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Global gradient estimates for elliptic and parabolic equations in variable exponent Lebesgue spaces

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Global gradient estimates for elliptic and parabolic equations in variable exponent Lebesgue spaces

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Abstract

We establish global Calderón-Zygmund theory for divergence type elliptic and parabolic equations in variable exponent Lebesgue spaces. We prove that the gradient of the unique weak solution to a given problem with the zero Dirichlet boundary condition is as integrable as the nonhomogeneous term of the problem in variable exponent Lebesgue space by deriving a suitable estimate. In this thesis we consider four equations: the linear elliptic equation, the linear parabolic equation, the nonlinear elliptic equation with variable growth and the nonlinear parabolic equation with variable growth. We also provide reasonable answers to minimal regularity assumptions on the variable exponents, the coefficients and the boundary of the domain to obtain the desired Calderón-Zygmund theory.

Key words: variable exponent Lebesgue space, gradient estimate, Calderón-Zygmund theory, BMO-space, Reifenberg domain
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Chapter 1

Introduction

Calderón-Zygmund theory is to make an investigation on the integrability of the gradient, or the Hessian, of solutions to partial differential equations, and is an important and classical regularity theory. For the Poisson equation

$$\operatorname{div}(Du) = \Delta u = \operatorname{div} F \quad \text{in } \mathbb{R}^n,\ n \geq 2.$$

Calderón and Zygmund [20] showed that

$$\|Du\|_{L^q} \leq c\|F\|_{L^q}, \quad \text{for every } q \in (1, \infty),$$

where $c > 0$ is the independent of u and F, which implies that

$$F \in L^q \quad \Longrightarrow \quad Du \in L^q. \tag{1.0.1}$$

For a bounded domain $\Omega \subset \mathbb{R}^n$ and a matrix function $\mathbf{A}(x)$ being bounded and uniformly elliptic, Byun and Wang [13, 14] extended the above estimate and relation to the following linear equation with the zero Dirichlet boundary condition:

$$\begin{cases} \operatorname{div}(\mathbf{A}(x)Du) = \operatorname{div} F & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.0.2}$$

with optimal regularity assumptions on $\mathbf{A}$ and the boundary of Ω.

On the other hand, Iwaniec [40] established Calderón-Zygmund theory for the nonlinear equations ; p-Laplace equations, see also [22] for p-Laplace systems. Precisely, the authors showed the following: for $1 < p < \infty$ if u is a weak solution to

$$\Delta_p u := \operatorname{div}\left(|Du|^{p-2}Du\right) = \operatorname{div}\left(|F|^{p-2}F\right),$$

then it holds that

$$|F|^p \in L^q \quad \Longrightarrow \quad |Du|^p \in L^q, \quad \text{for every } q \in (1,\infty). \tag{1.0.3}$$

After these pioneer works, there have been many research activities regarding the Calderón-Zygmund theory for nonlinear elliptic equations and systems with a constant p-growth, see [12, 15, 18, 42, 43] and references therein. It is worthy mentioning the paper [18], in which Caffarelli and Peral proved it for quite general type homogeneous equations with p-growth by using so called *maximal function technique*, or *good-λ-inequality*, which has been widely used in the proofs of Calderón-Zygmund type estimates. Also, Byun and Ryu [12] proved global Calderón-Zygmund theory for elliptic equations with the nonlinearity being in BMO(bounded mean oscillation) function space with respect to the space variables and the domain having a very rough boundary which may beyond the Lipschitz category.

For the parabolic p-Laplacian type problems Acerbi and Mingione [2] first proved the Calderón-Zygmund theory such that for $\frac{2n}{n+2} < p < \infty$ if u is a weak solution to

$$u_t - \operatorname{div}\left(a(x,t)|Du|^{p-2}Du\right) = \operatorname{div}\left(|F|^{p-2}F\right) \quad \text{in } \Omega_T,$$

then it holds the relation that

$$|F|^p \in L^q_{\text{loc}}(\Omega_T) \quad \Longrightarrow \quad |Du|^p \in L^q_{\text{loc}}(\Omega_T), \quad \text{for every } q \in (1,\infty), \tag{1.0.4}$$

where the coefficient function $a(x,t)$ is assumed to be discontinuous. We point out that the methods used in [40, 18] do not applicable to the parabolic problems with p-growth, $p \neq 2$, anymore. The main reason is that parabolic problems with p-growth, $p \neq 2$, do not have the following normalization property: if u is the solution to a given equation then λu, $\lambda > 0$, is also a solution to a equation having the same structure to the original equation. Note that the elliptic problems and the parabolic problems with the 2-growth have the previous normalization property. Hence, they created a new technique, so called *maximal function free technique* or *large-M-inequality* which is the only method to prove the Calderón-Zygmund theory for parabolic problems with p-growth, $p \neq 2$, and is well working to the elliptic problems. We also refer to [5, 8, 29, 49] and references therein for parabolic equations and systems with p-growth.

Recently, equations or systems with variable growth have been received many researchers attention. Some materials with inhomogeneities, for example electrorheological fluids, can be modeled with sufficient accuracy in the setting of variable exponent Lebesgue and Sobolev spaces, $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$, where $p(\cdot) : \Omega \to (1, \infty)$ is a variable function. Indeed, theoretical advances in the study for such variable exponent spaces have been made in the field of electrorheological fluids [51, 53], elastic mechanics [60], image restoration [19] and flows in porous media [3, 39]. The model equation of those is the $p(\cdot)$-Laplacian equation;

$$\Delta_{p(\cdot)} u := \operatorname{div} \left(|Du|^{p(x)-2} Du\right) = 0 \quad \text{in } \Omega,$$

where $p(\cdot) : \Omega \to (1, \infty)$ is a continuous function satisfying

$$1 < \inf_{x \in \Omega} p(x) \leq \sup_{x \in \Omega} p(x) < \infty.$$

The $p(\cdot)$-Laplace equation is the Euler-Lagrange equation of the functional

$$\int_\Omega \frac{1}{p(x)} |Dw|^{p(x)}\, dx.$$

For the above type problems, Acerbi and Mingione [1] obtained the following Calderón-Zygmund theory: if u is the weak solution to

$$\Delta_{p(\cdot)} u := \operatorname{div} \left(|Du|^{p(x)-2} Du\right) = \operatorname{div} \left(|F|^{p(x)-2} F\right) \quad \text{in } \Omega,$$

then we have the relation

$$|F|^{p(\cdot)} \in L^q_{\mathrm{loc}}(\Omega) \quad \Longrightarrow \quad |Du|^{p(\cdot)} \in L^q_{\mathrm{loc}}(\Omega), \quad \text{for every } q \in (1, \infty), \tag{1.0.5}$$

under the assumption on the variable exponent $p(\cdot)$ such that

$$\lim_{r \to 0} \omega(r) \log \left(\frac{1}{r}\right) = 0, \quad \text{where } |p(x) - p(y)| \leq \omega(|x - y|). \tag{1.0.6}$$

In the proof of the above result, the authors used the maximal function technique and obtained a suitable comparison estimate between $p(\cdot)$-Laplace equation and p_2-Laplace equation on a sufficiently small ball, where p_2 is the supremum of $p(\cdot)$ in the ball, by assuming (1.0.6). This work was extended by Baroni and Bögelein in [4] to parabolic systems of the form

$$u_t - \operatorname{div}\left(a(x,t)|Du|^{p(x,t)-2}Du\right) = \operatorname{div}\left(|F|^{p(x,t)-2}F\right) \quad \text{in } \Omega_T$$

with

$$\frac{2n}{n+2} < \inf_{(x,t)\in\Omega_T} p(x,t) \le \sup_{(x,t)\in\Omega_T} p(x,t) < \infty,$$

under the assumption that $a(x,t)$ is in VMO(vanishing mean oscillation) with respect to x. We also mention interesting works [30, 33, 36] where similar results were obtained for irregular obstacle problems and for higher order problems.

The main object of the thesis is to prove that the relations (1.0.1) and (1.0.5) still holds true when q changes a variable function $q(\cdot)$.

More precisely, let $q(\cdot) : \mathbb{R}^n \to (1,\infty)$ satisfy that

$$1 < \inf_{x\in\Omega} q(x) \le \sup_{x\in\Omega} q(x) < \infty.$$

Then we first prove that

$$F \in L^{q(\cdot)}(\Omega,\mathbb{R}^n) \quad \Longrightarrow \quad Du \in L^{q(\cdot)}(\Omega,\mathbb{R}^n)$$

for the linear elliptic equation (1.0.2), under minimal regularity assumptions on $q(\cdot)$, $\mathbf{A}$ and the boundary of Ω. This is the content of Chapter 3, in which we also treat the linear parabolic equations.(in fact, we will employ $p(\cdot)$, instead of $q(\cdot)$, as the variable exponent $p(\cdot)$ in Chapter 3.)

On the other hand, in chapter 4 we consider elliptic $p(\cdot)$-Laplacian type equations, and show that

$$|F|^{p(\cdot)} \in L^{q(\cdot)}(\Omega) \quad \Longrightarrow \quad |Du|^{p(\cdot)} \in L^{q(\cdot)}(\Omega),$$

under the suitable assumptions on $p(\cdot)$, $q(\cdot)$, the nonlinearity for the space variable and the boundary of Ω. In addition, in chapter 5 we consider parabolic $p(\cdot)$-Laplacian type equations, show that

$$|F|^{p(\cdot)} \in L^{q(\cdot)}(\Omega_T) \quad \Longrightarrow \quad |Du|^{p(\cdot)} \in L^{q(\cdot)}(\Omega_T),$$

under the suitable assumptions on $p(\cdot)$, $q(\cdot)$, the nonlinearity for the space variable and the boundary of Ω.

To the best of our knowledge, there is only one result related to the gradient estimate in variable exponent Lebesgue spaces. In [26] Diening, Lengeler and Růžička considered Poisson equations to obtain $L^{q(\cdot)}$-estimate, see also the monograph [25]. The main approach in [26] is totally based on the harmonic analysis frame, the boundedness of the associated kernel operators in the variable exponent Lebesgue space $L^{q(\cdot)}$ with the assumption that $q(\cdot)$ is *log-Hölder continuous*, see Definition 2.2.5. We note that the log-Hölder continuity of $q(\cdot)$ is an unavoidable condition studying variable exponent spaces. However this method dose not applicable to nonlinear problems and even linear problems with quit general circumstance, e.g. [13].

We point out that the methods used in the earlier works [1, 4, 10] to obtained L^q-estimates do not directly imply $L^{q(\cdot)}$-estimates. The main difficulty is that the integral identity formula

$$\int_U |Du|^{p(x)q}\,dx = q\int_0^\infty \lambda^{q-1}\left|\left\{x\in U : |Du|^{p(x)} > \lambda\right\}\right|\,d\lambda \tag{1.0.7}$$

can not be used when the constant q is replaced by a variable function $q(\cdot)$. Note that the maximal function technique and the maximal function free technique generally start with (1.0.7). To overcome this, we instead use its variant like

$$\int_B |Du|^{p(x)q(x)}\,dx = q_-\int_0^\infty \lambda^{q_--1}\left|\left\{x\in B : |Du|^{p(x)\frac{q(x)}{q_-}} > \lambda\right\}\right|\,d\lambda,$$

where $q_- = \inf_{x\in B} q(x)$ and B is a small ball. We then use the log-Hölder continuity of $p(\cdot)$ and $q(\cdot)$, a higher integrability result of $|Du|^{p(x)}$ and comparison estimates, to control the super-level sets of $|Du|^{p(x)\frac{q(x)}{q_-}}$, instead of those of $|Du|^{p(x)}$.

The rest of the thesis is organized as follows. In the next chapter, we introduce basic ingredients; notations, definitions and well known facts. In Chapter 3-5, as mentioned earlier, we prove the gradient estimate for the weak solutions to linear elliptic/parabolic equations(Chapter 3), nonlinear elliptic equations with variable $p(\cdot)$-growth(Chapter 4) and nonlinear parabolic equations with variable $p(\cdot)$-growth(Chapter 5).

Chapter 2

Preliminaries

2.1 Notations.

We denote basic notations used in throughout the thesis. Let $n \geq 2$ be a natural number, and Ω be a bounded domain in $\mathbb{R}^n$.

We first introduce the geometric notations. Let $y' = (y_2, \dots, y_n)$ be a point in $\mathbb{R}^{n-1}$, $y = (y_1, y') = (y_1, \dots, y_n)$ be a point in $\mathbb{R}^n$, and $r > 0$ be a positive number. We denote the open ball in $\mathbb{R}^n$(resp. $\mathbb{R}^{n-1}$) with the center y(resp. y') and the radius r by

$$B_r(y) := \{x \in \mathbb{R}^n : |x - y| < r\}(\text{resp. } B'_r(y') := \{x' \in \mathbb{R}^{n-1} : |x' - y'| < r\}),$$

and the cylinder in $\mathbb{R}^n$ with the center y and the radius r by

$$C_r(y) = \{x \in \mathbb{R}^n : |x_1 - y_1| < r, |x' - y'| < r\} = (y_1 - r, y_1 + r) \times B'_r(y').$$

For simplicity we shall write $B_r = B_r(0)$, $C_r = C_r(0)$, $B_r^+ := B_r \cap \{x_n > 0\}$, $C_r^+ := C_r \cap \{x_1 > 0\}$. Then, in Chapter 3 we define the parabolic cylinder with the center $\zeta = (y, s) \in \mathbb{R}^{n+1}$ and the radius $r > 0$ by

$$Q_r(\zeta) := C_r(y) \times (s - r^2, s + r^2),$$

and write

$$\Omega_r(y) := \Omega \cap C_r(y), \quad \Omega_r := \Omega \cap C_r, \quad \partial_w \Omega_r := \partial\Omega \cap C_r,$$

$$T_r := B'_r, \quad K_r(\zeta) := \Omega_r(y) \times (s - r^2, s + r^2), \quad Q_r^+ := C_r^+ \times (-r^2, r^2).$$

On the other hand, in Chapter 4 and Chapter 5 we define the parabolic cylinder with the center $w = (y, s) \in \mathbb{R}^{n+1}$ and the radius $r > 0$ by

$$Q_r(w) := B_r(y) \times (s - r^2, s + r^2),$$

and write

$$\Omega_r(y) := \Omega \cap B_r(y), \quad \Omega_r := \Omega \cap B_r, \quad \partial_w \Omega_r := \partial\Omega \cap B_r,$$

$$T_r := B_r \cap \{x_n = 0\}, \ K_r(w) := \Omega_r(y) \times (s - r^2, s + r^2), \ Q_r^+ := B_r^+ \times (-r^2, r^2).$$

In both cases, we simply write

$$K_r := K_r(0), \quad \partial_w K_r := \partial_w \Omega_r \times (-r^2, r^2).$$

In addition, we define $\Omega_T := \Omega \times (0, T]$ and the parabolic boundary of Ω_T by the bottom and side of Ω_T such that

$$\partial_p \Omega_T := \partial\Omega \times (0, T) \cup \Omega \times \{0\}.$$

For $Q_r(\zeta)$ and $K_r(\zeta)$ we also define the parabolic boundaries $\partial_p Q_r(\zeta)$ and $\partial_p K_r(\zeta)$ in the same way. Note that $\partial_p \Omega_T$ is called the *parabolic boundary* of Ω_T. Let $y, \tilde{y} \in \mathbb{R}^n$, $\tau, \tilde{\tau} \in \mathbb{R}$ and $w = (y, \tau), \tilde{w} = (\tilde{y}, \tilde{\tau}) \in \mathbb{R}^{n+1}$. We define the *parabolic distance* between w and $\tilde{w}$ by

$$d_{\mathcal{P}}(w, \tilde{w}) := \max\{|y - \tilde{y}|, \sqrt{|\tau - \tilde{\tau}|}\},$$

where $|\cdot|$ is the standard Euclidean norm, in $\mathbb{R}^N$, $N = 1, 2, \ldots$.

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^m)$, $N, m \in \mathbb{N}$, $\overline{f}_U = (f)_U$ is denoted by the integral average of f on a bounded subet U in $\mathbb{R}^N$, that is,

$$\overline{f}_U = (f)_U := \fint_U f \, dX = \frac{1}{|U|} \int_U f \, dX.$$

Finally $e \in \mathbb{R}$ is the Euler constant.

2.2 Variable exponent spaces.

Let $p(\cdot) : \mathbb{R}^N \to (1, \infty)$, $N \in \mathbb{N}$, be a positive function satisfying

$$1 < \inf_{X \in \mathbb{R}^N} p(X) \leq \sup_{X \in \mathbb{R}^N} p(X) < \infty, \tag{2.2.1}$$

for a bounded subset U in $\mathbb{R}^N$ the *variable exponent Lebesgue space* $L^{p(\cdot)}(U, \mathbb{R}^m)$, $m \in \mathbb{N}$, consists of all measurable functions $f : U \to \mathbb{R}^m$ satisfying

$$\int_U |f|^{p(X)}\, dX = \int_U |f(X)|^{p(X)}\, dX < \infty$$

with the following Luxemberg norm

$$\|f\|_{L^{p(\cdot)}(U,\mathbb{R}^m)} := \inf \left\{ a > 0 : \int_U \left| \frac{f(X)}{a} \right|^{p(X)} dX \leq 1 \right\},$$

and the *variable exponent Sobolev spaces* is

$$W^{1,p(\cdot)}(U, \mathbb{R}^m) := \left\{ f \in L^{p(\cdot)}(U, \mathbb{R}^m) : Df \in L^{p(\cdot)}(U, \mathbb{R}^{Nm}) \right\}$$

equipped with the norm

$$\|f\|_{W^{1,p(\cdot)}(U,\mathbb{R}^m)} := \|f\|_{L^{p(\cdot)}(U,\mathbb{R}^m)} + \|Df\|_{L^{p(\cdot)}(U,\mathbb{R}^{Nm})}.$$

We also denote $W_0^{1,p(\cdot)}(U, \mathbb{R}^m)$ by the closer of $C_0^\infty(U, \mathbb{R}^m)$ in $W^{1,p(\cdot)}(U, \mathbb{R}^m)$. Then, they are separable reflexive Banach spaces. For $m = 1$, we simply write $L^{p(\cdot)}(U)$, $W^{1,p(\cdot)}(U)$ and $W_0^{1,p(\cdot)}(\Omega)$. We also denote the Hölder conjugate exponent of $p(\cdot)$ by

$$p'(X) := \frac{p(X) - 1}{p(X)}.$$

Note that We we have the following norm-modular property:

$$\|f\|_{L^{p(\cdot)}(U)} \leq 1 \iff \int_U |f|^{p(x)}\, dx \leq 1. \tag{2.2.2}$$

We then give $p(\cdot)$ a crucial condition for $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ to have some important properties.

Definition 2.2.1. We say $p(\cdot)$ is *log-Hölder continuous* in U if

$$|p(X)-p(Y)| \leq \frac{L}{-\log|X-Y|} \quad \text{for all } X, Y \in \Omega \text{ with } |X-Y| \leq \frac{1}{2}, \tag{2.2.3}$$

for some constant $L > 0$.

We remark that, $p(\cdot)$ is log-Hölder continuous in U if and only if $p(\cdot)$ is modulus continuous, i.e., there is a nondecreasing continuous function $\omega : [0,\infty) \to [0,\infty)$ satisfying $\omega(0) = 0$ and

$$|p(X) - p(Y)| \leq \omega(|X-Y|), \tag{2.2.4}$$

for $X, Y \in U$, furthermore,

$$\omega(r) \log\left(\frac{1}{r}\right) \leq \tilde{L}, \quad \text{for all } r \leq \frac{1}{2}, \tag{2.2.5}$$

for some constant $\tilde{L} > 0$. If $p(\cdot)$ is log-Hölder continuous, then the Hardy-Littlewood maximal operator and the Sobolev imbedding on variable exponent spaces can be well understood and Poincaré's inequality also holds in $W_0^{1,p(\cdot)}(U)$, that is,

$$\|Du\|_{L^{p(\cdot)}(U)} \leq c(n,U)\|u\|_{L^{p(\cdot)}(U)},$$

for $u \in W_0^{1,p(\cdot)}(U)$. For properties about variable exponent spaces with log-Hölder continuous exponents, we refer to [25].

For a further discussion, we refer to [25, 24, 28, 38, 44, 55] and the references therein.

2.3 Technical background.

We start with a standard iteration lemma.

Lemma 2.3.1. *(Lemma 4.3 in [37]) Let ϕ be a bounded nonnegative function on $[r_1, r_2]$. Suppose that for any s_1, s_2 with $0 < r_1 \leq s_1 < s_2 \leq r_2$,*

$$\phi(s_1) \leq \kappa\phi(s_2) + \frac{P_1}{(s_2-s_1)^\beta} + P_2,$$

where $\beta, P_1, P_2 \geq 0$ and $\kappa \in (0,1)$. Then there holds

$$\phi(r_1) \leq \ c \left[\frac{P_1}{(r_2 - r_1)^{\theta_2}} + P_2 \right],$$

for some $c = c(\kappa, \beta) > 0$.

Lemma 2.3.2. *Let U be a bounded domain in $\mathbb{R}^{n+1}$. For $f \in L^q(U)$ with $q > 0$, we have*

$$\int_U |f|^q \, dz = \int_0^\infty q\lambda^{q-1} |\{z \in U : |f(z)| > \lambda\}| d\lambda. \tag{2.3.1}$$

For $f \in L^{q_2}(U)$ with $q_2 > q_1 > 0$, we have

$$\int_U |f(z)|^{q_2} \, dz = (q_2 - q_1) \int_0^\infty \lambda^{q_2 - q_1 - 1} \int_{\{z \in U : |f(z)| > \lambda\}} |f(z)|^{q_1} \, dz d\lambda. \tag{2.3.2}$$

We introduce the *Hardy-Littlewood maximal operators*. For $f \in L^1_{loc}(\mathbb{R}^n)$, we define

$$\mathcal{M}f(y) = \mathcal{M}(f)(y) := \sup_{r>0} \fint_{B_r(y)} |f(x)| \, dx$$

and

$$\mathcal{M}_q f(y) = \mathcal{M}_q(f)(y) := (\mathcal{M}(|f|^q)(y))^{\frac{1}{q}} = \sup_{r>0} \left(\fint_{B_r(y)} |f(x)|^q \, dx \right)^{\frac{1}{q}}, \quad q > 1.$$

Then we have the following properties.

Proposition 2.3.3. *(see [57])*

(1) Weak type (1,1) estimate : there exists a constant $c = c(n) > 0$ such that for any $\lambda > 0$

$$|\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}| \leq \frac{c}{\lambda} \int_{\mathbb{R}^n} |f| \, dx.$$

(2) For $1 < p < \infty$, there exists a constant $c = c(n,p) > 0$ such that

$$\int_{\mathbb{R}^n} |f|^p \, dx \leq \int_{\mathbb{R}^n} |\mathcal{M}f|^p \, dx \leq c \int_{\mathbb{R}^n} |f|^p \, dx.$$

(3) For $1 < q < p < \infty$, there exists a constant $c = c(n,p,q) > 0$ such that

$$\int_{\mathbb{R}^n} |f|^p\,dx \le \int_{\mathbb{R}^n} |\mathcal{M}_q f|^p\,dx \le c\int_{\mathbb{R}^n} |f|^p\,dx.$$

The following lemma is a certain of *Vitali covering lemma*, whose proof can be found in a similar way as in the proof of Theorem 3 in [59].

Lemma 2.3.4. *Suppose Ω is (δ, R)-Reifenberg flat. Consider $\Omega_{R_1} = \Omega_{R_1}(y)$, where $R_1 \le R$ and $y \in \Omega$. For $\epsilon \in (0,1)$, if the measurable subsets $C \subset D \subset \Omega_{R_1}$ satisfy that*

(i) $|C| \le \epsilon \frac{1}{(63)^n} |\Omega_{R_1}|$,

(ii) for any $\tilde{r} \le \frac{R_1}{63}$ and $\tilde{y} \in C$, if $|C \cap B_{\tilde{r}}(\tilde{y})| \ge \epsilon |B_{\tilde{r}}(\tilde{y})|$ then $\Omega_{\tilde{r}}(\tilde{y}) \subset D$,

then

$$|C| \le 5^n \left(\frac{2}{1-\delta}\right)^n \epsilon |D| \le \left(\frac{80}{7}\right)^n \epsilon |D|.$$

We will also use the following equivalent relation and estimate.

Lemma 2.3.5. *(see [17]) Let f be the measurable function in a bounded domain $U \subset \mathbb{R}^n$. Then, for $\lambda > 0$ and $q, A > 1$ we have*

$$f \in L^q(U) \iff S := \sum_{k \ge 1} A^{qk} \left|\left\{x \in U : |f(x)| > A^k \lambda\right\}\right| < \infty$$

with the estimate

$$c^{-1}\lambda^q S \le \int_U |f|^q\,dx \le c\lambda^q(|U| + S),$$

for some $c = c(A,q) > 0$.

We recall the following elementary inequality:

$$t^\beta \log t \le \max\left\{\frac{1}{e\beta}, 2^\beta \log 2\right\}, \quad \forall t \in (0,2]. \tag{2.3.3}$$

The next lemma is an estimate in $L\log L$-space which can be found in [1] and reference therein.

Lemma 2.3.6. *Let $\beta > 0$ and $p > 1$. For $f \in L^\sigma(U)$ we have*

$$\fint_U |f| \log^\beta \left(e + \frac{|f|}{\bar{f}_U} \right) dX \leq c \left(\fint_U |f|^\sigma \, dX \right)^{\frac{1}{\sigma}},$$

for some $c = c(N, \sigma, \beta) > 0$. Note that the constant $c(N, \sigma, \beta)$ is continuous with respect to β, where $\log^\beta t := (\log t)^\beta$ for $t > 1$.

Chapter 3

Gradient estimates for linear equations in variable exponent spaces

3.1 $W^{1,p(\cdot)}$-regularity for elliptic equations with measurable coefficients in nonsmooth domains.

3.1.1 Main result.

We recall the following linear elliptic equation in divergence form with Dirichlet boundary condition:

$$\begin{cases} \operatorname{div}(\mathbf{A}(x)Du) & = & \operatorname{div} F & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega, \end{cases} \tag{3.1.1}$$

where $\mathbf{A}(x)$ is an $n \times n$ matrix of the coefficients which is uniformly elliptic and bounded, see (3.1.4), and F is in $L^2(\Omega, \mathbb{R}^n)$. The aim of this chapter is to establish the well-posedness of the problem (3.1.1) in the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ under optimal conditions on $\mathbf{A}$ and $\partial\Omega$ by proving that

$$F \in L^{p(\cdot)}(\Omega; \mathbb{R}^n) \quad \Longrightarrow \quad Du \in L^{p(\cdot)}(\Omega; \mathbb{R}^n), \tag{3.1.2}$$

for every log-Hölder continuous function $p(\cdot) : \mathbb{R}^n \to (1, \infty)$ satisfying

$$2 < \gamma_1 \leq p(x) \leq \gamma_2 < \infty, \quad \forall x \in \mathbb{R}^n \text{ and } \exists \gamma_1, \gamma_2. \tag{3.1.3}$$

Since $p(\cdot)$ is log-Hölder continuous, there is a modulus continuity of $p(\cdot)$, $\omega : [0, \infty) \to [0, \infty)$, satisfying (2.2.5) with $\tilde{L}$ replaced by $m > 0$. The matrix $\mathbf{A}$ of the coefficients is supposed to be uniformly bounded and uniformly elliptic. That is, there exist $0 < \nu \leq \Lambda < +\infty$ such that

$$\mathbf{A} = [a_{ij}(x)] = [a_{ji}(x)] \text{ and } \nu|\xi|^2 \leq \mathbf{A}(x)\xi\xi \leq \Lambda|\xi|^2, \quad \forall x, \xi \in \mathbb{R}^n. \tag{3.1.4}$$

We say $u \in H_0^1(\Omega)$ is a weak solution of the Dirichlet problem (3.1.1) if it satisfies that

$$\int_\Omega \mathbf{A}(x) Du D\varphi \; dx = \int_\Omega F D\varphi \; dx, \quad \forall \varphi \in H_0^1(\Omega).$$

Then the Dirichlet problem (3.1.1) has a unique weak solution with the L^2-estimate

$$\int_\Omega |Du|^2 \; dx \leq c \int_\Omega |F|^2 \; dx, \tag{3.1.5}$$

where c is a constant depending only n, ν and Λ, see [34].

We now state the main assumptions on $\mathbf{A}(x)$ and Ω.

Definition 3.1.1. We say that $(\mathbf{A}, \Omega)$ is (δ, R_0)-*vanishing of codimension 1* if the following conditions hold. For each $y \in \overline{\Omega}$ and for each $r \in \left(0, \frac{1}{104\sqrt{2}} R_0\right]$, if $B_{(20\sqrt{2})r}(y) \subset \Omega$, then there exists a new coordinate system $\{z_1, \cdots, z_n\}$ in which the origin is y_i and

$$⨍_{C_{(20\sqrt{2})r}} \left| \mathbf{A}(z) - \mathbf{A}_{B'_{(20\sqrt{2})r}}(z_n) \right| \; dz \leq \delta.$$

On the other hand, if $dist(y, \partial\Omega) = |y - y_0| \leq (20\sqrt{2})r$ for some $y_0 \in \partial\Omega$, then there exists a new coordinate system $\{z_1, \cdots, z_n\}$ in which the origin lies somewhere in $B_{(104\sqrt{2})r\delta}(y_0)$ such that

$$C^+_{(104\sqrt{2})r} \subset \Omega_{(104\sqrt{2})r} \subset C_{(104\sqrt{2})r} \cap \left\{ z_n > -(208\sqrt{2})r\delta \right\}$$

and

$$\fint_{C^+_{(104\sqrt{2})r}} \left| \mathbf{A}(z) - \mathbf{A}_{B'_{(104\sqrt{2})r}}(z_n) \right| \, dz \leq \delta.$$

There are a few comments on the above definition.

1. Changed coordinate systems can be obtained by rotation and translation from the original coordinate system. Since the equation (3.1.1) is invariant under such rotation and translation, without loss of generality, in new coordinate systems, we still use the same notations used in original coordinate system, for example, x, $\mathbf{A}$ an F.

2. For sufficiently small regions $B_{\sqrt{2}r}(y)$, if $B_{\sqrt{2}r}(y)$ lies in Ω, then there exists one direction depending on y and r such that $\mathbf{A}$ is merely measurable in this direction and has a small BMO condition in the other directions. On the other hand, if $B_{\sqrt{2}r}(y)$ intersects the boundary of Ω, then there exists one direction which is normal to two parallel hyperplanes, one lying locally inside Ω and the other locally lying outside Ω near y_1 with the distance between $298\sqrt{2}\delta r$, such that $\mathbf{A}$ is merely measurable in this direction and has a small BMO condition in the other directions.

3. Only for a technical reason, we record the numbers 20 104 which can be easily changeable via a scaling. By the same reason, one can take R can be any positive number while δ is invariant under such a scaling.

One of main features of this domain is that it has the measure density condition like

$$\sup_{0<r\leq\frac{1}{104\sqrt{2}}R_0} \sup_{y\in\Omega} \frac{|B_r(y)|}{|\Omega\cap B_r(y)|} \leq \left(\frac{2}{1-\delta}\right)^n \leq \left(\frac{16}{7}\right)^n,$$

from which we discover that

$$\sup_{0<r\leq\frac{1}{104\sqrt{2}}R_0} \sup_{y\in\Omega} \frac{|C_r(y)|}{|\Omega\cap C_r(y)|} \leq \sup_{0<r\leq\frac{1}{104\sqrt{2}}R_0} \sup_{y\in\Omega} \frac{|B_{\sqrt{2}r}(y)|}{|\Omega\cap B_r(y)|} \leq \left(\frac{16\sqrt{2}}{7}\right)^n. \tag{3.1.6}$$

We next present some necessary auxiliary results which will be employed later on. We first provide a higher integrability result for (3.1.1). For the interior case, the proof relies on *Caccioppoli inequality*, *Poincaré inequality* and *Gehring Lemma*, see Proposition 1.1, p.122, in [35]. For the boundary case, additionally, we use the zero extension of weak solutions to the complement of the domain and the measure density condition (3.1.6).

Lemma 3.1.2. *(i) Interior case: Let* $u \in H^1(C_4)$ *be a weak solution of*

$$div\,(\mathbf{A}(x)Du) = div\ F \ \ in\ C_4 \subset \Omega.$$

Suppose $F \in L^{\gamma}(C_4)$ *for some* $\gamma > 2$*, then there exists a small positive constant* $\sigma_1 = \sigma_1(n, \nu, \Lambda, \gamma)$ *such that for all* $\sigma \leq \sigma_1$*,*

$$\fint_{C_1} |Du|^{2(1+\sigma)} dx \leq c \left[\left(\fint_{C_2} |Du|^2 dx \right)^{1+\sigma} + \fint_{C_2} |F|^{2(1+\sigma)} dx \right],$$

where $c = c(n, \nu, \Lambda, \gamma)$ *is a positive constant.*
(ii) Boundary case: Let Ω *satisfy the measure density condition (3.1.6), changed* $\frac{R_0}{100\sqrt{2}}$ *to* 4*, and* $u \in H^1(\Omega_4)$ *be a weak solution of*

$$\begin{cases} div\,(\mathbf{A}(x)Du) &= div\, F \quad in \quad \Omega_4, \\ u &= 0 \quad\quad on \quad \partial_w \Omega_4, \end{cases}$$

with

$$C_4^+ \ \subset\ \Omega_4 \ \subset\ C_4 \cap \{x^n > -8\delta\}.$$

Suppose $F \in L^{\gamma}(\Omega_4)$ *for some* $\gamma > 2$*, then there exists a small constant* $\sigma_2 = \sigma_2(n, \nu, \Lambda, \gamma) > 0$ *such that for all* $\sigma \leq \sigma_2$*,*

$$\fint_{\Omega_1} |Du|^{2(1+\sigma)} dx \leq c \left[\left(\fint_{\Omega_2} |Du|^2 dx \right)^{1+\sigma} + \fint_{\Omega_2} |F|^{2(1+\sigma)} dx \right],$$

for some positive constant $c = c(n, \nu, \Lambda, \gamma)$*. For the sake of simplicity, we write*

$$\sigma_0 = \min\{\sigma_1, \sigma_2\}.$$

The next Lemma shows *Lipschitz regularity* for solutions of linear elliptic equations with the coefficients which depends on only one variable. We refer to Lemma 5.1 and 5.6 in [14] for its proof.

Lemma 3.1.3. *Let* $\mathbf{A} = \mathbf{A}(x_n) : \mathbb{R} \to \mathbb{M}_n(\mathbb{R})$ *be a measurable matrix with (3.1.4). Then we have the following Lipschitz regularity.*
(i) Interior Case: Let $v \in H^1(C_2)$ *be a solution of*

$$div\,(\mathbf{A}(x_n)Dv) = 0 \text{ in } C_2.$$

Then $Dv \in L^\infty(C_1)$ *with the estimate*

$$\|Dv\|^2_{L^\infty(C_1)} \leq c \fint_{C_2} |Dv|^2\,dx,$$

where c *is a positive constant depending only* n, ν, Λ.
(ii) Boundary Case : Let $v \in H^1(C_2^+)$ *be a solution of*

$$\begin{cases} div\,(\mathbf{A}(x_n)Dv) &= 0 \quad in \quad C_2^+ \\ v &= 0 \quad on \quad T_2. \end{cases}$$

Then $Dv \in L^\infty(C_1^+)$ *with the estimate*

$$\|Dv\|^2_{L^\infty(C_1^+)} \leq c \fint_{C_2^+} |Dv|^2\,dx,$$

for some positive constant $c = c(n, \nu, \Lambda)$.

Now we state the result of $W^{1,p(\cdot)}$-regularity for (3.1.1).

Theorem 3.1.4. *Let* $R_0 > 0$. *Then there exists* $\delta = \delta(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot))$ *such that if* $(\mathbf{A}, \Omega)$ *is* (δ, R_0)*-vanishing codimension 1 and* $u \in H_0^1(\Omega)$ *is the weak solution of (3.1.1), then there holds (3.1.2) and we have the estimate*

$$\|Du\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^n)} \leq c\|F\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^n)}, \tag{3.1.7}$$

for some constant $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), R_0, |\Omega|)$.

3.1.2 Proof of Theorem 3.1.4.

We start this section under the a priori assumption that the unique weak solution $u \in H_0^1(\Omega)$ of (3.1.1) satisfies

$$\int_\Omega |Du|^{p(x)} dx < \infty. \tag{3.1.8}$$

This prescribed assumption can be removed by an approximation argument in the last section. We further assume that $(\mathbf{A}, \Omega)$ is (δ, R_0)-*vanishing codimension 1.* where R_0 is arbitrary given while δ is to be determined, see (3.1.42). But, without loss of generality, we may assume that $R_0 \leq 1$, because Definition 3.1.1 is stronger if R_0 is lager.

Our strategy is to obtain

$$\|Du\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^n)} \leq c(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), R_0, |\Omega|) \tag{3.1.9}$$

with the uniform assumption

$$\|F\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^n)} \leq 1. \tag{3.1.10}$$

In fact, for the solution u and the nonhomogeneous term F, consider

$$\tilde{u} = \frac{u}{\|F\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^n)}}, \ \tilde{F} = \frac{F}{\|F\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^n)}}.$$

Then, by applying (3.1.9) and (3.1.10) to $\tilde{u}$ and $\tilde{F}$, we get the required estimate (3.1.7).

Hereafter, for the sake of simplicity, we denote by c to mean a universal constant which can be computed only in terms of known data $n, \nu, \Lambda, \gamma_1, \gamma_2$ and $\omega(\cdot)$ (independent of $|\Omega|$ and R_0), and so its exact value varies depending on the lines.

We note from the norm-modulus unit ball property (2.2.2) the condition (3.1.10) is equivalent to

$$\int_\Omega |F|^{p(x)} dx \leq 1, \tag{3.1.11}$$

This and the standard L^2-estimate (3.1.5) yield that

$$\int_\Omega |F|^2 dx \leq \int_\Omega \left(|F|^{p(x)} + 1\right) dx \leq 1 + |\Omega| \ \text{ and } \ \int_\Omega |Du|^2 dx \leq c(1 + |\Omega|). \tag{3.1.12}$$

With the abbreviation

$$c^* = c^*(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), |\Omega|) \geq 104\sqrt{2}, \tag{3.1.13}$$

we let $R \in \left(0, \frac{R_0}{c^*}\right] \subset \left(0, \frac{R_0}{104\sqrt{2}}\right]$ and $x_0 \in \overline{\Omega}$, and localize our interest in the region $\Omega_{2R}(x_0)$. The choice of c^* will be clarified later in the context. We then fix any s_1 and s_2 with $1 \leq s_1 < s_2 \leq 2$. Under these assumptions and settings, we write

$$2 < \gamma_1 \leq p^- = \inf_{x \in \Omega_{2R}(x_0)} p(x) \leq p^+ = \sup_{x \in \Omega_{2R}(x_0)} p(x) \leq \gamma_2 < +\infty$$

and

$$\lambda_0 = \fint_{\Omega_{2R}(x_0)} \left(|Du|^{\frac{2p(x)}{p^-}} + \frac{1}{\delta} |F|^{\frac{2p(x)}{p^-}} \right) dx + \frac{1}{\delta} > 1. \tag{3.1.14}$$

Note that from $2 < \frac{2p(x)}{p^-} \leq p(x)$, (3.1.8) and $F \in L^{p(\cdot)}(\Omega; \mathbb{R}^n)$ the above integral is well defined. We next define an upper-level set

$$E(\lambda) = \left\{ x \in \Omega_{s_1 R}(x_0) : |Du(x)|^{\frac{2p(x)}{p^-}} > \lambda \right\}, \tag{3.1.15}$$

for λ large enough to satisfy

$$\lambda > \left(\frac{16\sqrt{2}}{7} \right)^n \left(\frac{400\sqrt{2}}{s_2 - s_1} \right)^n \lambda_0. \tag{3.1.16}$$

We observe from (3.1.15) that

$$\Omega_r(y) \subset \Omega_{2R}(x_0), \forall y \in E(\lambda) \text{ and } 0 < \forall r \leq (s_2 - s_1)R. \tag{3.1.17}$$

Then for any fixed $y \in E(\lambda)$, consider a continuous function $\Phi_y(r)$, defined by

$$\Phi_y(r) = \fint_{\Omega_r(y)} \left(|Du|^{\frac{2p(x)}{p^-}} + \frac{1}{\delta} |F|^{\frac{2p(x)}{p^-}} \right) dx, \ 0 < r \leq (s_2 - s_1)R. \tag{3.1.18}$$

When the choice of y is clear from the context, we frequently omit it and write $\Phi(r)$ instead of $\Phi_y(r)$. In light of (3.1.17) and (3.1.18), we have

$$\begin{aligned}
\Phi(r) &= \fint_{\Omega_r(y)} \left(|Du|^{\frac{2p(x)}{p^-}} + \frac{1}{\delta}|F|^{\frac{2p(x)}{p^-}} \right) dx \\
&\leq \frac{|\Omega_{2R}(x_0)|}{|\Omega_r(y)|} \fint_{\Omega_{2R}(x_0)} \left(|Du|^{\frac{2p(x)}{p^-}} + \frac{1}{\delta}|F|^{\frac{2p(x)}{p^-}} \right) dx \\
&\leq \frac{|C_{2R}(y)|}{|\Omega_r(y)|} \fint_{\Omega_{2R}(x_0)} \left(|Du|^{\frac{2p(x)}{p^-}} + \frac{1}{\delta}|F|^{\frac{2p(x)}{p^-}} \right) dx \\
&\leq \left(\frac{400\sqrt{2}}{s_2 - s_1} \right)^n \frac{|C_r(y)|}{|\Omega_r(y)|} \fint_{\Omega_{2R}(x_0)} \left(|Du|^{\frac{2p(x)}{p^-}} + \frac{1}{\delta}|F|^{\frac{2p(x)}{p^-}} \right) dx,
\end{aligned}$$

provided that

$$\frac{1}{200\sqrt{2}}(s_2 - s_1)R \leq r \leq (s_2 - s_1)R.$$

We recall the measure density condition (3.1.6), (3.1.14) and the selection (3.1.16), to find that

$$\Phi(r) < \lambda, \text{ for all } r \in \left[\frac{s_2 - s_1}{200\sqrt{2}} R, (s_2 - s_1)R \right].$$

On the other hand, the Lebesgue differentiation theorem implies that for almost every $y \in E(\lambda)$, $\lim_{r \to 0} \Phi(r) > \lambda$. Consequently, we conclude that for almost every $y \in E(\lambda)$, there exists $r_y = r(y) \in \left(0, \frac{s_2 - s_1}{200\sqrt{2}} R\right)$ such that

$$\Phi_y(r_y) = \lambda \text{ and } \Phi_y(r) < \lambda \text{ for all } r \in (r_y, (s_2 - s_1)R].$$

We thus infer the following lemma from the *Vitali covering lemma*.

Lemma 3.1.5. *Assume (3.1.16). Then there exists a disjoint family* $\{\Omega_{r_i}(y_i)\}_{i=1}^{\infty}$ *with* $y_i \in E(\lambda)$ *and* $r_i \in \left(0, \frac{s_2 - s_1}{200\sqrt{2}} R\right)$ *such that*

$$\Phi_{y_i}(r_i) = \lambda, \ \Phi_{y_i}(r) < \lambda \text{ for every } r \in (r_i, (s_2 - s_1)R] \tag{3.1.19}$$

and

$$E(\lambda) \subset \bigcup_{i=1}^{\infty} \Omega_{5r_i}(y_i).$$

As a consequence of Lemma 3.1.5, we estimate

$$\begin{aligned}|\Omega_{r_i}(y_i)| &= \frac{1}{\lambda}\left(\int_{\Omega_{r_i}(y_i)} |Du|^{\frac{2p(x)}{p^-}}\, dx + \frac{1}{\delta}\int_{\Omega_{r_i}(y_i)} |F|^{\frac{2p(x)}{p^-}} dx\right)\\ &\leq \frac{1}{\lambda}\Bigg(\int_{\Omega_{r_i}(y_i)\cap\left\{|Du|^{\frac{2p(x)}{p^-}}>\frac{\lambda}{4}\right\}} |Du|^{\frac{2p(x)}{p^-}} dx + \frac{\lambda}{4}|\Omega_{r_i}(y_i)|\\ &\qquad + \frac{1}{\delta}\int_{\Omega_{r_i}(y_i)\cap\left\{|F|^{\frac{2p(x)}{p^-}}>\frac{\lambda\delta}{4}\right\}} |F|^{\frac{2p(x)}{p^-}} dx + \frac{\lambda}{4}|\Omega_{r_i}(y_i)|\Bigg).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}|\Omega_{r_i}(y_i)| &\leq \frac{2}{\lambda}\Bigg(\int_{\Omega_{r_i}(y_i)\cap\left\{|Du|^{\frac{2p(x)}{p^-}}>\frac{\lambda}{4}\right\}} |Du|^{\frac{2p(x)}{p^-}}\, dx\\ &\qquad + \frac{1}{\delta}\int_{\Omega_{r_i}(y_i)\cap\left\{|F|^{\frac{2p(x)}{p^-}}>\frac{\lambda\delta}{4}\right\}} |F|^{\frac{2p(x)}{p^-}} dx\Bigg). \qquad (3.1.20)\end{aligned}$$

Proceeding from Lemma 3.1.5 and (3.1.20), we fix the point y_i and the scale r_i. Now there are two possible cases. One is the interior case that $B_{(20\sqrt{2})r_i}(y_i) \subset \Omega$. The other is the boundary case that $B_{(20\sqrt{2})r_i}(y_i) \not\subset \Omega$.

We first look at the interior case. Observe that $(20\sqrt{2})r_i < \frac{\sqrt{2}}{5}R < R \leq \frac{R_0}{104\sqrt{2}}$, by (3.1.13). Then since $\mathbf{A}$ is (δ, R_0)-vanishing of codimension $n-1$, we assume that in a new coordinate system $(z_1, \cdots, z_{n-1}, z_n)$, the origin is y_i and

$$⨍_{C_{(20\sqrt{2})r_i}} \left|\mathbf{A}(z) - \mathbf{A}_{B'_{(20\sqrt{2})r_i}}(z_n)\right|\, dz \leq \delta. \qquad (3.1.21)$$

We write

$$C_i^1 = C_{(5\sqrt{2})r_i},\ C_i^2 = C_{(10\sqrt{2})r_i},\ C_i^3 = C_{(20\sqrt{2})r_i}, \qquad (3.1.22)$$

$$p_i^- = \inf_{z\in C_i^3} p(z) \text{ and } p_i^+ = \sup_{z\in C_i^3} p(z).$$

We next recall (2.2.4) to see that

$$p_i^+ - p_i^- \leq \omega(40\sqrt{2}r_i). \qquad (3.1.23)$$

In light of (3.1.18) and (3.1.19), it follows from the invariance property under the change of the variables that

$$\fint_{C_i^3} |Du|^{\frac{2p(z)}{p^-}} dz \leq \frac{|C_{40r_i}|}{|C_{20\sqrt{2}r_i}|} \fint_{C_{40r_i}(y_i)} |Du|^{\frac{2p(x)}{p^-}} dx \leq \sqrt{2}^n \lambda, \tag{3.1.24}$$

and, similarly,

$$\fint_{C_i^3} |F|^{\frac{2p(z)}{p^-}} dz \leq \sqrt{2}^n \delta\lambda. \tag{3.1.25}$$

We next claim that

$$\fint_{C_i^3} |Du|^2 dz \leq c_0 \lambda^{\frac{p^-}{p_i^+}} \quad \text{and} \quad \fint_{C_i^3} |F|^2 dz \leq c_0 \lambda^{\frac{p^-}{p_i^+}} \delta^{\frac{\gamma_1}{\gamma_2}}, \tag{3.1.26}$$

for some universal constant $c_0 \geq 1$, being independent of i. To do this, we first observe that

$$\left(\fint_{C_i^3} |Du|^2 dz \right)^{p_i^+ - p_i^-} \leq c,$$

where $c \geq 1$ is a universal constant, being independent from the index i.

In fact, a direct computation yields that

$$\begin{aligned} \left(\fint_{C_i^3} |Du|^2 dz \right)^{p_i^+ - p_i^-} &= \left(\frac{1}{|C_i^3|} \right)^{p_i^+ - p_i^-} \left(\int_{C_i^3} |Du|^2 dz \right)^{p_i^+ - p_i^-} \\ &\overset{(3.1.22),(3.1.23)}{\leq} c \left(\frac{1}{40\sqrt{2}r_i} \right)^{n\omega(40\sqrt{2}r_i)} \left(\int_{C_i^3} |Du|^2 dz \right)^{p_i^+ - p_i^-} \\ &\overset{(2.2.5)}{\leq} c \left(\int_{C_i^3} |Du|^2 dz \right)^{p_i^+ - p_i^-}. \end{aligned}$$

On the other hand, By taking $c^* \geq |\Omega| + 1$ in (3.1.13), so $\frac{1}{40\sqrt{2}r_i} \geq \frac{1}{R} \geq \frac{c*}{R_0} \geq$

$|\Omega| + 1$, we find that

$$\begin{aligned}\left(\int_{C_i^3} |Du|^2 dz\right)^{p_i^+ - p_i^-} &\leq \left(\int_{\Omega} |Du|^2 dz\right)^{p_i^+ - p_i^-} \\ &\overset{(3.1.12)}{\leq} (c[|\Omega| + 1])^{p_i^+ - p_i^-} \\ &\overset{(3.1.23)}{\leq} (c[|\Omega| + 1])^{\omega(40\sqrt{2}r_i)} \\ &\leq c\left(\frac{1}{40\sqrt{2}r_i}\right)^{\omega(40\sqrt{2}r_i)} \overset{(2.2.5)}{\leq} c.\end{aligned}$$

Recall $\gamma_1 \leq p_i^+$ and use the above observation and the Jensen inequality, to obtain that

$$\begin{aligned}\fint_{C_i^3} |Du|^2 dz &= \left(\fint_{C_i^3} |Du|^2 dz\right)^{\frac{p_i^+ - p_i^-}{p_i^+}} \left(\fint_{C_i^3} |Du|^2 dz\right)^{\frac{p_i^-}{p_i^+}} \\ &\leq (c)^{\frac{1}{\gamma_1}} \left(\fint_{C_i^3} |Du|^{\frac{2p_i^-}{p^-}} dz\right)^{\frac{p^-}{p_i^+}} \leq c\left(\fint_{C_i^3} |Du|^{\frac{2p(z)}{p^-}} dz + 1\right)^{\frac{p^-}{p_i^+}}.\end{aligned}$$

Since $\lambda > 1$, by (3.1.24), we get the first inequality in (3.1.26).

Likewise, we find that

$$\begin{aligned}\fint_{C_i^3} |F|^2 dz &\leq c\left(\fint_{C_i^3} |F|^{\frac{2p(z)}{p^-}} dz + 1\right)^{\frac{p^-}{p_i^+}} \\ &\overset{(3.1.25)}{\leq} c(\delta\lambda + 1)^{\frac{p^-}{p_i^+}} \\ &\overset{(3.1.14)}{\leq} c(\delta\lambda + \delta\lambda_0)^{\frac{p^-}{p_i^+}} \\ &\leq c_0 \lambda^{\frac{p^-}{p_i^+}} \delta^{\frac{\gamma_1}{\gamma_2}},\end{aligned}$$

which is the second inequality in (3.1.26).

We define

$$\tilde{u}_i(y) = \frac{u([5\sqrt{2}]r_i y)}{[5\sqrt{2}]r_i \sqrt{c_0 \lambda^{\frac{p^-}{p_i^+}}}}, \ \tilde{F}_i(y) = \frac{F([5\sqrt{2}]r_i y)}{\sqrt{c_0 \lambda^{\frac{p^-}{p_i^+}}}} \text{ and } \widetilde{\mathbf{A}}_i(y) = \mathbf{A}([5\sqrt{2}]r_i y).$$

Then $\tilde{u}_i \in H^1(C_4)$ is a weak solution of

$$\operatorname{div}(\widetilde{\mathbf{A}}_i(y) D\tilde{u}_i) = \operatorname{div} \tilde{F}_i \text{ in } C_4. \tag{3.1.27}$$

It is easy to see from (3.1.21) and (3.1.26) that

$$\fint_{C_4} \left| \widetilde{\mathbf{A}}_i(y) - \widetilde{\mathbf{A}}_{iB_4'}(y_n) \right| \, dy \le \delta,$$

$$\fint_{C_2} |D\tilde{u}_i|^2 dy \le 1 \text{ and } \fint_{C_2} |\tilde{F}_i|^2 dy \le \delta^{\frac{\gamma_1}{\gamma_2}}.$$

Then one can obtain compare (3.1.27) with its limiting equation as $\delta \to 0$ by a perturbation argument. In fact, we recall Lemma 3.1.4 and Lemma 3.1.5, and apply Lemma 5.2 in [14], to discover that for any $\epsilon \in (0,1)$, there exists the $\tilde{v}_i \in H^1(C_2)$ of

$$\operatorname{div}(\widetilde{\mathbf{A}}_{iB_4'}(y_n) D\tilde{v}_i) = 0 \text{ in } C_2$$

and

$$\delta = \delta(\epsilon, n, \nu, \Lambda, \gamma_1, \gamma_2)$$

such that

$$\fint_{C_2} |D\tilde{u}_i - D\tilde{v}_i|^2 dy \le \epsilon \text{ and } \|\tilde{v}_i\|_{L^\infty(C_1)} \le c.$$

Scaling back and denoting v_i by the translated function of $\tilde{v}_i$, we conclude that

$$\fint_{C_i^2} |Du - Dv_i|^2 dz \le \epsilon c_0 \lambda^{\frac{p^-}{p_i^+}} \tag{3.1.28}$$

and

$$\|v_i\|^2_{L^\infty(C_i^1)} \le c_1 \lambda^{\frac{p^-}{p_i^+}}. \tag{3.1.29}$$

for some universal constant $c_1 \ge 1$, being independent of i.

We next consider the boundary case that $dist(y_i, \partial\Omega) = |y_i - y_0| \le 20\sqrt{2} r_i$ for some $y_0 \in \partial\Omega$. According to Definition 3.1.1, as $20\sqrt{2} r_i \le R \le \frac{R_0}{104\sqrt{2}}$, there exists a new coordinate system $\{z_1, \cdots, z_n\}$, after appropriate rotation and translation of the coordinates, in which we denote by z_0 and z_i to mean the points y_0 and y_i in the old system, respectively, such that the origin lies somewhere in $B_{(104\sqrt{2})r_i\delta}(z_0)$,

$$C^+_{(104\sqrt{2})r_i} \subset \Omega_{(104\sqrt{2})r_i} \subset C_{(104\sqrt{2})r_i} \cap \{z_n > -(208\sqrt{2})r_i\delta\},$$

$$\fint_{C^+_{(104\sqrt{2})r_i}} \left|\mathbf{A}(z) - \mathbf{A}_{B'_{(104\sqrt{2})r_i}}(z_n)\right| \, dz \le \delta.$$

By taking $\delta \le \frac{1}{104\sqrt{2}}$, we deduce that $|z_i| \le (20\sqrt{2}+1)r_i \le (21\sqrt{2})r_i$ so

$$\Omega_{(5\sqrt{2})r_i}(z_i) \subset \Omega_{(26\sqrt{2})r_i} \subset \Omega_{(104\sqrt{2})r_i} \subset C_{200r_i}(z_i). \tag{3.1.30}$$

We next write for $j = 1, 2, 3$,

$$\Omega_i^j = \Omega_{2^{j-1}(26\sqrt{2})r_i}, C_i^{j+} = C_i^j \cap \{z_n > 0\}, \tag{3.1.31}$$

and

$$p_i^- = \inf_{z\in\Omega_i^3} p(z), \; p_i^+ = \sup_{z\in\Omega_i^3} p(z).$$

Therefore, it follows from the above settings and (3.1.6) that

$$\begin{aligned}
&\fint_{\Omega_i^3} \left(|Du(z)|^{\frac{2p(z)}{p^-}} + \frac{1}{\delta}|F(z)|^{\frac{2p(z)}{p^-}}\right) dz \\
&\le \frac{|\Omega_{200r_i}(z_i)|}{|\Omega_{(104\sqrt{2})r_i}|} \fint_{\Omega_{200r_i}(z_i)} \left(|Du(z)|^{\frac{2p(z)}{p^-}} + \frac{1}{\delta}|F(z)|^{\frac{2p(z)}{p^-}}\right) dz \\
&\le \frac{|C_{200r_i}|}{|C^+_{(104\sqrt{2})r_i}|} \frac{|C_{(200\sqrt{2})r_i}|}{|\Omega_{200r_i}(z_i)|} \fint_{\Omega_{(200\sqrt{2})r_i}(y_i)} \left(|Du(x)|^{\frac{2p(x)}{p^-}} + \frac{1}{\delta}|F(x)|^{\frac{2p(x)}{p^-}}\right) dx \\
&\le c(n)\Phi_{y_i}((200\sqrt{2})r_i),
\end{aligned}$$

where $c(n)$ is a constant depending only n and we recall Φ_{y_i} from (3.1.18). Then by (3.1.19) in Lemma 3.1.5 we derive that

$$\fint_{\Omega_i^3} |Du(z)|^{\frac{2p(z)}{p^-}} dz \le c(n)\lambda \text{ and } \fint_{\Omega_i^3} |F(z)|^{\frac{2p(z)}{p^-}} dz \le c(n)\delta\lambda.$$

Once we have the above uniform bounds, one can find in the same spirit as in the interior case that for some universal constant c_2,

$$\fint_{\Omega_i^3} |Du|^2 dz \le c_2 \lambda^{\frac{p^-}{p_i^+}} \text{ and } \fint_{\Omega_i^3} |F|^2 dz \le c_2 \lambda^{\frac{p^-}{p_i^+}} \delta^{\frac{\gamma_1}{\gamma_2}}. \tag{3.1.32}$$

In light of (3.1.30)-(3.1.32) and in a similar way that we have used for the interior case, also see Lemma 5.8 in [14], to conclude that for any $\epsilon \in (0,1)$ there exist a small positive number

$$\delta = \delta(\epsilon, n, \nu, \Lambda, \gamma_1, \gamma_2)$$

and a function $v_i \in H^1(C_i^{2+})$ such that

$$\fint_{\Omega_i^2} |Du - Dv_i|^2 dz \leq \epsilon c_2 \lambda^{\frac{p^-}{p_i^+}} \tag{3.1.33}$$

and

$$\|v_i\|_{L^\infty(\Omega_i^1)} = \|v_i\|_{L^\infty(C_i^{1+})} \leq c_3 \lambda^{\frac{p^-}{p_i^+}} \tag{3.1.34}$$

for some universal constants c_2, c_3, being independent of i. Here we have extended v_i by the zero from C_i^{2+} to Ω_i^2.

