이학박사 학위논문

Oscillatory Integrals, Spectral Mutiplier Operators, Semilinear Elliptic Equations, and Pseudodifferential Calculus on Carnot Manifolds

(진동 적분, 분광 곱 작용소, 반선형 타원형 방정식, 그리고 캐놋 다양체위에서의 의미분 작용소 연산)

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Oscillatory Integrals, Spectral Mutiplier Operators, Semilinear Elliptic Equations, and Pseudodifferential Calculus on Carnot Manifolds

A dissertation
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by

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Abstract

Oscillatory Integrals, Spectral Mutiplier Operators, Semilinear Elliptic Equations, and Pseudodifferential Calculus on Carnot Manifolds

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This thesis consists of the following three parts; sharp estimates of linear operators, semiinear elliptic equations, and pseudodifferential calculus on Carnot manfiolds. These subjects are related to each other in direct and implicit ways.

The first part is based the three papers [Ch1, Ch2, Ch3] whose object are to obtain sharp estimates of some linear operators related to oscillatory integration and spectral multipliers. More precisely, in the first paper [Ch1], we obtain sharp L^2 and H^p boundedness of strongly singular operators and oscillating operators on Heisenberg groups by applying the oscillatory integral estimates for degenerate phases and the molecular decomposition for Hardy spaces. In the second paper [Ch2] we obtain a refined L^p bound for maximal functions of the multiplier operators on stratified groups and maximal functions of the multi-parameter multipliers on product spaces of stratified groups. As an application we find a refined L^p bound for maximal functions of joint spectral multipliers on Heisenberg group. In the third paper [Ch3], for a self-adjoint positive elliptic (-pseudo) differential operator P on a compact manifold M without boundary, we obtain a refined L^p bound of the maximal function of the multiplier operators associated to P satisfying the Hörmander-Mikhlin condition.

The second part is concerned with semilinear elliptic equations. It is based on the paper [Ch4] and the joint works [CKL, CKL2, ChS].

In [Ch4] we study strongly indefinite systems involving the fractional Laplacian on bounded domains. Explicitly, we obtain existence and non-existence results, *a priori* estimates of Gidas-Spruck type, and a symmetry result. In addition, we give a different proof for the *a priori* estimate for nonlinear elliptic problems with the fractional Laplacian obtained in [CT, T2].

In the paper [CKL] with S. Kim and K. Lee, we study the asymptotic behavior of least energy solutions and the existence of multiple bubbling solutions of nonlinear elliptic equations involving the fractional Laplacians and the critical exponents. This work can be seen as a nonlocal analog of the results of Han (1991) [H] and Rey (1990) [R].

In the paper [ChS] with J. Seok, we study a class of semilinear nonlocal elliptic equations posed on settings without compact Sobolev embedding. More precisely, we prove the existence of infinitely many solutions to the fractional Brezis-Nirenberg problems on bounded domain.

The last chapter of this part is based on the paper [CKL2] with S. Kim and K. Lee, The objective of this paper is to obtain qualitative characteristics of multi-bubble solutions to the Lane-Emden-Fowler equations with slightly subcritical exponents given any dimension $n \geq 3$. By examining the linearized problem at each m-bubble solution, we provide a number of estimates on the first $(n+2)m$-eigenvalues and their corresponding eigenfunctions. Specifically, we present a new proof of the classical theorem due to Bahri-Li-Rey (1995) [BLR] which states that if $n \geq 4$, then the Morse index of a multi-bubble solution is governed by a certain symmetric matrix whose component consists of a combination of Green's function, the Robin function, and their first and second derivatives. Our proof also allows us to handle the intricate case $n = 3$.

The third part is based on the joint works [CP1, CP2] with R. Ponge. In [CP1] we construct the tangent groupoid of a Carnot manifolds, i.e., a manifold equipped with a flag of sub-bundles $\{0\} = H_0 \subset H_1 \subset \cdots \subset H_r = TM$ of the tangent bundle. Based on the geometric study, we establish the calculus of Pseudo-differential operators on Carnot manifolds in the forthcoming paper [CP2]. We define the classes of $\Psi_H DO$s which are suitable for studying hypoelliptic operators and show that the class is invariant under the change of coordinates. Then, we obtain asymptotic symbolic calculus in the composition of $\Psi_H DO$s. As applications we can obtain the asymptotic expansion of kernels of Hörmander's sum of squares and the heat kernel asymptotics on Carnot manifolds.

Key words: semilinear elliptic equations, fractional Laplacians, oscillatory integrals, maximal multipliers, Carnot manifolds, pseudodifferential calculus
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Contents

Chapter 1

Introduction

During the last decades, the linear theoy in classical analysis has been largely developed and it also provided important tools for geometric problems and partial differential equations. This research flow motivated the works in this thesis. We shall rely on various techniques in mathematical analysis to study oscillatroy integral and spectral operators, semilinear elliptic equations, differential operators on Carnot Caratheorody spaces.

This thesis is written by collecting the following listed works of the author and the coworks with my advisor Prof. Raphaël Ponge, Dr. Seunghyeok Kim, Prof. Ki-Ahm Lee, and Prof. Jinmyoung Seok.

1. *L^2 and H^p boundedness of strongly singular operators and oscillating operators on Heisenberg groups*, to appear in Forum math (Online published).

2. *Maximal multiplier on Stratified groups,* to appear in Math. Nachr.

3. *Maximal functions of multipliers on compact manifolds without boundary,* arXiv:1207.0201, submitted.

4. *On strongly indefinite systems involving the fractional Laplacian,* to appear in Nonlinear Anal.

5. (with Raphaël Ponge) *Privileged coordinates and Tangent groupoid for Carnot manifolds,* in preparation.

6. (with Raphaël Ponge) *Pseudodifferential calculus on Carnot manifolds*, in preparation.

7. (with Seunghyeok Kim and Ki-Ahm Lee) *Asymptotic behavior of solutions for nonlinear elliptic problems with the fractional Laplacian,* J. Funct. Anal. 266 (2014), no. 11, 6531–6598.

8. (with Jinmyoung Seok) *Infinitely many solutions for semilinear nonlocal elliptic equations under noncompact settings,* arXiv:1404.1132, submitted.

9. (with Seunghyeok Kim and Ki-Ahm Lee) *Qualitative properties of multi-bubble solutions for nonlinear elliptic equations involving critical exponents,* arXiv:1408.2384, submitted.

In the following three sections, we shall explain the results and motivations of the above works. In Section 1 we introduce the linear estimates results of [Ch1, Ch2, Ch3]. Section 2 is devoted to introduce the results of [Ch4, CKL, CKL2, ChS] on semilinear elliptic equations. In Section 3 we introduce the results of [CP1, CP2] on groupoids and pseudodifferential caclulus on Carnot manifolds.

1.1 Oscillatory Integrals and Spectral Mutiplier Operators

This part is based on the papers [Ch1, Ch2, Ch3]. Oscillatory integral and Spectral multiplier operators are fundamental subjects in the linear theory and they also appear abundantly in geometry and partial differential equations. To handle those kind of operators, the theory has been established well for singular integral operators, oscillatory integral estimates, and interpolation method. In the same time, the theory has been extended to the geometric settings like the Riemannian manifolds and Carnot-Caratheodory spaces. In the first part of this thesis, we are concerned with two kind of problems of the linear estimates given on the geometric settings.

1.1.1 L^2 and H^p boundedness of strongly singular operators and oscillating operators on Heisenberg groups

In [Ch1] we study strongly singular operators on the Heisenberg group, which are convolution operators with kernels

$$K_{\alpha,\beta}(x,t) = \rho(x,t)^{-(2n+2+\alpha)} e^{i\rho(x,t)^{-\beta}} \chi(\rho(x,t)), \quad \alpha > 0, \quad \beta > 0,$$

where χ is a smooth bump function in a small neighborhood of the origin. These operators were first introduced by Laghi [Ly] and the result on L^2 boundedness was obtained in [Ly, LL], which was sharp only for some restricted cases. In [Ch1] the author obtained the sharp results on L^2 boundedness for any case using the oscillatory integral estimates for degenerated phases obtained by Pan-Sogge [PS] and Greenleaf-Seeger [GR]. In addition we obtain the sharp L^2 result in almost cases for oscillating convolution operators with the kernels

$$L_{\alpha,\beta}(x,t) = \rho(x,t)^{-(2n+2-\alpha)} e^{i\rho(x,t)^{\beta}} \chi(\rho(x,t)^{-1}), \quad \beta > 0.$$

We also provide the boundedness result on the Hardy space using the molecular decomposition of the Hardy space.

1.1.2 Maximal multiplier on Stratified groups and compact manifolds without boundary

In [Ch2, Ch3] we study the maximal functions of spectral multiplier operators. This work was motivated by the work of Grafakos-Honzik-Seeger [GHS] who obtained the sharp result on the maximal multipliers on the Euclidean spcae. They exploited the interaction of multiplier operators and martingale operators so that a good λ inequality of Chang-Wilson-Woff [CWW] for martingale operators can be applied. On the other hand, the multiplier theory has been extended to the various settings such as nilpotent Lie groups and Riemannian manifolds by many authors (see. e.g. [C1, MaM, SS]. We make use of the approach of [GHS] to obtain a refined estimate for the maximal multipliers on nilpotent Lie groups. This improves the previous result on the maximal multipliers on nilpotent Lie groups obtained in [MaM]. We also consider maximal multipliers on product spaces of nilpotent Lie groups and apply it to find a refined estimate for the maximal function of joint spectral multipliers on the Heisenberg group. A similar result was obtained in [Ch3] for multipliers on compact manifolds without boundary.

1.2 Semilinear Elliptic Equations and Fractional Laplacians

This part is based on the papers [Ch4, CKL, CKL2, ChS]. As we mentioned, the classical analysis provides various essential tools for studying partial differential equations, containing various time evolution equations like nonlinear Schrödinger equations and Gross-Pitaevskii equations. On the other hand, many properties of those equations are governed by travelling waves and stationary solutions which can be described by time independent semilinear elliptic equations. The second part of the thesis deals with this kind of equations. There are various topics in the theory of semilinear elliptic equations, and among those we first concentrate on problems involving the fractional Laplacian. The study of this topic has been boosted since the work of Caffarelli and Silvestre [CaS] where the authors developed a local interpretation of the fractional Laplacian given in $\mathbb{R}^n$ by considering a Neumann type operator in the extended domain $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^{n+1} : t > 0\}$. This observation made a significant influence on the study of related nonlocal problems. A similar extension was devised by Cabré and Tan [CT] on bounded domains and they studied the following type problem

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 < s < 1$, Ω is a smooth bounded domain of $\mathbb{R}^n$ and $(-\Delta)^s$ denotes the fractional Laplace operator $(-\Delta)^s$ in Ω with zero Dirichlet boundary values on $\partial\Omega$, defined in terms of the spectra of the Dirichlet Laplacian $-\Delta$ on Ω, and $f : \mathbb{R}^n \to \mathbb{R}$ is a certain function.

1.2.1 On strongly indefinite systems involving the fractional Laplacian,

Motivated by the work of [CT], the author [Ch1] studied the following nonlinear system

$$\begin{cases} (-\Delta)^s u = v^p & \text{in } \Omega, \\ (-\Delta)^s v = u^q & \text{in } \Omega, \\ u > 0, \ v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $0 < s < 1$, $p > 1$, and $q > 1$. This is a nonlocal version of the strongly indefinite system which has been studied during the last decades (see [FF, HV] and references therein). When $\Omega = \mathbb{R}^n$ the problem (1.1) was studied already by many authors (see e.g. [CLO, CLO2]). Let us define that a pair of exponents (p, q) is sub-critical if $\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2s}{n}$, critical if $\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2s}{n}$, and super-critical if $\frac{1}{p+1} + \frac{1}{q+1} < \frac{n-2s}{n}$. Then we obtain existence result for the sub-critical case and nonexistence result for the critical and super-critical case. We establish a moving plane argument and a maximum principle for the extended problem, to prove the following symmetry result. In addition *a priori* estimate of Gidas-Spruck type is obtained.

1.2.2 behavior of solutions for nonlinear elliptic problems with the fractional Laplacian

The collaboration works [CKL] with S. Kim and K. Lee and [CaS] with J. Seok are concerned with the Brezis-Nirenberg type problem

$$\begin{cases} (-\Delta)^s u = |u|^{2^*(s)-2-\epsilon} u + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $0 < s < 1$, $2^*(s) := \frac{2n}{n-2s}$, $\lambda > 0$.

In [CKL] we study the behavior of positive solutions as the parameter $\lambda = \epsilon$ goes to zero. We show that the least energy solution of (1.3) concentrates at a critical point of the Robin function of the fractional Laplacian $(-\Delta)^s$. Moreover, we construct multi-peak solutions by employing the Lyapunov-Schmidt reduction method. These two results are motivated by the work of Han [H] and Rey [R] on the classical local Brezis-Nirenberg problem, which dates back to Brezis and Peletier [BP],

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}} + \epsilon u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In the latter part of his paper, Rey [R] constructed a family of solutions for (1.4) which asymptotically blow up at a nondegenerate critical point of the Robin function. This result was extended in [MP], where Musso and Pistoia obtained the existence of multi-peak solutions for certain domains.

1.2.3 Infinitely many solutions for semilinear nonlocal elliptic equations under noncompact settings

In [CaS] we prove the existence of infinitely many solutions of the problem (1.3) for each fixed $\lambda > 0$. Due to the loss of compactness of critical Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ and $H_0^s(\Omega) \hookrightarrow L^{\frac{2n}{n-2s}}(\Omega)$, more careful analysis is required to construct nontrivial solutions to the equation (1.3) than equation with sub-critical nonlinearities. We employ Devillanova and Solimini's ideas in [DS]. The main strategy in these ideas is to consider approximating subcritical problems, In other words, we consider subcritical problems

$$\begin{cases} (-\Delta)^s u = |u|^{2^*(s)-2-\epsilon}u + \mu u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

for small $\epsilon > 0$. From the sub-criticality of the problems, one can verify by using standard variational methods that for every small $\epsilon > 0$, (1.5) admits infinitely many nontrivial solutions in a fractional Sobolev space $H_0^s(\Omega)$. By this reason we obtained the following compactness result to obtain nontrivial solutions to our original equation (1.3).

Theorem 1.2.1. *Assume $N > 6s$. Let $\{u_m\}$ be a sequence of solutions to* (1.5) *with $\epsilon = \epsilon_n \to 0$ as $n \to \infty$ and $\sup_{m\in\mathbb{N}} \|u_m\|_{H_0^s(\Omega)} < \infty$. Then $\{u_m\}$ converges strongly in $H_0^s(\Omega)$ up to a subsequence.*

1.2.4 Qualitative properties of multi-bubble solutions for nonlinear elliptic equations involving critical exponents

In paper [CKL2], we come back to study the local problem

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}-\epsilon} & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1_ϵ}$$

where p denotes the critical exponent $\frac{n+2}{n-2}$ here. We assume that $\{u_\epsilon\}_{\epsilon>0}$ is a family of solutions to the problem (1.1_ϵ) which blows up at m points in Ω. Our objective is to compute the Morse index of the blow up solutions. For this aim, we consider the linearized problem

$$\begin{cases} -\Delta v = \mu(p-\epsilon)u_\epsilon^{p-1-\epsilon}v & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Let $\mu_{\ell\epsilon}$ be the ℓ-th eigenvalue of (1.6) provided that the sequence of eigenvalues is arranged in nondecreasing order permitting duplication, and $v_{\ell\epsilon}$ the corresponding $L^\infty(\Omega)$-normalized eigenfunction (namely, $\|v_{\ell\epsilon}\|_{L^\infty(\Omega)} = 1$). We shall examine the behavior of eigenpairs $(\mu_{\ell\epsilon}, v_{\ell\epsilon})$ to the linearized problem (1.6) at u_ϵ for $1 \le \ell \le (n+2)m$. This extends the result of Grossi and Pacella [GP] for one-peak solutions.

1.3 Pseudodifferential Calculus on Carnot Manifolds

This part is based on the papers [CP1, CP2]. Among the topics of mathematical analysis related to geometry, an important one is the theory of pseudo-differential calculus. This theory is applied to obtain the inverse parametrix and heat kernel asymptotic expansion of elliptic operators on manifolds. These calculations provide important tools for the index theory of and the spectral asymptotics of elliptic operators on compact manifolds. For this reason, there has been lots of interest to extend the thoery for more general geometric settings which are not conatined the category of the Riemannian manifolds. One important setting is Carnot-Caratheodory spaces where the elliptic operators are replaced by hypoelliptic operators. An important example is the Heisenberg manifolds which contains CR manifolds and contact manifolds. Beals-Greiner [BG] and Taylor [Tay] established the pseudodifferential calculus on the Heisenberg manifolds. Ponge [P2] developed an intrinsic approach and extension to complex powers with presenting various applications. The last part of this thesis is aimed to establish the pseudodifferential calculus on Carnot manifolds, which stand for equi-regular Carnot-Caratheodory spaces. It is based on the joint works [CP1, CP2] with R. Ponge.

1.3.1 Privileged coordinates and Tangent groupoid for Carnot manifolds

In paper [CP1] we construct the tangent groupoid of a Carnot manifolds, i.e., a manifold equipped with a flag of sub-bundles $\{0\} = H_0 \subset H_1 \subset \cdots \subset H_r = TM$ of the tangent bundle. We find an intrinsic notion of tangent Lie group bundles of Carnot manifold and construct the tangent groupoid following the approach presented in [P1]. This was achieved with finding some intrinsic privileged coordinates and studying their properties.

1.3.2 Pseudodifferential calculus on Carnot manifolds

In the subsequent work [CP2], we define suitable classes of pseudodifferential operators on Carnot manifolds and establish the calculus containing the composition formula. Based on the calculus, we can discuss on the relation between the invertibility and the Rockland condition. As an application, we obtain the heat kernel asymptotic expansion and the spectral asymptotic for hypoelliptic operators.

Part I

Oscillatory Integrals and Spectral Mutiplier Operators

Chapter 2

L^2 and H^p boundedness of strongly singular operators and oscillating operators on Heisenberg groups [Ch1]

2.1 Introduction

The setting of this paper is the Heisenberg group $\mathbb{H}_a^n$, $a \in \mathbb{R}^*$, realized as $\mathbb{R}^{2n+1}$ equipped with the group law,

$$(x,t)\cdot(y,s) = (x+y, s+t-2ax^T Jy), \qquad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

For $K \in \mathcal{D}'(\mathbb{H}_a)$ we denote by T_K the convolution operator defined by K, i.e,

$$T_K f(x,t) := K * f(x,t) = \int_{\mathbb{H}^n} K\left((x,t)\cdot(y,s)^{-1}\right) f(y,s)dydx, \qquad f \in C_0^\infty(\mathbb{H}_a^n).$$

We say that the operator T_K is bounded on $L^p(\mathbb{H}^n)$ if there exist a $C > 0$ such that

$$\|T_K f\|_p \leq C\|f\|_p, \quad \text{for all } \; f \in C_0^\infty(\mathbb{H}^n).$$

A natural quasi-norm on the Heisenberg group is given by

$$\rho(x,t) = (|x|^4 + t^2)^{1/4}, \qquad (x,t) \in \mathbb{H}_a.$$

This quasi-norm satisfies $\rho(\lambda\cdot(x,t)) = \lambda\rho(x,t)$. For this quasi-norm, we define the strongly singular kernels,

$$K_{\alpha,\beta}(x,t) = \rho(x,t)^{-(2n+2+\alpha)} e^{i\rho(x,t)^{-\beta}} \chi(\rho(x,t)), \quad \alpha > 0, \quad \beta > 0,$$

where χ is a smooth bump function in a small neighborhood of the origin. This operator was introduced by Lyall [Ly] who showed that $T_{K_{\alpha,\beta}}$ is bounded when $\alpha \leq n\beta$. This result was obtained by using the Fourier transform on the Heisenberg group in combination with involved estimates on oscillatory integrals. Subsequently, Laghi-Lyall [LL] obtained sharp results in the special case $a^2 < C_\beta$ (where C_β is given by (2.1)) by using a version for the Heisenberg group of the L^2-boundedness theorem for non-degenerate oscillatory integral operators of Hörmander [Ho2]. In this paper, we shall consider the cases $a^2 \geq C_\beta$ and obtain sharp conditions using the theory for oscillatory integral operators with degenerate phases (see Section 2). Recall that the theory of the degenerate oscillatory integral operators was developed in depth to study X-ray transforms (see, e.g., Greenleaf-Seeger [GR2]).

Strongly singular convolution operators were originally considered on $\mathbb{R}^n$. Such operators correspond to suitable oscillating multipliers. They were first studied, by Fourier transform techniques, in the Euclidean setting with $\rho(x) = |x|$ by Hirschman [Hi] in the case d = 1, and in higher dimensions by Wainger [W], Fefferman [Fe], and Fefferman-Stein [FeS2].

Similar kind of convolution operators with kernels of the form $\frac{1}{|x|^{n-\alpha}} e^{i|x|^\beta}, \alpha, \beta > 0$, were introduced by Sjólin [Sj1, Sj2, Sj3]. Such kernels have no singularity near the origin, but they assume relatively small decaying property at infinity. Notice that the case $\beta = 1$ corresponds to the kernel of Bochner-Riesz means. For $\beta \neq 1$, the (L^p, L^q) estimates and Hardy space estimates hold (see Miyachi [Mi], Pan-Sampson [PSa] and Sjólin [Sj1, Sj2, Sj3]). The difference between the two cases comes from the fact that the phase kernel $|x - y|^\beta$ is degenerate only if $\beta = 1$. In this paper, we also consider the analogous problem on the Heisenberg groups for the following kernels,

$$L_{\alpha,\beta}(x, t) = \rho(x, t)^{-(2n+2-\alpha)} e^{i\rho(x,t)^\beta} \chi(\rho(x, t)^{-1}), \quad \beta > 0.$$

We denote by $T_{L_{\alpha,\beta}}$ the group convolution operators with the kernel $L_{\alpha,\beta}$. In the literature, the operators $T_{K_{\alpha,\beta}}$ (resp., $T_{L_{\alpha,\beta}}$) are called strongly singular operators (resp., oscillating convolution operators).

