G_x \times U_x$. One can easily check that ϕ is well-defined.

Recall that we assumed ϕ_1 to be injective and identify H_y as a subgroup of G_x. We observe the fiber $\Lambda\phi_0^{-1}(g, x)$ for a loop $g : x \to x$ in $\mathcal{G}$ in these local models.

In our local model, any objects in $\Lambda\phi_0^{-1}(g, x)$ can be written as $(g, [g', y])$ for some $g' \in G_x$. Suppose that $(g, [g_1, y])$ and $(g, [g_2, y])$ are distinct objects in $\Lambda\phi_0^{-1}(g, x)$. Now we want to find $k \in \Lambda\mathcal{H}_1$ which sends $(g, [g_1, y])$ to $(g, [g_2, y])$, i.e. k such that $k \cdot (g, [g_1, y]) = (g, [g_2, y])$ or, equvalently $(kgk^{-1}, [kg_1, y]) = (g, [g_2, y])$. This can be simply achieved by choosing $k = g_2 g_1^{-1}$. □

For general $\mathcal{G}$, $\Lambda\mathcal{H}$-action on $\Lambda\phi_0^{-1}(g, x)$ is not necessarily transitive. In the last paragraph of the proof of the lemma, the abelian assumption is crucial to find $k \in \Lambda\mathcal{H}_1$ satisfying $(kgk^{-1}, [kg_1, y]) = (g, [g_2, y])$. If G_x is not abelian, such k may not exist. One may try with $k = g_2 g_1^{-1}$ which sends $[g_1, y]$ to $[g_2, y]$, but the loop kgk^{-1} is different from g if k does not commute with g. See the following example.

Example 4.2.2. *Let G be the subgroup of* $\mathrm{SL}(2, \mathbb{C})$ *generated by*

$$a = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.2.4}$$

where $\rho = e^{\pi i/3}$. G is called the binary dihedral group of order 12. Consider its fundamental representation on $\mathbb{C}^2$. Relations on generators a and b are given by

$$a^6 = b^4 = 1, \quad bab^{-1} = a^{-1}, \quad a^3 = b^2.$$

Let V be the first coordinate axis in $\mathbb{C}^2$. Then, the subgroup H of G generated by a acts on V. Now,

$$G \ltimes (G \times_H V)$$

gives rise to an orbifold embedding into $[\mathbb{C}^2/G]$ whose image is the union of two coordinate axes in $\mathbb{C}^2$. Note that on the level of inertia, $(a, [e, 0])$ and $(a, [b, 0])$ in $\Lambda (G \ltimes (G \times_H V))$ are both mapped to $(a, (0, 0))$ in $\Lambda[\mathbb{C}^2/G]$ by the induced map between inertias.

We claim that there is no arrow between $(a, [e, 0])$ and $(a, [b, 0])$ in the inertia $\Lambda (G \ltimes (G \times_H V))$ and therefore, the induced map is not an orbifold embedding. Such an arrow would first send $[e, 0]$ to $[b, 0]$ and hence, it would be of the form bh for some $h \in H$. This arrow sends the loop a at $[e, 0]$ to the loop $(bh)\, a\, (bh)^{-1}$ at $[b, 0]$. However, for any $h \in H$, $(bh)\, a\, (bh)^{-1} = bab^{-1} = a^{-1}$ since H is abelian.

Finally we prove that $\Lambda\phi : \Lambda\mathcal{H} \to \Lambda\mathcal{G}$ satisfies the condition (2) of the orbifold embedding (4.1.1) under the abelian assumption.

Proposition 4.2.3. *Given an orbifold embedding $\phi : \mathcal{H} \to \mathcal{G}$, consider the induced map between inertia orbifolds $\Lambda\phi : \Lambda\mathcal{H} \to \Lambda\mathcal{G}$. If $\mathcal{G}$ is an abelian orbifold, then $\Lambda\phi$ is again an orbifold embedding.*

Proof. First of all, it is clear that $\Lambda\phi_0$ is an immersion. This follows from the fact that the sector fixed by the loop $h \in H_1$ should be mapped through ϕ_0 to the sector fixed by $\phi_1(h) \in G_1$. So $\Lambda\phi_{0*}$ is essentially the same as ϕ_{1*}, which sends tangent vector to the sector fixed by $h \in H_1$ to the one determined by $\phi_1(h)$.

The only non-trivial part is the second condition. For this, we can work on local charts. Suppose $\phi_0(y) = x$ for $y \in \mathcal{H}_0, x \in \mathcal{G}_0$, and we fix a neighborhood U_x of x in G_0 from the embedding property of ϕ, so that $H|_{\phi^{-1}(U_x)}$ can be identified with the action groupoid $G_x \ltimes (G_x \times_{H_y} V_y)$. We may identify H_y as a subgroup of G_x via the embedding map.

We fix an element $g \in H_y \subset G_x$. In general, the local chart of inertia orbifold $\Lambda\mathcal{G}$ near $(g, x) \in \Lambda\mathcal{G}_0$ can be written as

$$C_G(g) \ltimes (U_x^g \times \{g\}), \tag{4.2.5}$$

where U_x^g is the set of g-fixed points in U_x and $C_G(g) = \{h \in G_x | hg = gh\}$ acts on U_x^g by the left multiplication. We put $\{g\}$ in (4.2.5) to indicate the sector in the inertia orbifold $\Lambda\mathcal{G}$, and we will drop it for notational simplicity in the following.

In our case, $C_G(g) = G_x$ since $\mathcal{G}$ is abelian. We rewrite the local chart (4.2.5) as

$$G_x \ltimes U_x^g. \tag{4.2.6}$$

Choose an element $(g, [e, y])$ in $\Lambda\phi_0^{-1}(g, x)$. Then V_y^g is an open neighborhood of $(g, [e, y])$ and

$$\Lambda\mathcal{H}|_{V_y^g} \cong H_y \ltimes V_y^g.$$

Note that $H_y = C_H(g) := \{h \in H_y | hg = gh\}$, since H_y is a subgroup of the abelian group G_x.

We claim that the inverse image of the local chart (4.2.6) of (g, x) by $\Lambda\phi^{-1}$ can be written as

$$G_x \ltimes (G_x \times_{H_y} V_y^g).$$

To see this, we only need to show that the twisted g-sector in the local chart of $\mathcal{H}|_{\phi^{-1}(G_x \ltimes U_x)}$ is isomorphic to $G_x \ltimes (G_x \times_{H_y} V_y^g)$. It can be checked as follows:

If $[g', z] \in G_x \times_{H_y} V_y$ is a fixed point of g, then $[gg', z] = [g', z]$. By definition, this happens if $gg' = g'h$ and $hz = z$ for some $h \in H_y$. This is equivalent to $(g')^{-1}gg'z = z$, and by the abelian assumption on G_x, $gz = z$. Hence, objects in g-twist sector of $\mathcal{H}|_{\phi^{-1}(G_x \ltimes U_x)}$ are contained in $G_x \times_{H_y} V_y^g$. Conversely, using the condition that G_x is abelian and $g \in H_y$, it follows that any element $[g', z] \in G_x \times_{H_y} V_y^g$ is fixed by g.

By the definition of arrows in an inertia groupoid, $G_x \times (G_x \times_{H_y} V_y^g)$ is the arrow space of the g-twisted sector of $\mathcal{H}|_{\phi^{-1}(G_x \ltimes U_x)}$ with an obvious action map, and this proves the proposition. $\square$

4.3 Orbifold embeddings and equivariant immersions

In this section we show that equivariant immersions which are *strong* (in the sense that will be defined later) give rise to orbifold embeddings between orbifold quotients.

First, let us review orbifold quotients and its groupoid analogue, translation groupoids. Let G be a compact Lie group which acts on M smoothly. The quotient $[M/G]$ naturally has a structure of a translation groupoid.

Definition 4.3.1. *Suppose a Lie group G acts smoothly on a manifold M from the left. The translation groupoid $[G \ltimes M]$ assoicated to this group action is defined as follows. Let $(G \ltimes M)_0 := M$ and $(G \ltimes M)_1 := G \times M$, with $s : G \times M \to M$ the projection and $t : G \times M \to M$ the action. The other structure maps are defined in the natural way.*

In particular, we are interested in group actions which give rise to an orbifold groupoid structure.

Definition 4.3.2. *A G-action on M is said to be locally free if the isotropy groups G_p are discrete for all $p \in M$.*

Now we assume that the G-action on M is locally free. The compactness of G implies that G_x is finite for all $x \in M$. Since G acts on M locally freely, we have a representation of $[M/G]$ as an orbifold groupoids in the following manner which is called the slice representation in [MP].

Proposition 4.3.1. *For any translation groupoid $[M/G]$, there is an orbifold groupoid $\mathcal{G}$ with an equivalence groupoid homomorphism $p : \mathcal{G} \to [M/G]$.*

Proof. By the slice theorem, we can cover M by a collection of G-invariant open sets $\{U_i\}$ with G-equivariant diffeomorphisms

$$\psi_i : G \times_{G_i} V_i \to U_i$$

where V_i is a normal slice with local action of $G_i \leq G$. Define $\mathcal{G}$ as follows. Let $G_0 := \sqcup_i V_i$ be the disjoint union of all the V_i, and define a map $p : G_0 \to M$ as

$p(i, v) := \psi_i([1, v])$. Define G_1 as the pullback bundle of following diagram.

$$\begin{array}{ccc} G_1 & \longrightarrow & G \times M \\ \Big\downarrow{\scriptstyle (s,t)} & & \Big\downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow[(p,p)]{} & M \times M \end{array}$$

Then groupoid homomorphism $p : \mathcal{G} \to [M/G]$ is an equivalence. See the proof of Theorem 4.1 in [MP] for more details. □

The converse in general still remains as a conjecture. The conjecture was partially proven in the case of effective orbifold groupoids (Theorem 1.23 of [ALR]).

Conjecture 4.3.2. *Every orbifold groupoid can be represented by translation groupoid with locally free group action.*

Now, let us recall the definition of an equivariant immersion and introduce what we call strong equivariant immersion.

Definition 4.3.3. *Let N, M be G-manifolds. A G-equivariant immersion from N into M is a smooth map $\iota : N \to M$ such that*

1. *the derivative $d\iota : T_xN \to T_{\iota(x)}M$ is injective at every point in N;*
2. *$\iota(g \cdot x) = g \cdot \iota(x)$.*

When ι is an equivariant immersion, the inverse image of $p \in \iota(N) \subset M$ admits a natural G_p action. If $q \in N$ is a point in $\iota^{-1}(p)$, then for $g \in G_p$

$$\iota(g \cdot q) = g \cdot \iota(q) = g \cdot p = p. \tag{4.3.7}$$

Definition 4.3.4. *Suppose the G-action on N is locally free and $\iota : N \to M$ be a G-equivariant immersion. We call ι a* strong *G-equivariant immersion if for every $p \in M$, G_p action on $\iota^{-1}(p)$ is transitive.*

Here is an example. Let N be a submanifold of M, which may not be necessarily preserved by G-action. We take G copies of N, and denote it by $\widetilde{N}$. i.e. $\widetilde{N} = G \times N$. $\widetilde{N}$ admits a natural G-action

$$g : (h, x) \mapsto (gh, x) \tag{4.3.8}$$

for $g, h \in G$ and $x \in N$. An immersion $\iota : \widetilde{N} \to M$ defined by $\iota(g, x) = g \cdot x$ is then G-equivariant.

Lemma 4.3.3. *The G-equivariant immersion $\widetilde{\iota} : \widetilde{N} \to M$ obtained above is strong if and only if*

$$N \cap g \cdot N = N^g. \tag{4.3.9}$$

for all $g \in G$.

Proof. From the definition of ι, only the image under ι of a point in $h \cdot N \cap g \cdot N$ can have a multiple fiber. Up to the G-action, it suffices to consider a point, say $y \in N \cap g \cdot N$. Then there exists $x \in N$ such that $g \cdot x = y$. Observe that $(1, y)$ and (g, x) in $\widetilde{N}$ maps to the same point $y \in M$. For ι to be strong, there should be a group element mapping $(1, y)$ to (g, x), and from (4.3.8), this implies $x = y$. Therefore, $g \cdot x = x$ and, hence $x \in N^g$. □

When a nontrivial subgroup G_N of G preserves N but do not fix N, then condition (4.3.9) cannot be satisfied in general. However, we may try to use the minimal number of copies of N. Define G_N so that we have the property, $g \cdot N = h \cdot N$ if and only if $g^{-1}h \in G_N$. Thus, for an element α of the coset space G/G_N, αN is well defined. Let

$$\widetilde{N} = \bigcup_{\alpha \in G/G_N} \alpha N \times \{\alpha\}.$$

$\widetilde{N}$ is a G-space by letting

$$g : (x, \alpha) \mapsto (g \cdot x, g \cdot \alpha).$$

Obviously, the natural immersion $\iota : \widetilde{N} \to M$, $\iota(x, \alpha) = x$, is G-equivariant.

With this construction, we can interpret the orbifold diagonal for a global quotient orbifolds (cf. 4.1.2) as a strong equivariant immersion: Suppose a finite group G acts on M and let N be the diagonal submanifold of $M \times M$. Then, $G \times G/\Delta_G$ parametrizes sheets of the domain of the immersion where $\Delta_G = \{(g, g) | g \in G\}$. i.e.

$$\widetilde{N} = \bigcup_{\alpha \in G \times G/\Delta_G} \alpha N \times \{\alpha\}.$$

To see that $\widetilde{N} \to M \times M$ is strong, assume $(x, gx, [1, g])$ and $(x, hx, [1, h])$ are mapped to the same point z in $M \times M$ ($[1, g], [1, h] \in G \times G/\Delta_G$). Then, $(h^{-1}g, 1)$

belongs to the local isotropy of $z = (x, gx) = (x, hx) \in M \times M$ and it sends $(x, gx, [1, g])$ to $(x, hx, [1, h])$ since $h^{-1}gx = x$ and

$$(h^{-1}g, 1)[1, g] = [h^{-1}g, g] = [h^{-1}, 1] = [1, h].$$

One nice property which follows from the strong condition is that the strong equivariant immersions always induce injective maps between the quotient spaces.

Lemma 4.3.4. *If $\iota : N \to M$ is a strong G-equvariant immersion between two G-spaces N and M, then,*

$$|\iota| : |N/G| \to |M/G|$$

is injective.

Proof. Let $|\iota|(\overline{q_1}) = |\iota|(\overline{q_2})$ in $|M/G|$ for $\overline{q_i} \in |N/G|$. Then,

$$\iota(q_1) = g \cdot \iota(q_2)$$

for some $g \in G$. Denote $\iota(q_1)$ by p. We have to find $h \in G$ such that $h \cdot q_1 = q_2$. Observe that

$$\iota(g \cdot q_2) = g \cdot \iota(q_2) = \iota(q_1) = p,$$

which implies that $g \cdot q_2$ and q_1 lie over the same fiber $\iota^{-1}(p)$ of ι. Since ι is strong, there is $h' \in G_p$ such that $h' \cdot q_1 = g \cdot q_2$. By letting $h = g^{-1}h'$, we prove the claim. □

Next, we use the local model of strong G-equivariant immersion to construct an orbifold embedding.

Proposition 4.3.5. *Let $\iota : N \to M$ be a strong G-equivariant immersion between two G-manifolds with locally free G-actions. Then, there exist orbifold groupoid representations $\mathcal{H}$ and $\mathcal{G}$ of $[N/G]$ and $[M/G]$ respectively so that ι induces an orbifold embedding $\phi_\iota : \mathcal{H} \to \mathcal{G}$ whose underlying map between quotient spaces is injective.*

We will give a proof at the end of this section, after we discuss local models. The following lemma is an analogue of standard slice theorem.

Lemma 4.3.6. *Let M be a manifold on which a compact Lie group G acts locally freely. Suppose $\iota : N \to M$ is a G-equivariant immersion. For $q \in N$ and $p = \iota(q) \in M$, we can find a G-invariant neighborhood $\widetilde{U}_p$ of $G \cdot p$ in M and $\widetilde{V}_q$ of $G \cdot q$ in N with the following properties:*

(i) There are normal slices U_p and V_q to $G \cdot p$ and $G \cdot q$ at p and q respectively such that

$$\widetilde{U}_p \cong G \times_{G_p} U_p \quad \widetilde{V}_q \cong G \times_{G_q} V_q. \tag{4.3.10}$$

(ii) There is an G_q-equivariant embedding $e : V_q \to U_p$ such that the diagram

$$\begin{array}{ccc} \widetilde{V}_q & \xrightarrow{\iota} & \widetilde{U}_p \\ \downarrow{\cong} & & \downarrow{\cong} \\ G \times_{G_q} V_q & \xrightarrow{[id,e]} & G \times_{G_p} U_p \end{array}$$

commutes where the map on the second row is given by $(g, v) \mapsto (g, e(v))$.

Proof. This is a relative version of the slice theorem (see for example Theorem B.24 of [GGK]). We briefly sketch the construction of the slice here. Fix any G-invariant metric ξ on M. The exponential map E identifies a neighborhood of p in M with a neighborhood of 0 in T_pM. Moreover, ξ induces a decomposition

$$T_pM \cong T_p(G \cdot p) \oplus W \tag{4.3.11}$$

where W is normal to the orbit and hence, it is equipped with the linear G_p action (coming from the one on $T_pM/T_p(G \cdot p)$). Let $U_p \subset W$ be a G_p-invariant small disk in W around the origin on which E is a diffeomorphism. Now,

$$\psi : G \times_{G_p} U_p \to M, \quad [g, u] \mapsto g \cdot E(u)$$

is well defined and G-equivariant. Since ψ is a local diffeomorphism at the point $[e, o]$, G-equivariance implies that it is a local diffeomorphism at all points of the form $[g, 0]$. One can check that ψ is indeed injective if U_p is sufficiently small. (See the proof of Theorem B.24 in [GGK] for details.)

To get the relative version, we pull back ξ to N by ι. Since ι is an immersion, $\iota^*\xi$ gives a metric on N. From the G-equivariant injection $T_qN \xrightarrow{\iota_*} T_pM$, we can choose a decomposition compatible with (4.3.11):

$$T_qN \cong T_p(G \cdot q) \oplus W'$$

i.e. ι_* is decomposed as

$$\iota_* = (\iota_*^O, \iota_*^N) : T_q(G \cdot q) \oplus W' \to T_p(G \cdot p) \oplus W.$$

Note that $G_q \subset G_p$ and ι_*^N is G_q-equvariant. Let V_q be the inverse image of U_p by ι_*^N. We may assume that $G \times_{G_q} V_q$ is diffeomorphic to a neighborhood of $G \cdot q$ by shrinking U_p if necessary. Thus, we proved (i).

Finally, by letting e the restriction of ι_*^N to V_q, we get (ii). □

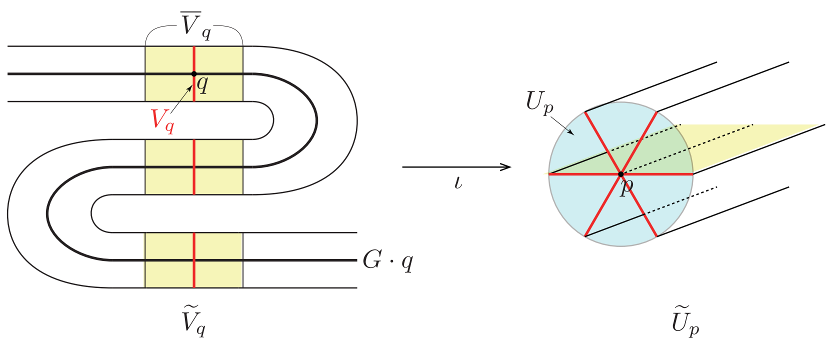


Figure 4.4: Slices

The following lemma provides the local model which is needed for the orbifold embedding.

Lemma 4.3.7. *Under the setting of Lemma 4.3.6, assume further that ι is strong. Then, we can find a G_p-invariant neighborhood $\overline{U}_p(\subset \widetilde{U}_p)$ of p in M and a G_q-invariant neighborhood $\overline{V}_q(\subset \widetilde{V}_q)$ of q in N such that there is a G_p-equivariant isomorphism*

$$\iota^{-1}(\overline{U}_p) \cong G_p \times_{G_q} \overline{V}_q \tag{4.3.12}$$

where the G_q-action on $G_p \times \overline{V}_q$ is given by

$$h \cdot (g, x) = (gh^{-1}, h \cdot x).$$

Proof. We take a *product type* neighborhood $\overline{V}_q(\subset \widetilde{V}_q)$ and $\overline{U}_p(\subset \widetilde{U}_p)$ of q and p as follows. We first identify the orbit $G \cdot q$ with G/G_q where q corresponds to the image of identity $[e]$ in G/G_q and $G \cdot p$ with G/G_p in a similar way. Then the tubular neighborhood $G \times_{G_q} V_q$ can be regarded as a fiber bundle over G/G_q. Take an open neighborhood O_q of $[e]$ in G/G_q which is invariant under the left G_q-action on G/G_q. As $[e]$ is fixed by this G_q-action, one can for instance choose left G_q-invariant metric on G/G_q and then, take O_q to be a small open ball around $[e]$. We may assume O_q is small enough so that

$$g \cdot O_q \cap O_q = \phi \tag{4.3.13}$$

for nontrivial $g \in G_p \setminus G_q$. (Note that G_p also acts on G/G_q from the left.) This is possible since G_p is finite. Let O_p be the image of O_q by the map $G/G_q \to G/G_p$. (4.3.13) implies that the map $O_q \to O_p$ is an embedding.