We are now ready to obtain gradient $L^{p(\cdot)}$-estimates on Ω_R. Recall c_1 in (3.1.29) and c_3 in (2.3.1) and write

$$c_4 = \max\{c_1, c_3\},$$

which is large universal constant, being large enough and independent of the index i. We also recall the assumption (3.1.16) on λ and the notation $E(\cdot)$ (3.1.15) for the upper-level sets. For the sake of simplicity and clearance, we also write

$$A = (4c_4)^{\frac{\gamma_2}{\gamma_1}}, \ B = \left(\frac{16\sqrt{2}}{7}\right)^n \left(\frac{400\sqrt{2}}{s_2 - s_1}\right)^n. \tag{3.1.35}$$

Then it is clear that $E(A\lambda) \subset E(\lambda)$. Thus Lemma 3.1.8 implies that $\{\Omega_{5r_i}(y_i)\}$ is an open covering $E(A\lambda)$. Therefore, we have

$$\begin{aligned} |E(A\lambda)| &= \left|\left\{x \in \Omega_{s_1 R} : |Du|^{\frac{2p(x)}{p^-}} > A\lambda\right\}\right| \\ &\leq \sum_{i=1}^{\infty} \left|\left\{x \in B_{5r_i}(y_i) : |Du|^2 > (A\lambda)^{\frac{p^-}{p(x)}}\right\}\right| \\ &\leq \sum_{i:\text{interior case}}^{\infty} \left|\left\{x \in \Omega_{5r_i}(y_i) : |Du|^2 > (A\lambda)^{\frac{p^-}{p(x)}}\right\}\right| \\ &\quad + \sum_{i:\text{boundary case}}^{\infty} \left|\left\{x \in \Omega_{5r_i}(y_i) : |Du|^2 > (A\lambda)^{\frac{p^-}{p(x)}}\right\}\right|. \end{aligned} \tag{3.1.36}$$

For the interior case, using the fact that $|Du|^2 \le 2|Du - Dv_i|^2 + 2|Dv_i|^2$ and recalling (3.1.22), (3.1.28), (3.1.29) and (3.1.35), we find that

$$\begin{aligned}\left|\left\{z \in C_i^1 : |Du|^2 > (A\lambda)^{\frac{p^-}{p(z)}}\right\}\right| &\le \left|\left\{z \in C_i^1 : |Du - Dv_i|^2 > c_0 \lambda^{\frac{p^-}{p_i^+}}\right\}\right| \\ &\quad + \left|\left\{z \in C_i^1 : |Dv_i|^2 > c_1 \lambda^{\frac{p^-}{p_i^+}}\right\}\right| \\ &\le \frac{c_1}{\lambda^{\frac{p^-}{p_i^+}}} \int_{C_i^1} |Du - Dv_i|^2 dz \\ &\le \epsilon |C_i^1| \le c\epsilon |C_{5\sqrt{2}r_i}|.\end{aligned}$$

Hence the invariance of the change of variables and the fact $B_{(5\sqrt{2})r_i} \subset C_i^1$ in the z-coordinate and $\Omega_{5r_i}(y_i) = C_{5r_i}(y_i) \subset B_{5\sqrt{2}r_i}(y_i)$ in the x-coordinate imply that

$$\left|\left\{x \in \Omega_{5r_i}(y_i) : |Du|^2 > (A\lambda)^{\frac{p^-}{p(x)}}\right\}\right| \le c\epsilon |\Omega_{r_i}(y_i)|. \tag{3.1.37}$$

For the boundary case, we recall (3.1.31), (3.1.33), (3.1.34) and (3.1.35). Then we carry out the same procedure in (3.1.36) to discover that

$$\left|\left\{z \in \Omega_i^1 : |Du|^2 > (A\lambda)^{\frac{p^-}{p(z)}}\right\}\right| \le c\epsilon |\Omega_i^1|.$$

Then using the measure density condition (3.1.6), $B_{(5\sqrt{2})r_i}(z_i) \cap \Omega \subset \Omega_i^1$ in the z-coordinate and $\Omega_{5r_i}(y_i) \subset B_{5\sqrt{2}r_i}(y_i) \cap \Omega$ in the x-coordinate, we return to the original x-coordinate to conclude that

$$\left|\left\{x \in \Omega_{5r_i}(y_i) : |Du|^2 > (A\lambda)^{\frac{p^-}{p(x)}}\right\}\right| \le c\epsilon |\Omega_{r_i}(y_i)|. \tag{3.1.38}$$

We next insert (3.1.37) and (3.1.38) into (3.1.36) and then apply (3.1.20).

Then we obtain that

$$\begin{aligned}|E(A\lambda)| &\leq c\epsilon\sum_{i=1}^{\infty}|\Omega_{r_i}(y_i)| \\ &\leq c\epsilon\frac{1}{\lambda}\sum_{i=1}^{\infty}\Bigg(\int_{\Omega_{r_i}(y_i)\cap\left\{|Du|^{\frac{2p(x)}{p^-}}>\frac{\lambda}{4}\right\}}|Du|^{\frac{2p(x)}{p^-}}dx \\ &\qquad +\frac{1}{\delta}\int_{\Omega_{r_i}(y_i)\cap\left\{|F|^{\frac{2p(x)}{p^-}}>\frac{\lambda\delta}{4}\right\}}|F|^{\frac{2p(x)}{p^-}}dx\Bigg).\end{aligned}$$

Then since $\{\Omega_{r_i}(y_i)\}$ are nonoverlapping in Ω_{s_2R}, we obtain the following result.

Lemma 3.1.6. *Under the same notations mentioned earlier, let $\|F\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^n)}\leq 1$ and $0<R\leq\frac{R_0}{c^*}<\frac{1}{4}$. Then for $0<\epsilon<1$ fixed, one can find a small number $\delta=\delta(n,\nu,\Lambda,\gamma_1,\gamma_2,\omega(\cdot),\epsilon)$ such that if $(\mathbf{A},\Omega)$ is (δ,R_0)-vanishing codimension 1 and $u\in H_0^1(\Omega)$ is the weak solution of (3.1.1), then for any $R\in\left(0,\frac{R_0}{c^*}\right]$ and for any $1\leq s_1<s_2\leq 2$ we have*

$$\begin{aligned}|E(A\lambda)| &\leq c\epsilon\frac{1}{\lambda}\Bigg(\int_{\Omega_{s_2R}\cap\left\{|Du|^{\frac{2p(x)}{p^-}}>\frac{\lambda}{4}\right\}}|Du|^{\frac{2p(x)}{p^-}}dx \\ &\qquad +\frac{1}{\delta}\int_{\Omega_{s_2R}\cap\left\{|F|^{\frac{2p(x)}{p^-}}>\frac{\lambda\delta}{4}\right\}}|F|^{\frac{2p(x)}{p^-}}dx\Bigg), \quad \textit{for all } \lambda>B\lambda_0,\end{aligned}$$

where c depends only on $n,\nu,\Lambda,\gamma_1,\gamma_2$ and $\omega(\cdot)$.

Now we begin to prove (3.1.9). First, we estimate in a local region $\Omega_R(x_0)$. We use the integral identity (2.3.1) when $f(x)=|Du(x)|^{\frac{2p(x)}{p^-}}$ and $q=\frac{p^-}{2}$. Direct calculation shows that

$$\begin{aligned}\int_{\Omega_{s_1R}}|Du|^{p(x)}dx &= \int_{\Omega_{s_1R}}\left(|Du|^{\frac{2p(x)}{p^-}}\right)^{\frac{p^-}{2}}dx=\frac{p^-}{2}\int_0^{\infty}\lambda^{\frac{p^-}{2}-1}|E(\lambda)|\,d\lambda \\ &= \frac{p^-}{2}A^{\frac{p^-}{2}}\int_0^{\infty}\lambda^{\frac{p^-}{2}-1}|E(A\lambda)|\,d\lambda \\ &\leq \underbrace{(A\cdot B\lambda_0)^{\frac{p^-}{2}}|\Omega_{s_1R}|}_{I_1}+\underbrace{\frac{p^-}{2}A^{\frac{p^-}{2}}\int_{B\lambda_0}^{\infty}\lambda^{\frac{p^-}{2}-1}|E(A\lambda)|d\lambda}_{I_2}. \qquad (3.1.39)\end{aligned}$$

We first estimation I_1 : We plug λ_0 in (3.1.14) and A, B in (3.1.35) into I_1. Then we have

$$\begin{aligned}
I_1 &\le c\frac{|\Omega_{2R}|}{(s_2-s_1)^{\frac{n\gamma_2}{2}}}\left\{\fint_{\Omega_{2R}}\left(|Du|^{\frac{2p(x)}{p^-}}+\frac{1}{\delta}|F|^{\frac{2p(x)}{p^-}}\right)dx+\frac{1}{\delta}\right\}^{\frac{p^-}{2}}\\
&\le c\frac{|\Omega_{2R}|}{(s_2-s_1)^{\frac{n\gamma_2}{2}}}\left(\fint_{\Omega_{2R}}|Du|^{2\frac{p^+}{p^-}}dx+1\right)^{\frac{p^-}{2}}\\
&\quad+\frac{c}{(s_2-s_1)^{\frac{n\gamma_2}{2}}}\left\{\left(\frac{1}{\delta}\right)^{\frac{p^-}{2}}\left(\int_{\Omega_{2R}}|F|^{p(x)}dx+|\Omega_{2R}|\right)\right\}\\
&\le c\frac{|\Omega_{2R}|}{(s_2-s_1)^{\frac{n\gamma_2}{2}}}\left(\fint_{\Omega_{2R}}|Du|^{2\frac{p^+}{p^-}}dx+1\right)^{\frac{p^-}{2}}+c(\delta)\frac{1}{(s_2-s_1)^{\frac{n\gamma_2}{2}}}\left(1+|\Omega_{2R}|\right),
\end{aligned}$$

where the inequality in the last line follow from (3.1.11).

We further assume on $0 < R \le \frac{R_0}{c^*}$ that

$$\omega(4\sqrt{2}R) \le \sigma_0\gamma_1,$$

by taking c^* large enough in (3.1.13), where σ_0 was given in Lemma 3.1.4. Then

$$\frac{p^+}{p^-} = 1+\frac{p^+-p^-}{p^-} \le 1+\frac{\omega(4\sqrt{2}R)}{\gamma_1} \le 1+\sigma_0 \le \gamma_1.$$

Then Lemma 3.1.2, (3.1.11) and (3.1.12) imply that

$$\begin{aligned}
\fint_{\Omega_{2R}}|Du|^{2\frac{p^+}{p^-}}dx &\le c\left(\fint_{\Omega_{4R}}|Du|^2dx\right)^{\frac{p^+}{p^-}}+c\fint_{\Omega_{4R}}|F|^{2\frac{p^+}{p^-}}dx\\
&\le c\left(\frac{1}{|\Omega_{4R}|}\int_{\Omega}|Du|^2dx\right)^{\frac{p^+}{p^-}}+c\left(\fint_{\Omega_{4R}}|F|^{p(x)}dx+1\right)\\
&\le c\left[\left(\frac{|\Omega|+1}{|\Omega_{4R}|}\right)^{\frac{p^+}{p^-}}+\frac{1}{|\Omega_{4R}|}+1\right] \le c\left(\frac{|\Omega|+1}{|\Omega_{4R}|}\right)^{\frac{p^+}{p^-}}.
\end{aligned}$$

Therefore, we derive from the measure density condition (3.1.6) that

$$\begin{aligned}
I_1 &\le \frac{c(\delta)}{(s_2-s_1)^{\frac{n\gamma_2}{2}}}\left[|\Omega_R|^{1-\frac{\gamma_2}{2}}(|\Omega|+1)^{\frac{\gamma_2}{2}}+|\Omega_R|+1\right]\\
&\le \frac{c(\delta)|\Omega_R|^{1-\frac{\gamma_2}{2}}(|\Omega|+1)^{\frac{\gamma_2}{2}}}{(s_2-s_1)^{\frac{n\gamma_2}{2}}}.
\end{aligned} \tag{3.1.40}$$

Using Lemma 3.1.6, we next estimate I_2:

$$\begin{aligned} I_2 \;&\leq\; c\, p^- A^{\frac{p^-}{2}} \int_0^\infty \lambda^{\frac{p^-}{2}-1} \frac{\epsilon}{\lambda}\Bigg(\int_{\Omega_{s_2 R}\cap\left\{|Du|^{\frac{2p(x)}{p^-}}>\frac{\lambda}{4}\right\}} |Du|^{\frac{2p(x)}{p^-}} dx \\ &\quad + \frac{1}{\delta}\int_{\Omega_{s_2 R}\cap\left\{|F|^{\frac{2p(x)}{p^-}}>\frac{\lambda\delta}{4}\right\}} |F|^{\frac{2p(x)}{p^-}} dx \Bigg) d\lambda \\ &\leq\; c\epsilon A^{\frac{\gamma_2}{2}} \Bigg\{ \underbrace{\int_0^\infty \left(\frac{\lambda}{4}\right)^{\frac{p^-}{2}-2} \left(\int_{\Omega_{s_2 R}\cap\left\{|Du|^{\frac{2p(x)}{p^-}}>\frac{\lambda}{4}\right\}} |Du|^{\frac{2p(x)}{p^-}} dx \right) d\left(\frac{\lambda}{4}\right)}_{I_3} \\ &\quad + \underbrace{\frac{1}{\delta^{\frac{\gamma_2}{2}}} \int_0^\infty \left(\frac{\lambda\delta}{4}\right)^{\frac{p^-}{2}-2} \left(\int_{\Omega_{s_2 R}\cap\left\{|F|^{\frac{2p(x)}{p^-}}>\frac{\lambda\delta}{4}\right\}} |F|^{\frac{2p(x)}{p^-}} dx \right) d\left(\frac{\lambda\delta}{4}\right)}_{I_4} \Bigg\}. \end{aligned}$$

We apply (2.3.2) to I_3 when $q = \frac{p^-}{2}$, $q_1 = 1$ and $|f(x)| = |Du(x)|^{\frac{2p(x)}{p^-}}$ with λ replaced by $\frac{\lambda}{4}$, to I_4 when $q = \frac{p^-}{2}$, $q_1 = 1$ and $|f(x)| = |F|^{\frac{2p(x)}{p^-}}$ with λ replaced by $\frac{\lambda\delta}{4}$, respectively, to discover

$$I_2 \leq c_5\epsilon \int_{\Omega_{s_2 R}} |Du|^{p(x)} dx + c(\epsilon,\delta) \int_{\Omega_{s_2 R}} |F|^{p(x)} dx$$

for some universal constant $c_5 = c_5(n,\nu,\Lambda,\gamma_1,\gamma_2,\omega(\cdot))$. But then (3.1.11) imply

$$I_2 \leq c_5\epsilon \int_{\Omega_{s_2 R}} |Du|^{p(x)} dx + c(\epsilon,\delta). \tag{3.1.41}$$

Combining (3.1.39), (3.1.40) and (3.1.41), we deduce that

$$\int_{\Omega_{s_1 R}} |Du|^{p(x)} dx \leq \frac{c(\delta)|\Omega_R|^{1-\frac{\gamma_2}{2}}(|\Omega|+1)^{\frac{\gamma_2}{2}}}{(s_2-s_1)^{\frac{n\gamma_2}{2}}} + c_5\epsilon \int_{\Omega_{s_2 R}} |Du|^{p(x)} dx + c(\epsilon,\delta).$$

We now select a sufficiently small $\epsilon = \epsilon((n,\nu,\Lambda,\gamma_1,\gamma_2,\omega(\cdot))$, in order to get

$$0 < c_5\epsilon < \frac{1}{2},$$

and so one can choose a corresponding

$$\delta = \delta(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)), \tag{3.1.42}$$

being sufficiently small, from Lemma 3.1.6, to obtain

$$\int_{\Omega_{s_1 R}} |Du|^{p(x)} dx \leq \frac{1}{2} \int_{\Omega_{s_2 R}} |Du|^{p(x)} dx + \frac{c|\Omega_R|^{1-\frac{\gamma_2}{2}} (|\Omega| + 1)^{\frac{\gamma_2}{2}}}{(s_2 - s_1)^{\frac{\gamma_2 n}{2}}} + c,$$

for all $1 \leq s_1 < s_2 \leq 2$. We then apply Lemma 5.2.9 when $\phi(s) = \int_{\Omega_{sR}} |Du|^{p(x)} dx$, $r_1 = 1$ and $r_2 = 2$, and then make simple computations along with the measure density condition (3.1.6), to finally derive

$$\int_{\Omega_R(x_0)} |Du|^{p(x)} dx \leq c|\Omega_R(x_0)|^{1-\frac{\gamma_2}{2}} (|\Omega| + 1)^{\frac{\gamma_2}{2}}, \tag{3.1.43}$$

where c depends only on $n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)$.

We next derive the global estimates. It can be obtained by using standard covering argument along with the local estimates (3.1.43). Since $\overline{\Omega}$ is compact, there exists finitely many points $x_0^k \in \overline{\Omega}$ and numbers $R_k = R_k(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), R_0, |\Omega|) \leq \frac{R_0}{c^*}$, $k = 1, \cdots, N$, such that $\Omega = \bigcup_{k=1}^N \Omega_{R_k}(x_0^k)$ and

$$\int_{\Omega_{R_k}(x_0^k)} |Du|^{p(x)} \, dx \leq c|\Omega_{R_k}(x_0^k)|^{1-\frac{\gamma_2}{2}} (|\Omega| + 1)^{\frac{\gamma_2}{2}} \qquad (k = 1, 2, \cdots, N),$$

where c is independent of k. Consequently, we have that for some universal constant $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), R_0, |\Omega|)$,

$$\int_\Omega |Du|^{p(x)} dx \leq \sum_{k=1}^N \int_{\Omega_{R_k(x_0^k)}} |Du|^{p(x)} \, dx \leq c,$$

which is the required one (3.1.9).

Proof of Theorem 3.1.4. We have established the global $W^{1,p(\cdot)}$-estimate under the a priori assumption (3.1.8) that $|Du| \in L^{p(\cdot)}(\Omega)$. We next remove this assumption via an approximation procedure to complete the proof Theorem 3.1.4.

Let $\{F^k\}_{k=1}^{\infty}$ be a sequence in $C_0^\infty(\Omega;\mathbb{R}^n)$ converging to F in $L^{p(\cdot)}(\Omega;\mathbb{R}^n)$, needless to say, $F^k \in L^{\gamma_2}(\Omega;\mathbb{R}^n)$. According to the earlier work [14], the unique weak solution $u^k \in H_0^1(\Omega)$ of

$$\begin{cases} \operatorname{div}\left(\mathbf{A}(x)Du^k\right) &= \operatorname{div} F^k \quad \text{in} \quad \Omega, \\ u^k &= 0 \quad\quad\;\; \text{on} \quad \partial\Omega \end{cases}$$

satisfies the global W^{1,γ_2}-regularity under the assumption that $(\mathbf{A},\Omega)$ is (δ, R_0)-vanishing of codimension 1. More precisely, we have

$$|Du^k| \in L^{\gamma_2}(\Omega) \subset L^{p(\cdot)}(\Omega).$$

As a consequence, we have

$$\|Du^k\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^n)} \leq c\|F^k\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^n)} \leq c\|F\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^n)},$$

where c is independent of k. From this estimate we observe that there exists $\bar{u} \in W_0^{1,p(\cdot)}(\Omega)$ which is the weak limit of $\{u^k\}$ in $W_0^{1,p(\cdot)}(\Omega)$ such that

$$\|D\bar{u}\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^n)} \leq c\|F\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^n)}.$$

Then it is easy to check that this $\bar{u}$ is the weak solution of the original problem (3.1.1). So by the uniqueness, we conclude that $u = \bar{u}$ almost everywhere in Ω. This completes our approximation procedure. $\square$

Remark 3.1.7. *One can consider the general case that* $1 < \gamma_1 \leq p(x) \leq \gamma_2 < +\infty$. *In fact, since* $|F| \in L^{p(\cdot)}(\Omega) \in L^{\gamma_1}(\Omega) \subset L^{\gamma}(\Omega)$ *for* $1 < \gamma < \gamma_1$, *we know that* $|Du| \in L^{\gamma}(\Omega)$ *from [14]. Then with the same spirit in the case that* $2 < \gamma_1 \leq p(x) \leq \gamma_2 < +\infty$, *one treat weak solutions defined in* $L^{\gamma}(\Omega)$ *to consider* $\left\{x \in \Omega_R(x_0) : |Du|^{\frac{\gamma p(x)}{p^-}} > \lambda\right\}$.

3.2 Optimal gradient estimates for parabolic equations in variable exponent spaces.

3.2.1 Main result.

We consider the following initial and boundary value problem for a divergence type parabolic equation

$$\begin{cases} u_t - \operatorname{div}(\mathbf{A}(x,t)Du) &= \operatorname{div} F \quad \text{in} \quad \Omega_T, \\ u &= 0 \quad \text{on} \quad \partial_p\Omega_T, \end{cases} \tag{3.2.1}$$

to show that the following relation

$$F \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^n) \quad \Longrightarrow \quad Du \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^n) \tag{3.2.2}$$

holds for each variable exponent $p(\cdot) = p(x,t)$ under an optimal regularity assumption on the coefficient matrix $\mathbf{A} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n^2}$ and a minimal geometric assumption on the bounded parabolic cylinder Ω_T.

Note that for $f = f(x,t) : \mathbb{R}^{n+1} \to \mathbb{R}$ and $F = F(x,t) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ we denote by $Df = D_x f$ to mean the spatial gradient vector of f, by $\operatorname{div}F = \operatorname{div}_x F$ the divergence of F with respect to spatial variables, and by f_t the derivative with respect to time variable.

We return to the parabolic problem (3.2.1). The basic structural condition on $\mathbf{A}$ is the following:

$$\nu|\xi|^2 \le (\mathbf{A}(x,t)\xi)\cdot\xi \quad \text{and} \quad |\mathbf{A}(x,t)| \le \Lambda, \quad \text{for all } x, \xi \in \mathbb{R}^n,\ t \in \mathbb{R} \tag{3.2.3}$$

and for some constants $0 < \nu \le \Lambda$. $F : \Omega_T \to \mathbb{R}^n$ is a vector valued function in $L^2(\Omega_T; \mathbb{R}^n)$. We then say $u \in C^0(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ is a weak solution of (3.2.1) if it satisfies that

$$\int_{\Omega_T} u\varphi_t \, dz - \int_{\Omega_T} (\mathbf{A}(z)Du)\cdot D\varphi \, dz = \int_{\Omega_T} F \cdot D\varphi \, dz \tag{3.2.4}$$

for all $\varphi \in C_0^\infty(\Omega_T)$ with $\varphi = 0$ when $t = T$, and $u(\cdot, 0) = 0$ in $L^2(\Omega)$. It is well known that (3.2.1) has a unique weak solution with the estimate

$$\sup_{0\le t\le T} \|u(\cdot,t)\|^2_{L^2(\Omega)} + \int_{\Omega_T} |Du|^2 dx \le c_0 \int_{\Omega_T} |F|^2 dx, \tag{3.2.5}$$

where c_0 depends only on n, ν and Λ. We refer to [34, 41] for the basic theories mentioned above.

Suppose that $p(\cdot)$ is log-Hölder continuous. We define

$$W_{\mathcal{P}}^{1,p(\cdot)}(\Omega_T) := W^{p(\cdot)}(\Omega_T) \quad \text{and} \quad W_{\mathcal{P},0}^{1,p(\cdot)}(\Omega_T) := W_0^{p(\cdot)}(\Omega_T)$$

with $\|f\|_{W_{\mathcal{P}}^{1,p(\cdot)}(\Omega_T)} = \|f\|_{W_{\mathcal{P},0}^{1,p(\cdot)}(\Omega_T)} := \|f\|_{L^{p(\cdot)}(\Omega_T)} + \|Df\|_{L^{p(\cdot)}(\Omega_T,\mathbb{R}^n)}$. Let $W_{\mathcal{P}}^{-1,p(\cdot)}(\Omega_T)$ be the dual space of $W_{\mathcal{P},0}^{1,p'(\cdot)}(\Omega_T)$ with the pairing $\langle\langle\cdot,\cdot\rangle\rangle$ between $W_{\mathcal{P}}^{-1,p(\cdot)}(\Omega_T)$ and $W_{\mathcal{P},0}^{1,p'(\cdot)}(\Omega_T)$. Then for any $v \in W_{\mathcal{P}}^{-1,p(\cdot)}(\Omega_T)$ there exist $G \in L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)$ and $g \in L^{p(\cdot)}(\Omega_T)$ such that

$$\langle\langle v,w\rangle\rangle = \int_{\Omega_T} (G\cdot Dw + gw)\, dz \;\text{ for all } w \in W_{\mathcal{P},0}^{1,p'(\cdot)}(\Omega_T) \tag{3.2.6}$$

and $\|v\|_{W_{\mathcal{P}}^{-1,p(\cdot)}(\Omega_T)} = \inf\left\{\|G\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)} + \|g\|_{L^{p(\cdot)}(\Omega_T)}\right\}$, where the infimum runs over all the functions G and g satisfying (3.2.6). We next introduce

$$W_*^{1,p(\cdot)}(\Omega_T) = \left\{u \in W_{\mathcal{P}}^{1,p(\cdot)}(\Omega_T) : u_t \in W_{\mathcal{P}}^{-1,p(\cdot)}(\Omega_T)\right\},$$

where we understand $u_t \in W_{\mathcal{P}}^{-1,p(\cdot)}(\Omega_T)$ in the sense of distribution, that is,

$$\langle\langle u_t,\varphi\rangle\rangle = -\int_{\Omega_T} u\varphi_t\, dz,$$

for every $\varphi \in C_0^\infty(\Omega_T)$ with $\varphi = 0$ for $t = T$, with

$$\|u\|_{W_*^{1,p(\cdot)}(\Omega_T)} = \|u\|_{W_{\mathcal{P}}^{1,p(\cdot)}(\Omega_T)} + \|u_t\|_{W_{\mathcal{P}}^{-1,p(\cdot)}(\Omega_T)}.$$

Remark 3.2.1. *Note that for the special case $p(\cdot) \equiv 2$, we obtain $W_{\mathcal{P}}^{1,2}(\Omega_T) = L^2(0,T;H^1(\Omega))$, $W_{\mathcal{P},0}^{1,2}(\Omega_T) = L^2(0,T;H_0^1(\Omega))$ and $W_{\mathcal{P}}^{-1,2}(\Omega_T) = L^2(0,T;H^{-1}(\Omega))$, where H^{-1} is the dual space of H_0^1. Then one can observe that the weak formulation (3.2.4) can be written as*

$$\langle\langle u_t,\varphi\rangle\rangle + \int_{\Omega_T} (\mathbf{A}(z)Du)\cdot D\varphi\, dz = -\int_{\Omega_T} F\cdot D\varphi\, dz,$$

for all $\varphi \in L^2(0,T;H_0^1(\Omega))$, and we have from (3.2.5) and (3.2.6) that

$$\begin{aligned}
\|u\|_{W_*^{1,2}(\Omega_T)} &= \|u\|_{L^2(0,T,H^{1,0}(\Omega)} + \|u_t\|_{W_{\mathcal{P}}^{-1,2}(\Omega_T)} \\
&\le c\left(\|Du\|_{L^2(\Omega_T;\mathbb{R}^n)} + \|F\|_{L^2(\Omega_T;\mathbb{R}^n)} + \|ADu\|_{L^2(\Omega_T;\mathbb{R}^n)}\right) \\
&\le c\|F\|_{L^2(\Omega_T,\mathbb{R}^n)},
\end{aligned} \tag{3.2.7}$$

where the positive constant c is independent of u and F. Furthermore, the weak solution u satisfies that, for almost every time $t \in [0,T]$,

$$\langle u_t(\cdot,t), \psi\rangle + \int_\Omega \mathbf{A}(x,t)Du(x,t)D\psi(x)\ dx = -\int_\Omega F(x,s)D\psi(x)\ dx, \quad (3.2.8)$$

for all $\psi \in H_0^1(\Omega)$, where $\langle\cdot,\cdot\rangle$ is the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

We now state the regularity assumptions on the coefficient matrix $\mathbf{A}$ and the boundary $\partial\Omega$ of Ω under which we essentially obtain the relation (3.2.2). On $\mathbf{A}$ we impose a small BMO condition in (x',t) variables while allow to be merely measurable in x_1 variable, depending on a point and a scale chosen. On $\partial\Omega$ we impose a δ-flatness which means the boundary can be locally trapped into two hyperplanes. These conditions on $\mathbf{A}$ and $\partial\Omega$ were reported in the earlier paper [11] where the same problem (3.2.1) is considered in the constant exponent Lebesgue space.

In order to measure the oscillation of $\mathbf{A}$ in (x',t) over $Q'_r(y',s)$, we consider

$$\overline{\mathbf{A}}_{Q'_r(y',s)}(x_1) = \fint_{Q'_r(y',s)} \mathbf{A}(x_1,x',t)dx'dt = \frac{1}{|Q'_r(y',s)|}\int_{Q'_r(y',s)} \mathbf{A}(x_1,x',t)dx'dt,$$

which is the integral average of $\mathbf{A}(x_1,\cdot,\cdot)$ in (x',t) over $Q'_r(y',s)$ for each x_1 slice.

Definition 3.2.2. Let $\delta < \frac{1}{8}$. We say that $(\mathbf{A}, \Omega\times\mathbb{R})$ is (δ,R)-*vanishing codimension 1* if the followings hold: for each $(y,s) \in \Omega\times\mathbb{R}$ and $0 < r \le R$,

1. if $\mathrm{dist}(y,\partial\Omega) : \inf\{|y-y^1| : y^1 \in \partial\Omega\} > \sqrt{2}r$, then reorienting and translating the axes, we find a new coordinate system, $\bar{z} = (\bar{x},\bar{t}) = (\bar{x}_1,\bar{x}',\bar{t})$ variables, with the origin at (y,s) such that in this system

$$\fint_{Q_r} |\mathbf{A}(\bar{z}) - \overline{\mathbf{A}}_{Q'_r}(\bar{x}_1)|dz \le \delta,$$

2. on the other hand, if $dist(y,\partial\Omega) \le \sqrt{2}r$, say $y^1 \in \partial\Omega \cap B_{\sqrt{2}r}(y)$ with $|y^1-y| = dist(y,\partial\Omega)$, then reorienting and translating the axes again, we find a new coordinate system, $\bar{z} = (\bar{x},\bar{t}) = (\bar{x}_1,\bar{x}',\bar{t})$ variables, with the origin at somewhere on $\partial B_{24r\delta}(y^1)\times\{s\}$ such that

$$C^+_{24r} \ \subset\ C_{24r}\cap\Omega \ \subset\ C_{24r}\cap\{\bar{x}_1 > -48\delta r\}$$

and
$$\fint_{Q^{+}_{24r}} |\mathbf{A}(\bar{z}) - \overline{\mathbf{A}}_{Q'_{24r}}(\bar{x}_1)|\, dz \le \delta.$$

There are a few comments on the above definition.

1. Changed coordinate systems can be obtained by rotation with respect to x variables and translation with respect to (x,t) variables from the original coordinate system. Since the equation (3.2.1) is invariant under such rotation and translation, without loss of generality, in new coordinate systems, we still use the same notations used in original coordinate system, for example, variables x and t, $\mathbf{A}$ an F.

2. For sufficiently small regions $B_{\sqrt{2}r}(y) \times (s - r^2, s + r^2)$, if $B_{\sqrt{2}r}(y)$ lies in Ω, then there exists one direction along one of the spatial variables, depending on y and r, such that $\mathbf{A}$ is merely measurable in this direction and has a small BMO condition in the other directions. On the other hand, if $B_{\sqrt{2}r}(y)$ intersects the boundary of Ω, then there exists one direction along one of the spatial variables, which is normal to two parallel hyperplanes, one lying locally inside Ω and the other locally lying outside Ω near y_1 with the distance between $48\delta r$, such that $\mathbf{A}$ is merely measurable in this direction and has a small BMO condition in the other directions.

3. Only for a technical reason, we record the number 24 which can be easily changeable via a scaling. By the same reason, one can take R can be any positive number while δ is invariant under such a scaling.

Remark 3.2.3. *If $(\mathbf{A}, \Omega \times \mathbb{R})$ is (δ, R)-vanishing codimension 1, one can observe that Ω is $(\delta, 24R)$-Reifenberg flat domain, [52], that is, for each $y \in \partial\Omega$ and each $0 < r < 24R$, there exists a coordinate system $y = (y_1, \dots, y_n)$ with the origin at y such that*

$$B_r \cap \{y_n > -\delta r\} \;\subset\; B_r \cap \Omega \;\subset\; B_r \cap \{y_n > \delta r\}\,.$$

Since $(\delta, 24R)$-Reifenberg flat domains have the following property :

$$\frac{|B_r(y)|}{|B_r(y) \cap \Omega|} \le \left(\frac{2}{1-\delta}\right)^n, \quad \text{for all } y \in \Omega \text{ and } r \in (0, 24R),$$

see [13], we observe that

$$\sup_{y\in\Omega}\sup_{0<r<24R}\frac{|C_r(y)|}{|C_r(y)\cap\Omega|}\leq\sup_{y\in\Omega}\sup_{0<r<24R}\frac{\sqrt{2}^n|B_r(y)|}{|B_r(y)\cap\Omega|}\leq\left(\frac{16\sqrt{2}}{7}\right)^n. \tag{3.2.9}$$

We refer to [13] for a further discussion on the δ-flatness condition.

Let $p(\cdot)=p(z):\mathbb{R}^{n+1}\to(2,\infty)$ with

$$2<\gamma_1\leq p(z)\leq\gamma_2<\infty \tag{3.2.10}$$

be log-Hölder continuous, hence there exists a modulus continuity of $p(\cdot)$ with respect to parabolic distance, $\omega:[0.\infty)\to[0,\infty)$, satisfying (5.1.1) and (5.1.2) with $\tilde{L}$ replaced by α_p. We rewrite the log-Hölder condition such that

$$|p(\zeta^1)-p(\zeta^2)|\leq\omega(d_{\mathcal{P}}(\zeta^1,\zeta^2)), \tag{3.2.11}$$

for all $\zeta^1,\zeta^2\in\mathbb{R}^{n+1}$, and

$$\omega(r)\log\left(\frac{1}{r}\right)\leq\alpha_p\quad\left(\Longleftrightarrow\quad\left(\frac{1}{r}\right)^{\omega(r)}\leq e^{\alpha_p}\right),\quad\text{for all } r\in(0,1]. \tag{3.2.12}$$

Theorem 3.2.4. *Let $R_0>0$ be a given fixed number. Then there exists a small $\delta=\delta(n,\nu,\Lambda,\gamma_1,\gamma_2,\omega(\cdot))$ such that if $(\mathbf{A},\Omega\times\mathbb{R})$ is (δ,R_0)-vanishing codimension 1 and $F\in L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)$ and $u\in C^0(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ is the unique weak solution of (3.2.1), then we have the estimate*

$$\|Du\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)}\leq c\|F\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)}, \tag{3.2.13}$$

for some $c=c\,(n,\nu,\Lambda,\gamma_1,\gamma_2,\omega(\cdot),\Omega,T,R_0)$.

Remark 3.2.5. *We recall the space $W_*^{1,p(\cdot)}(\Omega_T)$ mentioned in the previous subsection. Then we see from* (3.2.13) *that*

$$u\in W_*^{1,p(\cdot)}(\Omega_T)$$

with the estimate

$$\|u\|_{W_*^{1,p(\cdot)}(\Omega_T)}\leq c\|F\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)}, \tag{3.2.14}$$

for some $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), \Omega, T, R_0)$. *Indeed, using Poincaré's type inequality in parabolic Sobolev space, see Lemma 3.9 in [30],* (3.2.5) *and* (3.2.13), *we deduce*

$$\begin{aligned}\|u\|_{W_*^{1,p(\cdot)}(\Omega_T)} &= \|u\|_{W_{\mathcal{P}}^{1,p(\cdot)}(\Omega_T)} + \|u_t\|_{W_{\mathcal{P}}^{-1,p(\cdot)_{\mathcal{P}}}(\Omega_T)} \\ &\leq c\left(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), \Omega, T, R_0, \|Du\|_{L^{p(\cdot)}(\Omega_T)}, \|F\|_{L^{p(\cdot)}(\Omega_T)}\right) \\ &\leq c\left(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), \Omega, T, R_0, \|F\|_{L^{p(\cdot)}(\Omega_T)}\right).\end{aligned}$$

Therefore, using a standard normalization argument, we obtain (3.2.14).

3.2.2 Proof of Theorem 3.2.4.

Comparison Estimates.

In what follows, we denote the letter c to mean any constants which can be computed in terms of $n, \nu, \Lambda, \gamma_1, \gamma_2$ and $\omega(\cdot)$, and so c may be different line by line.

We now introduce higher integrability of spatial gradient of weak solutions of (3.2.1). The interior case was discussed in [35] when $F = 0$, using Caccioppoli estimate Sobolev-Poincaré's inequality and Gehring Lemma. Even if $F \neq 0$ and $F \in L^p(\Omega \times (0,T))$, $p > 2$, one can also obtain higher integrability in the same way. Furthermore, if Ω is a (δ, R)-Reifenberg flat domain, which enjoys the measure density condition, that is, for all $0 < r \leq R$ and for each $y \in \partial\Omega$, $\frac{|\Omega^c \cap B_r(y)|}{|B_r(y)|} \geq c(n)$ for some $\alpha(n) \in (0,1)$, and so Sobolev Poincaré's inequality can be applicable to zero extension of the solution of (3.2.1) in a neighborhood the boundary point. Consequently, we have higher integrability, as we now state.

Lemma 3.2.6. *Suppose that* Ω *is* (δ, R)*-Reifenberg flat and* $p > 2$. *Let* $\zeta = (y, s) \in \mathbb{R}^{n+1}$, *where* $y \in \Omega$ *and* $r \leq \frac{R}{2}$. *There exists* $\sigma_0 = \sigma_0(n, \nu, \Lambda, p) \leq \frac{p}{2} - 1$ *such that, for* $F \in L^p(K_{2r}(\zeta))$ *and a weak solution*

$$u \in C^0\left(s-(2r)^2, s+(2r)^2; L^2(\Omega_{2r}(y))\right) \cap L^2\left(s-(2r)^2, s+(2r)^2; H^1(\Omega_{2r}(y))\right)$$

of

$$\begin{cases} u_t - div\ (\mathbf{A}(z)Du) = div\, F & in \quad K_{2r}(\zeta), \\ u = 0 & on \quad \partial_w K_{2r}(\zeta),\ \ if\ C_{2r}(y) \not\subset \Omega, \end{cases}$$

there holds $Du \in L^{2(1+\sigma_0)}(K_r(\zeta))$ *and we have the estimate*

$$\fint_{K_r(\zeta)} |Du|^{2(1+\sigma)} dz = c \left(\fint_{K_{2r}(\zeta)} |Du|^2 \, dz \right)^{1+\sigma} + c \fint_{K_{2r}(\zeta)} |F|^{2(1+\sigma)} dz,$$

for every $\sigma \in (0, \sigma_0]$ *and for some positive constant* $c = c(n, \nu, \Lambda, p)$.

We next study local estimates of solutions of the original equations (3.2.1) by comparison with solutions of the limiting equations, which is the case when δ goes 0 and the coefficients depend on only one of spatial variables.

Lemma 3.2.7. *Let* $\epsilon \in (0, 1)$. *There exists a small* $\delta = \delta(\epsilon, n, \nu, \Lambda) > 0$ *such that the followings hold.*

1. *Interior estimates: If* $u \in C^0(-4^2, 4^2; L^2(C_4)) \cap L^2(-4^2, 4^2; H^1(C_4))$ *is a weak solution of*

   $$u_t - div(\mathbf{A}(z) Du) = div\, F \text{ in } Q_4$$

   with

   $$\fint_{Q_4} |Du|^2 \, dz \leq 1 \text{ and } \fint_{Q_4} \left(|\mathbf{A}(z) - \overline{\mathbf{A}}_{Q_4'}(x_1)| + |F|^2 \right) dz \leq \delta,$$

   then there exists a weak solution $v \in C^0(-2^2, 2^2; L^2(C_2)) \cap L^2(-2^2, 2^2; H^1(C_2))$ *of*

   $$v_t - div\left(\overline{\mathbf{A}}_{Q_4'}(x_1) Dv \right) = 0 \text{ in } Q_2,$$

   with

   $$\fint_{Q_4} |Dv|^2 \, dz \leq c$$

   such that

   $$\fint_{Q_2} |Du - Dv|^2 \, dz \leq \epsilon.$$

2. *Boundary estimates: Suppose* Ω *is a* $(\delta, 4)$*-Reifenberg flat domain. If* $u \in C^0(-4^2, 4^2; L^2(\Omega_4)) \cap L^2(-4^2, 4^2; H^1(\Omega_4))$ *is a weak solution of*

   $$\begin{cases} u_t - div\ (\mathbf{A}(z) Du) = 0 & in \quad K_4, \\ u = 0 & on \quad \partial_w K_4, \end{cases}$$

with

$$C_4^+ \subset \Omega_4 \subset C_4 \cap \{x_1 > -8\delta\},$$

$$\fint_{K_4} |Du|^2 \, dz \leq 1 \text{ and } \fint_{K_4} |\mathbf{A}(z) - \overline{\mathbf{A}}_{Q_4'}(x_1)| + |F|^2 \, dz \leq \delta,$$

then there exists a weak solution $v \in C^0(-3^2, 3^2; L^2(\Omega_3)) \cap L^2(-3^2, 3^2; H^1(\Omega_3))$ *of*

$$\begin{cases} v_t - \operatorname{div}\left(\overline{\mathbf{A}}_{Q_4'}(x_1) Dv\right) = 0 & \text{in } K_3, \\ v = 0 & \text{on } \partial_w K_3, \end{cases}$$

with

$$\fint_{K_3} |Dv|^2 \, dz \leq c,$$

such that

$$\fint_{K_3} |Du - Dv|^2 \, dz \leq \epsilon.$$

Proof. We only prove the boundary case. The interior case can be proved in a similar way. Let $w \in C^0(-4^2, 4^2; L^2(\Omega_4)) \cap L^2(-4^2, 4^2; H^1(\Omega_4))$ be the weak solution of

$$\begin{cases} w_t - \operatorname{div}\left(\mathbf{A}(z) Dw\right) = 0 & \text{in } K_4, \\ w = u & \text{on } \partial_p K_4, \end{cases} \tag{3.2.15}$$

and then let $v \in C^0(-3^2, 3^2; L^2(\Omega_3)) \cap L^2(-3^2, 3^2; H^1(\Omega_3))$ be the weak solution of

$$\begin{cases} v_t - \operatorname{div}\left(\overline{\mathbf{A}}_{Q_4'}(x_1) Dv\right) = 0 & \text{in } K_3, \\ v = w & \text{on } \partial_p K_3. \end{cases} \tag{3.2.16}$$

We next observe that $u - w \in C^0(-4^2, 4^2; L^2(\Omega_4)) \cap L^2(-4^2, 4^2; H_0^1(\Omega_4))$ is the weak solution of

$$\begin{cases} (u-w)_t - \operatorname{div}\left(\mathbf{A}(z) D(u-w)\right) = \operatorname{div} F & \text{in } K_4 \\ u - w = 0 & \text{on } \partial_p K_4 \end{cases}$$

Then from standard L^2 estimates for $u - w$ and the smallness assumption for F, we find

$$\fint_{K_4} |Du - Dw|^2 \, dz \leq c \fint_{K_4} |F|^2 \, dz \leq c\delta, \tag{3.2.17}$$

which also implies

$$\fint_{K_4} |Dw|^2\,dz \le c\left(\fint_{K_4} |Du|^2\,dz + \delta\right) \le c. \qquad (3.2.18)$$

On the other hand, $w - v \in C^0(-3^2, 3^2; L^2(\Omega_3)) \cap L^2(-3^2, 3^2; H_0^1(\Omega_3))$ is the weak solution of

$$\begin{cases} (w-v)_t - \operatorname{div}\left(\overline{\mathbf{A}}_{Q_4'}(x_1)D(w-v)\right) = \operatorname{div}\left(\left(\mathbf{A} - \overline{\mathbf{A}}_{Q_4'}(x_1)\right)Dw\right) & \text{in} \quad K_3, \\ w - v = 0 & \text{on} \quad \partial_p K_3, \end{cases}$$

and so we have

$$\fint_{K_3} |Dw - Dv|^2\,dz \le c\fint_{K_3} |\mathbf{A} - \overline{\mathbf{A}}_{Q_4'}(x_1)|^2 |Dw|^2\,dz.$$

We now recall higher integrability, Lemma 3.2.6, to see that $Dw \in L^{2(1+\sigma)}(K_2)$ for some $\sigma = \sigma(n, \nu, \Lambda)$ with the estimate

$$\fint_{K_3} |Dw|^{2(1+\sigma)}dz \le c\left(\fint_{K_4} |Dw|^2 dz\right)^{1+\sigma} \le c,$$

where we have used (3.2.18). Then it follows from Hölder's inequality, the smallness assumption for $\mathbf{A}$ and (3.2.3) that

$$\begin{aligned} &\fint_{K_3} |Dw - Dv|^2\,dz \\ &\quad \le c\left(\fint_{K_3} |\mathbf{A} - \overline{\mathbf{A}}_{Q_4'}(x_1)|^{2\frac{1+\sigma}{\sigma}}\,dz\right)^{\frac{\sigma}{1+\sigma}} \left(\fint_{K_3} |Dw|^{2(1+\sigma)}\,dz\right)^{\frac{1}{1+\sigma}} \\ &\quad \le \frac{c}{|K_3|}\left(\int_{K_3 \setminus Q_3^+} |\mathbf{A} - \overline{\mathbf{A}}_{Q_4'}(x_1)|^{2\frac{1+\sigma}{\sigma}}\,dz + \int_{Q_3^+} |\mathbf{A} - \overline{\mathbf{A}}_{Q_4'}(x_1)|^{2\frac{1+\sigma}{\sigma}}\,dz\right)^{\frac{\sigma}{1+\sigma}} \\ &\quad \le c\left(\frac{|K_4 \setminus Q_4^+|}{|Q_4^+|} + \fint_{Q_4^+} |\mathbf{A} - \overline{\mathbf{A}}_{Q_4'}(x_1)|\,dz\right)^{\frac{\sigma}{1+\sigma}} \\ &\quad \le c\delta^{\frac{\sigma}{1+\sigma}}. \end{aligned} \qquad (3.2.19)$$

We then combine (3.2.17) with (3.2.19) to derive

$$\fint_{K_3} |Du - Dv|^2\,dz \le c\left(\delta + \delta^{\frac{\sigma}{1+\sigma}}\right) \le \epsilon, \quad \text{and so} \quad \fint_{K_3} |Dv|^2\,dz \le c,$$

by choosing a sufficiently small δ so that the last inequality holds true. This completes the proof. □

We next discuss a local Lipschitz regularity for a fixed limiting problem with x_1-dependent coefficients on the flat boundary.

Lemma 3.2.8. *For any $\epsilon \in (0,1)$ there exists a small $\delta = \delta(\epsilon, n, \nu, \Lambda) > 0$ such that for any weak solution $v \in C^0(-3^2, 3^2; L^2(\Omega_3)) \cap L^2(-3^2, 3^2; H^1(\Omega_3))$ of*

$$\begin{cases} v_t - div\left(\overline{\mathbf{A}}_{Q_4'}(x_1)Dv\right) &= 0 \quad in \quad K_3, \\ v &= 0 \quad on \quad \partial_w K_3, \end{cases}$$

with

$$C_3^+ \subset \Omega_3 \subset C_3 \cap \{x_1 > -8\delta\}$$

and

$$\fint_{K_3} |Dv|^2 \, dz \leq \tilde{c},$$

for some $\tilde{c} > 0$, there exists a weak solution $\overline{v} \in C^0(-2^2, 2^2; L^2(C_2^+)) \cap L^2(-2^2, 2^2; H^1(C_2^+))$ of

$$\begin{cases} \overline{v}_t - div\left(\overline{\mathbf{A}}_{Q_4'}(x_1)D\overline{v}\right) &= 0 \quad in \quad Q_2^+, \\ \overline{v} &= 0 \quad on \quad Q_2 \cap \{x_1 = 0\} \end{cases} \tag{3.2.20}$$

such that

$$\fint_{K_2} |Dv - D\overline{v}|^2 \, dz \leq \epsilon\tilde{c}, \tag{3.2.21}$$

where we extend $\overline{v}$ from Q_2^+ to Q_2 by zero.

Proof. We prove this lemma by contradiction. If not, there exists $\epsilon_0 > 0$ so that for each sufficiently large $l \in \mathbb{N}$, there exist Ω^l and v_l such that

$$C_3^+ \subset \Omega_3^l \subset C_3 \cap \left\{x_1 > -\frac{8}{l}\right\}, \tag{3.2.22}$$

$v_l \in C^0(-3^2, 3^2; L^2(\Omega_3^l)) \cap L^2(-3^2, 3^2; H^1(\Omega_3^l))$ is a weak solution of

$$\begin{cases} (v_l)_t - \text{div } \left(\overline{\mathbf{A}}_{Q_4'}(x_1)Dv_l\right) &= 0 \quad \text{in} \quad K_3^l = \Omega_3^l \times (-3^2, 3^2), \\ v_l &= 0 \quad \text{on} \quad \partial_w K_3^l \end{cases} \tag{3.2.23}$$

and

$$\fint_{K_3^l} |Dv_l|^2 \, dz \leq \tilde{c}, \tag{3.2.24}$$

but

$$\fint_{K_2^l} |Dv_l - D\overline{v}|^2 \; dz > \epsilon_0 \tilde{c}, \tag{3.2.25}$$

for every weak solutions $\overline{v}$ of (3.2.20). In view of (3.2.22)-(3.2.24), we observe

$$\int_{Q_3^+} |Dv_l|^2 \, dz \leq \int_{K_3^l} |Dv_l|^2 \, dz \leq \tilde{c}.$$

Applying Poincaré's inequality to v_l for each time slice, we find

$$\int_{Q_3^+} |v_l|^2 \, dz \leq \int_{K_3^l} |v_l|^2 \, dz \leq c \int_{K_3^l} |Dv_l|^2 \, dz \leq c\tilde{c}.$$

From Section 2.3., we know $v_l \in W_*^{1,2}(Q_3^+)$, and by the above inequalities $\{v_l\}$ is uniformly bounded in $W_*^{1,2}(Q_3^+)$, hence there exists a function $v_0 \in W_*^{1,2}(Q_3^+)$ such that

$$\left\{ \begin{array}{ccccc} v_l & \rightarrow & v_0 & \text{strongly in} & L^2(Q_3^+) \\ Dv_l & \rightharpoonup & Dv_0 & \text{weakly in} & L^2(Q_3^+) \\ (v_l)_t & \rightharpoonup & (v_0)_t & \text{weakly}^* \text{ in} & L^2(0,T;H^{-1}(B_3^+)) \end{array} \right\} \text{ as } l \rightarrow \infty. \tag{3.2.26}$$

Note that the strong convergence in $L^2(Q_3^+)$ can be obtained by Aubin-Lions Theorem, see Proposition 1.3 in [56], which is that $W_*^{1,2}(\Omega_T)$ is compactly embedded in $L^2(\Omega_T)$. Needless to say, this v_0 is a weak solution of (3.2.20). We next extend v_l and v_0 to Q_3 by zero to claim that

$$Dv_l \;\longrightarrow\; Dv_0 \;\text{ as }\; l \rightarrow \infty \;\text{ strongly in } L^2(Q_2). \tag{3.2.27}$$

But then this is contradict to (3.2.25). Thus it remains to show the above claim. To this end, select a standard cut off function $\phi = \phi(x) \in C_0^\infty(C_{\frac{5}{2}})$ satisfying

$$0 \leq \phi \leq 1, \quad \phi \equiv 1 \text{ in } C_2, \quad |D\phi| \leq c,$$

and a function $\rho = \rho(t) \in C_0^\infty((-\left(5/2\right)^2, \left(5/2\right)^2))$ satisfying

$$0 \leq \rho \leq 1, \; \phi \equiv 1 \text{ in } (-2^2, 2^2), \; |\rho'| \leq \frac{8}{9}.$$

Observe that for almost everywhere $t \in [-3^2, 3^2]$, $\phi(\cdot)\rho(t)(v_l(\cdot,t) - v_0(\cdot,t)) \in H_0^1(\Omega_3^l)$, and so we use this one as a test function in the weak formulations like (3.2.8) of the equations (3.2.23), to discover that

$$\int_{\Omega_3^l} \overline{\mathbf{A}}(x_n) Dv_l(x,t) D\left(\phi(x)\rho(t)(v_l(x,t) - v_0(x,t))\right) dx = \langle (v_l)_t(\cdot,t), \phi(\cdot)\rho(t)(v_l(\cdot,t) - v_0(\cdot,t)) \rangle$$

for almost every $t \in (-3^2, 3^2)$ and for all sufficiently large l. Integrating the above equality for t, we obtain

$$\begin{aligned}
I_1 &:= \int_{Q_3} \overline{\mathbf{A}}_{Q_4'}(x_1) D(v_l - v_0) \cdot D(v_l - v_0) \phi\rho \, dz \\
&= \int_{K_3^l} \overline{\mathbf{A}}_{Q_4'}(x_1) Dv_l D(\phi\rho(v_l - v_0)) \, dz - \int_{K_3^l} \overline{\mathbf{A}}_{Q_4'}(x_1) Dv_0 D(\phi\rho(v_l - v_0)) \, dz \\
&\quad - \int_{K_3^l} \overline{\mathbf{A}}_{Q_4'}(x_1) D(v_l - v_0) D\phi \, (v_l - v_0)\rho \, dz \\
&= \int_{-3^2}^{3^2} \langle (v_l)_t, \phi\rho(v_l - v_0) \rangle \, dt - \int_{K_3^l} \overline{\mathbf{A}}_{Q_4'}(x_1) Dv_0 D(\phi\rho(v_l - v_0)) \, dz \\
&\quad - \int_{K_3^l} \overline{\mathbf{A}}_{Q_4'}(x_1) D(v_l - v_0) D\phi \, (v_l - v_0)\rho \, dz \\
&= \int_{-3^2}^{3^2} \langle (v_l)_t, \phi\rho v_l \rangle \, dt - \int_{-3^2}^{3^2} \langle (v_l)_t - (v_0)_t, \phi\rho v_0 \rangle \, dt - \int_{-3^2}^{3^2} \langle (v_0)_t, \phi\rho v_0 \rangle \, dt \\
&\quad - \int_{Q_3^+} \overline{\mathbf{A}}_{Q_4'}(x_1) Dv_0 D(\phi\rho(v_l - v_0)) \, dz \\
&\quad - \int_{K_3^l} \overline{\mathbf{A}}_{Q_4'}(x_1) D(v_l - v_0) D\phi \, (v_l - v_0)\rho \, dz \\
&=: I_2 - I_3 - I_4 - I_5 - I_6.
\end{aligned}$$

Here, we note that the underlying domain of $\langle , \rangle$ in I_2, respectively I_3 and I_4, is Ω_3^l, respectively B_3^+. From (3.2.3), we have

$$I_1 \geq \nu \int_{Q_2} |Dv_l - Dv_0|^2 \, dz.$$

From (3.2.26), we see that $I_3, I_5 \to 0$ as $l \to \infty$. To estimate I_2, for $0 < h < 1$, we consider v_l^h, the Steklov average of v_l, for a sufficiently small $h > 0$. We then recall $\rho(t) = 0$ in $-3^2 \leq t \leq -\frac{5^2}{2^2}$ and $\frac{5^2}{2^2} \leq t \leq 3^2$, and use the definition of a weak solution involving Steklov average, $(v_l)_t$, to discover that

$$\begin{aligned} I_2 &= -\int_{-3^2}^{3^2}\int_{\Omega_3^l} \overline{\mathbf{A}} Dv_l D(\phi\rho v_l)\,dz \\ &= -\lim_{h\to 0}\int_{-3^2}^{3^2}\int_{\Omega_3^l} (\overline{\mathbf{A}} Dv_l)^h D(\phi\rho v_l^h)\,dz = \lim_{h\to 0}\int_{-3^2}^{3^2}\int_{\Omega_3^l} v_l^h \partial_t(\phi\rho v_l^h)\,dz \end{aligned}$$

and that

$$\int_{-3^2}^{3^2}\int_{\Omega_3^l} v_l^h \partial_t(\phi\rho v_l^h)\,dz = \frac{1}{2}\int_{-3^2}^{3^2}\int_{\Omega_3^l} (v_l^h)^2\phi\rho'\,dz.$$

Therefore, by sending h to zero, we have

$$I_2 = \frac{1}{2}\int_{-3^2}^{3^2}\int_{\Omega_3^l} (v_l)^2\phi\rho'\,dz = \frac{1}{2}\int_{K_3^l} (v_l)^2\phi\rho'\,dz.$$

Similarly, we also obtain

$$I_4 = \frac{1}{2}\int_{-3^2}^{3^2}\int_{C_3^+} (v_0)^2\phi\rho'\,dz = \frac{1}{2}\int_{Q_3^+} (v_0)^2\phi\rho'\,dz.$$

Thus

$$I_2 - I_4 = \frac{1}{2}\left\{\int_{Q_3^+} \left((v_l)^2 - (v_0)^2\right)\phi\rho'\,dz + \int_{K_3^l\setminus Q_3^+} (v_l)^2\phi\rho'\,dz\right\}.$$

From the first convergence in (3.2.26), the first integral on right-hand side goes to zero as l goes to infinity. To estimate the second term, we note from higher integrability for (3.2.23) that $|Dv_l| \in L^{2(1+\sigma)}(K_{\frac{5}{2}}^2)$, where σ is independent of l. We then use Poincaré's inequality and (3.2.24) to find that

$$\int_{K_{\frac{5}{2}}^l} (v_l)^{2(1+\sigma)} dz \leq c\tilde{c}.$$

Thus,

$$\begin{aligned}\left|\int_{K_3^l\setminus Q_3^+}(v_l)^2\phi\rho'\,dz\right| &\le c\int_{K_{\frac52}^l\setminus Q_{\frac52}^+}(v_l)^2\,dz\\ &\le c\left(\int_{K_{\frac52}^l}(v_l)^{2(1+\sigma)}\,dz\right)^{\frac{1}{1+\sigma}}\left|K_{\frac52}^l\setminus Q_{\frac52}^+\right|^{\frac{\sigma}{1+\sigma}}\\ &\to 0 \text{ as } l\to\infty \text{ by (3.2.22)},\end{aligned}\tag{3.2.28}$$

and so $I_2 - I_4 \to 0$ as $l \to 0$.

We now estimate I_6.

$$\begin{aligned}|I_6| &\le \left|\int_{Q_3^+}\overline{\mathbf{A}}_{Q_4'}(x_1)D(v_l-v_0)D\phi\ (v_l-v_0)\rho\,dz\right| + c\int_{K_{\frac52}^l\setminus Q_{\frac52}^+}|Dv_l||v_l|\,dz\\ &=: I_{6a}+I_{6b}.\end{aligned}$$

In view of (3.2.26), I_{6a} goes to zero as l goes to infinity. We use Hölder's inequality, higher integrability, Poincaré's inequality and (3.2.24), to discover

$$\begin{aligned}I_{6b} &\le \left(\int_{K_{\frac52}^l\setminus Q_{\frac52}^+}|Dv_l|^{2(1+\sigma)}dz\right)^{\frac{1}{2(1+\sigma)}}\left(\int_{K_{\frac52}^l\setminus Q_{\frac52}^+}|v_l|^2dz\right)^{\frac12}\left|K_{\frac52}^l\setminus Q_{\frac52}^+\right|^{\frac{\sigma}{1+\sigma}}\\ &\le c\left|K_{\frac52}^l\setminus Q_{\frac52}^+\right|^{\frac{\sigma}{1+\sigma}} \to 0 \text{ as } l\to\infty.\end{aligned}$$

Consequently, we conclude that

$$\lim_{l\to\infty}\int_{Q_2}|Dv_l-Dv_0|^2\,dz = 0,$$

which is (3.2.27). □

From Lemma 3.2.7 and 3.2.8, we also have

$$⨍_{Q_2^+}|D\overline{v}|^2\,dz \le 4\left(⨍_{K_2}|D\overline{v}-Dv|^2\,dz + ⨍_{K_2}|Dv|^2\,dz\right)\le c. \tag{3.2.29}$$

The next lemma give Lipschitz regularity of weak solutions for limiting equations. For its proof, we refer to [11].

Lemma 3.2.9.

1. *Interior case: If v is a weak solution of*

$$v_t - div\left(\overline{\mathbf{A}}_{Q_4'}(x_1)Dv\right) = 0 \text{ in } Q_2,$$

then there holds $Dv \in L^\infty(Q_1)$ and we have the estimate

$$\|Dv\|_{L^\infty}(Q_1) \le c \fint_{Q_2} |Dv|^2\, dz.$$

2. *Boundary case: If $\overline{v}$ is a weak solution of*

$$\begin{cases} \overline{v}_t - div\left(\overline{\mathbf{A}}_{Q_4'}(x_1)D\overline{v}\right) &= 0 \quad in \quad Q_2^+, \\ \overline{v} &= 0 \quad on \quad Q_2 \cap \{x_1 = 0\}, \end{cases}$$

then there holds and we have $D\overline{v} \in L^\infty(Q_1^+)$ the estimate

$$\|D\overline{v}\|_{L^\infty}(Q_1^+) \le c \fint_{Q_2^+} |D\overline{v}|^2\, dz.$$

We end this section with the following comparison estimates.

Lemma 3.2.10. *Under the assumptions and conclusions of Lemma 3.2.7, 3.2.8 and 3.2.9, we find*

1. *Interior case:*

$$\|Dv\|_{L^\infty}(Q_1) \le c \text{ and } \fint_{Q_1} |Du - Dv|^2\, dz \le \epsilon.$$

2. *Boundary case:*

$$\|D\overline{v}\|_{L^\infty}(K_1) \le c \text{ and } \fint_{K_1} |Du - D\overline{v}|^2\, dz \le \epsilon.$$

Approximation.