In the first part of this paper, we shall find the optimal ranges of α and β where the convolution operators associated with $K_{\alpha,\beta}$ and $L_{\alpha,\beta}$ are bounded on $L^2(\mathbb{H}^n)$.

For $a^2 \geq C_\beta$, the phase doesn't satisfy the non-degeneracy condition anymore. Therefore, we need to deal with oscillatory integral operators with degenerate phases. A theory for this kind of operators has been developped by considering various conditions on phase functions to give different decaying properties (see [GR2]). We shall rely on the results of Greenleaf-Seeger [GR] and Pan-Sogge [PS]. To use such theory we shall carefully investigate the folding type for our phases. Interestingly enough, we have different folding types according to the values of the parameters a and β. Before stating our results, we recall the previous results of Laghi-Lyall [LL] and Lyall [Ly]. Set

$$C_\beta = \frac{\beta + 2}{2}(2\beta + 5 + \sqrt{(2\beta + 5)^2 - 9}). \tag{2.1}$$

Then we have

Theorem (Laghi-Lyall [LL], Lyall [Ly])**.**

1. *$T_{K_{\alpha,\beta}}$ is bounded on $L^2(\mathbb{H}^n)$ if $\alpha \le n\beta$.*
2. *If $0 < a^2 < C_\beta$, then $T_{K_{\alpha,\beta}}$ is bounded on $L^2(\mathbb{H}^n)$ if and only if $\alpha \le (n + 1/2)\beta$.*

The first main result of this paper gives sharp L^2 boundedness results for $T_{K_{\alpha,\beta}}$ when $a^2 \ge C_\beta$.

Theorem 2.1.1.

1. *If $a^2 > C_\beta$, then $T_{K_{\alpha,\beta}}$ is bounded on $L^2(\mathbb{H}^n_a)$ if and only if $\alpha \le (n + \frac{1}{3})\beta$.*
2. *If $a^2 = C_\beta$, then $T_{K_{\alpha,\beta}}$ is bounded on $L^2(\mathbb{H}^n_a)$ if and only if $\alpha \le (n + \frac{1}{4})\beta$.*

For the operators $T_{L_{\alpha,\beta}}$, we also have the sharp L^2 boundedness results except when $\beta = 1$ and $\beta = 2$.

Theorem 2.1.2.

1. *$0 < \beta < 1$, then $T_{L_{\alpha,\beta}}$ is bounded on L^2 if and only if one of the following condition holds.*
   - *(i) $a^2 < C_\beta$ and $\alpha \le (n + \frac{1}{2})\beta$,*
   - *(ii) $a^2 = C_\beta$ and $\alpha \le (n + \frac{1}{4})\beta$,*
   - *(iii) $a^2 > C_\beta$ and $\alpha \le (n + \frac{1}{3})\beta$.*
2. *If $1 < \beta < 2$, then $T_{L_{\alpha,\beta}}$ is bounded on L^2 if and only if $\alpha \le (n + \frac{1}{3})\beta$.*
3. *If $2 < \beta$, then $T_{L_{\alpha,\beta}}$ is bounded on L^2 if and only if $\alpha \le (n + \frac{1}{2})\beta$.*

In [LL] Laghi-Lyall reduced the boundedness problem for operators on the Heisenberg group to that for the local operators and used a version of Hömander's L^2-boundedness theorem on the Heisenberg group. However, as we shall show, we may view the operators on the Heisenberg group as operators on Euclidean space $\mathbb{R}^{2n+1}$. This will enable us to use the oscillatory integral estimates of Greenleaf-Seeger [GR] and Pan-Sogge [PS] on Euclidean space.

For the cases $\beta = 1$ or $\beta = 2$, we also can obtain the sharp results for some value a where the phase becomes non-degenerate or has folds of type 2. However, in these cases, higher order types of folds than 3 appear for some values of a and the degenerate oscillatory integral estimates have not been obtained optimally yet for these cases. The theory have been established optimally only for phases with one or two types of folds (see Greenleaf-Seeger [GR] and Pan-Sogge [PS]).

For $p > 1$, L^p boundedness can be obtained by interpolation between the L^2 boundedness estimates and some L^1 boundedness estimates for dyadic-piece operator. We refer to Laghi [LL, Theorem 5] for the case $a^2 < C_\beta$ except the endpoint. Using this typical interpolation technique, it is also possible to obtain the L^p boundedness in the case $a^2 \ge C_\beta$.

In the second part of this paper, we turn our attention to the boundedness on Hardy spaces H^p $(p \leq 1)$ of the operators $T_{K_{\alpha,\beta}}$ and $T_{L_{\alpha,\beta}}$.

For the analogous operators on $\mathbb{R}^n$, the boundedness on Hardy spaces was proved up to the endpoint cases by Sjólin [Sj1, Sj3]. In this case, the operator can be thought as a multiplier operator $Tf = (m\widehat{f})^\vee$ and we have the relation $c_p \sum_{j=1}^n \|R_j f\|_{L^p} \leq \|f\|_{H^p} \leq C_p \sum_{j=1}^n \|R_j f\|_{L^p}$ and we see that derivatives of the symbol $\frac{\xi_j}{|\xi|}m(\xi)$ of the multiplier $R_j m(D)$ are pointwisely bounded by the derivatives of the symbol $m(\xi)$. These things make it possible to calculate the H^p norm accurately to obtain the sharp boundedness result including for the endpoint cases (see Miyachi [Mi]).

The above outline seems difficult to adapt to the Heisenberg group. Instead we shall rely on the molecular decomposition for Hardy spaces. This approach can be adapted to similar oscillating convolution operators on (stratified) nilpotent Lie groups.

Theorem 2.1.3. *Let $p \in (0,1)$ and let α and β be real numbers such that $(\frac{1}{p}-1)(2n+2)\beta + \alpha < 0$. Then*

1. *The operator $T_{K_{\alpha,\beta}}$ is bounded on H^p space.*
2. *For $\beta \neq 1$, the operator $T_{L_{\alpha,\beta}}$ is bounded on H^p space.*

These conditions are optimal except for the endpoint case $(\frac{1}{p}-1)(2n+2)\beta + \alpha = 0$.

This paper is organized as follows. In Section 2, we reduce our problem on the Heisenberg group to a local oscillatory integral estimates on Euclidean space. In Section 3, we recall some essential results for the oscillatory integral operators with degenerate phase functions and study geometry of the canonical relation and projection maps associated with the phase functions of the reduced operators, which will complete the proof of Theorem 2.1.1 and Theorem 2.1.2. In section 4, we recall some background on hardy spaces on the Heisenberg group and its basic properties. In section 5, we prove Theorem 2.1.3. In Section 6, we show that the conditions of Theorem 2.1.3 are sharp except the endpoint cases.

Notation

We will use the notation $\lesssim$ instead of $\leq C$ when the constant C depends only on the fixed parameters such as a, α, β and n. In addition, we will use the notation $A \sim B$ when both inequalities $A \lesssim B$ and $A \gtrsim B$ hold.

2.2 Dyadic decomposition and Localization

In this section we reduce our problems to some oscillatory integral estimates problem on Euclidean space $\mathbb{R}^{2n+1}$. This reduction is well-known for operators on Euclidean space (see [St]). The issue of this reduction on the Heisenberg group is to control the localized operators $\tilde{T}_j^{k,l}$ in

(2.5) uniformly for (g_k, g_l) such that $\rho(g_k \cdot g_l^{-1}) \leq 2$. Note that the cut-off functions $\eta(\rho((x,t) \cdot g_k^{-1}))\, \eta(\rho((y,s) \cdot g_l^{-1}))$ have no uniform bound for their derivatives. Nevertheless we get the uniformity after a value-preserving change of coordinates (see (2.7)).

We decompose the kernels $K_{\alpha,\beta}$ and $L_{\alpha,\beta}$ as

$$K_{\alpha,\beta}(x,t) \;=\; \sum_{j=1}^{\infty} K^j_{\alpha,\beta}, \quad K^j_{\alpha,\beta} := \eta(2^j \rho(x,t)) K_{\alpha,\beta}(x,t), \tag{2.1}$$

and

$$L_{\alpha,\beta}(x,t) \;=\; \sum_{j=1}^{\infty} L^j_{\alpha,\beta}, \quad L^j_{\alpha,\beta} := \eta(2^{-j} \rho(x,t)) L_{\alpha,\beta}(x,t), \tag{2.2}$$

where $\eta \in C_0^\infty(\mathbb{R})$ is a bump function supported in $[\frac{1}{2}, 2]$ such that $\sum_{j=0}^{\infty} \eta(2^j r) = 1$ for all $0 < r \leq 1$. For notational convenience, we omit the index α and β from now on.
Set $T_j f = K^j_{\alpha,\beta} * f$ and $S_j f = L^j_{\alpha,\beta} * f$. Then we have

Lemma 2.2.1. *For each $N \in \mathbb{N}$, there exist constants $C_N > 0$ and $c_\beta > 0$ such that*

$$\|T_j^* T_{j'}\|_{L^2 \to L^2} + \|T_j T_{j'}^*\|_{L^2 \to L^2} \leq C_N 2^{-max\{j,j'\}N}$$
$$\|S_j^* S_{j'}\|_{L^2 \to L^2} + \|S_j S_{j'}^*\|_{L^2 \to L^2} \leq C_N 2^{-max\{j,j'\}N}$$

holds for all j and j' satisfying $|j - j'| \geq c_\beta$.

Proof. The proof follows from the integration parts technique in the typical way, so we omit the details. See [Ly] where the proof for T_j is given. □

By Cotlar-Stein Lemma, we only need to show that there is a constant $C > 0$ such that

$$\|T_j\|_{L^2 \to L^2} + \|S_j\|_{L^2 \to L^2} \leq C \quad \forall j \in \mathbb{N}.$$

We consider the dilated kernels

$$\begin{aligned} \tilde{K}^j_{\alpha,\beta}(x,t) =& K^j_{\alpha,\beta}(2^{-j} \cdot (x,t)) = \eta(\rho(x,t)) 2^{j(Q+\alpha)} \rho(x,t)^{-Q-\alpha} e^{i 2^{j\beta} \rho(x,t)^{-\beta}}, \\ \tilde{L}^j_{\alpha,\beta}(x,t) =& L^j_{\alpha,\beta}(2^{-j} \cdot (x,t)) = \eta(\rho(x,t)) 2^{-j(Q-\alpha)} \rho(x,t)^{-Q+\alpha} e^{i 2^{j\beta} \rho(x,t)^{\beta}}. \end{aligned} \tag{2.3}$$

We define $\tilde{T}_j$ and $\tilde{S}_j$ to be the convolution operators with kernels given by $\tilde{K}^j_{\alpha,\beta}$ and $\tilde{L}^j_{\alpha,\beta}$. Set $f_j(x,t) = f(2^{-j} \cdot (x,t))$. Then $K^j_{\alpha,\beta} * f(2^{-j} \cdot (x,t)) = 2^{-jQ} (\tilde{K}^j_{\alpha,\beta} * f_j)(x,t)$, and we have

$$\begin{aligned} \|T_j f\|_{L^2} = \|K^j_{\alpha,\beta} * f(x,t)\|_{L^2} =& 2^{-jQ/2} \|K_{\alpha,\beta} * f(2^{-j} \cdot (x,t))\|_{L^2} \\ \leq& 2^{-jQ/2} \cdot 2^{-jQ} \|\tilde{K}^j_{\alpha,\beta} * f_j(x,t)\|_{L^2} \\ \leq& 2^{-jQ/2} \cdot 2^{-jQ} \|\tilde{T}_j\|_{L^2 \to L^2} \|f_j\|_{L^2} \\ \leq& 2^{-jQ} \|\tilde{T}_j\|_{L^2 \to L^2} \|f\|_{L^2}. \end{aligned} \tag{2.4}$$

Similarly, we have $\|S_j f\|_{L^2} \leq 2^{jQ}\|\tilde{S}_j\|_{L^2\to L^2}\|f\|_{L^2}$. It follows that it is enough to prove that $\|\tilde{T}_j\|_{L^2\to L^2} \lesssim 2^{jQ}$ and $\|\tilde{S}_j\|_{L^2\to L^2} \lesssim 2^{-jQ}$.

Now, we further modify our operators to some operators defined locally using the fact that the kernels of $\tilde{T}_j$ and $\tilde{S}_j$ are supported in $\{(x,t) : \rho(x,t) \leq 2\}$. To do this we find a set of point $G = \{g_k : k \in \mathbb{N}\}$ such that $\bigcup_{k\in\mathbb{N}} B(g_k, 2) = \mathbb{H}^n_a$ and each $B(g_k, 4)$ contains only d_n's other g_l members in G.

We can split $f = \sum_{k=1}^{\infty} f_k$ with each f_k supported in $B(g_k, 2)$. Define

$$\tilde{T}^{k,l}_j f(x,t) = \int \tilde{K}^j_{\alpha,\beta}\left((x,t)\cdot(y,s)^{-1}\right)\cdot\eta\left(\rho\left((x,t)\cdot g_k^{-1}\right)\right)\eta\left(\rho\left((y,s)\cdot g_l^{-1}\right)\right) f(y,s)dyds. \tag{2.5}$$

Then,

$$\begin{aligned}
\|\tilde{T}_j * f\|^2_{L^2(\mathbb{H}^n_a)} &\leq \sum_{k=1}^{\infty}\|\tilde{T}_j * f\|^2_{L^2(B(g_k,2))}\\
&\leq \sum_{k=1}^{\infty}\|\tilde{T}_j * \sum_{l=1}^{\infty} f_l\|^2_{L^2(B(g_k,2))}\\
&\leq \sum_{k=1}^{\infty}\|\tilde{T}_j * \sum_{\{l:\rho(g_l\cdot g_k^{-1})\leq 2\}} f_l\|^2_{L^2((B(g_k,2)))}\\
&\lesssim \sum_{k=1}^{\infty}\sum_{l:\rho(g_l\cdot g_k^{-1})\leq 2}\|\tilde{T}^{k,l}_j\|_{L^2\to L^2}\|f_l\|^2_{L^2}\\
&\lesssim \sup_{\rho(g_l\cdot g_k^{-1})\leq 2}\|\tilde{T}^{k,l}_j\|_{L^2\to L^2}\|f\|^2_{L^2}.
\end{aligned} \tag{2.6}$$

We note that

$$\det\left(D_{x,t}\left((x,t)\cdot g\right)\right) = 1 \text{ for all } g \in \mathbb{H}^n_a. \tag{2.7}$$

Then, using the coordinate change $(y,s) \to ((y,s)\cdot g_k)$ and substituting $(x,t) \to ((x,t)\cdot g_k)$ in (2.5), we get

$$\begin{aligned}
&\tilde{T}^{k,l}_j f\left((x,t)\cdot g_k\right)\\
&\qquad = \int \tilde{K}^j_{\alpha,\beta}\left((x,t)\cdot(y,s)^{-1}\right)\eta(\rho(x,t))\eta(\rho((y,s)\cdot(g_k\cdot g_l^{-1})))f((y,s)\cdot g_k)dyds.
\end{aligned} \tag{2.8}$$

Notice that $\rho(g_k\cdot g_l^{-1}) \lesssim 1$. Set $\psi\left((x,t),(y,s)\right) = \eta(\rho(x,t))\eta(\rho((y,s)\cdot(g_k\cdot g_l^{-1})))$ and write f just for $f(()\cdot g_k)$. Then $\sup_{\rho(g_l\cdot g_k^{-1})\leq 2}\|\tilde{T}^{k,l}_j\|$ will be achieved if we prove $\|\mathcal{T}_j\|_{L^2\to L^2} \lesssim 2^{jQ}$ for

$$\mathcal{T}_j f(x,t) = \int \tilde{K}^j_{\alpha,\beta}\left((x,t)\cdot(y,s)^{-1}\right)\psi\left((x,t),(y,s)\right) f(y,s)dyds \tag{2.9}$$

with a compactly supported smooth function ψ. Finally we set

$$\begin{aligned} A_j(x,t) =& 2^{j\alpha}\mu(x,t)e^{i2^{j\beta}\rho(x,t)^{-\beta}}, \\ B_j(x,t) =& 2^{-j\alpha}\mu(x,t)e^{i2^{j\beta}\rho(x,t)^{\beta}}, \end{aligned} \tag{2.10}$$

where μ is a smooth function supported on the set $\{(x,t) \in \mathbb{R}^{2n+1} : \frac{1}{10} \leq \rho(x,t) \leq 10\}$. We define the operators L_{A_j} and L_{B_j} by

$$L_{A_j}f(x,t) = \int A_j\left((x,t)\cdot(y,s)^{-1}\right)\psi\left((x,t),(y,s)\right)f(y,s)dyds,$$
$$L_{B_j}f(x,t) = \int B_j\left((x,t)\cdot(y,s)^{-1}\right)\psi\left((x,t),(y,s)\right)f(y,s)dyds.$$

We shall deduce Theorem 2.1.1 and Theorem 2.1.2 from the following propositions.

Proposition 2.2.2.

1. *If $a^2 > C_\beta$, then*
$$\|L_{A_j}\|_{L^2\to L^2} \lesssim 2^{j(\alpha-(n+\frac{1}{3})\beta)}, \qquad \forall j \in \mathbb{N}.$$
2. *If $a^2 = C_\beta$, then*
$$\|L_{A_j}\|_{L^2\to L^2} \lesssim 2^{j(\alpha-(n+\frac{1}{4})\beta)}, \qquad \forall j \in \mathbb{N}.$$

Proposition 2.2.3.

1. *If $0 < \beta < 1$, then,*
   (i) For $a^2 < C_\beta$,
$$\|L_{B_j}\|_{L^2\to L^2} \lesssim 2^{j(\alpha-(n+\frac{1}{2})\beta)} \qquad \forall j \in \mathbb{N}.$$
   (ii) For $a^2 = C_\beta$,
$$\|L_{B_j}\|_{L^2\to L^2} \lesssim 2^{j(\alpha-(n+\frac{1}{4})\beta)} \qquad \forall j \in \mathbb{N}.$$
   (iii) For $a^2 > C_\beta$,
$$\|L_{B_j}\|_{L^2\to L^2} \lesssim 2^{j(\alpha-(n+\frac{1}{3})\beta)} \qquad \forall j \in \mathbb{N}.$$
2. *If $1 < \beta < 2$, then*
$$\|L_{B_j}\|_{L^2\to L^2} \lesssim 2^{j(\alpha-(n+\frac{1}{3})\beta)} \qquad \forall j \in \mathbb{N}.$$
3. *If $2 < \beta$, then*
$$\|L_{B_j}\|_{L^2\to L^2} \lesssim 2^{j(\alpha-(n+\frac{1}{2})\beta)} \qquad \forall j \in \mathbb{N}.$$

We get the first main result of this paper assuming these propositions:

Proof of Theorem 2.1.1 and Theorem 2.1.2. From the reductions (2.4), (2.6) and (2.8), in order to prove Theorem 2.1.1 it is enough to prove that $\|\mathcal{T}_j\|_{L^2\to L^2} \lesssim 2^{jQ}$ for the operators T_j given in (2.9). From (2.3) and (2.10) we have $T_j = 2^{jQ}L_{A_j}$ with a suitable function μ, and so $\|\mathcal{T}_j\|_{L^2\to L^2} = 2^{jQ}\|L_{A_j}\|_{L^2\to L^2}$. Therefore, the estimates of Proposition 2.2.2 yield Theorem 2.1.1. In the same way, Proposition 2.2.3 establishes Theorem 2.1.2. □

In the next section, we shall briefly review on the theory related to the operators L_{A_j} and L_{B_j}. We will make use of geometric properties of the phase function $\rho(x,t)^\beta$ to prove Proposition 2.2.2 and Proposition 2.2.3.

2.3 L^2 estimates

We begin with the $L^2 \to L^2$ theory for oscillatory integral operators. The operators we are concern with are of the form

$$T_\lambda^\phi f(x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)}a(x,y)f(y)dy,$$

where $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $a \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Suppose that the phase function ϕ satisfies $\det\left(\frac{\partial^2\phi}{\partial x_i \partial y_j}\right) \neq 0$ on the support of a, we say that ϕ is non-degenerate. We say that ϕ is degenerate if there is some point (x_0, y_0) where $\det\left(\frac{\partial^2}{\partial x_i\partial y_j}\right)\Big|_{(x_0,y_0)}$ equals to zero. For non-degenerate phases, we have the fundamental theorem of Hörmander.

Theorem 2.3.1 ([Ho2])**.** *Suppose that the phase function ϕ is non-degenerate. Then we have*

$$\|T_\lambda^\phi\|_{L^2\to L^2} \lesssim \lambda^{-\frac{n}{2}} \quad \forall\lambda \in [1,\infty).$$

This theorem gives sharp decaying rate of the norm $\|T_\lambda^\phi\|_{L^2\to L^2}$ in terms of λ. However, the phase functions of our operators L_{A_j} and L_{B_j} can become degenerate according to the values of a and β (see Lemma 2.3.4 and Lemma 2.3.5). For a degenerate phase function ϕ, the optimal number κ_ϕ for which the inequality $\|T_\lambda\|_{L^2\to L^2} \lesssim \lambda^{-\kappa_\phi}$ holds would be less than $\frac{n}{2}$. The number κ_ϕ's are related to the type of fold of the phase ϕ (see Definition 2.3.2). For phases whose types of folds are ≤ 3, the sharp numbers κ_ϕ were obtained by Greenleaf-Seeger [GR] and Pan-Sogge [PS]. We shall use the results. The sharp results for folding types ≤ 3 in [GR] are the best known results and there are no optimal results for folding types > 3 except some special cases (see [Cm]).