Finally, we define $\overline{V}_q$ and $\overline{U}_p$ to be open neighborhoods of q and p, respectively, such that the following diagrams are cartesian. (See Figure 4.4.)

$$\begin{array}{ccc} \overline{V}_q & \hookrightarrow & G \times_{G_q} V_q \\ \downarrow & & \downarrow \\ O_q & \hookrightarrow & G/G_q \end{array} \qquad \begin{array}{ccc} \overline{U}_p & \hookrightarrow & G \times_{G_p} V_p \\ \downarrow & & \downarrow \\ O_p & \hookrightarrow & G/G_p \end{array} \tag{4.3.14}$$

Then, by (4.3.13) we have

$$g \cdot \overline{V}_q = \begin{cases} \overline{V}_q & \text{if } g \in G_q \\ \text{disjoint from } \overline{V}_q & \text{if } g \in G_p \setminus G_q \end{cases} \tag{4.3.15}$$

(More precisely, $\overline{V}_q$ and $\overline{U}_p$ are image of these fiber products under the isomorphisms shown in (i) of the previous lemma.) Observe that $\iota|_{\overline{V}_q} : \overline{V}_q \to \overline{U}_p$ is an embedding since both $O_q \to O_p$ and $V_q \to U_p$ are embeddings.

Since the G_p-action on $\iota^{-1}(p)$ is transitive, there is $|G_p|/|G_q|$-open subsets of N (isomorphic to $\overline{V}_q$) which are mapped to $\overline{U}_p$. By (4.3.15), $\iota^{-1}(\overline{U}_p)$ is the disjoint union of these open subsets of N.

Now, define $\tilde{\phi} : G_p \times \overline{V}_q \to \iota^{-1}(\overline{U}_p)$ by

$$\begin{array}{rcl} \tilde{\phi} : G_p \times \overline{V}_q & \longrightarrow & \iota^{-1}(\overline{U}_p) \\ (g, x) & \longmapsto & g \cdot x \end{array}$$

This map is well defined because ι is G_p-equivariant and $\overline{U}_p$ is G_p-invariant subset of M. Furthermore, $\tilde{\phi}$ is surjective (and hence a submersion) by the strong condition of ι. It remains to show that it is injective up to G_q-action.

Suppose $\tilde{\phi}$ sends (g, x) and (g', x') in $G_p \times \overline{V}_q$ to the same point in $\iota^{-1}(\overline{U}_p)$. Then $g \cdot x = g' \cdot x'$, equivalently $(g')^{-1} g \cdot x = x'$. Note that both x and x' belong to $\overline{V}_q$ and $(g')^{-1} g \in G_p$. From the dichotomy (4.3.15), we have $(g')^{-1} g = h$ for some $h \in G_q$. Therefore, $g' = gh^{-1}$ and $x' = h \cdot x$ for $h \in G_q$. We conclude that $\tilde{\phi}$ is indeed a principal G_q-bundle and the isomorphism (4.3.12) follows. □

Remark 4.3.8. *Note that the induced map $\phi : G_p \times_{G_q} \overline{V}_q \to \iota^{-1}(\overline{U}_p)$ is G_p-equivariant by definition.*

Proof of proposition 4.3.5. Suppose we have a strong G-equivariant immersion $\iota : N \to M$. By equivariance, ι induces a map $\phi'_\iota : [N/G] \to [M/G]$ and $|\phi'_\iota| : N/G \to M/G$ is clearly injective.

Consider a point $\bar{p}$ in $|M/G|$ and let $\pi_M(p) = \bar{p}$ for the quotient map $\pi_M : M \to |M/G|$. From the definition of strong equivariant immersion, the group action on $\iota^{-1}(p)$ is transitive. If $q \in N$ maps to p, then there exists a G_p-invariant product type neighborhood $\overline{U}_p$ of p in M and G_q-invariant $\overline{V}_q$ of q in N which satisfies (4.3.12) from the lemma 4.3.7. We add all such slices U_p and V_q into the slice representations of $[N/G]$ and $[M/G]$ to get orbifold groupoids $\mathcal{H}$ and $\mathcal{G}$.

Now the previous remark implies that

$$\mathcal{H}|_{\phi_0^{-1}(U_p)} \cong G_p \ltimes \left(G_p \times_{G_q} V_q\right).$$

Essential injectivity of the resulting orbifold morphism follows directly from Lemma 4.3.4. Since the other conditions in Definition 4.1.1 are automatic, we get an orbifold embedding $\phi : \mathcal{H} \to \mathcal{G}$. □

Section 4.4 will be devoted to prove the converse of this proposition.

4.4 Construction of equivariant immersions from orbifold embeddings

Let $\mathcal{H}$ be an orbifold groupoid, and $\phi : \mathcal{H} \to [M/G]$ be a groupoid morphism which factors through an orbifold embedding ψ and a slice representation p :

$\mathcal{G} \to [M/G]$.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\phi} & [M/G] \\ & {\scriptstyle \psi:orb.emb.} \searrow \quad \nearrow {\scriptstyle p:\cong} & \\ & \mathcal{G} & \end{array} \tag{4.4.16}$$

With this assumption in this section, we construct a G-equivariant immersion map $\iota : N \to M$ for some G-manifold N.

Proposition 4.4.1. *Consider $\mathcal{H}, [M/G], \phi, \psi$ as above.*

Then, there exists G-space N and a strong G-equivariant immersion $\iota : N \to M$ such that

- *$[N/G]$ is Morita equivalent to $\mathcal{H}$.*
- *the induced map $[N/G] \to [M/G]$, again denoted by ι, fits into the following diagram of Lie groupoid homomorphisms*

$$\begin{array}{ccc} [N/G] & \xrightarrow{\iota} & [M/G] \\ {\scriptstyle Morita \cong} \downarrow & \nearrow {\scriptstyle \phi} & \\ \mathcal{H} & & \end{array} \tag{4.4.17}$$

and ι is a G-equivariant immersion.

Proof. From the groupoid structure of $[M/G]$, we have smooth maps

$$\phi_0 : H_0 \to M \qquad \phi_1 : H_1 \to G \times M$$

which are compatible with the structure maps of an orbifold groupoid. As in [PS], we interpret ϕ as a Hilsum-Skandalis-type map. So, we define a bibundle R_ϕ as

$$R_\phi := (G \times M) \times_{s,M,\phi_0} H_0.$$

Note that R_ϕ is a smooth manifold since s is a submersion.

$$\begin{array}{ccc} R_\phi & \xrightarrow{\pi_1} & G \times M \\ {\scriptstyle \pi_2} \downarrow & & \downarrow {\scriptstyle s} \\ H_0 & \xrightarrow{\phi_0} & M \end{array} \tag{4.4.18}$$

This space is first of all smooth and has two maps to H_0 and M,

$$H_0 \overset{\pi_2}{\longleftarrow} R_\phi \overset{\widetilde{\iota}:=t\circ\pi_1}{\longrightarrow} M$$

which will be used as anchor maps below. (We denote $t \circ \pi_1$ by $\widetilde{\iota}$.)

We define a right $\mathcal{H}$-action and a left G-action on R_ϕ as follows: Write an element of R_ϕ by $(g, \phi_0(y), y)$ which indicates a point y in H_0 and an arrow $g \in G \times M$ whose source is $\phi_0(y)$. Then,

- For an arrow $h \in H_1$,

$$(g, \phi_0(y), y) \cdot h := (g \circ \phi_1(h), \phi_0(s(h)), s(h))\,;$$

- For $g \in G$,

$$g' \cdot (g, \phi_0(y), y) := (g' \circ g, \phi_0(y), y)\,.$$

R_ϕ is a right $\mathcal{H}$-space and a left G-space as the figure 4.5 below shows. As mentioned, the corresponding anchor maps are π_2 and $\tilde{\iota}$, respectively. Indeed, π_2 is a principal left G-bundle (π_2 is a submersion since s in the diagram (4.4.18) is submersion).

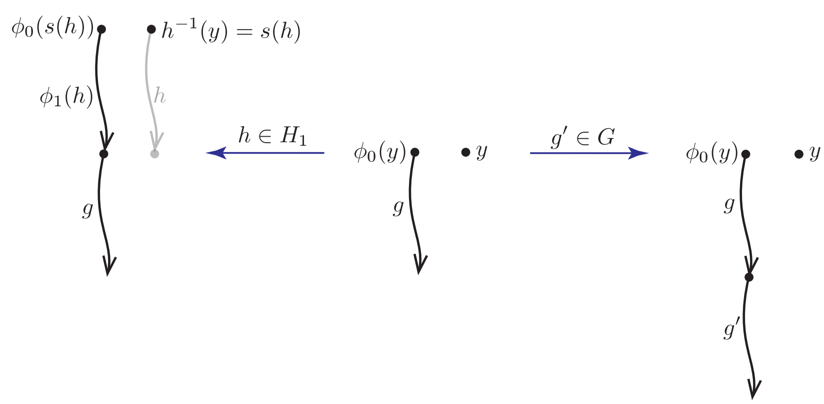


Figure 4.5: the right $\mathcal{H}$-action and the left G-action on R^*_ϕ

Now, the following are clear from the definition of both actions.

Lemma 4.4.2. *Two actions defined above have the following properties:*

(1) The right $\mathcal{H}$-action we have defined is free;

(2) The left G-action and the right $\mathcal{H}$-action commute;

(2) $\widetilde{\iota}$ is a G-equivariant map which is invariant under the $\mathcal{H}$-action.

Proof. We only show (1) and the others follow from the definition. Suppose h fixes $((g, \phi_0(y)), y) \in R_\phi$. Then h should be an element of H_y and $g \cdot \phi_1(h) = g$, where H_y is a local isotropy group of $y \in V_y$ for some local chart $H_y \ltimes V_y$ of $\mathcal{H}$. Thus h lies in the kernel of the group homomorphism

$$\phi_1|_{H_y} : H_y \to G.$$

Note that equivalence map $p : \mathcal{G} \to [M/G]$ preserves isotropy groups. More precisely,

$$Hom_{\mathcal{G}}(x, z) \cong Hom_{[M/G]}(p(x), p(z))$$

for all $x, z \in G_0$. Now it follows that $\phi_1| = p_1 \circ \psi_1|$ is injective, because $\psi_1|$ is injective from the definition of orbifold embedding and p_1 preserves isotropy groups. Hence, the h is the identity. □

We denote by N the quotient space of R_ϕ by the right $\mathcal{H}$-action and by $\pi : R^*_\phi \to N$ the quotient map.

Lemma 4.4.3. *N is a smooth manifold.*

Proof. This follows directly from the fact that the right $\mathcal{H}$-action on R_ϕ is free ((2) of Lemma 4.4.2) and proper (because $\mathcal{H}$ itself is étale and hence proper). □

From (2) of Lemma 4.4.2, N admits a left G-action which is induced by G-action on R_ϕ. Since the G-action on M is locally free, so is it on N. Therefore, we get a global quotient orbifold $[N/G]$ from the orbifold embedding $\phi : \mathcal{H} \to [M/G]$.

Lemma 4.4.4. *$[N/G]$ is Morita equivalent to $\mathcal{H}$.*

Proof. Note that we have a Hilsum-Skandalis map $\mathcal{H} \to [N/G]$ (or, $\bar{\pi}_2 : [N/G] \to \mathcal{H}$ from $\pi_2 : R_\phi \to H_0$):

$$\begin{array}{ccc} R_\phi & \xrightarrow{\pi_2} & H_0 \\ {\scriptstyle \pi}\downarrow & & \\ N & & \end{array}$$

We have shown that π is principal in (2) of Lemma 4.4.2. It is also obvious from the Hilsum-Skandalis construction that π_2 is principal. So the Hilsum-Skandalis map from R_ϕ is a Morita equivalence. □

From (3) of Lemma 4.4.2, we can observe that $\widetilde{\iota}$ factors through the quotient space H. Since $\widetilde{\iota}$ is a G-equivariant, we get a G-equivariant map $\iota : N \to M$. Furthermore,

Lemma 4.4.5. *ι is a G-equivariant immersion.*

$$\begin{array}{ccc} R_\phi & \xrightarrow{\widetilde{\iota}} & M \\ {\scriptstyle \pi}\downarrow & \nearrow_{\iota} & \\ N & & \end{array} \tag{4.4.19}$$

Proof. Since being an immersion is a local property, it suffices to prove it locally. However, we have a nice local model of ι from (4.1.1). Thus, it is enough to prove it with $\mathcal{H} = G_x \ltimes (G_x \times_{H_y} V_y)$, $\mathcal{G} = G_x \ltimes U_x$, $\psi : G_x \ltimes (G_x \times_{H_y} V_y) \to G_x \ltimes U_x$ and $p : G_x \ltimes U_x \to G \ltimes \widetilde{U}_x$ for $\widetilde{U}_x \cong G \times_{G_x} U_x$ as in the Lemma 4.3.6.

Then, $R_\phi = \left(G \times \widetilde{U}_x\right) \times_{s, \widetilde{U}_x, \phi_0} \left(G_x \times_{H_y} V_y\right)$. We mod it out by the right G_x-action (considered as a local H_1-action) to get the local shape of N, again denoted by N in this proof. Recall that this G_x-action is given by

$$((k_1, a), [g_1, b]) \cdot g = [(k_1 g, g^{-1} \cdot a), [g^{-1} g_1, b]]$$

for $g \in G_x$, $k_1 \in G$ and $g_1 \in G_x$ where $a = g_1 \cdot \phi_0(b)$. And $\tilde{\iota}$ on $\mathcal{R}_\phi$ which projects down to ι on N is defined as

$$\tilde{\iota}((k_1, a), [g_1, b]) = k_1 \cdot a = k_1 g_1 \cdot \phi_0(b) \in \widetilde{U}_x. \tag{4.4.20}$$

where $\phi = p \circ \psi : \mathcal{H} \to [M/G]$.

For given $z \in \widetilde{U}_x$, we check how many points in N are mapped to z. Suppose

$$\iota[(k_1, a_1), [g_1, b_1]] = z \quad \text{and} \quad \iota[(k_2, a_2), [g_2, b_2]] = z.$$

Up to the G_x-action, we may assume that $g_1 = g_2 = 1$ (recall $N = G_x \setminus R_\phi$). Therefore, we have $k_1\phi_0(b_1) = k_2\phi_0(b_2) = z$ by (4.4.20). This implies that $k_2^{-1}k_1 \in G_x$ since both $\phi_0(b_1)$ and $\phi_0(b_2)$ belong to the normal slice at x. As G_x is finite, there are finitely many k_2 with this property.

Since p_0 is an embedding and every fiber of ψ_0 is finite, $\phi_0 = p_0 \circ \psi_0$ is an immersion whose fibers are all finite as well. Finally, as b_2 lies in the fiber $\phi_0^{-1}(k_2^{-1}k_1b_1)$, there can exist only finitely many such b_2's. □

Finally, we show in the following lemma that the resulting equivaraint immersion is strong which finishes the proof of the proposition □

Lemma 4.4.6. *$\iota : N \to M$ constructed above is strong.*

Proof. Note that $|N/G| \cong |\mathcal{H}|$ and $|M/G| \cong |\mathcal{G}|$. From the construction in Section 4.5 (or Section 4.4), we have

$$\begin{array}{ccc} |N/G| & \xrightarrow{|\iota|} & |M/G| \\ \cong \big\downarrow & & \big\downarrow \cong \\ |\mathcal{H}| & \xrightarrow[|\phi|]{} & |\mathcal{G}| \end{array}$$

Since $|\phi|$ is injective from the definition of the orbifold embedding, $|\iota|$ is injective. □

4.5 General case

So far, we have considered a translation groupoid $[M/G]$ as our target space. The construction can be generalized to the case of general orbifolds which we will discuss from now on. We state this as a theorem, first.

Theorem 4.5.1. *Let $\phi : \mathcal{H} \to \mathcal{G}$ be an orbifold embedding, where $\mathcal{G}$ is Morita equivalent to a translation groupoid $[M/G]$. Then, there exist a manifold N on which the Lie group G acts locally freely such that*

(i) $\mathcal{H} \simeq [N/G]$ *and* $\mathcal{G} \simeq [M/G]$

(ii) there exists a G-equivariant immersion $\iota : N \to M$ which makes the diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\quad\phi\quad} & \mathcal{G} \\ {\scriptstyle Morita}\updownarrow & & \updownarrow{\scriptstyle Morita} \\ [N/G] & \xrightarrow[\quad\iota\quad]{} & [M/G] \end{array} \tag{4.5.21}$$

commute.

Remark 4.5.2. *The diagram in (ii) of the theorem can be regarded as a diagram of morphisms in the category of Lie groupoids where we can invert equivalances. (See Definition 2.2.9, or [ALR] for the precise definition of the morphisms in the category of groupoids.)*

We proceed the poof of theorem 4.5.1 as follows:

After fixing a Morita equivalence map

$$\mathcal{G} \overset{\psi\,\simeq}{\longleftarrow} \mathcal{G}' \overset{\sigma\,\simeq}{\longrightarrow} [M/G]$$

for some Lie groupoid $\mathcal{G}'$, we pull back the equivalence map $\psi : \mathcal{G}' \to \mathcal{G}$ to $\mathcal{H}$ to get $\phi^*\mathcal{G}' = \mathcal{H} \times_{\mathcal{G}} \mathcal{G}'$. Recall that

$$(\phi^*\mathcal{G}')_0 = H_0 \times_{\phi_0, G_0, s} G_1 \times_{t, G_0, \psi_0} G'_0$$
$$(\phi^*\mathcal{G}')_1 = H_1 \times_{s\phi_1, G_0, s} G_1 \times_{t, G_0, s\psi_1} G'_1$$

We denote the composition $\sigma \circ pr_2 : \phi^*\mathcal{G}' \to [M/G]$ by $\tilde{\phi}$. Then,

$$\tilde{\phi}_0 = \sigma_0 \circ (pr_2)_0 \quad \tilde{\phi}_1 = \sigma_1 \circ (pr_2)_1$$

where pr_2 is the projection from $\phi^*\mathcal{G}' = \mathcal{H} \times_{\mathcal{G}} \mathcal{G}'$ to $\mathcal{G}'$. We will apply the construction in the previous section to $\tilde{\phi}$.

$$\begin{array}{ccc} & & [M/G] \\ & \overset{\tilde{\phi}}{\nearrow} & \Big\uparrow {\scriptstyle \sigma:equiv.} \\ \phi^*\mathcal{G}' & \xrightarrow{pr_2} & \mathcal{G}' \\ {\scriptstyle pr_1}\Big\downarrow & & \Big\downarrow {\scriptstyle \psi:equiv.} \\ \mathcal{H} & \xrightarrow[\phi:orb.emb.]{} & \mathcal{G} \end{array} \tag{4.5.22}$$

First of all $\phi^*\mathcal{G}'$ is equivalent to $\mathcal{H}$. Pull-back of any equivalence is again an equivalence as shown in Lemma 2.2.5.

To construct a G-equivariant immersion from $\tilde{\phi} : \phi^*\mathcal{G}' \to [M/G]$, we introduce the Hilsum-Skandalis bibundle associated to $\tilde{\phi}$ as we did in the previous section. Recall

$$R_{\tilde{\phi}} = (G \times M) \times_{s,M,\tilde{\phi}_0} (\phi^*\mathcal{G}')_0.$$

An element of $R^*_{\tilde{\phi}}$ consists of the following data.

$$\begin{array}{c} \overset{m}{\odot} \\ \Big\downarrow {\scriptstyle a} \\ \end{array} \qquad \star\left(= [x,\ \ \phi_0(x) \xrightarrow{g} \psi_0(z),\ \ z]\right)$$

where $m \in M$, $a \in G$, $x \in H_0$, $z \in G'_0$ and $\sigma_0(z) = m$. Write $\mathbf{r}$ for this element. Then, $pr_2(\mathbf{r}) = z$ and the G-equivariant map $\tilde{\iota} : R_{\tilde{\phi}} \to M$ is given by $\tilde{\iota}(\mathbf{r}) = a \cdot \sigma_0(z)$. Recall

$$\begin{array}{ccc} R_{\tilde{\phi}} & \xrightarrow{\tilde{\iota}} & M \\ \Big\downarrow & \overset{\iota}{\nearrow} & \\ N & & \end{array} \tag{4.5.23}$$

where N is obtained from $R_{\tilde{\phi}}$ after taking a quotient by $\phi^*\mathcal{G}'$-action. Since local groups are preserved by equivalences, restriction of 1-level maps appearing in 4.5.22 to any local groups are all injective. Then, similar argument as in Lemma 4.4.2 shows the right $\phi^*\mathcal{G}'$-action on $R_{\tilde{\phi}}$ is free and proper. Note that $\phi^*\mathcal{G}$ is proper since it is equivalent to the proper Lie (indeed, étale) groupoid $\mathcal{G}$. Therefore, N is a smooth manifold.

It remains to show that the induced G-equivariant map $\iota : N \to M$ is indeed an immersion. We will directly compute the kernel of $d\tilde{\iota}$. For notational simplicity, we will write τ_* for the derivative $d\tau$ of a smooth map τ between two manifolds.