In this subsection we will show that Theorem 3.2.4 is proved if we once obtain theorem 3.2.4 for any weak solutions of (3.2.1) with $Du \in L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)$. This can be done by an approximation procedure.

To do this, we first assume that we already have estimate (3.2.14) for any weak solution of (3.2.1) with $Du \in L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)$ under a sufficiently small $\delta_1 > 0$. Note that, from the previous result in [11], we know that there exists $\delta_2 = \delta_2(n,\nu,\Lambda,\gamma_2)$ satisfying Theorem 3.2.4 for the particular case $p(\cdot) = \gamma_2$.

We next fix $\delta = \min\{\delta_1,\delta_2\}$, suppose $F \in L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)$, $(\mathbf{A},\Omega\times\mathbb{R})$ is (δ,R_0)-*vanishing codimension 1* and $u \in W^{1,2}_*(\Omega_T)$ is the weak solution of (3.2.1). Choose $F_k \in C^\infty(\Omega_T;\mathbb{R}^n)$, $k = 1,2,\dots$, such that

$$F_k \longrightarrow F \text{ as } k\to\infty \text{ in } L^{p(\cdot)}(\Omega_T;\mathbb{R}^n).$$

Let $u_k \in C^0(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ be the weak solution of

$$\begin{cases} (u_k)_t - \operatorname{div}(\mathbf{A}(z)Du_k) = \operatorname{div} F_k & \text{in } \Omega_T, \\ u_k = 0 & \text{on } \partial_p\Omega_T. \end{cases}$$

Since $F_k \in C^\infty(\Omega_T;\mathbb{R}^n) \subset L^{\gamma_2}(\Omega_T;\mathbb{R}^n) \subset L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)$, it follows from [11] that $Du_k \in L^{\gamma_2}(\Omega_T;\mathbb{R}^n) \subset L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)$. Therefore, by the a priori assumption, we know that

$$\|Du_k\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)} \le c\|F_k\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)},$$

and so

$$\|Du_k\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)} \le c\|F_k\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)} \le c\left(\|F\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)} + 1\right),$$

where c is independent of k. In view of Remark 3.2.5, $\{u_k\}$ is uniformly bounded in $W^{1,p(\cdot)}_*(\Omega_T)$ therefore there exists $u_0 \in W^{1,p(\cdot)}_*(\Omega_T)$ such that

$$u \rightharpoonup u_0 \text{ weakly in } W^{1,p(\cdot)}_*(\Omega_T) \subset W^{1,2}_*(\Omega_T)$$

as $k\to\infty$ up to a subsequence. In particular, Du_k is weakly convergent to Du_0 in $L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)$. The weak convergence of u_k in $W^{1,2}_*(\Omega_T)$ and the strong convergence of F_k in $L^2(\Omega_T)$ imply that u_0 is the weak solution of

(3.2.1), and so we conclude $u_0 = u$ from the uniqueness of the weak solution to (3.2.1). On the other hand, it follows from the weak convergence of Du_k in $L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)$ that

$$\|Du\|_{L^{p(\cdot)}(\Omega_T)} \leq \liminf_{k\to\infty} \|Du_k\|_{L^{p(\cdot)}(\Omega_T)} \leq \|F\|_{L^{p(\cdot)}(\Omega_T)}.$$

The a priori estimates.

We shall derive estimate (3.2.13) for the weak solution u of (3.2.1) under the a priori assumption

$$\int_{\Omega_T} |Du|^{p(z)} dz < \infty. \tag{3.2.30}$$

In particular, we treat only the case that

$$\|F\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)} \leq 1 \left(\iff \int_{\Omega_T} F^{p(z)} dz \leq 1 \right), \tag{3.2.31}$$

where the above equivalence relation comes from the Norm-modular unit ball property, see [25]. We then derive that

$$\int_{\Omega_T} |Du|^{p(z)} dz \leq c, \tag{3.2.32}$$

for some $c = c(n,\nu,\Lambda,\gamma_1,\gamma_2,\omega(\cdot),\Omega,T,R_0) > 1$ under a sufficiently small $\delta = \delta(n,\nu,\Lambda,\gamma_1,\gamma_2,\omega(\cdot)) > 0$. The general case for the estimate (3.2.13) directly comes from the normalization property of the problem (3.2.1). Then, we observe from (3.2.31) and (3.2.5) that

$$\int_{\Omega_T} |F|^2 dz \leq 1 + |\Omega_T| \text{ and so } \int_{\Omega_T} |Du|^2 dz \leq c_0(1 + |\Omega_T|). \tag{3.2.33}$$

We will obtain local estimates up to the lateral boundary. To do this, take $R > 0$ such that

$$R = \min\left\{ \frac{1}{4}, \frac{R_0}{4}, \frac{1}{1+|\Omega_T|} \right\}. \tag{3.2.34}$$

Choose a point $\zeta^0 = (y^0, s^0) \in \overline{\Omega} \times (0,T)$ satisfying $(s^0, -(2R)^2, s^0 + (2R)^2) \subset (0,T)$ and consider the region $K_{2R}(\zeta^0) \subset \Omega_T$.

For the sake of simplicity, we omit the center point to write $K_r(\zeta^0) = K_r$. We consider

$$p^+ := \sup_{z \in K_{2R}} p(z), \ p^- := \inf_{z \in K_{2R}} p(z)$$

and

$$\lambda_0 := \fint_{K_{2R}} \left[|Du|^{2\frac{p(z)}{p^-}} + \frac{1}{\delta} \left(|F|^{2\frac{p(z)}{p^-}} + 1 \right) \right] dz, \tag{3.2.35}$$

which is well-defined from (3.2.30), where $\delta \in \left(0, \frac{1}{8}\right)$ is to be determined later in a universal way.

We next fix $s_1$and s_2 with $1 \leq s_1 < s_2 \leq 2$ and consider $\lambda > 1$ satisfying

$$\lambda > \left(\frac{2180}{s_2 - s_1} \right)^{n+2} \left(\frac{16\sqrt{2}}{7} \right)^n \lambda_0 =: B\lambda_0, \quad B := \left(\frac{2180}{s_2 - s_1} \right)^{n+2} \left(\frac{16\sqrt{2}}{7} \right)^n \tag{3.2.36}$$

and the supper-level set

$$E(\lambda) := \left\{ z \in K_{s_1 R} : |Du|^{2\frac{p(z)}{p^-}} > \lambda \right\}.$$

The following covering lemma is a primary technical tool in our approach.

Lemma 3.2.11. *For each $\lambda > B\lambda_0$, there exist countable disjoint parabolic cylinders $\{Q_{r_i}(\zeta^i)\}_{i=1}^{\infty}$ with $\zeta^i = (y^i, s^i) \in E(\lambda)$ and $0 < r_i < \frac{s_2 - s_1}{1090} R$ such that*

$$E(\lambda) \ \subset \ \bigcup_{i=0}^{\infty} K_{5r_i}(\zeta^i)$$

(except a measure zero set),

$$\fint_{K_{r_i}(\zeta^i)} \left(|Du|^{2\frac{p(z)}{p^-}} + \frac{1}{\delta} |F|^{2\frac{p(z)}{p^-}} \right) dz = \lambda \tag{3.2.37}$$

and

$$\fint_{K_r(\zeta^i)} \left(|Du|^{2\frac{p(z)}{p^-}} + \frac{1}{\delta} |F|^{2\frac{p(z)}{p^-}} \right) dz < \lambda \ \ \textit{for every } r \in (r_i, (s_2 - s_1)R] \, . \tag{3.2.38}$$

Proof. We first observe that for almost every $\zeta \in E(\lambda)$

$$\lim_{r\to 0+} \fint_{K_r(\zeta)} \left(|Du|^{2\frac{p(z)}{p^-}} + \frac{1}{\delta}|F|^{2\frac{p(z)}{p^-}} \right) dz \geq |Du(\zeta)|^{2\frac{p(\zeta)}{p^-}} > \lambda.$$

Indeed, if r is sufficiently small, then one can apply Lebesgue differentiation theorem. On the other hand, for every $\zeta \in E(\lambda)$ and $r \in \left[\frac{s_2-s_1}{1090}R, (s_2-s_1)R\right]$, we see that $K_r(\zeta) \subset K_{2R}$. Then it follows from (3.2.9) and (3.2.36) that

$$\begin{aligned}
\fint_{K_r(\zeta)} \left(|Du|^{2\frac{p(z)}{p^-}} + \frac{1}{\delta}|F|^{2\frac{p(z)}{p^-}} \right) dz &\leq \frac{|Q_{2R}|}{|K_r(\zeta)|} \fint_{K_{2R}} \left(|Du|^{2\frac{p(z)}{p^-}} + \frac{1}{\delta}|F|^{2\frac{p(z)}{p^-}} \right) dz \\
&\leq \left(\frac{2180}{s_2-s_1} \right)^{n+2} \frac{|Q_r(\zeta)|}{|K_r(\zeta)|} \lambda_0 \\
&\leq \left(\frac{2180}{s_2-s_1} \right)^{n+2} \left(\frac{16\sqrt{2}}{7} \right)^{n} \lambda_0 < \lambda.
\end{aligned}$$

Consequently, we conclude that for almost every $\zeta \in E(\lambda)$, there exists $r_\zeta = r(\zeta) \in \left(0, \frac{s_2-s_1}{1090}R\right)$ such that

$$\fint_{K_{r_\zeta}(\zeta)} \left(|Du|^{2\frac{p(z)}{p_-}} + \frac{1}{\delta}|F|^{2\frac{p(z)}{p_-}} \right) dz = \lambda$$

and

$$\fint_{K_r(\zeta)} \left(|Du|^{2\frac{p(z)}{p_-}} + \frac{1}{\delta}|F|^{2\frac{p(z)}{p_-}} \right) dz, < \lambda \quad \text{for every } r \in (r_\zeta, (s_2-s_1)R]\,.$$

Applying the Vitali covering lemma to $\{Q_{r_\zeta}(\zeta)\}$, we finish the proof. □

Remark 3.2.12. *Owing to (3.2.37), we have*

$$\fint_{K_{r_i}(\zeta^i)} |Du|^{2\frac{p(z)}{p^-}} \, dz \geq \frac{\lambda}{2} \quad \text{or} \quad \frac{1}{\delta}\fint_{K_{r_i}(\zeta^i)} |F|^{2\frac{p(z)}{p^-}} \, dz \geq \frac{\lambda}{2},$$

and therefore

$$\begin{aligned}
|K_{r_i}(\zeta^i)| \leq \frac{4}{\lambda} \Bigg(&\int_{\left\{z\in K_{r_i}(\zeta^i) : |Du|^{2\frac{p(z)}{p_-}} > \frac{\lambda}{4}\right\}} |Du|^{2\frac{p(z)}{p_-}} \, dz \\
&+ \frac{1}{\delta} \int_{\left\{z\in K_{r_i}(\zeta^i) : |F|^{2\frac{p(z)}{p_-}} > \frac{\lambda\delta}{4}\right\}} |F|^{2\frac{p(z)}{p_-}} \, dz \Bigg) \qquad (3.2.39)
\end{aligned}$$

In what follows, we continue to consider two cases, **(1) Interior case** $\mathbf{B_{40r_i}(y^i) \subset \Omega}$ and **(2) Boundary case** $\mathbf{B_{40r_i}(y^i) \not\subset \Omega}$.

(1) Interior case $\mathbf{B_{40r_i}(y^i) \subset \Omega}$

We recall (A, Ω) is (δ, R_0)-vanishing codimension 1 and $40r_i \leq \frac{40(s_2-s_1)}{1090} R \leq R_0$ to discover from Definition 3.2.2 (1) that there exists a new coordinate system, $\bar{z} = (\bar{x}, \bar{t})$, with the origin at $\zeta^i = (y^i, s^i)$, such that

$$\fint_{Q_{20\sqrt{2}r_i}} \left| \mathbf{A}(\bar{z}) - \overline{\mathbf{A}}_{Q'_{20\sqrt{2}r_i}}(\bar{x}_1) \right| d\bar{z} \leq \delta. \tag{3.2.40}$$

In this coordinate system, we write

$$Q_i^1 := Q_{5\sqrt{2}r_i},\ Q_i^2 := Q_{20\sqrt{2}r_i},\ p_i^+ := \sup_{\bar{z} \in Q_i^2} p(\bar{z}) \text{ and } p_i^- := \inf_{\bar{z} \in Q_i^2} p(\bar{z})$$

and consider a localized equation of (3.2.1) in Q_i^2 like

$$u_{\bar{t}}(\bar{z}) - \operatorname{div}_{\bar{x}} \left(\mathbf{A}(\bar{z}) D_{\bar{x}} u(\bar{z}) \right) = \operatorname{div}_{\bar{x}} F(\bar{z}) \ \text{ in } Q_i^2, \tag{3.2.41}$$

Here, for simplicity, we abbreviate $\mathbf{A}(\Phi(\bar{z}))$, $u(\Phi(\bar{z}))$, $p(\Phi(\bar{z}))$ and $F(\Phi(\bar{z}))D\Phi(\bar{z})$ to $\mathbf{A}(\bar{z})$, $u(\bar{z})$, $p(\bar{z})$ and $F(\bar{z})$, respectively, where Φ is a mapping obtained by a proper translation and rotation from $\bar{z}$-coordinates to z-coordinates. We then point out that $\mathbf{A}(\bar{z})$ satisfies the addressed assumptions with replaced z by $\bar{z}$. We also point out from (3.2.11) that

$$p_i^+ - p_i^- \leq \sup_{z \in Q_{40r_i}(\zeta^i)} p(z) - \inf_{z \in Q_{40r_i}(\zeta^i)} p(z) \leq \omega(2 \cdot 40r_i) \text{ and } 80r_i \leq 2R.$$

Now according to (3.2.38) and the change of variable from $\bar{z} = (\bar{x}, \bar{t})$ to $z = (x, t)$, we find

$$\begin{aligned}
\fint_{Q_i^2} |Du(\bar{z})|^{2\frac{p(\bar{z})}{p^-}} d\bar{z} &= \frac{1}{|Q_{20\sqrt{2}r_i}|} \int_{-(20\sqrt{2}r_i)^2}^{(20\sqrt{2}r_i)^2} \int_{C_{20\sqrt{2}r_i}} |Du(\bar{x}, \bar{t})|^{\frac{p(\bar{x},\bar{t})}{p^-}} d\bar{x}\, d\bar{t} \\
&\leq \frac{1}{|Q_{20\sqrt{2}r_i}|} \int_{s^i-(40r_i)^2}^{s^i+(40r_i)^2} \int_{B_{40r_i}(y^i)} |Du(x,t)|^{\frac{p(x,t)}{p^-}} dxdt \\
&\leq \frac{|Q_{40r_i}(\zeta^i)|}{|Q_{20\sqrt{2}r_i}|} \fint_{K_{40r_i}(\zeta^i)} |Du|^{\frac{p(z)}{p^-}} dz \leq \sqrt{2}^{n+2} \lambda
\end{aligned} \tag{3.2.42}$$

and, likewise, we have

$$\fint_{Q_i^2} |F|^{2\frac{p(\bar{z})}{p_-}}\, d\bar{z} < \sqrt{2}^{n+2}\delta\lambda. \tag{3.2.43}$$

Then next claim that

$$\fint_{Q_i^2} |Du|^2\, d\bar{z} \le c_1 \lambda^{\frac{p^-}{p_i^+}} \text{ and } \fint_{Q_i^2} |F|^2\, d\bar{z} \le c_1 \delta^{\frac{\gamma_1}{\gamma_2}} \lambda^{\frac{p^-}{p_i^+}} \tag{3.2.44}$$

for some positive constant $c_1 > 1$ depending only $n, \nu, \Lambda, \gamma_1, \gamma_2$ and $\omega(\cdot)$. Indeed, we note that

$$\begin{aligned}\left(\fint_{Q_i^2} |Du|^2\, d\bar{z}\right)^{p_i^+ - p_i^-} &\le \left(\frac{1}{|Q_i^2|}\int_{\Omega_T} |Du|^2\, dz\right)^{p_i^+ - p_i^-} \\ &\overset{(3.2.33)}{\le} c\left(\frac{1+|\Omega_T|}{(20\sqrt{2}r_i)^{n+2}}\right)^{\omega(80r_i)} \overset{(3.2.34)}{\le} c\left(\frac{1}{R}\frac{1}{(20\sqrt{2}r_i)^{n+2}}\right)^{\omega(80r_i)} \\ &\le c\left(\frac{1}{80r_i}\right)^{(n+3)\omega(80r_i)} \overset{(3.2.12)}{\le} ce^{(n+3)\alpha_p} \le c.\end{aligned}$$

Here, we emphasize that constant c is independent of the index i. By the same reason, we have

$$\left(\fint_{Q_i^2} |F|^2\, d\bar{z}\right)^{p_i^+ - p_i^-} \le c\left(\frac{1+|\Omega_T|}{(20\sqrt{2}r_i)^{n+2}}\right)^{\omega(80r_i)} \le c.$$

In light of Jensen inequality and (3.2.42), we thus deduce

$$\begin{aligned}\fint_{Q_i^2} |Du|^2\, d\bar{z} &= \left(\fint_{Q_i^2} |Du|^2\, d\bar{z}\right)^{\frac{p_i^+ - p_i^-}{p_i^+}} \left(\fint_{Q_i^2} |Du|^2\, d\bar{z}\right)^{\frac{p_i^-}{p_i^+}} \\ &\le c\left(\fint_{Q_i^2} |Du|^{2\frac{p_i^-}{p^-}}\, d\bar{z}\right)^{\frac{p^-}{p_i^+}} \le c\left(\fint_{Q_i^2} |Du|^{2\frac{p(\bar{z})}{p_-}}\, d\bar{z} + 1\right)^{\frac{p_-}{p_2}} \\ &\le c\lambda^{\frac{p^-}{p_i^+}}.\end{aligned}$$

Similarly, we find that

$$\fint_{Q_i^2} |F|^2 \, d\bar{z} \le \left(\fint_{Q_i^2} |F|^{2\frac{p(\bar{z})}{p^-}} \, d\bar{z} + 1 \right)^{\frac{p^-}{p_i^+}} \le c(\delta\lambda)^{\frac{p^-}{p_i^+}} \le c\delta^{\frac{\gamma_1}{\gamma_2}} \lambda^{\frac{p^-}{p_i^+}},$$

where, we used the face $\delta\lambda > \delta\lambda_0 > 1$ from (3.2.35). This establishes (3.2.44).

Lemma 3.2.13. *For any $\epsilon > 0$, one can find a small $\delta = \delta(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), \epsilon) > 0$ and a function v_i with $Dv_i \in L^\infty(Q_i^1)$ such that*

$$\fint_{Q_i^1} |Du - Dv_i|^2 \, d\bar{z} \le \epsilon \lambda^{\frac{p^-}{p_i^+}} \tag{3.2.45}$$

and

$$\|Dv_i\|^2_{L^\infty(Q_i^1)} \le c_2 \lambda^{\frac{p^-}{p_i^+}}, \tag{3.2.46}$$

for some $c_2 > 1$ depending only on $n, \nu, \Lambda, \gamma_1, \gamma_2$ and $\omega(\cdot)$.

Proof. Let $\bar{x} := 5\sqrt{2} r_i \tilde{x}$, $\bar{t} := (5\sqrt{2} r_i)^2 \tilde{t}$ and $\tilde{z} := (\tilde{x}, \tilde{t})$. Define

$$\mathbf{A}_{i,\lambda}(\tilde{x}, \tilde{t}) = \mathbf{A}(5\sqrt{2} r_i \tilde{x}, (5\sqrt{2} r_i)^2 \tilde{t}) \text{ and } F_{i,\lambda}(\tilde{x}, \tilde{t}) = \frac{F(5\sqrt{2} r_i \tilde{x}, (5\sqrt{2} r_i)^2 \tilde{t})}{\sqrt{c_1 \lambda^{\frac{p^-}{p_i^+}}}}.$$

Then from (3.2.41) we see that

$$u_{i,\lambda}(\tilde{x}, \tilde{t}) = \frac{u(5\sqrt{2} r_i \tilde{x}, (5\sqrt{2} r_i)^2 \tilde{t})}{5\sqrt{2} r_i \sqrt{c_1 \lambda^{\frac{p^-}{p_i^+}}}} \in C^0(-4^2, 4^2; L^2(C_4)) \cap L^2(-4^2, 4^2; H^1(C_4))$$

is a weak solution of

$$(u_{i,\lambda})_{\tilde{t}} - \operatorname{div}_{\tilde{x}} (\mathbf{A}_{i,\lambda}(\tilde{z}) D u_{i,\lambda}) = \operatorname{div}_{\tilde{x}} F_{i,\lambda} \text{ in } Q_4.$$

We also check from (3.2.40) and (3.2.44) that

$$\fint_{Q_4} |\mathbf{A}_{i,\lambda}(\tilde{z}) - \overline{\mathbf{A}_{i,\lambda}}_4(\tilde{x}_1)| \, d\tilde{z} \le \delta, \ \fint_{Q_4} |Du_{i,\lambda}|^2 \, d\tilde{z} \le 1 \text{ and } \fint_{Q_4} |F_{i,\lambda}|^2 \, d\tilde{z} \le \delta^{\frac{\gamma_1}{\gamma_2}}.$$

We thus find by applying Lemma 3.2.10 (1) that there exists a small $\delta = \delta(n, \nu, \Lambda, \gamma_1, \gamma_2, \epsilon)$ and a function $v_{i,\lambda} \in C^0(-2^2, 2^2; L^2(C_2)) \cap L^2(-2^2, 2^2; H^1(C_2))$ such that

$$\fint_{Q_2} |Du_{i,\lambda} - Dv_{i,\lambda}|^2 \, d\tilde{z} \le \epsilon \text{ and } \|Dv_{i,\lambda}\|^2_{L^\infty(Q_1)} \le c.$$

Consequently, we get the conclusions after scaling reversely like

$$v_i(\bar{x}, \bar{t}) = 5\sqrt{2} r_i \sqrt{c_1 \lambda^{\frac{p_i^-}{p_i^+}}} \, v_{i,\lambda} \left(\frac{\bar{x}}{5\sqrt{2} r_i}, \frac{\bar{t}}{(5\sqrt{2} r_i)^2} \right).$$

This completes the proof. □

(2) boundary case $\mathbf{B_{40r_i}(y^i) \not\subset \Omega}$

In this case, we recall Definition 3.2.2 (2) to find that there exists a new coordinate system, $\bar{z} = (\bar{x}, \bar{t})$, with the origin at somewhere on $\partial B_{480\delta\sqrt{2}r_i}(y^{i,1}) \times \{s^i\}$, where $y^{i,1}$ is a point on $\partial\Omega \cap B_{40r_i}(y^i)$ satisfying $|y^{i,1} - y^i| = dist(y^i, \partial\Omega)$, so that

$$C^+_{480\sqrt{2}r_i} \subset C_{480\sqrt{2}r_i} \cap \Omega \subset C_{480\sqrt{2}r_i} \cap \left\{ \bar{x}_1 > -960\sqrt{2}\delta r_i \right\} \tag{3.2.47}$$

and

$$\fint_{Q^+_{480\sqrt{2}r_i}} |\mathbf{A}(\bar{z}) - \overline{\mathbf{A}}_{Q'_{480\sqrt{2}r_i}}(\bar{x}_1)| \, d\bar{z} \le \delta. \tag{3.2.48}$$

Like the interior case, we write

$$K_i^1 := K_{120\sqrt{2}r_i}, \; K_i^2 = K_{480\sqrt{2}r_i}, \; p_i^+ = \sup_{z \in K_i^2} p(\bar{z}) \text{ and } p_i^- = \inf_{z \in K_i^2} p(\bar{z})$$

and consider a localized problem of (3.2.1) near the lateral boundary

$$\begin{cases} u_{\bar{t}} - \operatorname{div}_{\bar{x}} (\mathbf{A}(\bar{z}) D_{\bar{x}} u(\bar{z})) = \operatorname{div}_{\bar{x}} F(\bar{z}) & \text{in} \quad K_i^2, \\ u = 0 & \text{on} \quad \partial_w K_i^2. \end{cases}$$

Let $\overline{y}^i$ be the changed point of y^i by the coordinate transformation and $\overline{\zeta}^i = (\overline{y}^i, 0)$. Then $|\overline{y}^i| \le 40r_i + 480\sqrt{2}\delta r_i \le (40 + 60\sqrt{2}) r_i$, and so

$$B_{40r_i}(\overline{y}^i) \subset B_{120\sqrt{2}r_i}(0) \subset C_{120\sqrt{2}r_i}(0) \text{ and } C_{480\sqrt{2}r_i}(0) \subset B_{960r_i}(0) \subset B_{1090r_i}(\overline{y}^i).$$

Consequently, $p_i^+ - p_i^- \le \omega(2 \cdot 1090 r_i)$ with $1090 r_i \le R$. By (3.2.38), (3.2.47) and the change of variables, we discover

$$\begin{aligned}
\fint_{K_i^2} |Du|^{2\frac{p(\bar z)}{p^-}}\, d\bar z &\le \frac{1}{|K_{480\sqrt{2}r_i}|} \int_{-(480\sqrt{2}r_i)^2}^{(480\sqrt{2}r_i)^2} \int_{B_{1090r_i}(\bar y^i)\cap\Omega} |Du|^{2\frac{p(\bar x,\bar t)}{p^-}}\, d\bar x\, d\bar t \\
&\le \frac{1}{|Q^+_{480\sqrt{2}r_i}|} \int_{s^i-(480\sqrt{2}r_i)^2}^{s^i+(480\sqrt{2}r_i)^2} \int_{B_{1090r_i}(y^i)\cap\Omega} |Du|^{2\frac{p(x,t)}{p^-}}\, dxdt \\
&\le 2\frac{|Q_{1090r_i}(\zeta^i)|}{|Q_{480\sqrt{2}r_i}|} \fint_{K_{1090r_i}(\zeta^i)} |Du|^{2\frac{p(z)}{p^-}}\, dz \le 2\left(\frac{109}{48\sqrt{2}}\right)^{n+2} \lambda.
\end{aligned}$$

Similarly, we have

$$\fint_{K_i^2} |F|^{2\frac{p(\bar z)}{p^-}}\, d\bar z \le 2\left(\frac{109}{48\sqrt{2}}\right)^{n+2} \delta\lambda.$$

We then in a similar way as in the interior case, we find after additionally using (3.2.47) that

$$\fint_{K_i^2} |Du|^2\, d\bar z \le c_3 \lambda^{\frac{p^-}{p_i^+}} \quad \text{and} \quad \fint_{K_i^2} |F|^2\, d\bar z \le c_3 \delta^{\frac{\gamma_1}{\gamma_2}} \lambda^{\frac{p^-}{p_i^+}}, \tag{3.2.49}$$

where the constant $c_3 > 1$ depends only on $n, \nu, \Lambda, \gamma_1, \gamma_2$ and $\omega(\cdot)$.

We now recall Lemma 3.2.10 (2) and use (3.2.49) via a proper scaling argument in the proof Lemma 3.2.13, to derive the following estimate.

Lemma 3.2.14. *For any $\epsilon > 0$, there exists $\delta = \delta(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), \epsilon) > 0$ and a function $\overline{v}_i$ with $Dv \in L^\infty(Q_i^{1+})$ and*

$$\overline{v}_i = 0 \ \text{on}\ T_{480r_i} = Q_{240\sqrt{2}r_i} \cap \{\bar x_1 = 0\},$$

in the trace sense, such that

$$\fint_{K_i^1} |Du - D\overline{v}_i|^2\, d\bar z \le \epsilon \lambda^{\frac{p^-}{p_i^+}} \tag{3.2.50}$$

and

$$\|D\overline{v}_i\|_{L^\infty(K_i^1)} = \|D\overline{v}_i\|_{L^\infty(Q_i^{1+})} \le c_4 \lambda^{\frac{p^-}{p_i^+}}, \tag{3.2.51}$$

where $\overline{v}_i$ is extended by zero from $Q^+_{480r_i}$ to $K_{480r_i} = K_i^2$ and the constant $c_4 > 1$ depends only on $n, \nu, \Lambda, \gamma_1, \gamma_2$ and $\omega(\cdot)$.

At the end, from the above Lemmas we derive the following estimate for the measure of the supper-level set $E(A\lambda)$, for some $A > 1$.

Lemma 3.2.15. *Let*

$$A := (4\max\{c_2, c_4\})^{\frac{\gamma_2}{\gamma_1}}. \tag{3.2.52}$$

Under the same assumptions and conclusions of Lemma 3.2.11, Lemma 3.2.13 and Lemma 3.2.14, we have that for any λ satisfying (3.2.36)

$$\begin{aligned} |E(A\lambda)| : &= \left|\left\{z \in K_{s_1R} : |Du|^{2\frac{p(z)}{p^-}} > A\lambda\right\}\right| \\ &\leq \frac{c\epsilon}{\lambda}\left(\iint_{\left\{z\in K_{s_2R}:|Du|^{2\frac{p(z)}{p^-}}>\frac{\lambda}{4}\right\}} |Du|^{2\frac{p(z)}{p^-}}dz \right. \\ &\qquad \left. +\frac{1}{\delta}\iint_{\left\{z\in K_{s_2R}:|F|^{2\frac{p(z)}{p^-}}>\frac{\lambda\delta}{4}\right\}} |F|^{2\frac{p(z)}{p^-}}dz\right), \end{aligned}$$

for some $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 1$.

Proof. From Lemma 3.2.11, we see that $E(A\lambda) \subset E(\lambda) \subset \bigcup_{i=1}^{\infty} K_{5r_i}(\zeta^i)$, and so follows that

$$|E(A\lambda)| \leq \sum_{i=1}^{\infty}\left|\left\{z \in K_{5r_i}(\zeta^i) : |Du(z)|^2 > (A\lambda)^{\frac{p^-}{p(z)}}\right\}\right|. \tag{3.2.53}$$

We first consider the case $B_{40r_i}(\zeta^i) \subset \Omega$ and recall the new coordinate system treated in **case (1)**. We write $\widetilde{Q}_i^1$ as the transformed one of $Q_{5r_i}(\zeta^i) = K_{5r_i}(\zeta^i)$. Then we observe that $\widetilde{Q}_i^1 \subset Q_i^1$ and $\sqrt{2}^{n+2}|\widetilde{Q}_i^1| = |Q_i^1|$. In the new coordinate system, by using (3.2.52) and the inequality that $|Du|^2 \leq 2|Du - Dv_i|^2 + 2|Dv_i|^2$, we estimate that

$$\begin{aligned} \left|\left\{\bar{z} \in Q_i^1 : |Du(\bar{z})|^2 > (A\lambda)^{\frac{p^-}{p(\bar{z})}}\right\}\right| &\leq \left|\left\{\bar{z} \in Q_i^1 : |Du(\bar{z})|^2 > (A\lambda)^{\frac{p^-}{p_i^+}}\right\}\right| \\ &\leq \left|\left\{\bar{z} \in Q_i^1 : |Du(\bar{z}) - Dv_i(\bar{z})|^2 > c_2\lambda^{\frac{p^-}{p_i^+}}\right\}\right| \\ &\quad + \left|\left\{\bar{z} \in Q_i^1 : |Dv_i|^2 > c_2\lambda^{\frac{p^-}{p_i^+}}\right\}\right|. \end{aligned}$$

According to (3.2.45) in Lemma 3.2.13, the second term on the right hand side of the previous inequality vanishes. So (3.2.46) in Lemma 3.2.13 imply that

$$\begin{aligned}\left|\left\{\bar{z}\in\widetilde{Q}_i^1:|Du(\bar{z})|^2>(A\lambda)^{\frac{p^-}{p(\bar{z})}}\right\}\right| &\le \frac{c}{\lambda^{\frac{p^-}{p_i^+}}}\int_{Q_i^1}|Du-Dv_i|^2d\bar{z}\le c\epsilon|Q_i^1|\\ &\le c\epsilon|\widetilde{Q}_i^1|.\end{aligned}$$

Therefore, in the original coordinate system, we discover

$$\left|\left\{z\in Q_{5r_i}(\zeta^i):|Du(z)|^2>(A\lambda)^{\frac{p^-}{p(z)}}\right\}\right|\le c\epsilon|Q_{5r_i}(\zeta^i)|\le c\epsilon|Q_{r_i}(\zeta^i)|. \tag{3.2.54}$$

On the other hand, if $B_{40r_i}\not\subset\Omega$, we apply (3.2.50) and (3.2.51), to find that

$$\left|\left\{\bar{z}\in\widetilde{K}_i^1:|Du(\bar{z})|^2>(A\lambda)^{\frac{p_-}{p(\bar{z})}}\right\}\right|\le c\epsilon|K_i^1|\le c\epsilon|\widetilde{K}_i^1|,$$

in the new coordinate system considered in **case (2)**, where $\widetilde{K}_i^1$ is the transformed one of $K_{5r_i}(\zeta^i)$ and we used the facts that $\widetilde{K}_i^1\subset K_i^1$ and $|K_i^1|\le c(n)|\widetilde{K}_i^1|$. Hence, in the original coordinate system, we have

$$\left|\left\{z\in K_{5r_i}(\zeta^i):|Du|^2>(A\lambda)^{\frac{p^-}{p(z)}}\right\}\right|\le c\epsilon|K_{5r_i}(\zeta^i)|\le c\epsilon|K_{r_i}(\zeta^i)|. \tag{3.2.55}$$

We now combine(3.2.53), (3.2.54) and (3.2.55) and recall (3.2.39), to derive

$$\begin{aligned}|E(A\lambda)| &\le c\epsilon\sum_{i=1}^{\infty}|K_{r_i}(\zeta^i)|\\ &\le \frac{c\epsilon}{\lambda}\sum_{i=1}^{\infty}\left(\int_{K_{r_i}(\zeta^i)\cap\left\{|Du|^{\frac{2p(z)}{p^-}}>\frac{\lambda}{4}\right\}}|Du|^{\frac{2p(z)}{p^-}}dz\right.\\ &\qquad\left.+\frac{1}{\delta}\int_{\Omega_{r_i}(\zeta^i)\cap\left\{|F|^{\frac{2p(z)}{p^-}}>\frac{\lambda\delta}{4}\right\}}|F|^{\frac{2p(z)}{p^-}}dz\right).\end{aligned}$$

The conclusion follows from the fact that $\{K_{r_i}(\zeta^i)\}_{i\ge1}$ are disjoint. □

We are ready to obtain the desired estimate (3.2.32).

Proof of (3.2.32). We first start with a local region $K_{s_1R} = K_{s_!R}(\zeta^0)$. Apply (2.3.1) to $f = |Du|^{2\frac{p(z)}{p^-}}$, $q = \frac{p^-}{2}$ and $U = K_{s_1R}$, to calculate as follows:

$$\begin{aligned}\int_{K_{s_1R}} |Du|^{p(z)} dz &= \int_0^\infty \frac{p^-}{2}(A\lambda)^{\frac{p^-}{2}-1} \left|\left\{z \in K_{s_1R} : |Du|^{\frac{2p(z)}{p^-}} > A\lambda\right\}\right| d(A\lambda) \\ &\leq (A \cdot B\lambda_0)^{\frac{p^-}{2}} |K_{s_1R}| + \int_{B\lambda_0}^\infty \frac{p^-}{2} A^{\frac{p^-}{2}} \lambda^{\frac{p^-}{2}-1} |E(A\lambda)|\, d\lambda \\ &=: I_2|K_{s_1R}| + I_3. \end{aligned} \tag{3.2.56}$$

We recall (3.2.35), (3.2.36) and (3.2.52) to calculate

$$\begin{aligned} I_2 &\leq \frac{c}{(s_2-s_1)^{\frac{p^-n}{2}}} \left\{ \fint_{K_{2R}} \left[|Du|^{\frac{2p(z)}{p^-}} + \frac{1}{\delta}\left(|F|^{\frac{2p(z)}{p^-}} + 1\right)\right] dz \right\}^{\frac{p^-}{2}} \\ &\leq \frac{c}{(s_2-s_1)^{\frac{\gamma_2 n}{2}}} \left\{ \left(\fint_{K_{2R}} |Du|^{\frac{2p^+}{p^-}} dz \right)^{\frac{p^-}{2}} + \frac{1}{\delta^{\frac{p^-}{2}}} \fint_{K_{2R}} \left(|F|^{p(z)} + 1\right) dz \right\}. \end{aligned}$$

We now impose R to satisfy

$$\omega(4\sqrt{2}R) \leq \sigma_0\gamma_1, \tag{3.2.57}$$

and so $\frac{p^+}{p^-} \leq 1 + \frac{p^+-p^-}{\gamma_1} \leq 1 + \sigma_0$. We then apply Lemma 3.2.6 for $\sigma = \frac{p^+}{p^-} - 1$, to discover

$$\left(\fint_{K_{2R}} |Du|^{\frac{2p^+}{p^-}} dz \right)^{\frac{p^-}{2}} \leq c \left\{ \left(\fint_{K_{4R}} |Du|^2 dz \right)^{\frac{p^+}{2}} + \left(\fint_{K_{4R}} |F|^{\frac{2p^+}{p^-}} dz \right)^{\frac{p^-}{2}} \right\}.$$

Since $\frac{2p^+}{p^-} \leq 2(1+\sigma_0) \leq \gamma_1 \leq p^-$, we apply Jensen inequality to the last term on the above inequality to find

$$\left(\fint_{K_{2R}} |Du|^{\frac{2p_+}{p_-}} dz \right)^{\frac{p_-}{2}} \leq c \left\{ \left(\fint_{K_{4R}} |Du|^2 dz \right)^{\frac{p_+}{2}} + \left(\fint_{K_{4R}} |F|^{p(z)} dz + 1 \right)^{\frac{p_+}{p_-}} \right\}.$$

From (3.2.9) we know that $\frac{|K_{4R}|}{|K_{2R}|} \leq \frac{2^{n+2}|Q_{2R}|}{|K_{2R}|} \leq c$. This and (3.2.33) imply that

$$\begin{aligned} I_2 &\leq \frac{c}{(s_2-s_1)^{\frac{\gamma_1 n}{2}}} \left\{ \left(\fint_{K_{4R}} |Du|^2 dz \right)^{\frac{p^+}{2}} + c(\delta) \left(\fint_{K_{4R}} |F|^{p(z)} dz + 1 \right)^{\frac{p^+}{p^-}} \right\} \\ &\leq \frac{c(\delta)}{(s_2-s_1)^{\frac{\gamma_1 n}{2}}} \left\{ \left(\frac{1+|\Omega_T|}{R^{n+2}} \right)^{\frac{p^+}{2}} + \left(\frac{1+|\Omega_T|}{R^{n+2}} \right)^{\frac{p^+}{p^-}} \right\} \\ &\leq \frac{c(\delta)}{(s_2-s_1)^{\frac{\gamma_1 n}{2}}} \left(\frac{1+|\Omega_T|}{R^{n+2}} \right)^{\frac{p^+}{2}}, \end{aligned} \tag{3.2.58}$$

for some constant $c(\delta)$ depending only on $n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)$ and δ.

On the other hand, using Lemma 3.2.15, it follows that

$$\begin{aligned} I_3 &= \int_{B\lambda_0}^{\infty} \frac{p_-}{2} A^{\frac{p_-}{2}} \lambda^{\frac{p_-}{2}-1} |E(A\lambda)| d\lambda \\ &\leq \frac{p_-}{2} A^{\frac{p_-}{2}} \int_0^{\infty} \lambda^{\frac{p_-}{2}-1} \frac{c\epsilon}{\lambda} \Bigg(\int_{\left\{z \in K_{s_2 R} : |Du|^{\frac{2p(z)}{p_-}} > \frac{\lambda}{4}\right\}} |Du|^{\frac{2p(z)}{p_-}} dz \\ &\quad + \frac{1}{\delta} \int_{\left\{z \in K_{s_2 R} : |F|^{\frac{2p(z)}{p_-}} > \frac{\lambda\delta}{4}\right\}} |F|^{\frac{2p(z)}{p_-}} dz \Bigg) d\lambda \\ &\leq c\epsilon A^{\frac{p_-}{2}} \Bigg[\int_0^{\infty} \left(\frac{\lambda}{4} \right)^{\frac{p_-}{2}-2} \int_{\left\{z \in K_{s_2 R} : |Du|^{\frac{2p(z)}{p_-}} > \frac{\lambda}{4}\right\}} |Du|^{\frac{2p(z)}{p_-}} dz \, d\left(\frac{\lambda}{4} \right) \\ &\quad + \frac{1}{\delta^{\frac{p_-}{2}}} \int_0^{\infty} \left(\frac{\lambda\delta}{4} \right)^{\frac{p_-}{2}-2} \int_{\left\{z \in K_{s_2 R} : |F|^{\frac{2p(z)}{p_-}} > \frac{\lambda\delta}{4}\right\}} |F|^{\frac{2p(z)}{p_-}} dz \, d\left(\frac{\lambda\delta}{4} \right) \Bigg]. \end{aligned}$$

Note that by (3.2.30) we know that $|Du| \in L^{2\frac{p(z)}{p_-}} \in L^{\frac{p_-}{2}}(K_{s_2 R})$. Then from (2.3.2) and (3.2.33), it follows that

$$\begin{aligned} I_3 &\leq c\epsilon A^{\frac{\gamma_2}{2}} \left(\int_{K_{s_2 R}} |Du|^{p(z)} dz + \frac{1}{\delta^{\frac{\gamma_2}{2}}} \int_{K_{s_2 R}} |F|^{p(z)} dz \right) \\ &\leq c_5 \epsilon \left(\int_{K_{s_2 R}} |Du|^{p(z)} dz + \frac{1}{\delta^{\frac{\gamma_2}{2}}} (1+|\Omega_T|) \right), \end{aligned} \tag{3.2.59}$$

for some positive constant $c_5 = c_5(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot))$.

We now select $\epsilon = \epsilon(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot))$ such that $c_5\epsilon \leq \frac{1}{2}$, and so find a corresponding $\delta = \delta(n, \nu, \Lambda, \gamma_1, \gamma_2, \omega(\cdot))$ from Lemma 3.2.15. We then combine (3.2.56), (3.2.58) and (3.2.59), to derive that

$$\begin{aligned}\int_{K_{s_1 R}} |Du|^{p(z)} dz \ \leq\ & \frac{1}{2}\int_{K_{s_2 R}} |Du|^{p(z)} dz + \frac{c}{(s_2 - s_1)^{\frac{\gamma_1 n}{2}}} \left(\frac{1 + |\Omega_T|}{R^{n+2}}\right)^{\frac{p^+}{2}} \\ & + c(1 + |\Omega_T|),\end{aligned}$$

for any s_1 and s_2 with $1 \leq s_1 < s_2 \leq 2$. We therefore apply Lemma 5.2.9 to $\phi(s) = \int_{\Omega_{sR}} |Du|^{p(z)} dz$, $r_1 = 1$ and $r_2 = 2$, to discover that

$$\int_{K_R} |Du|^{p(z)} dz \leq c \left(\frac{1 + |\Omega_T|}{R^{n+2}}\right)^{\frac{\gamma_2}{2}},$$

where R is determined from the restrictions (3.2.34) and (3.2.57). The final estimate (3.2.32) follows from standard covering argument and the following remark. □

Remark 3.2.16. *Since we have assumed that $K_{2R}(\zeta^0) \subset \Omega \times (0, T)$, the estimate can be obtained only in $\Omega \times (R, T - R)$. We therefore need to extend the equation (3.2.1) with respect to t-variable, in order to obtain the estimate in the hole region Ω_T. Indeed, we first let $F^* \in L^{p(\cdot)}(\Omega \times (-\infty, \infty); \mathbb{R}^n)$ be the vector valued function such that $F^* \equiv F$ in Ω_T and $F^* \equiv 0$ otherwise. We then consider a weak solution u^* of*

$$\begin{cases} u^*_t - div\,(\mathbf{A}(x,t) Du^*) = div\ F^* & in \quad \Omega \times (0, \infty), \\ u^* = 0 & in \quad \Omega \times (-\infty, 0], \\ u^* = 0 & on \quad \partial\Omega \times (-\infty, \infty). \end{cases}$$

Then by the uniqueness of weak solutions, we have that $u^* \equiv u$ in Ω_T. Applying the derived estimate to the above problem in $\Omega \times (-R, T + R)$, we conclude that

$$\|Du\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)} = \|Du^*\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)} \leq c\|F^*\|_{L^{p(\cdot)}(\Omega \times (-R, T+R);\mathbb{R}^n)}.$$

Since $\|F^*\|_{L^{p(\cdot)}(\Omega \times (-R,T+R);\mathbb{R}^n)} = \|F\|_{L^{p(\cdot)}(\Omega_T;\mathbb{R}^n)}$, we finally derive the required estimate (3.2.14) .

Chapter 4

Nonlinear elliptic equations with variable exponent growth in nonsmooth domains

Let $p(\cdot) = p(x) : \mathbb{R}^n \to (1, \infty)$, $n \geq 2$, be a given function satisfying

$$1 < \gamma_1 \leq p(x) \leq \gamma_2 < \infty, \tag{4.0.1}$$

for some constants γ_1, γ_2. The problem under consideration in this chapter is the following divergence type nonlinear elliptic equation with the zero boundary condition:

$$\begin{cases} \operatorname{div} \mathbf{a}(Du, x) = \operatorname{div} \left(|F|^{p(x)-2} F\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.0.2}$$

where Ω is a bounded open domain in $\mathbb{R}^n$, $\mathbf{a} = \mathbf{a}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function and $F = F(x) : \Omega \to \mathbb{R}^n$ is the nonhomogeneous term. We assume that $\mathbf{a}$ is differentiable with respect to ξ, measurable with respect to x and satisfies the following nonstandard growth and ellipticity conditions: there exist $0 < \nu \leq \Lambda < +\infty$ and $0 \leq \mu \leq 1$ such that

$$(\mu^2 + |\xi|^2)^{\frac{1}{2}} |D_\xi \mathbf{a}(\xi, x)| + |\mathbf{a}(\xi, x)| \leq \Lambda (\mu^2 + |\xi|^2)^{\frac{p(x)-1}{2}}, \tag{4.0.3}$$

$$\nu(\mu^2 + |\xi|^2)^{\frac{p(x)-2}{2}} |\eta|^2 \le \langle D_\xi \mathbf{a}(\xi, x)\eta, \eta\rangle, \tag{4.0.4}$$

whenever $x, \xi, \eta \in \mathbb{R}^n$(if $\mu = 0, \xi$ is selected in $\mathbb{R}^n \setminus \{0\}$). Here $D_\xi \mathbf{a} : \mathbb{R}^n \to \mathbb{R}^{n^2}$ is the gradient matrix of $\mathbf{a}$ with respect to ξ and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. A typical example is $p(x)$-*Laplace equation*

$$\Delta_{p(\cdot)} u := \text{div } \left(|Du|^{p(x)-2} Du\right) = 0 \tag{4.0.5}$$

for the case that $\mathbf{a}(\xi, x) = |\xi|^{p(x)-2}\xi$.

The objective of this chpter is to obtain a global Calderón-Zygmund type estimate for the problem (4.0.2) in the setting of variable exponent Lebesgue spaces. More precisely, we will prove that the following relation

$$|F|^{p(\cdot)} \in L^{q(\cdot)}(\Omega) \implies |Du|^{p(\cdot)} \in L^{q(\cdot)}(\Omega), \tag{4.0.6}$$

holds true for $q(\cdot) = q(x) : \Omega \to (1, \infty)$ satisfying

$$1 < \gamma_3 \le q(x) \le \gamma_4 < \infty, \tag{4.0.7}$$

for some constants γ_3 and γ_4, by essentially deriving the related estimate, see also Remark 4.1.9. We also present a reasonable answer as to what might be should regularity assumptions on $p(\cdot)$, $\mathbf{a}(\xi, \cdot)$ and the boundary of Ω for the relation (4.0.6) to be valid. As far as we are concerned, our regularity result reported here is the first one regarding Calderón-Zygmund type estimates for nonlinear problems with a variable growth in the frame of variable exponent Lebesgue spaces.

4.1 $W^{1,q(\cdot)}$-estimates for elliptic equations of $p(x)$-Laplacian type.

4.1.1 Main Result.

Definition 4.1.1. We say $u \in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution of (4.0.4) if

$$\int_\Omega \langle \mathbf{a}(Du, x), D\varphi\rangle \, dx \;=\; \int_\Omega \langle |F|^{p(x)-2}F, D\varphi\rangle \, dx, \quad \text{for all } \varphi \in W_0^{1,p(\cdot)}(\Omega).$$

We point out that from (4.0.4) one can derive the following monotonicity of $\mathbf{a}$:

$$\nu_1(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p(x)-2}{2}}|\xi - \eta|^2 \leq \langle \mathbf{a}(\xi, x) - \mathbf{a}(\eta, x), \xi - \eta \rangle, \tag{4.1.1}$$

for any $x, \xi, \eta \in \mathbb{R}^n$ and for some $\nu_1 = \nu_1(\nu, n, \gamma_1, \gamma_2)$. In particular, for the case $p(x) \geq 2$, it can be reduced to

$$2^{-\frac{\gamma_2 - 1}{2}}\nu_1|\xi - \eta|^{p(x)} \leq \langle \mathbf{a}(\xi, x) - \mathbf{a}(\eta, x), \xi - \eta \rangle. \tag{4.1.2}$$

By inserting 0 into η in (4.1.1) and using (4.0.3), we also have the following coercivity of $\mathbf{a}$:

$$\nu_2|\xi|^{p(x)} \leq \langle \mathbf{a}(\xi, x), \xi \rangle + \Lambda_1, \tag{4.1.3}$$

for any $x, \xi \in \mathbb{R}^n$ and for some ν_2 and Λ_1 depending only on $\Lambda, \nu, \gamma_1, \gamma_2$ and n.

By existence theory for nonlinear elliptic problems, see [56], it is well known that if $\mathbf{a}$ satisfies (4.1.1) and (4.1.3), $p(\cdot)$ is log-Hölder continuous and $F \in L^{p(\cdot)}(\Omega)$, then the problem (4.0.2) has a unique weak solution. Moreover, we have the estimate

$$\int_\Omega |Du|^{p(x)}\, dx \ \leq \ c_0 \int_\Omega \left[|F|^{p(x)} + 1\right]\, dx, \tag{4.1.4}$$

for some constant $c_0 = c_0(n, \nu, \Lambda, \gamma_1, \gamma_2) > 1$.

We additionally present our main assumptions on $p(\cdot)$, $\mathbf{a}(\xi, \cdot)$, Ω and $q(\cdot)$, under which the relation (4.0.6) holds.

Definition 4.1.2. Let $R > 0$ and $\delta \in \left(0, \frac{1}{8}\right)$. We say $(p(\cdot), \mathbf{a}, \Omega)$ is (δ, R)-*vanishing* if the following holds:

(1) Assumption on $p(\cdot)$.

$p(\cdot)$ has a modulus continuity $\omega : [0, \infty) \to [0, \infty)$, so that ω is nondecreasing, $\lim_{r\to 0} \omega(r) = 0$ and $|p(x) - p(y)| \leq \omega(|x - y|)$, and it satisfies that

$$\sup_{0<r\leq R} \omega(r) \log\left(\frac{1}{r}\right) \leq \delta. \tag{4.1.5}$$

(2) Assumption on $\mathbf{a}$.

For $U \subset \mathbb{R}^n$, write

$$\theta(\mathbf{a}, U)(x) = \sup_{\xi \in \mathbb{R}^n} \left| \frac{\mathbf{a}(\xi, x)}{(\mu^2 + \xi^2)^{\frac{p(x)-1}{2}}} - \overline{\left(\frac{\mathbf{a}(\xi, \cdot)}{(\mu^2 + |\xi|^2)^{\frac{p(\cdot)-1}{2}}} \right)}_{B_r(y)} \right|$$

(if $\mu = 0$, the above supremum runs over all $\xi \in \mathbb{R}^n \setminus \{0\}$). Then, $\mathbf{a}$ satisfies

$$\sup_{0<r\leq R} \sup_{y \in \mathbb{R}^n} \fint_{B_r(y)} \theta(\mathbf{a}, B_r(y))(x)\, dx \leq \delta. \tag{4.1.6}$$

(3) Assumption on $\partial\Omega$.

Ω is (δ, R)-*Reinfenberg flat*, that is, for each $y \in \partial\Omega$ and for each $r \in (0, R)$, there exists a coordinate system $\tilde{x} = \{\tilde{x}_1, \cdots, \tilde{x}_n\}$ with the origin at y such that

$$B_r(0) \cap \{\tilde{x}_n > \delta r\} \ \subset\ B_r(0) \cap \Omega \ \subset\ B_r(0) \cap \{\tilde{x}_n > -\delta r\}.$$

Remark 4.1.3. *$R > 0$ is arbitrary given while $\delta \in \left(0, \frac{1}{8}\right)$ will be selected in a universal way so that it can be independent of the solution u and F.*

Remark 4.1.4. *From (4.0.3), we see that $\theta \leq 2\Lambda$. The condition* (4.1.6) *means that for each $\xi \in \mathbb{R}^n$ the mapping $x \mapsto \frac{\mathbf{a}(\xi, \cdot)}{(\mu^2+|\xi|^2)^{\frac{p(\cdot)-1}{2}}}$ belongs to locally BMO and its seminorm is less than or equal to δ. We clearly point out that the condition* (4.1.6) *is invariant under translations and rotations of the coordinate system. Furthermore, one can readily check that in a new coordinate system obtained by a translation and a rotation from the old coordinate system, say $\tilde{x} = \{\tilde{x}_1, \ldots, \tilde{x}_n\}$,*

$$\sup_{0<r\leq R} \fint_{B_r^+} \theta(\mathbf{a}, B_r^+)(\tilde{x})\, d\tilde{x} \leq 4\delta. \tag{4.1.7}$$

Remark 4.1.5. *If Ω is a (δ, R)-Reifenberg flat, we have*

$$\sup_{0<r\leq R} \sup_{y \in \Omega} \frac{|B_r(y)|}{|\Omega \cap B_r(y)|} \leq \left(\frac{2}{1-\delta}\right)^n \leq \left(\frac{16}{7}\right)^n, \tag{4.1.8}$$

and

$$\inf_{0<r\leq R}\inf_{y\in\partial\Omega}\frac{|B_r(y)\cap\Omega^c|}{|B_r(y)|}\geq\left(\frac{1-\delta}{2}\right)^n\geq\left(\frac{7}{16}\right)^n, \tag{4.1.9}$$

see [13]. We also refer to [13, 52, 58] and references therein for general concepts and properties of Reifenberg flat domains.

In addition to (4.0.7), we assume that $q(\cdot)$ is log-Hölder continuous in Ω, namely, there exist a nondecreasing continuous function $\rho:[0,\infty)\to[0,\infty)$ satisfying $\lim_{r\to 0+}\rho(r)=0$ and $\rho(|x-y|)\leq|q(x)-q(y)|$, for $x,y\in\Omega$, and a constant $L_{q(\cdot)}>0$ such that

$$\rho(r)\log\left(\frac{1}{r}\right)\leq L_{q(\cdot)},\quad\text{for all } r\leq\frac{1}{2}. \tag{4.1.10}$$

For the sake of simplicity, we denote by "*data*" to mean all structure constants $n,\nu,\Lambda,\gamma_1,\gamma_2,\gamma_3,\gamma_4$ and $L_{q(\cdot)}$, and write

$$M=\int_\Omega\left[|F|^{p(x)}+1\right]\,dx+1,\quad\text{then by (4.1.4)}\quad\int_\Omega|Du|^{p(x)}dx\leq c_0M. \tag{4.1.11}$$

Theorem 4.1.6. *Let $R>0$ and $q(\cdot)$ satisfy (4.0.7) and (4.1.10). There exists $\delta=\delta(data)\in\left(0,\frac{1}{8}\right)$ such that if $(p(\cdot),\mathbf{a},\Omega)$ is (δ,R)-vanishing, $|F|\in L^{p(\cdot)}(\Omega)$ and $u\in W_0^{1,p(\cdot)}(\Omega)$ is the unique weak solution of (4.0.2), then we have the relation (4.0.6) with the following estimates: there exists $c_1=c_1(data,\omega(\cdot),\rho(\cdot),R)>1$ such that, for any $x_0\in\Omega$ and any $R_1\in(0,\frac{1}{c_1M}]$, we have*

$$\begin{aligned}&\fint_{\Omega_{R_1}(x_0)}|Du|^{p(x)q(x)}dx\\&\qquad\leq c\left\{\left(\fint_{\Omega_{4R_1}(x_0)}|Du|^{p(x)}dx\right)^{q_-}+\fint_{\Omega_{4R_1}(x_0)}|F|^{p(x)q(x)}\,dx+1\right\},\end{aligned} \tag{4.1.12}$$

for some $c=c(data)>0$, where $q_-=\inf\{q(x):x\in\Omega_{4R_1}(x_0)\}$. Moreover, it holds

$$\int_\Omega|Du|^{p(x)q(x)}\,dx\;\leq\;c_2\left(\int_\Omega|F|^{p(x)q(x)}\,dx+1\right)^{\frac{n(\gamma_4-1)+\gamma_4}{\gamma_3}}, \tag{4.1.13}$$

for some $c_2=c_2(data,\omega(\cdot),\rho(\cdot),R,\Omega)>0$.

Remark 4.1.7. *The local estimate* (4.1.12) *can be obtained only for a sufficiently small* R_1 *which depends on* M*. Here we present a relation between* R_1 *and* M *that* R_1 *is inverse proportional to* M*, see* (4.1.91)*. Thanks to this relation, we can derive the global estimate* (4.1.13) *with the constant* c_2 *independent of* M*.*

Remark 4.1.8. *In the global estimate* (4.1.13)*, the exponent* $\frac{n(\gamma_4-1)+\gamma_4}{\gamma_3} > 1$ *comes from a lack of normalization property of the equation* (4.0.2) *in the presence of variable exponent* $p(\cdot)$*, namely, a constant multiplication of a solution of even* $p(x)$*-Laplace equation* (4.0.5) *is not a solution of* (4.0.5) *anymore. A similar phenomenon occurs for parabolic equations with* p*-growth,* $p \neq 2$*, see [2]. We will show in Remark 4.1.19 that if the equation* (4.0.2) *has a normalization property, for instance* $p(\cdot) \equiv p$*, then we can obtain more natural estimates than* (4.1.13)*.*

Remark 4.1.9. *Suppose* $q(\cdot) = q(x) : \Omega \to (1, \infty)$ *satisfies*

$$1 < \gamma_3 < s(x) := \frac{q(x)}{p(x)} \leq \gamma_4 < \infty,$$

for some γ_3 *and* γ_4*. If* $q(\cdot)$ *is log-Hölder continuous in* Ω*, then we deduce from Theorem 4.1.6 that*

$$F \in L^{q(\cdot)}(\Omega, \mathbb{R}^n) \implies Du \in L^{q(\cdot)}(\Omega, \mathbb{R}^n). \tag{4.1.14}$$

Indeed, since $q(\cdot)$ *is log-Hölder continuous there exist a modulus continuity function of* $q(\cdot)$*,* $\rho : [0, \infty) \to [0, \infty)$*, and* $L_{q(\cdot)} > 0$ *satisfying (4.1.10). Then we easily check that*

$$|s(x) - s(y)| \leq \frac{\gamma_2 \rho(|x-y|) + \gamma_2 \gamma_4 \omega(|x-y|)}{\gamma_1^2} =: \tilde{\rho}(|x-y|),$$

for all $x, y \in \Omega$*. It follows that* $\tilde{\rho} : [0.\infty) \to [0, \infty)$ *is the modulus continuity of* $s(\cdot)$ *and satisfies (4.1.10) with* ρ *and* $L_{q(\cdot)}$ *replaced by* $\tilde{\rho}$ *and* $L_{s(\cdot)} := \frac{\gamma_2 L_{q(\cdot)} + \gamma_2 \gamma_4 L_{p(\cdot)}}{\gamma_1^2}$*, respectively, hence* $s(\cdot)$ *is log-Hölder continuous in* Ω*. Therefore by Theorem 4.1.6 the relation* (4.1.14) *follows from the relation* (4.0.6) *with* $q(\cdot)$ *replaced by* $s(\cdot)$*.*

Hereafter, we will denote by $c > 1$ any universal constant depending only on *data*, hence c may be different in any occurrence.

4.1.2 Auxiliary lemmas.

Higher Integrability.