It is well-known that the decaying property is strongly related to the geometry of the canonical relation,

$$C_\phi = \{(x, \partial_x\phi(x,y), y, -\partial_y\phi(x,y)) \; ; \; x, y \in \mathbb{R}^n\} \subset T^*(\mathbb{R}^n_x) \times T^*(\mathbb{R}^n_y). \tag{2.1}$$

Definition 2.3.2. Let M_1 and M_2 be smooth manifolds of dimension n, and let $f : M_1 \to M_2$ be a smooth map of corank ≤ 1. Let $S = \{P \in M_1 : \operatorname{rank}(Df) < n \text{ at } P\}$ be the singular set of f. Then we say that f has a $k-$type fold at a point $P_0 \in S$ if

1. $\operatorname{rank}(Df)|_{P_0} = n - 1$,

2. $\det(Df)$ vanishes of k order in the null direction at P_0.

Here, the null direction is the unique direction vector v such that $(D_v f)|_{P_0} = 0$.

Now we consider the two projection maps

$$\pi_L : C_\Phi \to T^*(\mathbb{R}^n_x) \quad \text{and} \quad \pi_R : C_\Phi \to T^*(\mathbb{R}^n_y). \tag{2.2}$$

Proposition 2.3.3 ([GR],[PS])**.** *Suppose that the projection maps π_L and π_R have 1-type folds (Whitney folds) singularities, then*

$$\|T_\lambda f\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-\frac{(n-1)}{2}-\frac{1}{3}}\|f\|_{L^2(\mathbb{R}^n)} \quad \forall \lambda \in [1, \infty).$$

If the projection maps π_L and π_R have 2-type folds singularities, then

$$\|T_\lambda f\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{-\frac{(n-1)}{2}-\frac{1}{4}}\|f\|_{L^2(\mathbb{R}^n)} \quad \forall \lambda \in [1, \infty).$$

In order to use Proposition 2.3.3, we shall study the projection maps (2.2) associated to the phase function of the operators L_{A_j} and L_{B_j}. Recall that $\rho(x, t) = (|x|^4 + t^2)^{1/4}$ and the phase function ϕ of the integral operators L_{A_j} and L_{B_j} is

$$\phi(x, t, y, s) = \rho^{-\beta}\left((x, t) \cdot (y, s)^{-1}\right).$$

To write the group law explicitly, we write $x = (x^1, x^2)$ and $y = (y^1, y^2)$ with $x^j, y^j \in \mathbb{R}^n$. Set $\Phi(x, t) = \rho(x, t)^{-\beta}$. Then

$$\phi(x, t, y, s) = \Phi\left(x^1 - y^1, x^2 - y^2, t - s - 2a(x^1y^2 - x^2y^1)\right). \tag{2.3}$$

For notational purpose set $t = x_{2n+1}$ and $s = y_{2n+1}$. To determine whether the phase function Φ is non-degenerate, we need to calculate the determinant of the matrix,

$$H = \left(\frac{\partial^2\phi(x, t, y, s)}{\partial y_i \partial x_j}\right).$$

The determinant is calculated in [LL]. However we give a somewhat simpler computation by considering the matrix L associated naturally with the matrix H (see below), which will also be useful in Lemma 2.3.6 and the proof of Proposition 2.2.2 and Proposition 2.2.3.

For simplicity, we write $(\mathbf{x}, \mathbf{t}) = (x, t) \cdot (y, s)^{-1}$. By the Chain Rule, for $1 \le i, j \le n$, we have

$$\frac{\partial}{\partial x_j}\phi(x,t,y,s) = \left[\partial_j + 2ay_{n+j}\partial_{2n+1}\right]\Phi(\mathbf{x},\mathbf{t}),$$
$$\frac{\partial}{\partial x_{j+n}}\phi(x,t,y,s) = \left[\partial_{j+n} - 2ay_j\partial_{2n+1}\right]\Phi(\mathbf{x},\mathbf{t}).$$

Using the Chain Rule once more, we get

$$\begin{aligned}
\frac{\partial}{\partial y_i}\frac{\partial}{\partial x_j}\phi(x,t,y,s) =& \left[(\partial_i + 2ax_{n+i}\partial_{\mathbf{t}})(\partial_j + 2ay_{n+j}\partial_{2n+1})\right]\Phi(\mathbf{x},\mathbf{t}),\\
\frac{\partial}{\partial y_{n+i}}\frac{\partial}{\partial x_j}\phi(x,t,y,s) =& \left[(\partial_{n+i} - 2ax_i\partial_{2n+1})(\partial_j + 2ay_{n+j}\partial_{2n+1})\right]\Phi(\mathbf{x},\mathbf{t}) + \left[2a\delta_{ij}\partial_{2n+1}\right]\Phi(\mathbf{x},\mathbf{t}),\\
\frac{\partial}{\partial y_i}\frac{\partial}{n+j}\phi(x,t,y,s) =& \left[(\partial_i + 2ax_{n+i}\partial_{2n+1})(\partial_{n+j} - 2ay_j\partial_{2n+1})\right]\Phi(\mathbf{x},\mathbf{t}) - \left[2a\delta_{ij}\partial_{2n+1}\right]\Phi(\mathbf{x},\mathbf{t}),\\
\frac{\partial}{\partial y_{n+i}}\frac{\partial}{\partial x_{n+j}}\phi(x,t,y,s) =& \left[(\partial_{n+i} - 2ax_i\partial_{2n+1})(\partial_{n+j} - 2ay_j\partial_{2n+1})\right]\Phi(\mathbf{x},\mathbf{t}).
\end{aligned} \tag{2.4}$$

Define

$$A_a(y) = \begin{pmatrix} I & 2aJy \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then we have

$$\begin{aligned}
H(x,t,y,s) =& A_a(x)\left(\partial_i\partial_j\Phi\right)(\mathbf{x},\mathbf{t})\,A_a(y)^T + 2a(\partial_{2n+1}\Phi)(\mathbf{x},\mathbf{t})\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}\\
=& A_a(x)\left[(\partial_i\partial_j\Phi) + 2a(\partial_{2n+1}\Phi)\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}\right](\mathbf{x},\mathbf{t})\,A_a(y)^T,
\end{aligned} \tag{2.5}$$

where the second equality holds because $A_a(x)\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}A_a(y)^T = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$. Set

$$L(x,t,y,s) = \left[(\partial_i\partial_j\Phi) + 2a(\partial_{2n+1}\Phi)\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}\right](\mathbf{x},\mathbf{t}). \tag{2.6}$$

Thus, to study the matrix H, it is enough to analyze the matrix L. Moreover we have $\det(A_a(x)) = \det(A_a(y)) = 1$ and it implies that $\det(H(x,t,y,s)) = \det(L(x,t,y,s))$. Therefore it is enough to calculate the determinant of L.

To find (2.6) we calculate the Hessian matrix of Φ. For $1 \le i, j \le 2n$,

$$\begin{aligned}
\partial_j\Phi(\mathbf{x},\mathbf{t}) &= -\frac{\beta}{4}(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-1}(4\mathbf{x}_j|\mathbf{x}|^2),\\
\partial_{2n+1}\Phi(\mathbf{x},\mathbf{t}) &= -\frac{\beta}{4}(|\mathbf{x}|^4 + \mathbf{t}^2)^{-\frac{\beta}{4}-1}(2\mathbf{t}),
\end{aligned}$$

and

$$\begin{aligned}
\partial_i\partial_j\Phi(\mathbf{x},\mathbf{t}) &= \beta(|\mathbf{x}|^4+\mathbf{t}^2)^{-\frac{\beta}{4}-2}\left[(\beta+4)|\mathbf{x}|^4-2(|\mathbf{x}|^4+\mathbf{t}^2)\right]\mathbf{x}_i\mathbf{x}_j-\beta(|\mathbf{x}|^4+\mathbf{t}^2)^{-\frac{\beta}{4}-1}\delta_{ij}|\mathbf{x}|^2,\\
\partial_i\partial_{2n+1}\Phi(\mathbf{x},\mathbf{t}) &= \beta(\beta+4)(|\mathbf{x}|^4+\mathbf{t}^2)^{-\frac{\beta}{4}-2}|\mathbf{x}|^2\mathbf{x}_i\cdot\frac{\mathbf{t}}{2},\\
\partial_{2n+1}^2\Phi(\mathbf{x},\mathbf{t}) &= \beta(\beta+4)(|\mathbf{x}|^4+\mathbf{t}^2)^{-\frac{\beta}{4}-2}\frac{\mathbf{t}}{2}\cdot\frac{\mathbf{t}}{2}-\beta(|\mathbf{x}|^4+\mathbf{t}^2)^{-\frac{\beta}{4}-1}\frac{1}{2}.
\end{aligned}$$

Set $D=(|\mathbf{x}|^2\mathbf{x},\frac{\mathbf{t}}{2})^T$. Then the above computations show that

$$\begin{aligned}
&\left[(\partial_i\partial_j\Phi)+2a(\partial_{2n+1}\Phi)\begin{pmatrix}J&0\\0&0\end{pmatrix}\right](\mathbf{x},\mathbf{t})\\
=&\beta(\beta+4)(|\mathbf{x}|^4+\mathbf{t}^2)^{-\frac{\beta}{4}-2}D\cdot D^T-\beta(|\mathbf{x}|^4+\mathbf{t}^2)^{-\frac{\beta}{4}-1}\begin{pmatrix}|\mathbf{x}|^2I+a\mathbf{t}J+2\mathbf{x}\cdot\mathbf{x}^T&0\\0&\frac{1}{2}\end{pmatrix}\\
=&-\beta(|\mathbf{x}|^4+\mathbf{t}^2)^{-\frac{\beta}{4}-1}(E+R),
\end{aligned}\tag{2.7}$$

where we set

$$B=|\mathbf{x}|^2I+a\mathbf{t}J,\quad K=\mathbf{x}\cdot\mathbf{x}^T,\quad E=\begin{pmatrix}B+2K&0\\0&\frac{1}{2}\end{pmatrix}\quad\text{and}\quad R=-\frac{(\beta+4)}{|\mathbf{x}|^4+\mathbf{t}^2}D\cdot D^T.\tag{2.8}$$

Then, from (2.6) and (9.87) we get

$$L(x,t,y,s)=[-\beta(|\mathbf{x}|^4+\mathbf{t}^2)^{-\frac{\beta}{4}-1}(E+R)]\,(\mathbf{x},\mathbf{t}).\tag{2.9}$$

Lemma 2.3.4. *We have*

$$\det H(x,t,y,s)=F((x,t)\cdot(y,s)^{-1}),$$

where $F(x,t)=c_{a,\beta}(|x|^4+a^2t^2)^{m_1}(|x|^4+t^2)^{m_2}f(x,t)$ *for some* $m_1,m_2,c_{a,\beta}\in\mathbb{R}$ *and* $f(x,t)=2(\beta+1)|x|^8+(3(\beta+2)-2a^2)|x|^4t^2+(\beta+2)a^2t^4$.

Proof. We write $(\mathbf{x},\mathbf{t})=(x,t)\cdot(y,s)^{-1}$ again. In view of (2.5), (2.6) and (2.9), it is enough to show that

$$\det[-\beta(|\mathbf{x}|^4+\mathbf{t}^2)^{-\frac{\beta}{4}-1}(E+R)]=F(\mathbf{x},\mathbf{t}).$$

Considering the form of the function F given, we only need to compute $\det(E+R)$. From (2.8) we have

$$E+R=\begin{pmatrix}B+2K&0\\0&\frac{1}{2}\end{pmatrix}-\frac{(\beta+4)}{|\mathbf{x}|^4+\mathbf{t}^2}D\cdot D^T.$$

For notational convenience, we shall use lower-case letters $f_1, \ldots, f_m$ to denote the rows of a given $m \times m$ matrix F. Notice that DD^T is of rank 1 and we have the following equality

$$\det(P+Q) = \det(P) + \sum_{j=1}^{m} \det(p_1^T, \ldots, p_{j-1}^T, q_j^T, p_{j+1}^T, \ldots, p_m^T), \tag{2.10}$$

for any $m \times m$ matrices P and Q with rank $Q = 1$. Recall that $B = |\mathbf{x}|^2 I + a\mathbf{t}J$ and $K = \mathbf{x} \cdot \mathbf{x}^T$, then direct calculations show that

$$\det(B) = (|\mathbf{x}|^4 + a^2\mathbf{t}^2)^n \tag{2.11}$$

and

$$\begin{aligned}
&\sum_{j=1}^{n} \mathbf{x}_j \det\left(b_1^T, \cdots, b_{j-1}^T, k_j^T, b_{j+1}^T, \cdots, b_{2n}^T\right) + \sum_{j=1}^{n} \mathbf{x}_{j+n} \det\left(b_1^T, \cdots, b_{j+n-1}^T, k_{j+n}^T, b_{j+n+1}^T, \cdots, b_{2n}^T\right) \\
&= \sum_{j=1}^{n} \mathbf{x}_j (|\mathbf{x}|^2 \mathbf{x}_j + \mathbf{x}_{n+j} a\mathbf{t})(|\mathbf{x}|^4 + a^2\mathbf{t}^2)^{n-1} + \sum_{j=1}^{n} \mathbf{x}_{j+n}(|\mathbf{x}|^2 \mathbf{x}_{j+n} - \mathbf{x}_j a\mathbf{t})(|\mathbf{x}|^4 + a^2\mathbf{t}^2)^{n-1} \\
&= (|\mathbf{x}|^4 + a^2\mathbf{t}^2)^{n-1} |\mathbf{x}|^4.
\end{aligned} \tag{2.12}$$

Thus, from (2.10), (2.11) and (2.12), we get

$$\begin{aligned}
\det(B+2K) &= (|\mathbf{x}|^4 + a^2\mathbf{t}^2)^n + 2|\mathbf{x}|^4 (|\mathbf{x}|^4 + a^2\mathbf{t}^2)^{n-1} \\
&= (|\mathbf{x}|^4 + a^2\mathbf{t}^2)^{n-1} (3|\mathbf{x}|^4 + a^2\mathbf{t}^2).
\end{aligned} \tag{2.13}$$

Using (2.10) once again, we obtain

$$\begin{aligned}
\det(E+R) &= \det(E) + \frac{1}{2} \sum_{j=1}^{2n} \det \begin{pmatrix} e_1 \\ \vdots \\ e_{j-1} \\ r_j \\ e_{j+1} \\ \vdots \\ e_{2n} \end{pmatrix} + \det \begin{pmatrix} e_1 \\ \vdots \\ e_{2n} \\ r_{2n+1} \end{pmatrix} \\
&=: S_1 + S_2 + S_3.
\end{aligned}$$

From (2.13) we have

$$S_1 = \det \begin{pmatrix} B+2K & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \det(B+2K) = \frac{1}{2} (|\mathbf{x}|^4 + a^2\mathbf{t}^2)^{n-1} (3|\mathbf{x}|^4 + a^2\mathbf{t}^2).$$

Using rank $K = 1$ we get

$$\det\begin{pmatrix} e_1 \\ \vdots \\ e_{j-1} \\ r_j \\ e_{j+1} \\ \vdots \\ e_{2n} \end{pmatrix} = \det\begin{pmatrix} b_1 + 2k_1 \\ \vdots \\ b_{j-1} + 2k_{j-1} \\ \frac{-(\beta+4)|\mathbf{x}|^4}{|\mathbf{x}|^4+\mathbf{t}^2} k_j \\ b_{j+1} + 2k_{j+1} \\ \vdots \\ b_{2n} + 2k_{2n} \end{pmatrix} = -\frac{(\beta+4)|\mathbf{x}|^4}{|\mathbf{x}|^4 + \mathbf{t}^2} \det\begin{pmatrix} b_1 \\ \vdots \\ b_{j-1} \\ k_j \\ b_{j+1} \\ \vdots \\ b_{2n} \end{pmatrix}.$$

Therefore,

$$S_2 = -\frac{1}{2}\left(\frac{(\beta+4)|\mathbf{x}|^4}{|\mathbf{x}|^4 + \mathbf{t}^2}\right)|\mathbf{x}|^4(|\mathbf{x}|^4 + a^2\mathbf{t}^2)^{n-1}.$$

Finally,

$$\begin{aligned} S_3 &= \det\begin{pmatrix} B + 2K & 0 \\ * & -\frac{\beta+4}{|\mathbf{x}|^4+\mathbf{t}^2}\frac{\mathbf{t}^2}{4} \end{pmatrix} = -\frac{\beta+4}{|\mathbf{x}|^4 + \mathbf{t}^2}\frac{\mathbf{t}^2}{4}\det(B + 2K) \\ &= -\frac{\beta+4}{|\mathbf{x}|^4 + \mathbf{t}^2}\frac{\mathbf{t}^2}{4}(|\mathbf{x}|^4 + a^2\mathbf{t}^2)^{n-1}(3|\mathbf{x}|^4 + a^2\mathbf{t}^2). \end{aligned}$$

Adding all these terms together, we get

$$\det(E + R) = p(|\mathbf{x}|^4 + a^2\mathbf{t}^2)\, q(|\mathbf{x}|^4 + \mathbf{t}^2)\, f(\mathbf{x}, \mathbf{t}),$$

where $p(r) = c_p r^{m_1}$, $q(r) = r^{m_2}$ for some $m_1, m_2, c_p \in \mathbb{R}$ and

$$f(\mathbf{x}, \mathbf{t}) = 2(\beta + 1)|\mathbf{x}|^8 + (3(\beta + 2) - 2a^2)|\mathbf{x}|^4\mathbf{t}^2 + (\beta + 2)a^2\mathbf{t}^4.$$

The proof is complete. □

Now, we should determine when the determinant of $H(x, t, y, s)$ can be zero for some values (x, t, y, s) with $\rho\left((x, t) \cdot (y, s)^{-1}\right) \sim 1$. Furthermore, to determine the type of folds in the degenerate cases, it is crucial to know the shape of the factorization.

Lemma 2.3.5. *There are nonzero constants γ, c, c_1, c_2, c_3 with $c_1 \neq c_2$ and $c_3 > 0$ that are determined by β and a such that:*

- *Case 1:*
  - · *If $\beta \in (-1, 0) \cup (0, \infty)$ and $a^2 < C_\beta$, then $f(x, t) > 0$.*
  - · *If $\beta \in (-1, 0) \cup (0, \infty)$ and $a^2 = C_\beta$, then $f(x, t) = \gamma(|x|^2 - ct^2)^2$.*

- *If $\beta \in (-1,0)\cup(0,\infty)$ and $a^2 > C_\beta$, then $f(x,t) = \gamma(|x|^2 - c_1 t)(|x|^2 + c_1 t)(|x|^2 - c_2 t)(|x|^2 + c_2 t)$.*

- *Case 2:*

  - *If $\beta \in (-2,-1)$, then $f(x,t) = \gamma(|x|^2 - c_1 t)(|x|^2 + c_1 t)(|x|^4 + c_3 t^2)$.*

- *Case 3:*

  - *If $\beta \in (\infty,-2)$, then $f(x,t) < 0$.*

Proof. Let $g(y,s) = 2(\beta+1)y^2 + (3(\beta+2)-2a^2)ys + (\beta+2)a^2 s^2$. Then $f(x,t) = g(|x|^4, t^2)$. Suppose $\beta \in (-1,0)\cup(0,\infty)$. First, we see that $f(x,t) > 0$ for $3(\beta+2) - 2a^2 > 0$. Secondly, we have $f(x,t) > 0$ if

$$\Delta := 4a^4 - 4(\beta+2)(2\beta+5)a^2 + 9(\beta+2)^2 < 0.$$

This holds if and only if

$$C_\beta^- < a^2 < C_\beta^+,$$

where

$$C_\beta^\pm = \frac{\beta+2}{2}\left(2\beta+5 \pm \sqrt{(2\beta+5)^2-9}\right).$$

Observe that

$$\begin{aligned} C_\beta^- = \frac{(\beta+2)}{2}(2\beta+5-\sqrt{(2\beta+5)^2-9}) &= \frac{(\beta+2)}{2}(2\beta+5-\sqrt{(2\beta+2)(2\beta+8)}) \\ &< \frac{(\beta+2)}{2}(2\beta+5-\sqrt{(2\beta+2)^2}) = \frac{3(\beta+2)}{2}. \end{aligned}$$

We can combine the above two conditions as $g(y,s) > 0$ for $a^2 < C_\beta^+$. For $a^2 = C_\beta$, we have $g(y,s) = \gamma(y-cs)^2$ for some $c > 0$. For $a^2 > C_\beta$, we have $g(y,s) = \gamma(y-c_1 s)(y-c_2 s)$ for some $c_1, c_2 > 0$ since $2(\beta+1)\cdot(\beta+2)a^2 > 0$.