A tangent vector on $R_{\tilde{\phi}}$ at $\mathbf{r}$ is given by the tuple

$$\mathbf{v} = [(v_l = v_a \oplus v_m, v_m), v_x, v_g, v_z]$$

where $v_l \in T(G \times M)$ and $v_a \in TG$ with the relations

- $s_*(v_l) = v_m$
- $(\sigma_0)_* (v_z) = v_m$,
- $s_*(v_g) = (\phi_0)_* (v_x) \quad t_*(v_g) = (\psi_0)_* (v_z)$.

Since $\mathcal{G}$ is étale and hence both

$$s_* : T_g G_1 \to T_{\phi_0(x)} G_0 \quad \text{and} \quad t_* : T_g G_1 \to T_{\psi_0(z)} G_0$$

are isomorphisms, we may rewrite the third relation as

$$\bullet \ \ s_*^{-1} (\phi_0)_* (v_x) = t_*^{-1} (\psi_0)_* (v_z) = v_g. \tag{4.5.24}$$

From the first relation, it suffices to represent $\mathbf{v}$ as

$$\mathbf{v} = [v_a \oplus v_m, v_x, v_g, v_z]$$

Note that v_m and v_g are determined by v_x and v_z. One can easily check that

$$\tilde{\iota}_*(\mathbf{v}) = t_*(v_a \oplus v_m) = (v_a)^\# + (L_a)_*(v_m),$$

where $(v_a)^\#$ is a vector field on M generated by the infinitesimal action of v_a on M. For simplicity, we assume that a is the identity element of G. Then,

$$\tilde{\iota}_*(\mathbf{v}) = (v_a)^\# + v_m.$$

Our goal is to compute the kernel of this map. If we can show that the kernel of $\tilde{\iota}_*$ lies in the tangent direction of $(\phi^*\mathcal{G}')$-orbit, then it will imply that ι is an immersion.

To do this, we first characterize the direction of $(\phi^*\mathcal{G}')$-orbit. By the definiton of the right $\phi^*\mathcal{G}'$ action on $R_{\tilde{\phi}}$, we should consider an arrow of $\phi^*\mathcal{G}'$ given by a pair $(h, k) \in H_1 \times G'_1$ such that

$$t(h) = x \text{ and } t(k) = z. \tag{4.5.25}$$

Considering the infinitesimal version of $\phi^*\mathcal{G}'$-action carefully, we have the following lemma.

Lemma 4.5.3. $\mathbf{v}$ *is tangent to* $(\phi^*\mathcal{G}')$*-orbit if and only if there exists* $(v_h, v_k) \in T_hH_1 \times T_kG'_1$ *such that*

$$t_*(v_h) = t_*(v_k) = 0 \tag{4.5.26}$$

and

$$\mathbf{v} = [(\sigma_1)_*(v_k), s_*(v_h), v_g, s_*(v_k)]. \tag{4.5.27}$$

(Here, we do not specify v_g *since they are completely determined by other components* (4.5.24)*.)*

Remark 4.5.4. *In the equation (4.5.26),* $t_*(v_h) = 0$ *implies* $v_h = 0$ *since* $\mathcal{H}$ *is étale. Then the equation (4.5.27) can be rewritten as*

$$\mathbf{v} = [(\sigma_1)_*(v_k), 0, 0, s_*(v_k)]. \tag{4.5.28}$$

Now, we are ready to prove the desired property of ι.

Lemma 4.5.5. $\iota : N \to M$ *above is an immersion.*

Proof. Let $\mathbf{v} = [(v_a, v_m), v_x, v_g, v_z] \in \ker \tilde{\iota}_*$. i.e. $t_*(v_a \oplus v_m) = (v_a)_\# + v_m = 0$. We should show that $v_h = 0$ and find v_k satisfying (4.5.26) and (4.5.28). First, we find v_k as follows:
From the condition $(\sigma_0)_*(v_z) = v_m = (-v_a)_\#$, we get

$$\exp(t \cdot (-v_a)) \cdot m = m(t),$$

where $m(t)$ is a curve in M with $m'(0) = v_m$. Since $\exp(-t \cdot v_a) = \exp(t \cdot v_a)^{-1}$,

$$m \equiv \exp(t \cdot v_a) \cdot m(t).$$

Note that $m(t) = \sigma_0(z(t))$ for some curve $z(t)$ in G'_0 with $z'(0) = v_z$. Since σ is an equivalence map and $(a(t), m(t)) \in G \times M$ is an arrow from $\sigma_0(z(t))$ to $\sigma_0(z)$

for each t, there is unique $k(t) \in G'_1$ which maps to $\sigma_1(k(t)) = (a(t), m(t))$ with $s(k(t)) = z(t)$ and $t(k(t)) \equiv z$. We define $v_k := k'(0) \in T_k G'_1$, then $(\sigma_1)_*(v_k) = (v_a, v_m) \in TG \times TM$ and $s_*(v_k) = v_z$.

Let γ be a curve in the $R^*_{\tilde{\phi}}$ such that $\gamma'(0) = \mathbf{v}$. Consider a component of γ, $g(t) \in G_1$ such that $g'(0) = v_g$. We claim that this curve $g(t)$ is a constant curve.

Note that $\psi_0 \circ t \circ k(t) \equiv \psi_0(z)$. Since ψ is an equivalence map and $\mathcal{G}$ is étale, $\psi_0 \circ s \circ k(t) \equiv \psi_0(z)$($\psi_1$ maps "infinitesimal action" on $\mathcal{G}'$ whose image of target is fixed to a "constant action" on $\mathcal{G}$). Note that $t \circ g(t) = \psi_0 \circ z(t) = \psi_0 \circ s \circ k(t) = \psi_0(z)$. Since $\mathcal{G}$ is étale and target points of $g(t)$ is fixed, $g(t)$ is a constant arrow in G_1. Since $\phi_0 \circ x(t) = s \circ g(t) = \phi_0(x)$ and ϕ_0 is an immersion map, $x(t) \equiv x$.

We conclude that, if $\tilde{\iota}_*(\mathbf{v}) = 0$, then there exist v_k such that

$$\mathbf{v} = [(\sigma_1)_*(v_k), s_*(0) = 0, v_g = 0, s_*(v_k)], \tag{4.5.29}$$

it proves that $\iota : N \to M$ is an immersion. □

Lastly, the equivariant immersion ι is strong by basically the same argument as in Lemma 4.4.6.

Chapter 5

Holomorphic orbi-spheres in elliptic $\mathbb{P}^1$ orbifolds and Diophantine equations[1]

In this chapter, we compute quantum cohomology ring of elliptic $\mathbb{P}^1$ orbifolds via orbi-sphere counting. The main technique is the classification theorem which relates holomorphic orbi-spheres with certain orbifold coverings. The counting of orbi-spheres are related to the integer solutions of Diophantine equations. This reproduces the computation of Satake and Takahashi in the case of $\mathbb{P}^1_{3,3,3}$ via different method.

5.1 Orbi-maps between two dimensional orbifolds

The main tool for our argument in this chapter is *an* orbifold covering theory established for orbi-maps in Definition 2.4.2 by Takeuchi [T]. Although orbifold quantum cohomology is defined by counting good maps given in (2) of Definition 2.2.2, in the case of elliptic $\mathbb{P}^1$ orbifolds, it turns out that maps given in (1) of Definition 2.2.2 satisfy axioms in Definition 2.4.2.

Remark 5.1.1. *From Lemma 2.2.2, a smooth map $f : X \to X'$ between two orbifolds X and X' is a good map with a unique compatible system up to isomorphism,*

[1] This chapter is based on [HS].

if the inverse image of the regular part of X' is an open dense and conneted subset of X. Note that a non-constant smooth map contributing to the quantum product for $\mathbb{P}^1_{a,b,c}$ automatically satisfies this property.

Consider two (good) orbifolds X and Y which admit manifold universal covering spaces $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$, respectively. Assume that the deck transformation action of p and q are orientaion preserving. Moreover, assume that both X and Y are two dimensional (which is the case of our main interest). Note that singular loci of X and Y are sets of isolated points.

Lemma 5.1.2. *With the setting as above, any non-constant smooth map $\phi : X \to Y$ satisfies the axioms in Definition 2.4.2 if* $\dim X = \dim Y = 2$*.*

Proof. We first show that there is a continuous map $\tilde{\phi} : \widetilde{X} \to \widetilde{Y}$ which lifts $\phi : X \to Y$.

$$\begin{array}{ccc} \widetilde{X} & \overset{\tilde{\phi}}{\dashrightarrow} & \widetilde{Y} \\ {\scriptstyle p}\downarrow & \overset{\phi\circ p}{\searrow} & \downarrow{\scriptstyle q} \\ X & \underset{\phi}{\longrightarrow} & Y \end{array} \tag{5.1.1}$$

What we want to have is basically a lift of the map $\phi \circ p : \widetilde{X} \to Y$. We claim that at each point $\tilde{x} \in \tilde{X}$ there is a local lifting of $\phi \circ p$. Let $x := p(\tilde{x})$. Then one can find a neighborhood of $\tilde{x}$ which uniformizes X locally around x. Since the same is true for any point in the inverse image $q^{-1}(\phi(x))$, we can find a local lifting of ϕ around $\tilde{x}$ by the properties of orbifold maps.

By gathering such a neighborhood for each $\tilde{x} \in \widetilde{X}$, we obtain a open covering $\widetilde{\mathcal{U}} = \left\{\widetilde{U}_i : \widetilde{U}_i \subset \widetilde{X}, i \in I\right\}$ of $\widetilde{X}$ which consists of open subsets of $\widetilde{X}$ on which ϕ can be locally lifted. For each $\widetilde{U}_i \in \widetilde{\mathcal{U}}$, we fix a local lifting $\tilde{\phi}_i$ of ϕ. On the intersection of two open subsets $\widetilde{U}_i$ and $\widetilde{U}_j$ in $\widetilde{\mathcal{U}}$, two local liftings $\tilde{\phi}_i$ and $\tilde{\phi}_j$ differ by an element g_{ij} of $\mathrm{Aut}(\widetilde{Y}, q) \cong \pi_1^{orb}(Y)$. i.e.

$$\tilde{\phi}_i|_{\widetilde{U}_i \cap \widetilde{U}_j} = g_{ij} \circ \tilde{\phi}_j|_{\widetilde{U}_i \cap \widetilde{U}_j}. \tag{5.1.2}$$

Note that $\{g_{ij} : i, j \in I\}$ satisfies the usual cocycle condition, that is,

$$g_{ij} g_{jk} g_{ki} = 1. \tag{5.1.3}$$

(5.1.3) follows from

$$\begin{aligned} \tilde{\phi}_i &= g_{ij} \circ \tilde{\phi}_j \\ &= (g_{ij} g_{jk}) \circ \tilde{\phi}_k \\ &= (g_{ij} g_{jk} g_{ki}) \circ \tilde{\phi}_i \end{aligned}$$

on $\widetilde{U}_i \cap \widetilde{U}_j \cap \widetilde{U}_k$ and the fact that the action of $\pi_1^{orb}(Y)$ on Y is free generically. (Recall that ϕ is a non-constant map.)

Therefore, $\{g_{ij}\}_{i,j \in I}$ defines a principal $\pi_1^{orb}(Y)$-bundle over $\widetilde{X}$ or equivalently a covering space of $\widetilde{X}$. Here, g_{ij} glues $\widetilde{U}_i \times \pi_1^{orb}(Y)$ and $\widetilde{U}_j \times \pi_1^{orb}(Y)$ by the left multiplication so that the resulting bundle admits the right action of $\pi_1^{orb}(Y)$. Since $\pi_1^{orb}(Y)$ is discrete and $\widetilde{X}$ is simply connected, this bundle should be trivial. Therefore, the cocycle $\{g_{ij}\}$ is also trivial up to coboundary. i.e. there exists a collection $\{(\epsilon_i, \widetilde{U}_i) \in \pi_1^{orb}(Y) \times \widetilde{\mathcal{U}} \ : \ i \in I\}$ of elements of $\pi_1^{orb}(Y)$ each of which is associated with an open subset in $\widetilde{\mathcal{U}}$ such that

$$g_{ij} = \epsilon_i^{-1} \epsilon_j. \tag{5.1.4}$$

(In other words, $\{\epsilon_i\}$ trivializes the principal bundle corresponding to the data $\{g_{ij}\}$.)

From (5.1.2) and (5.1.4), we have

$$\epsilon_i \tilde{\phi}_i = \epsilon_j \tilde{\phi}_j$$

on $\widetilde{U}_i \cap \widetilde{U}_j$. If we set $\tilde{\phi}'_i := \epsilon_i \tilde{\phi}_i$, then $\{\tilde{\phi}'_i\}_{i \in I}$ gives a collection of local liftings of ϕ any two of which agree on their common domain. Denote the resulting global lifting of ϕ by $\tilde{h}$.

We next check the second axiom of Definition 2.4.2. Let σ be a deck transformation of the covering $p : \widetilde{X} \to X$, that is, an element of $\mathrm{Aut}\,(\widetilde{X}, p)$. Then $\tilde{h} \circ \sigma$ is a lifting of $\phi \circ p$ because

$$\begin{aligned} q \circ \tilde{h} \circ \sigma &= (\phi \circ p) \circ \sigma \\ &= \phi \circ (p \circ \sigma) \\ &= \phi \circ p. \end{aligned}$$

Since both $\tilde{h}$ and $\tilde{h} \circ \sigma$ are liftings of $\phi \circ p$, one can find an element $\tau_{\tilde{x}}$ in $\mathrm{Aut}\,(Y, q)$ for each $\tilde{x} \in \widetilde{X}$ such that

$$\tilde{h} \circ \sigma(\tilde{x}) = \tau_{\tilde{x}}(\tilde{h}(\tilde{x})).$$

Since $\widetilde{X}$ is connected and $\pi_1^{orb}(Y)$ is discrete, $\tau_{\tilde{x}}$ has to be independent of $\tilde{x}$. This gives an element τ in the second property of orbi-maps.

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{h}\circ\sigma,\ \tilde{h}} & \widetilde{Y} \\ & \searrow^{\phi\circ p} & \downarrow q \\ & & Y \end{array} \tag{5.1.5}$$

Finally, the third condition of orbi-maps follows obviously since we are only considering non-constant morphisms between orbifolds with the same dimension. □

Remark 5.1.3. *From the proof, we see that the lemma also holds for a smooth map between two good orbifolds of general dimensions which does not send a whole open subset to a fixed locus.*

5.2 Holomorphic orbifold maps

As mentioned in the introduction, our main goal is to compute the (quantum) product structure of $H^*_{orb}(\mathbb{P}^1_{a,b,c}, \mathbb{Q})$ where (a, b, c) is one of $(3, 3, 3)$, $(2, 3, 6)$, and $(2, 4, 4)$. Throughout the section, $\mathbb{P}^1_{a,b,c}$ denotes one of elliptic orbifolds $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{2,3,6}$, and $\mathbb{P}^1_{2,4,4}$. In order to do this, we have to count holomorphic orbi-spheres in (or stable maps into) $\mathbb{P}^1_{a,b,c}$ with three markings. In this section, we first characterize these holomorphic maps and find their properties which are useful to classify holomorphic orbi-spheres in $\mathbb{P}^1_{a,b,c}$. We will see that if $(f, (\mathbb{P}^1, \boldsymbol{z}), \xi)$ is a non-constant orbifold stable map into elliptic $\mathbb{P}^1_{a,b,c}$ of type $\boldsymbol{x}$, $\boldsymbol{z}$ can not contain a smooth point. Thus, we may assume that $\boldsymbol{x}$ is a triple of twisted sectors. (See the discussion after the proof of Lemma 5.2.1 below.)

Recall that the type $\boldsymbol{x}$ determines the virtual dimension of a component of the moduli of orbifold stable maps containing $(f, (\mathbb{P}^1, \boldsymbol{z}), \xi)$ as well as the orbifold structure of domain orbi-Riemann sphere $(\mathbb{P}^1, \boldsymbol{z})$ (Remark 2.5.4). In fact, the virtual dimension is given as

$$\text{vir.dim}\,\overline{\mathcal{M}}_{0,3}(\mathbb{P}^1_{a,b,c}, J, \beta, \boldsymbol{x}) = 2 - 2\iota(\boldsymbol{x}).$$

As we only consider the 0-dimensional moduli for the quantum product, this gives a restriction on the type, that is, $\iota(\boldsymbol{x}) = 1$. If we impose an additional condition on the degrees of inputs for holomorphic orbi-sphere of type $\boldsymbol{x}$ with $\iota(\boldsymbol{x}) = 1$, we can show that it is actually an orbifold covering. This will be shown in Section 5.2.2.

As the first step, we show that there is no contribution to the quantum product from a degenerate orbi-sphere, which is an element lying on the boundary of $\overline{\mathcal{M}}_{0,3,\beta}(\mathbb{P}^1_{a,b,c})$.

5.2.1 Considerations on degenerate maps

Note that $\Delta^i_\circ$ are cohomology classes of nontrivial sectors of $\mathbb{P}^1_{a,b,c}$. We want to show that all the holomorphic orbi-spheres $u : (\Sigma, \boldsymbol{z} = (z_1, z_2, z_3)) \to \mathbb{P}^1_{a,b,c}$ of appropriate type $\boldsymbol{x} = (x_1, x_2, x_3)$, a triple of twisted sectors of $\mathbb{P}^1_{a,b,c}$, can not have any nodal singularity. More precisely, the above "appropriate" means that the $\boldsymbol{x}$ is a type with vir.dim $\overline{\mathcal{M}}_{0,3}(\mathbb{P}^1_{a,b,c}, J, \beta, \boldsymbol{x}) = 0$ for all β. Here, since $\mathbb{P}^1_{a,b,c}$ is elliptic, the virtual dimension does not depend on β (Remark 2.5.4).

Lemma 5.2.1. *There are no degenerate (i.e., nodal) holomorphic orbi-spheres which are non-constant and contribute to* $\langle \Delta^i_\circ, \Delta^j_\bullet, \Delta^k_\diamond \rangle^{\mathbb{P}^1_{a,b,c}}_{0,3}$ *for* $i + j + k = 1$.

Proof.

There are two classes of degenerate maps which are possibly contained in the boundary of the moduli space $\overline{\mathcal{M}}_{0,3}(\mathbb{P}^1_{a,b,c}, J, \beta, \boldsymbol{x})$:

1. $u_1 : \mathbb{P}^1_{\alpha,\beta,\bullet} \sqcup_\bullet \mathbb{P}^1_{\bullet,\delta} \to \mathbb{P}^1_{a,b,c}$,

2. $u_2 : \mathbb{P}^1_{\alpha,\beta,\delta,\circ} \sqcup_\circ \mathbb{P}^1_\circ \to \mathbb{P}^1_{a,b,c}$,

where $\bullet$ and $\circ$ are the order of local isotropy group of the nodal point. (See Figure 5.1.) Note that u_i $(i = 1, 2)$ is non-constant map when restricted to the second component of domain, since u_i should be stable. We claim that there can not exist such maps into $\mathbb{P}^1_{a,b,c}$.

First, consider the case of $u : \mathbb{P}^1_{\delta,\delta} \to \mathbb{P}^1_{a,b,c}$ ($\bullet = \delta$ in (1)). Since the quotient map $\pi : \mathbb{P}^1 \to \mathbb{P}^1_{\delta,\delta}$ (by the obvious $\mathbb{Z}_\delta$-action on $\mathbb{P}^1$) is holomorphic orbifold covering and $\pi_1(\mathbb{P}^1) = 0$, there is a holomorphic map $\widetilde{u}$ which makes following diagram commutes (using Proposition 2.4.2 and Lemma 5.1.2).

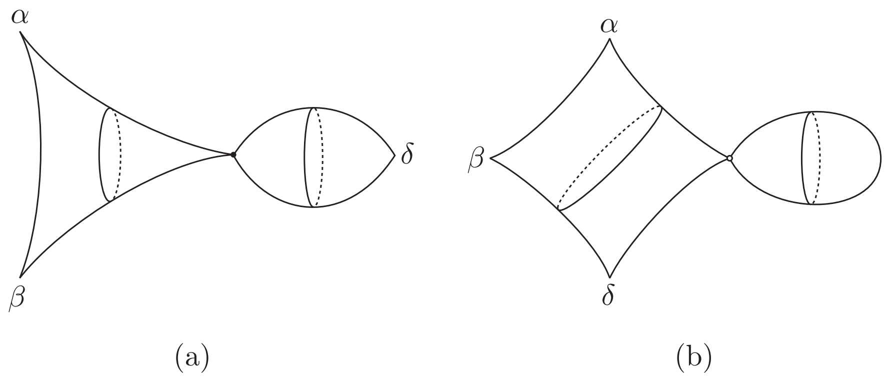


Figure 5.1: Domains of degenerate maps : (a) $\mathbb{P}^1_{\alpha,\beta,\bullet} \sqcup_\bullet \mathbb{P}^1_{\bullet,\delta}$ and (b) $\mathbb{P}^1_{\alpha,\beta,\delta,\circ} \sqcup_\circ \mathbb{P}^1_\circ$

$$\begin{array}{ccc} \mathbb{P}^1 & \overset{\exists \widetilde{u}}{\dashrightarrow} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ \mathbb{P}^1_{\delta,\delta} & \xrightarrow{u} & \mathbb{P}^1_{a,b,c} \end{array} \tag{5.2.6}$$

Note that the image of $\widetilde{u}$ must be homotopic to a constant map since $\pi_2(E) = 0$, and hence $\widetilde{u}$ is a constant map from the holomorphicity. This contradicts the stability of the map u, and hence there is no such holomorphic map u. A similar argument shows that $u : \mathbb{P}^1 \to \mathbb{P}^1_{a,b,c}$ ($\circ = 1$ in (2)) can not exist.