Lemma 4.1.10. *Let* $M_1 > 1$ *and* $|F|^{p(\cdot)} \in L^{\gamma_3}(\Omega)$, *where* γ_3 *is given by* (4.0.7). *Suppose that* $p(\cdot)$ *and* Ω *satisfy the assumptions* (1) *and* (3) *in Definition 4.1.2, respectively, and that* $R_0 > 0$ *satisfies*

$$R_0 \le \min\left\{\frac{R}{2}, \frac{1}{4}, \frac{1}{2M_1}\right\} \quad and \quad \omega(2R_0) \le \sqrt{\frac{n+1}{n}} - 1 < 1. \tag{4.1.15}$$

Then there exists a positive constant $\sigma_0 = \sigma_0(n, \nu, \Lambda, \gamma_1, \gamma_2, \gamma_3) \le \gamma_3 - 1$ *such that the following holds: for any* $0 < r \le R_0$ *and any* $y \in \Omega$, *if* $u \in W^{1,p(\cdot)}(\Omega_r(y))$ *is a weak solution of*

$$div\, \mathbf{a}(Du, x) = div\, (|F|^{p(x)-2}F) \quad in \quad B_r(y), \quad if \quad \Omega_r(y) = B_r(y) \subset \Omega,$$

or

$$\left\{\begin{array}{rcll} div\, \mathbf{a}(Du, x) & = & div\, (|F|^{p(x)-2}F) & in \quad \Omega_r(y), \\ u & = & 0 & on \quad \partial_w \Omega_r(y), \end{array}\right. \quad if \quad B_r(y) \not\subset \Omega, \tag{4.1.16}$$

satisfying

$$\int_{\Omega_r(y)} \left[|Du|^{p(x)} + 1\right] dx + 1 \le M_1,$$

then for any $\sigma \in (0, \sigma_0]$ *and any* $\Omega_{2\tilde{r}}(\tilde{y}) \subset \Omega_r(y)$ *with* $\tilde{r} \le \frac{r}{2}$ *we have*

$$\begin{aligned} &⨍_{\Omega_{\tilde{r}}(\tilde{y})} |Du|^{p(x)(1+\sigma)}\, dx \\ &\qquad \le c\left\{\left(⨍_{\Omega_{2\tilde{r}}(\tilde{y})} |Du|^{p(x)}\, dx\right)^{(1+\sigma)} + ⨍_{\Omega_{2\tilde{r}}(\tilde{y})} |F|^{p(x)(1+\sigma)}\, dx + 1\right\}, \end{aligned}$$

for some $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2) > 0$.

Proof. It suffices to prove for the case that $B_r(y) \not\subset \Omega$. Fix $B_{2\tilde{r}}(\tilde{y}) \subset B_r(y)$, $\tilde{y} \in \Omega$. For simplicity, we omit the center $\tilde{y}$ in our notation and write $p_1 = \inf_{x \in B_{2\tilde{r}}} p(x)$ and $p_2 = \sup_{x \in B_{2\tilde{r}}} p(x)$. We then have $p_2 - p_1 \le \omega(4\tilde{r})$.

If $B_{2\tilde{r}} \subset \Omega$, by taking $\eta^{p_2}(u - \overline{u}_{B_{2\tilde{r}}})$ as a test function in (4.1.16), where $\eta \in C_0^\infty(B_{2\tilde{r}})$ is a cut-off function with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{\tilde{r}}$ and $|D\eta| \le \frac{2}{\tilde{r}}$, we derive the following Caccioppoli type inequality

$$\fint_{B_{\tilde{r}}} |Du|^{p(x)}\,dx \le c\left(\fint_{B_{2\tilde{r}}} \left|\frac{u - \overline{u}_{B_{2\tilde{r}}}}{2\tilde{r}}\right|^{p_2} dx + \fint_{B_{2\tilde{r}}} |F|^{p(x)}\,dx + 1\right),$$

see the proof of Theorem 5 in [1] for details. We note from (4.1.15) that $\frac{p_2}{p_1} \le \sqrt{\frac{n+1}{n}} =: s$, hence $\frac{p_1}{s} \ge \frac{np_2}{n+1} \ge \frac{np_2}{n+p_2}$ and $\left(\frac{p_1}{s}\right)^* \ge \left(\frac{np_2}{n+p_2}\right)^* = p_2$. Applying Sobolev-Poincaré's inequality, see [48], we find that

$$\begin{aligned}\fint_{B_r} |Du|^{p(x)}\,dx &\le c\left(\fint_{B_{2r}} |Du|^{\frac{p_1}{s}}\,dx\right)^{\frac{sp_2}{p_1}} + c\left(\fint_{B_{2r}} |F|^{p(x)}\,dx + 1\right)\\ &\le c\left(\fint_{B_{2r}} \left[|Du|^{p(x)} + 1\right]\,dx\right)^{\frac{s(p_2-p_1)}{\gamma_1}} \left(\fint_{B_{2r}} |Du|^{\frac{p(x)}{s}}\,dx\right)^{s}\\ &\quad + c\left(\fint_{B_{2r}} |F|^{p(x)}\,dx + 1\right).\end{aligned}$$

We next show that

$$\left(\fint_{B_{2\tilde{r}}} \left[|Du|^{p(x)} + 1\right]\,dx\right)^{p_2-p_1} \le \left(\fint_{B_{2\tilde{r}}} \left[|Du|^{p(x)} + 1\right]\,dx\right)^{\omega(4\tilde{r})} \le c. \tag{4.1.17}$$

Indeed, since $4\tilde{r} \le 2r \le 2R_0 \le \min\left\{R, \frac{1}{M_1}\right\}$ and $\omega(4\tilde{r}) \le \omega(2r_0) \le 1$ by (4.1.15), we have from (4.1.5) that

$$\left(\frac{1}{|B_{2\tilde{r}}|}\right)^{\omega(4\tilde{r})} = \left(\frac{1}{|B_1|}\frac{1}{(2\tilde{r})^n}\right)^{\omega(4\tilde{r})} \le c\left(\frac{1}{4\tilde{r}}\right)^{\omega(4\tilde{r})n} \le ce^{\delta n} \le c$$

and

$$\left(\int_{B_{2\tilde{r}}} \left[|Du|^{p(x)} + 1\right]\,dx\right)^{\omega(4\tilde{r})} \le M_1^{\omega(4\tilde{r})} \le \left(\frac{1}{4\tilde{r}}\right)^{\omega(4\tilde{r})} \le e^{\delta} \le e.$$

Therefore, we obtain

$$\fint_{B_{\tilde{r}}} |Du|^{p(x)}\,dx \le c\left(\fint_{B_{2\tilde{r}}} |Du|^{\frac{p(x)}{s}}\,dx\right)^{s} + c\fint_{B_{2\tilde{r}}} \left[|F|^{p(x)} + 1\right]\,dx. \tag{4.1.18}$$

If $B_{2\tilde{r}} \not\subset \Omega$(without loss of generality we assume that $\tilde{y} \in \partial\Omega$), by taking $\eta^{p_2}u$ as a test function in (4.1.16) and using Sobolev-Poincaré's inequality with the measure density condition of Ω (4.1.9), we have

$$\begin{aligned}\fint_{\Omega_{\tilde{r}}} |Du|^{p(x)}\,dx \;&\leq\; c\left(\fint_{\Omega_{2\tilde{r}}} \left[|Du|^{p(x)}+1\right]\,dx\right)^{\frac{s\omega(4\tilde{r})}{\gamma_1}} \left(\fint_{\Omega_{2\tilde{r}}} |Du|^{\frac{p(x)}{s}}\,dx\right)^{s}\\ &\quad +c\left(\fint_{\Omega_{2\tilde{r}}} |F|^{p(x)}\,dx+1\right).\end{aligned}$$

In a similar way we have estimated (4.1.17), we have from (4.1.5), (4.1.8) and (4.1.15) that

$$\begin{aligned}\left(\fint_{\Omega_{2\tilde{r}}} \left[|Du|^{p(x)}+1\right]\,dx\right)^{\omega(4\tilde{r})} \;&\leq\; \left(\frac{1}{|\Omega_{2\tilde{r}}|}M_1\right)^{\omega(4\tilde{r})} \leq c\left(\frac{1}{4\tilde{r}}\right)^{(1+n)\omega(4\tilde{r})}\\ &\leq\; ce^{(1+n)\delta} \leq c.\end{aligned}$$

Consequently, we obtain

$$\fint_{\Omega_{\tilde{r}}} |Du|^{p(x)}\,dx \leq c\left(\fint_{\Omega_{2\tilde{r}}} |Du|^{\frac{p(x)}{s}}\,dx\right)^{s} + c\fint_{\Omega_{2\tilde{r}}} \left[|F|^{p(x)}+1\right]\,dx. \quad (4.1.19)$$

From (4.1.18) and (4.1.19), using Gehring's lemma, see Theorem 4 in [1] and references therein, we get the conclusion. □

Without loss of generality, σ_0 given in the previous lemma is supposed to

$$\sigma_0 \leq 4(\gamma_1 - 1), \quad (4.1.20)$$

where γ_1 is denoted by the lower bound of $p(\cdot)$ in (4.0.7).

Remark 4.1.11. *Note that in Lemma 4.1.10 the constant σ_0 related to higher integrability is determined in a universal way, that is, independent of M_1, but R_0 depends on M_1, which is the difference from Theorem 5 in [1]. Also, Lemma 4.1.10 still holds even $p(\cdot)$ satisfies the log-Hölder continuity, instead of the assumption $\mathbf{A}_{p(\cdot)}$. At that time, σ_0 depends on $L_{p(\cdot)}$ denoted in* (2.2.5).

Boundary Comparison Estimates.

We will drive local comparison estimates near the boundary by comparing the weak solution u with a solution of an associated limiting equation which can be obtained by $\delta \to +0$ in our main assumption on $(p(\cdot), \mathbf{a}, \Omega)$ in Definition 4.1.2.

Suppose that Ω is (δ, R)-Reifenberg flat. Let $R_0 > 0$ be a small number which will be determined in Lemma 4.1.12. We fix any $r \in (0, \frac{R_0}{4}]$, and then assume that a local region $\Omega_{4r}(0) = B_{4r}(0) \cap \Omega$ satisfies

$$B_{4r}^{+} \subset \Omega_{4r}(0) \subset B_{4r} \cap \{x_n > -8\delta r\}. \tag{4.1.21}$$

In this subsection, for the sake of simplicity, we omit the center point 0 in our notation and write

$$p_2 = \sup_{x \in \Omega_{4r}} p(x) \text{ and } p_1 = \inf_{x \in \Omega_{4r}} p(x).$$

We now introduce reference problems. Let $w \in W^{1,p(\cdot)}(\Omega_{4r})$ be the weak solution of

$$\begin{cases} \text{div } \mathbf{a}(Dw, x) &= 0 \quad \text{in} \quad \Omega_{4r}, \\ w &= u \quad \text{on} \quad \partial\Omega_{4r}, \end{cases} \tag{4.1.22}$$

where $u \in W_0^{1,p(\cdot)}(\Omega)$ is the weak solution of (4.0.2). Using $w - u \in W_0^{1,p(\cdot)}(\Omega_{4r})$ as a test function in (4.1.22), we find

$$\int_{\Omega_{4r}} |Dw|^{p(x)}\, dx \le c_3 \int_{\Omega_{4r}} \left[|Du|^{p(x)} + 1\right] dx, \tag{4.1.23}$$

for some $c_3 = c_3(n, \nu, \Lambda, \gamma_1, \gamma_2) > 1$. Then we have from (4.1.11) that

$$\int_{\Omega_{4r}} \left[|Dw|^{p(x)} + 1\right] dx \le c_3 \left(\int_{\Omega} \left[|Du|^{p(x)} + 1\right] dx + 1 \right) \le c_0 c_3 M. \tag{4.1.24}$$

From now on, we let

$$M_1 = c_0 c_3 M, \tag{4.1.25}$$

and suppose that $R_0 > 0$ satisfies (4.1.15) with (4.1.25).

Then, applying Lemma 4.1.10 to $u = w$ and $F = 0$, it follows that $|Dw|^{p(\cdot)} \in L^{(1+\sigma_0)}(\Omega_{3r})$ with the estimate

$$\fint_{\Omega_{3r}} |Dw|^{p(x)(1+\sigma)} dx \le c \left\{ \left(\fint_{\Omega_{4r}} |Dw|^{p(x)} dx \right)^{(1+\sigma)} + 1 \right\}, \tag{4.1.26}$$

for all $\sigma \le \sigma_0$. We further assume that R_0 satisfies

$$p_2 - p_1 \le \omega(8r) \le \omega(2R_0) \le \frac{\sigma_0}{4}. \tag{4.1.27}$$

Then for $x \in \Omega_{4r}$ we see that

$$p_2 \le p(x) \left(1 + \frac{p_2 - p_1}{\gamma_1} \right) \le p(x) \left(1 + \omega(8r)\right) \le p(x) \left(1 + \frac{\sigma_0}{4} \right) \tag{4.1.28}$$

and that, from (4.1.20),

$$\begin{aligned} p_2 \left(1 + \frac{\sigma_0}{4} \right) &\le (p(x) + p_2 - p_1) \left(1 + \frac{\sigma_0}{4} \right) \le p(x) \left(1 + \frac{\sigma_0}{4} \right) + (p_2 - p_1)\gamma_1 \\ &\le p(x) \left(1 + \frac{\sigma_0}{4} + (p_2 - p_1) \right) \le p(x) \left(1 + \frac{\sigma_0}{4} + \omega(8r) \right) \\ &\le p(x) \left(1 + \frac{\sigma_0}{2} \right). \end{aligned} \tag{4.1.29}$$

Therefore, it follows from (4.1.26) and (4.1.28) that $w \in W^{1,p_2}(\Omega_{3r})$ with the estimate

$$\begin{aligned} \fint_{\Omega_{3r}} |Dw|^{p_2}\, dx &\le \fint_{\Omega_{3r}} |Dw|^{p(x)(1+\omega(8r))}\, dx + 1 \\ &\le c \left\{ \left(\fint_{\Omega_{4r}} |Dw|^{p(x)}\, dx \right)^{1+\omega(8r)} + 1 \right\}. \end{aligned}$$

We note from (4.1.15) that $8r \le 2R_0 \le R$ and $M_1 \le \frac{1}{8r}$. Then (4.1.5), (4.1.21) and (4.1.24) imply that

$$\begin{aligned} \left(\fint_{\Omega_{4r}} |Dw|^{p(x)} dx \right)^{\omega(8r)} &\le \left(\frac{1}{|B_{4r}^+|} M_1 \right)^{\omega(8r)} \le c \left(\frac{1}{8r} \right)^{(n+1)\omega(8r)} \\ &\le c e^{(n+1)\delta} \le c, \end{aligned} \tag{4.1.30}$$

which yields

$$\fint_{\Omega_{3r}} |Dw|^{p_2}\,dx \le c\left(\fint_{\Omega_{4r}} |Dw|^{p(x)}\,dx + 1\right). \tag{4.1.31}$$

Additionally, we also have from (4.1.26), (4.1.29) and (4.1.30) that

$$\begin{aligned}
\fint_{\Omega_{3r}} |Dw|^{p_2\left(1+\frac{\sigma_0}{4}\right)}\,dx &\le \fint_{\Omega_{3r}} |Dw|^{p(x)\left(1+\frac{\sigma_0}{4}+\omega(8r)\right)}\,dx + 1 \\
&\le c\left\{\left(\fint_{\Omega_{4r}} |Dw|^{p(x)}\,dx\right)^{1+\frac{\sigma_0}{4}+\omega(8r)} + 1\right\} \\
&\le c\left\{\left(\fint_{\Omega_{4r}} |Dw|^{p(x)}\,dx\right)^{1+\frac{\sigma_0}{4}} + 1\right\}. \qquad (4.1.32)
\end{aligned}$$

We next define $\mathbf{b} = \mathbf{b}(\xi, x) : \mathbb{R}^n \times \Omega_{4r} \to \mathbb{R}^n$ by

$$\mathbf{b}(\xi, x) = \mathbf{a}(\xi, x)(\mu^2 + |\xi|^2)^{\frac{p_2 - p(x)}{2}}. \tag{4.1.33}$$

Then one can check that $\mathbf{b}$ satisfies

$$\begin{cases} (\mu^2 + |\xi|^2)^{\frac{1}{2}}|D_\xi \mathbf{b}(\xi, x)| + |\mathbf{b}(\xi, x)| \le 3\Lambda(\mu^2 + |\xi|^2)^{\frac{p_2-1}{2}}, \\ \langle D_\xi \mathbf{b}(\xi, x)\eta, \eta\rangle \ge \frac{\nu}{2}(\mu^2 + |\xi|^2)^{\frac{p_2-2}{2}}|\eta|^2, \end{cases} \tag{4.1.34}$$

for all $\xi, \eta \in \mathbb{R}^n$ and all $x \in \Omega_{4r}$, provided that

$$p_2 - p_1 \le \omega(8r) \le \omega(2R_0) \le \min\left\{1, \frac{\nu}{2\Lambda}\right\}. \tag{4.1.35}$$

Indeed, it follows from (4.0.3) and (4.1.47) that

$$|\mathbf{b}(\xi, x)| \le \Lambda(\mu^2 + |\xi|^2)^{\frac{p_2-1}{2}}. \tag{4.1.36}$$

A direct computation yields

$$\begin{aligned}
D_\xi(\mathbf{b}(\xi, x)) &= (p_2 - p(x))(\mu^2 + |\xi|^2)^{\frac{p_2-p(x)}{2}-1}\,\xi \otimes \mathbf{a}(\xi, x) \\
&\quad + (\mu^2 + |\xi|^2)^{\frac{p_2-p(x)}{2}} D_\xi(\mathbf{a}(\xi, x)). \qquad (4.1.37)
\end{aligned}$$

Then it follows from (4.0.3), (4.1.37) and (4.1.35) that

$$\begin{aligned}
|D_\xi(\mathbf{b}(\xi, x))| &\le (p_2 - p_1)(\mu^2 + |\xi|^2)^{\frac{p_2-p(x)}{2}-1}|\mathbf{a}(\xi, x)||\xi| \\
&\quad + (\mu^2 + |\xi|^2)^{\frac{p_2-p(x)}{2}}|D_\xi(\mathbf{a}(\xi, x))| \\
&\le \Lambda(p_2 - p_1 + 1)(\mu^2 + |\xi|^2)^{\frac{p_2-2}{2}} \le 2\Lambda(\mu^2 + |\xi|^2)^{\frac{p_2-2}{2}}.
\end{aligned}$$

It also follows from (4.0.4), (4.1.37) and (4.1.35) that

$$
\begin{aligned}
\langle D_\xi \mathbf{b}(\xi,x)\eta,\eta\rangle &= (p_2-p(x))(\mu^2+|\xi|^2)^{\frac{p_2-p(x)}{2}-1}\langle \xi\otimes \mathbf{a}(\xi,x)\eta,\eta\rangle \\
&\quad + (\mu^2+|\xi|^2)^{\frac{p_2-p(x)}{2}}\langle D_\xi(\mathbf{a}(\xi,x))\eta,\eta\rangle \\
&\geq -(p_2-p_1)(\mu^2+|\xi|^2)^{\frac{p_2-p(x)}{2}-1}|\mathbf{a}(\xi,x)||\xi||\eta|^2 \\
&\quad + \nu(\mu^2+|\xi|^2)^{\frac{p_2-p(x)}{2}}(\mu^2+|\xi|^2)^{\frac{p(x)-2}{2}}|\eta|^2 \\
&\geq [\nu-(p_2-p_1)\Lambda]\,(\mu^2+|\xi|^2)^{\frac{p_2-2}{2}}|\eta|^2 \\
&\geq \frac{\nu}{2}(\mu^2+|\xi|^2)^{\frac{p_2-2}{2}}|\eta|^2.
\end{aligned}
$$

We next denote $\overline{\mathbf{b}}=\overline{\mathbf{b}}(\xi):\mathbb{R}^n\to\mathbb{R}^n$ by the integral average of $\mathbf{b}(\cdot,\xi)$ on B_{4r}^+ such that

$$
\overline{\mathbf{b}}(\xi)=\fint_{B_{4r}^+}\mathbf{b}(\xi,x)\,dx.
$$

Then $\overline{\mathbf{b}}$ also satisfies the growth and elliptic conditions (4.1.34) with $\mathbf{b}(\xi,x)$ replaced by $\overline{\mathbf{b}}(\xi)$. Moreover, from a direct calculation we deduce that

$$
\sup_{\xi\in\mathbb{R}^n}\frac{|\mathbf{b}(\xi,x)-\overline{\mathbf{b}}(\xi)|}{(\mu^2+|\xi|^2)^{\frac{p_2-1}{2}}}=\theta(\mathbf{a},B_{4r}^+)(x), \tag{4.1.38}
$$

where θ is defined in Definition 4.1.2.

Let $v\in W^{1,p(\cdot)}(\Omega_{3r})$ be the weak solution of

$$
\begin{cases}
\operatorname{div}\overline{\mathbf{b}}(Dv) = 0 & \text{in } \Omega_{3r},\\
v = w & \text{on } \partial\Omega_{3r},
\end{cases} \tag{4.1.39}
$$

where w is the weak solution of (4.1.22) which belongs to $W^{1,p_2}(\Omega_{3r})$, see (4.1.26). Then it follows from the standard energy estimate that

$$
\fint_{\Omega_{3r}}|Dv|^{p_2}\,dx\le c\left(\fint_{\Omega_{3r}}|Dw|^{p_2}\,dx+1\right). \tag{4.1.40}
$$

Lemma 4.1.12. *Let $R_0>0$ satisfy (4.1.15), (4.1.27), (4.1.35) with (4.1.25). Fix any $\lambda>1$ and any $r\le\frac{R_0}{4}$. Suppose that Ω_{4r} satisfies (4.1.21). Then, for any $0<\epsilon<1$, there exists $\delta=\delta(n,\Lambda,\nu,\gamma_1,\gamma_2,\epsilon)>0$ such that if $(p(\cdot),\mathbf{a},\Omega)$ is (δ,R)-vanishing, $u\in W_0^{1,p(\cdot)}(\Omega)$, $w\in W^{1,p(\cdot)}(\Omega_{4r})\cap W^{1,p_2}(\Omega_{3r})$*

and $v \in W^{1,p_2}(\Omega_{3r})$ *are the weak solutions of* (4.0.2), (4.1.22) *and* (4.1.39), *respectively, with*

$$\fint_{\Omega_{4r}} |Du|^{p(x)}\,dx \le \lambda \quad \textit{and} \quad \fint_{\Omega_{4r}} |F|^{p(x)}\,dx \le \delta\lambda, \tag{4.1.41}$$

then we have

$$\fint_{\Omega_{4r}} |Dw|^{p(x)}\,dx + \fint_{\Omega_{3r}} |Dv|^{p_2}\,dx + \fint_{\Omega_{3r}} |Dw|^{p_2}\,dx \le c\lambda, \tag{4.1.42}$$

for some $c = c(n, \Lambda, \nu, \gamma_1, \gamma_2) > 0$,

$$\fint_{\Omega_{4r}} |Du - Dw|^{p(x)}\,dx \le \epsilon\lambda \quad \textit{and} \quad \fint_{\Omega_{3r}} |Dw - Dv|^{p_2}\,dx \le \epsilon\lambda. \tag{4.1.43}$$

Proof. In what follows, we will write κ_i, $i = 1, 2, 3, 4, 5$, as any number in $(0, 1)$ and $c(\kappa_i)$ as any constant depending only on $n, \nu, \Lambda, \gamma_1, \gamma_2$ and κ_i. Note that since $8r \le 2R_0 \le R$, we know from (4.1.5) and (4.1.7) that

$$\omega(8r) \log\left(\frac{1}{8r}\right) \le \delta \ \text{ and } \ \fint_{B^+_{4r}} \theta(\mathbf{a}, B^+_{4r})(x)\,dx \le 4\delta. \tag{4.1.44}$$

The estimate (4.1.42) directly follows from (4.1.23), (4.1.31), (4.1.40) and (4.1.41). We first derive the first inequality in (4.1.43). Since $u - w \in W_0^{1,p(\cdot)}(\Omega_{4r})$, we have from the equations (4.0.2) and (4.1.22) that

$$\fint_{\Omega_{4r}} \langle \mathbf{a}(Du, x) - \mathbf{a}(Dw, x), Du - Dw\rangle\,dx = \fint_{\Omega_{4r}} \langle |F|^{p(x)-2}F, Du - Dw\rangle\,dx. \tag{4.1.45}$$

By Young's inequality, the right hand side of (4.1.45) is estimated by

$$\fint_{\Omega_{4r}} \langle |F|^{p(x)-2}F, Du-Dw\rangle\,dx \le \kappa_1 \fint_{\Omega_{4r}} |Du-Dw|^{p(x)}\,dx + c(\kappa_1)\fint_{\Omega_{4r}} |F|^{p(x)}\,dx.$$

Note that if $p(x) \ge 2$, then by (4.1.2) we have

$$|Du - Dw|^{p(x)} \le c\langle \mathbf{a}(Du, x) - \mathbf{a}(Dw, x), Du - Dw\rangle,$$

and if $p(x) < 2$, then by Young's inequality and (4.1.1) we have

$$\begin{aligned} |Du - Dw|^{p(x)} &= \left(\mu^2 + |Du|^2 + |Dw|^2\right)^{\frac{p(x)(2-p(x))}{4}} \\ &\quad \times \left(\mu^2 + |Du|^2 + |Dw|^2\right)^{\frac{p(x)(p(x)-2)}{4}} |Du - Dw|^{p(x)} \\ &\leq \kappa_2 \left(\mu^2 + |Du|^2 + |Dw|^2\right)^{\frac{p(x)}{2}} \\ &\quad + c(\kappa_2) \left(\mu^2 + |Du|^2 + |Dw|^2\right)^{\frac{p(x)-2}{2}} |Du - Dw|^2 \\ &\leq \kappa_2 \left(1 + |Du|^{p(x)} + |Dw|^{p(x)}\right) \\ &\quad + c(\kappa_2) \langle \mathbf{a}(Du, x) - \mathbf{a}(Dw, x), Du - Dw \rangle. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} \fint_{\Omega_{4r}} |Du - Dw|^{p(x)}\, dx &\leq \kappa_2 \fint_{\Omega_{4r}} \left[1 + |Du|^{p(x)} + |Dw|^{p(x)}\right] dx \\ &\quad + c(\kappa_2) \fint_{\Omega_{4r}} \langle \mathbf{a}(Du, x) - \mathbf{a}(Dw, x), Du - Dw \rangle\, dx \\ &\leq \kappa_2 \fint_{\Omega_{4r}} \left[1 + |Du|^{p(x)} + |Dw|^{p(x)}\right] dx \\ + c(\kappa_2) \left(\kappa_1 \fint_{\Omega_{4r}} |Du - Dw|^{p(x)}\, dx \right. &\left. + c(\kappa_1) \fint_{\Omega_{4r}} |F|^{p(x)}\, dx \right). \end{aligned} \tag{4.1.46}$$

Thus, (4.1.41) and (4.1.42) imply that

$$\fint_{\Omega_{4r}} |Du - Dw|^{p(x)}\, dx \leq \kappa_2 c_4 \lambda + c(\kappa_2)\kappa_1 \fint_{\Omega_{4r}} |Du - Dw|^{p(x)}\, dx + c(\kappa_2)c(\kappa_1)\delta\lambda.$$

for some $c_4 = c_4(n, \nu, \Lambda, \gamma_1, \gamma_2) > 0$. By choosing

$$\kappa_2 = \frac{\epsilon}{4c_4}, \quad \kappa_1 = \frac{1}{2c(\kappa_2)} \quad \text{and} \quad \delta = \frac{\epsilon}{4c(\kappa_1)c(\kappa_2)},$$

we obtain the first inequality in (4.1.43).

We next derive the second inequality in (4.1.43). Since $w - v \in W_0^{1,p_2}(\Omega_{3r})$, in a similar way we have estimated (4.1.46), it follows from (4.1.22) and

(4.1.39) that

$$
\begin{aligned}
&\fint_{\Omega_{3r}} \langle \overline{\mathbf{b}}(Dw) - \overline{\mathbf{b}}(Dv), Dw - Dv\rangle \, dx \\
&\qquad = \fint_{\Omega_{3r}} \langle \overline{\mathbf{b}}(Dw) - \mathbf{b}(Dw, x), Dw - Dv\rangle \, dx \\
&\qquad\quad + \fint_{\Omega_{3r}} \langle \mathbf{b}(Dw, x) - \mathbf{a}(Dw, x), Dw - Dv\rangle \, dx \\
&\qquad =: I_1 + I_2
\end{aligned}
$$

and then

$$
\begin{aligned}
\fint_{\Omega_{3r}} |Dw - Dv|^{p_2} \, dx &\leq c_5\kappa_3\lambda + c(\kappa_3)\fint_{\Omega_{3r}} \langle \overline{\mathbf{b}}(Dw) - \overline{\mathbf{b}}(Dv), Dw - Dv\rangle \, dx \\
&= c_5\kappa_3\lambda + c(\kappa_3)(I_1 + I_2), \qquad (4.1.47)
\end{aligned}
$$

for some $c_5 = c_5(n, \nu, \Lambda, \gamma_1, \gamma_2) > 0$.

We now estimate I_1 and I_2. By Young's inequality and (4.1.38),

$$
\begin{aligned}
|I_1| &\leq \fint_{\Omega_{3r}} |\mathbf{b}(Dw, x) - \overline{\mathbf{b}}(Dw)||Dw - Dv| \, dx \\
&\leq \fint_{\Omega_{3r}} \theta(\mathbf{a}, B_{4r}^+) \left(\mu^2 + |Dw|^2\right)^{\frac{p_2-1}{2}} |Dw - Dv| \, dx \\
&\leq \kappa_4 \fint_{\Omega_{3r}} |Dw - Dv|^{p_2} \, dx + c(\kappa_4) \fint_{\Omega_{3r}} \theta^{\frac{p_2}{p_2-1}} (1 + |Dw|)^{p_2} \, dx.
\end{aligned}
$$

Using Hölder's inequality, (4.1.21), (4.1.32), (4.1.42) and (4.1.44),

$$
\begin{aligned}
&\fint_{\Omega_{3r}} \theta^{\frac{p_2}{p_2-1}} (1 + |Dw|)^{p_2} \, dx \\
&\quad \leq \left(\fint_{\Omega_{3r}} \theta^{\frac{p_2}{p_2-1}\frac{1+\sigma_0/4}{\sigma_0/4}} \, dx\right)^{\frac{\sigma_0/4}{1+\sigma_0/4}} \left(\fint_{\Omega_{3r}} (1 + |Dw|)^{p_2(1+\sigma_0/4)} \, dx\right)^{\frac{1}{1+\sigma_0/4}} \\
&\quad \leq c\left(\fint_{B_{3r}^+} \theta^{\frac{p_2}{p_2-1}\frac{1+\sigma_0/4}{\sigma_0/4}} \, dx + (2\Lambda)^{\frac{p_2}{p_2-1}\frac{1+\sigma_0/4}{\sigma_0/4}} \frac{|\Omega_{3r} \backslash B_{3r}^+|}{|B_{3r}^+|}\right)^{\frac{\sigma_0/4}{1+\sigma_0/4}} \lambda \\
&\quad \leq c\left\{\left(\frac{4}{3}\right)^n (2\Lambda)^{\frac{\gamma_1}{\gamma_1-1}\frac{1+\sigma_0/4}{\sigma_0/4}-1} \fint_{B_{4r}^+} \theta \, dx + \delta\right\}^{\frac{\sigma_0/4}{1+\sigma_0/4}} \lambda \leq c\delta^{\frac{\sigma_0}{4+\sigma_0}} \lambda.
\end{aligned}
$$

Thus, we obtain

$$|I_1| \leq \kappa_4 \fint_{\Omega_{3r}} |Dw - Dv|^{p_2}\, dx + c(\kappa_4)\delta^{\frac{\sigma_0}{4+\sigma_0}}\lambda. \tag{4.1.48}$$

We next estimate I_2. We note from (4.0.3) and (4.1.38) that if $\mu^2 + |Dw(x)|^2 = 0$, $\mu = |Dw(x)| = 0$, then $\mathbf{a}(Dw(x), x) = \mathbf{b}(Dw(x), x) = 0$. Set $\tilde{\Omega}_{3r} = \{x \in \Omega_{3r} : \mu^2 + |Dw(x)|^2 > 0\}$. Then, by Young's inequality, we have

$$\begin{aligned} |I_2| &\leq \frac{1}{|\Omega_{3r}|} \int_{\tilde{\Omega}_{3r}} \left|(\mu^2 + |Dw|^2)^{\frac{p_2 - p(x)}{2}} - 1\right| |\mathbf{a}(Dw, x)||Dw - Dv|\, dx \\ &\leq \frac{1}{|\Omega_{3r}|}\Bigg(\kappa_5 \int_{\tilde{\Omega}_{3r}} |Dw - Dv|^{p_2}\, dx \\ &\qquad + c(\kappa_5) \int_{\tilde{\Omega}_{3r}} \left[\left|(\mu^2 + |Dw|^2)^{\frac{p_2 - p(x)}{2}} - 1\right| |\mathbf{a}(Dw, x)|\right]^{\frac{p_2}{p_2 - 1}} dx\Bigg). \end{aligned}$$

For each $x \in \tilde{\Omega}_{3r}$, in view of the Mean Value Theorem to $(\mu^2 + |Dw|^2)^{\frac{p_2 - p(x)}{2}t}$, $t \in [0, 1]$, we have

$$(\mu^2 + |Dw|^2)^{\frac{p_2 - p(x)}{2}} - 1 = \frac{p_2 - p(x)}{2}\left(\mu^2 + |Dw|^2\right)^{t_x \frac{p_2 - p(x)}{2}} \log\left(\mu^2 + |Dw|^2\right)$$

for some $t_x \in (0, 1)$, and so

$$\begin{aligned} &\left|(\mu^2 + |Dw|^2)^{\frac{p_2 - p(x)}{2}} - 1\right| |\mathbf{a}(Dw, x)| \\ &\qquad \leq \frac{\omega(8r)}{2}\Lambda \left(\mu^2 + |Dw|^2\right)^{\frac{t_x(p_2 - p(x)) + p(x) - 1}{2}} \left|\log\left(\mu^2 + |Dw|^2\right)\right|. \end{aligned}$$

From a direct calculation, we know that $t^\beta |\log t| \leq \max\left\{\frac{1}{e\beta}, 2^\beta \log 2\right\}$ for all $t \in (0, 2]$, where $\beta > 0$. Hence, for $x \in \tilde{\Omega}_{3r}$ with $|Dw(x)| \leq 1$, we obtain

$$\left|(\mu^2 + |Dw|^2)^{\frac{p_2 - p(x)}{2}} - 1\right| |\mathbf{a}(Dw, x)| \leq \frac{\omega(8r)\Lambda}{2} \max\left\{\frac{2}{e(\gamma_1 - 1)}, 2^{\frac{\gamma_2 - 1}{2}} \log 2\right\}.$$

Thus,

$$\begin{aligned}
|I_2| &\leq \kappa_5 \fint_{\Omega_{3r}} |Dw - Dv|^{p_2}\,dx + c(\kappa_5)\frac{\omega(8r)^{\frac{p_2}{p_2-1}}}{|\Omega_{3r}|} \times \\
&\quad \left(|\tilde{\Omega}_{3r} \cap \{|Dw| \leq 1\}| + \int_{\tilde{\Omega}_{3r}\cap\{|Dw|>1\}} |Dw|^{p_2} \left[\log\left(e + |Dw|\right)\right]^{\frac{p_2}{p_2-1}}\,dx \right) \\
&\leq \kappa_5 \fint_{\Omega_{3r}} |Dw - Dv|^{p_2}\,dx \\
&\quad + c(\kappa_5)\omega(8r)^{\frac{p_2}{p_2-1}} \Bigg(\underbrace{\fint_{\Omega_{3r}} |Dw|^{p_2} \left[\log\left(e + |Dw|^{p_2}\right)\right]^{\frac{p_2}{p_2-1}}\,dx}_{I_3} + 1 \Bigg).
\end{aligned}$$

To estimate I_3, we first apply the inequality $\log(e+ab) \leq \log(e+a) + \log(e+b)$, $a, b > 0$, in order to get

$$\begin{aligned}
I_3 &\leq c\Bigg\{ \fint_{\Omega_{3r}} |Dw|^{p_2} \left[\log\left(e + \frac{|Dw|^{p_2}}{\overline{(|Dw|^{p_2})}_{\Omega_{3r}}}\right)\right]^{\frac{p_2}{p_2-1}}\,dx \\
&\qquad + \fint_{\Omega_{3r}} |Dw|^{p_2} \left[\log\left(e + \overline{(|Dw|^{p_2})}_{\Omega_{3r}}\right)\right]^{\frac{p_2}{p_2-1}}\,dx \Bigg\} \\
&=: c(I_4 + I_5).
\end{aligned}$$

Applying Lemma 2.3.6 to $f = |Dw|^{p_2}$, $\beta = \frac{p_2}{p_2-1} \in \left[\frac{\gamma_2}{\gamma_2-1}, \frac{\gamma_1}{\gamma_1-1}\right]$ and $\sigma = 1+\frac{\sigma_0}{4}$, and using (4.1.32) and (4.1.42), we find that

$$I_4 \leq c\left(\fint_{\Omega_{3r}} |Dw|^{p_2\left(1+\frac{\sigma_0}{4}\right)}\,dx\right)^{\frac{1}{1+\frac{\sigma_0}{4}}} \leq c\lambda.$$

On the other hand, since $\frac{1}{8r} \geq \frac{1}{2R_0} \geq \{M_1, 2\}$ by (4.1.15), we obtain from (4.1.24) and (4.1.31) that

$$\begin{aligned}
\log\left(e + \overline{(|Dw|^{p_2})}_{\Omega_{3r}}\right) &\leq \log\left\{e + c\left(\fint_{\Omega_{3r}} |Dw|^{p(x)}\,dx + 1\right)\right\} \\
&\leq c\left\{\log\frac{1}{|\Omega_{3r}|} + \log\left(1 + \int_{\Omega_{3r}} \left[|Dw|^{p(x)} + 1\right]\,dx\right) + 1\right\} \\
&\leq c\left(\log\frac{1}{8r} + \log(M_1) + 1\right) \leq c\log\frac{1}{8r},
\end{aligned}$$

which, together with (4.1.42), implies

$$I_5 \leq c\left(\log \frac{1}{8r}\right)^{\frac{p_2}{p_2-1}} \lambda.$$

Consequently, from (4.1.44) we have the following estimate for I_2 :

$$\begin{aligned}|I_2| &\leq \kappa_5 \fint_{\Omega_{3r}} |Dw - Dv|^{p_2}\, dx + c(\kappa_5)\left(\omega(8r)\log\frac{1}{8r}\right)^{\frac{p_2}{p_2-1}} \lambda \\ &\leq \kappa_5 \fint_{\Omega_{3r}} |Dw - Dv|^{p_2}\, dx + c(\kappa_5)\delta^{\frac{\gamma_2}{\gamma_2-1}}\lambda. \qquad (4.1.49)\end{aligned}$$

Inserting (4.1.48) and (4.1.49) into (4.1.47), we have

$$\begin{aligned}\fint_{\Omega_{3r}} |Dw - Dv|^{p_2}\, dx \leq c_5\kappa_3\lambda + c(\kappa_3)\bigg(\kappa_4 \fint_{\Omega_{3r}} |Dw - Dv|^{p_2}\, dx + c(\kappa_4)\delta^{\frac{\sigma_0}{4+\sigma_0}}\lambda \\ +\kappa_5 \fint_{\Omega_{3r}} |Dw - Dv|^{p_2}\, dx + c(\kappa_5)\delta^{\frac{\gamma_2}{\gamma_2-1}}\lambda\bigg).\end{aligned}$$

Finally, choosing $\kappa_3 = \frac{\epsilon}{9c_5}$, $\kappa_4 = \kappa_5 = \frac{1}{3c(\kappa_3)}$ and

$$\delta \leq \min\left\{\left(\frac{\epsilon}{9c(\kappa_3)c(\kappa_4)}\right)^{\frac{4+\sigma_0}{\sigma_0}}, \left(\frac{\epsilon}{9c(\kappa_3)c(\kappa_5)}\right)^{\frac{\gamma_2-1}{\gamma_2}}\right\},$$

we get the second inequality in (4.1.43). This completes the proof. □

Remark 4.1.13. *In Lemma 4.1.12, the selection of δ is independent of R_0, the choices of $r \leq \frac{R_0}{4}$ and $\lambda > 1$.*

Lemma 4.1.14. *For any $0 < \epsilon < 1$, there exists $\delta = \delta(\epsilon, n, \Lambda, \nu, p_2)$ such that if Ω_{4r} satisfies (4.1.21) and if $v \in W^{1,p_2}(\Omega_{3r})$ is a weak solution of*

$$\begin{cases} \mathit{div}\ \overline{\mathbf{b}}(Dv) = 0 & \mathit{in} \quad \Omega_{3r}, \\ v = 0 & \mathit{on} \quad \partial_w\Omega_{3r}, \end{cases}$$

with

$$\fint_{\Omega_{3r}} |Dv|^{p_2} dx \leq \lambda,$$

then there exists a weak solution $\overline{v} \in W^{1,p_2}(B_{3r}^+)$ *of*

$$\begin{cases} div\, \overline{\mathbf{b}}(D\overline{v}) = 0 & in \quad B_{3r}^+, \\ \overline{v} = 0 & on \quad T_{3r}, \end{cases} \tag{4.1.50}$$

with

$$\fint_{B_{3r}^+} |D\overline{v}|^{p_2} dx \leq \lambda, \tag{4.1.51}$$

such that

$$\fint_{\Omega_{2r}} |Dv - D\overline{v}|^{p_2} dx \leq \epsilon\lambda,$$

where $\overline{v}$ *is extended by zero from* B_{2r}^+ *to* Ω_{2r} *and* $c = c(\epsilon, n, \Lambda, \nu, p_2) > 0$.

Proof. We argue by contradiction. Suppose that there exists $\epsilon_0 > 0$ the following hold: for each $k = 8, 9, 10, \ldots$ there exist Ω^k, $v_k \in W^{1,p_2}(\Omega_{3r})$ such that

$$B_{4r}^+ \subset \Omega_{4r}^k(0) \subset B_{4r} \cap \{x_n > -\frac{8}{k}\},$$

v_k is a weak solution to

$$\begin{cases} \text{div}\, \overline{\mathbf{b}}(Dv_k) = 0 & \text{in} \quad \Omega_{3r}^k, \\ v = 0 & \text{on} \quad \partial_w \Omega_{3r}^k, \end{cases} \tag{4.1.52}$$

with

$$\fint_{\Omega_{3r}^k} |Dv_k|^{p_2}\, dx \leq 2\lambda,$$

but

$$\fint_{B_{2r}} |Dv_k - D\overline{v}|^{p_2} dx > \epsilon_0\lambda, \tag{4.1.53}$$

for every weak solution $\overline{v} \in W^{1,p_2}(B_{3r}^+)$ to (4.1.50) with (4.1.51). Here we extend v_k(resp. $\overline{v}$) by zero from Ω_{3r}^k(resp. $\overline{v}$) to B_{3r}.

Since

$$\fint_{B_{3r}} |Dv_k|^{p_2}\, dx \leq \fint_{\Omega_{3r}^k} |Dv_k|^{p_2}\, dx \leq \lambda,$$

Poincaré's inequality implies that $\{v_k\}$ is bounded in $W^{1,p_2}(B_{3r})$ hence there exists $v_0 \in W^{1,p_2}(B_{3r})$ such that

$$\left\{ \begin{array}{lll} v_k \to v_0 & \text{strongly in} & L^{p_2}(B_{3r}) \\ Dv_k \rightharpoonup Dv_0 & \text{weakly in} & L^{p_2}(B_{3r}) \end{array} \right\} \text{as } k \to \infty.$$

Note that since $v_k \equiv 0$ in $B_{3r} \setminus \Omega_{3r}^k$, we have $v_0 \equiv 0$ in $B_{3r} \cap \{x_n < 0\}$, which implies that $v_0 = 0$ on T_{3r} in the trace sense and

$$\fint_{B_{3r}^+} |Dv_0|^{p_2}\, dx \le 2\fint_{B_{3r}} |Dv_0|^{p_2}\, dx \le 2 \liminf_{k\to\infty} \fint_{B_{3r}} |Dv_k|^{p_2}\, dx \le 2\lambda.$$

From the above result we see that v_0 is a weak solution to (4.1.50) with $\overline{v}$ replaced by v_0, see [50]. We next show that

$$\lim_{k\to\infty} \fint_{B_{2r}} \left(\mu^2 + |Dv_k|^2 + |Dv_0|^2\right)^{\frac{p_2-2}{2}} |Dv_k - Dv_0|^2\, dx = 0, \tag{4.1.54}$$

which implies

$$\lim_{k\to\infty} \fint_{B_{2r}} |Dv_k - Dv_0|^{p_2}\, dx = 0,$$

hence is a contradiction to (4.1.64). Let $\phi \in C_0^\infty(B_{3r})$ be a cut off function such that $\phi \equiv 1$ in B_{2r}. Then, by the monotonicity of $\overline{\mathbf{b}}$ we have

$$\begin{aligned}
&\fint_{B_{2r}} \left(\mu^2 + |Dv_k|^2 + |Dv_0|^2\right)^{\frac{p_2-2}{2}} |Dv_k - Dv_0|^2\, dx \\
&\le \fint_{B_{3r}} \phi \left(\mu^2 + |Dv_k|^2 + |Dv_0|^2\right)^{\frac{p_2-2}{2}} |Dv_k - Dv_0|^2\, dx \\
&\le c\fint_{B_{3r}} \phi \langle \overline{\mathbf{b}}(Dv_k) - \overline{\mathbf{b}}(Dv_0), D(v_k - v_0)\rangle\, dx \\
&\le c\fint_{\Omega_{3r}^k} \langle \overline{\mathbf{b}}(Dv_k), D(\phi(v_k - v_0))\rangle\, dx + c\fint_{B_{3r}} \langle \overline{\mathbf{b}}(Dv_0), D(\phi(v_k - v_0))\rangle\, dx \\
&\qquad + c\fint_{B_{3r}} |\overline{\mathbf{b}}(Dv_k) - \overline{\mathbf{b}}(Dv_0)||D\phi||v_k - v_0|\, dx
\end{aligned}$$

The first term on the right hand side is zero since v_k is a weak solution to (4.1.52) and $\phi(v_k - v_0) \in W_0^{1,p_2}(\Omega_{3r}^k)$. The second and third term on the right hand side go to zero as $k \to \infty$ since $D(\phi(v_k - v_0))$ is weakly converge to zero in $L^{p_2}(B_{3r})$, $\overline{\mathbf{b}}(v_k)$ and $\overline{\mathbf{b}}(v_0)$ are bounded in $L^{\frac{p_2}{p_2-1}}(B_{3r})$, and v_k is strongly converge to v_k in $L^{p_2}(B_{3r})$. Hence we obtain (4.1.54). □

Furthermore, in view of [45, 46], we see that $D\overline{v} \in L^\infty(B_r^+)$ with the following estimate

$$\|D\overline{v}\|_{L^\infty(\Omega_r)}^{p_2} = \|D\overline{v}\|_{L^\infty(B_r^+)}^{p_2} \le c\left(\fint_{B_{2r}^+} |D\overline{v}|^{p_2}\, dx + 1\right) \le c\lambda. \tag{4.1.55}$$

Interior Comparison Estimates.

We derive similar comparison estimates on a interior region in the same way. Suppose that $R_0 > 0$ satisfies (4.1.15), (4.1.27) and (4.1.35) and fix any $r \le \frac{R_0}{4}$ with $B_{4r}(y) = \Omega_{4r}(y) \subset \Omega$. Write

$$p_1 = \inf_{x \in B_{4r}(y)} p(x) \quad \text{and} \quad p_2 = \sup_{x \in B_{4r}(y)} p(x).$$

For the weak solution $u \in W_0^{1,p(\cdot)}(\Omega)$ of (4.0.2), let $w \in W^{1,p(\cdot)}(B_{4r}(y))$ be the weak solution of

$$\begin{cases} \operatorname{div} \mathbf{a}(Dw, x) &= 0 \quad \text{in} \quad B_{4r}(y), \\ w &= u \quad \text{on} \quad \partial B_{4r}(y). \end{cases} \tag{4.1.56}$$

Then in a similar argument as in the boundary case, we have

$$\int_{B_{4r}(y)} \left[|Dw|^{p(x)} + 1 \right] dx \le M_1,$$

where M_1 is given by (4.1.25), and then $w \in W^{1,p_2}(B_{3r}(y))$. Let $v \in W^{1,p_2}(B_{3r}(y))$ be the weak solution of

$$\begin{cases} \operatorname{div} \overline{\mathbf{b}}(Dv) &= 0 \quad \text{in} \quad B_{3r}(y), \\ v &= w \quad \text{on} \quad \partial B_{3r}(y). \end{cases} \tag{4.1.57}$$

In the interior case, $\overline{\mathbf{b}} : \mathbb{R}^n \to \mathbb{R}^n$ is denoted by

$$\overline{\mathbf{b}}(\xi) = \fint_{B_4(y)} \mathbf{b}(\xi, x)\, dx = \fint_{B_4(y)} \mathbf{a}(\xi, x)(\mu^2 + |\xi|^2)^{\frac{p_2 - p(x)}{2}}\, dx.$$

Lemma 4.1.15. *Let $R_0 > 0$ satisfy (4.1.15), (4.1.27), (4.1.35) with (4.1.25). Fix any $\lambda > 1$ and any $r \le \frac{R_0}{4}$ with $B_{4r}(y) \subset \Omega$. Then, for any $0 < \epsilon < 1$, there exists $\delta = \delta(n, \Lambda, \nu, \gamma_1, \gamma_2, \epsilon) > 0$ such that if $p(\cdot)$ and $\mathbf{a}$ satisfy the assumptions (1) and (2) in Definition 4.1.2, respectively, and if $u \in W_0^{1,p(\cdot)}(\Omega)$, $w \in W^{1,p(\cdot)}(B_{4r}(y)) \cap W^{1,p_2}(B_{3r}(y))$ and $v \in W^{1,p_2}(B_{3r}(y))$ are the weak solutions of (4.0.2), (4.1.56) and (4.1.57), respectively, with*

$$\fint_{B_{4r}(y)} |Du|^{p(x)}\, dx \le \lambda, \quad \fint_{B_{4r}(y)} |F|^{p(x)}\, dx \le \delta\lambda,$$

then

$$\fint_{B_{4r}(y)} |Dw|^{p(x)}\,dx + \fint_{B_{3r}(y)} |Dw|^{p_2}\,dx + \fint_{B_{3r}(y)} |Dv|^{p_2}\,dx \le c\lambda,$$

for some $c = c(n, \Lambda, \nu, \gamma_1, \gamma_2) > 0$,

$$\fint_{B_{4r}(y)} |Du - Dw|^{p(x)}\,dx \le \epsilon\lambda \quad \text{and} \quad \fint_{\Omega_{3r}(y)} |Dw - Dv|^{p_2}\,dx \le \epsilon\lambda. \tag{4.1.58}$$

Proof. It is exactly same to the proof of Lemma 4.1.12. □

Also, in view of [45, 46], $Dv \in L^\infty(B_r(y))$ and we have

$$\|Dv\|^{p_2}_{L^\infty(B_r(y))} \le c\left(\fint_{B_{3r}(y)} |Dv|^{p_2}\,dx + 1\right) \le c\lambda. \tag{4.1.59}$$

4.1.3 Proof of Theorem 4.1.6.

Suppose that $(p(\cdot), \mathbf{a}, \Omega)$ is (δ, R)-vanishing. We first select $R_0 > 0$ to satisfy

$$\begin{cases} R_0 \le \min\left\{\frac{R}{2}, \frac{1}{4}, \frac{1}{2M_1}\right\}, \\ \omega(2R_0) \le \min\left\{\sqrt{\frac{n+1}{n}} - 1, \frac{\sigma_0}{4}, \frac{\nu}{2\Lambda}, 1\right\}, \\ \rho(2R_0) \le \min\left\{\frac{\sigma_0\gamma_3}{8}, \frac{\sigma_1\gamma_3}{2}, \frac{\gamma_3^2}{4\gamma_4}, \frac{\sigma_1}{2}, \frac{1}{2}\right\}, \end{cases} \tag{4.1.60}$$

where σ_0 and M_1 are given by Lemma 4.1.10 with (4.1.20) and (4.1.25), respectively, and

$$\sigma_1 := \min\left\{\frac{\gamma_3 - 1}{2}, 1\right\}. \tag{4.1.61}$$

Note that R_0 satisfies that (4.1.15), (4.1.27) and (4.1.35). Fix any $x_0 \in \Omega$ and any $R_1 \le \frac{R_0}{4}$ and consider $\Omega_{4R_1}(x_0)$. For the sake of simplicity, we omit the center x_0 in our notations and write

$$q_+ := \sup_{x\in\Omega_{4R_1}} q(x), \quad q_- := \inf_{x\in\Omega_{4R_1}} q(x),$$

$$\mathcal{M}[Du](x) := \mathcal{M}\left(|Du|^{p(\cdot)\frac{q(\cdot)}{q_-}}\chi_{\Omega_{2R_1}}\right)(x)$$

and

$$\mathcal{M}_{1+\sigma_1}[F](x) := \mathcal{M}_{1+\sigma_1}\left(\left[|F|^{p(\cdot)\frac{q(\cdot)}{q_-}}+1\right]\chi_{\Omega_{2R_1}}\right)(x),$$

where $\mathcal{M}$, $\mathcal{M}_{1+\sigma_1}$ are the Hardy-Littlewood maximal operators and $\chi_{\Omega_{2R_1}}$ is the characteristic function on Ω_{2R_1}, namely, $\chi_{\Omega_{2R_1}} \equiv 1$ on Ω_{2R_1} and $\chi_{\Omega_{2R_1}} \equiv 0$ otherwise. For $\epsilon \in (0,1)$ and $A > 1$, we define

$$\lambda_0 := \frac{1}{\epsilon}\left\{ \fint_{\Omega_{4R_1}} |Du|^{p(x)}\,dx + \left(\fint_{\Omega_{4R_1}} |F|^{p(x)(1+\sigma_1)}\,dx\right)^{\frac{1}{1+\sigma_1}} + 1\right\} \tag{4.1.62}$$

and supper-level sets : for $k = 0, 1, 2, \ldots$

$$C_k := \left\{x \in \Omega_{R_1} : \mathcal{M}[Du](x) > A^{k+1}\lambda_0\right\},$$

$$D_k := \left\{x \in \Omega_{R_1} : \mathcal{M}[Du](x) > A^{k}\lambda_0\right\} \cup \left\{x \in \Omega_{R_1} : \mathcal{M}_{1+\sigma_1}[F](x) > \delta A^{k}\lambda_0\right\}.$$

Note that ϵ and A will be determined later as universal constants depending only on *data*.

Lemma 4.1.16. *There exists $A = A(data) > 1$ such that*

$$|C_k| \le \frac{\epsilon}{(63)^n}|\Omega_{R_1}|,$$

for all $k = 0, 1, 2, \ldots$.

Proof. Since $C_k \subset C_0$ for all $k = 0, 1, 2, \ldots$, it suffices to show for the case $k = 0$. By the weak type (1,1)-estimate, Proposition 2.3.3 (1), and (4.1.8),

$$|C_0| \le \frac{c}{A\lambda_0}\int_{\Omega_{2R_1}} |Du|^{p(x)\frac{q(x)}{q_-}}\,dx \le \frac{c|\Omega_{R_1}|}{A\lambda_0}\left(\fint_{\Omega_{2R_1}} |Du|^{p(x)(1+\rho(8R_1))}\,dx + 1\right).$$

Since $\rho(8R_1) \le \rho(2R_0) \le \sigma_0$ by (4.1.60), Lemma 4.1.10 with $2\tilde{r} = r = 4R_1 \le R_0$ implies

$$|C_0| \le \frac{c|\Omega_{R_1}|}{A\lambda_0}\left\{\left(\fint_{\Omega_{4R_1}} |Du|^{p(x)}\,dx\right)^{1+\rho(8R_1)} + \fint_{\Omega_{4R_1}} |F|^{p(x)(1+\rho(8R_1))}\,dx + 1\right\}.$$

We note from (4.1.24) and (4.1.60) that $M \le c_0 M \le M_1$ and $8R_1 \le 2R_0 \le \min\{\frac{1}{2}, \frac{1}{M_1}\}$, then we see from (4.1.4), (4.1.8) and (4.1.10) that

$$\left(\fint_{\Omega_{4R_1}} |Du|^{p(x)}\, dx \right)^{\rho(8R_1)} \le c \quad \text{and} \quad \left(\fint_{\Omega_{4R_1}} |F|^{p(x)}\, dx \right)^{\rho(8R_1)} \le c. \tag{4.1.63}$$

Indeed,

$$\begin{aligned} \left(\fint_{\Omega_{4R_1}} |Du|^{p(x)}\, dx \right)^{\rho(8R_1)} &\le c \left(\frac{1}{|\Omega_{4R_1}|} c_0 M \right)^{\rho(8R_1)} \le c \left(\frac{1}{8R_1} \right)^{(n+1)\rho(8R_1)} \\ &\le c e^{(n+1)L_{q(\cdot)}} \le c, \end{aligned}$$

and the second estimate in (4.1.63) can be obtained in the same way.

Furthermore, since $\rho(8R_1) \le \rho(2R_0) \le \frac{\sigma_1}{2}$ by (4.1.60), Hölder's inequality and (4.1.63) imply that

$$\begin{aligned} \fint_{\Omega_{4R_1}} |F|^{p(x)(1+\rho(8R_1))}\, dx &\le \fint_{\Omega_{4R_1}} |F|^{\frac{p(x)}{2}} |F|^{\frac{p(x)}{2}(1+2\rho(8R_1))}\, dx \\ &\le \left(\fint_{\Omega_{4R_1}} |F|^{p(x)}\, dx \right)^{\frac{1}{2}} \left(\fint_{\Omega_{4R_1}} |F|^{p(x)(1+2\rho(8R_1))}\, dx \right)^{\frac{1}{2}} \\ &\le c \left(\fint_{\Omega_{4R_1}} |F|^{p(x)}\, dx \right)^{\frac{1-2\rho(8R_1)}{2}} \left(\fint_{\Omega_{4R_1}} |F|^{p(x)(1+\sigma_1)}\, dx \right)^{\frac{1+2\rho(8R_1)}{2(1+\sigma_1)}} \\ &\le c \left(\fint_{\Omega_{4R_1}} |F|^{p(x)(1+\sigma_1)}\, dx \right)^{\frac{1}{1+\sigma_1}}. \end{aligned}$$

Consequently, we have

$$|C_0| \le \frac{c_6 |\Omega_{R_1}|}{A\lambda_0} \left\{ \fint_{\Omega_{4R_1}} |Du|^{p(x)}\, dx + \left(\fint_{\Omega_{4R_1}} |F|^{p(x)(1+\sigma_1)}\, dx \right)^{\frac{1}{1+\sigma_1}} + 1 \right\},$$

for some $c_6 = c_6(data) > 0$. Applying (4.1.62), we finally obtain

$$|C_0| \le \frac{c_6}{A} \epsilon |\Omega_{R_1}| \le \frac{\epsilon}{(63)^n} |\Omega_{R_1}|,$$

by taking large $A = A(data) > 1$. □

Lemma 4.1.17. *There exist $A = A(data) > 1$ and $\delta = \delta(data, \epsilon) \in \left(0, \frac{1}{8}\right)$ such that for any $y_0 \in C_k$ and $r_0 \leq \frac{R_1}{63}$ if*

$$|C_k \cap B_{r_0}(y_0)| > \epsilon |B_{r_0}(y_0)|, \tag{4.1.64}$$

then $B_{r_0}(y_0) \subset D_k$.

Proof. Since the proof is rather long, we divide it into four steps.
Step 1. We argue by contradiction. Suppose there exist $y_0 \in C_k$ and $r_0 \leq \frac{R_1}{63}$ such that (4.1.64) holds but

$$B_{r_0}(y_0) \not\subset D_k. \tag{4.1.65}$$

We simply write

$$\lambda_k = A^k \lambda_0.$$

From (4.1.65) we can find $y_1 \in B_{r_0}(y_0)$ such that $y_1 \notin D_k$, thus

$$\fint_{B_r(y_1)} |Du|^{p(x)\frac{q(x)}{q_-}} \chi_{\Omega_{2R_1}}\, dx \leq \lambda_k \tag{4.1.66}$$

and

$$\left(\fint_{B_r(y_1)} \left[|F|^{p(x)\frac{q(x)}{q_-}} + 1 \right]^{1+\sigma_1} \chi_{\Omega_{2R_1}}\, dx \right)^{\frac{1}{1+\sigma_1}} \leq \delta \lambda_k, \tag{4.1.67}$$

for all $r > 0$. We consider the two cases, the interior case $B_{9r_0}(y_1) \subset \Omega$ and the boundary case $B_{9r_0}(y_1) \not\subset \Omega$, separately.