Finally, if $\beta \in (-2,-1)$, then $2(\beta+1)(\beta+2)a^2 < 0$, and so $g(y,s) = \gamma(y-c_1 s)(y+c_2 s)$. If $\beta \in (-\infty,-2)$, then $2(\beta+1) < 0$, $3(\beta+2) - 2a^2 < 0$ and $\beta+2 < 0$. Thus $g(y,s) < 0$. This completes the proof. □

Lemma 2.3.6. *Let $L_1(x,t,y,s)$ be the upper left $(2n)\times(2n)$ block matrix of $L(x,t,y,s)$ and suppose that (x,t,y,s) is contained in S. If $\beta \neq -4$, then*

$$\det L_1(x,t,y,s) \neq 0.$$

Proof. For simplicity, set $(z,w) := (x,t)\cdot(y,s)^{-1}$. In view of (2.8) and (2.9), except the nonzero common facts, we only need to check that the determinant of

$$M(z,w) = \left(|z|^2 I + awJ + 2z\cdot z^T - (\beta+4)\tfrac{|z|^4}{|z|^4+w^2} x\cdot z^T\right),$$

is nonzero for $(z,w)\neq(0,0)$. This determinant can be calculated in the same way as the determinant of L by using (2.11) and (2.13). We find

$$\begin{aligned}\det(M(z,w)) =&(|z|^4+a^2w^2)^n + (|z|^4+a^2w^2)^{n-1}|z|^4\left(2-(\beta+4)\frac{|z|^4}{|z|^4+w^2}\right)\\ =&\frac{(|z|^4+a^2w^2)^{n-1}}{|z|^4+w^2}\left[-(\beta+1)|z|^8+(a^2+3)|z|^4w^2+a^2w^4\right].\end{aligned} \tag{2.14}$$

Notice that (z,w) is in S and satisfies

$$2(\beta+1)|z|^8 + (3(\beta+2)-2a^2)|z|^4w^2 + (\beta+2)a^2w^4 = 0. \tag{2.15}$$

From (2.14) and (2.15) we get

$$\det(M(z,w)) = \frac{(|z|^4+a^2w^2)^{n-1}}{|z|^4+w^2}\frac{w^2}{2}(\beta+4)\left[3|z|^4+a^2w^2\right].$$

If $w = 0$, then z becomes zero in (2.15). Because $(z,w)\neq(0,0)$, w should be nonzero. Thus $\det(M(z,w))\neq 0$. The Lemma is proved. □

We are now ready to prove our first main theorems by studying the canonical relation (2.1) associated to the phase Φ,

$$C_\Phi = \{\left((x,t),\Phi_{(x,t)},(y,s),-\Phi_{(y,s)}\right)\}\subset T^*(\mathbb{R}^{2n+1})\times T^*(\mathbb{R}^{2n+1}),$$

and the associated projection maps $\pi_L : C_\Phi \to T^*(\mathbb{R}^{2n+1})$ and $\pi_R : C_\Phi\to T^*(\mathbb{R}^{2n+1})$.

Proof of Proposition 2.2.2 Proposition 2.2.3. Let

$$S = \{(x,t,y,s) : \det H(x,t,y,s) = 0\}.$$

In view of Proposition 2.3.3, it is enough to show that on the hypersurface S,

1. If $\beta\in(-2,-1)$ or $\beta\in(-1,0)\cup(0,\infty)$ and $a^2 > C_\beta$, then both projections π_L and π_R have 1-type folds singularities.
2. If $\beta\in(-1,0)\cup(0,\infty)$ and $a^2 = C_\beta$, then both π_L and π_R have 2-type folds singularities.

We will only prove (1). The second case can be proved in the same way, the only difference is the form of factorizations in Lemma 2.3.5 which determine the order of types. We need to show that on the hypersurface S, both π_L and π_R have 1-type folds singularities. Rcall from Lemma 2.3.4 that S is a subset of $\mathbb{R}^{2n+1}$ consisting of $(x,t,y,s) \in \mathbb{R}^{2(2n+1)}$ such that

$$F\left((x,t)\cdot(y,s)^{-1}\right) = F\left(x-y, s-t+2ax^T Jy\right) = 0 \quad \text{and} \quad \rho\left((x,t)\cdot(y,s)^{-1}\right) \sim 1.$$

From the form of F and the fact that $((x,t)\cdot(y,s)^{-1}) \neq 0$, we have

$$S = \{(x,t,y,s) \in \mathbb{R}^{2(2n+1)} \mid f\left(x-y, s-t+2ax^T Jy\right) = 0, \quad \rho\left((x,t)\cdot(y,s)^{-1}\right) \sim 1\}.$$

From Theorem 2.3.5, we have

$$f(x,t) = \gamma(|x|^2 - c_1 t)(|x|^2 + c_1 t)(|x|^2 - c_2 t)(|x|^2 + c_2 t).$$

for some two different constants $c_1, c_2 > 0$.

Note that Lemma 2.3.6 implies the condition (1) of Definition 2.3.2 is satisfied. Therefore, it is enough to show the second condition, i.e., at each point $P_0 \in S$ the determinant of Df vanishes with order 1 in each null direction of $d\pi_L$ and $d\pi_R$ at P_0. Fix a point $P_0 = (x,t,y,s) \in \mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}$ and assume that P_0 is contained in

$$S_1 =: \{(x,t,y,s) \in \mathbb{R}^{2(2n+1)} \mid |x-y|^2 - c_1(s-t+2ax^T Jy) = 0\}.$$

We may identify $C_\Phi = \{\left((x,t), \Phi_{(x,t)}, (y,s), -\Phi_{(y,s)}\right)\}$ with an open set in $\mathbb{R}^{(2n+1)} \times \mathbb{R}^{(2n+1)}$ by the diffeomorphsim $\psi : \mathbb{R}^{(2n+1)} \times \mathbb{R}^{(2n+1)} \to S$ given by

$$\psi(x,t,y,s) = \left((x,t), \Phi_{(x,t)}, (y,s), -\Phi_{(y,s)}\right).$$

Let $v_L \in \mathbb{R}^{2(2n+1)}$ be a null direction of $d\pi_L$ at P_0. i.e.,

$$\begin{pmatrix} I & 0 \\ \frac{\partial^2\Phi}{\partial_{(x,t)}\partial_{(x,t)}} & \frac{\partial^2\Phi}{\partial_{(y,s)}\partial_{(x,t)}} \end{pmatrix} v_L^T = 0.$$

Thus, v_L is of the form $v_L = (0,0,z,w)$ with $w \in \mathbb{R}^{2n}$ and $s \in \mathbb{R}$ such that

$$\frac{\partial^2\Phi}{\partial_{(y,s)}\partial_{(x,t)}} \begin{pmatrix} z^T \\ w \end{pmatrix} = 0. \tag{2.16}$$

To check that $\det H(x,t,y,s)$ vanishes of order 1 in the direction v_L, it is enough to show that v_L is not orthogonal to the gradient vector v_g of $\det H(x,t,y,s)$ at P_0. By a direct calculation we see that the gradient vector v_g is equal to

$$D_{(x,t),(y,s)}\Phi\left((x,t)\cdot(y,s)^{-1}\right)\Big|_p = \left(2(x-y) - 2ac_1 aJy,\ -c_1,\ -2(x-y) - 2ac_1 x^T J,\ c_1\right)$$

Suppose with a view to contradiction that v_L and v_g are orthogonal. It means that

$$-2(x-y)\cdot z - 2ac_1 x^T J \cdot z + c_1 w = 0. \tag{2.17}$$

From (2.5), we have

$$\frac{\partial^2 \Phi}{\partial_{(y,s)}\partial_{(x,t)}}\begin{pmatrix} z^T \\ w \end{pmatrix} = A_a(y)\left[(\partial_i\partial_j\Phi) - 2a(\partial_{2n+1}\Phi)\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}\right](\mathbf{x},\mathbf{t})\, A_a(x)^T \cdot \begin{pmatrix} z^T \\ w \end{pmatrix}. \tag{2.18}$$

A simple calculation shows that

$$\begin{aligned} A_a(x)^T \cdot \begin{pmatrix} z^T \\ w \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & \vdots \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ 2ax_{n+1} & \cdots & -2ax_1 & \cdots & -2ax_n & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n} \\ w \end{pmatrix} \\ &= \begin{pmatrix} z_1, & z_2, & \cdots, & z_{2n}, & 2a(x_{n+1}z_1 + \cdots + x_{2n}z_n - x_1 z_{n+1} - \cdots - x_n z_{2n}) + w \end{pmatrix}^T. \end{aligned}$$

On the other hand, from the orthogonal assumption (2.17) we get

$$2a(x_{n+1}z_1 + \cdots - x_n z_{2n}) + w = \frac{2(x-y)\cdot z}{c_1}.$$

Thus,

$$A_a(x)^T \cdot \begin{pmatrix} z^T \\ w \end{pmatrix} = \begin{pmatrix} z_1, & z_2, & \cdots, & z_{2n}, & \frac{2(x-y)\cdot z}{c_1} \end{pmatrix}^T.$$

Recall that

$$\left[(\partial_i\partial_j\Phi) - 2a(\partial_{2n+1}\Phi)\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}\right](x,t) = (\beta+4)\begin{pmatrix} |x|^4 x_1^2 & \cdots & |x|^4 x_1 x_n & |x|^2 x_1 \frac{t}{2} \\ \vdots & \ddots & \vdots & \vdots \\ |x|^4 x_n x_1 & \cdots & |x|^4 x_n^2 & |x|^2 x_n \frac{t}{2} \\ |x|^2 \frac{t}{2} x_1 & \cdots & |x|^2 \frac{t}{2} x_n & \frac{t^2}{4} \end{pmatrix} - (|x|^4 + t^2)\begin{pmatrix} J & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Substituting $x-y$ for x and $t - s + 2ax^T Jy = \frac{|x-y|^2}{c_1}$ for t, where the equality holds since the point P_0 is on the surface S_1. Then, from $(2n+1)$-th equality in (2.16) with (2.18), we have

$$(\beta+4)\left[|x-y|^2 \cdot \frac{1}{2}\frac{|x-y|^2}{c_{\beta,1}}(x-y)\cdot z + \frac{|x-y|^4}{c_{\beta,1}^2}\cdot\frac{2}{c_{\beta,1}}(x-y)\cdot z\right] - \frac{1}{2}(|x-y|^4 + \frac{|x-y|^4}{c_{\beta,1}^2})\frac{2}{c_{\beta,1}}(x-y)\cdot z = 0.$$

Rearranging it, we obtain

$$\left[\frac{\beta+2}{2c_{\beta,1}} + \frac{1}{c_{\beta,1}^3}\right]|x-y|^4\,(x-y)\cdot z = 0.$$

Thus $(x-y)\cdot z = 0$, and hence

$$A_a(x)^T \cdot \begin{pmatrix} z \\ w \end{pmatrix} = \Big(z_1, z_2, \cdots, z_{2n}, 0\Big)^T \quad \text{and} \quad L_1(x,t,y,s)\cdot \Big((z_1, z_2, \cdots, z_{2n})\Big)^T = 0.$$

Now from $\det L_1 \neq 0$ in Lemma 2.3.6 we have $z = 0$ and so $w = 0$ from (2.17). This is a contradiction since v_L should be a nonzero direction vector. Therefore v_L and v_R can not be orthogonal.

Now we shall prove the same conclusion for $d\pi_R$ without repeating the calculations. Note that the above argument for $d\pi_L$ is exactly to show that there is no nontrivial solution (z, w) of the system of equation $S(a, x, y)$:

$$(\frac{\partial^2}{\partial x_i \partial y_j}\Phi)\begin{pmatrix} z^T \\ w \end{pmatrix} = A_a(y)\left[(\partial_i\partial_j\Phi) - 2a(\partial_{2n+1}\Phi)\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}\right] A_a(x)^T \cdot \begin{pmatrix} z^T \\ w \end{pmatrix} = 0,$$

and

$$(-2(x-y) - 2ac_{\beta,1}x^T J, c_{\beta,1})\cdot(z, w) = 0.$$

On the other hand, to show the folding type condition for the projection π_R, it is enough to show that there is no nontrivial solution $v_R = (z_0, w_0, 0, 0)$ which satisfies the system of equations :

$$(\frac{\partial^2}{\partial y_i \partial x_j}\Phi)\begin{pmatrix} z_0^T \\ w_0 \end{pmatrix} = A_a(x)\left[(\partial_i\partial_j\Phi) + 2a(\partial_{2n+1}\Phi)\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}\right] A_a(y)^T \cdot \begin{pmatrix} z_0^T \\ w_0 \end{pmatrix} = 0,$$

and

$$\Big(2(x-y) + 2ac_{\beta,1}y^T J,\ -c_{\beta,1}\Big)\cdot(z_0, w_0) = 0.$$

Because $A_{-a}(-x) = A_a(x)$ and $A_{-a}(-y) = A_a(y)$, the above system can be written as follows.

$$(\frac{\partial^2}{\partial y_i \partial x_j}\Phi)\begin{pmatrix} z_0^T \\ w_0 \end{pmatrix} = A_{-a}(-x)\left[(\partial_i\partial_j\Phi) - 2(-a)(\partial_{2n+1}\Phi)\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}\right] A_{-a}(-y)^T \cdot \begin{pmatrix} z_0^T \\ w_0 \end{pmatrix} = 0,$$

and

$$\Big(-2((-y) - (-x)) - 2(-a)c_{\beta,1}(-y)^T J,\ c_{\beta,1}\Big)\cdot(z_0, w_0) = 0.$$

We now see that (z_0, w_0) satisfies the system $S(-a, -y, -x)$. Since the above argument for proving nonexistence of nontrivial solution of $S(a, x, y)$ does not depend on specific values of a, x and y, the same conclusion holds for the system $S(-a, -y, -x)$. This completes the proof. □

Remark 2.3.7. On $\mathbb{R}^n$, the oscillating kernel is of the form $|x|^{-\gamma}e^{i|x|^\beta}$ with $\beta \neq 0$. The behavior for the phases $|x|^\beta$ depends only on whether $\beta \neq 1$ or $\beta = 1$. Precisely, for $\beta \neq 1$, we have $\det\left(\frac{\partial^2}{\partial x \partial y}|x-y|^\beta\right) \neq 0$ for any (x, y) with $x \neq y$, but $\det\left(\frac{\partial^2}{\partial x \partial y}|x-y|\right) = 0$ for any (x, y) with $x \neq y$ and this case correspond to Bochner-Riesz means operators, which still remains as a conjecture. On hand, the phase $\rho((x,t)\cdot(y,s)^{-1})^\beta$ has fold of the highest order type when $\beta = 1$ or $\beta = 2$, which also remains open in this paper. In order to establish the sharp L^2 estimate for these cases, we would need to improve the current theory of oscillatory integral estimates for degenerate phases to higher orders (see [Cm, GR, GR2]).

Remark 2.3.8. We note that from Lemma 2.3.6 and the case 3 in Lemma 2.3.5,

$$\|L_{A_j}\|_{L^2\to L^2} + \|L_{B_j}\|_{L^2\to L^2} \lesssim 2^{j(\alpha-n\beta)} \tag{2.19}$$

holds for all cases. It will be sufficient to use this weaker bound for the Hardy spaces estimates in Section 5.

2.4 Hardy spaces on the Heisenberg groups

In this section we recall some properties of Hardy spaces on the Heisenberg group. We refer [CW2, **?**] for the details. From now on, we shall write $\rho(x)$ (resp., $x \cdot y$) just as $|x|$ (resp., xy) for the notational convenience. It is known that $|x\cdot y| \leq |x| + |y|$ holds for all $x, y \in \mathbb{H}_a$ (see [Lin]).

The left-invariant vector fields on $\mathbb{H}^n_a$ is spanned by $T = \frac{\partial}{\partial t}$ and $X_j = \frac{\partial}{\partial x_j} + 2ax_{n+j}\frac{\partial}{\partial t}$, $\quad X_{j+n} = \frac{\partial}{\partial x_{j+n}} - 2ax_n\frac{\partial}{\partial t}$, $1 \leq j \leq n$. Let $Y_j = X_j$ for $1 \leq j \leq 2n$ and $Y_{2n+1} = T$. We say that the right-invariant differential operator $Y^I = Y_1^{i_1}\cdots Y_{2n+1}^{i_{2n+1}}$ has homogeneous degree $d(I) = i_1 + i_2 + \cdots + i_{2n} + 2i_{2n+1}$. For $a \in \bar{\mathbb{N}}$, we define $\mathcal{P}_a$ to be the set of all homogeneous polynomials of degree a.

Suppose that $x \in \mathbb{H}^n$, $a \in \bar{\mathbb{N}}$, and f is a function whose distributional derivatives $Y^I f$ are continuous in a neighborhood of x for $d(I) \leq a$. The homogeneous right *Taylor polynomial* of f at x of degree a is the unique $P_{f,x} \in \mathcal{P}_a$ such that $Y^I P_{f,x}(0) = Y^I f(x)$ for $d(I) \leq a$.

Proposition 2.4.1 ([FoS]). *Suppose that $f \in C^{k+1}$, $T \in \mathcal{S}'$, and $P_{f,x}(y) = \sum_{d(I)\leq k} a_I(x)\eta^I(y)$ is the right Taylor polynomial of f at x of homogeneous degree k. Then a_I is a linear combination of the $Y^J f$ for $d(J) \leq k$,*

$$|f(yx) - P_{f,x}(y)| \leq C_k|y|^{k+1} \sup_{\substack{d(I)=k+1 \\ |z|\leq b^{k+1}|y|}} |Y^I f(zx)|. \tag{2.1}$$

We will use some properties for H^p functions including the atomic decomposition and the molecular characterization. For $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, $s \in \mathbb{Z}$ and $s \geq [(2n+2)(1/p-1)]$, we say that the triple (p, q, s) is admissible.

Definition 2.4.2. For an admissible triple (p, q, s), we define (p, q, s)-atom centered at x_0 as a function $a \in L^q(\mathbb{H}^n)$ supported on a ball $B \subset \mathbb{H}^n$ with center x_0 in such way that

(i) $\|a\|_q \leq |B|^{1/q-1/p}$.

(ii) $\int_{\mathbb{H}^n} a(x)P(x)dx = 0$ for all $P \in \mathcal{P}_s$.

Later, we will choose $q = 2$ to use the L^2 boundedness (2.19) obtained in Section 3.

Proposition 2.4.3 (Atomic decomposition in $\mathbb{H}^p$; see [CW2])**.** *Let (p, q, s) be an admissible triple. Then any f in H^p can be represented as a linear combination of (p, q, s)-atoms,*

$$f = \sum_{i=1}^{\infty} \lambda_i f_i, \qquad \lambda_i \in \mathbb{C},$$

where the f_i are (p, q, s)-atoms and the sum converges in H^p. Moreover, $\|f\|_{H^p}^p \sim \inf\{\sum_{i=1}^{\infty} |\lambda_i|^p$: $\sum \lambda_i f_i$ is a decomposition of f into (p,q,s)-atoms$\}$.

For an admissible triple (p, q, s), we choose an arbitrary real number $\epsilon > \max\{s/(2n+2), 1/p-1\}$. Then we call (p, q, s, ϵ) an admissible quadruple. Now we introduce the molecules.

Definition 2.4.4. Let (p, q, s, ϵ) be an admissible quadruple. We set

$$a = 1 - 1/p + \epsilon, \quad b = 1 - 1/q + \epsilon. \tag{2.2}$$

A (p, q, s, ϵ)-molecule centered at x_0 is a function $M \in L^q(\mathbb{H}^n)$ such that

1. $M(x) \cdot |x_0^{-1}x|^{(2n+2)b} \in L^q(\mathbb{H}^n)$.
2. $\|M\|_q^{a/b} \cdot \|M(x) \cdot |x_0^{-1}x|^{(2n+2)b}\|_q^{1-a/b} \equiv \mathcal{N}(M) < \infty$.
3. $\int_{\mathbb{H}^n} M(x)P(x)dx = 0$ for every $P \in \mathcal{P}_s$.

Theorem 2.4.5.

1. *Every (p, q, s')-atom f is a (p, q, s, ϵ)-molecule for any $\epsilon > \max\{s/(2n+2), 1/p-1\}$, $s \leq s'$ and $\mathcal{N}(f) \leq C_1$, where the constant C_1 is independent of the atom.*
2. *Every (p, q, s, ϵ)-molecule M is in H^p and $\|M\|_{H^p} \leq C_2\mathcal{N}(M)$, where the constant C_2 is independent of the molecule.*

Thanks to this Theorem, in order to verify that T is bounded on H^p it is enough to show that, for all p-atoms f, the function Tf is a p-molecule and $\mathcal{N}(Tf) \leq C$ for some constant C independent of f.

2.5 H^p estimates

We start with a lemma which will be useful in the proofs of the sequel.

Lemma 2.5.1.

1. *Suppose that* $d < 0$*,* $c + d < 0$ *and* $B > 1$*. Then*

$$\sum_{j=1}^{\infty} 2^{cj} \min\{1, B2^{dj}\} \lesssim 1 + (\log B)B^{-\frac{c}{d}}.$$

2. *Suppose that* $c < 0$*,* $d > 0$ *and* $B < 1$*. Then*

$$\sum_{j=1}^{\infty} 2^{cj} \min\{1, B2^{dj}\} \lesssim B + |\log B|B^{-\frac{c}{d}}.$$

Proof. Set $K = \sum_{j=1}^{\infty} 2^{cj} \min\{1, B2^{dj}\}$. Then,

$$K = \sum_{B2^{dj} \leq 1} 2^{(c+d)j} + \sum_{B2^{dj} > 1} 2^{cj}.$$

A straighforward calculation gives the bound for K. Suppose that $d < 0, c + d > 0$ and $B > 1$. Then

- $K \lesssim 1$ for $c < 0$,
- $K \lesssim \log B$ for $c = 0$,
- $K \lesssim B^{-\frac{c}{d}}$ for $c > 0$.

In any case we see that $K \lesssim 1 + (\log B)B^{-\frac{c}{d}}$. Suppose now that $c < 0, d > 0$ and $B < 1$. Then

- $K \lesssim B$ for $c + d < 0$,
- $K \lesssim \log B \cdot B$ for $c + d = 0$,
- $K \lesssim B^{-\frac{c}{d}}$ for $c + d > 0$.