The remaining case is when the second component of the domain is not a good orbifold, $u : \mathbb{P}^1_{mp,mq} \to \mathbb{P}^1_{a,b,c}$ for some natural numbers p, q, and m satisfying $\gcd(p,q) = 1$ and $pq \neq 1$. Consider the holomorphic quotient map $\pi : \mathbb{P}^1_{p,q} \to \mathbb{P}^1_{mp,mq}$ and the holomorphic map $v := u \circ \pi$.

$$\begin{array}{ccc} \mathbb{P}^1_{p,q} & & \\ \downarrow{\scriptstyle \pi} & \searrow{\scriptstyle v} & \\ \mathbb{P}^1_{mp,mq} & \xrightarrow{u} & \mathbb{P}^1_{a,b,c} \end{array}$$

Let $x \in \mathbb{P}^1_{mp,mq}$ be an orbi-singular point and $\widetilde{x} \in \mathbb{P}^1_{p,q}$ be the element in $\pi^{-1}(x)$. We may assume that $\widetilde{x}$ and $u(x)$ have isotropy groups $\mathbb{Z}_p$ for some $p \neq 1$ and $\mathbb{Z}_a$, respectively. Then from the definition of an orbifold map, the map v should be lifted locally to an equivariant map $\widetilde{v}$ on the local uniformizing charts

$$\begin{array}{ccc} U_{\widetilde{x}} & \xrightarrow{\widetilde{v}} & V_{v(\widetilde{x})} \\ \downarrow & & \downarrow \\ U_{\widetilde{x}}/\mathbb{Z}_p & \xrightarrow{v} & V_{v(\widetilde{x})}/\mathbb{Z}_a, \end{array} \tag{5.2.7}$$

and the induced group homomorphism between isotropy groups $\phi_v : \mathbb{Z}_p \to \mathbb{Z}_a$ should be injective since $\phi_v = \phi_u \circ \phi_\pi : \mathbb{Z}_p \overset{\times m}{\to} \mathbb{Z}_{mp} \to \mathbb{Z}_a$ is a composition of two injective morphism. (The injectivity of the second map comes from the definition of an orbifold map. See Definition 2.2.2.) Hence the generator g of $\pi_1^{orb}(U_{\widetilde{x}}/\mathbb{Z}_p)$ should be mapped to an order p element of $\pi_1^{orb}(V_{v(\widetilde{x})}/\mathbb{Z}_a)$. However, from the van Kampen's theorem, $\pi_1^{orb}(\mathbb{P}^1_{p,q}) = \{0\}$, so the image of g in $\pi_1^{orb}(\mathbb{P}^1_{p,q})$ is zero whereas $\pi_1^{orb}(\mathbb{P}^1_{a,b,c}) = \left\langle g_1, g_2, g_3 \,\middle|\, g_1^a = g_2^b = g_3^c = g_1g_2g_3 = 1 \right\rangle$ is nontrivial. Note that the homomorphism $\pi_1^{orb}(V_{v(\widetilde{x})}/\mathbb{Z}_a) \to \pi_1^{orb}(\mathbb{P}^1_{a,b,c})$ induced from the inclusion map $\iota : V_{v(\widetilde{x})} \to \mathbb{P}^1_{a,b,c}$ is injective, since $\mathbb{P}^1_{a,b,c}$ is a good orbifold (See for example [D, Prop.1.18]). This gives a contradiction. □

Consider an orbifold stable map $(f, (\mathbb{P}^1, \boldsymbol{z}), \xi)$ with three makings of type $\boldsymbol{x}$. If there is a smooth point in $\boldsymbol{z}$, f can be thought of as a map from an orbi-sphere with two singular points. Then exactly the same argument in the proof of Lemma 5.2.1 implies that f is indeed a constant map.

5.2.2 Orbifold coverings of $\mathbb{P}^1_{a,b,c}$ contributing to the quantum product

In this section, we prove that holomorphic orbi-spheres satisfying certain properties become orbifold covering maps. Most of holomorphic orbi-spheres contributing to the quantum product of elliptic $\mathbb{P}^1_{a,b,c}$ will turn out to satisfy these properties, later. (There is only one exceptional case for $\mathbb{P}^1_{2,3,6}$ where non-trivial holomorphic orbi-spheres from the hyperbolic orbifold $\mathbb{P}^1_{3,6,6}$ contribute to the quantum product of $\mathbb{P}^1_{2,3,6}$.)

Let u be a holomorphic orbi-sphere from $\mathbb{P}^1_{\alpha,\beta,\delta}$ to $\mathbb{P}^1_{a,b,c}$ and consider the universal covering map $\pi : \widetilde{\mathbb{P}^1_{\alpha,\beta,\delta}} \to \mathbb{P}^1_{\alpha,\beta,\delta}$ and $p : \mathbb{C} \to \mathbb{P}^1_{a,b,c}$. Here, p is a holomorphic map since the complex structure on $\mathbb{P}^1_{a,b,c}$ comes from that on its universal cover.

From orbifold covering theory, we obtain a lifting $\widetilde{u}$ of $u \circ \pi$ of the underlying orbifold morphism $u : \mathbb{P}^1_{\alpha,\beta,\delta} \to \mathbb{P}^1_{a,b,c}$:

$$\begin{array}{ccc} \widetilde{\mathbb{P}^1_{\alpha,\beta,\delta}} & \xrightarrow{\exists \widetilde{u}} & \mathbb{C} \\ \downarrow \pi & \searrow^{u\circ\pi} & \downarrow p \\ \mathbb{P}^1_{\alpha,\beta,\delta} & \xrightarrow{u} & \mathbb{P}^1_{a,b,c}. \end{array} \tag{5.2.8}$$

For each equivalence class $[u] \in \mathcal{M}^{\mathrm{reg}}_{0,3,\beta}(\mathbb{P}^1_{a,b,c})$, if we choose a representative u of $[u]$ by fixing the location of three special points on the domain (denoted by $\mathbb{P}^1_{\alpha,\beta,\delta}$), there is no further equivalence relation since there is a unique automorphism which sends given three point to the other. For such u, the lifting $\widetilde{u}$ is holomorphic, since it is locally holomorphic.

To avoid notational complexity, let us write X for $\mathbb{P}^1_{a,b,c}$, and consider a triple of twisted sectors $\boldsymbol{x} = (X_{(g_1)}, X_{(g_2)}, X_{(g_3)})$. Let G_i be an isotropy group of a point in $X_{(g_i)}$ which is defined up to conjugacy. Since X is one dimensional, the age of an element of G_i is given by $\iota_{(g_i)} = \frac{l_i}{|G_i|}$ for some $l_i \in \{1, \cdots, |G_i| - 1\}$. Since we only count 0-dimensional strata of moduli of orbifold stable maps for the quantum product, we assume that

$$\sum_{i=1}^{3} \iota_{(g_i)} = 1. \tag{5.2.9}$$

From Definition 2.4.1, we see that the necessary condition for u to be an orbifold covering map is that

$$\iota^{-1}_{(g_i)} \in \mathbb{Z} \quad i = 1, 2, 3, \tag{5.2.10}$$

or equivalently $l_i \mid |G_i|$ for $i = 1, 2, 3$. Nonetheless, this condition (5.2.10) is indeed sufficient to guarantee that u is an orbifold covering map:

Lemma 5.2.2. *If u is a non-constant holomorphic orbifold stable map from $(\mathbb{P}^1, \boldsymbol{z}, \boldsymbol{m})$ to X of the type $\boldsymbol{x}$ satisfying* (5.2.9) *and* (5.2.10)*, then u is an orbifold covering map. Here, $\boldsymbol{z} = (z_1, z_2, z_3)$ is a triple of marked points, and $\boldsymbol{m} = (m_1, m_2, m_3)$ is the triple of orders of isotropy groups of $\boldsymbol{z}$.*

Proof. Recall that any non-constant holomorphic map between Riemann surfaces is a branched covering. We want to show that this map is an orbifold covering (See Definition 2.4.1).

Consider three orbi-singular points $\{w_1, w_2, w_3\}$ in the target space and their inverse image $u^{-1}(w_i)$. Since $\boldsymbol{x}$ consists of twisted sectors, there is a function $I : \{1,2,3\} \to \{1,2,3\}$ such that $u(z_i) = w_{I(i)} \in X_{(g_i)}$. We denote the number of points in $u^{-1}(w_i) - \{z_1, z_2, z_3\}$ by $m(w_i)$.

Let U_i be an open neighborhood of z_i with uniformizing system $(\widetilde{U}_i, \mathbb{Z}_{m_i}, br_i)$, where $br_i : z \mapsto z^{m_i}$ and let V be an open neighbood of $w_{I(i)}$ uniformized by $(\widetilde{V}, \mathbb{Z}_{|G_i|}, br)$ for $br : z \mapsto z^{|G_i|}$. From the definition of orbifold morphism, there is a local holomorphic lifting $\widetilde{u}$ of u such that

$$\begin{array}{ccc} \widetilde{U}_i & \xrightarrow{\widetilde{u}} & \widetilde{V} \\ {\scriptstyle br_i}\downarrow & & \downarrow{\scriptstyle br} \\ U_i & \xrightarrow[u]{} & V \end{array}$$

commutes. Then, from Equation (5.2.10) and the injectivity of morphism $\mathbb{Z}_{m_i} \to \mathbb{Z}_{|G_j|}$ which maps 1 to l_i, we can check that $m_i l_i = |G_i|$ and $\widetilde{u}(z) = z^{|G_i|a_i+1}$ for some $a_i \in \mathbb{N}_{\geq 0}$. Then $u(w) = w^{|G_i|a_i l_i + l_i}$ with respect to the local holomorphic coordinate $w = z^{m_i}$ of the underlying space.

Since any orbifold Riemann surface $(\Sigma_g, \boldsymbol{z})$ is analytically isomorphic to a smooth Riemann surface Σ_g, the orbifold map u can be regarded as a branched covering map between underlying spaces $\mathbb{C}\mathrm{P}^1$. Hence, the ramification index at z_i is $|G_i|a_i l_i + l_i$. Except these three orbi-singular points $\boldsymbol{z}$, other point in the inverse image $u^{-1}(w_1)$ has ramification index which is a multiple of a, say ae_j for some $e_j \in \mathbb{N}$ $(j = 1, \cdots, m(w_1))$. Similarly, in $u^{-1}(w_2)$ and $u^{-1}(w_3)$, there are $m(w_2)$ and $m(w_3)$ number of points with ramification index bf_k and cg_l for $1 \leq k \leq m(w_2)$ and $1 \leq l \leq m(w_3)$, respectively.

We apply the Riemann-Hurwitz formula to u so that

$$2 \leq 2d - \left\{ \sum_{i=1}^{3} (|G_i|a_i l_i + l_i - 1) + \sum_{j=1}^{m(w_1)} (ae_j - 1) + \sum_{k=1}^{m(w_2)} (bf_k - 1) + \sum_{l=1}^{m(w_3)} (cg_l - 1) \right\} \tag{5.2.11}$$

where d is the degree of u. (Here, 2 in the left hand side is the topological Euler characteristic of $\mathbb{P}^1_{a,b,c}$.) If u does not have any branching outside $u^{-1}(w_1) \cup u^{-1}(w_2) \cup u^{-1}(w_3)$, then the equality holds in (5.2.11). Since d is the weighted count of the number of points in the fiber $u^{-1}(w_i)$ of u, we have

$$\begin{aligned} d &= \sum_{i\in I^{-1}(1)} (|G_i|a_il_i + l_i) + a\sum_{j=1}^{m(w_1)} e_j = \sum_{i\in I^{-1}(2)} (|G_i|a_il_i + l_i) + b\sum_{k=1}^{m(w_2)} f_k \\ &= \sum_{i\in I^{-1}(3)} (|G_i|a_il_i + l_i) + c\sum_{l=1}^{m(w_3)} g_l, \end{aligned} \tag{5.2.12}$$

and hence by inserting (5.2.12) in the (5.2.11),

$$d \le 1 + \sum_{j=1}^{3} m(w_j). \tag{5.2.13}$$

(5.2.12) together with $e_j \ge 1$ implies that

$$d \ge \sum_{i\in I^{-1}(1)} (|G_i|a_il_i + l_i) + am(w_1), \tag{5.2.14}$$

and similar inequalities also hold for w_2 and w_3.

Combining (5.2.13) and (5.2.14),

$$\begin{aligned} & abc\left(\sum_{i=1}^{3} m(w_i)\right) + bc\sum_{i\in I^{-1}(1)} (|G_i|a_il_i + l_i) \\ & + ca\sum_{i\in I^{-1}(2)} (|G_i|a_il_i + l_i) + ab\sum_{i\in I^{-1}(3)} (|G_i|a_il_i + l_i) \\ & \le (bc + ca + ab)d = abcd \end{aligned} \tag{5.2.15}$$

where the last equality follows from $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$.

Note that $\sum_{i=1}^{3} \frac{l_i}{|G_i|} = 1$ from the condition (5.2.9). Hence

$$bc\sum_{i\in I^{-1}(1)} l_i + ca\sum_{i\in I^{-1}(2)} l_i + ab\sum_{i\in I^{-1}(3)} l_i = abc. \tag{5.2.16}$$

Since $|G_i| = a$ if $i \in I^{-1}(1)$, (and with similar equalities for other two cases)

$$bc\sum_{i\in I^{-1}(1)} |G_i|a_il_i + ca\sum_{i\in I^{-1}(2)} |G_i|a_il_i + ab\sum_{i\in I^{-1}(3)} |G_i|a_il_i = abc\sum_{i=1}^{3} a_il_i. \tag{5.2.17}$$

Combining (5.2.15) with (5.2.13), (5.2.16), and (5.2.17), it follows that

$$abc\left(\sum_{i=1}^{3} m(w_i) + \sum_{i=1}^{3} a_i l_i + 1\right) \leq abc\left(1 + \sum_{j=1}^{3} m(w_j)\right), \tag{5.2.18}$$

hence $a_1 = a_2 = a_3 = 0$ from the non-negativity of a_i's.

If we do not use the inequality (5.2.14) and proceed, we have the following more precise estimate

$$abc\left(\sum_{j=1}^{m(w_1)} e_j + \sum_{k=1}^{m(w_2)} f_k + \sum_{l=1}^{m(w_3)} g_l + \sum_{i=1}^{3} a_i + 1\right) \leq abc\left(1 + \sum_{j=1}^{3} m(w_j)\right)$$

which implies $e_j = f_k = g_l = 1$ for all j, k, l. Therefore, u is an orbifold covering. □

Remark 5.2.3. *For the case of* $c_1(TX) < 0$ *(i.e.,* $X = \mathbb{P}^1_{a,b,c}$ *is hyperbolic), the same lemma also holds since* (5.2.15) *is still valid. However, if* $c_1(TX) > 0$ *(i.e., X is spherical), the argument in Lemma 5.2.2 is not true any more.*

5.2.3 Regularity of holomorphic maps

Now we prove the Fredholm regularity of holomorphic orbi-spheres in elliptic $\mathbb{P}^1_{a,b,c}$, which are orbifold coverings. This will imply that the relevant moduli space of holomorphic orbi-spheres is smooth, and hence, justify our *direct* counting of holomorphic orbi-spheres, later.

Proposition 5.2.4. *Let X be an elliptic orbi-sphere and* $(\Sigma, \mathbf{z})$ *be a domain Riemann orbi-curve. If* $u : \Sigma \to X$ *is a holomorphic orbi-sphere satisfying* (5.2.9) *and* (5.2.10)*, then u is regular.*

Proof. Consider the pullback orbi-bundle $u^*TX \to \Sigma$ and the linearized $\overline{\partial}$-operator $D\overline{\partial}_J(u) : C^\infty(u^*TX) \to \Omega^{0,1}(u^*TX)$. Since J is integrable, $D\overline{\partial}_J(u) = \overline{\partial}_J$. Hence it is sufficient to show that $H^{0,1}_{\overline{\partial}}(X, u^*TX) = 0$.

Note that the first Chern number of the tangent bundle of X is zero, and for any orbifold covering u, that of u^*TX is also zero. From this, we can see that the desingularized bundle (See Section 2.5.2) of u^*TX, $|u^*TX|$ has (desingularized) Chern number -1 from Equation (5.2.9), i.e., $c_1(|u^*TX|) = -1$. Since X is a complex orbifold and u is a holomorphic orbi-map, u^*TX is a holomorphic orbi-bundle over Σ. The desingularization of holomorphic orbi-bundle is also holomorphic. From the Lemma 3.5.1 in [MS1], this implies that the holomorphic line bundle $|u^*TX|$ has vanishing cohomology group $H^{0,1}_{\bar\partial}(|\Sigma|, |u^*TX|)$. As the sheaf of holomorphic sections of $|u^*TX|$ is the same as the sheaf of (orbifold) holomorphic sections of u^*TX on Σ, we have the vanishing of $H^{0,1}_{\bar\partial}(\Sigma, u^*TX)$. More precisely,

$$\begin{aligned} H^{0,1}_{\bar\partial}(|\Sigma|, |u^*TX|) &\cong H^1(|\Sigma|, \mathcal{O}(|u^*TX|)) \\ &\cong H^1(\Sigma, \mathcal{O}(u^*TX)) \\ &\cong H^{0,1}_{\bar\partial}(\Sigma, u^*TX). \end{aligned}$$

For the last isomorphism, note that the two term complex $\bar\partial_J : C^\infty(u^*TX) \to \Omega^{0,1}(u^*TX)$ is a fine resolution for the sheaf of holomorphic sections $\mathcal{O}(u^*TX)$ as in the smooth case. □

Remark 5.2.5. *Even if u is not an orbifold covering, we still have $c_1(u^*TX) \geq c_1(TX)$. (Indeed, we can improve this inequality by considering the degree of u.) Therefore, the above proposition also holds as long as X has a non-negative first Chern number.*

For example, when we calculate the quantum cohomology of $\mathbb{P}^1_{2,3,6}$, we need to count holomorphic orbi-spheres $u : \mathbb{P}^1_{3,6,6} \to \mathbb{P}^1_{2,3,6}$ which are not orbifold covering maps. These orbi-spheres are also Fredholm regular exactly by the same argument.

5.3 The quantum cohomology ring of $\mathbb{P}^1_{3,3,3}$

In this section, we explicitly compute the product structure on $QH^*_{orb}(\mathbb{P}^1_{3,3,3}, \mathbb{Q})$, which proves Theorem 1.0.3. For this, we first classify holomorphic orbi-spheres in $\mathbb{P}^1_{3,3,3}$ (Section 5.3.1). Recall from Lemma 5.2.1 and Proposition 5.2.4 that these stable maps, in fact, are maps from a single orbi-sphere component and are regular. Thus by counting holomorphic orbi-spheres inside $\mathbb{P}^1_{3,3,3}$, we obtain the 3-fold Gromov-Witten invariant for $\mathbb{P}^1_{3,3,3}$ which combined with the constant map contributions (Section 5.3.4) gives rise to the quantum product on $QH^*_{orb}(\mathbb{P}^1_{3,3,3}, \mathbb{Q})$. One interesting feature is that one can relate these orbi-spheres with the solutions of a certain Diophantine equation.

It will turn out in Section 5.3.1 that only

$$\langle \Delta_1^{1/3}, \Delta_2^{1/3}, \Delta_3^{1/3} \rangle_{0,3}^{\mathbb{P}^1_{3,3,3}} \quad \text{and} \quad \langle \Delta_i^{1/3}, \Delta_i^{1/3}, \Delta_i^{1/3} \rangle_{0,3}^{\mathbb{P}^1_{3,3,3}}$$

for $i = 1, 2, 3$ are nontrivial, which precisely give the coefficients f_0 and f_1 of cubic terms for the Gromov-Witten potential in [ST, Theorem 3.1].

Remark 5.3.1. *We will write the details on the classification orbi-spheres in* $\mathbb{P}^1_{3,3,3}$ *as concrete as possible. For other two cases,* $\mathbb{P}^1_{2,3,6}$ *and* $\mathbb{P}^1_{2,4,4}$*, we will find similar classification results in Section 5.4, but without much details as the arguments are not very much different from the one for* $\mathbb{P}^1_{3,3,3}$.

5.3.1 Classification of orbi-spheres in $\mathbb{P}^1_{3,3,3}$

From the expected dimension formula and representability of orbifold stable map, the only possible domain orbi-sphere in 0-dimensional components of the moduli space $\overline{\mathcal{M}}_{0,3,\beta}(\mathbb{P}^1_{3,3,3})$ is $\mathbb{P}^1_{3,3,3}$ itself. Since there exists a unique biholomorphism $\phi : (\mathbb{P}^1_{3,3,3}, z) \to (\mathbb{P}^1_{3,3,3}, z')$ sending any triple of orbi-points $z = (z_1, z_2, z_3)$ to another $z' = (z_1', z_2', z_3')$, there is no domain parameter in the relevant moduli $\overline{\mathcal{M}}_{0,3,\beta}(\mathbb{P}^1_{3,3,3})$. Hence from now on, we take the domain orbi-sphere to be $\mathbb{P}^1_{3,3,3}$ with the fixed conformal structure which is induced by the quotient map $E \to \mathbb{P}^1_{3,3,3}$ in the Section 2.5.4.