Step 2. Let $B_{9r_0}(y_1) \subset \Omega$. Since $9r_0 \leq 63 r_0 \leq R_1$, we see that $B_{8r_0}(y_0) \subset B_{9r_0}(y_1) \subset \Omega_{2R_1}$. Set

$$p_1 := \inf_{x \in B_{8r_0}(y_0)} p(x), \quad p_2 := \sup_{x \in B_{8r_0}(y_0)} p(x)$$

$$q_1 := \inf_{x \in B_{8r_0}(y_0)} q(x) \ \text{ and } \ q_2 := \sup_{x \in B_{8r_0}(y_0)} q(x).$$

We claim that

$$\fint_{B_{8r_0}(y_0)} |Du|^{p(x)}\, dx \leq c_a \lambda_k^{\frac{q_-}{q_2}} \ \text{ and } \ \fint_{B_{8r_0}(y_0)} |F|^{p(x)}\, dx \leq c_a \delta^{\frac{\gamma_3}{\gamma_4}} \lambda_k^{\frac{q_-}{q_2}}, \tag{4.1.68}$$

for some $c_a = c_a(data) > 1$. Indeed, since $16r_0 \leq 2R_1 \leq 2R_0 \leq \min\{\frac{1}{2}, \frac{1}{M_1}\}$ by (4.1.60), in the same way we have estimated (4.1.63), we see that

$$\left(\fint_{B_{8r_0}(y_0)} |Du|^{p(x)}\,dx\right)^{\rho(16r_0)} \leq c, \quad \left(\fint_{B_{8r_0}(y_0)} |F|^{p(x)}\,dx\right)^{\rho(16r_0)} \leq c.$$

Then Jensen's inequality, (4.1.66) and (4.1.67) imply that

$$\begin{aligned}\fint_{B_{8r_0}(y_0)} |Du|^{p(x)}\,dx &= \left(\fint_{B_{8r_0}(y_0)} |Du|^{p(x)}\,dx\right)^{\frac{q_2-q_1}{q_2}} \left(\fint_{B_{8r_0}(y_0)} |Du|^{p(x)}\,dx\right)^{\frac{q_1}{q_2}} \\ &\leq \left(\fint_{B_{8r_0}(y_0)} |Du|^{p(x)}\,dx + 1\right)^{\frac{\rho(16r_0)}{\gamma_3}} \left(\fint_{B_{8r_0}(y_0)} |Du|^{p(x)\frac{q_1}{q_-}}\,dx\right)^{\frac{q_-}{q_2}} \\ &\leq c\left(\fint_{B_{9r_0}(y_1)} |Du|^{p(x)\frac{q(x)}{q_-}}\,dx + 1\right)^{\frac{q_-}{q_2}} \leq c\lambda_k^{\frac{q_-}{q_2}}\end{aligned}$$

and

$$\begin{aligned}\fint_{B_{8r_0}(y_0)} |F|^{p(x)}\,dx &\leq c\left(\fint_{B_{9r_0}(y_1)} \left[|F|^{p(x)\frac{q(x)}{q_-}} + 1\right]\,dx\right)^{\frac{q_-}{q_2}} \\ &\leq c\left(\fint_{B_{9r_0}(y_1)} \left[|F|^{p(x)\frac{q(x)}{q_-}} + 1\right]^{1+\sigma_1}\,dx\right)^{\frac{1}{1+\sigma_1}\frac{q_-}{q_2}} \leq c\delta^{\frac{\gamma_3}{\gamma_4}}\lambda_k^{\frac{q_-}{q_2}}.\end{aligned}$$

Applying Lemma 4.1.15 and (4.1.59) with λ, r, ϵ and δ replaced by $c_a\lambda_k^{\frac{q_-}{q_2}}, 2r_0, \eta$ and $\delta^{\frac{\gamma_3}{\gamma_4}}$, respectively, we can find $\delta = \delta(data, \eta)$ such that

$$\fint_{B_{4r_0}(y_0)} |Dw|^{p(x)}\,dx \leq c\lambda_k^{\frac{q_-}{q_2}},$$

$$\fint_{B_{2r_0}(y_0)} |Du - Dw|^{p(x)}\,dx \leq \eta\lambda_k^{\frac{q_-}{q_2}}, \quad \fint_{B_{2r_0}(y_0)} |Dw - Dv|^{p_2}\,dx \leq \eta\lambda_k^{\frac{q_-}{q_2}}$$

and

$$\|Dv\|_{L^\infty(B_{2r_0}(y_0))} \leq c\lambda_k^{\frac{q_-}{q_2}\frac{1}{p_2}},$$

for some $c = c(data) > 0$, where w and v are the weak solutions of (4.1.56) and (4.1.57) with $r = 2r_0$, respectively. Using the above results we can derive that

$$\fint_{B_{2r_0}(y_0)} |Du - Dv|^{p(x)\frac{q(x)}{q_-}}\, dx \leq \tilde{c}_a \eta^{\frac{1}{4}} \lambda_k \tag{4.1.69}$$

and

$$\left\| |Dv|^{p(\cdot)\frac{q(\cdot)}{q_-}} \right\|_{L^\infty(B_{2r_0}(y_0))} \leq \tilde{c}_a \lambda_k, \tag{4.1.70}$$

for some $\tilde{c}_a = \tilde{c}_a(data) > 1$. Since we need some technical computations to derive (4.1.69) and (4.1.70), we will show it later in Step 4.

We now estimate $|C_k \cap B_{r_0}(y_0)|$. We assert that if

$$A \geq \max\left\{2^{\frac{\gamma_2\gamma_4}{\gamma_3}-1}(1+\tilde{c}_a), 3^n\right\}, \tag{4.1.71}$$

then

$$C_k \cap B_{r_0}(y_0) \subset \{x \in B_{r_0}(y_0) : \mathcal{M}^*[Du - Dv](x) > \lambda_k\}. \tag{4.1.72}$$

Indeed, let $y \in C_k \cap B_{r_0}(y_0)$. If $r < r_0$ we know that $B_r(y) \subset B_{2r_0}(y_0)$. From the elementary inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, $a, b > 0$ and $p > 1$, we have

$$|Du(x)|^{p(x)\frac{q(x)}{q_-}} \leq 2^{p(x)\frac{q(x)}{q_-}-1}\left(|Du(x) - Dv(x)|^{p(x)\frac{q(x)}{q_-}} + |Dv(x)|^{p(x)\frac{q(x)}{q_-}}\right),$$

for almost every $x \in B_r(y)$. Integrating both sides on $B_r(y)$ and using (4.1.70), we obtain

$$\fint_{B_r(y)} |Du|^{p(x)\frac{q(x)}{q_-}}\, dx \leq 2^{\frac{\gamma_2\gamma_4}{\gamma_3}-1}\left(\mathcal{M}^*[u-v](y) + \tilde{c}_a\lambda_k\right), \tag{4.1.73}$$

where

$$\mathcal{M}^*[Du - Dv](y) := \mathcal{M}\left(|Du - Dv|^{p(\cdot)\frac{q(\cdot)}{q_-}} \chi_{B_{2r_0}(y_0)}\right)(y).$$

On the other hand, if $r \geq r_0$ we know that $B_r(y) \subset B_{2r}(y_0) \subset B_{3r}(y_1)$. Then, we have from (4.1.66) that

$$\fint_{B_r(y)} |Du|^{p(x)\frac{q(x)}{q_-}}\, dx \leq 3^n \fint_{B_{3r}(y_1)} |Du|^{p(x)\frac{q(x)}{q_-}}\, dx \leq 3^n \lambda_k. \tag{4.1.74}$$

Therefore (4.1.73) and (4.1.74) imply that

$$\mathcal{M}[Du](y) \leq \max\left\{2^{\frac{\gamma_4}{\gamma_3}\gamma_2 - 1}\left(\mathcal{M}^*[Du - Dv](y) + \tilde{c}_a\lambda_k\right), 3^n\lambda_k\right\}.$$

Since $A\lambda_k = \lambda_{k+1}$ and $C_k = \{x \in \Omega_{R_1} : \mathcal{M}[Du](x) > \lambda_{k+1}\}$, we easily check that (4.1.71) implies (4.1.72).

In view of the weak type $(1,1)$ estimate, Proposition 2.3.3 (1), and (4.1.69), we finally obtain from (4.1.72) that

$$|C_k \cap B_{r_0}(y_0)| \leq \frac{c}{\lambda_k}\int_{B_{2r_0}(y_0)} |Du - Dv|^{p(x)\frac{q(x)}{q_-}}\,dx \leq c_7\eta^{\frac{1}{4}}|B_{r_0}(y_0)|,$$

for some $c_7 = c_7(data) > 0$. By taking sufficiently small $\eta = \eta(data, \epsilon) > 0$, hence $\delta = \delta(data, \epsilon) > 0$ is also determined, we have

$$|C_k \cap B_{r_0}(y_0)| \leq \epsilon|B_{r_0}(y_0)|,$$

which is the contradiction to (4.1.64).

Step 3. Let $B_{9r_0}(y_1) \not\subset \Omega$. Since $63r_0 \leq R_1 \leq \frac{R_0}{4} \leq \frac{R}{8}$, from the assumption (3) in Definition 4.1.2, one can find coordinate system, denoted by still $x = (x_1, \ldots, x_n)$ variables, such that

$$B^+_{52r_0} \ \subset\ \Omega_{52r_0}(0) \ \subset B_{52r_0}(0) \cap \{x_n > -104\delta R_0\}. \tag{4.1.75}$$

Then, in this new coordinate system, we have $|y_1| \leq 10r_0$ and so

$$\Omega_{2r_0}(y_0) \subset \Omega_{3r_0}(y_1) \subset \Omega_{13r_0}(0), \quad \Omega_{52r_0}(0) \subset \Omega_{62r_0}(y_1) \subset \Omega_{R_1}(y_0) \subset \Omega_{2R_1}. \tag{4.1.76}$$

Set

$$p_1 := \inf_{x\in B_{52r_0}(0)} q(x), \quad p_2 := \sup_{x\in B_{52r_0}(0)} p(x),$$
$$q_1 := \inf_{x\in B_{52r_0}(0)} q(x) \ \text{ and } \ q_2 := \sup_{x\in B_{52r_0}(0)} q(x).$$

In a similar way we have estimated (4.1.68), we infer from (4.1.10), (4.1.66), (4.1.67), (4.1.75) and (4.1.76) that

$$\fint_{\Omega_{52r_0}(0)} |Du|^{p(x)}\,dx \leq c_b\lambda_k^{\frac{q_-}{q_2}} \ \text{ and } \ \fint_{\Omega_{52r_0}(0)} |F|^{p(x)}\,dx \leq c_b\delta^{\frac{3}{4}}\lambda_k^{\frac{q_-}{q_2}},$$

for some $c_b = c_b(data) > 1$. Therefore, Applying Lemma 4.1.12, Lemma 4.1.14 and (4.1.55) with λ, r, ϵ and δ replaced by $c_b\lambda_k^{\frac{q_-}{q_2}}, 13r_0, \eta$ and $\delta^{\frac{\gamma_3}{\gamma_4}}$, we can find $\delta = \delta(data, \eta)$ and $\overline{v} \in W^{1,\infty}(\Omega_{13r_0})$ such that

$$\fint_{\Omega_{52r_0}(0)} |Dw|^{p(x)}\,dx \le c\lambda_k^{\frac{q_-}{q_2}}, \tag{4.1.77}$$

$$\fint_{\Omega_{13r_0}(0)} |Du-Dw|^{p(x)}\,dx \le \eta\lambda_k^{\frac{q_-}{q_2}}, \quad \fint_{\Omega_{13r_0}(0)} |Dw-D\overline{v}|^{p_2}\,dx \le \eta\lambda_k^{\frac{q_-}{q_2}} \tag{4.1.78}$$

and

$$\|D\overline{v}\|_{L^\infty(\Omega_{13r_0}(0))} \le c\lambda_k^{\frac{q_-}{q_2}\frac{1}{p_2}}, \tag{4.1.79}$$

for some $c = c(data) > 0$, where w is the weak solutions of (4.1.22) with $r = 13r_0$. Using the above results we can derive that

$$\fint_{\Omega_{13r_0}(0)} |Du - Dv|^{p(x)\frac{q(x)}{q_-}}\,dx \le \tilde{c}_b\eta^{\frac{1}{4}}\lambda_k \tag{4.1.80}$$

and

$$\left\| |Dv|^{p(\cdot)\frac{q(\cdot)}{q_-}} \right\|_{L^\infty(\Omega_{13r_0}(0))} \le \tilde{c}_b\lambda_k, \tag{4.1.81}$$

for some $\tilde{c}_b = \tilde{c}_b(data) > 1$. We will also show (4.1.80) and (4.1.81) in Step 4.

Proceeding as in Step 2, it follows from (4.1.66) and (4.1.81) that

$$C_k \cap \Omega_{r_0}(y_0) \subset \{y \in \Omega_{r_0}(y_0) : \mathcal{M}^*[Du - D\overline{v}](y) > \lambda_k\},$$

by selecting sufficiently large $A = A(data) > 1$. Here we write

$$\mathcal{M}^*[Du - D\overline{v}](y) := \mathcal{M}\left(|Du - D\overline{v}|^{p(\cdot)\frac{q(\cdot)}{q_-}}\chi_{\Omega_{2r_0}(y_0)}\right)(y).$$

Therefore, in view of the weak type (1,1) estimate, Proposition 2.3.3 (1), (4.1.75) and (4.1.80) we obtain

$$\begin{aligned} |C_k \cap B_{r_0}(y_0)| &= |\{y \in \Omega_{r_0}(y_0) : \mathcal{M}^*[Du - D\overline{v}](y) > \lambda_k\}| \\ &\le \frac{c}{\lambda_k}\int_{\Omega_{2r_0}(y_0)} |Du - D\overline{v}|^{p(x)\frac{q(x)}{q_-}}\,dx \\ &\le \frac{c|B_{r_0}|}{\lambda_k}\fint_{\Omega_{13r_0}(0)} |Du - D\overline{v}|^{p(x)\frac{q(x)}{q_-}}\,dx \le c_8\eta^{\frac{1}{4}}|B_{r_0}(y_0)|, \end{aligned}$$

for some $c_8 = c_8(data) > 0$. By choosing $\eta = \eta(data, \epsilon) > 0$ sufficiently small, hence $\delta = \delta(data, \epsilon) > 0$ is also determined, we have

$$|C_k \cap B_{r_0}(y_0)| \le \epsilon |B_{r_0}(y_0)|,$$

which is the contradiction to (4.1.64).

Step 4. It remains to derive the estimates (4.1.69) and (4.1.70) in Step 2 and (4.1.80) and (4.1.81) in Step 3. We only prove the estimates (4.1.80) and (4.1.81). The estimates (4.1.69) and (4.1.70) can be obtained in the same way. In this step, for the sake of simplicity, we write $\Omega_{13jr_0} = \Omega_{13jr_0}(0)$, $j = 1, 2$.

Note that the estimate (4.1.81) directly follows from (4.1.79). We now derive (4.1.80). Since $104r_0 \le 2R_0 \le \min\{\frac{1}{2}, \frac{1}{M_1}\}$ by (4.1.60), in a similar way we have estimated (4.1.30) and (4.1.63), one can find that

$$\left(\fint_{\Omega_{52r_0}} \left[|Du|^{p(x)} + |Dw|^{p(x)} \right] dx \right)^{\rho(104r_0)} \le c. \tag{4.1.82}$$

By Hölder's inequality, we have

$$\begin{aligned} \fint_{\Omega_{13r_0}} |Du - Dw|^{p(x)\frac{q(x)}{q_-}} \, dx &= \fint_{\Omega_{13r_0}} |Du - Dw|^{\frac{p(x)}{2} + p(x)\left(\frac{q(x)}{q_-} - \frac{1}{2}\right)} dx \\ \le \left(\fint_{\Omega_{13r_0}} |Du - Dw|^{p(x)} \, dx \right)^{\frac{1}{2}} &\left(\fint_{\Omega_{13r_0}} |Du - Dw|^{p(x)\left(2\frac{q(x)}{q_-} - 1\right)} dx \right)^{\frac{1}{2}} \end{aligned} \tag{4.1.83}$$

Applying (4.1.78) and (4.1.82), the first term on the right-hand side of (4.1.83) is estimated by

$$\begin{aligned} &\left(\fint_{\Omega_{13r_0}} |Du - Dw|^{p(x)} \, dx \right)^{\frac{1}{2}} \\ &= \left(\fint_{\Omega_{13r_0}} |Du - Dw|^{p(x)} \, dx \right)^{\frac{1}{2} - \frac{q_2}{q_-}\frac{q_2 - q_1}{q_1}} \left(\fint_{\Omega_{13r_0}} |Du - Dw|^{p(x)} \, dx \right)^{\frac{q_2}{q_-}\frac{q_2 - q_1}{q_1}} \\ &\le c \left(\eta \lambda_k^{\frac{q_-}{q_2}} \right)^{\frac{1}{2} - \frac{q_2}{q_-}\frac{q_2 - q_1}{q_1}} \left(\fint_{\Omega_{52r_0}} \left[|Du|^{p(x)} + |Dw|^{p(x)} \right] dx + 1 \right)^{\frac{\gamma_4}{\gamma_3^2}\rho(104r_0)} \\ &\le c \eta^{\frac{1}{2} - \frac{q_2}{q_-}\frac{q_2 - q_1}{q_1}} \lambda_k^{\frac{1}{2}\frac{q_-}{q_2} - \frac{q_2 - q_1}{q_1}} \le c \eta^{\frac{1}{4}} \lambda_k^{\frac{1}{2}\frac{q_-}{q_2} - \frac{q_2 - q_1}{q_1}}. \end{aligned}$$

For the last inequality, we have used the fact $\frac{q_2}{q_-}\frac{q_2-q_1}{q_1} \leq \frac{\gamma_4}{\gamma_3^2}\rho(104r_0) \leq \frac{\gamma_4}{\gamma_3^2}\rho(2R_0) \leq \frac{1}{4}$ which is obtained by (4.1.60).

We next estimate the second term on the right-hand side of (4.1.83). We note from (4.1.60) that $\rho(8R_1) \leq \rho(2R_0) \leq \min\left\{\frac{\gamma_3}{2}\sigma_0, \frac{\gamma_3}{8}\sigma_1\right\}$, and so

$$2\frac{q_2}{q_-} - 1 \leq 1 + 2\frac{q_+ - q_-}{q_-} \leq 1 + 2\frac{\rho(8R_1)}{\gamma_3} \leq \min\left\{1 + \frac{\sigma_0}{4}, 1 + \sigma_1\right\}.$$

Then, applying Lemma 4.1.10 to u and w and Hölder's inequality, we obtain from (4.1.67), (4.1.68), (4.1.75) and (4.1.77) that

$$\begin{aligned}
&\fint_{\Omega_{13r_0}} |Du - Dw|^{p(x)\left(2\frac{q(x)}{q_-}-1\right)}\,dx \\
&\leq c\left(\fint_{\Omega_{13r_0}} \left[|Du|^{p(x)\left(2\frac{q_2}{q_-}-1\right)} + |Dw|^{p(x)\left(2\frac{q_2}{q_-}-1\right)}\right]\,dx + 1\right) \\
&\leq c\left\{\left(\fint_{\Omega_{26r_0}} |Du|^{p(x)} + |Dw|^{p(x)}\,dx\right)^{2\frac{q_2}{q_-}-1} + \fint_{\Omega_{26r_0}} |F|^{p(x)\left(2\frac{q_2}{q_-}-1\right)}\,dx + 1\right\} \\
&\leq c\left\{\lambda_k^{2-\frac{q_-}{q_2}} + \left(\fint_{\Omega_{36r_0}(y_1)} |F|^{p(x)\frac{q_1}{q_-}(1+\sigma_1)}\,dx\right)^{\frac{1}{1+\sigma_1}\frac{q_-}{q_1}\left(2\frac{q_2}{q_-}-1\right)} + 1\right\} \\
&\leq c\left(\lambda_k^{2-\frac{q_-}{q_2}} + \lambda_k^{2\frac{q_2}{q_1}-\frac{q_-}{q_1}} + 1\right).
\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}
\fint_{\Omega_{13r_0}} |Du - Dw|^{p(x)\frac{q(x)}{q_-}}\,dx &\leq c\epsilon^{\frac{1}{4}}\lambda_k^{\frac{1}{2}\frac{q_-}{q_2}-\frac{q_2-q_1}{q_1}}\left(\lambda_k^{1-\frac{1}{2}\frac{q_-}{q_2}} + \lambda_k^{\frac{q_2}{q_1}-\frac{1}{2}\frac{q_-}{q_1}} + 1\right) \\
&= c\epsilon^{\frac{1}{4}}\left(\lambda_k^{1-\frac{q_2-q_1}{q_1}} + \lambda_k^{1-\frac{1}{2}\frac{q_-}{q_1}+\frac{1}{2}\frac{q_-}{q_2}} + \lambda_k^{\frac{1}{2}\frac{q_-}{q_2}-\frac{q_2-q_1}{q_1}}\right) \\
&\leq c\epsilon^{\frac{1}{4}}\lambda_k. \qquad (4.1.84)
\end{aligned}$$

On the other hand, we obtain from Hölder's inequality and (4.1.78) that

$$\begin{aligned}
&\fint_{\Omega_{13r_0}} |Dw - D\overline{v}|^{p(x)\frac{q(x)}{q_-}}\,dx \\
&\le \left(\fint_{\Omega_{13r_0}} |Dw - D\overline{v}|^{p_2}\,dx\right)^{\frac{1}{2}} \left(\fint_{\Omega_{13r_0}} |Dw - D\overline{v}|^{p_2\left(2\frac{p(x)}{p_2}\frac{q(x)}{q_-}-1\right)}\,dx\right)^{\frac{1}{2}} \\
&\le c\left(\eta\lambda^{\frac{q_-}{q_2}}\right)^{\frac{1}{2}} \left(\fint_{\Omega_{13r_0}} \left[|Dw|^{p_2\left(2\frac{q_2}{q_-}-1\right)} + |D\overline{v}|^{p_2\left(2\frac{q_2}{q_-}-1\right)}\right]dx + 1\right)^{\frac{1}{2}}. \quad (4.1.85)
\end{aligned}$$

We know from (4.1.20) and (4.1.60) that $\rho(8R_1) \le \rho(2R_0) \le \frac{\gamma_3\sigma_0}{8} \le \frac{\gamma_3(\gamma_1-1)}{2}$ and $\omega(104r_0) \le \omega(2R_0) \le \frac{\sigma_0}{4}$, so that we find that

$$\begin{aligned}
p_2\left(2\frac{q_2}{q_-}-1\right) &\le p(x)\left(2\frac{q_2}{q_-}-1\right) + (p_2-p_1)\left(1+2\frac{\rho(8R_1)}{\gamma_3}\right) \\
&\le p(x)\left(2\frac{q_2}{q_-}-1\right) + \omega(104r_0)\gamma_1 \\
&\le p(x)\left(2\frac{q_2}{q_-}-1+\omega(104r_0)\right) \le p(x)\left(1+\frac{\sigma_0}{2}\right).
\end{aligned}$$

Thus (4.1.26), (4.1.30), (4.1.75) and (4.1.77) imply that

$$\begin{aligned}
\fint_{\Omega_{13r_0}} |Dw|^{p_2\left(2\frac{q_2}{q_-}-1\right)}\,dx &\le \fint_{\Omega_{39r_0}} |Dw|^{p(x)\left(2\frac{q_2}{q_-}-1+\omega(104r_0)\right)}\,dx + 1 \\
&\le c\left\{\left(\fint_{\Omega_{52r_0}} |Dw|^{p(x)}\,dx\right)^{2\frac{q_2}{q_-}-1+\omega(104r_0)} + 1\right\} \\
&\le c\left\{\left(\fint_{\Omega_{52r_0}} |Dw|^{p(x)}\,dx\right)^{2\frac{q_2}{q_-}-1} + 1\right\} \le c\lambda_k^{2-\frac{q_-}{q_2}}. \quad (4.1.86)
\end{aligned}$$

Also, we have from (4.1.79) that

$$\fint_{\Omega_{13r_0}} |D\overline{v}|^{p_2\left(2\frac{q_2}{q_-}-1\right)}\,dx \le c\lambda_k^{2-\frac{q_-}{q_2}}. \quad (4.1.87)$$

Inserting (4.1.86) and (4.1.87) into (4.1.85), we obtain

$$\fint_{\Omega_{13r_0}} |Dw - D\overline{v}|^{p(x)\frac{q(x)}{q_-}}\,dx \le c\left(\eta\lambda^{\frac{q_-}{q_2}}\right)^{\frac{1}{2}}\left(\lambda_k^{2-\frac{q_-}{q_2}}\right)^{\frac{1}{2}} \le c\eta^{\frac{1}{2}}\lambda_k. \quad (4.1.88)$$

Consequently, (4.1.84) and (4.1.88) imply the estimate (4.1.80). □

In view of Lemma 2.3.4, Lemma 4.1.16 and Lemma 4.1.17, we finally have the following power decay estimate.

Lemma 4.1.18. *There exist $A = A(data) > 1$ and $\delta = \delta(data, \epsilon) \in \left(0, \frac{1}{8}\right)$ such that*

$$|C_k| \leq \epsilon \left(\frac{80}{7}\right)^n |D_k|, \quad k = 0, 1, 2, \ldots.$$

Moreover, by an iteration argument we have

$$\left|\left\{x \in \Omega_{R_1} : \mathcal{M}[Du](x) > A^k \lambda_0\right\}\right| \leq \epsilon_1^k \left|\{x \in \Omega_{R_1} : \mathcal{M}[Du](x) > \lambda_0\}\right| + \sum_{i=1}^{k} \epsilon_1^i \left|\left\{x \in \Omega_{R_1} : \mathcal{M}_{1+\sigma_1}[F](x) > \delta A^{k-i} \lambda_0\right\}\right|, \quad (4.1.89)$$

where $\epsilon_1 = \epsilon \left(\frac{80}{7}\right)^n$.

Now, we prove the estimates in Theorem (4.1.6).

Local Estimates : Proof of (4.1.12)**.**

Consider

$$S := \sum_{k=1}^{\infty} A^{kq_-} \left|\left\{x \in \Omega_R : \mathcal{M}[Du](x) > A^k \lambda_0\right\}\right|,$$

then according to (4.1.89),

$$\begin{aligned} S &\leq \sum_{k=1}^{\infty} A^{kq_-} \epsilon_1^k \left|\{x \in \Omega_{R_1} : \mathcal{M}[Du](x) > \lambda_0\}\right| \\ &\quad + \sum_{k=1}^{\infty} A^{kq_-} \sum_{i=1}^{k} \epsilon_1^i \left|\left\{x \in \Omega_{R_1} : \mathcal{M}_{1+\sigma_1}[F](x) > \delta A^{k-i} \lambda_0\right\}\right| \\ &\leq |\Omega_{R_1}| \sum_{k=1}^{\infty} \left(A^{q_-} \epsilon_1\right)^k \\ &\quad + \sum_{i=1}^{\infty} (A^{q_-} \epsilon_1)^i \sum_{k=i}^{\infty} A^{q_-(k-i)} \left|\left\{x \in \Omega_{R_1} : \mathcal{M}_{1+\sigma_1}[F](x) > A^{k-i} \delta \lambda_0\right\}\right|. \end{aligned}$$

Applying Lemma 2.3.5 and (2) of Proposition 2.3.3, we have

$$\begin{aligned}\sum_{k=i}^{\infty} A^{q_-(k-i)} \left|\left\{x \in \Omega_{R_1} : \mathcal{M}_{1+\sigma_1}[F](x) > A^{k-i}\delta\lambda_0\right\}\right| \\ &\le |\Omega_{R_1}| + \frac{c}{(\delta\lambda_0)^{q_-}} \int_{\mathbb{R}^n} \left(\mathcal{M}_{1+\sigma_1}[F]\right)^{q_-}\, dx \\ &\le |\Omega_{R_1}| + \frac{c}{(\delta\lambda_0)^{q_-}} \int_{\mathbb{R}^n} \left[\left(|F|^{p(x)\frac{q(x)}{q_-}} + 1\right)\chi_{\Omega_{2R}}\right]^{q_-}\, dx \\ &\le |\Omega_{R_1}| + \frac{c}{(\delta\lambda_0)^{q_-}} \int_{\Omega_{2R_1}} \left[|F|^{p(x)q(x)} + 1\right]\, dx.\end{aligned}$$

Thus

$$S \le c\left(\frac{1}{\delta^{\gamma_4}\lambda_0^{q_-}} \int_{\Omega_{2R_1}} \left[|F|^{p(x)q(x)} + 1\right]\, dx + |\Omega_{R_1}|\right) \sum_{k=1}^{\infty} \left(A^{\gamma_4}\epsilon_1\right)^k.$$

At this stage, we take $\epsilon = \epsilon(data) > 0$ such that

$$\epsilon\left(\frac{80}{7}\right)^n A^{\gamma_4} = \epsilon_1 A^{\gamma_4} = \frac{1}{2},$$

hence we also determine the constant δ depending only on $data$ by Lemma 4.1.18. Consequently, we obtain

$$S \le c\left(\frac{1}{\lambda_0^{q_-}} \int_{\Omega_{2R_1}} \left[|F|^{p(x)q(x)} + 1\right]\, dx + |\Omega_{R_1}|\right). \tag{4.1.90}$$

By Lemma 2.3.5, (4.1.62) and (4.1.90), we get

$$\begin{aligned}\int_{\Omega_{R_1}} |Du|^{p(x)q(x)}\, dx &\le \int_{\Omega_{R_1}} \mathcal{M}[Du]^{q_-}\, dx \le c\lambda_0^{q_-}\left(|\Omega_{R_1}| + S\right) \\ &\le c\left(|\Omega_{R_1}|\lambda_0^{q_-} + \int_{\Omega_{2R_1}} \left[|F|^{p(x)q(x)} + 1\right]\, dx\right) \\ &\le c|\Omega_{R_1}|\Bigg\{\left(\fint_{\Omega_{4R_1}} |Du|^{p(x)}\, dx\right)^{q_-} + \left(\fint_{\Omega_{4R_1}} |F|^{p(x)(1+\sigma_1)}\, dx\right)^{\frac{q_-}{1+\sigma_1}} \\ &\qquad + \fint_{\Omega_{4R_1}} |F|^{p(x)q(x)}\, dx + 1\Bigg\}.\end{aligned}$$

Since $1+\sigma_1 \le \gamma_3 \le q_-$ by (4.1.61), we finally obtain

$$\fint_{\Omega_{R_1}} |Du|^{p(x)q(x)}\,dx \le c\left\{\left(\fint_{\Omega_{4R_1}} |Du|^{p(x)}\,dx\right)^{q_-} + \fint_{\Omega_{4R_1}} |F|^{p(x)q(x)}\,dx + 1\right\},$$

for some $c = c(data) > 0$, which is the desired estimate (4.1.12). Furthermore, we see from (4.1.25) and (4.1.60) that the above estimate holds for any $R_1 \in (0, \frac{1}{c_1 M}]$ with

$$\frac{1}{c_1(data,\omega(\cdot),\rho(\cdot),R)} := \min\left\{\frac{R}{2},\frac{1}{4},\frac{\omega^{-1}(d_1)}{2},\frac{\rho^{-1}(d_2)}{2}\right\}\frac{1}{8c_0c_3}, \tag{4.1.91}$$

where

$$d_1 = \min\left\{\sqrt{\frac{n+1}{n}}-1,\frac{\sigma_0}{4},\frac{\nu}{2\Lambda}\right\},\quad d_2 = \min\left\{\frac{\sigma_0\gamma_3}{8},\frac{\sigma_1\gamma_3}{2},\frac{\gamma_3^2}{4\gamma_4},\frac{\sigma_1}{2},\frac{1}{2}\right\}$$

$$\omega^{-1}(\theta) := \sup\{0<r<1 : \omega(r)\le\theta\} \text{ and } \rho^{-1}(\theta) := \sup\{0<r<1 : \rho(r)\le\theta\},$$

for $\theta > 0$. Note that ω^{-1} and ρ^{-1} are well defined since ω and ρ are nondecreasing continuous functions with $\lim_{r\to 0+}\omega(r) = \lim_{r\to 0+}\rho(r) = 0$.

Global Estimates : Proof of (4.1.13)**.**

The estimates (4.1.13) can be obtained by using a standard covering argument with the local estimates (4.1.12). We first construct a covering of Ω. Let $R_1 = \frac{1}{c_1 M}$ where c_1 is given by Theorem 4.1.6. Since $\overline{\Omega}$ is compact, we can find a finite covering which consists of balls centered in Ω with radius $\frac{R_1}{3}$. Then Vitali's covering lemma implies that there exists a disjoint set $\left\{B_{\frac{R_1}{3}}(y_k)\right\}_{k=1}^{N}$, $N \in \mathbb{N}$ and $y_k \in \Omega$ such that $\{B_{R_1}(y_k)\}_{k=1}^{N}$ covers $\overline{\Omega}$. Note that there exists $c(n)$ depending only on the dimension n such that $\sum_{k=1}^{N}\int_{\Omega_{4R_1}(y_k)} f\,dx \le c(n)\int_{\Omega} f\,dx$.

Then, applying the estimate (4.1.12) with $x_0 = y_k$, $k = 1,2,\ldots,N$, we have

$$\begin{aligned}\int_\Omega |Du|^{p(x)q(x)}dx &\le \sum_{k=1}^N \int_{\Omega_{R_1}(y_k)} |Du|^{p(x)q(x)}dx\\ &\le \sum_{k=1}^N c\left\{R_1^n\left(\fint_{\Omega_{4R_1}(y_k)}\left[|Du|^{p(x)}+1\right]dx\right)^{\gamma_4}\right.\\ &\qquad\qquad\left.+\int_{\Omega_{4R_1}(y_k)}\left[|F|^{p(x)q(x)}+1\right]dx\right\}\\ &\le c\left\{R_1^{n(1-\gamma_4)}\left(\int_\Omega\left[|Du|^{p(x)}+1\right]dx\right)^{\gamma_4}+\int_\Omega\left[|F|^{p(x)q(x)}+1\right]dx\right\}.\end{aligned}$$

Replacing R_1 by $\frac{1}{c_1 M}$ and using (4.1.4) and Hölder's inequality, we finally obtain

$$\begin{aligned}&\int_\Omega |Du|^{p(x)q(x)}\,dx\\ &\quad\le c\left\{c_1^{n(\gamma_4-1)}\left(\int_\Omega\left[|F|^{p(x)}+1\right]\,dx\right)^{n(\gamma_4-1)+\gamma_4}+\int_\Omega\left[|F|^{p(x)q(x)}+1\right]\,dx+1\right\}\\ &\quad\le c\left\{|\Omega|^{\frac{(\gamma_3-1)(n(\gamma_4-1)+\gamma_4)}{\gamma_3}}\left(\int_\Omega\left[|F|^{p(x)}+1\right]^{\gamma_3}\,dx\right)^{\frac{n(\gamma_4-1)+\gamma_4}{\gamma_3}}\right.\\ &\qquad\qquad\left.+\int_\Omega\left[|F|^{p(x)q(x)}+1\right]\,dx+1\right\}\\ &\quad\le c\left(|\Omega|^{\frac{(\gamma_3-1)(n(\gamma_4-1)+\gamma_4)}{\gamma_3}}+1\right)\left(\int_\Omega\left[|F|^{p(x)q(x)}+1\right]\,dx+1\right)^{\frac{n(\gamma_4-1)+\gamma_4}{\gamma_3}}.\\ &\quad= c_2\left(\int_\Omega\left[|F|^{p(x)q(x)}+1\right]\,dx+1\right)^{\frac{n(\gamma_4-1)+\gamma_4}{\gamma_3}},\end{aligned}$$

for some $c_2 = c_2(data,\omega(\cdot),\rho(\cdot),R,\Omega) > 0$. This completes the proof of Theorem 4.1.6.

Before ending the chapter, we would like to mention two remarks.

Remark 4.1.19. *If $p(\cdot)$ is a constant function, we can derive a natural estimate in terms of norms on $L^{p(\cdot)q(\cdot)}$-spaces. We first note that, when $p(x) =$*

$\gamma_1 = \gamma_2 =: p$, *the equation* (4.0.2) *has the following scaling property: for any* $\alpha > 0$ *if we let*

$$\tilde{u}(x) = \frac{u(x)}{\alpha}, \quad \tilde{F}(x) = \frac{F(x)}{\alpha} \quad \textit{and} \quad \tilde{\mathbf{a}}(\xi, x) = \frac{\mathbf{a}(\alpha\xi, x)}{\alpha^{p-1}},$$

where $u \in W_0^{1,p}(\Omega)$ *is the weak solution of* (4.0.2), *then we see that* $\tilde{\mathbf{a}}$ *satisfies* (4.0.3) *and* (4.0.4) *with* $\mathbf{a}$, $p(x)$ *and* μ *replaced by* $\tilde{\mathbf{a}}$, p *and* $\tilde{\mu} := \frac{\mu}{\alpha}$, *respectively,* $(\tilde{\mathbf{a}}, p, \Omega)$ *is* (δ, R)*-vanishing, Definition 4.1.2 with* μ *replaced by* $\tilde{\mu}$, *and* $\tilde{u} \in W_0^{1,p}(\Omega)$ *is the unique weak solution of*

$$\begin{cases} \textit{div}\ \tilde{\mathbf{a}}(D\tilde{u}, x) &= \textit{div}\left(|\tilde{F}|^{p-2}\tilde{F}\right) \quad \textit{in} \quad \Omega, \\ \tilde{u} &= 0 \quad \textit{on} \quad \partial\Omega. \end{cases} \tag{4.1.92}$$

Set

$$\alpha = \|F\|_{L^{pq(\cdot)}(\Omega,\mathbb{R}^n)} + \mu$$

(without loss of generality we assume $\|F\|_{L^{pq(\cdot)}(\Omega)} > 0$*). Then we know* $0 \leq \tilde{\mu} \leq 1$ *and* $\|F\|_{L^{pq(\cdot)}(\Omega)} \leq 1$. *From the following fact*

$$\min_{\gamma\in\{p\gamma_3, p\gamma_4\}} \left(\int_\Omega |\mathbf{f}|^{pq(x)}\, dx\right)^{\frac{1}{\gamma}} \leq \|\mathbf{f}\|_{L^{pq(\cdot)}(\Omega,\mathbb{R}^n)} \leq \max_{\gamma\in\{p\gamma_3, p\gamma_4\}} \left(\int_\Omega |\mathbf{f}|^{pq(x)}\, dx\right)^{\frac{1}{\gamma}}, \tag{4.1.93}$$

see lemma 3.2.5 in [25], we have $\int_\Omega |F|^{pq(x)}\, dx \leq 1$. *Consequently, applying Theorem 4.1.6 to the equation* (4.1.92), *we obtain*

$$\int_\Omega |D\tilde{u}|^{pq(x)}\, dx \leq c,$$

for some constant $c = c(n, \nu, \Lambda, p, \gamma_3, \gamma_4, L_{q(\cdot)}, \rho(\cdot), R, \Omega) > 1$. *Note that all the constants and estimates in Theorem 4.1.6 are independent of the choice of* $\mu \in [0, 1]$. *Consequently, by* (4.1.93) *we have*

$$\|D\tilde{u}\|_{L^{pq(\cdot)}(\Omega,\mathbb{R}^n)} \leq c^{\frac{1}{p\gamma_3}},$$

equivalently,

$$\|Du\|_{L^{pq(\cdot)}(\Omega,\mathbb{R}^n)} \leq c^{\frac{1}{p\gamma_3}} \left(\|F\|_{L^{pq(\cdot)}(\Omega,\mathbb{R}^n)} + \mu\right).$$

Moreover, in view of Poincaré's inequality in $W_0^{1,pq(\cdot)}(\Omega)$*, we also have*

$$\|u\|_{W^{1,pq(\cdot)}(\Omega)} \leq c\left(\|F\|_{L^{pq(\cdot)}(\Omega,\mathbb{R}^n)} + \mu\right),$$

for some $c > 0$ *depending only on* $n, \nu, \Lambda, p, \gamma_3, \gamma_4, L_{q(\cdot)}, \rho(\cdot), R$ *and* Ω.

Remark 4.1.20. *Theorem 4.1.6 can be extended to* $p(\cdot)$*-Laplace systems. Let* $\mathbf{F} : \Omega \to \mathbb{R}^{nm}$ *belong to* $L^{p(\cdot)}(\Omega, \mathbb{R}^{nm})$ *and* $a : \mathbb{R}^n \to \mathbb{R}$ *satisfy*

$$0 < \nu \leq a(x) \leq \Lambda < \infty.$$

Consider the unique weak solution $\mathbf{u} \in W_0^{1,p(\cdot)}(\Omega, \mathbb{R}^m)$ *of the following* $p(x)$*-Laplace system*

$$\begin{cases} div\left(a(x)|D\mathbf{u}|^{p(x)-2}D\mathbf{u}\right) = div\left(|\mathbf{F}|^{p(x)-2}\mathbf{F}\right) & in \quad \Omega, \\ \mathbf{u} = 0 & on \quad \partial\Omega. \end{cases} \tag{4.1.94}$$

Then, with the same spirit as in the (4.0.2)*, one can obtain the same result to Theorem 4.1.6 for* (4.1.94)*, that is,*

$$\mathbf{F} \in L^{p(\cdot)q(\cdot)}(\Omega, \mathbb{R}^{nm}) \implies D\mathbf{u} \in L^{p(\cdot)q(\cdot)}(\Omega, \mathbb{R}^{nm})$$

with the estimates (4.1.12) *and* (4.1.13) *replaced* u *and* F *by* $\mathbf{u}$ *and* $\mathbf{F}$*, respectively, under the sufficiently small* $\delta > 0$*. At that case, we denote by*

$$M = \int_\Omega \left[|\mathbf{F}|^{p(x)} + 1\right] dx + 1,$$

and the assumption (2) *in Definition 4.1.2 is replaced by*

$$\sup_{y\in\mathbb{R}^n} \sup_{0<r\leq R_0} \fint_{B_r(y)} |a(x) - \overline{a}_{B_r(y)}|\, dx \leq \delta.$$

4.2 Global gradient estimates for elliptic equations of $p(x)$-Laplacian type with BMO nonlinearity.

We consider the sepecial case of the result in Section 4.1 that the variable function $q(\cdot)$ is a constant, $q(\cdot) \equiv q$. In this case, we present a different proof.

4.2.1 Main result.

We recall notations used in Section 4.1.1.

Theorem 4.2.1. *[9] Let $u \in W_0^{1,p(\cdot)}(\Omega)$ be the weak solution of* (4.0.2)*. Suppose that $|F|^{p(\cdot)} \in L^q(\Omega)$ for some $q \in (1,\infty)$. Then there exists $\tilde{\sigma} > 0$ depending only on $n, \gamma_1, \gamma_2, \nu, \Lambda, q, \omega(\cdot), M$, such that the following holds: For every $\sigma \in (0, \tilde{\sigma})$, there exists $\delta = \delta(n, \gamma_1, \gamma_2, \nu, \Lambda, q) > 0$ such that if $(\mathbf{a}, \Omega)$ is (δ, R)-vanishing for some $R > 0$ and $p(\cdot)$ satisfies*

$$\lim_{r\to 0} \omega(r) \log\left(\frac{1}{r}\right) = 0,$$

then we have that $|Du|^{p(\cdot)} \in L^q(\Omega)$ with the estimates

$$\left(\int_\Omega |Du|^{p(x)q}\, dx\right)^{1/q} \leq cM^\sigma \left(\int_\Omega \left[|F|^{p(x)q} + 1\right] dx\right)^{1/q},$$

where $c = c(n, \nu, \Lambda, \gamma_1, \gamma_1, \omega(\cdot), R, q, \sigma, \Omega)$.

4.2.2 Proof of Theorem 4.2.1.

In this section let us fix any point $x_0 \in \Omega$ and concentrate on the small region

$$\Omega_{R_0} = \Omega_{R_0}(x_0) = \Omega \cap B_{R_0}(x_0),$$

by assuming that

$$R_0 \leq \min\left\{\frac{R}{2}, \frac{1}{4}, \frac{1}{2M_1}\right\} \quad \text{and} \quad \omega(2R_0) \leq \sqrt{\frac{n+1}{n}} - 1 < 1. \tag{4.2.1}$$

Covering argument.

We write

$$\lambda_0 = \fint_{\Omega_{2R_0}} \left[|Du|^{p(x)} + \frac{1}{\delta}|F|^{p(x)}\right] dx + 1 \tag{4.2.2}$$

and

$$E(\lambda) = \{y \in \Omega_{R_0} : |Du(y)|^{p(y)} > \lambda\}, \tag{4.2.3}$$

where λ is any number satisfying

$$\lambda \geq \left(\frac{4800}{7}\right)^n \lambda_0, \tag{4.2.4}$$

where δ is to be determined later. Given a fixed point $y \in E(\lambda)$, we define a continuous function $G_y : (0, R_0] \to [0, \infty)$ by

$$G_y(\rho) = \fint_{\Omega_\rho(y)} \left[|Du|^{p(x)} + \frac{1}{\delta}|F|^{p(x)}\right] dx. \tag{4.2.5}$$

Then it follows from Lebesgue's theorem, (4.2.3) and (4.2.5) that

$$\lim_{\rho \to 0} G_y(\rho) = |Du(y)|^{p(y)} + \frac{1}{\delta}|F(y)|^{p(y)} \geq |Du(y)|^{p(y)} > \lambda, \tag{4.2.6}$$

for almost everywhere $y \in E(\lambda)$. On the other hand, from (4.1.8) and (4.2.4), for any $\rho \in [\frac{R_0}{150}, R_0]$

$$\begin{aligned} G_y(\rho) &= \fint_{\Omega_\rho(y)} \left[|Du|^{p(x)} + \frac{1}{\delta}|F|^{p(x)}\right] dx \\ &\leq \frac{|\Omega_{2R_0}|}{|\Omega_\rho(y)|} \cdot \fint_{\Omega_{2R_0}} \left[|Du|^{p(x)} + \frac{1}{\delta}|F|^{p(x)}\right] dx \\ &\leq \frac{|B_{2R_0}|}{|B_\rho(y)|}\frac{|B_\rho(y)|}{|\Omega_\rho(y)|}\lambda_0 < \left(\frac{2R_0}{\rho}\right)^n \left(\frac{16}{7}\right)^n \lambda_0 \\ &\leq \left(\frac{4800}{7}\right)^n \lambda_0 \ \leq \ \lambda. \end{aligned} \tag{4.2.7}$$

Since G_y is a continuous map, by (4.2.6) and (4.2.7), one can see that for almost every $y \in E(\lambda)$ there exists a number $\rho_y \in (0, \frac{R_0}{150})$ such that

$$G_y(\rho_y) = \lambda \ \text{ and } \ G_y(\rho) < \lambda \text{ for all } \rho \in (\rho_y, R_0].$$

Applying Vitali's covering Lemma, we obtain the following:

Lemma 4.2.2. *Under the same notation and assumptions in (4.2.1)-(4.2.5), there exists a family of disjoint balls* $\{\Omega_{\rho_i}(y_i)\}_{i=1}^\infty$ *with* $y_i \in E(\lambda)$ *and* $\rho_i = \rho_{y_i} \in \left(0, \frac{R_0}{150}\right)$ *such that*

$$G_{y_i}(\rho_i) = \lambda \ \ \text{and} \ \ G_{y_i}(\rho) < \lambda \ \ \text{for all} \ \ \rho \in (\rho_i, R_0]. \tag{4.2.8}$$

and

$$E(\lambda) \subset \bigcup_{i\geq 1} \Omega_{5\rho_i}(y_i) \cup \textit{negligible set.}$$

We now focus on the points $y_i \in E(\lambda)$ and $\rho_i \in \left(0, \frac{R_0}{150}\right)$ selected in Lemma 4.2.2. In the following we concentrate our attention on $\Omega_{\rho_i}(y_i)$. Firstly, let us estimate $|\Omega_{\rho_i}(y_i)|$. According to (4.2.5) and (4.2.8),

$$\begin{aligned} \lambda|\Omega_{\rho_i}(y_i)| &= \int_{\Omega_{\rho_i}(y_i)} \left[|Du|^{p(x)} + \frac{1}{\delta}|F|^{p(x)}\right] dx \\ &\leq \int_{\{x\in\Omega_{\rho_i}(y_i):|Du(x)|^{p(x)}>\frac{\lambda}{4}\}} |Du(x)|^{p(x)}dx + \frac{\lambda}{4}|\Omega_{\rho_i}(y_i)| \\ &\quad +\frac{1}{\delta}\int_{\{x\in\Omega_{\rho_i}(y_i):|F(x)|^{p(x)}>\frac{\delta\lambda}{4}\}} |F(x)|^{p(x)}dx + \frac{\lambda}{4}|\Omega_{\rho_i}(y_i)|. \end{aligned}$$

Consequently,

$$|\Omega_{\rho_i}(y_i)| \leq \frac{2}{\lambda}\left(\int_{\{x\in\Omega_{\rho_i}(y_i):|Du|^{p(x)}>\frac{\lambda}{4}\}} |Du|^{p(x)}dx + \frac{1}{\delta}\int_{\{x\in\Omega_{\rho_i}(y_i):|F|^{p(x)}>\frac{\delta\lambda}{4}\}} |F|^{p(x)}dx\right). \tag{4.2.9}$$

Our next argument for comparison estimates depends on

$$\text{whether } B_{20\rho_i}(y_i) \subset \Omega, \text{ or, } B_{20\rho_i}(y_i)\cap\Omega \neq B_{20\rho_i}(y_i).$$

The former is the interior case and the latter is the boundary case.

For the interior case, for simplicity, we use the following notation:

$$B_i^0 = B_{\rho_i}(y_i) = \Omega_{\rho_i}(y_i),\ B_i^j = B_{5j\rho_i}(y_i) = \Omega_{5j\rho_i}(y_i), \quad j = 1, 2, 3, 4. \tag{4.2.10}$$

Note that $0 < \rho_i < 5j\rho_i \leq 20\rho_i < \frac{2R}{15}$, to see from (4.2.5) and (4.2.8) that

$$\fint_{B_i^j} \left[|Du|^{p(x)} + \frac{1}{\delta}|F|^{p(x)}\right] dx < \lambda.$$

For the boundary case, one can find a boundary point $\overline{y_i}$ with

$$\overline{y_i} \in B_{20\rho_i}(y_i) \cap \partial\Omega \text{ and } \Omega_{5\rho_i}(y_i) \subset \Omega_{25\rho_i}(\overline{y_i}). \tag{4.2.11}$$

Then from the (δ, R_0)-Reifenberg flatness condition, see (3) of Definition 4.1.2 when $r = 150\rho_i \leq R \leq R_0$, there exists a new coordinate system in $z = z(i) = (z^1, \cdots, z^n)$-variables so that in this coordinate system,

$$\begin{cases} y_i = z_i,\ \overline{y_i} + 125\delta\rho_i(0, \cdots, 1) \text{ is the origin}, \\ B^+_{125\rho_i} \ \subset \ \Omega_{125\rho_i} \ \subset \ B_{125\rho_i} \cap \{z^n > -250\delta\rho_i\}. \end{cases} \tag{4.2.12}$$

We select δ so small, in order to get

$$\delta \ \leq \ \frac{1}{250}, \tag{4.2.13}$$

which assures that

$$\Omega_{5\rho_i}(z_i) \subset \Omega_{25\rho_i}(0), \tag{4.2.14}$$

in this z-coordinate system.

We next write

$$\Omega_i^0 = \Omega_{\rho_i}(z_i),\ \Omega_i^j = \Omega_{25j\rho_i}(0), \quad j = 1, 2, 3, 4, 5, \tag{4.2.15}$$

and observe from (4.2.11)-(4.2.15) that

$$B_i^{j+} = B^+_{25j\rho_i} \ \subset \ \Omega_i^j \ \subset \ B_{25j\rho_i} \cap \{z^n > -250\delta\rho_i\} \tag{4.2.16}$$

and

$$\Omega_i^j \subset \Omega_{125\rho_i} \subset \Omega_{150\rho_i}(z_i). \tag{4.2.17}$$

Then it follows from (4.2.15) and (4.2.17) that

$$\begin{aligned} &\fint_{\Omega_i^j} \left[|Du(z)|^{p(z)} \ + \ \frac{1}{\delta}|F(z)|^{p(z)} \right] dz \\ &\qquad \leq \frac{|\Omega_{150\rho_i(z_i)}|}{|\Omega_{25\rho_i}|} \fint_{\Omega_{150\rho_i}(z_i)} \left[|Du|^{p(z)} + \frac{1}{\delta}|F|^{p(z)} \right] dz \\ &\qquad \leq \frac{|B_{150\rho_i}(z_i)|}{|B^+_{25\rho_i}|} \fint_{\Omega_{150\rho_i}(z_i)} \left[|Du|^{p(z)} + \frac{1}{\delta}|F|^{p(z)} \right] dz \\ &\qquad \leq 2 \cdot 5^n \fint_{\Omega_{150\rho_i}(z_i)} \left[|Du|^{p(z)} + \frac{1}{\delta}|F|^{p(z)} \right] dz. \end{aligned}$$

Employing (4.2.8) and change of variable like (4.2.12), we deduce

$$\fint_{\Omega_i^j} \left[|Du(z)|^{p(z)} + \frac{1}{\delta}|F(z)|^{p(z)} \right] dz \leq 2 \cdot 5^n \lambda. \tag{4.2.18}$$

Comparison maps.

We now mainly treat the boundary case. For the interior case, one can find similar results by applying, in a similar but a much simpler way, the ideas and the techniques that will be used for the boundary case. In this case, with a proper translation and rotation of the original coordinates, as in (4.2.11)-(4.2.12), the related quantities are still invariant under such rotation and translation. For this reason, we keep using the same coordinates.

We select σ so small that

$$\sigma \leq \tilde{\sigma} := \min \left\{ \sigma_0,\ 4(\gamma_1 - 1),\ \frac{\log 2}{q \log (c_0 M)} \right\}, \tag{4.2.19}$$

where $c_0 M$ is given by (4.1.11). Recall the conditions on $\omega(\cdot)$, the modulus continuity of $p(\cdot)$, to choose R small enough that

$$\omega(2R) \leq \frac{\sigma}{4} \leq \gamma_1 - 1. \tag{4.2.20}$$

We set

$$p_1 = \min_{x \in \overline{\Omega_i^4}} p(x),\ p_2 = \max_{x \in \overline{\Omega_i^4}} p(x) \text{ and } r_i = 100\rho_i \ \left(\leq \frac{2}{3} R \right). \tag{4.2.21}$$

Then we have

$$p_2 - p_1 \leq \omega(2r_i) \leq \omega(2R) \leq \frac{\sigma}{4}. \tag{4.2.22}$$

A direct computation yields, for every $x \in \Omega_i^4$,

$$p_2 = p_1 + (p_2 - p_1) \leq p_1(1 + (p_2 - p_1)) \leq p(x)\,(1 + \omega(2r_i)) \leq p(x) \left(1 + \frac{\sigma}{4}\right). \tag{4.2.23}$$

Moreover, it follows from (4.2.20)-(4.2.23) that

$$\begin{aligned} p_2 \left(1 + \frac{\sigma}{4}\right) &\leq \Big(p_1 + \omega(2r_i)\Big) \left(1 + \frac{\sigma}{4}\right) \\ &\leq p_1 \left(1 + \frac{\sigma}{4}\right) + \omega(2r_i)\gamma_1 \\ &\leq p(x) \left(1 + \frac{\sigma}{4} + \omega(2r_i)\right) \\ &\leq p(x) \left(1 + \frac{\sigma}{2}\right). \end{aligned} \tag{4.2.24}$$

For the boundary case that $B_{20\rho_i} \subset \Omega$, by similar a computation in Lemma 4.1.15 and (4.1.59), we have the following lemma. For the detail proof, see [9].

Lemma 4.2.3. *Let u be the weak solution of (4.0.2). Then for any $0 < \epsilon < 1$, there exists $\delta = \delta(\epsilon, n, \gamma_1, \gamma_2, \nu, \Lambda)$, $R = R(\epsilon, n, \gamma_1, \gamma_2, \nu, \Lambda, \sigma, \omega(\cdot))$, $w_i \in W^{1,p(\cdot)}(\Omega_i^4) \cap W^{1,p_2}(\Omega_i^3)$, $v_i \in W^{1,p_2}(\Omega_i^3) \cap W^{1,\infty}(\Omega_i^1)$ such that*

$$\fint_{B_i^4} |Du - Dw_i|^{p(x)} dx + \fint_{B_i^3} |Dw_i - Dv_i|^{p_2} dx \leq \epsilon K_i^{\sigma} \lambda$$

and

$$\|D\overline{v}_i\|_{L^{\infty}(B_i^1)}^{p_2} \leq \overline{\lambda}_1 K_i^{\sigma} \lambda,$$

where

$$K_i = \int_{B_i^4} |Du|^{p(x)} dx + 1,$$

and the universal constant $\overline{\lambda}_1 \in (1, \infty)$ is independent of u, F and i.

On the other hand, for the boundary case that $B_{20\rho_i} \not\subset \Omega$, by similar computations in Lemma 4.1.12 and Lemma 4.1.14 and (4.1.55), we have the following lemma. For the detail proof, see [9].

Lemma 4.2.4. *Let u be the weak solution of (4.0.2). Then for any $0 < \epsilon < 1$, there exists $\delta = \delta(\epsilon, n, \gamma_1, \gamma_2, \nu, \Lambda)$, $R = R(\epsilon, n, \gamma_1, \gamma_2, \nu, \Lambda, \sigma, \omega(\cdot))$, $w_i \in W^{1,p(\cdot)}(\Omega_i^4) \cap W^{1,p_2}(\Omega_i^3)$, $v_i \in W^{1,p_2}(\Omega_i^3)$ and $\overline{v}_i \in W^{1,p_2}(\Omega_i^2) \cap W^{1,\infty}(\Omega_i^1)$ such that*

$$\fint_{\Omega_i^4} |Du - Dw_i|^{p(x)} dx + \fint_{\Omega_i^3} |Dw_i - Dv_i|^{p_2} dx + \fint_{\Omega_i^2} |Dv_i - D\overline{v}_i|^{p_2} dx \leq \epsilon K_i^{\sigma} \lambda$$

and

$$\|Dv_i\|_{L^{\infty}(\Omega_i^1)}^{p_2} \leq \lambda_1 K_i^{\sigma} \lambda,$$

where

$$K_i = \int_{\Omega_i^4} |Du|^{p(x)} dx + 1$$

and the universal constant $\lambda_1 \in (1, \infty)$ is independent from u, F and i.

A priori estimates.

We prove the a priori estimate. More precisely, we essentially obtain

$$\int_\Omega |Du|^{p(x)q}\,dx \;\leq\; c\,M^{\sigma q}\int_\Omega \big(|F|^{p(x)q}+1\big)\,dx, \tag{4.2.25}$$

under the a priori assumption

$$\int_\Omega |Du|^{p(x)q}\,dx < +\infty, \tag{4.2.26}$$

where $\sigma \leq \tilde{\sigma}$ as in (4.2.19) and M is the number given by (4.1.11). This assumption can be removed by a standard approximation argument, see cite-BOR2.

We recall the standard inequality

$$|\{x\in E : g(x)>\lambda\}| \;\leq\; \frac{1}{\lambda}\int_E g(x)dx. \tag{4.2.27}$$

We now fix any point $x\in\Omega$. Then we select a universal constant

$$R_0 = R_0(\epsilon, n, \gamma_1, \gamma_2, \nu, \Lambda, \sigma, \omega(\cdot), R) > 0 \tag{4.2.28}$$

so that the prescribed conditions (4.2.1), (4.2.20), Lemma 4.2.3 and Lemma 4.2.4 hold true for such a small R_0. Hereafter we mainly focus on the domain $\Omega_{2R_0} = \Omega_{2R_0}(x)$.

In the previous section we have made a covering argument on the λ-upper level set of $|Du(\cdot)|^{p(\cdot)}$ for any sufficiently large number λ with (4.2.4) and have made comparison estimates there based on the regularity assumptions on the nonlinearity and the boundary of the domain.