In any case we have $K \lesssim B + |\log B|B^{-\frac{c}{d}}$. The Lemma is proved. □

Theorem 2.5.2. *Assume* $p \leq 1$ *and* $(\frac{1}{p} - 1)(2n + 2)\beta + \alpha < 0$*. Then* $T_{K_{\alpha,\beta}}$ *is bounded on* H^p*.*

Proof. From the decompostion of kernel (2.1), we have

$$\|K_{\alpha,\beta} * f\|_{H^p}^p \le \sum_{j\ge 1} \|K_{\alpha,\beta}^j * f\|_{H^p}^p.$$

We shall bound the norm $\|K_{\alpha,\beta}^j * f\|_{H^p}$ for each $j \in \mathbb{N}$ by some constant multiple of $\|f\|_{H^p}$. Notice that $K_j(x,t) = \rho(x,t)^{-(2n+2+\alpha)} e^{i\rho(x,t)^{-\beta}} \chi(2^j \rho(x,t))$. From the atomic decomposition for H^p space, it is enough to establish the estimate for any atom f supported on $B(0,R)$ with some $R > 0$ such that

$$\begin{aligned} &- \|f\|_{L^2} \le R^{(2n+2)(\frac{1}{2}-\frac{1}{p})}, \\ &- \int f(x) x^\alpha dx = 0, \quad \text{for all } |\alpha| \le s = [(2n+2)(\frac{1}{p} - 1)]. \end{aligned} \tag{2.1}$$

In view of part (2) of Theorem 2.4.5 it suffices to bound $\mathcal{N}(K_j * f)$. For an admissible quadruple, we choose an $\epsilon > \max\{\frac{s}{2n+2}, \frac{1}{p} - 1\} = \frac{1}{p} - 1$ and set $\epsilon = \frac{1}{p} - 1 + \delta$ with some $\delta > 0$. Then we have $a = \delta$ and $b = \frac{1}{p} - \frac{1}{2} + \delta$ in (2.2). We will choose δ sufficiently small later. Recall that $\mathcal{N}(K_j * f) = \|K_j * f\|_2^{a/b} \cdot \|K_j * f(x) \cdot |x|^{(2n+2)b}\|_2^{1-a/b}$. From the L^2 estimate (2.19) we get

$$\|K_j * f\|_2 \lesssim 2^{j(\alpha - n\beta)} \|f\|_2. \tag{2.2}$$

We have

$$\|K_j * f(x) \cdot |x|^{(2n+2)b}\|_2^2 = \int_{\mathbb{H}^n} |K_j * f(x)|^2 \cdot |x|^{2(2n+2)b}\, dx = I_1 + I_2,$$

where

$$I_1 = \int_{|x|\le 2R} |K_j * f(x)|^2 \cdot |x|^{2(2n+2)b}\, dx \quad \text{and} \quad I_2 = \int_{|x|>2R} |K_j * f(x)|^2 \cdot |x|^{2(2n+2)b}\, dx.$$

Then

$$\begin{aligned} \sum_{j\ge 1} \|K_j * f\|_{H^p}^p &\lesssim \sum_{j\ge 1} \mathcal{N}(K_j * f)^p \\ &\lesssim \sum_{j\ge 1} \Big(\|K_j * f\|_2^{a/b} \cdot (I_1^{1/2(1-a/b)} + I_2^{1/2(1-a/b)}) \Big)^p \\ &\lesssim \sum_{j\ge 1} \|K_j * f\|_2^{pa/b} \cdot I_1^{p/2(1-a/b)} + \sum_{j\ge 1} \|K_j * f\|_2^{pa/b} \cdot I_2^{p/2(1-a/b)} \end{aligned} \tag{2.3}$$

Set $S_1 = \sum_{j\ge 1} \|K_j * f\|_2^{pa/b} \cdot I_1^{p/2(1-a/b)}$ and $S_2 = \sum_{j\ge 1} \|K_j * f\|_2^{pa/b} \cdot I_2^{p/2(1-a/b)}$. Then it is enough to show that $S_1 \lesssim 1$ and $S_2 \lesssim 1$. We use (2.19) and (2.1) to bound I_1 as follows.

$$\begin{aligned} I_1 &\lesssim \int_{\mathbb{H}^n} |f * K_j(x)|^2 dx \cdot R^{2(2n+2)b} \lesssim 2^{2j(\alpha - n\beta)} \|f\|_2^2 \cdot R^{2(2n+2)b} \\ &\lesssim 2^{2j(\alpha - n\beta)} R^{2(2n+2)b} \cdot R^{(2n+2)(1-2/p)} \\ &\lesssim 2^{2j(\alpha - (n+1/2)\beta)} R^{2(2n+2)\delta}, \end{aligned} \tag{2.4}$$

where the last inequality comes from (2.1). From (2.2) and (2.4) we have

$$\begin{aligned}\|K_j * f\|_2^{a/b} \cdot I_1^{1/2(1-a/b)} &\lesssim \left\{2^{j(\alpha-n\beta)} R^{(2n+2)(1/2-1/p)}\right\}^{a/b} \cdot \left\{2^{j(\alpha-n\beta)} \cdot R^{(2n+2)\delta}\right\}^{(1-a/b)} . \\ &= 2^{j(\alpha-n\beta)},\end{aligned}$$

where the equality comes from the calculation $(\frac{1}{2}-\frac{1}{p})\frac{a}{b}+a(1-\frac{a}{b}) = \frac{a}{b}(\frac{1}{2}-\frac{1}{p}-a)+a = \frac{a}{b}(-b)+a = 0$. Thus we have $S_1 \lesssim \sum_{j\geq 1} 2^{j(\alpha-n\beta)p} \lesssim 1$.

Now we consider I_2 and S_2. We have $I_2 = 0$ for $R > 1$ since the support of $K_j * f$ is contained in the subset $\{x : |x| \leq 1 + R\}$ which is a subset of $\{x : |x| < 2R\}$ for $R > 1$. Thus we may only consider the case $R \leq 1$. In the following integral expression

$$(K_j * f)(x) = \int K_j(xy^{-1}) f(y) dy,$$

We have $|xy^{-1}| \leq 2^{-j}$ and $|y| \leq R$. These imply $|x| \leq |xy^{-1}| + |y| \leq 2^{-j} + R$. It means that $I_2 = 0$ for $2^{-j} < R$. Thus we only need to consider $j \in \mathbb{N}$ such that $2^{-j} \geq R$, for which we have $|x| \leq 2^{-j+1}$ for $x \in \mathrm{Supp}(K_j * f)$. Then we get

$$I_2 = \int_{|x|>2R} |f * K_j(x)|^2 \cdot |x|^{2(2n+2)b} dx \lesssim \int_{|x|>2R} |f * K_j(x)|^2 dx \cdot 2^{-2(2n+2)bj}. \tag{2.5}$$

From Proposition 2.4.1, for any $I \in \mathbb{N}_0$, there is a polynomial P_j^x of degree $\leq I$ such that

$$\begin{aligned}|K_j(xy^{-1}) - P_j^x(y)| &\lesssim |y|^{I+1} \sup_{|\alpha|\leq I+1} |X^\alpha K_j(xy^{-1})| \\ &\lesssim |y|^{I+1} 2^{j(2n+2+\alpha)} 2^{j(\beta+1)(I+1)}.\end{aligned} \tag{2.6}$$

From (2.1) we get the identity for $0 \leq I \leq s$,

$$K_j * f(x) = \int (K_j(xy^{-1}) - P_j^x(y)) f(y) dy.$$

Note that $f(y)$ has support in $|y| \leq R$, then from (2.1) and (2.6) we get

$$\begin{aligned}|K_j * f(x)| &\lesssim R^{I+1} 2^{j(2n+2+\alpha)} 2^{j(\beta+1)(I+1)} \int_{|y|\leq R} |f(y)| dy \\ &\lesssim R^{I+1} 2^{j(2n+2+\alpha)} 2^{j(\beta+1)(I+1)} R^{\frac{1}{2}(2n+2)} \|f\|_2 \\ &\lesssim 2^{j(2n+2+\alpha)} (R 2^{j(\beta+1)})^{(I+1)} R^{(2n+2)(1-\frac{1}{p})}.\end{aligned}$$

Now we can estimate (2.5) as

$$\begin{aligned}I_2 &\lesssim 2^{-2(2n+2)bj} 2^{-j(2n+2)} \left\{2^{j(2n+2+\alpha)} (R 2^{j(\beta+1)})^{(I+1)} R^{(2n+2)(1-\frac{1}{p})}\right\}^2 . \\ &= 2^{2j\{(2n+2)(1-\frac{1}{p}-\delta)+\alpha\}} (R 2^{j(\beta+1)})^{2(I+1)} R^{2(2n+2)(1-\frac{1}{p})}.\end{aligned}$$

Here we may choose $I = 0$ or $I = s$, which gives

$$I_2 \lesssim 2^{2j\{(2n+2)(1-\frac{1}{p}-\delta)+\alpha\}} R^{2(2n+2)(1-\frac{1}{p})} \min\{1, (R2^{j(\beta+1)})^{2(s+1)}\}.$$

Now we have

$$\begin{aligned} \|K_j * f\|_2^{a/b} \cdot I_2^{\frac{1}{2}(1-a/b)} &\lesssim \{2^{j(\alpha-n\beta)} R^{(2n+2)(1/2-1/p)}\}^{a/b} \\ &\cdot \{2^{j\{(2n+2)(1-\frac{1}{p}-\delta)+\alpha\}} R^{(2n+2)(1-\frac{1}{p})} \min\left(1, (R2^{j(\beta+1)})^{(s+1)}\right)\}^{(1-a/b)}. \end{aligned} \tag{2.7}$$

From $p \leq 1$ and $\alpha < 0$ we have $(2n+2)(1-\frac{1}{p}-\delta)+\alpha < 0$. Thus, if $\min(1, (R2^{j(\beta+1)})^{s+1}) = 1$ the exponent of 2^j is smaller than zero provided a is small enough. Recall that $R \leq 1$. Then, using (2) in Lemma 2.5.1 we get

$$\sum_{j\geq 1} \|K_j * f\|_2^{pa/b} \cdot I_2^{\frac{p}{2}(1-a/b)} \quad \lesssim \quad R^{p\mu_\delta} + |\log R| \cdot R^{p\kappa_\delta},$$

where

$$\begin{aligned} R^{p\mu_\delta} &= (R^{(2n+2)(1/2-1/p)})^{pa/b} \cdot (R^{(2n+2)(1-\frac{1}{p})+(s+1)})^{p(1-a/b)}, \\ R^{p\kappa_\delta} &= \left[R^{-\frac{1}{\beta+1}[\alpha-(n+1/2)\beta]} R^{(2n+2)(1/2-1/p)}\right]^{p\delta/b} \cdot \left[R^{-\frac{1}{\beta+1}[(2n+2)(1-1/p-\delta)+\alpha]} R^{(2n+2)(1-1/p)}\right]^{p(1-\delta/b)}. \end{aligned}$$

Observe that

$$\mu_0 = \{(2n+2)(1-\frac{1}{p}) + (s+1)\} > 0,$$

and

$$\kappa_0 = -\frac{1}{\beta+1}[(2n+2)\beta(\frac{1}{p}-1)+\alpha] > 0.$$

Thus, for δ small enough, we have $\mu_\delta, \kappa_\delta > 0$ and since $R \leq 1$,

$$\sum_{j\geq 1} \|K_j * f\|_2^{pa/b} \cdot I_2^{\frac{p}{2}(1-a/b)} \lesssim R^{p\mu_\delta} + |\log R| \cdot R^{p\kappa_\delta} \leq 1. \tag{2.8}$$

We then conclude that $S_2 \lesssim 1$. The proof is complete. □

We now consider $T_{L_{\alpha,\beta}}$. Observe that the oscillating term $e^{i\rho(x,t)^\beta}$ exhibits different behavior whether $0 < \beta < 1$ or $\beta > 1$. As ρ goes to infinity, the oscillation becomes faint if for the case $0 < \beta < 1$. In contrary, the oscillation grows to infinity for $\beta > 1$. Hence we deal with the two cases seperately.

Theorem 2.5.3. *Assume $0 < \beta < 1$ and $p \leq 1$ and $(\frac{1}{p}-1)(2n+2)\beta + \alpha < 0$. Then the operator $T_{L_{\alpha,\beta}}$ is bounded on H^p space.*

Proof. From (2.2) we have

$$\|L_{\alpha,\beta} * f\|_{H^p}^p \leq \sum_{j\geq 1} \|L_{\alpha,\beta}^j * f\|_{H^p}^p. \tag{2.9}$$

We now estimate each norm $\|L_{\alpha,\beta}^j * f\|_{H^p}$ by $\|f\|_{H^p}$. From the atomic decomposition for H^p space, we may choose f as an atom supported on $B(0,R)$ with some $R > 0$, which satisfies

$$\begin{aligned} &- \|f\|_{L^2} \leq R^{(2n+2)(\frac{1}{2}-\frac{1}{p})}, \\ &- \int f(x)x^\alpha dx = 0, \quad \text{for all } |\alpha| \leq s = [(2n+2)(\frac{1}{p}-1)]. \end{aligned} \tag{2.10}$$

From (b) in Theorem 2.4.5, it suffices to estimate $\mathcal{N}(L_j * f)$. For an admissible quadruple (p,q,s,ϵ) we may choose any $\epsilon > \max\{\frac{s}{2n+2}, \frac{1}{p}-1\} = \frac{1}{p}-1$. Simply we let $\epsilon = \frac{1}{p}-1+\delta$ with some $\delta > 0$. Then we have $a = \delta$ and $b = \frac{1}{p} - \frac{1}{2} + \delta$. for (2.2). We will choose δ sufficiently small later.

From (2.19) we have

$$\|L_j * f\|_2 \lesssim 2^{j(\alpha-n\beta)}\|f\|_2.$$

We have

$$\|L_j * f(x)\cdot |x|^{(2n+2)b}\|_2^2 = \int_{\mathbb{H}^n} |L_j * f(x)|^2 \cdot |x|^{2(2n+2)b} dx = I_1 + I_2, \tag{2.11}$$

where

$$I_1 = \int_{|x|\leq 2R} |L_j * f(x)|^2 \cdot |x|^{2(2n+2)b}\, dx \quad \text{and} \quad I_2 = \int_{|x|>2R} |L_j * f(x)|^2 \cdot |x|^{2(2n+2)b}\, dx.$$

Then,

$$\begin{aligned} \sum_{j\geq 1}\|L_j * f\|_{H^p}^p &\lesssim \sum_{j\geq 1} \mathcal{N}(L_j * f)^p \\ &\lesssim \sum_{j\geq 1}\left(\|L_j * f\|_2^{a/b}\cdot (I_1^{1/2(1-a/b)} + I_2^{1/2(1-a/b)})\right)^p \\ &\lesssim \sum_{j\geq 1}\|L_j * f\|_2^{pa/b}\cdot I_1^{p/2(1-a/b)} + \sum_{j\geq 1}\|L_j * f\|_2^{pa/b}\cdot I_2^{p/2(1-a/b)} \end{aligned} \tag{2.12}$$

Set $S_1 = \sum_{j\geq 1}\|L_j * f\|_2^{pa/b}\cdot I_1^{p/2(1-a/b)}$ and $S_2 = \sum_{j\geq 1}\|L_j * f\|_2^{pa/b}\cdot I_2^{p/2(1-a/b)}$. Then it is enough to show that $S_1 \lesssim 1$ and $S_2 \lesssim 1$. First we estimate I_1 with L^2estimates (2.19) as follows

$$\begin{aligned} I_1 &\lesssim \int_{\mathbb{H}^n} |f * L_j(x)|^2 dx \cdot R^{2(2n+2)b} \lesssim 2^{2j(\alpha-n\beta)}\|f\|_2^2 \cdot R^{2(2n+2)b} \\ &\leq 2^{2j(\alpha-n\beta)} R^{2(2n+2)b}\cdot R^{(2n+2)(1-2/p)} = 2^{2j(\alpha-(n+1/2)\beta)}R^{2(2n+2)\delta}. \end{aligned}$$

Thus we can bound $\|L_j * f\|_2^{a/b} \cdot I_1^{\frac{1}{2}(1-a/b)}$ as

$$\begin{aligned} \|L_j * f\|_2^{a/b} \cdot I_1^{1/2(1-a/b)} &\lesssim \left\{2^{j(\alpha-n\beta)} R^{(2n+2)(1/2-1/p)}\right\}^{a/b} \cdot \left\{2^{j(\alpha-n\beta)} \cdot R^{(2n+2)\delta}\right\}^{(1-a/b)} \\ &= 2^{j(\alpha-n\beta)}, \end{aligned}$$

and we have $S_1 \lesssim \sum_{j\geq 1} 2^{j(\alpha-n\beta)p} \lesssim 1$.

For I_2 we consider the two cases $R > 1$ and $R \leq 1$.

Case (*i*): Suppose $R > 1$. In the integral

$$(L_j * f)(x) = \int L_j(xy^{-1}) f(y) dy,$$

we have $|xy^{-1}| \leq 2^j$ and $|y| \leq R$, which imply $|x| \leq |xy^{-1}| + |y| \leq 2^j + R$. Therefore, in (2.11), we have that $I_2 = 0$ for $2^j < R$. Thus we only need to consider j with $2^j \geq R$. Then we have $|x| \leq 2^{j+1}$ for x in the support of $L_j * f$, and so

$$I_2 \lesssim \int_{|x|>2R} |f * L_j(x)|^2 dx \cdot 2^{2(2n+2)bj}. \tag{2.13}$$

By (2.1) we have

$$\begin{aligned} |L_j(xy^{-1}) - P_j^x(y)| &\lesssim |y|^{I+1} \sup_{|\alpha|\leq I+1} |X^\alpha L_j(xy^{-1})| \\ &\lesssim |y|^{I+1} 2^{-j(2n+2-\alpha)} 2^{j(\beta-1)(I+1)}. \end{aligned}$$

Since $f(y)$ has support in $|y| \leq R$ and (2.10), we have

$$\begin{aligned} |L_j * f(x)| &\lesssim R^{I+1} 2^{-j(2n+2-\alpha)} 2^{j(\beta-1)(I+1)} \int_{|y|\leq R} |f(y)| dy \\ &\lesssim R^{I+1} 2^{-j(2n+2-\alpha)} 2^{-j(\beta-1)(I+1)} R^{\frac{1}{2}(2n+2)} \|f\|_2. \\ &\lesssim 2^{-j(2n+2-\alpha)} (R 2^{-j(\beta-1)})^{(I+1)} R^{(2n+2)(1-\frac{1}{p})}. \end{aligned}$$

Thus we can estimate (2.13) as

$$\begin{aligned} I_2 &\lesssim 2^{2(2n+2)bj} 2^{j(2n+2)} \left\{2^{-j(2n+2-\alpha)} (R 2^{-j(\beta-1)})^{(I+1)} R^{(2n+2)(1-\frac{1}{p})}\right\}^2. \\ &= 2^{2j\{(2n+2)(1/p+1+\delta)+\alpha\}} (R 2^{j(\beta-1)})^{2(I+1)} R^{2(2n+2)(1-\frac{1}{p})}. \end{aligned}$$

Here we may choose $I = 0$ and $I = s$, which gives

$$I_2 \lesssim 2^{2j\{(2n+2)(1/p-1+\delta)+\alpha\}} R^{2(2n+2)(1-\frac{1}{p})} \min\{1, (R 2^{j(\beta-1)})^{2(s+1)}\}.$$

Thus,

$$\begin{aligned} \|L_j * f\|_2^{a/b} \cdot I_2^{\frac{1}{2}(1-a/b)} \lesssim \{&2^{j(\alpha-n\beta)} R^{(2n+2)(1/2-1/p)}\}^{a/b} \\ &\cdot \{2^{j\{(2n+2)(1/p-1+\delta)+\alpha\}} R^{(2n+2)(1-\frac{1}{p})} \min\left(1, (R 2^{j(\beta-1)})^{(s+1)}\right)\}^{(1-a/b)}. \end{aligned} \tag{2.14}$$

Provided δ is small enough, we have

$$\begin{aligned}(2n+2)(\frac{1}{p}-1+\delta)+\alpha+(\beta-1)(s+1) &= (2n+2)(\frac{1}{p}-1+\delta)+\alpha+(\beta-1)([(2n+2)(\frac{1}{p}-1)]+1)\\ &< (2n+2)(\frac{1}{p}-1+\delta)+\alpha+(\beta-1)(2n+2)(\frac{1}{p}-1)\\ &= (2n+2)(\frac{1}{p}-1)\beta+\alpha+(2n+2)\delta<0.\end{aligned}$$

Therefore the index of 2^j in (2.14) with $(R2^{j(\beta-1)})^{s+1}$ is negative for small $\delta > 0$. Remind that $R > 1$. Then, from (1) in Lemma 2.5.1 we have

$$\sum_{j\geq 1}\|L_j * f\|_2^{pa/b}\cdot I_2^{\frac{p}{2}(1-a/b)} \lesssim R^{p\mu_\delta}+\log(R+1)R^{p\kappa_\delta},$$

where

$$\begin{aligned}R^{p\mu_\delta} &= R^{(2n+2)(1/2-1/p)\frac{pa}{b}+(2n+2)(1-1/p)p(1-a/b)},\\ R^{p\kappa_\delta} &= [R^{-\frac{1}{1-\beta}[\alpha-n\beta]}R^{(2n+2)(1/2-1/p)}]^{p\delta/b}\cdot[R^{\frac{1}{1-\beta}\{(2n+2)(1/p-1+\delta)+2\alpha\}}\cdot R^{(2n+2)(1-1/p)}]^{p(1-a/b)}.\end{aligned}$$

Because $p \leq 1$, we easily see that $\mu_\delta \leq 0$. Moreover,

$$\kappa_0 = \frac{1}{1-\beta}\{\beta(2n+2)(\frac{1}{p}-1)+\alpha\}<0.$$

From this, we get $\kappa_\delta < 0$ for δ small enough. Therefore we have

$$S_2 \lesssim R^{\mu_\delta}+\log(R+1)R^{\kappa_\delta} \lesssim 1.$$

Case (*ii*): Suppose $R \leq 1$. We see that $\min(1,(R2^{j(\beta-1)(s+1)})) = R2^{j(\beta-1)(s+1)}$and (2.14) becomes

$$\sum_{j\geq 1}\|L_j * f\|_2^{pa/b}\cdot I_2^{\frac{p}{2}(1-a/b)} \lesssim \{2^{j(\alpha-n\beta)}R^{(2n+2)(1/2-1/p)}\}^{pa/b}$$
$$\{2^{j(2n+2)(1/p-1+\delta)+\alpha}R^{(2n+2)(1-\frac{1}{p})}\cdot(R2^{j(\beta-1)})^{(s+1)}\}^{p(1-a/b)}.$$

Because the power of 2^j is negative, provided δ is small enough, we get

$$\begin{aligned}\sum_{j\geq 1}\|L_j * f\|_2^{pa/b}\cdot I_2^{\frac{p}{2}(1-a/b)} &\lesssim R^{(2n+2)(1/2-1/p)\frac{pa}{b}}\cdot R^{\{(2n+2)(1-\frac{1}{p})+(s+1)\}p(1-\frac{a}{b})}\\ &=: R^{p\mu_\delta}.\end{aligned}$$

Observe that

$$\mu_0 = (2n+2)(1-\frac{1}{p})+(s+1) = (2n+2)(1-\frac{1}{p})+([(2n+2)(\frac{1}{p}-1)]+1)>0.$$

Thus we have $\mu_\delta > 0$ for δ small enough. Now we get

$$\sum_{j\geq 1}\|L_j * f\|_2^{pa/b}\cdot I_2^{\frac{p}{2}(1-a/b)} \lesssim R^{p\mu_\delta} \leq 1.$$

We then conclude that $S_2 \lesssim 1$. The proof is complete. □

We now establish the same result for the case $\beta > 1$.