By degree reason, $\langle \Delta_\circ^i, \Delta_\bullet^j, \Delta_\diamond^k \rangle_{0,3}^{\mathbb{P}^1_{3,3,3}}$ is trivial unless $i + j + k = 1$. Note that by the obvious symmetry on $\mathbb{P}^1_{3,3,3}$, it is enough to consider only the following three cases:

1. $\langle \Delta_1^{1/3}, \Delta_1^{1/3}, \Delta_1^{1/3} \rangle_{0,3}$,
2. $\langle \Delta_1^{1/3}, \Delta_2^{1/3}, \Delta_3^{1/3} \rangle_{0,3}$,
3. $\langle \Delta_1^{1/3}, \Delta_2^{1/3}, \Delta_2^{1/3} \rangle_{0,3}$.

Namely, we may assume that the first marked point z_1 in the domain orbi-sphere is mapped to the orbi-singular point w_1 in $\mathbb{P}^1_{3,3,3}$ associated with $\Delta_1^{1/3}$.

Let u be a holomorphic orbi-sphere from $\mathbb{P}^1_{3,3,3}$ to $\mathbb{P}^1_{3,3,3}$, which contributes to $\langle \Delta_1^{1/3}, \Delta_\circ^{1/3}, \Delta_\bullet^{1/3} \rangle_{0,3}$. We fix base points of the domain orbi-sphere and the target orbi-sphere of u and their universal coverings as follows: $x_0 := z_1 \in \mathbb{P}^1_{3,3,3}$ and $\widetilde{x_0} := 0 \in p^{-1}(x)(\subset \mathbb{C})$ for the domain $\mathbb{P}^1_{3,3,3}$, and $y_0 := w_1 \in \mathbb{P}^1_{3,3,3}$ and $\widetilde{y_0} := 0 \in p^{-1}(w_1)(\subset \mathbb{C})$. Recall that $p : \mathbb{C} \to \mathbb{P}^1_{3,3,3}$ is the orbifold universal covering, and $u(x_0) = y_0$. Thus, we obtain a unique lifting $\widetilde{u}$ of $u \circ p$ for the underlying holomorphic orbi-sphere $u : \mathbb{P}^1_{3,3,3} \to \mathbb{P}^1_{3,3,3}$:

$$\begin{array}{ccc} \mathbb{C} & \overset{\exists \widetilde{u}}{\dashrightarrow} & \mathbb{C} \\ \downarrow p & & \downarrow p \\ \mathbb{P}^1_{3,3,3} & \xrightarrow{u} & \mathbb{P}^1_{3,3,3}. \end{array} \tag{5.3.19}$$

Note that the conditions in the lemma 5.2.2 are automatic in this case. Therefore, $u : \mathbb{P}^1_{3,3,3} \to \mathbb{P}^1_{3,3,3}$ is an orbifold covering, and its lifting $\widetilde{u}$ (5.3.19) has a particularly nice shape.

Proposition 5.3.2. *If u is a non-constant holomorphic orbi-sphere contributing to $\langle \Delta_1^{1/3}, \Delta_\circ^{1/3}, \Delta_\bullet^{1/3} \rangle_{0,3}$, then $\widetilde{u}(z) = \lambda z$ for some $\lambda \in \mathbb{Z}[\tau]$, where $\mathbb{Z}[\tau] := \{a + b\tau \mid a, b \in \mathbb{Z}\}$*

Proof. Because u is an orbifold universal covering, so is the composition $u \circ p$. Now, by the uniqueness of orbifold universal covering, $\widetilde{u}$ should be a homeomorphism. Note that $\widetilde{u}$ is an entire proper holomorphic map, since $\widetilde{u}$ is a homeomorphism and is a lifting of the holomorphic map $u \circ p$. It is well-known that any entire and proper holomorphic map on $\mathbb{C}$ is a polynomial. Since $\widetilde{u}$ is invertible, we conclude that $\widetilde{u}$ is a linear map $\widetilde{u}(z) = \lambda z$ for some $\lambda \in \mathbb{C}$. Here, $\widetilde{u}$ does not have a constant term because the lifting preserves the base points, $\widetilde{u}(\widetilde{x_0}) = \widetilde{y_0}$ (i.e., $\widetilde{u}(0) = 0$).

Recall from Subsection 2.5.4 that $p^{-1}(x_0)$ gives a lattice $\mathbb{Z}\langle 1, \tau \rangle = \mathbb{Z}[\tau]$ ($\because \tau^2 = -\tau - 1$) in $\mathbb{C} \cong \mathbb{R}^2$. (See the left side of Figure 5.2.) Since $1 \in \mathbb{C}$ is an element of this lattice, $\widetilde{u}$ should map 1 to a point in $p^{-1}(y_0)$ which is also the same lattice $\mathbb{Z}[\tau]$ in $\mathbb{C}$. Thus, $\lambda = \widetilde{u}(1)$ has to lie in $\mathbb{Z}[\tau]$, which finishes the proof. □

It is obvious from the picture that the fundamental domain of the domain orbi-sphere covers that of the target orbi-sphere $|\lambda|^2$-times by the holomorphic map

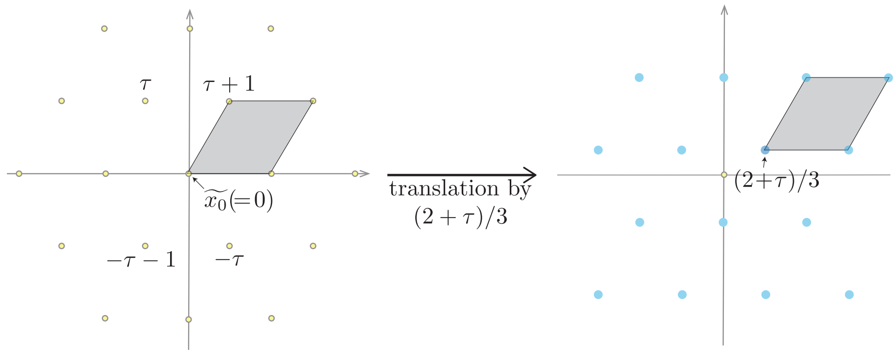


Figure 5.2: The shaded regions show the images of degree-3 maps contributing to $\langle \Delta_1^{1/3}, \Delta_1^{1/3}, \Delta_1^{1/3} \rangle_{0,3}$ (LHS) and $\langle \Delta_2^{1/3}, \Delta_2^{1/3}, \Delta_2^{1/3} \rangle_{0,3}$ (RHS)

induced by the linear map $z \mapsto \lambda z$ between the universal covers. Another way to see this is to consider the energy $\int |du|^2$ which equals the symplectic area of the holomorphic orbi-sphere u. Note that for $\tilde{u}(z) = \lambda z$, $|d\tilde{u}|^2 = |\lambda|^2$. Therefore, such a map induces a term containing $q^{|\lambda|^2}$ in the (3-fold) Gromov-Witten potential.

Conversely, any linear map $\widetilde{u} = \lambda z$ with a coefficient λ in $\mathbb{Z}[\tau]$ induces a $\mathbb{Z}_3$-equivariant holomorphic map between the middle level torus E. The equivariance implies that $\widetilde{u}$ descends to a holomorphic map $u : \mathbb{P}^1_{3,3,3} \to \mathbb{P}^1_{3,3,3}$. u is a well-defined orbifold morphism between $\mathbb{P}^1_{3,3,3}$, as it is represented by an equivariant map between E, and $\mathbb{P}^1_{3,3,3} = [E/\mathbb{Z}_3]$.

Later, we will establish the one-to-one correspondence between linear maps with $\mathbb{Z}[\tau]$-coefficients modulo "certain equivalences" and orbi-spheres in $\mathbb{P}^1_{3,3,3}$ contributing to $\langle \Delta_1^{1/3}, \Delta_\circ^{1/3}, \Delta_\bullet^{1/3} \rangle_{0,3}$ modulo equivalences given in (2.5.5).

Remark 5.3.3. *Let σ be the cyclic permutation $(1, 2, 3)$ in the permutation group S_3 on 3-letters $\{1, 2, 3\}$. If u contributes to $\langle \Delta_1^{1/3}, \Delta_i^{1/3}, \Delta_j^{1/3} \rangle_{0,3}$, then it is easy to see from Figure 5.2 that the translation of $\widetilde{u}$ by an element of $\mathbb{Z}\langle \frac{2+\tau}{3}, \frac{1+2\tau}{3} \rangle$ induces a map $v : \mathbb{P}^1_{3,3,3} \to \mathbb{P}^1_{3,3,3}$ contributing to $\langle \Delta_{\sigma^k(1)}^{1/3}, \Delta_{\sigma^k(i)}^{1/3}, \Delta_{\sigma^k(j)}^{1/3} \rangle_{0,3}$ for some $0 \leq k \leq 2$. This explains the coincidence between various 3-fold Gromov-Witten invariants appearing in [ST, Theorem 3.1]. Note that $\mathbb{Z}[\tau]$ is a sub-lattice of $\mathbb{Z}\langle \frac{2+\tau}{3}, \frac{1+2\tau}{3} \rangle$.*

5.3.2 Symmetries of the lifting of orbi-maps

We have proved that any orbi-spheres with orbi-insertion $\Delta_1^{1/3}, \Delta_i^{1/3}, \Delta_j^{1/3}$ can be lifted to a linear map $z \mapsto \lambda z$ for $\lambda \in \mathbb{Z}[\tau]$. We next investigate a natural equivalence relation $\sim$ on $\{\widetilde{u} = \lambda z \,|\, \lambda \in \mathbb{Z}[\tau]\}$ such that if $\widetilde{u}_1 \sim \widetilde{u}_2$, then these maps induce a pair of equivalent holomorphic orbi-spheres. Let us denote the set of linear maps $\{z \mapsto \lambda z | \lambda \in \mathbb{Z}[\tau]\}$ by $L(\mathbb{C})$. We now find the equivalence relation on $L(\mathbb{C})$ such that the set of equivalence classes of $L(\mathbb{C})$ corresponds bijectively to the moduli space $\overline{\mathcal{M}}_{0,3}(\mathbb{P}^1_{3,3,3}; \Delta_1^{1/3}, \Delta_i^{1/3}, \Delta_j^{1/3})$ for $1 \leq i, j \leq 3$.

Recall that positions of three orbi-markings as well as the domain $\mathbb{P}^1_{3,3,3}$ itself are fixed by regarding $\mathbb{P}^1_{3,3,3}$ as a quotient of $E = \mathbb{C}/\mathbb{Z}\langle 1, \tau\rangle$ via the $\mathbb{Z}/3$-action. Therefore, we do not have an equivalence from a domain reparametrization and, it is enough to find the condition for two linear maps $\widetilde{u}_i(z) = \lambda_i z \,(\lambda_i \in \mathbb{Z}[\tau], (i = 1, 2))$ inducing the same map on the quotient orbifold.

Denote the induced orbi-spheres from $\widetilde{u}_i$ by u_i for $i = 1, 2$, and suppose that they have orbi-insertions $\Delta_1^{1/3}$, $\Delta_i^{1/3}$ and $\Delta_j^{1/3}$ at z_1, z_2 and z_3, respectively. Since u_1 and u_2 are the same orbifold morphism and both of them send z_1 to w_1, their local liftings should be related by the local isotropy group at w_1, which is isomorphic to $\mathbb{Z}_3$ and is generated by the τ-multiplication. On the level of universal covers, this local group can be realized as the local isotropy group at the origin (lying in $p^{-1}(w_1)$) of $\mathbb{C}$. Local groups at other points in the fiber $p^{-1}(w_i)$ can not relate $\widetilde{u}_1$ and $\widetilde{u}_2$ since they do not preserve the origin.Consequently, $\lambda_1 = \tau^k \lambda_2$ for some $k = 0, 1, 2$, and this gives the desired equivalence relation on $L(\mathbb{C})$.

In summary, we have obtained an identification

$$[L(\mathbb{C})/\mathbb{Z}_3] \cong \bigcup_{i,j,d} \overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{3,3,3}; \Delta_1^{1/3}, \Delta_i^{1/3}, \Delta_j^{1/3}) \tag{5.3.20}$$

where the degree on the right hand side corresponds to $|\lambda|^2 = a^2 - ab + b^2$ on the left hand side (see the discussion below the proof of Proposition 5.3.2). It will turn out in Section 5.3.3 that only possible (i, j)'s are $(1, 1)$, $(2, 3)$ and $(3, 2)$.

For computational simplicity, we consider another type of symmetry on $\mathbb{P}^1_{3,3,3}$, which is induced from the $(1+\tau)$-multiplication on $\mathbb{C}$ (Note that $(1+\tau)^2 = \tau$). This action gives rise to an action of $\mathbb{Z}_6$ on $L(\mathbb{C})$ (and hence $\mathbb{Z}_2$-action on $[L(\mathbb{C})/\mathbb{Z}_3]$) in an obvious way. In view of holomorphic orbi-spheres corresponding to elements of $L(\mathbb{C})$, this action switches two orbi-insertions $\Delta_i^{1/3}$ and $\Delta_j^{1/3}$ without changing the degree. Thus, for example, the $(1+\tau)$-multiplication gives rise to the one-to-one correspondence

$$\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{3,3,3};\Delta_1^{1/3},\Delta_2^{1/3},\Delta_3^{1/3}) \longleftrightarrow \overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{3,3,3};\Delta_1^{1/3},\Delta_3^{1/3},\Delta_2^{1/3})$$

both of which are components of the right hand side of (5.3.20).

In Section 5.3.3, we will count elements in $L(\mathbb{C})$ whose underlying holomorphic orbi-spheres contributing $\langle\Delta_1^{1/3},\Delta_2^{1/3},\Delta_3^{1/3}\rangle_{0,3}$ or $\langle\Delta_1^{1/3},\Delta_3^{1/3},\Delta_2^{1/3}\rangle_{0,3}$ simultaneously. Then, dividing the number of such linear maps by the order of the group generated by the $(1+\tau)$-multiplication which is 6, we find the presentation of

$$f_0(q) = \sum_{d\in H_2(X,\mathbb{Z})} \langle\Delta_1^{1/3},\Delta_2^{1/3},\Delta_3^{1/3}\rangle^X_{0,3,d}q^d.$$

5.3.3 Identification of inputs

We have shown that a degree-d non-constant holomorphic orbi-sphere in $\mathbb{P}^1_{3,3,3}$ contributing to $\langle\Delta_1^{1/3},\Delta_i^{1/3},\Delta_j^{1/3}\rangle_{0,3,d}$ has one-to-one correspondence with a linear map $z\mapsto\lambda z$ for some $\lambda = a+b\tau\in\mathbb{Z}[\tau]$ with $|\lambda|^2 = d(\neq 0)$. (Recall that this d is really the degree of the corresponding holomorphic orbi-sphere. See the discussion below the proof of Proposition 5.3.2.) We subdivide this set of holomorphic orbi-spheres in terms of their orbi-insertions. Orbi-insertions of the holomorphic orbi-sphere corresponding to $z\mapsto\lambda z$ can be determined in the following way. Note that the triangle with vertices 0, $\frac{1+2\tau}{3}$ and $\frac{2+\tau}{3}$ in the universal cover of the domain $\mathbb{P}^1_{3,3,3}$ gives a fundamental domain for the upper hemisphere of $\mathbb{P}^1_{3,3,3}$. Thus, we can think of $\frac{1+2\tau}{3}$ and $\frac{2+\tau}{3}$ as (liftings of) the second and the third markings of the domain $\mathbb{P}^1_{3,3,3}$, respectively. See the shaded region in the left side of Figure 5.3.

Since $\lambda\in\mathbb{Z}[\tau]$, the images $\lambda\cdot\left(\frac{1+2\tau}{3}\right)$ and $\lambda\cdot\left(\frac{2+\tau}{3}\right)$ will lie in the lattice $\mathbb{Z}\langle\frac{1+2\tau}{3},\frac{2+\tau}{3}\rangle$ in the universal cover of the target $\mathbb{P}^1_{3,3,3}$. It is clear that the types of these two lattice points determine the orbi-insertions of the orbi-sphere associated with λ, i.e., if $\lambda\cdot\left(\frac{1+2\tau}{3}\right)$ lies in $\frac{1+2\tau}{3}+\mathbb{Z}\langle 1,\tau\rangle$, then the second orbi-insertion is $\Delta_2^{1/3}$ and so on. See Figure 5.3. (Note that $z\mapsto\lambda z$ always send the origin to the origin, which is related to the fact that we fix the first orbi-insertion as $\Delta_1^{1/3}$ using the symmetry.)

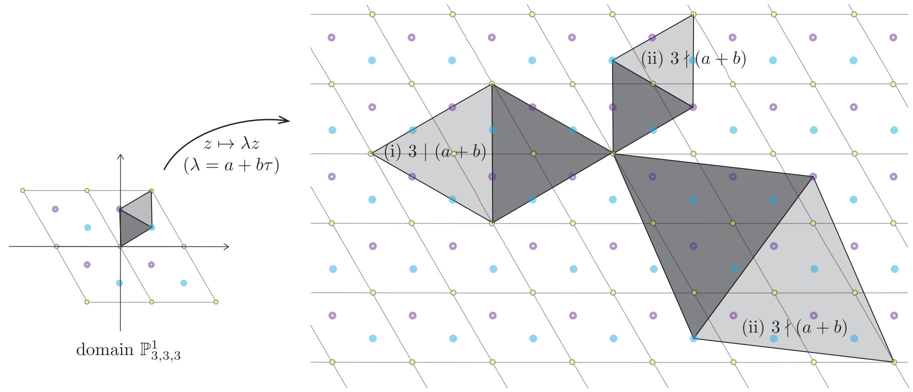


Figure 5.3: Images of holomorphic orbi-spheres in $\mathbb{P}^1_{3,3,3}$ visualized in its universal cover

Observe that

$$\lambda \cdot \left(\frac{1+2\tau}{3}\right) = (a+b\tau)\left(\frac{1+2\tau}{3}\right) = \frac{(a-2b)+(2a-b)\tau}{3}$$

and

$$\lambda \cdot \left(\frac{2+\tau}{3}\right) = \frac{(2a-b)+(a+b)\tau}{3}.$$

(Here, we used the relation $\tau^2 = -\tau - 1$.) Using

$$a - 2b \equiv a + b \mod 3,$$

we see that there are only two possibilities:

(i) $3 \mid (a+b)$, for which both $\lambda \cdot \left(\frac{1+2\tau}{3}\right)$ and $\lambda \cdot \left(\frac{2+\tau}{3}\right)$ correspond to the insertion $\Delta_1^{1/3}$;

(ii) $3 \nmid (a+b)$, for which both $\lambda \cdot \left(\frac{1+2\tau}{3}\right)$ and $\lambda \cdot \left(\frac{2+\tau}{3}\right)$ correspond to two different insertions $\Delta_2^{1/3}$ and $\Delta_3^{1/3}$;

We remark that the in case (ii), insertions $(\Delta_2^{1/3}, \Delta_3^{1/3})$ can be located at either (z_2, z_3) or (z_3, z_2). See the discussion at the end of Section 5.3.2.

Note that

$$d = |\lambda|^2 = a^2 - ab + b^2 = (a+b)^2 - 3ab \equiv (a+b)^2 \mod 3. \tag{5.3.21}$$

Therefore, if $d \equiv 0 \mod 3$, then the corresponding holomorphic spheres contribute to

$$\langle \Delta_1^{1/3}, \Delta_1^{1/3}, \Delta_1^{1/3} \rangle_{0,3},$$

and if $d \equiv 1 \mod 3$, they contribute to

$$\langle \Delta_1^{1/3}, \Delta_2^{1/3}, \Delta_3^{1/3} \rangle_{0,3}.$$

Let $F(q)$ denote the power series

$$F(q) := \sum_{a,b \in \mathbb{Z}} q^{a^2 - ab + b^2} \tag{5.3.22}$$

See (5.5.32) for first few terms of F.

By (5.3.21), the power of any nontrivial term in $F(q)$ should be either 0 (mod 3) or 1 (mod 3). Thus, we can decompose F as $F = F_{0,3} + F_{1,3}$ according to the remainder of the power of q by 3. Then the above discussion directly implies that

$$f_0(q) = \sum_{d \in H_2(X,\mathbb{Z})} \langle \Delta_1^{1/3}, \Delta_2^{1/3}, \Delta_3^{1/3} \rangle_{0,3,d}^X q^d = \frac{1}{6} F_{1,3}(q)$$

since there can not be contributions from constant maps. Here, $\frac{1}{6}$ is responsible for the group $\mathbb{Z}_6 \cong \langle 1 + \tau \rangle$ which is discussed at the end of Subsection 5.3.2.

For $f_1(q) = \sum_{d \in H_2(X,\mathbb{Z})} \langle \Delta_1^{1/3}, \Delta_1^{1/3}, \Delta_1^{1/3} \rangle_{0,3,d}^X q^d$, there is an additional contribution from the constant map (see Subsection 5.3.4, (5.3.24)) so that $f_1(q) = \frac{1}{3} F_{0,3}(q)$, where $\frac{1}{3}$ again comes from $\mathbb{Z}_3$, the isotropy group at w_1 (or the origin in $\mathbb{C}$).