According to Lemma 4.2.2, there exists a family of disjoint members $\{\Omega_{\rho_i}(y_i)\}_{i=1}^{\infty}$ such that $E(\lambda)\subset\bigcup_{i\geq 1}\Omega_{5\rho_i}(y_i)$ without a measure zero set. In the interior case that $B_i^4 = B_{20\rho_i}(y_i)\subset\Omega$, we find from Lemma 4.2.3 that for any $\epsilon\in(0,1)$, there exists a small $\delta=\delta(\epsilon,\nu,\Lambda,n,\gamma_1,\gamma_2)>0$ such that

$$\fint_{B_i^4}|Du-Dw_i|^{p(x)}dx + \fint_{B_i^2}|Dw_i-D\overline{v}_i|^{p_{i_2}}dx \leq \epsilon K_i^{\sigma}\lambda \tag{4.2.29}$$

and

$$\|D\overline{v}_i\|_{L^\infty(B_i^1)} \leq \overline{\lambda}_1 K_i^{\sigma}\lambda, \tag{4.2.30}$$

where $p_{i_2} = \max_{x \in \overline{B_i^4}} p(x)$. On the other hand, for the boundary case $B_i^4 \not\subset \Omega$, we find from Lemma 4.2.4 that for any $\epsilon \in (0,1)$, there exists a small $\delta = \delta(\epsilon, \nu, \Lambda, n, \gamma_1, \gamma_2) > 0$ such that

$$\fint_{\Omega_i^4} |Du - Dw_i|^{p(x)} dx + \fint_{\Omega_i^3} |Dw_i - Dv_i|^{p_2} dx + \fint_{\Omega_i^2} |Dv_i - D\overline{v}_i|^{p_2} dx \leq \epsilon K_i^\sigma \lambda \tag{4.2.31}$$

and

$$\|Dv_i\|_{L^\infty(\Omega_i^1)} \leq \lambda_1 K_i^\sigma \lambda, \tag{4.2.32}$$

where $p_{i2} = \max_{x \in \overline{\Omega_i^4}} p(x)$. Consequently, there exists

$$\delta = \delta(\epsilon, \nu, \Lambda, n, \gamma_1, \gamma_2) > 0 \tag{4.2.33}$$

such that (4.2.29) - (4.2.32) hold.

We now write

$$\lambda_2 = \max\{\overline{\lambda}_1, \lambda_1\} \quad \text{and} \quad K = \int_{\Omega_{2R}} \left(|Du|^{p(x)} + 1 \right) dx + 1. \tag{4.2.34}$$

Then we observe from the standard $L^{p(x)}$-estimate (4.1.11) that

$$K_i \leq c_0 M.$$

Also we observe that

$$E\left(5 \cdot 8^{\gamma_1 - 1} \lambda_2 \left(c_0 M\right)^\sigma \lambda\right) \subset E(\lambda) \subset \bigcup_{i \geq 1} \Omega_{5\rho_i}(y_i). \tag{4.2.35}$$

Using (4.2.34) and (4.2.35), we separate the resulting estimation into the interior and boundary cases, to derive that

$$\begin{aligned}
&|E\left(5 \cdot 8^{\gamma_1 - 1} \lambda_2 \left(c_0 M\right)^\sigma \lambda\right)| \\
&= |\{y \in \Omega_R : |Du(y)|^{p(y)} \geq 5 \cdot 8^{\gamma_1 - 1} \lambda_2 (c_0 M)^\sigma \lambda\}| \\
&\leq \sum_{i \geq 1} |\{y \in \Omega_{5\rho_i}(y_i) : |Du(y)|^{p(y)} \geq 5 \cdot 8^{\gamma_1 - 1} \lambda_2 (c_0 M)^\sigma \lambda\}| \\
&= \sum_{i:\text{interior case}} |\{y \in \Omega_{5\rho_i}(y_i) : |Du(y)|^{p(y)} \geq 5 \cdot 8^{\gamma_1 - 1} \lambda_2 K_i^\sigma \lambda\}| \\
&\qquad + \sum_{i:\text{boundary case}} |\{y \in \Omega_{5\rho_i}(y_i) : |Du(y)|^{p(y)} \geq 5 \cdot 8^{\gamma_1 - 1} \lambda_2 K_i^\sigma \lambda\}|.
\end{aligned} \tag{4.2.36}$$

In the interior case, it follows from (4.2.10), (4.2.27), (4.2.29), (4.2.30) and (4.2.34) that

$$\begin{aligned}
&|\{y \in B_i^1 : |Du(y)|^{p(y)} \geq 5 \cdot 8^{\gamma_1 - 1}\lambda_2 K_i^\sigma \lambda\}| \\
&\quad \leq |\{y \in B_i^1 : |Du(y)|^{p(y)} \geq 4^{\gamma_1}\overline{\lambda}_1 K_i^\sigma \lambda\}| \\
&\quad \leq |\{y \in B_i^1 : |Du - Dw_i|^{p(y)} \geq \overline{\lambda}_1 K_i^\sigma \lambda\}| \\
&\qquad + |\{y \in B_i^1 : |Dw_i - Dv_i|^{p_{i2}} \geq \overline{\lambda}_1 K_i^\sigma \lambda\}| + |\{y \in B_i^1 : |Dv_i|^{p_{i2}} \geq \overline{\lambda}_1 K_i^\sigma \lambda\}| \\
&\quad \leq \frac{1}{\overline{\lambda}_1 K_i^\sigma \lambda} \int_{B_i^1} \left(|Du - Dw_i|^{p(x)} + |Dw_i - Dv_i|^{p_{i2}}\right) dx \\
&\quad \leq \epsilon |B_i^4| = 20^n \epsilon \, |B_i^0|. \qquad (4.2.37)
\end{aligned}$$

In the boundary case, it follows from (4.2.14), (4.2.15), (4.2.27), (4.2.31), (4.2.32) and (4.2.34) that

$$\begin{aligned}
&|\{y \in \Omega_{5\rho_i}(y_i) : |Du(y)|^{p(y)} \geq 5 \cdot 8^{\gamma_1 - 1}\lambda_2 K_i^\sigma \lambda\}| \\
&\quad = |\{z \in \Omega_{5\rho_i}(z_i) : |Du(z)|^{p(z)} \geq 5 \cdot 8^{\gamma_1 - 1}\lambda_2 K_i^\sigma \lambda\}| \\
&\quad \leq |\{z \in \Omega_i^1 : |Du(z)|^{p(z)} \geq 5 \cdot 8^{\gamma_1 - 1}\lambda_1 K_i^\sigma \lambda\}| \\
&\quad \leq |\{z \in \Omega_i^1 : |Du - Dw_i|^{p(z)} \geq \lambda_1 K_i^\sigma \lambda\}| \\
&\qquad + |\{z \in \Omega_i^1 : |Dw_i - Dv_i|^{p_{i2}} \geq \lambda_1 K_i^\sigma \lambda\}| \\
&\qquad + |\{z \in \Omega_i^1 : |Dv_i - D\overline{v}_i|^{p_{i2}} \geq \lambda_1 K_i^\sigma \lambda\}| + |\{z \in \Omega_i^1 : |D\overline{v}_i|^{p_{i2}} \geq \lambda_1 K_i^\sigma \lambda\}| \\
&\quad \leq \frac{1}{\lambda_1 K_i^\sigma \lambda} \int_{\Omega_i^1} \left(|Du - Dw_i|^{p(z)} + |Dw_i - Dv_i|^{p_{i2}} + |Dv_i - D\overline{v}_i|^{p_{i2}}\right) dz \\
&\qquad \leq \epsilon |\Omega_i^5| \leq \epsilon |B_{125\rho_i}| = 125^n \epsilon \, |B_{\rho_i}| \leq 125^n \left(\frac{16}{7}\right)^n \epsilon |\Omega_i^0|. \qquad (4.2.38)
\end{aligned}$$

At the last inequality, we recall the measure density condition on the *Reifenberg flat* domain, see (4.1.8). We now combine (4.2.36), (4.2.37) and (4.2.38), to derive that

$$E(5 \cdot 8^{\gamma_1 - 1}\lambda_2 (c_0 M)^\sigma \lambda) \ \leq \ c\epsilon \sum_{i \geq 1} |\Omega_{\rho_i}(y_i)|.$$

Using Lemma 4.2.2 and (4.2.9), we conclude that

$$\begin{aligned} E(5\cdot 8^{\gamma_1-1}\lambda_2(c_0M)^\sigma\lambda) &\leq c\epsilon\frac{1}{\lambda}\sum_{i\geq 1}\int_{\Omega_{\rho_i}(y_i)\cap\{|Du|^{p(x)}\geq\frac{\lambda}{4}\}}|Du|^{p(x)}dx \\ &\quad + c\epsilon\frac{1}{\delta\lambda}\sum_{i\geq 1}\int_{\Omega_{\rho_i}(y_i)\cap\{|F|^{p(x)}\geq\frac{\delta\lambda}{4}\}}|F|^{p(x)}dx \\ &\leq c\epsilon\frac{1}{\lambda}\int_{\{x\in\Omega_{2R_0}:|Du|^{p(x)}\geq\frac{\lambda}{4}\}}|Du|^{p(x)}dx \\ &\quad + c\epsilon\frac{1}{\delta\lambda}\int_{\{x\in\Omega_{2R_0}:|F|^{p(x)}\geq\frac{\delta\lambda}{4}\}}|F|^{p(x)}dx, \quad (4.2.39) \end{aligned}$$

where we have used the fact that $\{\Omega_{\rho_i}(y_i)\}$ is disjoint for the last inequality. For the sake of simplicity, we write

$$A = 5\cdot 8^{\gamma_1-1}\lambda_2\,(c_0M)^\sigma \text{ and } B = \left(\frac{4800}{7}\right)^n. \tag{4.2.40}$$

Now, we start the local estimate of $|Du|^{p(x)q}$. By (2.3.1), we have

$$\begin{aligned} \int_{\Omega_{R_0}}|Du|^{p(x)q}dx &= \int_{\Omega_{R_0}}\left(|Du|^{p(x)}\right)^q dx \\ &\leq \int_0^\infty q(A\lambda)^{q-1}\left|\left\{x\in\Omega_{R_0}:|Du|^{p(x)}\geq A\lambda\right\}\right|\,d(A\lambda) \\ &\leq A^q\int_0^\infty q\lambda^{q-1}\left|\left\{x\in\Omega_{R_0}:|Du|^{p(x)}\geq A\lambda\right\}\right|d\lambda \\ &\leq A^q\int_0^{B\lambda_0} q\lambda^{q-1}\left|\left\{x\in\Omega_{R_0}:|Du|^{p(x)}\geq A\lambda\right\}\right|d\lambda \\ &\qquad +A^q\int_{B\lambda_0}^\infty q\lambda^{q-1}\ \left|\left\{x\in\Omega_{R_0}:|Du|^{p(x)}\geq A\lambda\right\}\right|d\lambda \\ &= II_1+II_2. \end{aligned} \tag{4.2.41}$$

Estimate for II_1:

$$II_1 = A^q\int_0^{B\lambda_0} q\lambda^{q-1}\left|\left\{x\in\Omega_{R_0}:|Du|^{p(x)}\geq A\lambda\right\}\right|d\lambda \leq (A\cdot B\cdot\lambda_0)^q|\Omega_{R_0}|. \tag{4.2.42}$$

Estimate for II_2: Applying in turn (2.3.2) and (4.2.39), we derive

$$\begin{aligned} II_2 &= A^q \int_{B\lambda_0}^{\infty} q\lambda^{q-1} \left|\left\{x \in \Omega_{R_0} : |Du|^{p(x)} \geq A\lambda\right\}\right| d\lambda \\ &\leq cA^q \epsilon \int_0^{\infty} \lambda^{q-2} \left(\int_{\{x\in\Omega_{2R_0}:|Du|^{p(x)}\geq\frac{\lambda}{4}\}} |Du|^{p(x)} dx \right) d\lambda \\ &\quad + cA^q \epsilon \int_0^{\infty} \lambda^{q-2} \left(\int_{\{x\in\Omega_{2R_0}:|F|^{p(x)}\geq\frac{\delta\lambda}{4}\}} \frac{|F|^{p(x)}}{\delta}\, dx \right) d\lambda \\ &\leq cA^q \left(\epsilon \int_{\Omega_{2R_0}} |Du|^{p(x)q} dx + c(\epsilon,\delta) \int_{\Omega_{2R_0}} |F|^{p(x)q} dx \right). \qquad (4.2.43) \end{aligned}$$

We combine (4.2.40), (4.2.41), (4.2.42) and (4.2.44), to derive

$$\begin{aligned} \int_{\Omega_{R_0}} |Du|^{p(x)q} dx &\leq cM^{\sigma q} \epsilon \int_{\Omega_{R_0}} |Du|^{p(x)q} dx \\ &\quad + cK^{\sigma q}[\lambda_0]^q |\Omega_{R_0}| + c(\epsilon,\delta) cM^{\sigma q} \int_{\Omega_{2R_0}} |F|^{p(x)q} dx. \end{aligned}$$

From (4.2.19), we observe that

$$M^{\sigma q} \leq (c_0 M)^{\tilde{\sigma} q} \leq 2,$$

We then take $\epsilon = \epsilon(n, q, \gamma_1, \gamma_2, \nu, \Lambda) > 0$ small enough, in order to obtain

$$\begin{aligned} \int_{\Omega_{R_0}} |Du|^{p(x)q} dx &\leq \frac{1}{2} \int_{\Omega_{2R_0}} |Du|^{p(x)q} dx \\ &\quad + cK^{\sigma q}[\lambda_0]^q |\Omega_{R_0}| + cK^{\sigma q} \int_{\Omega_{2R_0}} |F|^{p(x)q} dx. \qquad (4.2.44) \end{aligned}$$

Once the selection of ϵ is made, one can find a corresponding $\delta = \delta(n, q, \gamma_1, \gamma_2, \nu, \Lambda)$ and $R = R(n, q, \gamma_1, \gamma_2, \nu, \Lambda, \sigma, \omega(\cdot), R_0)$, see (4.2.28) and (4.2.33), for which the relevant results in the previous section hold.

We now replace R and $2R$ by $s_1 R$ and $s_2 R$, respectively, where $1 \leq s_1 < s_2 \leq 2$, and repeat the procedure we have made for (4.2.44), as shown in [7], to discover

$$\begin{aligned}\int_{\Omega_{s_1R}} |Du|^{p(x)q}dx &\leq \frac{1}{2}\int_{\Omega_{s_2R}} |Du|^{p(x)q}dx \\ &\quad + cK^{\sigma q}\frac{(\lambda_0)^q}{(s_2-s_1)^{nq}}|\Omega_{R_0}| + cK^{\sigma q}\int_{\Omega_{s_2R}} |F|^{p(x)q}dx.\end{aligned}$$

By Lemma 5.2.9 and (4.2.2), we find

$$\begin{aligned}\int_{\Omega_{R_0}} |Du|^{p(x)q}dx &\leq cK^{\sigma q}[\lambda_0]^q|\Omega_{R_0}| + cK^{\sigma q}\int_{\Omega_{2R_0}} |F|^{p(x)q}dx \\ &\leq cK^{\sigma q}\left(\fint_{\Omega_{2R_0}} |Du|^{p(x)}dx + \fint_{\Omega_{2R_0}} |F|^{p(x)}dx + 1\right)^q |\Omega_{R_0}| \\ &\quad + cK^{\sigma q}\int_{\Omega_{2R_0}} |F|^{p(x)q}dx,\end{aligned}$$

from which we arrive at

$$\begin{aligned}&\left(\fint_{\Omega_{R_0}} |Du|^{p(x)q}dx\right)^{\frac{1}{q}} \\ &\leq cK^{\sigma}\left[\fint_{\Omega_{2R_0}} |Du|^{p(x)}dx + \left(\fint_{\Omega_{2R_0}} \left(|F|^{p(x)q}+1\right)dx\right)^{\frac{1}{q}}\right].\end{aligned} \tag{4.2.45}$$

This is the local estimate up to the boundary. Note that the constant c in the above inequality is dependent only on $n, q, \gamma_1, \gamma_2, \nu$, and Λ.

We next use the standard covering argument, to obtain the global a priori estimate (4.2.25). Since Ω is bounded in $\mathbb{R}^n$, there exists $N \in \mathbb{N}$ and $x_k \in \Omega$ for $k = 1, \cdots, N$ such that

$$\Omega \subset \bigcup_{k=1}^{N} B_R(x_k).$$

Then we have

$$\int_{\Omega} |Du|^{p(x)q}dx \leq \sum_{k=1}^{N}\int_{\Omega_{R_0}(x_k)} |Du|^{p(x)q}dx. \tag{4.2.46}$$

Now the estimate (4.2.45) implies

$$
\begin{aligned}
&\int_{\Omega_{R_0}(x_k)} |Du|^{p(x)q} dx \\
&\quad \leq cM^{\sigma q}\Bigg[|\Omega_{R_0}(x_k)|^{1-q}\Big(\int_{\Omega_{2R_0}(x_k)} |Du|^{p(x)} dx\Big)^q \\
&\qquad\qquad + \int_{\Omega_{2R_0}(x_k)} \big(|F|^{p(x)q}+1\big)\, dx\Bigg] \\
&\quad \leq cM^{\sigma q}\Bigg[|\Omega_{R_0}(x_k)|^{1-q}\Big(\int_{\Omega} |Du|^{p(x)} dx\Big)^q + \int_{\Omega} \big(|F|^{p(x)q}+1\big)\, dx\Bigg] \\
&\quad \leq cM^{\sigma q}\Bigg[|\Omega_{R_0}(x_k)|^{1-q}\Big(\int_{\Omega} \big(|F|^{p(x)}+1\big)\, dx\Big)^q \\
&\qquad\qquad + \int_{\Omega} \big(|F|^{p(x)q}+1\big)\, dx\Bigg] \\
&\quad \leq cM^{\sigma q}\big(|\Omega_{R_0}(x_k)|^{1-q}|\Omega|^{q-1}+1\big)\int_{\Omega} \big(|F|^{p(x)q}+1\big)\, dx. \qquad (4.2.47)
\end{aligned}
$$

We finally combine (4.2.46) and (4.2.47), to discover from (4.1.8) that

$$
\begin{aligned}
\int_{\Omega} |Du|^{p(x)q} dx &\leq \sum_{k=1}^{N} cM^{\sigma q}\big(|\Omega_{R_0}(x_k)|^{1-q}|\Omega|^{q-1}+1\big)\int_{\Omega} \big(|F|^{p(x)q}+1\big)\, dx \\
&\leq cNM^{\sigma q}\big(|B_R|^{1-q}|\Omega|^{q-1}+1\big)\int_{\Omega} \big(|F|^{p(x)q}+1\big)\, dx \\
&\leq cM^{\sigma q}\int_{\Omega} \big(|F|^{p(x)q}+1\big)\, dx,
\end{aligned}
$$

where $c = c(n,\nu,\Lambda,\gamma_1,\gamma_2,\omega(\cdot),R,q,\sigma,\Omega)$. This claims (4.2.25).

Chapter 5

Nonlinear parabolic equations with variable exponent growth in nonsmooth domains

In this chapter, we are concerned with the following divergence type Dirichlet parabolic problem with variable growth:

$$\begin{cases} u_t - \operatorname{div} \mathbf{a}(Du, x, t) = \operatorname{div}\left(|F|^{p(x,t)-2}F\right) & \text{in} \quad \Omega_T, \\ u = 0 & \text{on} \quad \partial_p\Omega_T. \end{cases} \tag{5.0.1}$$

Here, we assume that the variable function $p(z) = p(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\frac{2n}{n+2} < \gamma_1 \leq p(z) \leq \gamma_2 < \infty \tag{5.0.2}$$

for some constants γ_1 and γ_2, the nonlinearity $\mathbf{a}(\xi, z) = \mathbf{a}(\xi, x, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is measurable and differentiable almost everywhere with respect to ξ, and has the following variable growth and elliptic conditions: There exist $0 < \nu \leq \Lambda < \infty$ and $0 \leq \mu \leq 1$ such that

$$(\mu^2 + |\xi|^2)^{\frac{1}{2}}|D_\xi \mathbf{a}(\xi, z)| + |\mathbf{a}(\xi, z)| \leq \Lambda(\mu^2 + |\xi|^2)^{\frac{p(z)-1}{2}}, \tag{5.0.3}$$

$$\nu(\mu^2 + |\xi|^2)^{\frac{p(z)-2}{2}}|\eta|^2 \leq (D_\xi \mathbf{a}(\xi, z)\eta) \cdot \eta, \tag{5.0.4}$$

whenever $z \in \mathbb{R}^{n+1}$, $\xi, \eta \in \mathbb{R}^n$(if $\mu = 0$, ξ is selected in $\mathbb{R}^n \setminus \{0\}$), and $F(z) : \Omega_T \to \mathbb{R}^n$ belongs to the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega_T)$. We will introduce the definition and some properties of variable exponent Lebesgue spaces later in Section 5.1.1.

We establish global Calderón-Zygmund theory in variable exponent Lebesgue spaces to the parabolic equation (5.0.1), that is, we show the following relation:

$$|F|^{p(\cdot)} \in L^{q(\cdot)}(\Omega_T) \implies |Du|^{p(\cdot)} \in L^{q(\cdot)}(\Omega_T), \tag{5.0.5}$$

by deriving a corresponding estimate, where $q(z) = q(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies

$$1 < \gamma_3 \leq q(z) \leq \gamma_4 < \infty \tag{5.0.6}$$

for some constants γ_3 and γ_4, with an estimate. We also present minimal assumptions on $p(\cdot)$, $q(\cdot)$, $\mathbf{a}(\xi, \cdot, t)$ and the boundary of Ω to satisfy (5.0.5).

5.1 Main Result.

5.1.1 Notations and log-Hölder continuity for parabolic problems.

We start with introducing some basic notations which will be used in this chpater. Let $y \in \mathbb{R}^n$, $\tau \in \mathbb{R}$, $w = (y, \tau) \in \mathbb{R}^{n+1}$, $r > 0$. $B_r(y)$ is the open ball in $\mathbb{R}^n$ centered at y with radius r. the *parabolic cylinder* $Q_r(w)$ is denoted by $Q_r(w) := B_r(y) \times (\tau - r^2, \tau + r^2)$, and for $\lambda \geq 1$, the *intrinsic parabolic cylinder* $Q_r^\lambda(w)$ is denoted by

$$Q_r^\lambda(w) := B_r(y) \times (\tau - \lambda^{\frac{2-p(w)}{p(w)}} r^2, \tau + \lambda^{\frac{2-p(w)}{p(w)}} r^2).$$

Recalling the underlying domain $\Omega \subset \mathbb{R}^n$, we write

$$\Omega_r(y) := B_r(y) \cap \Omega, \quad K_r(w) := Q_r(w) \cap \Omega_T, \quad K_r^\lambda(w) := Q_r^\lambda(w) \cap \Omega_T,$$

$$\partial_{\mathrm{w}} \Omega_r(y) := B_r(y) \cap \partial\Omega, \quad \partial_{\mathrm{w}} K_r(w) := K_r(w) \cap \{\partial\Omega \times (0,T)\}$$
$$\text{and } \ \partial_{\mathrm{w}} K_r^\lambda(w) := K_r^\lambda(w) \cap \{\partial\Omega \times (0,T)\}.$$

In particular, for the sake of convenience, if $w = 0$ we simply write $B_r = B_r(0)$, $\Omega_r = \Omega_r(0)$, $K_r^\lambda = K_r^\lambda(0)$ and so on. Moreover we write

$$B_r^+ := B_r \cap \{x_n > 0\}, \quad Q_r^{\lambda+} := B_r^+ \times (-\lambda^{\frac{2-p(0)}{p(0)}} r^2, \lambda^{\frac{2-p(0)}{p(0)}} r^2)$$
$$\text{and} \quad T_r^\lambda := (B_r \cap \{x_n = 0\}) \times (-\lambda^{\frac{2-p(0)}{p(0)}} r^2, \lambda^{\frac{2-p(0)}{p(0)}} r^2).$$

We next check that the log-Hölder continuity of $p(\cdot)$ in U is equivalent to the following condition: there exists a nondecreasing continuous function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ such that

$$|p(z) - p(\tilde{z})| \leq \omega(d_p(z, \tilde{z})), \quad \text{for any } z, \tilde{z} \in U, \tag{5.1.1}$$

and

$$\sup_{0<r\leq\frac{1}{2}} \omega(r) \ln \frac{1}{r} < \tilde{L}, \quad \text{for some } \tilde{L} > 0. \tag{5.1.2}$$

Indeed, we first suppose that $p(\cdot)$ satisfies (2.2.3). Set a nondecreasing continuous function ω such that

$$\omega(r) := \sup\{|p(z) - p(\tilde{z})| : d_p(z, \tilde{z}) \leq r, \quad z, \tilde{z} \in U\}$$

then it holds (5.1.1). Furthermore, for $r \leq \frac{1}{4}$, consider $z, \tilde{z} \in U$ such that $\tilde{r} := d_p(z, \tilde{z}) \leq r$, then we have $|z - \tilde{z}| \leq \frac{\sqrt{5}}{2}\tilde{r}$ and so $\ln \frac{1}{r} \leq \ln \frac{1}{\tilde{r}} \leq \ln \frac{1}{|z-\tilde{z}|} + \ln \frac{\sqrt{5}}{2}$. Consequently applying (2.2.3) we get

$$|p(z) - p(\tilde{z})| \ln \frac{1}{r} \leq |p(z) - p(\tilde{z})| \left(\ln \frac{1}{|z - \tilde{z}|} + \ln \frac{\sqrt{5}}{2} \right) \leq L + \omega(1/4) \ln \frac{\sqrt{5}}{2}$$

hence

$$\omega(r) \ln \frac{1}{r} \leq L + \omega(1/4) \ln \frac{\sqrt{5}}{2}, \quad \text{for any } r \leq \frac{1}{4},$$

which implies (5.1.2). Conversely, suppose that there is $\omega : (0, \infty) \to (0, \infty)$ satisfying (5.1.1) and (5.1.2). Then one can find $\tilde{L}_1 > 0$ such that

$$\omega(r) \ln \frac{1}{r} \leq \tilde{L}_1, \quad \text{for any } r \leq \frac{1}{\sqrt{2}}.$$

For $z, \tilde{z} \in U$ with $|z - \tilde{z}| = r \leq \frac{1}{2}$, since $d_p(z, \tilde{z}) \leq \sqrt{r}$, we have

$$|p(z) - p(\tilde{z})| \ln \frac{1}{|z - \tilde{z}|} \leq \omega(\sqrt{r}) \ln \frac{1}{r} = 2\omega(\sqrt{r}) \ln \frac{1}{\sqrt{r}} \leq 2\tilde{L}_1.$$

Note that ω is actually modulus of continuity of $p(\cdot)$, and the condition (5.1.2) can be replaced by

$$\limsup_{r\to 0+} \omega(r)\ln\frac{1}{r} < \infty.$$

5.1.2 Main result.

We denote a Banch space $W^{p(\cdot)}(\Omega_T)$ by

$$W^{p(\cdot)}(\Omega_T) := \left\{ f \in L^{p(\cdot)}(\Omega_T) : Df \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^n) \right\}$$

equipped the following norm:

$$\|f\|_{W^{p(\cdot)}(\Omega_T)} := \|f\|_{L^{p(\cdot)}(\Omega)} + \|Df\|_{L^{p(\cdot)}(\Omega_T, \mathbb{R}^n)},$$

and its subspace $W_0^{p(\cdot)}(\Omega_T) := W^{p(\cdot)}(\Omega, \mathbb{R}^N) \cap L^1(0,T; W_0^{1,1}(\Omega))$. We then say $u \in C(0,T;L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ is a weak solution of (5.0.1) if it holds

$$\int_{\Omega_T} u\varphi_t \, dz - \int_{\Omega_T} \mathbf{a}(Du, z)\cdot D\varphi \, dz = \int_{\Omega_T} |F|^{p(z)-2} F \cdot D\varphi \, dz$$

for every $\varphi \in C_0^\infty(\Omega_T)$, and $u(\cdot,0) \equiv 0$. In the next section we will discuss about the weak solutions of parabolic equations with variable growth.

Let $q(\cdot) : \mathbb{R}^{n+1} \to (1,\infty)$ satisfy (5.0.6). In addition, we assume that $q(\cdot)$ is log-Hölder continuous in Ω_T, hence there is a nondecreasing continuous function $\rho : [0,\infty) \to [0,\infty)$ with $\rho(0) = 0$ such that

$$|q(z) - q(\tilde{z})| \le \rho(d_p(z,\tilde{z})), \quad \text{for every } z, \tilde{z} \in \Omega_T, \tag{5.1.3}$$

and

$$\sup_{0<\rho\le\frac{1}{2}} \rho(r)\ln\frac{1}{r} \le L_1, \quad \text{for some } L_1 > 0. \tag{5.1.4}$$

We next introduce the regularity assumptions on $p(\cdot)$, $\mathbf{a}(\xi,\cdot,t)$ and the boundary of Ω.

Definition 5.1.1. Let $\delta \in (0,1/8)$ and $R \in (0,1)$. We say $(p(\cdot), \mathbf{a}(\xi,\cdot,t), \Omega)$ is (δ,R)-*vanishing* if

(1) For $p(\cdot)$, there exists a modulus of continuity $\omega : (0,\infty) \to (0,\infty)$ of $p(\cdot)$ such that

$$\sup_{0<r\le R} \omega(r) \ln \frac{1}{r} \le \delta. \tag{5.1.5}$$

(2) The nonlinearity $\mathbf{a}$ satisfies

$$\sup_{t_1,t_2\in\mathbb{R},t_1<t_2} \sup_{y\in\mathbb{R}^n} \sup_{0<r\le R} \fint_{t_1}^{t_2} \fint_{B_r(y)} \theta(\mathbf{a}, B_r(y))(x,t)\, dx\, dt \le \delta,$$

where

$$\theta(\mathbf{a}, U)(x,t) := \sup_{\xi\in\mathbb{R}^n} \left| \frac{\mathbf{a}(\xi,x,t)}{(\mu^2+|\xi|^2)^{\frac{p(x,t)-1}{2}}} - \left(\frac{\mathbf{a}(\xi,\cdot,t)}{(\mu^2+|\xi|^2)^{\frac{p(\cdot,t)-1}{2}}} \right)_U \right|$$

and $U \subset \mathbb{R}^n$.

(3) For each $y \in \partial\Omega$ and $0 < r < R$ there exists a coordinate system, still say $x = (x_1, \ldots, x_n)$ coordinate, with the origin at y such that

$$B_r \cap \{x_n > \delta r\} \;\subset\; \Omega_r \;\subset\; B_r \cap \{x_n > -\delta r\}.$$

Here are some remarks related to the above definition..

Remark 5.1.2.

(1) For $0 < r_1 < r_2$, if $(p(\cdot), \mathbf{a}(\xi,\cdot,t), \Omega)$ is (δ, r_2)-vanishing then it is (δ, r_1)-vanishing.

(2) It is easy to check that if $p(\cdot)$ satisfies the condition (1) of Definition 5.1.1 then it is log-Hölder continuous in Ω_T.

(3) Generally speaking, the condition (2) of Definition 5.1.1 means that the map $x \mapsto \frac{\mathbf{a}(\xi,\cdot,t)}{(\mu^2+|\xi|^2)^{\frac{p(\cdot,t)-1}{2}}}$ has a small BMO semi-norm that is less than or equal to δ uniformly on $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

(4) If Ω satisfies the condition (3) of Definition 5.1.1, then we sat Ω is a (δ, R)-Reifenberg flat domain. Note that (δ, R)-Reifenberg flat domains have the following measure density conditions:

$$\sup_{y\in\Omega} \sup_{r\le R} \frac{|B_r(y)|}{|\Omega_r(y)|} \le \left(\frac{2}{1-\delta}\right)^n \le \left(\frac{16}{7}\right)^n, \tag{5.1.6}$$

$$\inf_{y\in\partial\Omega}\inf_{r\le R}\frac{|\Omega^c\cap B_r(y)|}{|B_r(y)|}\ge\left(\frac{1-\delta}{2}\right)^n\ge\left(\frac{7}{16}\right)^n. \tag{5.1.7}$$

We refer to [13, 52, 58] for further discussion about Reifenberg flat domains.

We now state the main results in this chapter. We first define

$$d_w=d(w)=d(p(w)):=\begin{cases}\frac{p(w)}{2} & \text{if } \ p(w)\ge 2,\\ \frac{2p(w)}{p(w)(n+2)-2n} & \text{if } \ \frac{2n}{n+2}<p(w)<2,\end{cases} \tag{5.1.8}$$

$$d_M:=\sup_{w\in\Omega_T} d(w), \tag{5.1.9}$$

$$\alpha:=\min\left\{1,\gamma_1\frac{n+2}{4}-\frac{n}{2}\right\}, \tag{5.1.10}$$

and

$$M:=\int_{\Omega_T}\left(|F|^{p(z)q(z)}+1\right)\,dx+1. \tag{5.1.11}$$

Theorem 5.1.3. *Let $p(\cdot)$ satisfy* (5.0.2). *Assume that $q(\cdot)$ satisfies* (5.0.6), (5.1.3) *and* (5.1.4). *There exist small $\delta=\delta(n,\nu,\Lambda,\gamma_1,\gamma_2,\gamma_3,\gamma_4,L_1)>0$ and $\tilde\delta=\tilde\delta(n,\nu,\Lambda,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\omega(\cdot),\rho(\cdot))>0$ such that if $(p(\cdot),\mathbf{a}(\xi,\cdot,t),\Omega)$ is (δ,R)-vanishing for some $R\in(0,1)$, $|F|^{p(\cdot)}\in L^{q(\cdot)}(\Omega_T)$ and u is a weak solution of* (5.0.1), *then the following hold:*

(1) (Local estimate) For any $r\le\min\{\tilde\delta R^{\frac{1}{\alpha}}M^{-(\frac{n+3}{\alpha}+1)},\sqrt{T/16}\}$ and $w=(y,\tau)\in\Omega_T$ we have

$$\fint_{K_r(w)}|Du|^{p(z)q(z)}\,dz\le c\left\{\fint_{K_{4r}(w)}|Du|^{p(z)}\,dz+\left(\fint_{K_{4r}(w)}|F|^{p(z)q(z)}\,dz\right)^{\frac{1}{q_w}}+1\right\}^{1+d_w(q(w)-1)}, \tag{5.1.12}$$

for some $c=c(n,\nu,\Lambda,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\rho(\cdot))>0$.

(2) (Global estimate) We have

$$\int_{\Omega_T}|Du|^{p(z)q(z)}\,dz\ \le\ c\left\{\left(\int_{\Omega_T}|F|^{p(z)q(z)}\,dz\right)^{d_0}+1\right\}, \tag{5.1.13}$$

for some $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \omega(\cdot), \rho(\cdot), R, \Omega, T)$, *where*

$$d_0 := (n+2)\left[\frac{n+3}{\alpha}+1\right]\left[\frac{\gamma_4(1+d_M(\gamma_3-1))}{\gamma_3}-1\right]+\frac{\gamma_4(1+d_M(\gamma_3-1))}{\gamma_3^2} > 1.$$

Therefore, there holds (5.0.5).

Remark 5.1.4. *Suppose* $s : \mathbb{R}^{n+1} \to (1, \infty)$ *is bounded and strictly bigger than* $p(\cdot)$, *that is, there exist* $1 < \gamma_3 \le \gamma_4 < \infty$ *such that*

$$1 < \gamma_3 \le \frac{s(z)}{p(z)} \le \gamma_4 < \infty.$$

If $s(\cdot)$ *is log-Hölder continuous in* Ω_T, *then Theorem 5.1.3 implies*

$$|F| \in L^{s(\cdot)}(\Omega_T) \quad \Longrightarrow \quad |Du| \in L^{s(\cdot)}(\Omega_T)$$

with the estimates (5.1.12) *and* (5.1.13), *where* $q(\cdot) = \frac{s(\cdot)}{p(\cdot)}$. *Indeed, since* $s(\cdot)$ *is log-Hölder continuous, there exists a modulus of continuity of* $s(\cdot)$, $\tilde{\rho} : (0, \infty) \to (0, \infty)$, *satisfying* (5.1.4) *with* ρ *replaced by* $\tilde{\rho}$. *Set* $q(x) := \frac{s(x)}{p(x)}$ *then we easily check that* $1 < \gamma_3 \le q(x) \le \gamma_4$ *and*

$$|q(z) - q(\tilde{z})| \le \frac{\gamma_2(\tilde{\rho}(|z-\tilde{z}|) + \gamma_4\omega(|z-\tilde{z}|))}{\gamma_1^2} =: \rho(|z-\tilde{z}|),$$

for all $z, \tilde{z} \in \Omega_T$. *Consequently,* $\rho : (0.\infty) \to (0, \infty)$ *satisfies* (5.1.4).

Remark 5.1.5. *We will prove the estimate* (5.1.12) *only for the regions* $K^\lambda_{4r}(w)$ *satisfying* $(\tau - 16r^2, \tau + 16r^2) \subset (0, T)$. *In fact, if* $Q^\lambda_{4r}(w)$ *touches the bottom or the top of* Ω_T, *i.e.,* $(\tau - 16r^2, \tau + 16r^2) \not\subset (0, T)$, *we consider the extended equation such that*

$$\begin{cases} \tilde{u}_t - \mathit{div}\,\tilde{\mathbf{a}}(D\tilde{u}, x, t) = \mathit{div}\left(|\tilde{F}|^{p(x,t)-2}\tilde{F}\right) & \text{in} \quad \Omega \times (-T, 2T] \\ \tilde{u} = 0 & \text{on} \quad \partial_p(\Omega \times (-T, 2T], \end{cases} \tag{5.1.14}$$

where

$$\mathbf{a}(\xi, x, t) := \mathbf{a}(\xi, x, 2T-t),\ \tilde{p}(x,t) := p(x, 2T-t),\ \tilde{q}(x,t) := q(x, 2T-t),\ \text{if } t > T,$$

$$\tilde{\mathbf{a}}(\xi, x, t) := \mathbf{a}(\xi, x, t), \quad \tilde{p}(x,t) := p(x,t), \quad \tilde{q}(x,t) := q(x,t), \quad \text{if } \ t \le T,$$

and

$$\begin{cases} \tilde{u}(x,t) := u(x, 2T-t), & \tilde{F}(x,t) := F(x,2T-t), & \text{if } T < t \le 2T, \\ \tilde{u}(x,t) := u(x,t), & \tilde{F}(x,t) := F(x,t), & \text{if } 0 < t \le T, \\ \tilde{u}(x,t) := 0, & \tilde{F}(x,t) := 0, & \text{if } -T < t \le 0. \end{cases}$$

Note that $\tilde{\mathbf{a}}$, $\tilde{p}(\cdot)$, $\tilde{p}(\cdot)$ *and* $\tilde{F}$ *satisfy the same assumptions of* $\mathbf{a}$, $p(\cdot)$, $p(\cdot)$ *and* F *in Theorem 5.1.3, and* $(\tau - 16r^2, \tau + 16r^2) \subset (-T, 2T)$. *Consequently, applying* (5.1.12) *in Theorem 5.1.3 to the equation* (5.1.14) *for the region* $\tilde{K}_r(w) := Q_r(w) \cap \{\Omega \times (-T, 2T]\}$ *with* $w \in \Omega_T$, *we have*

$$\begin{aligned} &\fint_{\tilde{K}_r(w)} |D\tilde{u}|^{\tilde{p}(z)\tilde{q}(z)}\, dz \\ &\qquad \le c \left\{ \fint_{\tilde{K}_{4r}(w)} |D\tilde{u}|^{\tilde{p}(z)}\, dz + \left(\fint_{\tilde{K}_{4r}(w)} |\tilde{F}|^{\tilde{p}(z)\tilde{q}(z)}\, dz \right)^{\frac{1}{q_w}} + 1 \right\}^{1 + d_w(q(w)-1)}, \end{aligned}$$

where $\tilde{K}_{4r}(w) := Q_{4r}(w) \cap \{\Omega \times (-T, 2T]\}$, *which implies the estimate* (5.1.12) *for the regions* $K_{4r}(w)$ *with* $(\tau - 16r^2, \tau + 16r^2) \not\subset (0, T)$.

5.2 Preliminaries.

5.2.1 Parabolic Sobolev spaces and P.D.E. with variable exponents.

We introduce parabolic spaces with a variable exponent and existence of the weak solutions of parabolic Cauchy-Dirichlet equations with variable growth. For details, we refer to [27, 31, 32]. In this subsection, we assume that Ω has a *fat complement*; there exists $c > 0$ such that for any $x \in \Omega$ there holds $|B_{2d(x,\partial\Omega)}(x) \cap \Omega^c| \ge c |B_{2d(x,\partial\Omega)}(x)|$, and $p(\cdot)$ satisfies (5.0.2), and is log-Hölder continuous. Note that (δ, R)-Reifenberg flat domain has a fat complement.

Let $W^{p(\cdot)}(\Omega_T)'$ be the dual space of $W_0^{p(\cdot)}(\Omega_T)$, and then $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{\Omega_T}$ be the pairing between $W^{p(\cdot)}(\Omega_T)'$ and $W_0^{p(\cdot)}(\Omega_T)$. Then for each $g \in W^{p(\cdot)}(\Omega_T)'$ there exist $g_0 \in L^{p'(\cdot)}(\Omega_T)$ and $G \in L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n)$ such that

$$\langle g, f \rangle = \int_{\Omega_T} f g_0\, dz + \int_{\Omega_T} Df \cdot G\, dz, \quad \text{for any } f \in W_0^{p(\cdot)}(\Omega_T), \tag{5.2.1}$$

where $p'(\cdot) := \frac{p(\cdot)-1}{p(\cdot)}$. Moreover, we have

$$\|g\|_{W^{p(\cdot)}(\Omega_T)'} = \inf\left(\|g_0\|_{L^{p'(\cdot)}(\Omega_T)} + \|G\|_{L^{p'(\cdot)}(\Omega_T,\mathbb{R}^n)}\right),$$

where the infimum in the previous equality runs for $g_0 \in L^{p'(\cdot)}(\Omega_T)$ and $G \in L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n)$ satisfying (5.2.1). We further define

$$W_{p(\cdot)}(\Omega_T) := \left\{f \in W^{p(\cdot)}(\Omega_T) : f_t \in W^{p(\cdot)}(\Omega_T)'\right\}$$

with $\|f\|_{W_{p(\cdot)}} := \|f\|_{W^{p(\cdot)}(\Omega_T)} + \|f_t\|_{W^{p(\cdot)}(\Omega_T)'}$. Here, $f_t \in W^{p(\cdot)}(\Omega_T)'$ is understood in the distributional sense that

$$\langle f_t, \varphi\rangle := \int_T f\partial_t\varphi\, dz, \quad \text{for any } \varphi \in C_0^\infty(\Omega_T),$$

where $\partial_t\varphi$ means the classical time derivative of φ.

Remark 5.2.1. *When $p(\cdot)$ is a constant function, $p(\cdot) \equiv p$, the above function spaces return to well known classical parabolic Sobolev spaces, precisely,*

$$W^p(\Omega_T) = L^p(0,T;W^{1,p}(\Omega)), \quad W_0^p(\Omega_T) = L^p(0,T;W_0^{1,p}(\Omega))$$

and

$$W^p(\Omega_T)' = L^p(0,T;W_0^{1,p}(\Omega))' = L^{p'}(0,T;W^{-1,p'}(\Omega)).$$

For the above spaces, we have the following property.

Proposition 5.2.2. *[27, 32] $W_{p(\cdot)}(\Omega_T) \cap L^1(0,T;W_0^{1,1}(\Omega))$ is continuously embedded in $C(0,T;L^2(\Omega))$, that is, for any $f \in W_p(\cdot)(\Omega_T) \cap L^1(0,T;W_0^{1,1}(\Omega))$ we have*

$$\|f\|_{C(0,T;L^2(\Omega))} \leq c\|f\|_{W_{p(\cdot)}(\Omega_T)}$$

for some $c > 0$ independent of f, moreover there holds

$$\langle f_t, f\rangle_{\Omega\times[t_1,t_2]} = \frac{1}{2}\left(\|f(\cdot,t_2)\|_{L^2(\Omega)} - \|f(\cdot,t_1)\|_{L^2(\Omega)}\right),$$

for any $0 \leq t_1 < t_2 \leq T$.

Now consider the following parabolic equation:

$$\begin{cases} u_t - \operatorname{div} \tilde{\mathbf{a}}(Du,z) &= \operatorname{div}\left(|F|^{p(z)-2}F\right) \quad \text{in} \quad \Omega_T, \\ u &= f \quad \text{on} \quad \partial\Omega\times(0,T], \\ u(\cdot,0) &= f_0 \quad \text{on} \quad \Omega, \end{cases} \tag{5.2.2}$$

where

$$F \in L^{p(\cdot)}(\Omega_T), \quad f_0 = L^2(\Omega), \tag{5.2.3}$$

$$f \in C(0,T;L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T), \quad f_t \in W^{p(\cdot)}(\Omega_T)', \tag{5.2.4}$$

and $\tilde{\mathbf{a}} : \mathbb{R}^n \times \mathbb{R}^{n+1}$ satisfies that

$$\begin{cases} |\tilde{\mathbf{a}}(\xi,z)| \leq \Lambda_1(1+|\xi|)^{p(z)-1}, \\ \tilde{\mathbf{a}}(\xi,z)\cdot\xi \geq \nu_1|\xi|^{p(z)} - \nu_2, \\ (\tilde{\mathbf{a}}(\xi_1,z)-\tilde{\mathbf{a}}(\xi_2,z))\cdot(\xi_1-\xi_2) \geq \nu_3(\mu^2+|\xi_1|^2+|\xi_2|^2)^{\frac{p(z)-2}{2}}|\xi_1-\xi_2|^2, \end{cases} \tag{5.2.5}$$

for every $z \in \mathbb{R}^{n+1}$, $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$ and for some positive constants $\nu_1, \nu_2, \nu_3, \Lambda_1$ and $\mu \in [0,1]$. We then say $u \in L^1(\Omega_T)$ is a weak solution of (5.2.2) if

$$u \in C(0,T;L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T) \ \text{ with } \ u_t \in W^{p(\cdot)}(\Omega_T)',$$

$$u - f \in L^1(0,T;W_0^{1,1}(\Omega)), \quad u(\cdot,0) = f_0 \ \text{ in } \ L^2(\Omega),$$

and there holds

$$\int_{\Omega_T} u\varphi_t\,dz - \int_{\Omega_T} \tilde{\mathbf{a}}(Du,z)\cdot D\varphi\,dz = \int_{\Omega_T} |F|^{p(z)-2}F\cdot D\varphi\,dz,$$

for every $\varphi \in C_0^\infty(\Omega_T)$. We point out that if u is a weak solution to (5.2.2) then we have

$$\langle u_t, \varphi\rangle_{\Omega\times[t_1,t_2]} + \int_{\Omega\times[t_1,t_2]} \tilde{\mathbf{a}}(Du,z)\cdot D\varphi\,dz = -\int_{\Omega\times[t_1,t_2]} |F|^{p(z)-2}F\cdot D\varphi\,dz,$$

for every $\varphi \in W_0^{p(\cdot)}(\Omega_T)$ and every $0 \leq t_1 < t_2 \leq T$.

Theorem 5.2.3. *(Existence and Uniqueness) Under the above assumptions* (5.2.3)-(5.2.5), *there exists a unique weak solution u to* (5.2.2). *Moreover, if $f \equiv 0$, then we have the estimate*

$$\sup_{0\leq t\leq T} \|u(\cdot,t)\|^2_{L^2(\Omega)} + \int_{\Omega_T} |Du|^{p(z)}dz \leq c\left(\int_{\Omega_T}\left[|F|^{p(z)}+1\right]\,dz + \|f_0\|^2_{L^2(\Omega)}\right),$$

for some $c = c(n,\Lambda,\nu,\gamma_1,\gamma_2) > 0$.

Returning to the main equation (5.0.1), we know from (5.0.3) and (5.0.4) that the nonlinearity **a** satisfies

$$\mathbf{a}(\xi, z) \cdot \xi \geq \nu_1 |\xi|^{p(z)} - \nu_2$$

and

$$(\mathbf{a}(\xi_1, z) - \mathbf{a}(\xi_2, z)) \cdot (\xi_1 - \xi_2) \geq \nu_3 (\mu^2 + |\xi_1|^2 + |\xi_2|^2)^{\frac{p(z)-2}{2}} |\xi_1 - \xi_2|^2, \quad (5.2.6)$$

for every $z \in \mathbb{R}^{n+1}$, $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$ and for some ν_1, ν_2, ν_3 depending only on $n, \Lambda, \nu, \gamma_1, \gamma_2$.

Corollary 5.2.4. *There exists a unique weak solution u to* (5.0.1)*. Moreover, we have*

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\|^2_{L^2(\Omega)} + \int_{\Omega_T} |Du|^{p(z)} dz \leq c_0 \int_{\Omega_T} \left[|F|^{p(z)} + 1 \right] dz, \qquad (5.2.7)$$

for some $c_0 = c_0(n, \Lambda, \nu, \gamma_1, \gamma_2) > 0$.

5.2.2 Self improving integrability.

We introduce self improving integrability results for the gradient of the weak solutions of nonlinear parabolic equations with variable growth. In [6] Bögelein and Duzaar showed local self improving integrability for parabolic systems with variable growth assuming that the variable exponent $p(\cdot)$ is log-Hölder continuous. This result can be naturally extended to a global one on Reifenberg flat domains with the zero boundary condition, since they have the measure density conditions (5.1.6) and (5.1.7). Note that, by following the proof in [6], one can find a exact relation between the radii of parabolic cylinders and the constant $M_1 > 1$ satisfying (5.2.9), see (5.2.8).

Lemma 5.2.5. *Let $M_1, \gamma > 1$, $p(\cdot)$ satisfy* (5.0.2), $(p(\cdot), \Omega)$ *be* (δ, R)*-vanishing, and $|F|^{p(\cdot)} \in L^\gamma(\Omega_T)$. Then there exist $\sigma_\gamma = \sigma_\gamma(n, \nu, \Lambda, \gamma_1, \gamma_2, \gamma) \in (0, \gamma - 1]$ and $c_h = c_h(n, \gamma_1, \gamma_2, \omega(\cdot)) \geq 1$ such that the following holds: for any*

$$r < \min\left\{ c_h^{-1} M_1^{-\frac{1}{\alpha}}, R \right\} \qquad (5.2.8)$$

and any $w = (y, \tau) \in \Omega_T$, *if* u *is a weak solutions of*

$$u_t - div\,\mathbf{a}(Du, z) = div(|F|^{p(z)-2}F) \quad in \quad Q_r(w) = K_r(w), \ \ if\ B_r(y) \subset \Omega,$$

or

$$\begin{cases} u_t - div\,\mathbf{a}(Du, z) &= div\left(|F|^{p(z)-2}F\right) \quad in \quad K_r(w), \\ u &= 0 \qquad\qquad\qquad\quad on \quad \partial_{\mathrm{w}} K_r(w), \end{cases} \quad if\ B_r(y) \not\subset \Omega,$$

satisfying

$$\int_{K_r(w)} \left[|Du|^{p(z)} + |F|^{p(z)} + 1\right] dz \leq M_1, \tag{5.2.9}$$

then $|Du|^{p(\cdot)} \in L^{1+\sigma_\gamma}(K_{\frac{r}{2}}(w))$ *and we have*

$$\begin{aligned} \fint_{K_{\frac{r}{2}}(w)} |Du|^{p(z)(1+\sigma)}\, dz \ &\leq \ c\left(\fint_{K_r(w)} \left[|Du|^{p(z)} + |F|^{p(z)}\right] dz\right)^{1+\sigma d(p(w))} \\ &\quad + c\fint_{K_r(w)} |F|^{p(z)(1+\sigma)}\, dz + c \end{aligned} \tag{5.2.10}$$

for every $0 < \sigma \leq \sigma_\gamma$ *and for some* $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2) > 0$.

Note that if $p(\cdot)$ is just log-Hölder continuous, then σ_γ in the above lemma depends also on $p(\cdot)$, see [6, Theorem 2.2], however using (δ, R)-vanishing condition on $p(\cdot)$, $\delta \leq 1/8$, we can obtain σ_γ independent of $p(\cdot)$.

Using a scaling argument, we deduce a homogeneous self improving integrability estimate on intrinsic parabolic cylinders from the above lemma. We note the assumptions (5.2.13) and (5.2.14) in the below corollary will be clarified in the proof of Theorem 5.1.3 later.

Corollary 5.2.6. *Let* $M_1, \gamma, \Gamma, c_a > 1$, $p(\cdot)$ *satisfy* (5.0.2), $(p(\cdot), \Omega)$ *be* (δ, R)*-vanishing, and* $|F|^{p(\cdot)} \in L^\gamma(\Omega_T)$. *Then there exists* $\tilde{\sigma}_\gamma = \tilde{\sigma}_\gamma(n, \nu, \Lambda, \gamma_1, \gamma_2, \gamma, c_a) \in (0, \gamma - 1]$ *such that, for any* $w = (y, \tau) \in \Omega_T$ *if* u *is a weak solutions of*

$$u_t - div\,\mathbf{a}(Du, z) = div\left(|F|^{p(z)-2}F\right) \quad in \quad Q_r^\lambda(w) = K_r^\lambda(w), \quad if\ B_r(y) \subset \Omega,$$

or

$$\begin{cases} u_t - div\,\mathbf{a}(Du, z) &= div\left(|F|^{p(z)-2}F\right) \quad in \quad K_r^\lambda(w), \\ u &= 0 \qquad\qquad\qquad\quad on \quad \partial_{\mathrm{w}} K_r^\lambda(w), \end{cases} \quad if\ B_r(y) \not\subset \Omega,$$

satisfying

$$r < \min\left\{1, \lambda^{\frac{p(w)-2}{2p(w)}}\right\} c_h^{-1} \left(c_a^{\frac{1}{\gamma_1(\gamma_1-1)}} M_1\right)^{-\frac{1}{\alpha}} \quad \textit{and} \quad r < R, \tag{5.2.11}$$

$$\int_{K_r^\lambda(w)} \left[|Du|^{p(z)} + |F|^{p(z)} + 1\right] dz \leq M_1, \tag{5.2.12}$$

$$\fint_{K_r^\lambda(w)} |Du|^{p(z)}\, dz + \left(\fint_{K_r^\lambda(w)} |F|^{p(z)\gamma}\, dz\right)^{\frac{1}{\gamma}} \leq \lambda, \tag{5.2.13}$$

and

$$p_2 - p_1 \leq 1, \quad \lambda^{p_2-p_1} \leq c_a, \tag{5.2.14}$$

where

$$p_1 := \inf_{z\in K_r^\lambda(w)} p(z) \quad \textit{and} \quad p_2 := \sup_{z\in K_r^\lambda(w)} p(z),$$

then $|Du|^{p(\cdot)} \in L^{1+\tilde{\sigma}_\gamma}(K^\lambda_{\frac{r}{2}}(w))$ *and we have*

$$\fint_{K^\lambda_{\frac{r}{2}}(w)} |Du|^{p(z)(1+\sigma)}\, dz \leq c\lambda^{1+\sigma}, \tag{5.2.15}$$

for every $\sigma \leq \tilde{\sigma}_\gamma$ *and for some* $c = c(n, \nu, \Lambda, \gamma_1, \gamma_2, \gamma, c_a) > 0$.