Theorem 2.5.4. *For $1 < \beta$, $p \le 1$, if $(\frac{1}{p} - 1)(2n+2)\beta + \alpha < 0$, the operator $T_{L_{\alpha,\beta}}$ is bounded on H^p space.*

Proof. By arguing as in (2.9)–(2.12) in the proof of Theorem 2.5.3 to obtain the following

$$\sum_{j\ge 1} \|L_j * f\|_{H^p}^p \lesssim \sum_{j\ge 1} \|L_j * f\|_2^{pa/b} \cdot I_1^{p/2(1-a/b)} + \sum_{j\ge 1} \|L_j * f\|_2^{pa/b} \cdot I_2^{p/2(1-a/b)}, \tag{2.15}$$

where I_1 and I_2 are defined as in (2.11). Because the estimate for I_1 is exactly same with the proof of Theorem 2.5.3, we only deal with I_2. As before, we have

$$\begin{aligned} \|L_j * f\|_2^{a/b} \cdot I_2^{\frac{1}{2}(1-a/b)} &\lesssim \{2^{j(\alpha-n\beta)} R^{(2n+2)(1/2-1/p)}\}^{a/b} \\ &\cdot \{2^{j\{(2n+2)(1/p-1+\delta)+\alpha\}} R^{(2n+2)(1-\frac{1}{p})} \min\left(1, (R2^{j(\beta-1)})^{(s+1)}\right)\}^{(1-a/b)} \end{aligned} \tag{2.16}$$

Case (*i*): Suppose $R > 1$. As for the case $\beta < 1$, we have $I_2 = 0$ if $2^j < R$ and we only need consider j with $2^j \ge R$. Since $R2^{j(\beta-1)} \ge 1$, we estimate I_2 as

$$I_2 \lesssim 2^{j\{2(2n+2)[1/p-1+\delta]+2\alpha\}} R^{2(2n+2)(1-1/p)}.$$

Note that

$$(2n+2)(\frac{1}{p} - 1) + \alpha < (2n+2)(\frac{1}{p} - 1)\beta + \alpha < 0. \tag{2.17}$$

Thus, if δ is sufficiently small, we have $(2n+2)(1/p - 1 + \delta) + \alpha < 0$ and we can sum (2.16) as

$$\sum_{j\ge 1} \|L_j * f\|_2^{pa/b} \cdot I_2^{\frac{p}{2}(1-a/b)} \lesssim R^{(2n+2)(1/2-1/p)\frac{pa}{b}} \cdot R^{\{(2n+2)(1-\frac{1}{p})\}p(1-\frac{a}{b})} \le 1, \tag{2.18}$$

where the last inequality holds because $p \le 1$ and $R > 1$.
Case (*ii*): Suppose $R \le 1$. From (2.17), using (1) in Lemma 2.5.1 we have

$$\sum_{j\ge 1} \|L_j * f\|_2^{pa/b} \cdot I_2^{\frac{1}{2}p(1-a/b)} \lesssim R^{p\mu_\delta} + |\log R|\, R^{p\kappa_\delta},$$

where

$$\begin{aligned} R^{\mu_\delta} &= R^{(2n+2)(1/2-1/p)\frac{a}{b}} \cdot R^{\{(2n+2)(1-1/p)+(s+1)\}(1-\frac{a}{b})}, \\ R^{\kappa_\delta} &= R^{(2n+2)(1/2-1/p)\frac{a}{b}} \cdot \{R^{\frac{1}{1-\beta}\{(2n+2)(1/p-1+\delta)+\alpha\}} R^{(2n+2)(1-1/p)}\}^{1-a/b}. \end{aligned}$$

Observe that

$$\mu_0 = (2n+2)(1 - \frac{1}{p}) + (s+1) = (2n+2)(1 - \frac{1}{p}) + [(2n+2)(\frac{1}{p} - 1)] + 1 > 0$$

and

$$\kappa_0 = \frac{1}{1-\beta}\{(2n+2)(1/p-1)+\alpha\} + (2n+2)(1-1/p) = \frac{1}{1-\beta}\{\beta(2n+2)(\frac{1}{p}-1)+\alpha\} > 0.$$

Therefore we have $\mu_\delta, \kappa_\delta > 0$ for δ small enough, and so

$$\sum_{j\geq 1} \|L_j * f\|_2^{a/b} \cdot I_2^{\frac{1}{2}(1-a/b)} \lesssim R^{\mu_\delta} + |\log R| \cdot R^{\kappa_\delta} \leq 1. \tag{2.19}$$

Now we conclude that $S_2 \lesssim 1$ from (2.18) and (2.19). The proof is complete. □

2.6 Necessary conditions

In this section we show that the Hardy space boundedness obtained in the previous section is sharp except for the endpoint cases. We only give an example for Theorem 2.5.2. Examples for the other theorems can be found similarly. We refer to [Sj3] for the Euclidean case.
We let $g(x)$ a function such that

$$\int_{\mathbb{R}} x^\alpha g(x)dx = 0 \quad \text{for } 0 \leq \alpha \leq k \quad \text{and} \quad \int_{\mathbb{R}} x^{k+1} g(x)dx \neq 0.$$

Let $h(x_2, \dots, x_{2n}, x_{2n+1})$ a function supported on the ball $B(0,1)$ such that $\int_{\mathbb{R}^{2n}} h \neq 0$ and let f be the function on $\mathbb{R}^{2n+1}$ defined by $f(x_1, \dots, x_{2n+1}) = g(x_1)h(x_2, \dots, x_{2n+1}) \forall (x_1, \cdots, x_{2n+1}) \in \mathbb{R}^{2n+1}$. Then

$$\int_{\mathbb{H}^n} x^\alpha f(x) = 0, \quad \text{if} \quad |\alpha| \leq k.$$

For $\epsilon > 0$ set $f_\epsilon(x) = \epsilon^{-(2n+2)/p} f(\frac{x}{\epsilon})$. We note that $\|f_\epsilon\|_{H^p} = C$ for all $\epsilon > 0$. Assume that $T_{K_{\alpha,\beta}}$ is bounded on H^p. Then $\|T_{K_{\alpha,\beta}}(f_\epsilon)\|_{H^p} \lesssim 1$. Note that $|y| \leq \epsilon$ for $y \in \operatorname{supp}(f_\epsilon)$. Then, for $|x| \geq C\epsilon$ with a large constant $C > 0$, we have

$$\begin{aligned}
K * f(x) &= \int K(xy^{-1}) f_\epsilon(y)dy \\
&= \int \left(K(xy^{-1}) - \sum_{|\alpha|\leq k+1} \frac{1}{\alpha!} D^\alpha K(x) y^\alpha \right) f_\epsilon(y)dy + \int \left(\sum_{|\alpha|\leq k+1} \frac{1}{\alpha!} D^\alpha K(x) y^\alpha \right) f_\epsilon(y)dy \\
&= \int D^{k+2} K(xy_*^{-1}) O(y^{k+2}) f_\epsilon(y)dy + C\partial_{x_1}^{k+1} K(x) \int_{\mathbb{R}} y_1^{k+1} f_\epsilon(y_1)dy_1, \qquad |y_*| \leq |y| \leq \epsilon \\
&= O(\epsilon^{(2n+2)+k+2-\frac{(2n+2)}{p}} |x|^{-(n+\alpha+(k+2)(\beta+1))}) + \epsilon^{k+1+(2n+2)-\frac{(2n+2)}{p}} \partial_{x_1}^{k+1} K(x).
\end{aligned}$$

Take $K(x) = |x|^{-2n-2-\alpha} e^{i|x|^{-\beta}} \chi(x)$. We see that $|\partial_{x_1}^{k+1} K(x)| \sim |x|^{-(2n+2)-\alpha-(k+1)(\beta+1)}$ for small x. For $\epsilon \lesssim |x|^{\beta+1}$ we have

$$\epsilon^{(2n+2)+k+2-\frac{2n+2}{p}} |x|^{-(2n+2+\alpha+(k+2)(\beta+1))} \lesssim \epsilon^{(2n+2)+(k+1)-\frac{(2n+2)}{p}} |x|^{-(2n+2)-\alpha-(k+1)(\beta+1)}.$$

Therefore we get

$$K_{\alpha,\beta} * f_\epsilon(x) \sim \epsilon^{(2n+2)+k+1-\frac{(2n+2)}{p}}|x|^{-(2n+2)-\alpha-(k+1)(\beta+1)} \quad \text{for } |x| \gtrsim \epsilon^{1/(\beta+1)}.$$

Then,

$$\begin{aligned} 1 \gtrsim \int_{\mathbb{H}^n} |K_{\alpha,\beta} * f_\epsilon(x)|^p dx &\gtrsim \epsilon^{p(2n+2)+kp+p-(2n+2)} \int_{c \geq |x| \gtrsim \epsilon^{1/(\beta+1)}} |x|^{-(2n+2)p-\alpha p-(k+1)(\beta+1)p} dx \\ &\gtrsim \epsilon^{p(2n+2)+kp+p-(2n+2)} \epsilon^{-\frac{(2n+2)p-(2n+2)+\alpha p}{\beta+1}-(k+1)p} \\ &= \epsilon^{\frac{-p}{\beta+1}\left[(\frac{1}{p}-1)(2n+2)\beta+\alpha\right]}. \end{aligned}$$

This implies that $(1-\frac{1}{p})(2n+2)\beta+\alpha$ must be ≤ 0. This shows that Theorem 2.5.2 is sharp except the endpoint case $(1-\frac{1}{p})(2n+2)\beta+\alpha = 0$.

Chapter 3

Maximal functions for multipliers on stratified groups [Ch2]

3.1 Introduction

Consider a stratified group G with homogeneous dimension Q and let L be a left invariant sub-Laplacian on G. Denote by $\{E(\lambda) : \lambda \geq 0\}$ the spectral resolution of L. Then, for a bounded function $m : [0, \infty) \to \mathbb{R}$, we can define the multiplier operator

$$m(L) = \int_0^\infty m(\lambda) dE(\lambda).$$

A sufficient conditions on a function m for $\|m(L)f\|_p \lesssim \|f\|_p$ was obtained by Christ [C1] and Mauceri-Meda [MaM] independently. They proved that if the function m satisfies the following condition

$$\sup_{\lambda \in \mathbb{R}^+} \|\phi(\cdot)m(\lambda \cdot)\|_{H^\alpha(\mathbb{R})} < \infty \qquad \text{for some } \alpha > \frac{Q}{2},$$

then $m(L)$ is bounded on $L^p(G)$ for any $1 < p < \infty$. Here ϕ is any nonzero function in $C^\infty([1, 2])$ and $H^\alpha(\mathbb{R})$ denotes the Sobolev space endowed with the norm $\|g\|^2_{H^\alpha(\mathbb{R})} := \int_{\mathbb{R}} (1 + |\xi|)^\alpha |\hat{g}(\xi)|^2 d\xi$. The index $\frac{Q}{2}$ is sharp when G is a Euclidean space, whereas Martini-Müller [MM] improved the condition to $\alpha > \frac{d}{2}$ for a class of 2-step stratified groups, where d is the topological dimension of G. We also refer to the related works [MRS, MRS2, MS]. In addition, multiplier theorems of this kind have been extended to various spaces. For example, Alexopoulos [Al] studied the multipliers on Lie groups of polynomial growth and Seeger-Sogge [SS] studied the multipliers on compact manifolds. Duong-Ouhabaz-Sikora [DOS] obtained a general result that the multiplier theorems follow from the appropriate estimates of the L^2-norm of the kernel of the multipliers and the Gaussian bounds for the corresponding heat kernel.

In this paper we are concerned with the maximal multiplier

$$\mathcal{M}_m f(x) := \sup_{t>0} |m(tL)f(x)|.$$

It is a challenging problem to find conditions that ensure that $\mathcal{M}_m$ is bounded on $L^p(G)$. Mauceri-Meda [MaM] proved that $\mathcal{M}_m$ is bounded on $L^p(G)$ if the condition

$$\sum_{k\in\mathbb{Z}} \|\phi(\cdot)m(2^k\cdot)\|_{H^s(\mathbb{R})} < \infty \tag{3.1}$$

holds for some $s > Q(\frac{1}{p} - \frac{1}{2}) + \frac{1}{2}$ when $p \in (1, 2]$, or for some $s > (Q-1)(\frac{1}{2} - \frac{1}{p}) + \frac{1}{2}$ when $p \in [2, \infty]$.

We aim to improve the summability condition (3.1). As a preliminary step we shall prove the following result on the maximal function of the multipliers satisfying a uniform bound.

Theorem 3.1.1. *For $1 \le r < 2$, suppose that there is $\alpha > Q/r$ such that $m_1, \dots, m_N$ satisfy the condition*

$$\sup_{\lambda\in\mathbb{R}} \|\phi(\cdot)m_i(\lambda\cdot)\|_{H^\alpha} \le B \qquad \text{for } i = 1, \cdots, N.$$

Then, for all $p \in (r, \infty)$, we have

$$\|\sup_{i=1,\dots,N} |m_i(L)f|\|_p \le C_{p,r} B \sqrt{\log(N+1)} \|f\|_p.$$

This result is a generalization of the result Grafakos-Honzik-Seeger [GHS] for the multipliers on the Euclidean space. The growth rate $\sqrt{\log(N+1)}$ was shown to be sharp in [CGHS] in the case of the Euclidean space.

Using Theorem 3.1.1 we will prove the following result on the maximal multipliers.

Theorem 3.1.2. *Suppose that*

$$\|\phi(\cdot)m(2^k\cdot)\|_{H^\alpha} \le \omega(k), \quad k \in \mathbb{Z},$$

holds for some α and suppose that the non-increasing rearrangement ω^* satisfies

$$\omega^*(0) + \sum_{l=1}^{\infty} \frac{\omega^*(l)}{l/\sqrt{\log l}} < \infty.$$

If $\alpha > Q/r + 1$ for some $1 \le r < 2$, then $\mathcal{M}_m$ is bounded on $L^p(G)$ for $p \in (r, \infty)$.

We remark that this result improves the summability condition (3.1) of the result obtained by Mauceri-Meda [MaM]. However we note that their result requires less number of derivatives in (3.1) than our result.

The second aim of this paper is to consider maximal functions of the multi-parameter multipliers on the product spaces of stratified groups. Let G be the product space of n-stratified groups $G_1, \cdots, G_n$. Consider sub-Laplcians L_j, $1 \le j \le n$ and their lifting to G denoted by $L_j^\sharp$. We recall that, under the following assumption on m:

$$\sup_{\xi \in \mathbb{R}^n} |(\xi_1 \partial_{\xi_1})^{\alpha_1} \cdots (\xi_n \partial_{\xi_n})^{\alpha_n} m(\xi_1, \cdots, \xi_n)| \le B \tag{3.2}$$

for all $\alpha_j \le M$, with M large enough, Müller-Ricci-Stein [MRS] proved the L^p boundedness property of the multiplier $m(L_1^\sharp, \cdots, L_s^\sharp)$. For maximal functions of these multipliers, we shall obtain the following boundedness result.

Theorem 3.1.3. *Suppose that functions $m_1, \ldots, m_N$ on $(\mathbb{R}_+)^n$ satisfy the condition* (3.2) *uniformly, i.e., for some $M \in \mathbb{N}$ large enough, there exists a positive number B such that*

$$\sup_{1 \le k \le N} \sup_{\xi \in \mathbb{R}^n} |(\xi_1 \partial_{\xi_1})^{\alpha_1} \cdots (\xi_n \partial_{\xi_n})^{\alpha_n} m_k(\xi_1, \cdots, \xi_n)| \le B \quad \textit{for all } \alpha_j \le M. \tag{3.3}$$

Then, for all $p \in (1, \infty)$, we have the inequality

$$\left\| \sup_{1 \le i \le N} |m_i(L_1^\sharp, \cdots, L_n^\sharp) f| \right\|_p \le C_p B (\log(N+1))^{n/2} \|f\|_p .$$

As an application of this theorem we also obtain a similar result for the joint spectral multipliers on the Heisenberg group $\mathbb{H}_n$ which is endowed with the sub-Laplacian Δ and the derivative $T = \frac{\partial}{\partial t}$. The L^p boundedness of the joint spectral multiplier $m(\Delta, iT)$ was obtained by Müller-Ricci-Stein [MRS]. Using Theorem 3.1.3 and the transference method of [CWW] we will prove the following theorem.

Theorem 3.1.4. *Suppose that functions $m_1, \ldots, m_N$ on $(\mathbb{R}_+)^2$ satisfy the condition* (3.2) *uniformly, i.e., for some $M \in \mathbb{N}$ large enough, there exists a positive number B such that*

$$\sup_{1 \le k \le N} \sup_{\xi \in \mathbb{R}^2} |(\xi_1 \partial_{\xi_1})^{\alpha_1} (\xi_2 \partial_{\xi_2})^{\alpha_2} m_k(\xi_1, \xi_2)| \le B \quad \textit{for all } \alpha_j \le M. \tag{3.4}$$

Then, for all $p \in (1, \infty)$, we have

$$\left\| \sup_{1 \le i \le N} |m_i(\Delta, iT) f| \right\|_p \le C_p B (\log(N+1)) \|f\|_p .$$

In order to prove Theorem 1.1 we shall use the argument of Grafakos-Honzik-Seeger [GHS] who make use of a good λ inequality for martingale operators due to Chang-Wilson-Wolff [CWW](see Lemma 3.3.2 below). We shall use the martingales constructed by Christ [C2] on the setting of homogeneous space. For applying the martingale theory to study multipliers, a basic but necessary step in [GHS] is to find cancellation property arising when we compose martingale operators and Littlewood-Paley projections. Due to the technical difficulties of the Fourier transform on stratified groups, we will show it in a different way by suitable partitioning the kernels of projections (see Lemma 3.3.3).

In stratified groups, it is also not as easy as on Euclidean space to know exaclty the kernels of multipliers. Nevertheless, a technique was developed by Folland and Stein [FoS] using the kernel of the heat semi-group $e^{-tL}, t > 0$. In addition, Christ [C1] and Mauceri-Meda [MaM] obtained a sharp estimate on the L^q norm ($1 < q \leq 2$) of the kernels by using the Plancherel formula on stratified groups (see Lemma 3.2.2). For our purpose we will extend it to the range $q > 2$. It will enable us to bound a multiplier operator with localized multiplier function pointwise by the Hardy-Littlewood maximal function (see Lemma 3.2.4).

For proving Theorem 3.1.3 we shall use an idea of Honzik [Ho1] who obtained a sharp boundedness result for maximal functions of Marcinkiewicz multipliers on $\mathbb{R}^n$ which correspond to the multi-parameter multipliers on the product space $G = \mathbb{R} \times \cdots \times \mathbb{R}$. However, it is difficult to follow his approach in a direct way because the approach uses crucially the L^p ($1 < p < \infty$) boundedness property of fourier multipliers whose multiplier functions are 1_Q for rectangles Q in $\mathbb{R}^n$ meanwhile this fact does not hold for mulitplier functions 1_Q when Q is a sphere. This is due to the well-known result of Fefferman [Fe2] that the ball multipliers on $\mathbb{R}^n$ for $n \geq 2$ are not bounded in L^p space when $p \neq 2$. For this reason, we will generalize the argument in [GHS] in a different way.