Remark 5.3.4. *Number theoretic aspects of F such as an explicit description of its Fourier coefficients will be given in Section 5.5. In particular, we will describe the Fourier coefficients of F in terms of the prime factorization of the exponent of q.*

5.3.4 Contribution from constant maps

The constant map whose image lies in a single singular point also contributes to the quantum product. Indeed, these constant maps induce the product structure "$\cdot$" of the Chen-Ruan cohomology ring [CR1] of $\mathbb{P}^1_{3,3,3}$, and the quantum product deforms this structure analogously to the relation between cup products and quantum products for smooth symplectic manifolds.

Let us consider one of singular points w_i and constant maps from an orbi-sphere with three markings onto this point. The computation is essentially the same for all $i = 1, 2, 3$ because of the symmetry. Obviously, there are two constant maps with image w_i whose domain orbi-spheres are $\mathbb{P}^1_{3,3,3}$ and $\mathbb{P}^1_{3,3}$. We denote these maps by c_1 and c_2. Here, the markings for c_2 are located at two singular points and a chosen smooth point. We remark that the second map does not violate Lemma 5.2.1 since it only holds for non-constant holomorphic orbi-spheres.

c_1 and c_2 give rise to classical parts

$$\langle \Delta_i^{1/3}, \Delta_i^{1/3}, \Delta_i^{1/3} \rangle^X_{0,3,d=0} \quad \text{and} \quad \langle \Delta_i^{1/3}, \Delta_i^{2/3}, 1 \rangle^X_{0,3,d=0}$$

of the 3-fold Gromov-Witten invariant on $X = \mathbb{P}^1_{3,3,3}$. Both of these numbers are $\frac{1}{3}$, where the fraction comes from the definition of the orbifold integration [ALR] (see Section 2.5.4). Therefore, the Chen-Ruan cup product for $\mathbb{P}^1_{3,3,3}$ is given as follows.

$$\Delta_i^{1/3} \cdot \Delta_i^{2/3} = \frac{1}{3} PD(1) = \frac{1}{3}[pt] \tag{5.3.23}$$

$$\Delta_i^{1/3} \cdot \Delta_i^{1/3} = \frac{1}{3} PD(\Delta_i^{1/3}) = \Delta_i^{2/3} \tag{5.3.24}$$

Here, we used $PD(\Delta_i^{1/3}) = 3 \times \Delta_i^{2/3}$ (Remark 2.5.5). (5.3.24) completes the computation of f_1.

Remark 5.3.5. *In fact, to verify* (5.3.23)*, it remains to show that there are no other contributions than constants. However, we have shown in Lemma 5.2.1 that there are no holomorphic orbi-spheres in* $\mathbb{P}^1_{3,3,3}$ *which has only two orbifold markings.*

5.4 Further applications : (2,3,6), (2,4,4)

In this section, we proves Theorem 1.0.4 and Proposition 1.0.5. We slightly modify the classification of holomorphic orbi-spheres in $\mathbb{P}^1_{3,3,3}$ in order to compute the quantum cohomology rings of two other orbifold projective lines with three singular point: $\mathbb{P}^1_{2,3,6}$ and $\mathbb{P}^1_{2,4,4}$. For a certain product in $QH^*_{orb}(\mathbb{P}^1_{2,3,6})$, we use a heuristic argument, so the proof is incomplete (see Conjecture 5.4.3). In fact, all 3-point correlators including what remains as a conjecture here are computed in [MR]. We hope to rediscover this missing part by classifying holomorphic orbi-spheres whose domain admits a hyperbolic structure, and leave it to future investigation.

5.4.1 The product on $QH^*_{orb}(\mathbb{P}^1_{2,3,6})$

We set the notation for generators of $H^*_{orb}(\mathbb{P}^1_{2,3,6})$ as follows. Recall that E is the elliptic curve associated with the lattice $\mathbb{Z}\langle 1, \tau\rangle$ in $\mathbb{C}$ where $\tau = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$. Then $\mathbb{P}^1_{2,3,6}$ is obtained as the global quotient $[E/\mathbb{Z}_6]$, where $\mathbb{Z}_6 \cong \langle 1+\tau\rangle$ acts on E by the complex multiplication. There are three cone points on $\mathbb{P}^1_{2,3,6}$ and we use the same notation w_1, w_2, and w_3 for these singular point as we did for $\mathbb{P}^1_{3,3,3}$, where w_1, w_2 and w_3 have isotropy groups $\mathbb{Z}_2, \mathbb{Z}_3$, and $\mathbb{Z}_6$, respectively. The inertia orbifold $I\mathbb{P}^1_{2,3,6}$ consists of the smooth sector, $B\mathbb{Z}_2$, $B\mathbb{Z}_3$, and $B\mathbb{Z}_6$. The $\mathbb{Q}$-basis of $H^*_{orb}(\mathbb{P}^1_{2,3,6}, \mathbb{Q})$ is given as $1, \Delta_1^{1/2}, \Delta_2^{1/3}, \Delta_2^{2/3}, \Delta_3^{1/6}, \cdots, \Delta_3^{5/6}$, [pt] as follows.

The basis of smooth sector are

$$H^0_{orb}(\mathbb{P}^1_{2,3,6}, \mathbb{Q}) = \mathbb{Q}\cdot 1, \qquad H^2_{orb}(\mathbb{P}^1_{2,3,6}, \mathbb{Q}) = \mathbb{Q}\cdot[\mathrm{pt}].$$

For twist sectors, let $\Delta_1^{1/2} \in H^1_{orb}(\mathbb{P}^1_{2,3,6}, \mathbb{Q})$, $\Delta_2^{j/3} \in H^{\frac{2j}{3}}_{orb}(\mathbb{P}^1_{2,3,6}, \mathbb{Q})(j = 1, 2)$, and $\Delta_3^{k/6} \in H^{\frac{2k}{6}}_{orb}(\mathbb{P}^1_{2,3,6}, \mathbb{Q})(k = 1, \cdots, 5)$ which are supported at singular points w_1, w_2, and w_3, respectively. From the virtual dimension formula of $\overline{\mathcal{M}}_{0,3,d}(\mathbb{P}^1_{2,3,6})$, we can classify all possible orbi-insertions with expected dimension 0 and the corresponding domain orbi-sphere as in the following list.

(a) $\mathbb{P}^1_{2,3,6} : \langle\Delta_1^{1/2}, \Delta_2^{1/3}, \Delta_3^{1/6}\rangle, \langle\Delta_3^{3/6}, \Delta_2^{1/3}, \Delta_3^{1/6}\rangle\ \langle\Delta_1^{1/2}, \Delta_3^{2/6}, \Delta_3^{1/6}\rangle, \langle\Delta_3^{3/6}, \Delta_3^{2/6}, \Delta_3^{1/6}\rangle,$

(b) $\mathbb{P}^1_{3,3,3} : \langle\Delta_3^{2/6}, \Delta_3^{2/6}, \Delta_3^{2/6}\rangle, \langle\Delta_2^{1/3}, \Delta_3^{2/6}, \Delta_3^{2/6}\rangle, \langle\Delta_2^{1/3}, \Delta_2^{1/3}, \Delta_3^{2/6}\rangle, \langle\Delta_2^{1/3}, \Delta_2^{1/3}, \Delta_2^{1/3}\rangle$

(c) $\mathbb{P}^1_{3,6,6}$(hyperbolic) : $\langle\Delta_3^{1/6}, \Delta_3^{1/6}, \Delta_3^{4/6}\rangle, \langle\Delta_2^{2/3}, \Delta_3^{1/6}, \Delta_3^{1/6}\rangle$

(d) $\mathbb{P}^1_{2,2} : \langle 1, \Delta_3^{3/6}, \Delta_3^{3/6}\rangle, \langle 1, \Delta_1^{1/2}, \Delta_3^{3/6}\rangle, \langle 1, \Delta_1^{1/2}, \Delta_1^{1/2}\rangle$

(e) $\mathbb{P}^1_{3,3} : \langle 1, \Delta_2^{1/3}, \Delta_3^{4/6}\rangle, \langle 1, \Delta_2^{2/3}, \Delta_3^{2/6}\rangle$

From Lemma 5.2.1, there are no nontrivial maps which contribute to the type of (4) and (5). Thus, if we denote $\boldsymbol{t} := \sum t_{j,i}\Delta_j^i$, the genus 0-Gromov-Witten potential of $\mathbb{P}^1_{2,3,6}$ can be written up to order of t^3 as follows:

$$\begin{aligned} F_0^{\mathbb{P}^1_{2,3,6}}(\boldsymbol{t}) = \;& \tfrac{1}{2}t_0^2 \log q + t_0(\tfrac{1}{2}t_{1,\frac{1}{2}}t_{1,\frac{1}{2}} + \tfrac{1}{6}t_{3,\frac{3}{6}}t_{3,\frac{3}{6}}) + +(t_{1,\frac{1}{2}}t_{2,\frac{1}{3}}t_{3,\frac{1}{6}}) \cdot h_0(q) \\ & +(t_{3,\frac{3}{6}}t_{2,\frac{1}{3}}t_{3,\frac{1}{6}}) \cdot h_1(q) + (t_{1,\frac{1}{2}}t_{3,\frac{2}{6}}t_{3,\frac{1}{6}}) \cdot h_2(q) + (t_{3,\frac{3}{6}}t_{3,\frac{2}{6}}t_{3,\frac{1}{6}}) \cdot h_3(q) \\ & +\tfrac{1}{6}t^3_{3,\frac{2}{6}} \cdot h_4(q) + \tfrac{1}{2}t^2_{3,\frac{2}{6}}t_{2,\frac{1}{3}} \cdot h_5(q) + \tfrac{1}{2}t_{3,\frac{2}{6}}t^2_{2,\frac{1}{3}} \cdot h_6(q) + \tfrac{1}{6}t^3_{2,\frac{1}{3}} \cdot h_7(q) \\ & +\tfrac{1}{2}t^2_{3,\frac{1}{6}}t_{3,\frac{4}{6}} \cdot h_8(q) + \tfrac{1}{2}t^2_{3,\frac{1}{6}}t_{2,\frac{2}{3}} \cdot h_9(q) + \tfrac{1}{2}t_{2,\frac{2}{3}}t^2_{3,\frac{1}{6}} \cdot h_{10}(q) + O(t^4), \end{aligned} \tag{5.4.25}$$

where the precise expressions of $h_i(q)$ for $0 \leq i \leq 10$ will be given, later.

For holomorphic orbi-spheres of type (a) and (b), we choose the presentations of domain orbi-spheres as $[E/\mathbb{Z}_6]$ and $[E/\mathbb{Z}_3]$, respectively. Here, E is the elliptic curve corresponding to the $\mathbb{Z}$-lattice $\langle 1, \tau\rangle$ in $\mathbb{C}$, where $\tau = \exp \frac{2\pi\sqrt{-1}}{3}$.

Observe that any holomorphic orbi-sphere with the orbi-insertion condition in (a) or (b) satisfies the condition in the lemma 5.2.2. So, we can lift such maps $u : \mathbb{P}^1_{2,3,6} \to \mathbb{P}^1_{2,3,6}$ and $u : \mathbb{P}^1_{3,3,3} \to \mathbb{P}^1_{2,3,6}$ to a linear map between universal coverings $\widetilde{u} : \mathbb{C} \to \mathbb{C}$. Below, we will count these holomorphic orbi-spheres with help of the lattice structures of inverse image of orbi-singular points in $\mathbb{C}$. As in the case of $\mathbb{P}^1_{3,3,3}$, it will turn out that the counting matches the number of solutions of a certain Diophantine equations. The regularity of these holomorphic orbi-spheres are guaranteed by Lemma 5.2.4.

To clarify the orbi-insertions by looking at the lifted linear map $\tilde{u} : \mathbb{C} \to \mathbb{C}$, we explicitly identify the lattice structure on $\mathbb{C}$ coming from the universal orbifold covering map $p : \mathbb{C} \to \mathbb{P}^1_{2,3,6}$ as follows:

$$\begin{aligned} p^{-1}(w_1) &= \mathbb{Z}\left\langle \tfrac{1}{2}, \tfrac{\tau}{2} \right\rangle + p^{-1}(w_3) \\ &= \left\{ \tfrac{a}{2} + \tfrac{b}{2}\tau \mid a, b \in \mathbb{Z} \text{ and } a \text{ or } b \text{ is an odd integer} \right\}, \\ p^{-1}(w_2) &= \mathbb{Z}\left\langle \tfrac{2+\tau}{3}, \tfrac{1+2\tau}{3} \right\rangle + p^{-1}(w_1), \\ p^{-1}(w_3) &= \mathbb{Z}\langle 1, \tau\rangle \; (\ni 0). \end{aligned} \tag{5.4.26}$$

In particular, w_3 is set to be a base point associated with the universal covering $(\mathbb{C}, 0)$. (See Figure 5.4)

The universal cover $\mathbb{C}$ of the domain $\mathbb{P}^1_{2,3,6}$ also has the same lattice structure, and the lattices on $\mathbb{C}$ from the domain $\mathbb{P}^1_{3,3,3}$ are given as in Section 5.3.3.

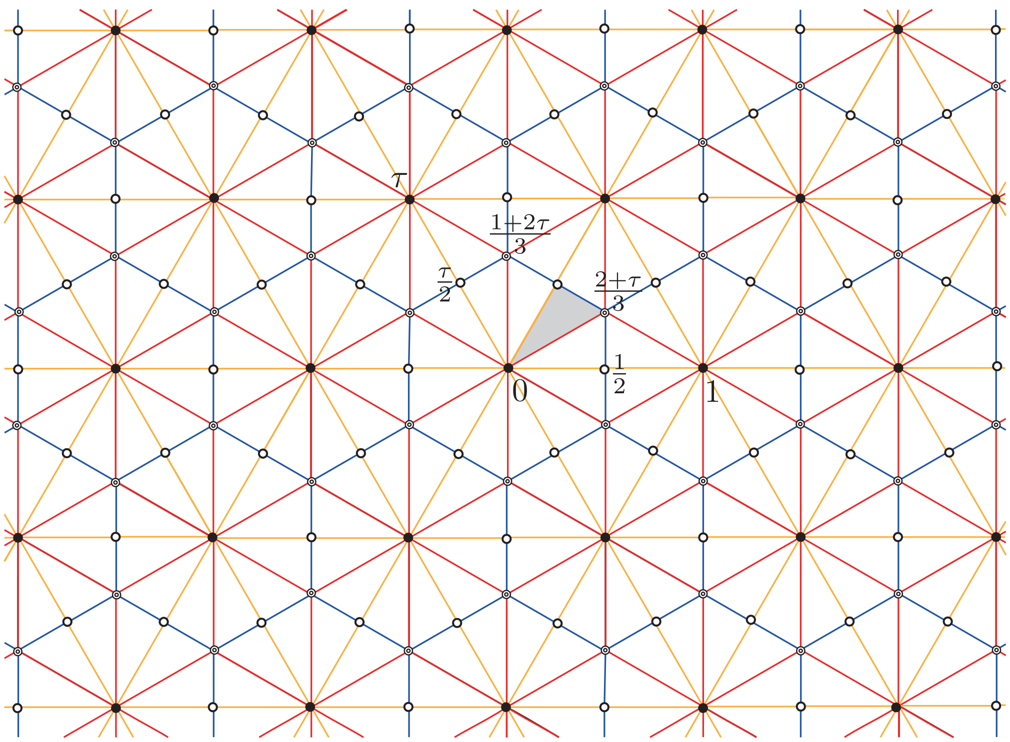


Figure 5.4: Lattices on the universal cover of $\mathbb{P}^1_{2,3,6}$: $p^{-1}(w_1) = \{\circ\}$, $p^{-1}(w_2) = \{\circledcirc\}$ and $p^{-1}(w_3) = \{\bullet\}$

Case (a) with the domain orbi-sphere $\mathbb{P}^1_{2,3,6}$: h_i for $0 \leq i \leq 3$

Let z_1, z_2, and z_3 be the three orbi-points in the domain $\mathbb{P}^1_{2,3,6}$, whose orders of singularities are 2, 3, and 6, respectively. If u is a holomorphic map from $\mathbb{P}^1_{2,3,6}$ to itself with the orbi-insertion condition as in (a), then u is a orbifold covering map by Lemma 5.2.2, so one can find the lifting $\widetilde{u} : \mathbb{C} \to \mathbb{C}$ with $\widetilde{u}(0) = 0$:

$$\begin{array}{ccc} \mathbb{C} & \overset{\widetilde{u}}{\dashrightarrow} & \mathbb{C} \\ \downarrow p & & \downarrow p \\ \mathbb{P}^1_{2,3,6} & \xrightarrow{u} & \mathbb{P}^1_{2,3,6}. \end{array} \tag{5.4.27}$$

Since any holomorphic orbi-sphere u contributing to (a) maps z_3 to w_3 (by the arrangement of insertions in (a)), $\widetilde{u}(z) = \lambda z$ for some $\lambda \in \mathbb{Z}[\tau]$. Conversely, it is clear from Figure 5.4 that any such linear map $\widetilde{u}$ descends to a holomorphic orbi-sphere with insertions as in (a). Since for $\lambda = a + b\tau$ $(a, b \in \mathbb{Z})$, the degree of the underlying map of $\widetilde{u}(z) = \lambda z$ is $N := |\lambda|^2 = \lambda\bar{\lambda} = a^2 - ab + b^2$, the above discussion shows that

$$h_0(q) + h_1(q) + h_2(q) + h_3(q) = \frac{1}{6}F(q),$$

where $F(q)$ is defined by the equation (5.3.22). Here, $\frac{1}{6}$ in the right hand side comes from the symmetry between linear maps which induce the same holomorphic orbi-sphere. By the same argument as in Section 5.3.2, we see that the symmetry among these linear maps is generated by the $(1 + \tau)$-multiplication, which is nothing but the action of the isotropy group of w_3 (isomorphic to $\mathbb{Z}_6$).

Note that the triangle whose vertices are $0 \in p^{-1}(z_3)$, $\frac{2+\tau}{3} \in p^{-1}(z_2)$, and $\frac{1+\tau}{2} \in p^{-1}(z_1)$ gives the fundamental domain of the upper-hemisphere of (the domain) $\mathbb{P}^1_{2,3,6}$ (see the shaded region in Figure 5.4 and compare it with Figure 5.5). As in the case of $\mathbb{P}^1_{3,3,3}$, we classify the orbi-insertion condition by chasing the images of $\frac{2+\tau}{3}$ and $\frac{1+\tau}{2}$ in the domain. For $\lambda = a + b\tau$,

$$\lambda \cdot \frac{2+\tau}{3} = \frac{2a-b}{3} + \frac{a+b}{3}\tau$$

and

$$\lambda \cdot \frac{1+\tau}{2} = \frac{a-b}{2} + \frac{a}{2}\tau.$$

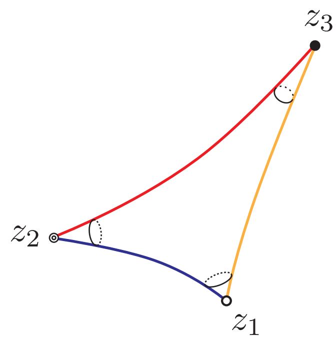


Figure 5.5: $\mathbb{P}^1_{2,3,6}$

First, note that

$$\lambda \cdot \frac{1+\tau}{2} \in p^{-1}(w_1) \quad \Longleftrightarrow \quad a \text{ or } b \text{ is odd,}$$

$$\lambda \cdot \frac{1+\tau}{2} \in p^{-1}(w_3) \quad \Longleftrightarrow \quad a \text{ and } b \text{ is even.}$$

From ($3 \mid 2a - b \iff 3 \mid a + b$), it can be easily checked that

$$\lambda \cdot \frac{2+\tau}{3} \in p^{-1}(w_3) \quad \Longleftrightarrow \quad 3 \mid (a+b),$$

$$\lambda \cdot \frac{2+\tau}{3} \in p^{-1}(w_2) \quad \Longleftrightarrow \quad 3 \nmid (a+b).$$

Hence using the equation (5.3.21), there are two possible orbi-insertions at the marked point corresponding to $\lambda \cdot \frac{2+\tau}{3}$:

$\Delta_3^{2/6}$ $3 \mid (a+b) \iff N \equiv 0 \mod 3,$

$\Delta_2^{1/3}$ $3 \nmid (a+b) \iff N \equiv 1 \mod 3.$

Similarly, two possible orbi-insertions at the marked point corresponding to $\lambda \cdot \frac{1+\tau}{2}$ are

$\Delta_3^{3/6}$ a and b is even $\iff N \equiv 0 \mod 2,$

$\Delta_1^{1/2}$ a or b is odd $\iff N \equiv 1 \mod 2.$

Summarizing the above discussion, we conclude from the Chinese remainder theorem that u contributes to

$$\langle \Delta_1^{1/2}, \Delta_2^{1/3}, \Delta_3^{1/6} \rangle \iff \deg u \equiv 1 \mod 6,$$
$$\langle \Delta_3^{3/6}, \Delta_2^{1/3}, \Delta_3^{1/6} \rangle \iff \deg u \equiv 4 \mod 6.$$
$$\langle \Delta_1^{1/2}, \Delta_3^{2/6}, \Delta_3^{1/6} \rangle \iff \deg u \equiv 3 \mod 6,$$
$$\langle \Delta_3^{3/6}, \Delta_3^{2/6}, \Delta_3^{1/6} \rangle \iff \deg u \equiv 0 \mod 6.$$

Recall $N = |\lambda|^2 = \deg u$, which equals the exponent of q for the term in the Gromov-Witten potential that u contributes to. Therefore, we obtain

$$h_0(q) = \frac{1}{6} \sum_{\substack{N=1 \\ N\equiv 1 \bmod 6}}^{\infty} \sum_{\substack{m^2-mn+n^2=N \\ m,n\in\mathbb{Z}}} q^N = \frac{1}{6} F_{1,6}(q),$$
$$h_1(q) = \frac{1}{6} \sum_{\substack{N=1 \\ N\equiv 4 \bmod 6}}^{\infty} \sum_{\substack{m^2-mn+n^2=N \\ m,n\in\mathbb{Z}}} q^N = \frac{1}{6} F_{4,6}(q),$$
$$h_2(q) = \frac{1}{6} \sum_{\substack{N=1 \\ N\equiv 3 \bmod 6}}^{\infty} \sum_{\substack{m^2-mn+n^2=N \\ m,n\in\mathbb{Z}}} q^N = \frac{1}{6} F_{3,6}(q),$$
$$h_3(q) = \frac{1}{6} \left(1 + \sum_{\substack{N=1 \\ N\equiv 0 \bmod 6}}^{\infty} \sum_{\substack{m^2-mn+n^2=N \\ m,n\in\mathbb{Z}}} q^N \right) = \frac{1}{6} + \frac{1}{6} F_{0,6}(q).$$

where $F_{i,6}$ is the sum of terms in F whose exponents of q is i modulo 6. Here, the constant term of h_3 can be obtained from a similar argument in the subsection 5.3.4.