Proof. For the sake of convenience, we may assume $w = 0$ and write $p_0 = p(0)$. We first consider the case $p_0 \geq 2$. In this case, we define the rescaling functions such that for $(x,t) \in \mathbb{R}^{n+1}$

$$\tilde{p}(x,t) := p(x, \lambda^{\frac{2-p_0}{p_0}} t), \quad \tilde{\mathbf{a}}(\xi, x, t) := \lambda^{\frac{1-p_0}{p_0}} \mathbf{a}(\lambda^{\frac{1}{p_0}}\xi, x, \lambda^{\frac{2-p_0}{p_0}} t),$$

and for $(x,t) \in K_r$

$$\tilde{F}(x,t) := \lambda^{\frac{1-p_0}{p_0(\tilde{p}(x,t)-1)}} F(x, \lambda^{\frac{2-p_0}{p_0}} t), \quad \tilde{u}(x,t) := \lambda^{-\frac{1}{p_0}} u(x, \lambda^{\frac{2-p_0}{p_0}} t).$$

Then, for $z_1 = (x_1, t_1), z_2 = (x_2, t_2) \in \mathbb{R}^{n+1}$, $p(\cdot)$ satisfies

$$\begin{aligned} |\tilde{p}(z_1) - \tilde{p}(z_2)| &= |p(x_1, \lambda^{\frac{2-p_0}{p_0}} t_1) - p(x_2, \lambda^{\frac{2-p_0}{p_0}} t_2)| \\ &\leq \omega\left(\max\{|x_1 - x_2|, \lambda^{\frac{2-p_0}{2p_0}} \sqrt{|t_1 - t_2|}\}\right) \\ &\leq \omega(d_p(z_1, z_2)), \end{aligned}$$

hence $\tilde{p}(\cdot)$ is (δ, R)-vanishing, and $\tilde{\mathbf{a}}$ satisfies (5.0.3) and (5.0.4) with $(\mathbf{a}, \nu, \Lambda, \mu, p(\cdot))$ replaced by $(\tilde{\mathbf{a}}, c_a^{-1}\nu, c_a\Lambda, \lambda^{-\frac{1}{p_0}}\mu, \tilde{p}(\cdot))$, by using (5.2.14). Moreover, $\tilde{u}$ is a weak solution of

$$\tilde{u}_t - \operatorname{div}\, \tilde{\mathbf{a}}(D\tilde{u}, z) = \operatorname{div}\,(|\tilde{F}|^{\tilde{p}(z)-2}\tilde{F}) \quad \text{in} \quad Q_r = K_r, \quad \text{if } B_r \subset \Omega,$$

or

$$\begin{cases} \tilde{u}_t - \operatorname{div}\, \tilde{\mathbf{a}}(D\tilde{u}, z) &= \operatorname{div}\left(|\tilde{F}|^{\tilde{p}(z)-2}\tilde{F}\right) \quad \text{in} \quad K_r, \\ \tilde{u} &= 0 \quad \text{on} \quad \partial_{\mathrm{w}} K_r, \end{cases} \qquad \text{if } B_r \not\subset \Omega.$$

with $|\tilde{F}|^{\tilde{p}(\cdot)} \in L^\gamma(K_r)$ and, by (5.2.12),

$$\begin{aligned} &\int_{K_r} \left[|D\tilde{u}|^{\tilde{p}(z)} + |\tilde{F}|^{\tilde{p}(z)} + 1\right] dz \\ &\qquad = \lambda^{\frac{p_0-2}{2p_0}} \int_{K_r^\lambda} \left[\lambda^{-\frac{p(z)}{p_0}}|Du|^{p(z)} + \lambda^{\frac{p(z)(1-p_0)}{p_0(p(z)-1)}}|F|^{p(z)} + 1\right] dz \\ &\qquad = \int_{K_r^\lambda} \left[\lambda^{-\frac{1}{2}-\frac{1}{p_0}+\frac{p_0-p(z)}{p_0}}|Du|^{p(z)} + \lambda^{-\frac{1}{2}-\frac{p_0-1}{p_0(p(z)-1)}}|F|^{p(z)} + 1\right] dz \le M_1. \end{aligned}$$

Since r satisfies (5.2.11) and so (5.2.8), applying Lemma 5.2.5 to the above equations, one can find $\tilde{\sigma}_\gamma = \tilde{\sigma}_\gamma(n, \nu, \Lambda, \gamma_1, \gamma_2, \gamma, c_a) \in (0, \gamma-1]$ such that $|Du|^{p(\cdot)} \in L^{1+\tilde{\sigma}_\gamma}(K_r)$ and the estimate (5.2.10) holds for every $\sigma \le \tilde{\sigma}_\gamma$ with $u, F, p(\cdot)$ replaced by $\tilde{u}, \tilde{F}, \tilde{p}(\cdot)$, respectively. Therefore, for any $\sigma \le \tilde{\sigma}_\gamma$

$$\begin{aligned} &⨍_{K^\lambda_{\frac{r}{2}}} |Du|^{p(z)(1+\sigma)}\, dz \le ⨍_{K_{\frac{r}{2}}} \lambda^{\frac{\tilde{p}(z)}{p_0}(1+\sigma)}|D\tilde{u}|^{\tilde{p}(z)(1+\sigma)}\, dz \\ &\quad \le \lambda^{1+\sigma} ⨍_{K_{\frac{r}{2}}} |D\tilde{u}|^{\tilde{p}(z)(1+\sigma)}\, dz \\ &\quad \le c\lambda^{1+\sigma}\left\{\left(⨍_{K_r}\left[|D\tilde{u}|^{\tilde{p}(z)} + |\tilde{F}|^{\tilde{p}(z)}\right] dz\right)^{1+\sigma d(p_0)} + ⨍_{K_r} |F|^{\tilde{p}(z)(1+\sigma)}\, dz + 1\right\} \\ &\quad \le c\lambda^{1+\sigma}\left\{\left(⨍_{K_r^\lambda}\left[\lambda^{-\frac{p(z)}{p_0}}|Du|^{p(z)} + \lambda^{-\frac{p(z)(p_0-1)}{p_0(p(z)-1)}}|F|^{p(z)}\right] dz\right)^{1+\sigma d(p_0)}\right. \\ &\qquad\qquad\qquad \left. + ⨍_{K_r} \lambda^{-\frac{p(z)(p_0-1)(1+\sigma)}{p_0(p(z)-1)}}|F|^{p(z)(1+\sigma)}\, dz + 1\right\}. \end{aligned}$$

Observe that $\frac{p(z)}{p_0} \geq 1 - \frac{p_2-p_1}{\gamma_1}$ and $\frac{p(z)(p_0-1)}{p_0(p(z)-1)} = 1 + \frac{p_0-p(z)}{p_0(p(z)-1)} \geq 1 - \frac{p_2-p_1}{\gamma_1(\gamma_1-1)}$. Using these inequalities, (5.2.13) and (5.2.14) we have

$$\begin{aligned}
\fint_{K^{\lambda}_{\frac{r}{2}}} |Du|^{p(z)(1+\sigma)}\,dz &\leq c\Bigg\{\lambda^{\sigma-\sigma d(p_0)}\left(\fint_{K^{\lambda}_r}\left[|Du|^{p(z)}+|F|^{p(z)}\right]\,dz\right)^{1+\sigma d(p_0)} \\
&\qquad + \left(\fint_{K_r} |F|^{p(z)\gamma}\,dz\right)^{\frac{1+\sigma}{\gamma}} + 1\Bigg\} \\
&\leq c\lambda^{1+\sigma}.
\end{aligned}$$

We next consider the case $p_0 < 2$. In this case, let $\tilde{r} := \lambda^{\frac{2-p_0}{2p_0}} r$, and define the rescaling functions such that for $(x,t) \in \mathbb{R}^{n+1}$

$$\tilde{p}(x,t) := p(\lambda^{\frac{p_0-2}{2p_0}}x,t), \quad \tilde{\mathbf{a}}(\xi,x,t) := \lambda^{\frac{1-p_0}{p_0}}\mathbf{a}(\lambda^{\frac{1}{p_0}}\xi, \lambda^{\frac{p_0-2}{2p_0}}x,t),$$

and for $(x,t) \in K_{\tilde{r}}$

$$\tilde{F}(x,t) := \lambda^{\frac{1-p_0}{p_0(\tilde{p}(x,t)-1)}}F(\lambda^{\frac{p_0-2}{2p_0}}x,t), \quad \tilde{u}(x,t) := \lambda^{-\frac{1}{2}}u(\lambda^{\frac{p_0-2}{2p_0}}x,t).$$

Then, in a similar argument as in the case $p_0 \geq 2$, we see from the fact $\frac{2n}{n+2} < p_0$ and (5.2.14) that

$$|\tilde{p}(z_1) - \tilde{p}(z_2)| \leq \omega(d_p(z_1,z_2))$$

for every $z_1 = (x_1,t_1), z_2 = (x_2,t_2) \in \mathbb{R}^{n+1}$, and $\tilde{\mathbf{a}}$ satisfies the condition (5.0.3) and (5.0.4) with $(\mathbf{a},\nu,\Lambda,\mu,p(\cdot))$ replaced by $(\tilde{\mathbf{a}}, c_a^{-1}\nu, c_a\Lambda, \lambda^{-\frac{1}{p_0}}\mu, \tilde{p}(\cdot))$. Moreover, $\tilde{u}$ is a weak solution to

$$\tilde{u}_t - \operatorname{div}\tilde{\mathbf{a}}(D\tilde{u},z) = \operatorname{div}(|\tilde{F}|^{\tilde{p}(z)-2}\tilde{F}) \quad \text{in} \quad Q_{\tilde{r}} = K_{\tilde{r}}, \quad \text{if } B_{\tilde{r}} \subset \tilde{\Omega},$$

or

$$\begin{cases} \tilde{u}_t - \operatorname{div}\tilde{\mathbf{a}}(D\tilde{u},z) = \operatorname{div}\left(|\tilde{F}|^{\tilde{p}(z)-2}\tilde{F}\right) & \text{in} \quad K_{\tilde{r}}, \\ \tilde{u} = 0 & \text{on} \quad \partial_{\mathrm{w}}K_{\tilde{r}}, \end{cases} \quad \text{if } B_{\tilde{r}} \not\subset \tilde{\Omega},$$

with $|\tilde{F}|^{\tilde{p}(\cdot)} \in L^{\gamma}(K_{\tilde{r}})$ and

$$\begin{aligned}
&\int_{K_{\tilde{r}}} \left[|D\tilde{u}|^{\tilde{p}(z)} + |\tilde{F}|^{\tilde{p}(z)} + 1 \right] dz \\
&\qquad = \lambda^{n\frac{2-p_0}{2p_0}} \int_{K^{\lambda}_{\tilde{r}}} \left[\lambda^{-\frac{p(z)}{p_0}} |Du|^{p(z)} + \lambda^{\frac{p(z)(1-p_0)}{p_0(p(z)-1)}} |F|^{p(z)} + 1 \right] dz \\
&\qquad \leq \lambda \int_{K^{\lambda}_{r}} \left[\lambda^{-1+\frac{p_2-p_1}{\gamma_1}} |Du|^{p(z)} + \lambda^{-1+\frac{p_2-p_1}{\gamma_1(\gamma_1-1)}} |F|^{p(z)} + 1 \right] dz \leq c_a^{\frac{1}{\gamma_1(\gamma_1-1)}} M_1,
\end{aligned}$$

where $\tilde{\Omega} := \left\{ x \in \mathbb{R}^n : \lambda^{\frac{p_0-2}{2p_0}} x \in \Omega \right\}$. Note that $\tilde{\Omega}$ is $(\delta, \lambda^{\frac{2-p_0}{2p_0}} R)$-vanishing and, by (5.2.11), $\tilde{r} > 0$ satisfies

$$\tilde{r} < \min \left\{ c_h^{-1} \left(c_a^{\frac{1}{\gamma_1(\gamma_1-1)}} M_1 \right)^{-\frac{1}{\alpha}}, \lambda^{\frac{2-p_0}{2p_0}} R \right\}.$$

Therefore, applying Lemma 5.2.5, scaling back from u to $\tilde{u}$ and repeating similar computations as in the case $p_0 \geq 2$, we obtain (5.2.15). □

5.2.3 Lipschitz regularity.

We recall L^{∞}-estimates up to the flat boundary for the gradient of the weak solutions of homogeneous parabolic equations of p-Laplacian type with a nonlinearity independent of the space variable x. DiBenedetto and Friedman showed the interior gradient bound for parabolic systems, see [23] and the monograph [21], and in [47] Lieberman extended these results up to the boundary for parabolic equations.

Let $\overline{\mathbf{b}} = \overline{\mathbf{b}}(\xi, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a vector valued function satisfying the condition (5.0.3) and (5.0.4) with $\mathbf{a}(\xi, z)$ and $p(z)$ replaced by $\mathbf{b}(\xi, t)$ and p, respectively, where $p \in \left(\frac{2n}{n+2}, \infty\right)$ is a fixed constant. In this subsection, we temporally denote the intrinsic parabolic cylinder such that

$$Q^{\lambda}_r(w) := B_r(y) \times (\tau - \lambda^{\frac{2-p}{p}} r^2, \tau + \lambda^{\frac{2-p}{p}} r^2) \text{ and } Q^{\lambda+}_r := B^+_r \times (-\lambda^{\frac{2-p}{p}} r^2, \lambda^{\frac{2-p}{p}} r^2).$$

Lemma 5.2.7. *(1) Let v be a weak solution to*

$$v_t - div\, \overline{\mathbf{b}}(Dv, t) = 0, \quad in\ Q^{\lambda}_r(w)$$

with

$$\fint_{Q_r^\lambda(w)} |Dv|^p \, dz \leq \lambda$$

for some $\lambda > 1$. *Then* $Du \in L^\infty(Q_{\frac{r}{2}}^\lambda(w))$ *with the estimate*

$$\|Du\|^p_{L^\infty(Q_{\frac{r}{2}}^\lambda(w))} \leq c\lambda,$$

for some $c = c(n, \nu, \Lambda, p) > 0$.
(2) Let $\overline{v}$ *be a weak solution to*

$$\begin{cases} \overline{v}_t - div\, \overline{\mathbf{b}}(D\overline{v}, t) & = & 0 & in & Q_r^{\lambda+} \\ \overline{v} & = & 0 & on & T_r^\lambda, \end{cases}$$

with

$$\fint_{Q_r^{\lambda+}(w)} |D\overline{v}|^p \, dz \leq \lambda,$$

for some $\lambda > 1$. *Then* $Du \in L^\infty(Q_{\frac{r}{2}}^{\lambda+})$ *with the estimate*

$$\|Du\|^p_{L^\infty(Q_{\frac{r}{2}}^{\lambda+})} \leq c\lambda,$$

for some $c = c(n, \nu, \Lambda, p) > 0$.

5.2.4 Technical tools.

We start with a Vitali type covering lemma for intrinsic parabolic cylinders. A similar one can be found in [6]. Here, we impose more strong restriction on the scales of intrinsic parabolic cylinders than one in [6, Lemma 7.1], to get a rather simple proof.

Lemma 5.2.8. *Let* $\lambda, c_a > 1$. *If* $\mathcal{F} := \{Q_{r_j}^\lambda(w_j)\}_{j \in \mathcal{J}}$ *is the family of intrinsic parabolic cylinders satisfying that* $\bigcup_{j \in \mathcal{J}} Q_{r_j}^\lambda(w_j)$ *is bounded in* $\mathbb{R}^{n+1}$ *and*

$$\lambda^{p_2^j - p_1^j} \leq c_a \quad for\ any\ j \in \mathcal{J}, \tag{5.2.16}$$

where

$$p_j^+ := \sup_{w \in Q_{r_j}^\lambda(w_j)} p(w) \quad and \quad p_j^- := \inf_{w \in Q_{r_j}^\lambda(w_j)} p(w),$$

then there exists a countable disjoint subcollection $\mathcal{G} = \{Q^{\lambda}_{r_i}(w_i)\}_{i\in\mathcal{I}}$, $\mathcal{I} \subset \mathcal{J}$, *satisfying*

$$\bigcup_{j\in\mathcal{J}} Q^{\lambda}_{r_j}(w_j) \ \subset\ \bigcup_{i\in\mathcal{I}} Q^{\lambda}_{\chi r_i}(w_i), \quad \textit{where}\ \ \chi := \max\left\{5, 2\sqrt{2}c_a^{\frac{1}{\gamma_1^2}}\right\}.$$

Proof. For $k = 1, 2, \dots,$ we define

$$\mathcal{F}_k := \left\{Q^{\lambda}_{r_j}(w_j) \in \mathcal{F} : \frac{r_0}{2^k} < r_i \le \frac{r_0}{2^{k-1}}\right\}, \quad \text{where} \ \ \sup_{j\in\mathcal{J}} r_j = r_0,$$

and $\mathcal{G}_1 \subset \mathcal{F}_1$ by any maximal disjoint subcollection of $\mathcal{F}_1$. We inductively select $\mathcal{G}_k \subset \mathcal{F}_k$ by any maximal disjoint subcollection of

$$\left\{Q \in F_k : Q \cap \hat{Q} = \emptyset \text{ for all } \hat{Q} \in \bigcup_{l=1}^{k-1} \mathcal{G}_l\right\}.$$

Note that, since the measure of $\bigcup_{j\in\mathcal{J}} Q^{\lambda}_{r_j}(w_j)$ is finite by the assumption of the lemma, $\mathcal{G}_k$ has a finite element. Finally, we define

$$\mathcal{G} := \bigcup_{k=1}^{\infty} \mathcal{G}_k.$$

Then $\mathcal{G}$ is countable and its elements are mutually disjoint. Therefore, it suffices to show that for any $Q^{\lambda}_{r_j}(w_j) \in \mathcal{F}$, there exists $Q^{\lambda}_{r_i}(w_i) \in \mathcal{G}$ such that $Q^{\lambda}_{r_j}(w_j) \subset Q^{\lambda}_{\chi r_i}(w_i)$. If $Q^{\lambda}_{r_j}(w_j) \in \mathcal{G}$ there is nothing to prove. Fix $Q^{\lambda}_{r_j}(w_j) \in \mathcal{F} \setminus \mathcal{G}$ then $Q^{\lambda}_{r_j}(w_j) \in F_k$ for some k. By the maximality of $\mathcal{G}_k$, there exists $Q^{\lambda}_{r_i}(w_i) \in \bigcup_{l=1}^{k} \mathcal{G}_l$ such that $Q^{\lambda}_{r_j}(w_j) \cap Q^{\lambda}_{r_i}(w_i) \neq \emptyset$. We note

$$r_j \le \frac{r_0}{2^{k-1}} < 2r_i. \tag{5.2.17}$$

Let $w_i = (y_i, \tau_i)$, $w_j = (y_j, \tau_j)$ and $w_0 \in Q^{\lambda}_{r_j}(w_j) \cap Q^{\lambda}_{r_i}(w_i)$. Then (5.2.17) directly implies

$$B_{r_j}(y_j) \subset B_{5r_i}(y_i) \subset B_{\chi r_i}(y_i).$$

We next show that

$$(\tau_j - \lambda^{\frac{2-p(w_j)}{p(w_j)}} r_j^2, \tau_j + \lambda^{\frac{2-p(w_j)}{p(w_j)}} r_j^2) \subset (\tau_i - \lambda^{\frac{2-p(w_i)}{p(w_i)}} (\chi r_i)^2, \tau_i + \lambda^{\frac{2-p(w_i)}{p(w_i)}} (\chi r_i)^2). \tag{5.2.18}$$

If $p(w_j) \geq p(w_i)$, (5.2.17) implies

$$\lambda^{\frac{2-p(w_j)}{p(w_j)}} r_j^2 \leq \lambda^{\frac{2-p(w_i)}{p(w_i)}} (2r_i)^2 = 4\lambda^{\frac{2-p(w_i)}{p(w_i)}} r_i^2,$$

and so

$$(\tau_j - \lambda^{\frac{2-p(w_j)}{p(w_j)}} r_j^2, \tau_j + \lambda^{\frac{2-p(w_j)}{p(w_j)}} r_j^2) \subset (\tau_i - 9\lambda^{\frac{2-p(w_i)}{p(w_i)}} r_i^2, \tau_i + 9\lambda^{\frac{2-p(w_i)}{p(w_i)}} r_i^2).$$

On the other hand, if $p(w_j) < p(w_i)$,

$$\begin{aligned} \lambda^{\frac{2-p(w_j)}{p(w_j)}} r_j^2 &= \lambda^{\frac{p(w_i)-p(w_j)}{p(w_j)p(w_i)}} \lambda^{\frac{2-p(w_i)}{p(w_i)}} r_j^2 \leq \lambda^{\frac{p(w_i)-p(w_0)}{\gamma_1^2}} \lambda^{\frac{p(w_0)-p(w_j)}{\gamma_1^2}} \lambda^{\frac{2-p(w_i)}{p(w_i)}} r_j^2 \\ &\leq \lambda^{\frac{p_i^+-p_i^-}{\gamma_1^2}} \lambda^{\frac{p_j^+-p_j^-}{\gamma_1^2}} \lambda^{\frac{2-p(w_i)}{p(w_i)}} r_j^2, \end{aligned}$$

hence using (5.2.16) and (5.2.17) we obtain

$$\lambda^{\frac{2-p(w_j)}{p(w_j)}} r_i^2 \leq 4c_a^{\frac{2}{\gamma_1^2}} \lambda^{\frac{2-p(w_i)}{p(w_i)}} r_i^2,$$

so that

$$(\tau_j - \lambda^{\frac{2-p(w_j)}{p(w_j)}} r_j^2, \tau_j + \lambda^{\frac{2-p(w_j)}{p(w_j)}} r_j^2) \subset (\tau_i - 8c_a^{\frac{2}{\gamma_1^2}} \lambda^{\frac{2-p(w_i)}{p(w_i)}} r_i^2, \tau_i + 8c_a^{\frac{2}{\gamma_1^2}} \lambda^{\frac{2-p(w_i)}{p(w_i)}} r_i^2).$$

Consequently, we have (5.2.18). □

Lemma 5.2.9. *[37, Lemma 4.3] Let ϕ be a bounded nonnegative function on $[r_1, r_2]$. Suppose that for any s_1, s_2 with $0 < r_1 \leq s_1 < s_2 \leq r_2$,*

$$\phi(s_1) \leq \kappa\phi(s_2) + \frac{P_1}{(s_2 - s_1)^\beta} + P_2,$$

where $\beta, P_1, P_2 \geq 0$ and $\kappa \in (0, 1)$. Then there holds

$$\phi(r_1) \leq c\left[\frac{P_1}{(r_2 - r_1)^{\theta_2}} + P_2\right],$$

for some $c = c(\kappa, \beta) > 0$.

The following inequality can be found in [1] and references therein. For $U \subset \mathbb{R}^{n+1}$, $\beta > 0$, $\sigma > 1$ and $f \in L^\sigma(U)$, we have

$$\fint_U |f| \ln^\beta \left(e + \frac{|f|}{f_U} \right) dz \le c(\sigma, \beta) \left(\fint_U |f|^\sigma \, dz \right)^{\frac{1}{\sigma}}, \tag{5.2.19}$$

for some $c(\sigma, \beta) > 0$, where we write $\ln^\beta t := (\ln t)^\beta$. Note that the constant $c(\sigma, \beta)$ is continuous with respect to β. We end this section with the elementary inequality

$$t^\beta |\ln t| \le \max \left\{ \frac{1}{e\beta}, 2^\beta \ln 2 \right\}, \quad {}^\forall t \in (0, 2]. \tag{5.2.20}$$

5.3 Comparison Estimates.

In this section, we obtain interior and boundary comparison estimates on intrinsic parabolic cylinders, under a circumstance satisfying several conditions described in (5.3.2), (5.3.3), (5.3.4), (5.3.14) and (5.3.21). We start with setting various parameters and stating required conditions.

Let u be the unique weak solution of (5.0.1), $p(\cdot)$ satisfy (5.0.2), and $(p(\cdot), \mathbf{a}(\xi, \cdot, t), \Omega)$ is (δ, R)-vainshing for some $R \in (0, 1)$. Here $\delta \in (0, 1/8)$ is a sufficiently small number that will be determined from Lemma 5.3.1, Lemma 5.3.2 and Lemma 5.3.3. We fix any $\lambda > 1$ and sufficiently small $r \in (0, R/8)$. Then for $w = (y, \tau) \in \Omega_T$ we set

$$p_0 := p(w), \quad p_1 := \inf_{z \in K^\lambda_{4r}(w)} p(z) \quad \text{and} \quad p_2 := \sup_{z \in K^\lambda_{4r}(w)} p(z).$$

In this section, we only consider the *interior case* that $Q^\lambda_{4r}(w) = K^\lambda_{4r}(w) \subset \Omega_T$ or the *boundary case* that

$$\begin{cases} B^+_{4r} \subset \Omega_{4r}(y) \subset B_{4r}(0) \cap \{x_n > -10\delta r\}, \\ (\tau - \lambda^{\frac{2-p_0}{p_0}}(4r)^2, \tau + \lambda^{\frac{2-p_0}{p_0}}(4r)^2) \subset (0, T). \end{cases} \tag{5.3.1}$$

In addition, we assume that the following hold:

$$p_2 - p_1 \le \min \left\{ \frac{\tilde{\sigma}_2}{4(\gamma_1 - 1)}, \frac{\nu}{2\Lambda}, 1 \right\}, \tag{5.3.2}$$

$$\lambda^{\frac{2}{p_0}} \le \Gamma r^{-(n+2)}, \quad p_2 - p_1 \le \omega(\Gamma(4r)^\alpha), \quad \lambda^{p_2-p_1} \le c_a, \tag{5.3.3}$$

for some $\Gamma, c_a > 1$, and

$$\fint_{K_r^\lambda(w)} |Du|^{p(z)}\, dz \le \lambda \quad \text{and} \quad \fint_{K_r^\lambda(w)} |F|^{p(z)}\, dz \le \delta\lambda. \tag{5.3.4}$$

We remark that Γ and c_a will be determined in Section 5.2. Finally, for the sake of convenience, we write κ_i, $i = 1, \dots 5$, to be any constant in $(0,1)$, c to be any positive constant depending only on $n, \Lambda, \nu, \gamma_1, \gamma_2, c_a$, and $c(\kappa_i)$ to be any positive constant depending only on $n, \Lambda, \nu, \gamma_1, \gamma_2, c_a, \kappa_i$.

Let h be the weak solution to

$$\begin{cases} h_t - \operatorname{div} \mathbf{a}(Dh, z) = 0 & \text{in} \quad K_{4r}^\lambda(w), \\ h = u & \text{on} \quad \partial_p K_{4r}^\lambda(w). \end{cases} \tag{5.3.5}$$

Lemma 5.3.1. *For any $\epsilon \in (0,1)$ there exists a small $\delta = \delta(n, \Lambda, \nu, \gamma_1, \gamma_2, \epsilon) > 0$ such that*

$$\fint_{K_{4r}^\lambda(w)} |Dh|^{p(z)}\, dz \le c_1 \lambda \quad \textit{and} \quad \fint_{K_{4r}^\lambda(w)} |Du - Dh|^{p(z)}\, dz \le \epsilon\lambda \tag{5.3.6}$$

for some $c_1 = c_1(n, \Lambda, \nu, \gamma_1, \gamma_2) > 1$.

Proof. Without loss of generality, we assume that $w = 0$. Since $u - h \in W_0^{p(\cdot)}(K_{4r}^\lambda)$, $u - h$ is allowed to be a test function in the weak formulations of (5.0.1) and (5.3.5) hence we have

$$\begin{aligned} \langle (u-h)_t, u-h \rangle_{K_{4r}^\lambda} + \int_{K_{4r}^\lambda} (\mathbf{a}(Du, z) - \mathbf{a}(Dh, z)) \cdot (Du - Dh)\, dz \\ = -\int_{K_{4r}^\lambda} |F|^{p(z)-2} F \cdot (Du - Dh)\, dz. \end{aligned} \tag{5.3.7}$$

We note by Proposition 5.2.2 that

$$\langle (u-h)_t, u-h \rangle_{K_{4r}^\lambda} \ge 0. \tag{5.3.8}$$

From (5.2.6) we observe that

$$\nu_3 (\mu^2 + |Du|^2 + |Dh|^2)^{\frac{p(z)-2}{2}} |Du - Dh|^2 \le (\mathbf{a}(Du, z) - \mathbf{a}(Dh, z)) \cdot (Du - Dh).$$

Here, if $p(z) \geq 2$ a directly computation yields

$$\frac{\nu_3}{2}|Du - Dh|^{p(z)} \leq (\mathbf{a}(Du, z) - \mathbf{a}(Dh, z)) \cdot (Du - Dh),$$

and if $p(z) < 2$ Young's inequality implies

$$\begin{aligned}
&|Du - Dh|^{p(z)} \\
&\leq \kappa_1 \left(1 + |Du|^{p(z)} + |Dh|^{p(z)}\right) + c(\kappa_1) \left(\mu^2 + |Du|^2 + |Dh|^2\right)^{\frac{p(z)-2}{2}} |Du - Dh|^2 \\
&\leq \kappa_1 \left(1 + |Du|^{p(z)} + |Dh|^{p(z)}\right) + c(\kappa_1) \left(\mathbf{a}(Du, z) - \mathbf{a}(Dh, z)\right) \cdot (Du - Dh).
\end{aligned}$$

Therefore we deduce

$$\begin{aligned}
\int_{K_{4r}^{\lambda}} |Du - Dh|^{p(z)}\, dz \quad &\leq \quad \kappa_1 \int_{K_{4r}^{\lambda}} \left(1 + |Du|^{p(z)} + |Dh|^{p(z)}\right) dz \\
&\quad + c(\kappa_1) \int_{K_{4r}^{\lambda}} (\mathbf{a}(Du, z) - \mathbf{a}(Dh, z)) \cdot (Du - Dh)\, dz. \quad (5.3.9)
\end{aligned}$$

Applying Young's inequality to the right hand side on (5.3.7) we have

$$\begin{aligned}
\left| \int_{K_{4r}^{\lambda}} |F|^{p(z)-2} F \cdot (Du - Dh)\, dz \right| \quad &\leq \quad \kappa_2 \int_{K_{4r}^{\lambda}} |Du - Dh|^{p(z)}\, dz \\
&\quad + c(\kappa_2) \int_{K_{4r}^{\lambda}} |F|^{p(z)}\, dz. \quad (5.3.10)
\end{aligned}$$

Combining (5.3.7)-(5.3.10) we find

$$\begin{aligned}
\fint_{K_{4r}^{\lambda}} |Du - Dh|^{p(z)}\, dz \quad &\leq \quad \kappa_1 \fint_{\Omega_T} \left(1 + |Du|^{p(z)} + |Dh|^{p(z)}\right) dz \\
&\quad + \kappa_2 c(\kappa_1) \fint_{K_{4r}^{\lambda}} |Du - Dh|^{p(z)}\, dz + c(\kappa_1) c(\kappa_2) \int_{\Omega_T} |F|^{p(z)}\, dz. \quad (5.3.11)
\end{aligned}$$

We first choose κ_1 and κ_2 sufficiently small to find

$$\fint_{K_{4r}^{\lambda}} |Dh|^{p(z)}\, dz \leq c \left(\fint_{K_{4r}^{\lambda}} |Du|^{p(z)}\, dz + \fint_{K_{4r}^{\lambda}} |F|^{p(z)}\, dz + 1 \right). \quad (5.3.12)$$

Then inserting the assumption in (5.3.4) into the previous inequality we obtain the first estimate in (5.3.6). Returning to (5.3.11), applying (5.3.4) and the first estimate in (5.3.6) and taking

$$\kappa_1 = \frac{\epsilon}{4(2 + c_1)}, \quad \kappa_2 = \frac{1}{2c(\kappa_1)} \quad \text{and} \quad \delta = \frac{\epsilon}{4c(\kappa_1)(\kappa_2)},$$

we get the second estimate in (5.3.6). □

We next observe the self improving integrability of Dv. We first note from the estimate (5.1.11), (5.2.7) and (5.3.12) that

$$\int_{K^\lambda_{4r}} |Dh|^{p(z)}\,dz \le \tilde{c}_0 M$$

for some $\tilde{c}_0 = \tilde{c}_0(n,\Lambda,\nu,\gamma_1,\gamma_2) > c_0 + 1$, and then define

$$M_1 := \tilde{c}_0 M. \tag{5.3.13}$$

We now further assume that $r > 0$ satisfies

$$4r < \min\left\{1, \lambda^{\frac{p(w)-2}{2p(w)}}\right\} c_h^{-1}\left(c_a^{\frac{1}{\gamma_1(\gamma_1-1)}} M_1\right)^{-\frac{1}{\alpha}}. \tag{5.3.14}$$

Then, in view of Corollary 5.2.6 with $F = 0$, $\gamma = 2$ and u replaced by v, we have $|Dh|^{p(\cdot)} \in L^{1+\tilde{\sigma}_2}(K^\lambda_{3r}(w))$ with the estimate

$$\fint_{K^\lambda_{3r}(w)} |Dh|^{p(z)(1+\sigma)}\,dz \le c\lambda^{1+\sigma}, \tag{5.3.15}$$

for every $0 < \sigma < \tilde{\sigma}_2$. Note from the first restriction for $p_2 - p_1$ in (5.3.2) that

$$p_0 \le p'_w(p_2-1) \le \frac{p_1(p_2-1)}{p_1-1} \le p(z)\left(1 + \frac{p_2-p_1}{\gamma_1-1}\right) \le p(z)\left(1+\frac{\tilde{\sigma}_2}{4}\right)$$

and

$$\begin{aligned} p_0\left(1+\frac{\tilde{\sigma}_2}{4}\right) &\le p'_w(p_2-1)\left(1+\frac{\tilde{\sigma}_2}{4}\right) \le p(z)\left(1+\frac{p_2-p_1}{\gamma_1-1}\right)\left(1+\frac{\tilde{\sigma}_2}{4}\right) \\ &\le p(z)\left(1+\tilde{\sigma}_2\right) \end{aligned}$$

for every $z \in K^\lambda_{4r}(w)$, where $p'_0 := \frac{p_0}{p_0-1}$. Then, using the third inequality (5.3.3) and (5.3.15), we see

$$\begin{aligned} \fint_{K^\lambda_{3r}(w)} |Dh|^{p_0}\,dz &\le \fint_{K^\lambda_{3r}(w)} |Dh|^{p'_w(p_2-1)}\,dz + 1 \\ &\le \fint_{K^\lambda_{3r}(w)} |Dh|^{p(z)\left(1+\frac{p_2-p_1}{\gamma_1-1}\right)}\,dz + 2 \\ &\le c\lambda^{1+\frac{p_2-p_1}{\gamma_1-1}} \le c\lambda \end{aligned} \tag{5.3.16}$$

and

$$\begin{aligned}\fint_{K_{3r}^{\lambda}(w)} |Dh|^{p_0\left(1+\frac{\tilde{\sigma}_2}{4}\right)}\,dz &\le \fint_{K_{3r}^{\lambda}(w)} |Dh|^{p'_w(p_2-1)\left(1+\frac{\tilde{\sigma}_2}{4}\right)}\,dz+1\\ &\le \fint_{K_{3r}^{\lambda}(w)} |Dh|^{p(z)\left(1+\frac{p_2-p_1}{\gamma_1-1}\right)\left(1+\frac{\tilde{\sigma}_2}{4}\right)}\,dz+2\\ &\le c\lambda^{1+\frac{\tilde{\sigma}_2}{4}+\frac{p_2-p_1}{\gamma_1-1}\left(1+\frac{\tilde{\sigma}_2}{4}\right)} \le c\lambda^{1+\frac{\tilde{\sigma}_2}{4}}. \qquad (5.3.17)\end{aligned}$$

We next define a vector-valued fuction $\mathbf{b} : \mathbb{R}^n \times K_{4r}^{\lambda}(w) \to \mathbb{R}^n$ by

$$\mathbf{b}(\xi, z) := \mathbf{a}(\xi, z)\left(\mu^2+|\xi|^2\right)^{\frac{p_0-p(z)}{2}}.$$

Then direct computations yield

$$|\mathbf{b}(\xi,z)| \le L(\mu^2+|\xi|^2)^{\frac{p_0-1}{2}}, \quad |D_\xi \mathbf{b}(\xi,z)| \le L(p_2-p_1+1)(\mu^2+|\xi|^2)^{\frac{p_0-1}{2}}$$

and

$$(D_\xi \mathbf{b}(\xi,z)\eta)\cdot\eta \ge \{\nu-(p_2-p_1)\Lambda\}(\mu^2+|\xi|^2)^{\frac{p_0-2}{2}}|\eta|^2.$$

Then applying (5.3.2) we have

$$\begin{cases} |\mathbf{b}(\xi,z)| + (\mu^2+|\xi|^2)^{\frac{1}{2}}|D_\xi \mathbf{b}(\xi,z)| \le 3\Lambda(\mu^2+|\xi|^2)^{\frac{p_0-1}{2}}, \\ \frac{\nu}{2}(\mu^2+|\xi|^2)^{\frac{p_0-2}{2}}|\eta|^2 \le (D_\xi \mathbf{b}(\xi,z)\eta)\cdot\eta, \end{cases} \qquad (5.3.18)$$

for every $z \in K_{4r}^{\lambda}(w)$ and $\eta, \xi \in \mathbb{R}^n$, provided that $p_2 - p_1$ satisfies the second and third condition in (5.3.2).

For the interior case that $K_{4r}^{\lambda}(w) = Q_{4r}^{\lambda}(w) \subset \Omega_T$, we define $\overline{\mathbf{b}} : \mathbb{R}^n \times (\tau - \lambda^{\frac{2-p_0}{p_0}}(4r)^2, \tau + \lambda^{\frac{2-p_0}{p_0}}(4r)^2) \to \mathbb{R}^n$ by

$$\overline{\mathbf{b}}(\xi,t) := \fint_{B_{4r}(y)} \mathbf{b}(\xi,x,t)\,dx.$$

Then, we have from (2) in Definition 5.1.1 that

$$\fint_{K_{4r}^{\lambda}(w)} \sup_{\xi\in\mathbb{R}^n} \frac{|\overline{\mathbf{b}}(\xi,t)-\mathbf{b}(\xi,z)|}{(\mu^2+|\xi|^2)^{\frac{p_0-1}{2}}}\,dz = \fint_{Q_{4r}^{\lambda}(w)} \theta(\mathbf{a}, B_{4r}(y))(z)\,dz \le \delta.$$

On the other hand, for the bounary case that $K^\lambda_{4r}(w)$ satisfies (5.3.1), we define $\overline{\mathbf{b}} : \mathbb{R}^n \times (\tau - \lambda^{\frac{2-p_0}{p_0}}(4r)^2, \tau + \lambda^{\frac{2-p_0}{p_0}}(4r)^2) \to \mathbb{R}^n$ by

$$\overline{\mathbf{b}}(\xi, t) := \fint_{B^+_{4r}} \mathbf{b}(\xi, x, t)\, dx.$$

Then again by (2) in Definition 5.1.1 we see that

$$\begin{aligned} \fint_{Q^{\lambda+}_{4r}} \sup_{\xi \in \mathbb{R}^n} \frac{|\overline{\mathbf{b}}(\xi, t) - \mathbf{b}(\xi, z)|}{(\mu^2 + |\xi|^2)^{\frac{p_0-1}{2}}}\, dz &= \fint_{Q^{\lambda+}_{4r}} \theta(\mathbf{a}, B^+_{4r})(z)\, dz \\ &\leq 4 \fint_{Q^\lambda_{4r}} \theta(\mathbf{a}, B_{4r})(z)\, dz \leq 4\delta. \end{aligned} \tag{5.3.19}$$

In both cases, let v be the weak solution to

$$\begin{cases} v_t - \operatorname{div} \overline{\mathbf{b}}(Dv, t) = 0 & \text{in} \quad K^\lambda_{3r}(w), \\ v = h & \text{on} \quad \partial_p K^\lambda_{3r}(w). \end{cases} \tag{5.3.20}$$

Note that $\overline{\mathbf{b}}$ satisfies (5.3.18) with $\mathbf{b}(\xi, z)$ replaced by $\overline{\mathbf{b}}(\xi, t)$, and by (5.3.16) the equation (5.3.20) is well defined.

Lemma 5.3.2. *Suppose that $r > 0$ satisfies (5.3.14) and*

$$r \leq \min\left\{ e^{-1} \Gamma^{-\left(\frac{n+3}{\alpha}+1\right)}, (\Gamma^{-1} R)^{\frac{1}{\alpha}} \right\}. \tag{5.3.21}$$

For any $\epsilon \in (0,1)$ there exists a small $\delta = \delta(n, \Lambda, \nu, \gamma_1, \gamma_2, c_a, \epsilon) > 0$ such that

$$\fint_{K^\lambda_{3r}(w)} |Dv|^{p_0}\, dz \leq c_2 \lambda \quad \text{and} \quad \fint_{K^\lambda_{3r}(w)} |Dh - Dv|^{p_0}\, dz \leq \epsilon \lambda \tag{5.3.22}$$

for some $c_2 = c_2(n, \Lambda, \nu, \gamma_1, \gamma_2, c_a) > 0$.

Proof. We assume $w = 0$. We prove the lemma only if $K^\lambda_{4r}(w)$ is the boundary region. The proof when $K^\lambda_{4r}(w)$ is the interior region is exactly same with the one of case that $K^\lambda_{4r}(w)$ is the boundary region. In this proof, $\langle\ ,\ \rangle$ means the paring between $L^{p_0}(-\lambda^{\frac{2-p_0}{p_0}}(3r)^2, \lambda^{\frac{2-p_0}{p_0}}(3r)^2; W^{1,p_0}_0(\Omega_{3r}))$ and its dual space.

We note from (5.3.16) that $|Dh| \in L^{p'_w(p_2-1)}(K^\lambda_{3r}$, hence $|\mathbf{a}(Dh, z)| \in L^{p'_w}(K^\lambda_{3r})$. Then by an approximation argument we have

$$\langle h_t, \varphi \rangle = -\int_{K^\lambda_{3r}} \mathbf{a}(Dh, z) \cdot D\varphi\, dz$$

for every $\varphi \in L^{p_0}(-\lambda^{\frac{2-p_0}{p_0}}(3r)^2, \lambda^{\frac{2-p_0}{p_0}}(3r)^2; W_0^{1,p_0}(\Omega_{3r}))$. From this and (5.3.20) we obtain

$$\begin{aligned}\langle (h-v)_t, h-v\rangle &= \int_{K_{3r}^\lambda} \overline{\mathbf{b}}(Dv,t)\cdot(Dh-Dv)\,dz\\ &\quad -\int_{K_{3r}^\lambda} \mathbf{a}(Dh,z)\cdot(Dh-Dv)\,dz.\end{aligned}$$

In view of Proposition 5.2.2 with the case $p(\cdot)\equiv p_0$, we have

$$\begin{aligned} I_1 &:= \fint_{K_{3r}^\lambda} (\overline{\mathbf{b}}(Dh,t)-\overline{\mathbf{b}}(Dv,t))\cdot(Dh-Dv)\,dz\\ &\leq \fint_{K_{3r}^\lambda} (\overline{\mathbf{b}}(Dh,t)-\mathbf{b}(Dh,z))\cdot(Dh-Dv)\,dz\\ &\quad +\fint_{K_{3r}^\lambda} (\mathbf{b}(Dh,z)-\mathbf{a}(Dh,z))\cdot(Dh-Dv)\,dz\\ &=: I_2+I_3.\end{aligned}$$

We now estimate I_1, I_2 and I_3. In a similar argument we estimated (5.3.9), we see from Young's inequality, (5.2.6) and (5.3.16) that

$$\begin{aligned}\fint_{K_{3r}^\lambda} |Dh-Dv|^{p_0}\,dz &\leq \kappa_3 \fint_{K_{3r}^\lambda}\Big[|Dh|^{p_0}+|Dv|^{p_0}+1\Big]\,dz + c(\kappa_3)I_1\\ &\leq \kappa_3\left(\fint_{K_{3r}^\lambda}|Dv|^{p_0}\,dz+\lambda\right)+c(\kappa_3)I_1. \qquad (5.3.23)\end{aligned}$$

For I_2, Young's inequality yields

$$I_2 \leq \kappa_4 \fint_{K_{3r}^\lambda}|Dh-Dv|^{p_0}\,dz + c(\kappa_4)\fint_{K_{3r}^\lambda}|\overline{\mathbf{b}}(Dh,t)-\mathbf{b}(Dh,z)|^{p_w'}\,dz.$$

Using Hölder's inequality, the first condition in (5.3.18) replaced $\mathbf{b}(\xi,z)$ by

$\overline{\mathbf{b}}(\xi,t)$, (5.3.17) and (5.3.19), we have

$$\begin{aligned}
&\fint_{K_{3r}^{\lambda}} |\overline{\mathbf{b}}(Dh,t)-\mathbf{b}(Dh,z)|^{p_w'}\,dz \le \fint_{K_{3r}^{\lambda}} \theta(\mathbf{a},B_{4r}^{+})^{p_w'}(1+|Dh|)^{p_0}\,dz \\
&\le \left(\fint_{K_{3r}^{\lambda}} \theta(\mathbf{a},B_{4r}^{+})^{p_w'\frac{4+\sigma_0}{\sigma_0}}\,dz\right)^{\frac{\sigma_0}{4+\sigma_0}} \left(\fint_{K_{3r}^{\lambda}} (1+|Dh|)^{p_0(1+\frac{\sigma_0}{4})}\,dz\right)^{\frac{1}{1+\sigma_0/4}} \\
&\le \left((2\Lambda)^{\gamma_1'\frac{4+\sigma_0}{\sigma_0}-1}\left(\frac{3}{4}\right)^{n+2}\fint_{Q_{4r}^{\lambda+}} \theta(\mathbf{a},B_{4r}^{+})\,dz + (2\Lambda)^{\gamma_1'\frac{4+\sigma_0}{\sigma_0}}\frac{|K_{3r}^{\lambda}\setminus Q_{3r}^{\lambda+}|}{|Q_{3r}^{\lambda+}|}\right)^{\frac{\sigma_0}{4+\sigma_0}}\lambda \\
&\le c\delta^{\frac{\sigma_0}{4+\sigma_0}}\lambda.
\end{aligned}$$

Hence, we obtain

$$I_2 \le \kappa_4 \fint_{K_{3r}^{\lambda}} |Dh-Dv|^{p_0}\,dz + c(\kappa_4)\delta^{\frac{\sigma_0}{4+\sigma_0}}\lambda. \tag{5.3.24}$$

As for I_3, we first set $E := \left\{z\in K_{3r}^{\lambda} : \mu^2+|Dh(z)|^2>0\right\}$. Then Young's inequality implies

$$\begin{aligned}
I_3 &\le \frac{1}{|K_{3r}^{\lambda}|}\int_E \left|(\mu^2+|Dh|^2)^{\frac{p_0-p(z)}{2}}-1\right| |\mathbf{a}(Dh,z)||Dh-Dv|\,dz \\
&\le \kappa_5 \fint_{K_{3r}^{\lambda}} |Dh-Dv|^{p_0}\,dz \\
&\quad + \frac{c(\kappa_5)}{|K_{3r}^{\lambda}|}\int_E \left[\left|(\mu^2+|Dh|^2)^{\frac{p_0-p(z)}{2}}-1\right|(\mu^2+|Dh|^2)^{\frac{p(z)-1}{2}}\right]^{p_0'}\,dz.
\end{aligned}$$

For each $z\in E$, applying the mean value theorem to the function $f(\theta)=(\mu^2+|Dh|^2)^{\theta\frac{p_0-p(z)}{2}}$ on the interval $[0,1]$, we find

$$\begin{aligned}
&\left|(\mu^2+|Dh(z)|^2)^{\frac{p_0-p(z)}{2}}-1\right|(\mu^2+|Dh(z)|^2)^{\frac{p(z)-1}{2}} \\
&\qquad = \frac{|p_0-p(z)|}{2}(\mu^2+|Dh(z)|^2)^{s_z\frac{p_0-p(z)}{2}+\frac{p(z)-1}{2}}\left|\ln(\mu^2+|Dh(z)|^2)\right|
\end{aligned}$$

for some $s_z\in(0,1)$. Note that if $|Dh(z)|\le 1$, (5.2.20) implies

$$(\mu^2+|Dh(z)|^2)^{s_z\frac{p_0-p(z)}{2}+\frac{p(z)-1}{2}}\left|\ln(\mu^2+|Dh(z)|^2)\right| \le c.$$

Therefore, we obtain

$$\begin{aligned}&\int_{E_2}\left[\left|(\mu^2+|Dh|^2)^{\frac{p_0-p(z)}{2}}-1\right|(\mu^2+|Dh|^2)^{\frac{p(z)-1}{2}}\right]^{p_0'}dz\\&\qquad\le c|p_2-p_1|^{p_w'}\left(\int_{E_2\cap\{|Dh(z)|\ge 1\}}\left[|Dh|^{(p_2-1)}\ln(e+|Dh|)\right]^{p_w'}dz+|K_{3r}^\lambda|\right)\\&\qquad\le c|p_2-p_1|^{p_w'}\left(\int_{K_{3r}^\lambda}|Dh|^{p_0'(p_2-1)}\ln^{p_w'}(e+|Dh|^{p_0'(p_2-1)})\,dz+|K_{3r}^\lambda|\right),\end{aligned}$$

and so, by the second inequality in (5.3.3),

$$I_3\le\kappa_5\fint_{K_{3r}^\lambda}|Dh-Dv|^{p_0}\,dz+c(\kappa_5)\omega(\Gamma(4r)^\alpha)^{p_w'}(I_{3a}+1),$$

where

$$I_{3a}:=\fint_{K_{3r}^\lambda}|Dh|^{p_0'(p_2-1)}\ln^{p_w'}(e+|Dh|^{p_0'(p_2-1)})\,dz.$$

The elementary inequality $\ln(e+ab)=\ln(e+a)+\ln(e+b)$, $a,b>0$, implies

$$\begin{aligned}I_{3a}\quad\le&\quad\fint_{K_{3r}^\lambda}|Dh|^{p_0'(p_2-1)}\ln^{p_w'}\left(e+\frac{|Dh|^{p_0'(p_2-1)}}{\overline{(|Dh|^{p_0'(p_2-1)})}_{K_{3r}^\lambda}}\right)dz\\&+\fint_{K_{3r}^\lambda}|Dh|^{p_0'(p_2-1)}\ln^{p_w'}\left(e+\overline{(|Dh|^{p_0'(p_2-1)})}_{K_{3r}^\lambda}\right)dz.\end{aligned}$$

Applying (5.2.19) to $f=|Dh|^{p_0'(p_2-1)}$, $U=K_{3r}^\lambda$, $\beta=p_w'$ and $\sigma=1+\frac{\sigma_0}{4}$ and using (5.3.17), we have

$$\fint_{K_{3r}^\lambda}|Dh|^{p_0'(p_2-1)}\ln^{p_w'}\left(e+\frac{|Dh|^{p_0'(p_2-1)}}{\overline{(|Dh|^{p_0'(p_2-1)})}_{K_{3r}^\lambda}}\right)dz\le c\lambda.$$

On the other hand, by (5.3.16) and the first inequality in (5.3.3) we know

$$\begin{aligned}\ln\left(e+\overline{(|Dh|^{p_0'(p_2-1)})}_{K_{3r}^\lambda}\right)\quad&\le\quad\ln(e+c\lambda)\ \le\ c\left(\ln\lambda^{\frac{2}{p_0}}+1\right)\\&\le\quad c\{\ln(\Gamma(4r)^{-n-2})+1\},\end{aligned}$$

which together with (5.3.16) implies

$$\fint_{K^\lambda_{3r}} |Dh|^{p_0'(p_2-1)} \ln^{p_w'}\left(e + \overline{(|Dh|^{p_0'(p_2-1)})}_{K^\lambda_{3r}}\right)\, dz \le c\{\ln(\Gamma(4r)^{-n-2}) + 1\}^{p_0'}\lambda.$$

Therefore we obtain

$$I_{3a} \le c\{\ln(\Gamma(4r)^{-n-2}) + 1\}^{p_0'}\lambda,$$

and so

$$I_3 \le \kappa_5 \fint_{K^\lambda_{3r}} |Dh - Dv|^{p_0}\, dz + c(\kappa_5)\omega(\Gamma(4r)^\alpha)^{p_w'}\{\ln(\Gamma(4r)^{-n-2}) + 1\}^{p_0'}\lambda.$$

Note that (5.3.21) and (5.1.5) yield

$$\begin{aligned} \omega(\Gamma(4r)^\alpha)^{p_w'}\{\ln(\Gamma(4r)^{-n-2}) + 1\}^{p_0'} &= \omega(\Gamma(4r)^\alpha)^{p_w'} \ln^{p_0'}(4re\Gamma(4r)^{-n-3}) \\ &\le \omega(\Gamma(4r)^\alpha)^{p_w'} \ln^{p_0'}(\Gamma^{-\frac{n+3}{\alpha}}(4r)^{-n-3}) \\ &\le c\left\{\omega(\Gamma(4r)^\alpha)\ln\left(\frac{1}{\Gamma(4r)^\alpha}\right)\right\}^{p_0'} \\ &\le c\delta^{\gamma_2'}, \end{aligned}$$

to discover that

$$I_3 \le \kappa_5 \fint_{K^\lambda_{3r}} |Dh - Dv|^{p_0}\, dz + c(\kappa_5)\delta^{\gamma_2'}\lambda. \tag{5.3.25}$$

Combining (5.3.23), (5.3.24) and (5.3.25) we obtain

$$\begin{aligned} \fint_{K^\lambda_{3r}} |Dh - Dv|^{p_0}\, dz \;\; &\le \;\; \kappa_3\left(\fint_{K^\lambda_{3r}} |Dv|^{p_0}\, dz + \lambda\right) \\ &\quad + \kappa_4 c(\kappa_3)\fint_{K^\lambda_{3r}} |Dh - Dv|^{p_0}\, dz + c(\kappa_3)c(\kappa_4)\delta^{\frac{\sigma_0}{4+\sigma_0}}\lambda \\ &\quad + \kappa_5 c(\kappa_3)\fint_{K^\lambda_{3r}} |Dh - Dv|^{p_0}\, dz + c(\kappa_3)c(\kappa_5)\delta^{\gamma_2'}\lambda. \end{aligned} \tag{5.3.26}$$

Consequently, we get the first inequality in (5.3.22) by choosing $\kappa_3, \kappa_4, \kappa_5$ sufficiently small in (5.3.26), and then get the second inequality (5.3.22), by choosing

$$\kappa_3 = \frac{\epsilon}{9(c_b + 1)}, \quad \kappa_4 = \kappa_5 = \frac{1}{3c(\kappa_3)}$$

and

$$\delta = \min\left\{\left(\frac{\epsilon}{9c(\kappa_3)c(\kappa_4)}\right)^{-\frac{\sigma_0}{4+\sigma_0}}, \left(\frac{\epsilon}{9c(\kappa_3)c(\kappa_5)}\right)^{-\gamma_2'}\right\}$$

in (5.3.26). □

For the interior case, we know from (1) in Lemma 5.2.7 that $|Dv|$ is bounded in $K_r^\lambda(w) = Q_r^\lambda(w)$. However, for the boundary region, we can not ensure that $|Dv|$ is bounded in $K_r^\lambda(w)$, since the the boundary of Ω might be extremely rough. Hence we need to find a function whose gradient is bounded and sufficiently close to Dv in L^{p_0}-sense for the boundary case. The next lemma ensures existence of the desired one $\overline{v}$.

Lemma 5.3.3. *Suppose that K_{4r}^λ satisfies (5.3.1). For any $\epsilon \in (0,1)$, there exists $\delta = \delta(n, \Lambda, \nu, \gamma_1, \gamma_2, c_a, \epsilon) > 0$ and a weak solution $\overline{v}$ of*

$$\begin{cases} \overline{v}_t - div\, \overline{\mathbf{b}}(D\overline{v}, t) &= 0 \quad in \quad Q_{3r}^{\lambda+}, \\ \overline{v} &= 0 \quad on \quad T_{3r}^\lambda \end{cases}$$

with

$$\fint_{Q_{3r}^{\lambda+}} |D\overline{v}|^{p_0} \le c_2\lambda,$$

where c_2 is given in Lemma 5.3.2, such that

$$\fint_{K_{2r}^\lambda} |Dv - D\overline{v}|^{p_0} \le \epsilon\lambda.$$

Here we extend $\overline{v}$ from $Q_{3r}^{\lambda+}$ to K_{3r}^λ by zero.

Proof. We first Define

$$\tilde{\Omega} := \{x \in \mathbb{R}^n : rx \in \Omega\},$$

$$\tilde{K}_{\tilde{r}} := (\tilde{\Omega} \cap B_{\tilde{r}}) \times (-\tilde{r}^2, \tilde{r}^2) \quad \text{and} \quad \partial_{\mathrm{w}} \tilde{K}_{\tilde{r}} := (\partial\tilde{\Omega} \cap B_{\tilde{r}}) \times (-\tilde{r}^2, \tilde{r}^2),$$

for $\tilde{r} > 0$, and

$$v_{\lambda,r}(x,t) := (\lambda^{\frac{1}{p_0}} r)^{-1} u(rx, \lambda^{\frac{2-p_0}{p_0}} r^2 t) \text{ and } \overline{\mathbf{b}}_{\lambda,r}(t,\xi) := \lambda^{\frac{1-p_0}{p_0}} \mathbf{a}(\lambda^{\frac{2-p_0}{p_0}} r^2 t, \lambda^{\frac{1}{p_0}} \xi),$$

for $(x,t) \in \tilde{K}_3$. Then we see from (5.3.20) and (5.3.22) that $v_{\lambda,r}$ is a weak solution of

$$\begin{cases} (v_{\lambda,r})_t - \operatorname{div} \overline{\mathbf{b}}_{\lambda,r}(Dv_{\lambda,r},t) &= 0 \quad \text{in} \quad \tilde{K}_3, \\ v_{\lambda,r} &= 0 \quad \text{on} \quad \partial_{\mathrm{w}}\tilde{K}_3. \end{cases}$$

and

$$\fint_{\tilde{K}_3} |Dv_{\lambda,r}|^{p_0}\, dz \le c_2.$$

Moreover we have from (5.3.1) that

$$B_3^+ \ \subset\ \tilde{\Omega}_3 \ \subset\ B_3 \cap \{x_n > -10\delta\}$$

In the same argument as in [8, Lemma 3.8], there exist $\delta = \delta(n,\Lambda,\nu,\gamma_1,\gamma_2,c_a,\epsilon) > 0$ and a weak solution $\overline{v}_{\lambda,r}$ to

$$\begin{cases} (\overline{v}_{\lambda,r})_t - \operatorname{div} \overline{\mathbf{b}}_{\lambda,r}(D\overline{v}_{\lambda,r},t) &= 0 \quad \text{in} \quad Q_3^+, \\ \overline{v} &= 0 \quad \text{on} \quad Q_3 \cap \{x_n = 0\}, \end{cases}$$

such that

$$\fint_{Q_3^+} |D\overline{v}_{\lambda,r}|^{p_0}\, dz \le c_2,$$

and

$$\fint_{\tilde{K}_2} |Dv_{\lambda,r} - D\overline{v}_{\lambda,r}|^{p_0} \le \epsilon.$$

Then $\overline{v}(x,t) := \lambda^{\frac{1}{p_0}} r \overline{v}_{\lambda,r}(r^{-1}x, \lambda^{\frac{p_0-2}{p_0}} r^{-2}t)$, $(x,t) \in K_{3r}^{\lambda}(w)$, becomes the desired function. □

5.4 Gradient Estimate in the Variable Exponent Lebesgue Spaces.

We devote this section to the proof of Theorem 5.1.3. Hence, let $p(\cdot)$ satisfy (5.0.2), $q(\cdot)$ satisfy (5.0.6), (5.1.3) and (5.1.4), that is, $q(\cdot)$ is log-Hölder continuous, $(p(\cdot), \mathbf{a}(\xi,\cdot,t), \Omega)$ be (δ,R)-vanishing for some $R \in (0,1)$, $|F|^{p(\cdot)} \in L^{q(\cdot)}(\Omega_T)$, and u is a weak solution of (5.0.1). Note that $\delta \in (0,1/8)$ will be selected as a positive small constant depending only on $n,\Lambda,\nu,\gamma_1,\gamma_2,\gamma_3,\gamma_4,L_1$, see Remark 5.4.8. We recall the specific constants $c_h, \sigma_\gamma, \tilde{\sigma}_\gamma, \alpha, M, M_1$ defined in Lemma 5.2.5, Corollary 5.2.6, (5.1.10), (5.1.11) and (5.3.13), and define

$$\overline{\sigma} := \min\left\{\frac{\gamma_3 - 1}{4}, 1\right\}. \tag{5.4.1}$$

Choose $R_0 > 0$ such that

$$\begin{cases} 4R_0 \le \min\left\{ \begin{array}{c} c_h^{-1}\left(c_a^{\frac{1}{\gamma_1(\gamma_1-1)}} M_1\right)^{-\frac{1}{\alpha}},\ e^{-1}\Gamma^{-\left(\frac{n+3}{\alpha}+1\right)}, \\ (\Gamma^{-1}R)^{\frac{1}{\alpha}},\ \Gamma^{-\frac{2}{\alpha}},\ \frac{1}{M},\ R \end{array}\right\}, \\ \omega(8R_0) \le \min\left\{\frac{\tilde{\sigma}_2}{4(\gamma_1-1)},\ \frac{\nu}{2\Lambda},\ \frac{\gamma_1\tilde{\sigma}_2}{2},\ 1\right\}, \\ \rho(8R_0) \le \min\left\{\gamma_3\sigma_{\gamma_3},\ \gamma_3\overline{\sigma},\ \frac{\gamma_3\tilde{\sigma}_2}{4},\ \frac{\gamma_3\tilde{\sigma}_{1+\overline{\sigma}}}{2},\ 1\right\}, \end{cases} \tag{5.4.2}$$

where the constants Γ and c_a will be denoted in Remark 5.4.2. We fix any $r \le R_0$ and any $w_0 = (y_0, \tau_0) \in \Omega_T$ with $(\tau_0 - (4r)^2, \tau_0 + (4r)^2) \subset (0, T)$, and consider a local region $K_r(w_0) = Q_r(w_0) \cap \Omega_T$. For the sake of convenience, we assume $w_0 = 0$ and write

$$d_0 := d(w_0),\quad d^+ := \sup_{z\in Q_{4r}} d(p(z)),\quad p^- := \inf_{z\in Q_{4r}} p(z),\quad p^+ := \sup_{z\in Q_{4r}} p(z),$$

$$q^- := \inf_{z\in Q_{4r}} q(z) \quad\text{and}\quad q^+ := \sup_{z\in Q_{4r}} q(z).$$

Then we have

$$p^+ - p^- \le \omega(8r) \le \omega(8R_0) \quad\text{and}\quad q^+ - q^- \le \rho(8r) \le \rho(8R_0).$$

From now on, the constant c is denoted by any constant which depends only on $n, \Lambda, \nu, \gamma_1, \gamma_2, \gamma_3, \gamma_4, L_1$.

Since $4r \le 4R_0$ satisfies (5.2.8) by the first condition on (5.4.2), we observe from Lemma 5.2.5 that

$$\begin{aligned} \fint_{K_{2r}} |Du|^{p(z)(1+\sigma)}\,dz \ \le\ & c\left\{ \left(\fint_{K_{4r}} \left[|Du|^{p(z)} + |F|^{p(z)}\right] dz\right)^{1+\sigma d_0} \right. \\ & \left. + \fint_{K_{4r}} |F|^{p(z)(1+\sigma)}\,dz + 1\right\}, \end{aligned} \tag{5.4.3}$$

for any $0 < \sigma \leq \sigma_{\gamma_3}$. Recalling the third condition on (5.4.2), we have $q^+ - q^- \leq \rho(8r) \leq \rho(8R_0) \leq \min\{\gamma_3\sigma_{\gamma_3}, \gamma_3\overline{\sigma}\}$, which together with (5.4.1) implies that

$$\frac{p(z)q(z)}{q^-} \leq p(z)\left(1 + \frac{\rho(8r)}{\gamma_3}\right) \leq \min\{p(z)(1+\sigma_{\gamma_3}), p(z)(1+\overline{\sigma})\} \quad (5.4.4)$$

and

$$\frac{p(z)q(z)(1+\overline{\sigma})}{q^-} \leq p(z)(1+3\overline{\sigma}) < p(z)\gamma_3, \quad (5.4.5)$$

for any $z \in K_{2r}$. Consequenty, from (5.4.3), (5.4.4) and (5.4.5), we have

$$|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}, |F|^{\frac{p(\cdot)q(\cdot)(1+\overline{\sigma})}{q^-}} \in L^1(K_{2r}).$$

The next lemma will play a crucial role in estimates on intrinsic parabolic cylinders.