This paper is organized as follows. In section 2, we study kernels of the multiplier operators on homogeneous spaces. In section 3, we exploit the cancellation property between the martingales and the Littlewood-Paley operators. Then we will prove Theorem 3.1.1 and Theorem 3.1.2. In Section 4 we study the maximal multipliers on the product spaces. We shall study multi-expectation operators on prouduct spaces. Applying it to the multi-parameter multipliers we will prove Theorem 3.1.3. Then we will use a transference argument to complete the proof of Theorem 3.1.4. In Section 5, we discuss how one can apply Theorem 3.1.3 to study the multi-parameter maximal multipliers.

We denote by C a generic constant depending only on the background spaces and the index p of the space L^p used in the inequality. Also, we shall use the notation $A \lesssim B$ to denote an inequality $A \leq CB$.

3.2 Kernels of multipliers on Stratified groups

In this section we shall begin with a brief review on the stratified groups spectral multipliers defined on those groups. Then we shall study integration property of kernels of multiplier operators in terms of smoothness of multiplier functions. In the last part, we shall estimate multipliers by the Hardy-Littlewood maximal functions.

We denote by $\mathfrak{g}$ a finite-dimensional nilpotent Lie algebra of the form

$$\mathfrak{g} = \bigoplus_{i=1}^{s} \mathfrak{g}_i$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i, j, and by G the associated simply connected Lie group. Then, its homogeneous dimension is $Q = \sum_j j \cdot \dim(\mathfrak{g}_\mathrm{j})$. We call it a stratified group when $\mathfrak{g}_g$ generates $\mathfrak{g}$ as a Lie algebra. Throughout the paper, G stands for a stratified group.

We denote by $\{\delta_r : r > 0\}$ a family of dilations of the Lie algebra $\mathfrak{g}$ which satisfy $\delta_r X = r^j X$ for $X \in \mathfrak{g}_j$, and is extended by linearity. We shall also denote by $\{\delta_r : r > 0\}$ the induced family of dilations of G. They are group automorphisms. We define a homogeneous norm of G to be a continuous function $|\cdot| : G \longrightarrow [0, \infty)$ which is, C^∞ away from 0, and satisfies $|x| = 0 \Leftrightarrow x = 0$ and $|\delta_r x| = r|x|$ for all $r \in \mathbb{R}^+$, $x \in G$.

We denote by $S(G)$ the space of Schwartz functions in G. Now we choose any finite subset $\{X_k\}$ of $\mathfrak{g}_1$ which spans $\mathfrak{g}_1$. We may identify each X_k with a unique left-invariant vector field on G. We also denote it by X_k. Then we define a sub-Laplacian as $L = -\sum X_k^2$, which is a left-invariant second-order differential operator. $L^p(G)$ is defined with respect to a bi-invariant Haar measure. As an operator on $\{f \in L^2(G) : Lf \in L^2(G)\}$, L is self-adjoint. Therefore it admits a spectral resolution $L = \int_0^\infty \lambda dP_\lambda$. For a bounded Borel function m on $[0, \infty)$, we define the bounded operator $m(L)$ on L^2 by

$$m(L) = \int_0^\infty m(\lambda) dP_\lambda.$$

By the Schwartz kernel theorem, there exists a tempered distribution k_m on G satisfying $m(L)f = f * k_m$ for all functions in G. For a tempered distribution k on G, we always denote by $k_{(t)}$ for $t > 0$ the distribution satisfying

$$\langle k_{(t)}, f \rangle = \langle k, f \circ \delta_t \rangle$$

for all $f \in S(G)$. If k is a measurable function on G, then $k_{(t)}(x) = \frac{1}{t^Q} f(\frac{x}{t})$.

The heat semigroup $\{e^{-tL}\}_{t>0}$ on G can be defined as

$$e^{-tL} = \int_0^\infty e^{-\lambda t} dP_\lambda$$

and we set $h_t(x)$ be the heat kernel satisfying $e^{-tL} f = f * h_t$ for all $f \in L^2$. Simply we write $h(x)$ for $h_1(x)$. Then we have $h_t(x) = h_{(\sqrt{t})}(x)$ and it was proved in [JS] that there exist $c_0, C \in \mathbb{R}^+$ such that

$$|h(x)| \leq C e^{-c_0 |x|^2}. \tag{3.1}$$

The next lemma is from [FoS, Lemma 6.29].

Lemma 3.2.1. *If M is a bounded Borel function on $(0,\infty)$, let K be the distribution kernel of $M(L)$. Then for any $t>0$, if $M_{(t)}(\lambda)=M(t\lambda)$, the distribution kernel of $M_{(t)}(L)$ is $K_{\sqrt{t}}$.*

We recall Lemma 1.2 in [MaM] on boundedness of the kernels of multipliers.

Lemma 3.2.2. *Let $\alpha \geq 0$ and $1 \leq p \leq 2$. Suppose that $s>0$ satisfies $s>\alpha/p+Q(1/p-1/2)$. Then, for each multi-index I there exists a constant $C_I>0$ such that any function $m \in H^s_2(\mathbb{R}_+)$ with its support in $(1/2,2)$ and the distribution kernel k of $m(L)$ satisfy*

$$\int_G |x|^\alpha |X^I k(x)|^p dx \leq C_I \|m\|^p_{H^s}. \tag{3.2}$$

For each $s>\frac{\alpha}{2}$, taking $p=2$ in the previous lemma, we see that for any multi-index I there is a positive constant $C_I>0$ such that

$$\int_G |x|^\alpha |X^I k(x)|^2 dx \leq C\|m\|_{H^s}, \qquad \forall\, m \in H^s(\mathbb{R}_+) \quad \text{such that } \operatorname{supp}(m)\subset(1/2,2). \tag{3.3}$$

Employing this estimate, we shall prove the following lemma.

Lemma 3.2.3. *Suppose that m is a function in $H^s(\mathbb{R}_+)$ supported in $(1/2,2)$ with $s>\alpha/2$. Let k be the distribution kernel of $m(L)$. Then, for any multi-index I we have*

$$\sup_{x\in G}(1+|x|)^{\frac{\alpha}{2}}|X^I k(x)| \lesssim \|m\|_{H^s}. \tag{3.4}$$

and

$$\int_G |x|^{\alpha\frac{q}{2}}|X^I k(x)|^q dx \lesssim \|m\|^q_{H^s}. \tag{3.5}$$

for each $q>2$.

Proof. Set $m_1(\lambda)=e^\lambda m(\lambda)$ and K_1 be the distribution kernel of $m_1(L)$. Note that H^s_2 norms of M and M_1 are comparable because the support of m is contained in $(1/2,2)$. Since $m(L)=e^{-L}m_1(L)=m_1(L)e^{-L}$ we have $K=h*K_1=K_1*h$. Hence K is C^∞ and $X^IK=K_1*X^Ih$ for any multi-index I. Since $h\in S(G)$ we have $\||x|^N X^I h(x)\|_{L^2}\lesssim 1$ for any $N>0$. Thus using (3.3) and the triangle inequality we get

$$\begin{aligned}(1+|x|)^{\alpha/2}|X^IK(x)| &\lesssim \int (1+|y|)^{\alpha/2}|K_1(y)|(1+|y^{-1}x|)^{\alpha/2}|X^Ih(y^{-1}x,1)|dy\\ &\lesssim \left(\int (1+|y|)^\alpha|K_1(y)|^2dy\right)^{1/2}\left(\||x|^{\alpha/2}X^Ih(x)\|_{L^2}\right)\\ &\lesssim \|m\|_{H^s},\end{aligned}$$

which proves (3.4). Next, for $q>2$ we combine (3.3) and (3.4) using Hölder's inequality to get

$$\begin{aligned}\int_G |x|^{\alpha\frac{q}{2}}|X^Ik(x)|^qdx &\lesssim \sup_{x\in G}|x|^{\alpha\frac{q-2}{2}}|X^Ik(x)|^{q-2}\int_G|x|^\alpha|X^Ik(x)|^2dx.\\ &\lesssim \|m\|^{q-2}_{H^s}\|m\|^2_{H^s}=\|m\|^q_{H^s},\end{aligned}$$

which is the desired estimate (3.5). The proof is finished. ◻

Next we split the support of multipliers into dyadic pieces. For this aim, we take a bump function $\phi \in C^\infty(0,\infty)$ supported on $[\frac{1}{2}, 2]$ satisfying $\sum_{j\in\mathbb{Z}} \phi(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^+$ and then

$$m(L) = \sum_{j\in\mathbb{Z}} m_j(L), \tag{3.6}$$

where $m_j(\xi) = \phi(2^{-j}\xi)m(\xi)$.

We also define $\widetilde{m}_j(\xi) := m_j(2^j\xi)$ and denote by $M_r f(x) = (M(|f|^r)(x))^{1/r}$ the higher order maximal functions for each $r > 1$. In the following lemma, we state the main result of this section.

Lemma 3.2.4. $|m_k(L)f(x)| \lesssim M_r f(x) \cdot \|\widetilde{m}_k\|_{H^s}, \quad s > Q/r, \quad r \leq 2.$

Proof. In the proof, we denote by K_k (resp. $\widetilde{K}_k$) the kernel of the operator $m_k(L)$(resp. $\widetilde{m}_k(L)$). Then, we see from Lemma 3.2.1 that

$$\widetilde{K}_k(x) = K_{k(\sqrt{2^k})}(x) = 2^{-kQ/2} K_k(\frac{x}{2^{k/2}}).$$

Observing $r' > 2$, we apply Lemma 3.2.3 to get

$$\int_G |x|^{\alpha\frac{r'}{2}} |\widetilde{K}_k(x)|^{r'} dx \lesssim \|\widetilde{m}_k\|_{H^s}^{r'}, \quad \text{for all } 0 \leq \alpha < 2s. \tag{3.7}$$

Letting further $\widetilde{K}_{k,l}(x) = \widetilde{K}_k(x) \cdot 1_{\{2^{l-1} \leq |x| < 2^l\}}$ for $l \in \mathbb{N}$ and $\widetilde{K}_{k,0}(x) = \widetilde{K}_k(x) \cdot 1_{\{|x|<1\}}$, we see directly from (3.7) that

$$\sup_{l\geq 0} 2^{l\alpha\frac{r'}{2}} \int |\widetilde{K}_{k,l}(x)|^{r'} dx \lesssim \|\widetilde{m}_k\|_{H^s}^{r'} \quad \text{for} \quad 0 \leq \alpha < 2s. \tag{3.8}$$

Since $\frac{2Q}{r} < 2s$ we can find a small $\epsilon > 0$ such that $\alpha_0 = \frac{2Q}{r} + \epsilon < 2s$. In particular, we will use estimate (3.8) with this α_0.

Let us rewrite $m_k(L)$ as

$$m_k(L)f(x) = \int_G 2^{kQ/2} \widetilde{K}_k(2^{k/2}y) f(xy^{-1}) dy = \sum_{l=0}^\infty \int_G 2^{kQ/2} \widetilde{K}_{k,l}(2^{k/2}y) f(xy^{-1}) dy,$$

and apply Hölder's ineqaulity to obtain

$$\begin{aligned}
|m_k(L)f(x)| &\lesssim \sum_{l=0}^\infty \left(\int_G 2^{kQ/2} |\widetilde{K}_{k,l}(2^{k/2}(y))|^{r'} dy \right)^{1/r'} \left(2^{kQ/2} \int_{|y|\leq 2^{l-k/2}} |f(xy^{-1})|^r dy \right)^{1/r} \\
&\lesssim \sum_{l=0}^\infty 2^{lQ/r} (M(|f|^r)(x))^{1/r} \left(\int_G |\widetilde{K}_{k,l}(y)|^{r'} dy \right)^{1/r'}.
\end{aligned}$$

In the right hand side, we use estimate (3.8) with $\alpha = \alpha_0$ to get

$$\begin{aligned}
|m_k(L)f(x)| &\lesssim \|\widetilde{m}_k\|_{H^s} \sum_{l=0}^\infty 2^{lQ/r} 2^{-l\alpha_0/2} (M(|f|^r)(x))^{1/r} = \|\widetilde{m}_k\|_{H^s} \sum_{l=0}^\infty 2^{-\frac{l\epsilon}{2}} (M(|f|^r)(x))^{1/r} \\
&\lesssim \|\widetilde{m}_k\|_{H^s} (M(|f|^r)(x))^{1/r}.
\end{aligned}$$

It proves the lemma. □

3.3 Martingales on homogeneous space and its application to maximal multipliers

In this section, we first recall the martingales on homogeneous space from [C2]. Then we shall study some cancellation property arising when the martingale operators are composed with Littlewood-Paley projections. In the last part, the proof of Theorem 3.1.1 and Theorem 3.1.2 will be given.

In what follows, open set Q_α^k will role as dyadic cubes of side-lengths 2^{-k} (or more precisely, δ^k) with the two conventions: 1. For each k, the index α will run over some unspecified index set dependent on k. 2. For two sets with $Q_\alpha^{k+1} \subset Q_\beta^k$, we say that Q_β^k is a parent of Q_α^{k+1}, and Q_α^{k+1} a child of Q_β^k.

Theorem 3.3.1 (Theorem 14 in [C2])**.** *Let X be a space of homogeneous type. Then there exists a family of subset $Q_\alpha^k \subset X$, defined for all integers k, and constants $\delta, \rho > 0, C < \infty$ such that*

- $\mu(X \setminus \cup_\alpha Q_\alpha^k) = 0 \ \forall k$
- *for any α, β, k, l with $l \geq k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$*
- *each Q_α^k has exactly one parent for all $k \geq 1$*
- *each Q_α^k has at least one child*
- *if $Q_\alpha^{k+1} \subset Q_\beta^k$ then $\mu(Q_\alpha^{k+1}) \geq \rho\mu(Q_\beta^k)$*
- *for each (α, k) there exists $x_{\alpha,k} \in X$ such that $B(x_{\alpha,k}, \delta^k) \subset Q_\alpha^k \subset B(x_{\alpha,k}, C\delta^k)$.*

Moreover,

$$\mu\{y \in Q_\alpha^k : \rho(y, X \setminus Q_\alpha^k) \leq t\delta^k\} \leq Ct^\epsilon \mu(Q_\alpha^k) \ for\ 0 < t \leq 1,\ for\ all\ \alpha, k. \tag{3.1}$$

Now we define the expectation operator

$$\mathbb{E}_k f(x) = \mu(Q_\alpha^k)^{-1} \int_{Q_\alpha^k} f d\mu \quad \text{for } x \in Q_\alpha^k,$$

the martingale operator $\mathbb{D}_k f(x) = \mathbb{E}_{k+1} f(x) - \mathbb{E}_k f(x)$ and set the square function

$$S(f) = (\sum_{k\geq 1} |\mathbb{D}_k f(x)|^2)^{1/2}.$$

Next we state the following good λ inequality.

[Ch2]

Lemma 3.3.2 ([CWW])**.** *There are constants $C > 0$ and $C_1 > 0$ such that for all $\lambda > 0,\ 0 < \epsilon < \frac{1}{2}$, the following inequality holds.*

$$meas(\{x : \sup_{k\geq 1} |\mathbb{E}_k g(x) - \mathbb{E}g(x)| > 2\lambda, S(g) < \epsilon\lambda\}) \tag{3.2}$$

$$\leq C\exp(-\frac{C_1}{\epsilon^2}) meas(\{x : \sup_{k\geq 1} |\mathbb{E}_k g(x)| > \lambda\}). \tag{3.3}$$

Although this lemma was proved in [CWW] for the Euclidean setting, the proof is applicable for the homogeneous group setting as well.

Recall the functions ϕ_j and m_j defined in (3.6). We choose a bump function $\psi \in C_0^\infty$ which is supported on $[\frac{1}{4}, 4]$ and equal to 1 on $[\frac{1}{2}, 2]$, and let $\psi_j(\xi) = \psi(2^{-j}\xi)$. Then, since $m_j(\xi) = \phi(2^{-j}\xi)m(\xi)$ is supported on $[\frac{1}{2}, 2]$, it holds that $m_j(\xi) = \psi_j^2(\xi)m_j(\xi)$, which leads to the identity

$$m_j(L) = \psi_j(L)m_j(L)\psi_j(L).$$

Consequently,

$$\mathbb{D}_k(m(L)f) \;=\; \mathbb{D}_k\Big(\sum_{j\in\mathbb{Z}} m_j(L)f\Big) = \sum_{j\in\mathbb{Z}} \mathbb{D}_k(\psi_j(L)m_j(L)\psi_j(L)f). \tag{3.4}$$

For $n \in \mathbb{Z}$ we denote by $K_n : G \to \mathbb{R}$ the kernel of $\psi_n(L)$, i.e.,

$$\psi_n(L)f = K_n * f \qquad \forall f \in S(G),$$

and denote K_1 by K for notational simplicity. By Lemma 3.2.1, we have $K_n(x) = 2^{Qn/2}K(2^{n/2}x)$. Also it holds that $\int_G K(x)dx = 0$ since the support of ψ is away from the zero. In addition, we know from Lemma 6.36 in [FoS] that

$$K(x) \lesssim (1 + |x|)^{-N} \quad \text{ for any } N > 0. \tag{3.5}$$

In the next lemma we exploit certain cancellation property arising in composition of the projections and the martingale operators.

Lemma 3.3.3.

(i) There exists a constant $\gamma > 0$ such that $|\mathbb{E}_k(\psi_n(L)f)(x)| \lesssim 2^{(-(\log_2\delta)k - n/2)\gamma} M_q f(x)$ holds uniformly for $n/2 > (-\log_2\delta)k + 10$.

(ii) There exists a constant $\gamma > 0$ such that $|\mathbb{D}_k(\psi_n(L)f)(x)| \lesssim 2^{((\log_2\delta)k + n/2)\gamma} M_q f(x)$ holds uniformly for $n/2 < (-\log_2\delta)k - 10$.

In particular, these two estimates imply

$$|\mathbb{D}_k(\psi_n(L)f)(x)| \lesssim 2^{-|(\log_2\delta)k + n/2|} M_q f(x), \quad \forall (n, k) \in \mathbb{Z}^2.$$

Proof. For each $x \in G$ we find a unique Q_α^k such that $x \in Q_\alpha^k$ and then,

$$\begin{aligned}\mathbb{E}_k(\psi_n(L)f)(x) =& \frac{1}{\mu(Q_\alpha^k)} \int_{Q_\alpha^k} (\psi_n(L)f)(y)dy \\ =& \frac{1}{\mu(Q_\alpha^k)} \int_{Q_\alpha^k} \left[\int_G 2^{Qn/2} K(2^{n/2}(y \cdot z^{-1})) f(z) dz \right] dy \\ =& \frac{1}{\mu(Q_\alpha^k)} \int_G \left[\int_{Q_\alpha^k} 2^{Qn/2} K(2^{n/2}(y \cdot z^{-1})) dy \right] f(z) dz.\end{aligned} \tag{3.6}$$

For simplicity, we let $d(k,n) := \frac{n}{2} + (\log_2 \delta)k$.

(i) *Case* $d(k,n) > 10$.

In order to estimate (3.6) we partition the domain of the variable z, the whole space G, into the following disjoint subsets:

- $B = \{z : \mathrm{dist}(z, \partial Q_\alpha^k) \le 2^{-[(-\log_2 \delta)k + \frac{d(k,n)}{2}]}\}$
- $A_1 = Q_\alpha^k \cap B^c$
- $A_2 = (Q_\alpha^k)^c \cap B^c$,

which satisfy $G = B \cup A_1 \cup A_2$. Then we have $f = f_{A_1} + f_{A_2} + f_B := f\chi_{A_1} + f\chi_{A_2} + f\chi_B$, and hence

$$\mathbb{E}_k(\psi_n(L)f)(x) = \mathbb{E}_k(\psi_n(L)f_{A_1})(x) + \mathbb{E}_k(\psi_n(L)f_{A_2})(x) + \mathbb{E}_k(\psi_n(L)f_B)(x).$$

In order to estimate $\mathbb{E}_k(\psi_n(L)f)(x)$, we are going to estimate each of the above three terms seperately.

$\cdot$ *Estimate for* f_{A_1}.

We begin with the formula (3.6) with replacing f by f_{A_1},

$$\begin{aligned}\mathbb{E}_k(\psi_n(L)f_{A_1}(x)) =& \frac{1}{\mu(Q_\alpha^k)} \int_{Q_\alpha^k} \left[\int_G 2^{Qn/2} K(2^{n/2}(y \cdot z^{-1})) 1_{A_2}(z) f(z) dz \right] dy \\ =& \frac{1}{\mu(Q_\alpha^k)} \int_G \left[\int_{Q_\alpha^k} 2^{Qn/2} K(2^{n/2}(y \cdot z^{-1})) dy \right] \chi_{A_1}(z) f(z) dz.\end{aligned} \tag{3.7}$$

Using $\int_G K = 0$ and observing that $|y \cdot z^{-1}| \ge 2^{-[(-\log_2 \delta)k + \frac{d(k,n)}{2}]}$ holds for any $z \in A_1 = Q_\alpha^k \cap B^c$ and $y \in (Q_\alpha^k)^c$, we deduce

$$\begin{aligned}\left| \int_{Q_\alpha^k} 2^{Qn/2} K(2^{n/2}(y \cdot z^{-1})) dy \right| =& \left| \int_{(Q_\alpha^k)^c} 2^{Qn/2} K(2^{n/2}(y \cdot z^{-1}))\, dy \right| \\ \le& \int_{(Q_\alpha^k)^c} 2^{Qn/2} |K(2^{n/2}(y \cdot z^{-1}))|\, dy \\ \le& \int_{|w| \ge 2^{-[(-\log_2 \delta)k + \frac{d(k,n)}{2}]}} 2^{Qn/2} |K(2^{n/2} w)|\, dw \\ \le& \int_{|w| \ge 2^{m/2}} |K(w)|\, dw \le 2^{-m/2c},\end{aligned} \tag{3.8}$$

where the last inequality follows from (3.5). Plugging this estimate into (3.7), we obtain

$$|\mathbb{E}_k(\psi_n(L)f(x))| \le \frac{1}{\mu(Q_\alpha^k)} \int_G 2^{-mc/2} 1_{A_1}(z) f(z) dz \le 2^{-mc/2} Mf(x). \tag{3.9}$$

$\cdot$ *Estimate for* f_{A_2}.