Case (b) **with the domain orbi-sphere** $\mathbb{P}^1_{3,3,3}$

We first show that holomorphic orbi-spheres with orbi-insertions as in case (b) can be lifted to the one on $\mathbb{P}^1_{3,3,3}$. Let $\pi : \mathbb{P}^1_{3,3,3} \to \mathbb{P}^1_{2,3,6}$ be the 2-fold orbifold covering map which comes from the action on $[E/\langle\tau\rangle]$ generated by the $(1+\tau)$-multiplication, as drawn in Figure 5.6. Write w_i' and w_i ($i = 1, 2, 3$) for orbi-points in $\mathbb{P}^1_{3,3,3}$ and $\mathbb{P}^1_{2,3,6}$, respectively and let π send both w_1' and w_2' to w_2, and w_3' to w_3.

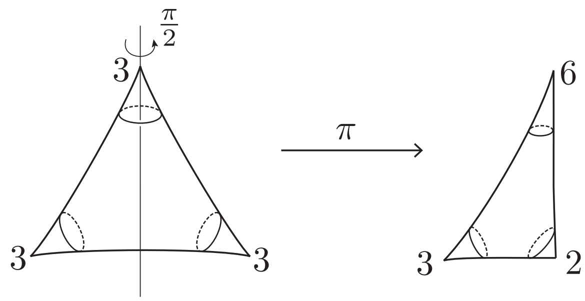


Figure 5.6: The 2-fold covering $\pi : \mathbb{P}^1_{3,3,3} \to \mathbb{P}^1_{2,3,6}$

After fixing base points of $\mathbb{P}^1_{3,3,3}$ and $\mathbb{P}^1_{2,3,6}$,

$$\pi_1^{orb}(\mathbb{P}^1_{3,3,3}) = \left\langle \rho_1, \rho_2, \rho_3 \mid (\rho_1)^3 = (\rho_2)^3 = (\rho_3)^3 = \rho_1\rho_2\rho_3 = 1 \right\rangle,$$
$$\pi_1^{orb}(\mathbb{P}^1_{2,3,6}) = \left\langle \lambda_1, \lambda_2, \lambda_3 \mid (\lambda_1)^2 = (\lambda_2)^3 = (\lambda_3)^6 = \lambda_1\lambda_2\lambda_3 = 1 \right\rangle.$$

(see Section 2.3) and π induces a group homomorphism $\pi_* : \pi_1^{orb}(\mathbb{P}^1_{3,3,3}) \to \pi_1^{orb}(\mathbb{P}^1_{2,3,6})$. We see from Figure 5.6 that the images of ρ_1 and ρ_2 under π_* lie in the conjugacy class of λ_2, and the image of the other generator ρ_3 lies in that of $(\lambda_3)^2$. (Here, conjugacy classes depend on the choice of base points.) It follows that $\pi_*\left(\pi_1^{orb}(\mathbb{P}^1_{3,3,3})\right)$ contains λ_2 and $(\lambda_3)^2$, as it is a normal subgroup of $\pi_1^{orb}(\mathbb{P}^1_{2,3,6})$.

Lemma 5.4.1. *For a given (b)-type holomorphic orbi-sphere $u : \mathbb{P}^1_{3,3,3} \to \mathbb{P}^1_{2,3,6}$, there exists a holomorphic orbi-sphere $\widetilde{u}$ which make the following diagram commute:*

$$\begin{array}{ccc} & & \mathbb{P}^1_{3,3,3} \\ & \overset{\exists \widetilde{u}}{\nearrow} & \downarrow \pi \\ \mathbb{P}^1_{3,3,3} & \xrightarrow{u} & \mathbb{P}^1_{2,3,6}. \end{array}$$

Proof. Observe that only two kinds of orbi-insertions $\Delta_2^{1/3}$ and $\Delta_3^{2/6}$ appear in (b). Hence u_* maps a generator of $\pi_1^{orb}(\mathbb{P}^1_{3,3,3})$ to an element in the conjugacy class of λ_2 or λ_3^2. (Indeed, if we choose base points and the generators ρ_1 and λ_2 as in (b) of Figure 2.2, then u_* sends ρ_1 exactly to λ_2, and similar happens for ρ_2 and ρ_3.) Thus, $u_*\left(\pi_1^{orb}(\mathbb{P}^1_{3,3,3})\right)$ is contained in $\pi_*\left(\pi_1^{orb}(\mathbb{P}^1_{3,3,3})\right)$. From Proposition 2.4.2, there exists an orbi-map $\widetilde{u} : \mathbb{P}^1_{3,3,3} \to \mathbb{P}^1_{3,3,3}$ which lifts u. □

Let z be the one of three orbi-points of the domain $\mathbb{P}^1_{3,3,3}$ for a holomorphic orbi-sphere u of type (b). If u sends z to $w_2 \in \mathbb{P}^1_{2,3,6}$ with the orbi-insertion $\Delta_2^{1/3}$, then the corresponding orbi-insertion of the lifting $\widetilde{u} : \mathbb{P}^1_{3,3,3} \to \mathbb{P}^1_{3,3,3}$ is $\Delta_1^{1/3}$ or $\Delta_2^{1/3}$. Similarly, if $u(z) = w_3$ with the insertion $\Delta_3^{2/6}$, then the corresponding orbi-insertion of a lifting $\widetilde{u}$ is $\Delta_3^{1/3}$. Here, we abused the notation for orbi-insertions of $\mathbb{P}^1_{3,3,3}$ and $\mathbb{P}^1_{2,3,6}$.

For each holomorphic orbi-sphere $u : \mathbb{P}^1_{3,3,3} \to \mathbb{P}^1_{2,3,6}$ with orbi-insertion of type (b), there are two liftings $\tilde{u} : \mathbb{P}^1_{3,3,3} \to \mathbb{P}^1_{3,3,3}$. Two liftings of u are related by the $\mathbb{Z}_2$-action (i.e. the action of the deck transformation group) which switches w'_1 and w'_2. Therefore, if one lifting has orbi-insertion $\Delta_1^{1/3}$, then the other lifting has orbi-insertion $\Delta_2^{1/3}$.

In summary, Lemma 5.4.1 gives rise to the following one-to-two correspondences:

$$\begin{aligned}
\langle \Delta_3^{2/6}, \Delta_3^{2/6}, \Delta_3^{2/6} \rangle^{\mathbb{P}^1_{2,3,6}} &\overset{1:2}{\longleftrightarrow} \langle \Delta_3^{1/3}, \Delta_3^{1/3}, \Delta_3^{1/3} \rangle^{\mathbb{P}^1_{3,3,3}} \\
\langle \Delta_2^{1/3}, \Delta_2^{1/3}, \Delta_3^{2/6} \rangle^{\mathbb{P}^1_{2,3,6}} &\overset{1:2}{\longleftrightarrow} \langle \Delta_1^{1/3}, \Delta_2^{1/3}, \Delta_3^{1/3} \rangle^{\mathbb{P}^1_{3,3,3}} + \langle \Delta_2^{1/3}, \Delta_1^{1/3}, \Delta_3^{1/3} \rangle^{\mathbb{P}^1_{3,3,3}} \\
&\qquad = 2\langle \Delta_1^{1/3}, \Delta_2^{1/3}, \Delta_3^{1/3} \rangle^{\mathbb{P}^1_{3,3,3}} \\
\langle \Delta_2^{1/3}, \Delta_2^{1/3}, \Delta_2^{1/3} \rangle^{\mathbb{P}^1_{2,3,6}} &\overset{1:2}{\longleftrightarrow} \langle \Delta_1^{1/3}, \Delta_1^{1/3}, \Delta_1^{1/3} \rangle^{\mathbb{P}^1_{3,3,3}} + \langle \Delta_2^{1/3}, \Delta_2^{1/3}, \Delta_2^{1/3} \rangle^{\mathbb{P}^1_{3,3,3}} \\
&\qquad = 2\langle \Delta_1^{1/3}, \Delta_1^{1/3}, \Delta_1^{1/3} \rangle^{\mathbb{P}^1_{3,3,3}}.
\end{aligned}$$

($\langle \Delta_2^{1/3}, \Delta_3^{2/6}, \Delta_3^{2/6} \rangle^{\mathbb{P}^1_{2,3,6}}$ vanishes since there are no corresponding liftings.)

Therefore, h_i for $4 \leq i \leq 7$ is given as follows:

Proposition 5.4.2. *Let $f_0^{\mathbb{P}^1_{3,3,3}}(q)$ and $f_1^{\mathbb{P}^1_{3,3,3}}(q)$ be the coefficient of the $t_1t_2t_3$ and t_i^3 of $F_0^{\mathbb{P}^1_{3,3,3}}$, respectively. Then*

$$\begin{aligned}
h_4(q) &= \frac{1}{2} f_1^{\mathbb{P}^1_{3,3,3}}(q^2) = \frac{1}{6} + q^6 + q^{18} + q^{24} + 2q^{42} + O(q^{48}), \\
h_5(q) &= 0, \\
h_6(q) &= f_0^{\mathbb{P}^1_{3,3,3}}(q^2) = q^2 + q^8 + 2q^{14} + 2q^{26} + q^{32} + 2q^{38} + O(q^{48}), \\
h_7(q) &= f_1^{\mathbb{P}^1_{3,3,3}}(q^2) = \frac{1}{3} + 2q^6 + 2q^{18} + 2q^{24} + 4q^{42} + O(q^{48}).
\end{aligned}$$

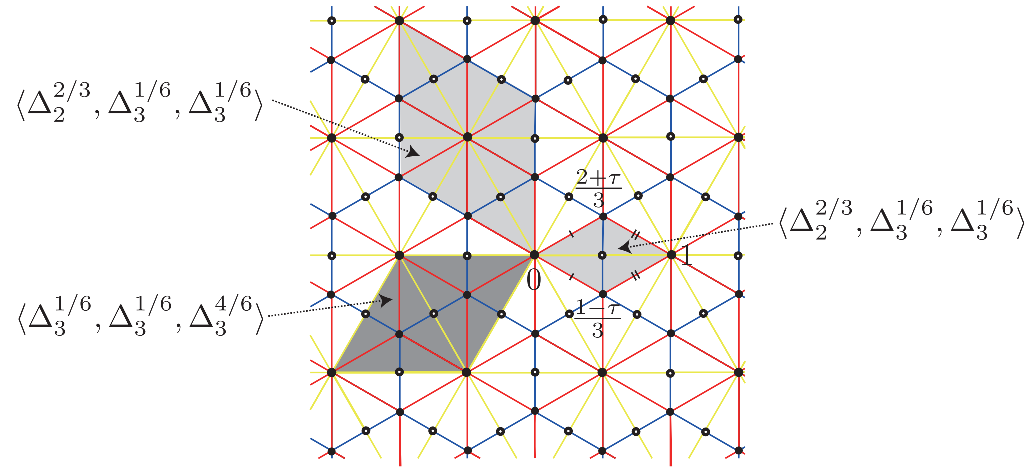


Figure 5.7: Images of holomorphic orbi-spheres $\mathbb{P}^1_{3,6,6} \to \mathbb{P}^1_{2,3,6}$ visualized in the universal cover of $\mathbb{P}^1_{2,3,6}$

Case (c) **with the domain orbi-sphere** $\mathbb{P}^1_{3,6,6}$

For these kind of contributions, the lifting of holomorphic orbi-spheres on the universal cover level is no longer a linear map, since the domain orbi-sphere is hyperbolic. Hence we can not use our classification argument any more. However, we may try to find such maps directly by looking at their image on the universal cover $\mathbb{C}$ of the target $\mathbb{P}^1_{2,3,6}$.

For this, we consider rhombi in the universal covering of $\mathbb{P}^1_{2,3,6}$ whose vertices lie in the $p^{-1}(w_1, w_2, w_3)$. For example, observe that the rhombus v whose set of vertices are $\left\{0, \frac{2+\tau}{3}, 1, \frac{1-\tau}{3}\right\}$ gives one of contribution from $\mathbb{P}^1_{3,6,6}$ to $\langle \Delta_2^{2/3}, \Delta_3^{1/6}, \Delta_2^{1/6} \rangle$. (See the rightmost Rhombus in Figure 5.7.) One can visualize this holomorphic orbi-sphere by folding this rhombus along its longer diagonal. Pairs of identified edges after this process are drawn in Figure 5.7.

There are various such rhombi, and their corresponding orbi-insertions can be classified in the following way. Note that these rhombi are images of the smallest rhombus v given above by linear maps $z \mapsto \lambda z$ for $\lambda \in \mathbb{Z}[\tau]$. (This means, we regard the vertices $\frac{2+\tau}{3}$ and $\frac{1-\tau}{3}$ of v as markings z_2 and z_3 of the order 6 in the domain $\mathbb{P}^1_{3,6,6}$, respectively.) Since $(1+\tau) \cdot \frac{1-\tau}{3} = \frac{2+\tau}{3}$, insertions at z_2 and z_3 are the same, and if $\lambda \cdot \left(\frac{2+\tau}{3}\right)$ is contained in $p^{-1}(w_3)$ (resp. $p^{-1}(w_2)$), the corresponding insertions are $\langle \Delta_3^{1/6}, \Delta_3^{1/6}, \Delta_3^{4/6}\rangle$ (resp. $\langle \Delta_2^{2/3}, \Delta_3^{1/6}, \Delta_3^{1/6}\rangle$). Recall that h_8 counts $\langle \Delta_3^{1/6}, \Delta_3^{1/6}, \Delta_3^{4/6}\rangle$ whereas h_9 counts $\langle \Delta_2^{2/3}, \Delta_3^{1/6}, \Delta_3^{1/6}\rangle$.

Using the identity $\lambda \cdot \frac{2+\tau}{3} = \frac{2a-b}{3} + \frac{a+b}{3}\tau$ and proceeding as in case (a), we have

$$\lambda \cdot \frac{2+\tau}{3} \in p^{-1}(w_3) \iff 3 \mid N(= a^2 - ab + b^2)$$

and

$$\lambda \cdot \frac{2+\tau}{3} \in p^{-1}(w_2) \iff 3 \nmid N(= a^2 - ab + b^2)$$

It is easy to see that six rhombi related by $\mathbb{Z}_6$-rotation at the origin represent the same map, and the degrees of these rhombi are also given by $|\lambda|^2$. Comparing with the decomposition of F in terms of q-th power (mod 3) as in Section 5.3.3, it follows that $h_8(q) = \frac{1}{6}F_{0,3}(q^2)$ and $h_9(q) = \frac{1}{6}F_{1,3}(q^2)$, if one can prove that there are *no* other contributions.

Conjecture 5.4.3. *We conjecture that there are no contributions from $\mathbb{P}^1_{3,6,6,}$ other that these rhombi, or equivalently,*

$$\begin{aligned} h_8(q) &= \frac{1}{6}F_{0,3}(q^2) = \frac{1}{2}f_1^{\mathbb{P}^1_{3,3,3}}(q^2) \\ &= \frac{1}{6} + q^6 + q^{18} + q^{24} + 2q^{42} + q^{54} + q^{72} + 2q^{78} + O(q^{96}), \\ h_9(q) &= \frac{1}{6}F_{1,3}(q^2) = f_0^{\mathbb{P}^1_{3,3,3}}(q^2) \\ &= q^2 + q^8 + 2q^{14} + 2q^{26} + q^{32} + 2q^{38} + O(q^{48}). \end{aligned}$$

Remark 5.4.4. *One way to see that the conjecture holds true is the following. It can be shown that the conjectural answer is modular on* $\Gamma(6)$ *and first few terms match with the one given in [MR] (or [MSh]). Then by modularity, they must be identically the same. (Hence, it is not a conjecture in honest sense.) We left it as a conjecture as our purpose is direct classification of holomorphic orbi-spheres which we failed for this particular* 3*-point invariants.*

There is a nontrivial algebraic relation between $h_8(q)$ and $h_9(q)$ which basically comes from the Frobenius structure on $QH^*_{orb}(\mathbb{P}^1_{2,3,6};\mathbb{Q})$. This can be obtained as follows: firstly,

$$\begin{aligned} h_8 &= \left(\Delta_3^{1/6} * \Delta_3^{1/6}, \Delta_3^{4/6}\right) \\ &= \left(\Delta_3^{1/6} * \Delta_3^{1/6}, \tfrac{1}{6}(h_6)^{-1}\Delta_2^{1/3} * \Delta_2^{1/3} - \tfrac{1}{2}(h_6)^{-1}h_7\Delta_2^{2/3}\right) \\ &= -\tfrac{1}{2}(h_6)^{-1}h_7h_9 + \tfrac{1}{6}(h_6)^{-1}\left(\Delta_3^{1/6} * \Delta_3^{1/6}, \Delta_2^{1/3} * \Delta_2^{1/3}\right) \end{aligned} \tag{5.4.28}$$

where (,) is the Poincarè paring and in the second equality, we used

$$\Delta_2^{1/3} * \Delta_2^{1/3} = 6h_6\Delta_3^{4/6} + 3h_7\Delta_2^{2/3}$$

which is completely known from cases (b) and (c).

The last term in (5.4.28) can be computed with the help of the Frobenius structure:

$$\begin{aligned}
\left(\Delta_3^{1/6} * \Delta_3^{1/6}, \Delta_2^{1/3} * \Delta_2^{1/3}\right) &= \left(\Delta_3^{1/6}, \Delta_3^{1/6} * \Delta_2^{1/3} * \Delta_2^{1/3}\right) \\
&= \left(\Delta_3^{1/6}, (6h_1\Delta_3^{3/6} + 2h_0\Delta_1^{1/2}) * \Delta_2^{1/3}\right) \\
&= 6h_1\left(\Delta_3^{1/6}, \Delta_3^{3/6} * \Delta_2^{1/3}\right) + 2h_0\left(\Delta_3^{1/6}, \Delta_1^{1/2} * \Delta_2^{1/3}\right) \\
&= 6(h_1)^2 + 2(h_0)^2.
\end{aligned}$$

Plugging it into (5.4.28), we obtain the relation

$$6h_6h_8 = -3h_7h_9 + 6(h_1)^2 + 2(h_0)^2. \tag{5.4.29}$$

One can check (5.4.29) numerically up to a higher enough order using Mathematica with our conjectural h_8 and h_9.

Remark 5.4.5. *A similar kind of lifting argument as in case (b) tells us that* $(3, 6, 6)$*-contributions for* $\mathbb{P}^1_{3,3,3}$ *is equivalent to a certain kind of* 4*-fold Gromov-Witten of* $\mathbb{P}^1_{3,3,3}$ *which counts holomorphic orbi-spheres* $\mathbb{P}^1_{3,3,3,3} \to \mathbb{P}^1_{3,3,3}$.

5.4.2 The product on $QH^*_{orb}(\mathbb{P}^1_{2,4,4})$

Let E' be the elliptic curve associated with the lattice $\mathbb{Z}\langle 1, i\rangle$, where $i = \sqrt{-1}$. (In fact, E and E' are isomorphic as symplectic manifolds.) Then the quotient of E' by the $\mathbb{Z}_4$-action which is generated by the $i(= \sqrt{-1})$-multiplication is the elliptic orbifold projective line $\mathbb{P}^1_{2,4,4} = [E'/\mathbb{Z}_4]$ with three singular points w_1, w_2, w_3. Here, w_1 is the point with the local group isomorphic to $\mathbb{Z}_2$, and w_2, w_3 have local groups isomorphic to $\mathbb{Z}_4$.