Lemma 5.4.1. *Let $c_b > 1$ and recall $\alpha \in (0,1]$ defined in* (5.1.10). *Then there exists $c_3 = c_3(n, \Lambda, \nu, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \geq 1$ such that, for any $\lambda > 1$, any $w = (y,\tau) \in K_{2r}$ and any*

$$\tilde{r} \leq \min\left\{R, (4c_3c_bM\delta^{-1})^{-\frac{2}{\alpha}}\right\} \quad (5.4.6)$$

satisfying $K^\lambda_{\tilde{r}}(w) \subset K_{2r}$, if

$$\lambda \leq c_b\left\{\fint_{K^\lambda_{\tilde{r}}(w)} |Du|^{\frac{p(z)q(z)}{q^-}}dz + \frac{1}{\delta}\left(\fint_{K^\lambda_{\tilde{r}}(w)} |F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}}dz\right)^{\frac{1}{1+\overline{\sigma}}}\right\}, \quad (5.4.7)$$

then we have

$$\lambda^{\frac{2}{p_0}} \leq \Gamma\tilde{r}^{-(n+2)}, \quad p_2 - p_1 \leq \omega(\Gamma\tilde{r}^\alpha), \quad \lambda^{p_2-p_1} \leq e^{\frac{3n\gamma_2}{\alpha}}, \quad (5.4.8)$$

and

$$q_2 - q_1 \leq \omega(\Gamma\tilde{r}^\alpha), \quad \lambda^{q_2-q_1} \leq e^{\frac{3n\gamma_4 L_1}{\alpha}}, \quad (5.4.9)$$

where

$$\Gamma := 4c_3c_bM\delta^{-1}, \quad p_1 := \inf_{x\in K^\lambda_{\tilde{r}}(w)} p(z), \quad p_2 := \sup_{x\in K^\lambda_{\tilde{r}}(w)} p(z),$$

$$q_1 := \inf_{x\in K^\lambda_{\tilde{r}}(w)} q(z) \quad \textit{and} \quad q_2 := \sup_{x\in K^\lambda_{\tilde{r}}(w)} q(z).$$

Proof. Fix $K^{\lambda}_{\tilde r}(w) \subset K_{2r}$ and write $p_0 := p(w)$. By (5.4.2) we know that

$$M \le \frac{1}{4R_0} \le \frac{1}{4r} \quad \text{and} \quad 4r \le 4R_0 \le 1.$$

Then (5.1.4), (5.4.3) and (5.4.5) imply

$$\begin{aligned}
&\fint_{K_{2r}} |Du|^{\frac{p(z)q(z)}{q^-}} dz + \frac{1}{\delta}\left(\fint_{K_{2r}} |F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}} dz\right)^{\frac{1}{1+\overline{\sigma}}} \\
&\quad \le \frac{1}{\delta}\fint_{K_{2r}} \left[|Du|^{p(z)\left(1+\frac{\rho(4r)}{\gamma_3}\right)} + |F|^{p(z)q(z)} + 1\right] dz \\
&\quad \le \frac{c}{\delta}\left\{\left(\fint_{K_{4r}} \left[|Du|^{p(z)} + |F|^{p(z)}\right] dz\right)^{1+\frac{\rho(4r)}{\gamma_3}d_0} + \fint_{K_{4r}} |F|^{p(z)q(z)} dz + 1\right\} \\
&\quad \le \frac{cM}{\delta|K_{4r}|}\left\{\left(\frac{M}{|K_{4r}|}\right)^{\frac{\rho(4r)}{\gamma_3}d_0} + 1\right\} \\
&\quad \le \frac{cM}{\delta|K_{4r}|}\left\{\left(\frac{1}{4r}\right)^{\rho(4r)\frac{(n+3)d_M}{\gamma_3}} + 1\right\} \le \frac{cM}{\delta|K_{4r}|},
\end{aligned}$$

where d_M is defined in (5.1.9). From (5.1.6), (5.4.7) and the previous estimate, we see

$$\begin{aligned}
\lambda^{\frac{2}{p_0}} &= \lambda^{\frac{2-p_0}{p_0}+1} \\
&\le \lambda^{\frac{2-p_0}{p_0}} c_b\left\{\fint_{K^{\lambda}_{\tilde r}(w)} |Du|^{\frac{p(z)q(z)}{q^-}} dz + \frac{1}{\delta}\left(\fint_{K^{\lambda}_{\tilde r}(w)} |F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}} dz\right)^{\frac{1}{1+\overline{\sigma}}}\right\} \\
&\le \frac{cc_b M\lambda^{\frac{2-p_0}{p_0}}}{\delta|K^{\lambda}_{\tilde r}(w)|} \le \frac{cc_b M}{\delta \tilde r^{n+2}},
\end{aligned} \tag{5.4.10}$$

which gives the first estimate in (5.4.8). Recalling the definitions of p_1 and p_2, and (5.1.3) we have

$$p_2 - p_1 \le \omega(2\tilde r + \sqrt{2}\lambda^{\frac{2-p_0}{2p_0}}\tilde r) \le \omega(2\tilde r + \sqrt{2}\lambda^{\frac{2-\gamma_1}{2p_0}}\tilde r).$$

If $\gamma_1 \ge 2$, it is clear from (5.4.6), inparticular $\tilde r \le R < 1$, that

$$p_2 - p_1 \le \omega(4\tilde r) \le \omega(\Gamma \tilde r^{\alpha}).$$

If $\frac{2n}{n+2} < \gamma_1 < 2$, (5.4.10) yields that

$$p_2 - p_1 \le \omega(4\lambda^{\frac{2-\gamma_1}{2p_0}}\tilde{r}) \le \omega(4(c_3 c_b M\delta^{-1})^{\frac{2-\gamma_1}{4}}\tilde{r}^{\gamma_1\frac{n+2}{4}-\frac{n}{2}}) \le \omega(\Gamma\tilde{r}^{\alpha}).$$

Hence we obtain the second estimate in (5.4.8). We next derive the last estimate in (5.4.8). We see from (5.4.6) that $\Gamma\tilde{r}^{\alpha} \le R\Gamma^{-1} \le R$. Then (5.1.5) implies

$$\Gamma^{p_2-p_1} \le \Gamma^{\omega(\Gamma\tilde{r}^{\alpha})} \le (R\Gamma^{-1})^{-\omega(R\Gamma^{-1})} \le e$$

and so

$$\tilde{r}^{-(p_2-p_1)} \le \tilde{r}^{-\omega(\Gamma\tilde{r}^{\alpha})} \le \Gamma^{\frac{\omega(\Gamma\tilde{r}^{\alpha})}{\alpha}}(\Gamma\tilde{r}^{\alpha})^{-\frac{\omega(\Gamma\tilde{r}^{\alpha})}{\alpha}} \le e^{\frac{2}{\alpha}},$$

which is the third estimate in (5.4.8). Combining the previous two estimates, we obtain the last estimate in (5.4.8) such that

$$\lambda^{p_2-p_1} \le \left(\Gamma\tilde{r}^{-(n+2)}\right)^{\frac{p_0}{2}(p_2-p_1)} \le e^{\frac{p_0}{2}\left(1+\frac{2(n+2)}{\alpha}\right)} \le e^{\frac{3n\gamma_2}{\alpha}}.$$

The estimates in (5.4.9) can be obtained in a similar way, by using (5.1.4) instead of (5.1.5). □

Remark 5.4.2. *Actually, from* (5.4.18) *and* (5.4.33)*, we will take* $c_b = 2(48)^{n+2}$ *in the previous lemma, hence define* Γ *is determined by*

$$\Gamma := 8(48)^{n+2}c_3 M\delta^{-1}.$$

On the other hand, from the third inequality in (5.4.8)*, we define* $c_a > 1$ *by*

$$c_a := \frac{3n\gamma_2}{\alpha}.$$

5.4.1 choice of intrinsic cylinders.

Let us first define λ_0 by

$$\lambda_0^{\frac{1}{d^+}} := \fint_{K_{2r}} |Du|^{\frac{p(z)q(z)}{q^-}}dz + \frac{1}{\delta}\left\{\left(\fint_{K_{2r}} |F|^{\frac{p(z)q(z)(1+\bar{\sigma})}{q^-}}dz\right)^{\frac{1}{1+\bar{\sigma}}} + 1\right\}, \quad (5.4.11)$$

and, for $\lambda > 1$ and $1 \leq s \leq 2$, the upper-level set of $|Du(\cdot)|^{\frac{p(\cdot)q(\cdot)}{q^-}}$ by

$$E(s,\lambda) := \left\{ z \in K_{sr} : |Du(z)|^{\frac{p(z)q(z)}{q^-}} > \lambda \right\}, \tag{5.4.12}$$

We fix any $1 \leq s_1 < s_2 \leq 2$, and consider $\lambda > 1$ satisfying

$$\lambda > A(s_1,s_2)\lambda_0, \quad \text{where } A(s_1,s_2) := \left\{ \left(\frac{16}{7}\right)^n \left(\frac{120\chi}{s_2 - s_1}\right)^{n+2} \right\}^{d^+}, \tag{5.4.13}$$

where $\chi \geq 5$ is defined in Lemma 5.2.8. Using Lemma 5.2.8 we obtain the following covering lemma.

Lemma 5.4.3. *Let λ satisfy* (5.4.13). *There exist $\{w_i\}_{i=1}^{\infty} = \{(y_i,\tau_i)\}_{i=1}^{\infty} \subset E(s_1,\lambda)$ and*

$$r_i \in \left(0, \min\{1, \lambda^{\frac{p_i-2}{2p_i}}\}\frac{(s_2-s_1)r}{60\chi}\right], \quad i = 1,2,\ldots,$$

where $p_i := p(w_i)$, such that $\{Q^{\lambda}_{r_i}(w_i)\}_{i=1}^{\infty}$ is mutually disjoint,

$$E(s_1,\lambda)\backslash N \subset \bigcup_{i=1}^{\infty} K^{\lambda}_{\chi r_i}(w_i) \subset K_{s_2 r}$$

for some Lebesgue measure zero set N, and for each i we have

$$\fint_{K^{\lambda}_{r_i}(w_i)} |Du|^{\frac{p(z)q(z)}{q^-}} dz + \frac{1}{\delta}\left(\fint_{K^{\lambda}_{r_i}(w_i)} |F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}} dz\right)^{\frac{1}{1+\overline{\sigma}}} = \lambda \tag{5.4.14}$$

and

$$\fint_{K^{\lambda}_{\tilde{r}}(w_i)} |Du|^{\frac{p(z)q(z)}{q^-}} dz + \frac{1}{\delta}\left(\fint_{K^{\lambda}_{\tilde{r}}(w_i)} |F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}} dz\right)^{\frac{1}{1+\overline{\sigma}}} < \lambda \tag{5.4.15}$$

for any $\tilde{r} \in \left(r_i, \min\{1, \lambda^{\frac{p_i-2}{2p_i}}\}(s_2-s_1)r\right]$.

Proof. For each $w \in E(s_1, \lambda)$, we denote the continuous function G_w on the interval $\left(0, \min\{1, \lambda^{\frac{p_0-2}{2p_0}}\}(s_2 - s_1)r\right]$ by

$$G_w(\tilde{r}) := \fint_{K_{\tilde{r}}^{\lambda}(w)} |Du|^{\frac{p(z)q(z)}{q^-}} dz + \frac{1}{\delta} \left(\fint_{K_{\tilde{r}}^{\lambda}(w)} |F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}} dz \right)^{\frac{1}{1+\overline{\sigma}}},$$

and write $p_0 := p(0)$ and $d_w := d(w)$. Then, for any

$$\tilde{r} \in \left(\min\{1, \lambda^{\frac{p_0-2}{2p_0}}\} \frac{(s_2 - s_1)}{60\chi} r \, , \, \min\{1, \lambda^{\frac{p_0-2}{2p_0}}\}(s_2 - s_1)r \right],$$

we have from (5.1.6), (5.4.11) and (5.4.13) that

$$\begin{aligned}
G_w(\tilde{r}) &\leq \frac{|Q_{2r}|}{|K_{\tilde{r}}^{\lambda}(w)|} \left\{ \fint_{K_{2r}} |Du|^{\frac{p(z)q(z)}{q^-}} dz + \frac{1}{\delta} \left(\fint_{K_{2r}} |F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}} dz \right)^{\frac{1}{1+\overline{\sigma}}} \right\} \\
&\leq \frac{|Q_{2r}|}{\lambda^{\frac{2-p_0}{p_0}} |K_{\tilde{r}}(w)|} \lambda_0^{\frac{1}{d^+}} \\
&\leq \left(\frac{16}{7} \right)^n \left(\frac{2r}{\tilde{r}} \right)^{n+2} \lambda^{\frac{p_0-2}{p_0}} \lambda_0^{\frac{1}{d^+}} \\
&\leq \left(\frac{16}{7} \right)^n \left(\frac{120\chi}{\min\{1, \lambda^{\frac{p_0-2}{2p_0}}\}(s_2 - s_1)} \right)^{n+2} \lambda^{\frac{p_0-2}{p_0}} \lambda_0^{\frac{1}{d^+}} \\
&< \frac{\lambda^{\frac{p_0-2}{2p_0}} d^{\frac{1}{d_+}}}{\min\{1, \lambda^{\frac{p_0-2}{2p_0}(n+2)}\}}.
\end{aligned}$$

If $p_0 \geq 2$, then $\min\{1, \lambda^{\frac{p_0-2}{2p_0}(n+2)}\} = 1$ and $\frac{1}{d^+} \leq \frac{1}{d_w} = \frac{2}{p_0}$, which implies $G_w(\tilde{r}) < \lambda$. If $\frac{2n}{n+2} < p_0 < 2$, then $\min\{1, \lambda^{\frac{p_0-2}{2p_0}}\} = \lambda^{\frac{p_0-2}{2p_0}}$ and $\frac{1}{d^+} \leq \frac{1}{d_w} = \frac{p_0(n+2)-2n}{2p_0}$, which also implies $G_w(\tilde{r}) < \lambda^{\frac{2n-p_0 n}{2p_0}} \lambda^{\frac{p_0(n+2)-2n}{2p_0}} = \lambda$. On the other hands, in view of the Lebesgue differentiation theorem, we have

$$\lim_{\tilde{r} \to 0} G_w(\tilde{r}) \geq |Du(w)|^{\frac{p_0 q(w)}{q^-}} > \lambda$$

for every Lebesgue's point w of $|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}$ in $E(s_1, \lambda)$.

Consequently, as G_w is continuous function, for every Lebesgue's point w of $|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}$ in $E(s_1,\lambda)$ we can find $r_w \in \left(0, \min\{1, \lambda^{\frac{p_0-2}{2p_0}}\}\frac{(s_2-s_1)}{4\chi}r\right]$ such that

$$G_w(r_w) = \lambda \text{ and } G_w(\tilde{r}) < \lambda, \quad {}^\forall \tilde{r} \in \left(r_w, \min\{1, \lambda^{\frac{p_0-2}{2p_0}}\}(s_2-s_1)r\right].$$

Since $\tilde{r} \le r$, we konw by (5.4.2) that $\tilde{r}$ satisfies the condition (5.4.7) in Lemma 5.4.1 for $c_b = 1$, hence applying Lemma 5.4.1 with $\tilde{r} = r_w$ and $c_b = 1$ we have $\lambda^{p_2^w - p_1^w} \le e^{\frac{3n\gamma_2}{\alpha}}$, where $p_2^w := \sup_{z\in Q^\lambda_{r_w}(w)} p(z)$ and $p_2^w := \inf_{z\in Q^\lambda_{r_w}(w)} p(z)$. Finally, applying Lemma 5.2.8 to $\{Q^\lambda_{r_w}(w)\}$ with $c_a = e^{\frac{3n\gamma_2}{\alpha}}$, we obtain the conclusion of the lemma. □

5.4.2 Comparison estimates.

For λ satisfying (5.4.13), we consider $Q^\lambda_{r_i}(w_i)$, $i = 1, 2, \ldots$, selected in Lemma 5.4.3. From the choice of r_i we have

$$60\chi r_i < \min\left\{1, \lambda^{\frac{p(w)-2}{2p(w)}}\right\}(s_2-s_1)r \le (s_2-s_1)r \le r. \tag{5.4.16}$$

For each i we note that there are the two possible cases that **Case 1**: $Q^\lambda_{4\chi r_i}(w_i) \subset \Omega_T$, and **Case 2**: $Q^\lambda_{4\chi r_i}(w_i) \not\subset \Omega_T$.

We first consider **Case 1**. For the sake of simplicity, we write

$$Q^\lambda_{i,j} := K^\lambda_{j\chi r_i}(w_i) = Q^\lambda_{j\chi r_i}(w_i), \quad j = 1, 2, 3, 4,$$

and

$$p_i := p(w_i), \quad p_i^- = \inf_{z\in Q^\lambda_{i,4}} p(z), \quad p_i^+ = \sup_{z\in Q^\lambda_{i,4}} p(z),$$

$$q_i^- = \inf_{z\in Q^\lambda_{i,4}} q(z), \quad q_i^+ = \sup_{z\in Q^\lambda_{i,4}} q(z).$$

Then from (5.4.16) we have $Q^\lambda_{i,4} \subset Q_{s_2 r} \subset Q_{2r} \subset Q_{4r}$ so that

$$p_i^+ - p_i^- \le p^+ - p^- \le \omega(8R_0) \quad \text{and} \quad q_i^+ - q_i^- \le q^+ - q^- \le \rho(8R_0). \tag{5.4.17}$$

Recalling (5.4.14) in Lemma 5.4.3, we have

$$\lambda < 4^{n+2}\left\{ \fint_{Q^{\lambda}_{i,4}} |Du|^{\frac{p(z)q(z)}{q^-}} dz + \frac{1}{\delta}\left(\fint_{Q^{\lambda}_{i,4}} |F|^{\frac{p(z)q(z)(1+\bar{\sigma})}{q^-}} dz \right)^{\frac{1}{1+\bar{\sigma}}} \right\}, \quad (5.4.18)$$

hence, in view of Lemma 5.4.1 with $\tilde{r} = 4\chi r_i$ and $w = w_i$, we obtain

$$\lambda^{\frac{2}{p_i}} \leq \Gamma \tilde{r}^{-(n+2)}, \quad p_i^+ - p_i^- \leq \omega(\Gamma(4\chi r_i)^{\alpha}), \quad \lambda^{p_i^+ - p_i^-} \leq c_a, \qquad (5.4.19)$$

and

$$q_i^+ - q_i^- \leq \omega(\Gamma(4\chi r_i)^{\alpha}), \quad \lambda^{q_i^+ - q_i^-} \leq e^{\frac{3n\gamma_4 L_1}{\alpha}}, \qquad (5.4.20)$$

where $\Gamma = 8(48)^{n+2} c_3 M \delta^{-1}$ and $c_a = e^{\frac{3n\gamma_2}{\alpha}}$. Now we claim that

$$\fint_{Q^{\lambda}_{i,4}} |Du|^{p(z)} dz \leq c_4 \lambda^{\frac{q^-}{q_i^+}} \text{ and } \fint_{Q^{\lambda}_{i,4}} |F|^{p(z)} dz \leq c_4 \delta^{\frac{\gamma_3}{\gamma_4}} \lambda^{\frac{q^-}{q_i^+}}, \qquad (5.4.21)$$

for some $c_4 = c_4(n, \Lambda, \nu, \gamma_1, \gamma_2, \gamma_3, \gamma_4, L_1) \geq 1$. Indeed, by (5.4.15) we have

$$\fint_{Q^{\lambda}_{i,4}} |Du|^{p(z)} dz \leq \lambda + 1 \leq 2\lambda \text{ and } \fint_{Q^{\lambda}_{i,4}} |F|^{p(z)} dz \leq \delta\lambda + 1 < 2\lambda.$$

This together with the third inequality in (5.4.20) we obtain

$$\left(\fint_{Q^{\lambda}_{i,4}} |Du|^{p(z)}\, dz \right)^{q_i^+ - q_i^-} + \left(\fint_{Q^{\lambda}_{i,4}} |F|^{p(z)}\, dz \right)^{q_i^+ - q_i^-} \leq c.$$

Therefore, the estimates in (5.4.21) follow from Hölder's inequality, (5.4.15) and the previous estimate such that

$$\begin{aligned} \fint_{Q^{\lambda}_{i,4}} |Du|^{p(z)}\, dz &\leq \left(\fint_{Q^{\lambda}_{i,4}} |Du|^{p(z)}\, dz \right)^{\frac{q_i^+ - q_i^-}{q_i^+}} \left(\fint_{Q^{\lambda}_{i,4}} |Du|^{p(z)}\, dz \right)^{\frac{q_i^-}{q_i^+}} \\ &\leq c \left(\fint_{Q^{\lambda}_{i,4}} |Du|^{\frac{p(z)q_i^-}{q^-}}\, dz \right)^{\frac{q^-}{q_i^-}} \\ &\leq c \left(\fint_{Q^{\lambda}_{i,4}} |Du|^{\frac{p(z)q(z)}{q^-}}\, dz + 1 \right)^{\frac{q^-}{q_i^+}} \leq c\lambda^{\frac{q^-}{q_i^+}}, \end{aligned}$$

and

$$\begin{aligned}
\fint_{Q^\lambda_{i,4}} |F|^{p(z)}\,dz &\le c\left(\fint_{Q^\lambda_{i,4}} |F|^{\frac{p(z)q(z)}{q^-}}\,dx + 1\right)^{\frac{q^-}{q_i^+}} \\
&\le c\left\{\left(\fint_{Q^\lambda_{i,4}} |F|^{\frac{p(z)q(z)(1+\bar\sigma)}{q^-}}\,dx\right)^{\frac{1}{1+\bar\sigma}} + 1\right\}^{\frac{q^-}{q_i^+}} \\
&\le c\delta^{\frac{\gamma_3}{\gamma_4}}\lambda^{\frac{q^-}{q_i^+}}.
\end{aligned}$$

In the last inequality, we have used the fact $\delta\lambda > 1$ obtained by (5.4.11).

We now see from (5.4.2), (5.4.16), (5.4.17), (5.4.19) and (5.4.21) that the assumptions in Section 5.3, (5.3.2), (5.3.3), (5.3.4), (5.3.14) and (5.3.21), are satisfied for the region $Q^\lambda_{i,4}$ with λ and δ replaced by $c_4\lambda^{\frac{q^-}{q_i^+}}$ and $\delta^{\frac{\gamma_3}{\gamma_4}}$. Therefore, by Lemma 5.3.1 and Lemma 5.3.2 and Lemma 5.2.7 (1), we have the following lemma.

Lemma 5.4.4. *For any $\epsilon \in (0,1)$, there exist $\delta = \delta(n,\Lambda,\nu,\gamma_1,\gamma_2,\gamma_3,\gamma_4,L_1,\epsilon) > 0$ and $v_i \in L^1(Q^\lambda_{i,1})$ with $Dv_i \in L^\infty(Q^\lambda_{i,1})$ for each i satisfying $Q^\lambda_{i,4} = Q^\lambda_{4\chi r_i}(w_i) \subset \Omega_T$ such that*

$$\fint_{Q^\lambda_{i,1}} |Du - Dh_i|^{p(z)}\,dz \le \epsilon\lambda^{\frac{q^-}{q_i^+}}, \quad \fint_{Q^\lambda_{i,1}} |Dh_i - Dv_i|^{p_i}\,dz \le \epsilon\lambda^{\frac{q^-}{q_i^+}}, \quad (5.4.22)$$

$$\fint_{Q^\lambda_{i,2}} |Dh_i|^{p(z)}\,dz \le c\lambda^{\frac{q^-}{q_i^+}} \quad \text{and} \quad \|Dv_i\|^{p_i}_{L^\infty(Q^\lambda_{i,1})} \le c\lambda^{\frac{q^-}{q_i^+}}, \qquad (5.4.23)$$

for some $c = c(n,\Lambda,\nu,\gamma_1,\gamma_2,\gamma_3,\gamma_4,L_1) \ge 1$, where h_i is the weak solution of

$$\begin{cases} (h_i)_t - \mathit{div}\,\mathbf{a}(Dh_i,z) = 0 & \text{in } Q^\lambda_{i,4}, \\ h_i = u & \text{on } \partial_p Q^\lambda_{i,4}. \end{cases}$$

Corollary 5.4.5. *Under the assumptions and conclusions as in Lemma 5.4.4, we have*

$$\fint_{Q^\lambda_{i,1}} |Du - Dv_i|^{\frac{p(z)q(z)}{q^-}}\,dz \le \epsilon\lambda \ \text{ and } \ |Dv_i(z)|^{\frac{p(z)q(z)}{q^-}} \le c_5\lambda \ \ \forall z \in Q^\lambda_{i,1}, \ (5.4.24)$$

for some $c_5 = c_5(n,\Lambda,\nu,\gamma_1,\gamma_2,\gamma_3,\gamma_4,L_1) \ge 1$.

Proof. The second inequality in (5.4.24) directly follows from (5.4.19) and the second estimate in (5.4.23). Indeed, for $z \in Q^{\lambda}_{i,1}$

$$\begin{aligned} |Dv_i(z)|^{\frac{p(z)q(z)}{q^-}} &\le |Dv_i(z)|^{\frac{p_i^+ q_i^+}{q^-}} + 1 \le c_2 \lambda^{\frac{p_i^+}{p_i}} + 1 \le c_2 \lambda^{1+\frac{p_i^+ - p_i^-}{\gamma_1}} + 1 \\ &\le \left(c_2 e^{\frac{3n\gamma_2}{\gamma_1 \alpha}} + 1 \right) \lambda. \end{aligned}$$

We then prove the first inequality in (5.4.24). We observe from (5.4.2) that

$$\frac{2q_i^+}{q^-} - 1 \le 1 + \frac{2\rho(8R_0)}{\gamma_3} \le \min\{1 + \tilde{\sigma}_{1+\overline{\sigma}}, 1 + \tilde{\sigma}_2\},$$

$$\frac{2q_i^+}{q^-} - \frac{p_i^-}{p_i^+} \le 1 + \frac{2\rho(8R_0)}{\gamma_3} + \frac{\omega(8R_0)}{\gamma_1} \le 1 + \tilde{\sigma}_2,$$

and from (5.4.15) that

$$\fint_{Q^{\lambda}_{i,2}} |Du|^{p(z)} dz + \left(\fint_{Q^{\lambda}_{i,2}} |F|^{p(z)(1+\overline{\sigma})}\, dz \right)^{\frac{1}{1+\overline{\sigma}}} < \lambda + 2 \le 3\lambda. \tag{5.4.25}$$

Hence applying 5.2.6 to the weak solutions u and h_i on $Q^{\lambda}_{i,1}$ with (5.4.25) and (5.4.23), respectively, we have

$$\fint_{Q^{\lambda}_{i,1}} \left[|Du|^{p(z)\left(\frac{2q_i^+}{q^-} - 1\right)} + |Dh_i|^{p(z)\left(\frac{2q_i^+}{q^-} - 1\right)} \right] dz \le c \left(\lambda^{\frac{2q_i^+}{q^-} - 1} + \lambda^{2 - \frac{q^-}{q_i^+}} \right) \tag{5.4.26}$$

and

$$\fint_{Q^{\lambda}_{i,1}} |Dh_i|^{p(z)\left(\frac{2q_i^+}{q^-} - \frac{p_i^-}{p_i^+}\right)} dz \le c \lambda^{2 - \frac{q^-}{q_i^+}\frac{p_i^-}{p_i^+}}. \tag{5.4.27}$$

We split the left hand side on the first inequality in (5.4.24) as follow:

$$\begin{aligned} &\fint_{Q^{\lambda}_{i,1}} |Du - Dv_i|^{\frac{p(z)q(z)}{q^-}} dz \\ &\qquad \le 2^{\frac{\gamma_2\gamma_4}{\gamma_3} - 1} \left(\fint_{Q^{\lambda}_{i,1}} |Du - Dh_i|^{\frac{p(z)q(z)}{q^-}} dz + \fint_{Q^{\lambda}_{i,1}} |Dh_i - Dv_i|^{\frac{p(z)q(z)}{q^-}} dz \right) \\ &\qquad =: 2^{\frac{\gamma_2\gamma_4}{\gamma_3} - 1} (I_4 + I_5). \end{aligned} \tag{5.4.28}$$

For I_4, Hölder's inequality implies

$$\begin{aligned} I_4 &= \fint_{Q_{i,1}^\lambda} |Du - Dh_i|^{\frac{p(z)}{2} + p(z)\left(\frac{q(z)}{q^-} - \frac{1}{2}\right)} dz \\ &\le \left(\fint_{Q_{i,1}^\lambda} |Du - Dh_i|^{p(z)} dz\right)^{\frac{1}{2}} \left(\fint_{Q_{i,1}^\lambda} |Du - Dh_i|^{p(z)\left(\frac{2q(z)}{q^-} - 1\right)} dz\right)^{\frac{1}{2}} \\ &=: (I_{4a})^{\frac{1}{2}} (I_{4b})^{\frac{1}{2}}. \end{aligned}$$

We estimate by the first estimates in (5.4.21), (5.4.22) and (5.4.23), and the second estimate in (5.4.20)

$$\begin{aligned} I_{4a} = \left(\fint_{Q_{i,1}^\lambda} |Du - Dh_i|^{p(z)} dz\right)^{1 - \frac{2q_i^+}{q^-}\frac{q_i^+ - q_i^-}{q_i^-}} & \left(\fint_{Q_{i,1}^\lambda} |Du - Dh_i|^{p(z)} dz\right)^{\frac{2q_i^+}{q^-}\frac{q_i^+ - q_i^-}{q_i^-}} \\ \le \left(\epsilon \lambda^{\frac{q^-}{q_i^+}}\right)^{1 - \frac{2q_i^+}{q^-}\frac{q_i^+ - q_i^-}{q_i^-}} & \left(2^{\gamma_2} \fint_{Q_{i,1}^\lambda} \left[|Du|^{p(z)} + |Dh_i|^{p(z)}\right] dz\right)^{\frac{2q_i^+}{q^-}\frac{q_i^+ - q_i^-}{q_i^-}} \\ \le c\epsilon^{\frac{1}{2}} \lambda^{\frac{q^-}{q_i^+} - \frac{2(q_i^+ - q_i^-)}{q_i^-}} & \lambda^{(q_i^+ - q_i^-)\frac{2}{\gamma_3}} \le c\epsilon^{\frac{1}{4}} \lambda^{\frac{q^-}{q_i^+} - \frac{2(q_i^+ - q_i^-)}{q_i^-}}, \end{aligned}$$

and by (5.4.26) that

$$\begin{aligned} I_{4b} &\le c\fint_{Q_{i,1}^\lambda} \left[|Du|^{p(z)\left(\frac{2q_i^+}{q^-} - 1\right)} + |Dh_i|^{p(z)\left(\frac{2q_i^+}{q^-} - 1\right)}\right] dz + 1 \\ &\le c\left(\lambda^{\frac{2q_i^+}{q^-} - 1} + \lambda^{2 - \frac{q^-}{q_i^+}}\right), \end{aligned}$$

hence we obtain

$$I_4 \le c\epsilon^{\frac{1}{4}} \lambda^{\frac{q^-}{2q_i^+} - \frac{q_i^+ - q_i^-}{q_i^-}} \left(\lambda^{\frac{q_i^+}{q^-} - \frac{1}{2}} + \lambda^{1 - \frac{q^-}{2q_i^+}}\right) \le c\epsilon^{\frac{1}{4}} \lambda. \tag{5.4.29}$$

We next estimate I_5. By Hölder's inequality, the second estimates in (5.4.19), (5.4.22) and (5.4.23), and (5.4.27)

$$
\begin{aligned}
I_5 &= \fint_{Q^\lambda_{i,1}} |Dh_i - Dv_i|^{\frac{p_i}{2}+\frac{p(z)q(z)}{q^-}-\frac{p_i}{2}}\,dz \\
&\le c\left(\fint_{Q^\lambda_{i,1}} |Dh_i - Dv_i|^{p_i}\,dz\right)^{\frac12}\left(\fint_{Q^\lambda_{i,1}} |Dh_i - Dv_i|^{p(z)\left(\frac{2q(z)}{q^-}-\frac{p_i}{p(z)}\right)}\,dz\right)^{\frac12} \\
&\le c\epsilon^{\frac12}\lambda^{\frac{q^-}{2q_i^+}} \\
&\quad\times\left(\fint_{Q^\lambda_{i,1}} |Dh_i|^{p(z)\left(\frac{2q_i^+}{q^-}-\frac{p_i^-}{p_i^+}\right)}\,dz + \fint_{Q^\lambda_{i,1}} |Dv_i|^{p_i\left(\frac{2q_i^+}{q^-}\frac{p_i^+}{p_i^-}-1\right)}\,dz + 1\right)^{\frac12} \\
&\le c\epsilon^{\frac12}\left(\lambda^{1-\frac{q^-}{2q_i^+}\frac{p_i^-}{p_i^+}} + \lambda^{\frac{p_i^+}{p_i^-}-\frac{q^-}{2q_i^+}} + 1\right) \\
&\le c\epsilon^{\frac12}\left(\lambda + \lambda^{\frac{p_i^+}{p_i^-}} + 1\right) \;\le\; c\epsilon^{\frac12}\lambda. \qquad (5.4.30)
\end{aligned}
$$

Inserting (5.4.29) and (5.4.30) into (5.4.28) we obtain

$$\fint_{Q^\lambda_{i,1}} |Du - Dv_i|^{\frac{p(z)q(z)}{q^-}}\,dz \le c\epsilon^{\frac14}\lambda.$$

Since $\epsilon \in (0,1)$ is arbitrary, we get the second inequality in (5.4.24) from the previous one. □

We next consider **Case 2** that $Q^\lambda_{4\chi r_i}(w_i) \not\subset \Omega_T$. Note that, since $dist(y_i, \partial\Omega) \le 4\chi r_i$, we can take $y_i' \in \partial\Omega$ such that $|y_i - y_i'| \le 4\chi r_i$. Since $56\chi r_i \le R$, in view of Definition 5.1.1 (3), there exists a spatial coordinate system, still denote $x = (x_1, \dots, x_n)$-coordinate, with the origin at y_i' such that

$$B_{56\chi r_i}(0) \cap \{x_n > 56\chi r_i\delta\} \;\subset\; \Omega_{56\chi r_i}(0) \;\subset\; B_{56\chi r_i}(0) \cap \{x_n > -56\chi\delta r_i\}.$$

Note from the fact $\delta \le 1/8$ that $B_{48\chi r_i}(56\chi\delta r_i e_n) \subset B_{56\chi r_i}(0)$, where $e_n = (0, \dots, 0, n)$. We then translate the spatial coordinate system to x_n-direction by $56\chi\delta r_i$, still denote $x = (x_1, \dots, x_n)$-coordinate, so that we have

$$B_r^+(0) \;\subset\; \Omega_r(0) \;\subset\; B_r(0) \cap \{x_n > -112\chi\delta r_i\}, \quad \text{for any } 0 < r < 48\chi r_i. \qquad (5.4.31)$$

We remark that, since the transformation is composed of only a transformation and a rotation, the basic structures of the problem (5.0.1) and the main assumption Definition 5.1.1 are invariant. Therefore, without loss of generality, we will continuously use the original symbols and notations in this new coordinate system.

After the above transformation of the spatial coordinate system, we set

$$w_i' := (0, t_i), \quad K_{i,j}^{\lambda} := K_{12j\chi r_i}^{\lambda}(w_i'), \quad j = 1, 2, 3, 4,$$

and

$$p_i := p(w_i), \quad p_i^- := \inf_{z \in K_{i,4}^{\lambda}} p(z), \quad p_i^+ := \sup_{z \in K_{i,4}^{\lambda}} p(z),$$

$$q_i^- := \inf_{z \in K_{i,4}^{\lambda}} q(z), \quad q_i^+ := \sup_{z \in K_{i,4}^{\lambda}} q(z).$$

Since $|y_i| \le |y_i - y_i'| + |y_i'| \le (4 + 56\delta)\chi r_i \le 11\chi r_i$, we have from (5.4.16) that

$$K_{\chi r_i}^{\lambda}(w_i) \subset K_{i,1}^{\lambda} \subset K_{i,4}^{\lambda} \subset K_{60\chi r_i}^{\lambda}(w_i) \subset K_{s_2 r_i}, \tag{5.4.32}$$

and so have the relations in (5.4.17).

We first observe from (5.4.14) in Lemma 5.4.3, (5.4.31) and (5.4.32) that

$$\lambda < 2(48)^{n+2} \left\{ \fint_{K_{i,4}^{\lambda}} |Du|^{\frac{p(z)q(z)}{q^-}} dz + \frac{1}{\delta} \left(\fint_{K_{i,4}^{\lambda}} |F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}} dz \right)^{\frac{1}{1+\overline{\sigma}}} \right\}. \tag{5.4.33}$$

Then, in a similar way to **Case 1**, using Lemma 5.1.10, (5.4.15) and (5.4.31), we obtain (5.4.19), (5.4.20) and the estimates

$$\fint_{K_{i,4}^{\lambda}} |Du|^{p(z)} dz \le c\lambda^{\frac{q_i^-}{q_i^+}} \quad \text{and} \quad \fint_{K_{i,4}^{\lambda}} |F|^{p(z)} dz \le c\delta^{\frac{\gamma_3}{\gamma_4}} \lambda^{\frac{q_i^-}{q_i^+}}.$$

Therefore, applying Lemma 5.3.1, Lemma 5.3.2, Lemma 5.3.3 and Lemma 5.2.7 (2), we have the following lemma and corollary.

Lemma 5.4.6. *For any $\epsilon \in (0,1)$, there exist $\delta = \delta(n, \Lambda, \nu, \gamma_1, \gamma_2, \gamma_3, \gamma_4, L_1, \epsilon) > 0$ and $\overline{v}_i \in L^1(K_{i,1}^{\lambda})$ with $D\overline{v}_i \in L^{\infty}(K_{i,1}^{\lambda})$ for each i satisfying $Q_{4\chi r_i}^{\lambda}(w_i) \not\subset \Omega_T$ such that*

$$\fint_{K_{i,1}^{\lambda}} |Du - Dh_i|^{p(z)}\, dz \le \epsilon\lambda^{\frac{q_i^-}{q_i^+}}, \quad \fint_{K_{i,1}^{\lambda}} |Dh_i - Dv_i|^{p_i}\, dz \le \epsilon\lambda^{\frac{q_i^-}{q_i^+}},$$

$$\fint_{K_{i,2}^{\lambda}} |Dh_i|^{p(z)}\, dz \leq c\lambda^{\frac{q^-}{q_i^+}} \quad and \quad \|Dv_i\|_{L^\infty(K_{i,1}^{\lambda})}^{p_i} \leq c\lambda^{\frac{q^-}{q_i^+}},$$

for some $c = c(n, \Lambda, \nu, \gamma_1, \gamma_2, \gamma_3, \gamma_4, L_1) \geq 1$, *where* h_i *is the weak solution of*

$$\begin{cases} (h_i)_t - div\, \mathbf{a}(Dh_i, z) &= 0 \quad in \quad K_{i,4}^{\lambda}, \\ h_i &= u \quad on \quad \partial_p K_{i,4}^{\lambda}. \end{cases}$$

Corollary 5.4.7. *Under the assumptions and conclusion of Lemma 5.4.6, we have*

$$\fint_{K_{i,1}^{\lambda}} |Du - Dv_i|^{\frac{p(z)q(z)}{q^-}}\, dz \leq \epsilon\lambda \quad and \quad |Dv_i(z)|^{\frac{p(z)q(z)}{q^-}} \leq c_6\lambda \quad {}^{\forall} z \in K_{i,1}, \tag{5.4.34}$$

for some $c_6 = c_6(n, \Lambda, \nu, \gamma_1, \gamma_2, \gamma_3, \gamma_4, L_1) \geq 1$.

5.4.3 Gradient estimates on upper-level sets.

Now, for each $1 \leq s_1 < s_2 \leq 2$ and each $\lambda > 1$ satisfying (5.4.13) we estimate the integration of $|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}$ on the upper-level sets $E(s_1, B\lambda)$ and

$$B := 2^{\frac{\gamma_2\gamma_4}{\gamma_3}} \max\{c_5, c_6\}. \tag{5.4.35}$$

We recall Lemma (5.4.3). Then we have $E(s_1, B\lambda) \subset E(s_1, \lambda) \subset \bigcup_{i=1}^{\infty} K_{\chi r_i}^{\lambda}(w_i) \setminus N$ and

$$\int_{E(s_1,B\lambda)} |Du|^{\frac{p(z)q(z)}{q^-}}\, dz \leq \sum_{i=1}^{\infty} \int_{E(s_1,B\lambda)\cap K_{\chi r_i}^{\lambda}(w_i)} |Du|^{\frac{p(z)q(z)}{q^-}}\, dz.$$

If $K_{4\chi r_i}^{\lambda}(w_i) \not\subset \Omega_T$(**Case 2**), we have from the second estimate in (5.4.34) and (5.4.35) that

$$\begin{aligned} |Du(z)|^{\frac{p(z)q(z)}{q^-}} &\leq 2^{\frac{\gamma_2\gamma_4}{\gamma_3}-1}\left(|Du(z) - Dv_i(z)|^{\frac{p(z)q(z)}{q^-}} + |Dv_i|^{\frac{p(z)q(z)}{q^-}}\right) \\ &\leq 2^{\frac{\gamma_2\gamma_4}{\gamma_3}-1}\left(|Du(z) - Dv_i(z)|^{\frac{p(z)q(z)}{q^-}} + c_6\lambda\right) \\ &\leq 2^{\frac{\gamma_2\gamma_4}{\gamma_3}-1}|Du(z) - Dv_i(z)|^{\frac{p(z)q(z)}{q^-}} + \frac{1}{2}|Du(z)|^{\frac{p(z)q(z)}{q^-}} \end{aligned}$$

for any $z \in E(s_1, B\lambda) \cap K^{\lambda}_{1,i}$, hence by (5.1.6), (5.4.32) and the first estimate in (5.4.34) we get

$$\int_{E(s_1,B\lambda)\cap K^{\lambda}_{\chi r_i}(w_i)} |Du|^{\frac{p(z)q(z)}{q^-}} \leq 2^{\frac{\gamma_2\gamma_4}{\gamma_3}} \int_{E(s_1,B\lambda)\cap K^{\lambda}_{i,1}} |Du - Dv_i|^{\frac{p(z)q(z)}{q^-}} \, dz$$
$$\leq 2^{\frac{\gamma_2\gamma_4}{\gamma_3}} \epsilon\lambda |K^{\lambda}_{i,1}| \leq 2^{\frac{\gamma_2\gamma_4}{\gamma_3}} \left(\frac{16}{7}\right)^n (10\chi)^{n+2} \epsilon\lambda |K^{\lambda}_{r_i}(w_i)|.$$

Similarly, if $K^{\lambda}_{4\chi r_i}(w_i) \subset \Omega_T$(**Case 1**), by using (5.4.24) and (5.4.35) we get

$$\int_{E(s_1,B\lambda)\cap K^{\lambda}_{\chi r_i}(w_i)} |Du|^{\frac{p(z)q(z)}{q^-}} \leq 2^{\frac{\gamma_2\gamma_4}{\gamma_3}} \chi^{n+2} \epsilon\lambda |K^{\lambda}_{r_i}(w_i)|.$$

Therefore, we see that

$$\int_{E(s_1,B\lambda)} |Du|^{\frac{p(z)q(z)}{q^-}} \, dz \leq \epsilon c_7 \lambda \sum_{i=1}^{\infty} |K^{\lambda}_{r_i}(w_i)|, \tag{5.4.36}$$

where $c_7 := 2^{\frac{\gamma_2\gamma_4}{\gamma_3}} \left(\frac{16}{7}\right)^n (10\chi)^{n+2}$.

On the other hand, we know from (5.4.14) that

$$\frac{\lambda}{2} \leq \fint_{K^{\lambda}_{r_i}(w_i)} |Du|^{\frac{p(z)q(z)}{q^-}} \, dz \quad \text{or} \quad \frac{\lambda}{2} \leq \frac{1}{\delta} \left(\fint_{K^{\lambda}_{r_i}(w_i)} |F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}} \, dz \right)^{\frac{1}{1+\overline{\sigma}}}. \tag{5.4.37}$$

The first case of (5.4.37) implies

$$|K^{\lambda}_{r_i}(w_i)| \leq \frac{4}{\lambda} \int_{\{z \in K^{\lambda}_{r_i}(w_i) : |Du|^{\frac{p(z)q(z)}{q^-}} > \frac{\lambda}{4}\}} |Du|^{\frac{p(z)q(z)}{q^-}} \, dz$$

and the second case of (5.4.37) gives

$$|K^{\lambda}_{r_i}(w_i)| \leq \left(\frac{4}{\delta\lambda}\right)^{1+\overline{\sigma}} \int_{\{z \in K^{\lambda}_{r_i}(w_i) : |F|^{\frac{p(z)q(z)}{q^-}} > \frac{\delta\lambda}{4}\}} |F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}} \, dz.$$

Therefore we get

$$\begin{aligned} |K^{\lambda}_{r_i}(w_i)| \;\; &\leq \;\; \frac{4}{\lambda} \int_{\{z \in K^{\lambda}_{r_i}(w_i) : |Du|^{\frac{p(z)q(z)}{q^-}} > \frac{\lambda}{4}\}} |Du|^{\frac{p(z)q(z)}{q^-}} \, dz \\ &\quad + \left(\frac{4}{\delta\lambda}\right)^{1+\overline{\sigma}} \int_{\{z \in K^{\lambda}_{r_i}(w_i) : |F|^{\frac{p(z)q(z)}{q^-}} > \frac{\delta\lambda}{4}\}} |F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}} \, dz. \end{aligned} \tag{5.4.38}$$

Plugging (5.4.38) into (5.4.36) and using the fact that the elements of $\{K_{r_i}^{\lambda}(w_i)\}_{i=1}^{\infty}$ are disjoint, we finally obtain

$$\int_{E(s_1,B\lambda)} |Du|^{\frac{p(z)q(z)}{q^-}}\,dz \le 4^{1+\overline{\sigma}} c_7 \epsilon \Bigg\{ \int_{E(s_2,\frac{\lambda}{4})} |Du|^{\frac{p(z)q(z)}{q^-}}\,dz + \frac{1}{\lambda^{\overline{\sigma}}\delta^{1+\overline{\sigma}}} \int_{\{z\in K_{s_2 r}:|F|^{\frac{p(z)q(z)}{q^-}}>\frac{\delta\lambda}{4}\}} |F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}}\,dz \Bigg\}. \quad (5.4.39)$$

5.4.4 Local gradient estimates in $L^{p(\cdot)q(\cdot)}$-space: the proof of (5.1.12).

For a sufficiently large $k \in \mathbb{N}$, we first define the truncations of $|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}$ such that

$$\left(|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}\right)_k := \min\{|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}, k\}.$$

Let $1 \le s_1 < s_2 \le 2$. By Fubini's theorem we have

$$\begin{aligned}
&\int_{K_{s_1 r}} \left(|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}\right)_k^{q^- -1} |Du|^{\frac{p(z)q(z)}{q^-}}\,dz \\
&\quad= \left(q^- - 1\right) \int_{K_{s_1 r}} \left[\int_0^{\left(|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}\right)_k} \lambda^{q^- -2}\,d\lambda\right] |Du|^{\frac{p(z)q(z)}{q^-}}\,dz \\
&\quad= \left(q^- - 1\right) \int_0^k \lambda^{q^- -2} \int_{K_{s_1 r}\cap\{|Du|^{\frac{p(z)q(z)}{q^-}}>\lambda\}} |Du|^{\frac{p(z)q(z)}{q^-}}\,dz\,d\lambda \\
&\quad= \left(q^- - 1\right) B^{q^- -1} \int_0^{k/B} \lambda^{q^- -2} \int_{E(s_1,B\lambda)} |Du|^{\frac{p(z)q(z)}{q^-}}\,dz\,d\lambda \\
&\quad= \left(q^- - 1\right) B^{q^- -1} \int_0^{A\lambda_0} \lambda^{q^- -2} d\lambda \int_{K_{s_1 R}} |Du|^{\frac{p(z)q(z)}{q^-}}\,dz \\
&\qquad+ \left(q^- - 1\right) B^{q^- -1} \int_{A\lambda_0}^{k/B} \lambda^{q^- -2} \int_{E(s_1,B\lambda)} |Du|^{\frac{p(z)q(z)}{q^-}}\,dz\,d\lambda \\
&\quad=: I_6 + I_7,
\end{aligned}$$

where λ_0, $A(s_1, s_2)$ and B are defined (5.4.11), (5.4.13) and (5.4.35), respectively. For I_3, a direct calculation yields that

$$\begin{aligned} I_6 &\le (A(s_1,s_2)B\lambda_0)^{q^- -1}\int_{K_{2r}} |Du|^{\frac{p(z)q(z)}{q^-}}\,dz \\ &\le \frac{c\lambda_0^{q^- -1}}{(s_2-s_1)^{(n+2)d^+}}\int_{K_{2r}} |Du|^{\frac{p(z)q(z)}{q^-}}\,dz. \end{aligned}$$

For I_7, inserting (5.4.39) into I_4 and using Fubini's theorem we have

$$\begin{aligned} I_7 &\le 4^{1+\overline{\sigma}}c_7(q^- -1)B^{q^- -1}\epsilon\Bigg\{\int_0^{4k}\lambda^{q^- -2}\int_{E(s_2,\frac{\lambda}{4})}|Du|^{\frac{p(z)q(z)}{q^-}}\,dz\,d\lambda \\ &\quad +\frac{1}{\delta^{1+\overline{\sigma}}}\int_0^\infty \lambda^{q^- -2-\overline{\sigma}}\int_{\{z\in K_{s_2r}:|F|^{\frac{p(z)q(z)}{q^-}}>\frac{\delta\lambda}{4}\}}|F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}}\,dz\,d\lambda\Bigg\} \\ &= 4^{q^- +\overline{\sigma}}c_7(q^- -1)B^{q^- -1}\epsilon\Bigg\{\int_0^{k}\lambda^{q^- -2}\int_{E(s_2,\lambda)}|Du|^{\frac{p(z)q(z)}{q^-}}\,dz\,d\lambda \\ &\quad +\frac{1}{4^{\overline{\sigma}}\delta^{q^-}}\int_0^\infty \lambda^{q^- -2-\overline{\sigma}}\int_{\{z\in K_{s_2r}:|F|^{\frac{p(z)q(z)}{q^-}}>\lambda\}}|F|^{\frac{p(z)q(z)(1+\overline{\sigma})}{q^-}}\,dz\,d\lambda\Bigg\} \\ &= 4^{\gamma_4+\overline{\sigma}}c_7B^{\gamma_4-1}\epsilon\int_{K_{s_2r}}\left(|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}\right)_k^{q^- -1}|Du|^{\frac{p(z)q(z)}{q^-}}\,dz \\ &\quad +c(\epsilon,\delta)\int_{K_{s_2r}}|F|^{p(z)q(z)}\,dz, \end{aligned}$$

where $c(\epsilon,\delta)$ is a positive constant depending only on $n, \Lambda, \nu, \gamma_1, \gamma_2, \gamma_3, \gamma_4, L_1$.

At this point, we choose $\epsilon = \epsilon(n, \Lambda, \nu, \gamma_1, \gamma_2, \gamma_3, \gamma_4, L_1) \in (0,1)$ sufficiently small such that

$$4^{\gamma_4+\overline{\sigma}}c_7B^{\gamma_4-1}\epsilon = \frac{1}{2}.$$

Remark 5.4.8. *From the choice of ϵ, $\delta \in (0, 1/8)$ is determined to be a sufficiently small constant depending only on $n, \Lambda, \nu, \gamma_1, \gamma_2, \gamma_3, \gamma_4, L_1$.*

Combining the above results we derive

$$\begin{aligned}\int_{K_{s_1 r}} \left(|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}\right)_k^{q^- -1} |Du|^{\frac{p(z)q(z)}{q^-}}\, dz \\ \le \frac{1}{2}\int_{K_{s_2 r}} \left(|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}\right)_k^{q^- -1} |Du|^{\frac{p(z)q(z)}{q^-}}\, dz \\ + \frac{c\lambda_0^{q^- -1}}{(s_2-s_1)^{(n+2)d^+}}\int_{K_{2r}} |Du|^{\frac{p(z)q(z)}{q^-}}\, dz + c\int_{K_{2r}} |F|^{p(z)q(z)}\, dz.\end{aligned}$$

Since $1 \le s_1 < s_2 \le 2$ are arbitrary, applying Lemma 5.2.9, we have

$$\begin{aligned}\int_{K_r} \left(|Du|^{\frac{p(\cdot)q(\cdot)}{q^-}}\right)_k^{q^- -1} |Du|^{\frac{p(z)q(z)}{q^-}}\, dz \\ \le c\lambda_0^{q^- -1}\int_{K_{2R}} |Du|^{\frac{p(z)q(z)}{q^-}}\, dz + c\int_{K_{2r}} |F|^{p(z)q(z)}\, dz.\end{aligned}$$

Passing to the limit $k \to \infty$ in the above inequality and applying Fatou's lemma we have

$$\int_{K_r} |Du|^{p(z)q(z)}\, dz \le c\lambda_0^{q^- -1}\int_{K_{2R}} |Du|^{\frac{p(z)q(z)}{q^-}}\, dz + c\int_{K_{2r}} |F|^{p(z)q(z)}\, dz,$$

which implies

$$\begin{aligned}⨍_{K_r} |Du|^{p(z)q(z)}\, dz \\ \le c\left\{⨍_{K_{4r}} |Du|^{p(z)} dz + \left(⨍_{K_{4r}} |F|^{p(z)q(z)}\, dz\right)^{\frac{1}{q^-}} + 1\right\}^{1+d^+(q^- -1)}.\end{aligned} \tag{5.4.40}$$

Indeed, we first observe from (5.1.4), (5.1.6), (5.2.7) and the fact $M < \frac{1}{4R_0} \le \frac{1}{4r}$ by (5.4.2) that

$$\begin{aligned}\left(⨍_{K_{2r}} \left[|Du|^{p(z)} + |F|^{p(z)q(z)}\right]\, dz\right)^{\rho(8r)} &\le c\left(\frac{M}{|K_{2r}|}\right)^{\rho(8r)} \\ &\le c\left(\frac{1}{(8r)^{n+3}}\right)^{\rho(8r)} \le c.\end{aligned}$$

From this, (5.4.3) and (5.4.4) we see

$$\begin{aligned}
&\fint_{K_{2r}} |Du|^{\frac{p(z)q(z)}{q^-}}\,dz \le \fint_{K_{2r}} |Du|^{p(z)(1+\frac{\rho(8r)}{\gamma_3})}\,dz + 1 \\
&\le c\left\{ \left(\fint_{K_{4r}} \left[|Du|^{p(z)} + |F|^{p(z)} \right] dz \right)^{1+\frac{\rho(8r)}{\gamma_3} d_0} + \fint_{K_{4r}} |F|^{p(z)(1+\frac{\rho(8r)}{\gamma_3})}\,dz + 1 \right\} \\
&\le c\left\{ \fint_{K_{4r}} \left[|Du|^{p(z)} + |F|^{p(z)} \right] dz + \left(\fint_{K_{4r}} |F|^{p(z)q^-}\,dz \right)^{\left(1+\frac{\rho(8r)}{\gamma_3}\right)\frac{1}{q^-}} + 1 \right\} \\
&\le c\left\{ \fint_{K_{4r}} |Du|^{p(z)}\,dz + \left(\fint_{K_{4r}} |F|^{p(z)q(z)}\,dz \right)^{\frac{1}{q^-}} + 1 \right\}.
\end{aligned}$$

Therefore, applying (5.4.11) and the previous estimate and Höler's inequality we obtain (5.4.40). Finally, using (5.1.4), (5.1.5) and the fact $M < \frac{1}{4R_0} \le \frac{1}{4r}$ by (5.4.2), we can replace d^+ and q^- in (5.4.40) by $d_w = d(w)$ and $q_w = q(w)$, which proves the desired estimate (5.1.12).

We end the subsection determining the constant $\tilde{\delta}$ in Theorem 5.1.3. In view of Remark 5.4.2 and Remark 5.4.8, we can rewrite (5.4.2) as

$$4R_0 \le \delta_1 R^{\frac{1}{\alpha}} M^{-(\frac{n+3}{\alpha}+1)}, \quad \omega(8R_0) \le \delta_2 \quad \text{and} \quad \rho(8R_0) \le \delta_3,$$

for some $\delta_1, \delta_2, \delta_3 \in (0,1)$ depending only on $n, \Lambda, \nu, \gamma_1, \gamma_2, \gamma_3, \gamma_4, L_1$. Hence we take R_0 such that

$$R_0 := \tilde{\delta} R^{\frac{1}{\alpha}} M^{-(\frac{n+3}{\alpha}+1)} \quad \text{where } \tilde{\delta} := \min\left\{ \frac{\delta_1}{4}, \frac{\omega^{-1}(\delta_2)}{8}, \frac{\rho^{-1}(\delta_3)}{8} \right\}. \tag{5.4.41}$$

Here $\omega^{-1}, \rho^{-1} : [0,\infty) \to [0,\infty]$ are denoted by

$$\omega^{-1}(s) := \sup\{\tilde{s} \in [0,\infty) : \omega(\tilde{s}) \le s\} \text{ and } \rho^{-1}(s) := \sup\{\tilde{s} \in [0,\infty) : \rho(\tilde{s}) \le s\}.$$

At the end, R_0 satisfies the condition (5.4.2) and then the estimate (5.1.12) holds for every $r \le R_0$.

5.4.5 Global gradient estimates in $L^{p(\cdot)q(\cdot)}$-space: the proof of (5.1.13).

We derive the global estimate (5.1.13) from the local estimates (5.1.12) by using a standard covering argument. Recall $R_0 > 0$ defined in (5.4.41)(without loss of generality we assume $R_0 \le \sqrt{T/16}$). From the local estimate (5.1.12) and (5.1.6) we have

$$\int_{K_{R_0}(w)} |Du|^{p(z)q(z)}\,dz \le c|Q_{4R_0}|^{1-d_1}\left\{|Q_{4R_0}|^{1-\gamma_4}\left(\int_{K_{4R_0}(w)} |Du|^{p(z)}\,dz\right)^{\gamma_4} + \int_{K_{4R_0}(w)} \left[|F|^{p(z)q(z)}+1\right]\,dz\right\}^{d_1}. \quad (5.4.42)$$

for every $w \in \Omega_T$, where $d_1 := \frac{1+d_M(\gamma_3-1)}{\gamma_3} > 1$. Since $\overline{\Omega_T}$ is compact in $\mathbb{R}^{n+1}$, using Vitali's covering lemma we can find $\{w_k\}_{k=1}^{k_0} \subset \Omega_T$, $k_0 \in \mathbb{N}$, such that all $Q_{R_0/3}(w_k)$, $k = 1, 2, \ldots, k_0$, are disjoint and $\{Q_{R_0}(w_k)\}$ covers Ω_T. Moreover one has

$$\sum_{k=1}^{k_0} \int_{K_{4R_0}(w_k)} f\,dz \le c(n) \int_{\Omega_T} f\,dz$$

for every $f \in L^1(\Omega_T)$, where $c(n) > 0$ depends only on n. Then we see from (5.4.41) and (5.4.42) that

$$\begin{aligned}
\int_{\Omega} |Du|^{p(z)q(z)}\,dz &\le \sum_{k=1}^{k_0} \int_{K_{R_0}(w_k)} |Du|^{p(z)q(z)}\,dz \\
&\le cM^{\alpha_0(n+2)(d_1-1)}\left\{M^{\alpha_0(n+2)(\gamma_4-1)}\left(\sum_{k=1}^{k_0}\int_{K_{4R_0}(w_k)} |Du|^{p(z)}\,dz\right)^{\gamma_4} + \sum_{k=1}^{k_0}\int_{K_{4R_0}(w_k)} \left[|F|^{p(z)q(z)}+1\right]\,dz\right\}^{d_1} \\
&\le cM^{\alpha_0(n+2)(d_1-1)}\left(M^{\alpha_0(n+2)(\gamma_4-1)+\frac{\gamma_4}{\gamma_3}} + M\right)^{d_1} \\
&\le cM^{\alpha_0(n+2)(\gamma_4 d_1-1)+\frac{\gamma_4 d_1}{\gamma_3}}.
\end{aligned}$$

This proves (5.1.13).

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국문초록

이 학위 논문에서는 변동지수르베그공간에서의 발산형 타원형 및 포물형 방정식들에 대한 대역적 칼데론-지그먼드 이론에 대하여 연구한다. 특히, 적절한 가늠을 유도함으로써 디리클레 형식의 경계값이 영인 방정식의 유일한 해의 그레디언트가 변동지수르베그공간에서 비동차항과 동등한 적분가능성을 가진다는 것을 증명한다. 본 연구에서는 선형 타원형 방정식, 선형 포물형 방정식, 변동 성장조건을 가지는 타원형 방정식, 변동 성장조건을 가지는 포물형 방정식등 네가지 형태의 방정식을 다룬다. 그리고 대역적 칼데론-지그먼드 이론을 얻기위한 변동지수, 계수함수, 경계영역의 최소 조건을 제시한다.

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