The inertial orbifold $I\mathbb{P}^1_{2,4,4}$ consists of the smooth sector together with a $B\mathbb{Z}_2$ and two $B\mathbb{Z}_4$'s. As usual, the $\mathbb{Q}$-basis of $H^*_{orb}(\mathbb{P}^1_{2,4,4}, \mathbb{Q})$ is taken as

$$1, \Delta_1^{1/2}, \Delta_2^{1/4}, \Delta_2^{2/4}, \Delta_2^{3/4}, \Delta_3^{1/4}, \Delta_3^{2/4}, \Delta_3^{3/4}, [\mathrm{pt}]$$

Then the cohomology of the smooth sector is given by

$$H^0_{orb}(\mathbb{P}^1_{2,4,4},\mathbb{Q}) = \mathbb{Q}\cdot 1, \qquad H^2_{orb}(\mathbb{P}^1_{2,4,4},\mathbb{Q}) = \mathbb{Q}\cdot[\mathrm{pt}].$$

For twist sectors, $\Delta_1^{1/2} \in H^1_{orb}(\mathbb{P}^1_{2,4,4},\mathbb{Q})$, $\Delta_k^{j/4} \in H^{\frac{2j}{4}}_{orb}(\mathbb{P}^1_{2,3,6},\mathbb{Q})(j = 1,2,3,\ k = 2,3)$ are generators supported at singular points w_1, w_2, w_3, respectively. In a similar way with $\mathbb{P}^1_{2,3,6}$ case, we classify all the triple orbi-insertions with expected dimension 0 and their domain orbifolds.

(a) $\mathbb{P}^1_{2,4,4} : \langle \Delta_1^{1/2}, \Delta_j^{1/4}, \Delta_k^{1/4}\rangle, \langle \Delta_j^{2/4}, \Delta_j^{1/4}, \Delta_k^{1/4}\rangle,$
$\langle \Delta_j^{2/4}, \Delta_k^{1/4}, \Delta_k^{1/4}\rangle$ for $j,k = 2,3$.

(b) $\mathbb{P}^1_{2,2} : \langle 1, \Delta_1^{1/2}, \Delta_1^{1/2}\rangle, \langle 1, \Delta_1^{1/2}, \Delta_k^{2/4}\rangle, \langle 1, \Delta_j^{2/4}, \Delta_k^{2/4}\rangle$ for $j,k = 2,3$.

(c) $\mathbb{P}^1_{4,4} : \langle 1, \Delta_j^{1/4}, \Delta_k^{3/4}\rangle$ for $j,k = 2,3$

Again, (b) and (c) do not occur because of Lemma 5.2.1. If we denote $\boldsymbol{t} := \sum t_{j,i}\Delta_j^i$, the genus-0 Gromov-Witten potential of $\mathbb{P}^1_{2,4,4}$ is written up to order of t^3 as

$$\begin{aligned} F_0^{\mathbb{P}^1_{2,4,4}}(\boldsymbol{t}) &= \tfrac{1}{2}t_0^2\log q + \tfrac{1}{2}t_0 t_{1,\frac12}t_{1,\frac12} \\ &+ \tfrac14 t_0 t_{2,\frac24}t_{2,\frac24} + t_{2,\frac14}t_{2,\frac34} + t_{3,\frac24}t_{3,\frac24} + t_{3,\frac14}t_{3,\frac34} + \tfrac12 t_{1,\frac12}(t_{2,\frac14}^2 + t_{3,\frac14}^2)\cdot g_0(q) \\ &+ t_{1,\frac12}t_{2,\frac14}t_{3,\frac14}\cdot g_1(q) + \tfrac12(t_{2,\frac24}t_{2,\frac14}^2 + t_{3,\frac24}t_{3,\frac14}^2)\cdot g_2(q) \\ &+ \tfrac12(t_{2,\frac24}t_{3,\frac14}^2 + t_{3,\frac24}t_{2,\frac14}^2)\cdot g_3(q) + (t_{2,\frac24}t_{2,\frac14}t_{3,\frac14} + t_{3,\frac24}t_{3,\frac14}t_{2,\frac14})\cdot g_4(q) + O(t^4) \end{aligned} \tag{5.4.30}$$

The classification in (a) shows that the domain orbi-sphere should have the same orbifold structure, that is, the contributions only come from maps $\mathbb{P}^1_{2,4,4} \to \mathbb{P}^1_{2,4,4}$. Let $p : \mathbb{C} \to \mathbb{P}^1_{2,4,4}$ be the universal covering which factors through the $\mathbb{Z}_4$-quotient map $E' \to \mathbb{P}^1_{2,4,4}$. We abuse the notation p for covering maps of both the domain and the target $\mathbb{P}^1_{2,4,4}$. From the obvious symmetry between w_2 and w_3, we may fix one of the orbi-insertions by $\Delta_2^{1/4}$, similarly to what we did for $\mathbb{P}^1_{3,3,3}$ case. So, we assume that our holomorphic orbi-sphere sends z_2 to w_2.

As before, any holomorphic orbi-sphere u in our concern can be lifted to a linear map $\tilde{u} : z \to \lambda z$ by Lemmas 5.2.2 and 5.2.4. We set the lattice structure

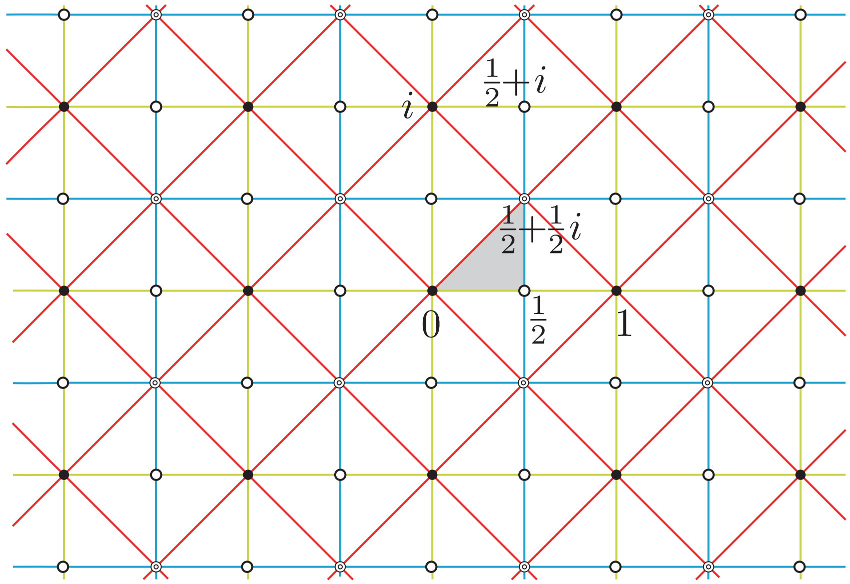


Figure 5.8: Lattices on the universal cover of $\mathbb{P}^1_{2,4,4}$: $p^{-1}(w_1) = \{\circ\}$, $p^{-1}(w_2) = \{\bullet\}$ and $p^{-1}(w_3) = \{\circledcirc\}$

on $\mathbb{C}$ induced by the covering $\mathbb{C} \to \mathbb{P}^1_{2,4,4}$ for both the domain and the target as follows:

$$p^{-1}(w_1) = \left\{ \frac{1}{2}(a+ib) \middle| \text{either } a \text{ or } b \text{ is an odd number, but not both.} \right\},$$

$$p^{-1}(w_2) = \mathbb{Z}\langle 1, i\rangle \ (\ni 0),$$

$$p^{-1}(w_3) = \left\{ \frac{1}{2}(a+ib) \middle| \text{both } a \text{ and } b \text{ are odd numbers} \right\}.$$

Here, we think of w_2 as the base point associated with the universal cover $(\mathbb{C}, 0)$. (See Figure 5.8.)

Since we have assumed that the orbifold singular point z_2 is mapped to w_2, the lifting $\widetilde{u}$ of $u : \mathbb{P}^1_{2,4,4} \to \mathbb{P}^1_{2,4,4}$ maps $p^{-1}(z_2)$ to $p^{-1}(w_2)$ fixing the origin. Therefore, $\widetilde{u}(z) = \lambda z$ for some $\lambda \in \mathbb{Z}[i]$. As mentioned, the degree of a holomorphic orbi-sphere u is $|\lambda|^2 = a^2 + b^2$ if the lifting of u is $\widetilde{u}(z) = \lambda z$ with $\lambda = a + bi$. Let $G(q)$ denote the power series

$$G(q) = \sum_{a,b\in\mathbb{Z}} q^{a^2+b^2}. \tag{5.4.31}$$

See (5.5.32) for first few terms of G. Note that if we divide a^2+b^2 by 4, then 3 can not appear as a remainder for any $a, b \in \mathbb{Z}$. Thus, we can decompose G into $G = G_{0,4} + G_{1,4} + G_{2,4}$ in accordance with the exponent of q modulo 4.

We determine the orbi-insertion for each $\lambda = a+bi$ by the same way as before. Note that the right-angled isosceles triangle whose vertices $\{0, \frac{1}{2}, \frac{1+i}{2}\}$ is one of the fundamental domain of the upper-hemisphere of $\mathbb{P}^1_{2,4,4}$. (See the shaded region in Figure 5.8.)

Observe that the two marked points other than the origin in this fundamental domain map to

$$(a+bi)\cdot\frac{1}{2} = \frac{a}{2}+\frac{b}{2}i \quad \text{and} \quad (a+bi)\cdot\frac{1+i}{2} = \frac{a-b}{2}+\frac{a+b}{2}i$$

by the linear map $z \mapsto (a+bi)z$. By proceeding as in the case of $\mathbb{P}^1_{2,3,6}$, we see that there are only three possibilities of the type of insertions, which are listed as follows:

(i) "$\frac{a}{2}+\frac{b}{2} \in p^{-1}(w_1)$ and $\frac{a-b}{2}+\frac{a+b}{2} \in p^{-1}(w_3)$" if and only if $a^2+b^2 \equiv 1 \mod 4$

(ii) "$\frac{a}{2}+\frac{b}{2} \in p^{-1}(w_2)$ and $\frac{a-b}{2}+\frac{a+b}{2} \in p^{-1}(w_2)$" if and only if $a^2+b^2 \equiv 0 \mod 4$

(iii) "$\frac{a}{2}+\frac{b}{2} \in p^{-1}(w_3)$ and $\frac{a-b}{2}+\frac{a+b}{2} \in p^{-1}(w_2)$" if and only if $a^2+b^2 \equiv 3 \mod 4$

By definition of coefficients g_i in (5.4.30), holomorphic orbi-spheres with insertions (i), (ii) and (iii) precisely give rise to g_1, g_2 and g_3, respectively. Therefore, we conclude that $g_0(q) = g_4(q) = 0$, and

$$\begin{aligned}
g_1(q) &= \frac{1}{4}\sum_{\substack{N=1\\ N\equiv 1 \bmod 4}}^{\infty}\sum_{\substack{a^2+b^2=N\\ a,b\in\mathbb{Z}}} q^N = \frac{1}{4}G_{1,4},\\
g_2(q) &= \frac{1}{4}\sum_{\substack{N=1\\ N\equiv 0 \bmod 4}}^{\infty}\sum_{\substack{a^2+b^2=N\\ a,b\in\mathbb{Z}}} q^N,\\
g_3(q) &= \frac{1}{4}\sum_{\substack{N=1\\ N\equiv 2 \bmod 4}}^{\infty}\sum_{\substack{a^2+b^2=N\\ a,b\in\mathbb{Z}}} q^N.
\end{aligned}$$

Again, $\frac{1}{4}$ is due to the $\mathbb{Z}_4$-symmetry at the origin in the universal cover $\mathbb{C}$ (from the action of the local group at w_2) which is generated by the i-multiplication.

5.5 Theta series

Recall that our results were expressed in terms of the following two power series

$$\begin{aligned} F(q) &= \textstyle\sum_{a,b\in\mathbb{Z}} q^{a^2-ab+b^2} \\ &= 1+6q+6q^3+6q^4+12q^7+6q^9 \\ &\quad +6q^{12}+12q^{13}+6q^{16}+12q^{19}+12q^{21}+O(q^{24}), \\ G(q) &= \textstyle\sum_{a,b\in\mathbb{Z}} q^{a^2+b^2} \\ &= 1+4q+4q^2+4q^4+8q^5+4q^8 \\ &\quad +4q^9+8q^{10}+8q^{13}+4q^{16}+8q^{17}+4q^{18}+o(q^{20}). \end{aligned} \tag{5.5.32}$$

In this section, we briefly explain several number theoretic feature of F and G. (For more details, see [B, Chapter 4] or [G].) We first provide a description of Fourier coefficients of F and G.

Proposition 5.5.1. *Write $F(q) = \sum_{N\geq 0} a_N q^N$ and $G(q) = \sum_{N\geq 0} b_N q^N$. Then*

$$a_N = 6(d_{1/3}(N) - d_{2/3}(N)),$$

$$b_N = 4(d_{1/4}(N) - d_{3/4}(N))$$

where $d_{j/3}(N)$ denotes the number of divisors of N which are j modulo 3, and $d_{j/4}(N)$ the number of divisors of N which are j modulo 4

Proof. We only prove the first identity for a_N, and refer readers to [G, Theorem 3] for G. The following is a simple modification of the argument given in [G], but we repeat it here for completeness.

Recall that for $\tau = e^{2\pi\sqrt{-1}/3}$, $|a+b\tau|^2 = a^2 - ab + b^2$. This gives a structure of Euclidean domain in $\mathbb{Z}[\tau]$ (this ring is usually called the ring of *Eisenstein integers* or *Eulerian integers*). In particular, $\mathbb{Z}[\tau]$ is a unique factorization domain, and hence a prime factorization in this ring makes sense up to units which are $= \{\pm 1, \pm\tau, \pm\tau^2\} = \{(1+\tau)^k \,|\, 0 \leq k \leq 5\}$ (and also up to the order of factors). It is known that a prime number in $\mathbb{Z}[\tau]$ is either a prime number in $\mathbb{Z}$ which is 2 modulo 3, or $a+b\tau$ whose modulus square $|a+b\tau|^2$ is a prime number in $\mathbb{Z}$. In latter case, $|a+b\tau|^2$ is alway 1 modulo 3 unless it is $3 = (1-\tau)\overline{(1-\tau)} = (1-\tau)(2+\tau)$ itself. (Of course, a prime number multiplied with a unit is also prime.)

Note that finding solutions of

$$a^2 - ab + b^2 = (a+b\tau)\overline{(a+b\tau)} = N \in \mathbb{Z} \tag{5.5.33}$$

is equivalent to finding factorizations $N = \alpha\beta$ of N in $\mathbb{Z}[\tau]$ such that $\beta = \bar{\alpha}$ where $\bar{\alpha}$ is the complex conjugation of α. Let $N = 3^f n_1 n_2$ with $n_1 = \prod_{p\equiv 1(\text{mod}3)} p^r$ and $n_2 = \prod_{q\equiv 2(\text{mod}3)} q^s$. Then the prime factorization of N in $\mathbb{Z}[\tau]$ can be written as

$$N = \{(1-\tau)(2+\tau)\}^f \prod_{\substack{c^2-cd+d^2=p \\ p\equiv 1(\text{mod}3)}} \{(c+d\tau)\overline{(c+d\tau)}\}^r \prod_{q\equiv 2(\text{mod}3)} q^s$$

where $c + d\tau$ and $\overline{c + d\tau}$ come in pair for each p since N is an integer. Now the condition $\beta = \bar{\alpha}$ forces them to be of the following forms:

$$\begin{aligned} \alpha &= (1+\tau)^t(1-\tau)^{f_1}(2+\tau)^{f_2} \prod\{(c+d\tau)^{r_1}\overline{(c+d\tau)}^{r_2}\} \prod q^{s_1} \\ \beta &= (1+\tau)^{-t}(1-\tau)^{f_2}(2+\tau)^{f_1} \prod\{(c+d\tau)^{r_2}\overline{(c+d\tau)}^{r_1}\} \prod q^{s_2} \end{aligned} \tag{5.5.34}$$

with $0 \le t \le 5$, $f_1 + f_2 = f$, $r_1 + r_2 = r$ and $s_1 + s_2 = s$. $\bar{\beta} = \alpha$ also implies that $s_1 = s_2$, so there is no solution to (5.5.33) if s is odd. Let us assume that s is even from now on. Then s_i's are uniquely determined (as the half of s). Observe that t has six choices and f_1, r_1 determine $f_2 = f - f_1$, $r_2 = r - r_1$ respectively. Thus, there are seemingly $6(f+1)\prod(r+1)$ number of choice for α and β satisfying (5.5.34). However, replacing one $(1-\tau)$ by $(2+\tau)$ for in the expression of α (5.5.34) affects α by multiplying a unit since $(2+\tau)/(1-\tau) = 1+\tau$ is a multiplicative generator of the group of units in $\mathbb{Z}[\tau]$. Getting rid of this redundancy, the number of pairs (α,β) satisfying $N = \alpha\beta$ and $\beta = \bar{\alpha}$ is given by $6\prod(r+1)$. It is easy to check that this number is same to $6(d_{1/3}(N) - d_{2/3}(N))$. □

Remark 5.5.2. *From the proof, we see that "6" in the expression of a_N is related to the number of units in the ring $\mathbb{Z}[\tau]$, which give the symmetries on the associated moduli space of orbi-spheres (see the last paragraph of Section* (5.3.2)*).*

We next describe F and G in terms of famous Jacobi theta functions. The definitions of related Jacobi theta functions are given as follows.

Definition 5.5.1. *The second and the third Jacobi theta functions are the power series θ_2 and θ_3 in q which are defined as follows:*

$$\theta_2(q) := \sum_{-\infty}^{\infty} q^{(n+1/2)^2},$$

$$\theta_3(q) := \sum_{-\infty}^{\infty} q^{n^2}.$$

Remark 5.5.3. *Originally, theta functions are two variable functions depending on z and q. Above θ_i is indeed obtained by putting $z = 0$.*

Let us now express $F(q)$ and $G(q)$ in terms of θ_i's ($i = 2, 3$). Firstly for F, observe that the number of integer solutions of $x^2 - xy + y^2 = N$ is equivalent to that of solutions of $(m^2 + 3n^2)/4 = N$. To see this, simply put $m = x + y$ and $n = x - y$ to $(m^2 + 3n^2)/4$. Note that m and n should have the same parity. Therefore,

$$\begin{aligned} F(q) &= \sum_{x,y\in\mathbb{Z}} q^{x^2-xy+y^2} \\ &= \sum_{m,n:\ even} q^{\frac{m^2+3n^2}{4}} + \sum_{m,n:\ odd} q^{\frac{m^2+3n^2}{4}} \\ &= \sum_{k,l\in\mathbb{Z}} q^{k^2+3l^2} + \sum_{k,l\in\mathbb{Z}} q^{((k+1)/2)^2+3((l+1)/2)^2} \\ &= \theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3) \end{aligned}$$

The expression of $G(q)$ is even simpler since

$$G(q) = \sum_{x,y\in\mathbb{Z}} q^{x^2+y^2} = (\theta_3(q))^2.$$

In general, the theta function associated with a binary quadratic form $Q(x, y) = ax^2 + bxy + cz^2$ is defined by

$$\theta_Q(z) = \sum_{(x,y)\in\mathbb{Z}^2} \exp(2\pi i z Q(x, y))$$

where we have used the substitution $q = \exp(2\pi i z)$ mostly in Chapter 5. In the Fourier expansion,

$$\theta_Q(z) = \sum_{N=0}^{\infty} R_Q(N) \exp(2\pi i N z),$$

the numbers $R_Q(N)$ are called the representation numbers of the form Q, and hence a_N and b_N above are given as $R_F(N)$ and $R_G(N)$, respectively.

These theta functions are known to be modular forms of weight 1 on (an appropriately defined subgroup of) the modular group. We believe that this modularity of F and G may help to compare our result with the one given in [ST] or [MR].

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국문초록

이 학위 논문에서는 라그랑지안 플로어 이론을 오비폴드 위로 확장하는데 필요한 오비폴드의 사교기하를 연구한다. 첫 번째로, 다발쌍의 마슬로프 지표를 L-직교 유니타리 접속의 곡률 적분을 이용하여 새롭게 정의하며, 이것이 기존의 정의와 일치함을 보인다. 이 새로운 정의를 이용해서 우리는 내부 특이점을 갖는 오비다발쌍의 마슬로프 지표를 자연스럽게 정의할 수 있다. 두 번째로, 우리는 오비폴드 묻기 개념을 연구한다. 묻기가 들어갈 오비폴드가 부드러운 다양체를 리군 G로 잘라서 만들어지는 몫공간일 경우, 오비폴드 묻기는 G-동등한 몰입과 등변임을 증명한다.

후반부에서는, 정칙 오비폴드 구면들을 분류하여 타원 오비폴드 구의 오비폴드 양자 코호몰로지를 계산한다. 흥미롭게도, 이러한 오비폴드 구면들과 특정한 디오판투스 방정식들의 해 사이의 대응관계를 찾을 수 있다. 이때의 디오판투스 방정식들은 타원 오비폴드 구의 보편덮개 위에 있는 특이점의 역상으로 인해 생기는 격자 구조와 관련이 있다.

주요어휘: 오비폴드, 사교기하, 마슬로프 지표, 오비폴드 묻기, 오비폴드 양자 코호몰로지, 디오판투스 방정식
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