Like (3.7) we have

$$\mathbb{E}_k(\psi_n(L)f_{A_2}(x)) \;=\; \frac{1}{\mu(Q_\alpha^k)} \int_{Q_\alpha^k} \left[\int_G 2^{Qn/2} K(2^{n/2}(y\cdot z^{-1})) 1_{A_2}(z) f(z) dz \right] dy. \tag{3.10}$$

By definition it holds that $|(y\cdot z^{-1})| \ge 2^{-[(-\log_2\delta)k + \frac{d(n,k)}{2}]}$ for any $z \in A_2 = (Q_\alpha^k)^c \cap B^c$ and $y \in Q_\alpha^k$, which leads to $|2^{n/2}(y\cdot z^{-1})| \ge 2^{n/2+(\log_2\delta)k - \frac{d(n,k)}{2}} = 2^{\frac{d(n,k)}{2}}$. Combining this and (3.5) we find

$$\sup_{y\in Q_\alpha^k} \int_{A_2} 2^{Qn/2} \left| K(2^{n/2}(y\cdot z^{-1})\right| dz \lesssim \int_{|x|\ge 2^{d(n,k)/2}} (1+|x|)^{-3N} dx \lesssim 2^{-d(n,k)N},$$

and in fact, we can also deduce

$$\sup_{y\in Q_\alpha^k} \int_G \left| 2^{Qn/2} K(2^{n/2}(y\cdot z^{-1})) 1_{A_2} f(z) \right| dz \lesssim Mf(x)\cdot 2^{-d(n,k)N},$$

for any large $N > 0$. Applying this estimate in (3.10) we obtain

$$\mathbb{E}_k(\psi_n(L)f_{K_2}(x)) \lesssim \frac{2^{-d(n,k)N}}{\mu(Q_\alpha^k)} \int_{Q_\alpha^k} Mf(x) dy = Mf(x)\cdot 2^{-d(n,k)N}. \tag{3.11}$$

$\cdot$ *Estimate for* f_B.

Having the formula like (3.7) again, we use elementary estimates to get

$$\begin{aligned} |\mathbb{E}_k(\phi_n(L)f_B)(x)| &= \frac{1}{\mu(Q_\alpha^k)} \left| \int_B \left[\int_{Q_\alpha^k} 2^{Qn/2} K(2^{n/2}(y\cdot z^{-1}) dy \right] f(z) dz \right| \\ &\le \frac{1}{\mu(Q_\alpha^k)} \int_B \left(\int_{Q_\alpha^k} 2^{Qn/2} |K(2^{n/2}(yz^{-1}))| dy \right) f(z) dz \\ &\le \frac{1}{\mu(Q_\alpha^k)} \int_B \left(\int_G 2^{Qn/2} |K(2^{n/2}(y))| dy \right) |f(z)| dz \\ &= \frac{C}{\mu(Q_\alpha^k)} \int_B |f(z)| dz. \end{aligned} \tag{3.12}$$

By the property (3.1) we have $\mu(B) \lesssim \mu(Q_\alpha^k) 2^{-\frac{d(n,k)}{2}\rho}$. Using this and Hölder's inequality we estimate (3.12) as

$$\begin{aligned} |\mathbb{E}_k(\phi_n(L)f_B)(x)| &\lesssim \frac{1}{\mu(Q_\alpha^k)} \int_B |f(z)| dz \lesssim \frac{\mu(B)^{\frac{1}{q'}}}{\mu(Q_\alpha^k)} \left(\int_B |f(z)|^q dz \right)^{1/q} \\ &\lesssim 2^{-\frac{\rho}{2q'} d(n,k)} \left(\frac{1}{\mu(Q_k^\alpha)} \int_B |f(z)|^q dx \right)^{1/q} \lesssim 2^{-\frac{\rho}{2q'} d(n,k)} M_q f(x). \end{aligned} \tag{3.13}$$

Combining (3.9), (3.11) and (3.13) we get the desired estimate

$$\begin{aligned}|\mathbb{E}_k(\phi_n(L)f)| &= |\mathbb{E}_k(\phi_n(L)(f_{A_1} + f_{A_2} + f_B))(x)| \\ &\lesssim 2^{-d(n,k)\gamma} M_q f(x),\end{aligned}$$

where $\gamma = \min(\frac{c}{2}, \frac{\rho}{2q'})$. It proves the lemma in the case $d(k,n) > 10$.
(ii) *Case* $d(k,n) < 10$.

By the definitions of $\mathbb{D}_k$ and $\mathbb{E}_k$ we have

$$\begin{aligned}&\mathbb{D}_k(\psi_n(L)f)(x) = \mathbb{E}_{k+1}(\psi_n(L)f) - \mathbb{E}_k(\psi_n(L)f) \\ &= \frac{1}{\mu(Q_\alpha^{k+1})}\int_{Q_\alpha^{k+1}} (\psi_n(L)f)(y)dy - \frac{1}{\mu(Q_\alpha^k)}\int_{Q_\alpha^k}(\psi_n(L)f)(y)dy \\ &= \int_G f(z)\left[\frac{1}{\mu(Q_\alpha^{k+1})}\int_{Q_\alpha^{k+1}} 2^{Qn/2}K(2^{n/2}(y\cdot z^{-1}))dy - \frac{1}{\mu(Q_\alpha^k)}\int_{Q_\alpha^k} 2^{Qn/2}K(2^{n/2}(y\cdot z^{-1}))dy\right]dz.\end{aligned}$$

Here, adding the integration of the identity $2^{Qn/2}K(2^{n/2}(x\cdot z^{-1})) - 2^{Qn/2}K(2^{n/2}(x\cdot z^{-1})) = 0$ over G, we get

$$\begin{aligned}&\mathbb{D}_k(\psi_n(L)f)(x) \\ &= \int_G f(z)\left[\frac{1}{\mu(Q_\alpha^{k+1})}\int_{Q_\alpha^{k+1}} 2^{Qn/2}\left[K(2^{n/2}(y\cdot z^{-1})) - K(2^{n/2}(x\cdot z^{-1}))\right]dy\right]dz \\ &\qquad - \int_G f(z)\left[\frac{1}{\mu(Q_\alpha^k)}\int_{Q_\alpha^k} 2^{Qn/2}\left[K(2^{n/2}(y\cdot z^{-1})) - K(2^{n/2}(x\cdot z^{-1}))\right]dy\right]dz \\ &:= A_1 + A_2.\end{aligned} \tag{3.14}$$

By the mean value theorem for stratified groups (see [**?**, Theorem 1.33]) there is a constant $\beta = \beta(G) > 0$ such that

$$\begin{aligned}&\left|K((2^{n/2}(yx^{-1})\cdot 2^{n/2}(xz^{-1})) - K(2^{n/2}(xz^{-1}))\right| \\ &\qquad \lesssim \sum_{j=1}^d |2^{n/2}(yx^{-1})|^{d_j} \sup_{|w|\le|\beta 2^{n/2}(yx^{-1})|}\left|X_jK(w2^{n/2}(xz^{-1})\right|.\end{aligned} \tag{3.15}$$

For $x, y \in Q_\alpha^k$ we have $|(yx^{-1})| \le \delta^k$, and so $|2^{n/2}(yx^{-1})| \le 2^{n/2}2^{(\log_2\delta)k} \le 2^{-10}$ by the assumption. Using this we deduce that

$$\begin{aligned}&\sum_{j=1}^d |2^{n/2}(yx^{-1})|^{d_j} \sup_{|w|\le|\beta 2^{n/2}(yx^{-1})|}\left|X_jK(w2^{n/2}(xz^{-1})\right| \\ &\lesssim \sum_{j=1}^d (2^{n/2}\delta^k)^{d_j}\sup_{|w|\le\beta 2^{-10}}\left(1+|w2^{n/2}(xz^{-1})|\right)^{-N} \lesssim (2^{n/2}\delta^k)\left(1+|2^{n/2}(xz^{-1})|\right)^{-N}.\end{aligned} \tag{3.16}$$

Combining (3.14), (3.15) and (3.16) we get

$$|A_1| \lesssim (2^{n/2}\delta^k) \int_G 2^{Qn/2} \left(1 + |2^{n/2}(xz^{-1})|\right)^{-N} f(z)dz$$
$$\lesssim (2^{n/2}\delta^k) Mf(x),$$

and the same argument shows that $|A_2| \lesssim (2^{n/2}\delta^k)Mf(x)$. Consequently, we can bound (3.14) as

$$|\mathbb{D}_k(\psi_n(L)f)(x)| \lesssim (2^{n/2}\delta^k)Mf(x) = 2^{-|(\log_2 \delta)k+n/2|}Mf(x),$$

which finishes the proof. □

We set $\mathcal{M} = M \circ M \circ M$ and

$$G_r(f) = (\sum_{k\in\mathbb{Z}} (\mathcal{M}(|L_k f|^r))^{2/r})^{1/2}. \tag{3.17}$$

Let us recall the inequality of Fefferman-Stein [FeS]:

$$\|G_r(f)\|_p \le C_{p,r}\|f\|_p, \qquad 1 < r < 2,\ r < p < \infty. \tag{3.18}$$

Now we apply Lemma 3.3.3 to prove the following lemma.

Lemma 3.3.4. *If* $1 < r \le \infty$ *and* $\alpha > \frac{Q}{r}$*, then we have*

$$S(m(L)f)(x) \lesssim \|m\|_{L_2^\alpha} G_r(f)(x) \quad \forall x \in G. \tag{3.19}$$

If we further assume that $m(\xi) = 0$ *for* $|\xi| \le N$*, then we have*

$$\mathbb{E}_0(m(L)f)(x) \lesssim 2^{-N}\|m\|_{L_2^\alpha} G_r(f)(x). \tag{3.20}$$

Proof. Using Lemma 3.3.3 we get

$$|\mathbb{B}_k(m(L)f)(x)| = \left|\sum_{n\in\mathbb{Z}} \mathbb{B}_k(\psi_n(L)m_n(L)\psi_n(L)f)(x)\right|$$
$$\lesssim \|m\|_{L_2^\alpha} \sum_{n\in\mathbb{Z}} 2^{-\left|k|\log_2\delta|-n\right|\gamma} M_r(\psi_n(L)f).$$

Apply the Cauchy-Schwartz inequality to get

$$|\mathbb{B}_k(m(L)f)(x)|^2 \lesssim \left(\sum_{n\in\mathbb{Z}} 2^{-\left|k|\log_2\delta|-n\right|\gamma}\right) \sum_{n\in\mathbb{Z}} 2^{-\left||k\log_2\delta|-n\right|\gamma} (M_r(\psi_n(L)f))^2$$
$$\lesssim \sum_{n\in\mathbb{Z}} 2^{-\left||k\log_2\delta|-n\right|\gamma} (M_r(\psi_n(L)f))^2.$$

Summing this over $k \in \mathbb{N}$ we get

$$\begin{aligned} S(m(L)f)(x)^2 &= \sum_{k=1}^{\infty} |\mathbb{B}_k(m(L)f)(x)|^2 \\ &\lesssim \sum_{k=1}^{\infty} \sum_{n\in\mathbb{Z}} 2^{-\|k\log_2\delta|-n|\gamma} |M_r(\psi_n(L)f)(x)|^2 \\ &\lesssim \sum_{n\in\mathbb{Z}} |M_r(\psi_n(L)f)(x)|^2. \end{aligned}$$

It proves the first inequality of the lemma.

Next we consider the case $m(\xi) = 0$ for $|\xi| \le N$. By this assumption and Lemma 3.3.3, we find

$$\begin{aligned} |\mathbb{E}_0(m(L)f)(x)| &= \left| \sum_{n\ge N-1} \mathbb{E}_0(\psi_n(L)m_n(L)\psi_n(L)f)(x) \right| \\ &\lesssim \|m\|_{L_2^\alpha} \sum_{n\ge N-1} 2^{-n} M_r(\psi_n(L)f)(x) \\ &\lesssim \|m\|_{L_2^\alpha} 2^{-N} M_r(\psi_n(L)f)(x), \end{aligned}$$

which completes the proof of the lemma. □

We are ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. For the proof, we combine Lemma 3.3.4 with the argument of Grafakos-Honzik-Seeger [GHS]. First we write

$$\left\| \sup_{1\le i\le N} |T_i f| \right\|_p = \left(p4^p \int_0^\infty \lambda^{p-1} \mathrm{meas}\left(\left\{ x : \sup_i |T_i f(x)| > 4\lambda \right\} \right) d\lambda \right)^{1/p}, \tag{3.21}$$

and we note that

$$\left| \left\{ x : \sup_{1\le i\le N} |T_i f(x)| > 4\lambda \right\} \right| \le |E_\lambda| + |F_\lambda|,$$

where $E_\lambda = \{x : \sup_{1\le i\le N} |T_i f(x) - \mathbb{E}_0 T_i f(x)| > 2\lambda\}$ and $F_\lambda = \{x : \sup_{1\le i\le N} |\mathbb{E}_0 T_i f(x)| > 2\lambda\}$. Concerning F_λ, we use estimate (3.20) to get the estimate

$$\left(p2^p \int_0^\infty \lambda^{p-1} \mathrm{meas}(F_\lambda) d\lambda \right)^{1/p} = \left\| \sup_{1\le i\le N} |\mathbb{E}_0 T_i f(x)| \right\|_p \le \sum_{1\le i\le N} \|\mathbb{E}_0 T_i f(x)\|_p \lesssim N2^{-N} \|M(|f|^r)\|_p . \tag{3.22}$$

Next, we bound $|E_\lambda|$ once more by $|E_\lambda| \le |E_{\lambda,1}| + |E_{\lambda,2}|$ where we set

$$\begin{aligned} E_{\lambda,1} &= \left\{ x : \sup_{1\le i\le N} |T_i f(x) - \mathbb{E}_0 T_i f(x)| > 2\lambda, G_r(f)(x) \le \varepsilon_N \lambda \right\}, \\ E_{\lambda,2} &= \{ x : G_r(f)(x) > \varepsilon_N \lambda \}, \end{aligned}$$

and

$$\epsilon_N := \left(\frac{C_1}{\log(N+1)}\right)^{1/2}.$$

It holds that $S(T_i f) \le A_r B G_r(f)$ by Lemma 3.3.4, and so we have

$$E_{\lambda,1} \subset \bigcup_{i=1}^{N} \{x : |T_i f(x) - \mathbb{E}_0 T_i f(x)| > 2\lambda, S(T_i f) \lesssim \varepsilon_N \lambda\}.$$

By using inequality (3.2) we deduce that

$$\begin{aligned} \text{meas}(E_{\lambda,1}) &\le \sum_{i=1}^{N} \text{meas}(\{x : |T_i f(x) - \mathbb{E}_0 T_i f(x)| > 2\lambda, S(T_i f) \le \varepsilon_N \lambda\}) \\ &\le \sum_{i=1}^{N} C \exp\left(-\frac{C_1}{\varepsilon_N^2}\right) \text{meas}\left(\left\{x : \sup_k |\mathbb{E}_k(T_i f)| > \lambda\right\}\right). \end{aligned}$$

Therefore

$$\begin{aligned} &\left(p \int_0^\infty \lambda^{p-1} \text{meas}(E_{\lambda,1}) d\lambda\right)^{1/p} \\ &\lesssim \left(\sum_{i=1}^{N} \exp\left(-\frac{C_1}{\varepsilon_N^2}\right) \| \sup_k |\mathbb{E}_k(T_i f)\|_p^p\right)^{1/p} \lesssim \left(\sum_{i=1}^{N} \exp\left(-\frac{C_1}{\varepsilon_N^2}\right) \|T_i f\|_p^p\right)^{1/p} \\ &\lesssim B\left(N \exp\left(-\frac{C_1}{\varepsilon_N^2}\right)\right)^{1/p} \|f\|_p \lesssim \|f\|_p. \end{aligned} \tag{3.23}$$

Performing a change of variables and using (3.18), we obtain

$$\begin{aligned} \left(p \int_0^\infty \lambda^{p-1} \text{meas}(E_{\lambda,2}) d\lambda\right)^{1/p} &= \frac{B}{\varepsilon_N} \|G_r(f)\|_p \\ &\lesssim B \sqrt{\log(N+1)} \|f\|_p. \end{aligned} \tag{3.24}$$

From (3.22), (3.23) and (3.24) we get the desired estimate for (3.21). The proof is completed. □

Now we shall prove Theorem 3.1.2.

Proof of Theorem 3.1.2. First we consider the dyadic maximal multiplier

$$\mathcal{M}_m^{\text{dyad}} f(x) = \sup_{k \in \mathbb{Z}} |m(2^k L) f(x)|,$$

where m is a function such that for some $\alpha > Q/p$ we have $\|\phi(\cdot) m(2^k \cdot)\|_{H^\alpha} \le \omega(k)$ for each $k \in \mathbb{Z}$ where the non-increasing rearrangement ω^* satisfies $\omega^*(0) + \sum_{l=1}^{\infty} \frac{\omega^*(l)}{l/\sqrt{\log l}} < \infty$. We set

$$\mathcal{I}_j = \{k \in \mathbb{Z} : w^*(2^{2^j}) < |\omega(k)| \le \omega^*(2^{2^{j-1}})\} \quad j \in \mathbb{N},$$

and split $m = \sum_{j=1}^{\infty} m_j$ so that m_j has support in the union of dyadic interval $\cup_{k\in\mathcal{I}_j}\{\xi : 2^{k-1} < |\xi| < 2^{k+1}\}$. For any $k \in \mathbb{Z}$ and $1 \le j < \infty$, we define $T_k^j f = m_j(2^k L)f$. Note that we have $\sup_{k\in\mathbb{Z}} \|\phi(\cdot)m_j(2^k\cdot)\|_{H^\alpha} \le w(k) \le w^*(2^{2^{j-1}})$ for each $1 \le j < \infty$.

Lemma 3.1 in [CGHS] guarantees that there exists a set of integers $B = \{b_i\}_{i\in\mathbb{N}} \subset \mathbb{Z}$ such that $\mathbb{Z} = \cup_{n=-4^{2^j+1}}^{4^{2^j+1}}(n + B)$ and elements of $\{b_i + \mathcal{I}_j\}_{i\in\mathbb{Z}}$ are pairwise disjoint for each $1 \le j < \infty$. Next we write

$$\left\|\mathcal{M}_{m_j}^{\text{dyad}} f(x)\right\|_p = \left\|\sup_k |T_k^j f|\right\|_p = \left\|\sup_{|n|\le 4^{2^j+1}} \sup_{i\in\mathbb{Z}} |T_{b_i+n}^j f|\right\|_p. \tag{3.25}$$

To estimate the right hand side, we shall use the L^p norm equivalence of Rademacher functions $\{r_i\}_{i=1}^{\infty}$

$$c_p\left(\sum_i a_i^2\right)^{1/2} \le \left(\int_0^1 \left|\sum_{i=1}^{\infty} r_i(s)a_i\right|^2 ds\right)^{1/p} \le C_p\left(\sum_i a_i^2\right)^{1/2},$$

where $\{a_i\}_{i\in\mathbb{Z}}$ is any set of real numbers (see e.g. [Su, p. 276]). Then

$$\begin{aligned}
\left\|\sup_{|n|\le 4^{2^j+1}} \sup_{i>0} |T_{b_i+n}^j f|\right\|_p &\le \left\|\sup_{|n|\le 4^{2^j+1}} \left(\sum_{i>0} |T_{b_i+n}^j f|^2\right)^{1/2}\right\|_p \\
&\le C_p \left\|\sup_{|n|\le 4^{2^j+1}} \left(\int_0^1 \left|\sum_{i=1}^{\infty} r_i(s) T_{b_i+n}^j f\right|^p ds\right)^{1/p}\right\|_p \\
&\le C_p \left\|\left(\int_0^1 \sup_{|n|\le 4^{2^j+1}} \left|\sum_{i=1}^{\infty} r_i(s) T_{b_i+n}^j f\right|^p dx\right)^{1/p}\right\|_p \\
&= C_p \left(\int_0^1 \left\|\sup_{|n|\le 4^{2^j}} \left|\sum_{i=1}^{\infty} r_i(s) T_{b_i+n}^j f\right|\right\|_p^p ds\right)^{1/p}.
\end{aligned} \tag{3.26}$$

Applying Theorem 3.1.1 we have

$$\left\|\sup_{|n|\le 4^{2^j}} \left|\sum_{i=1}^{\infty} r_i(s) T_{b_i+n}^j f\right|\right\|_p \le C\sqrt{\log(4^{2^j})}\|f\|_p. \tag{3.27}$$

Combining (3.25),(3.26), and (3.27) we obtain

$$\|\mathcal{M}_{m_j}^{\text{dyad}}\|_{L^p\to L^p} \lesssim 2^{j/2}\omega^*(2^{2^{j-1}}),$$

which yields that $\|\mathcal{M}_m^{\text{dyad}}\|_{L^p\to L^p} \lesssim \sum_{j=1}^{\infty} 2^{j/2}\omega^*(2^{2^{j-1}}) \lesssim 1$.

In order to obtain the bound of $\mathcal{M}_m$ using the property of $\mathcal{M}_m^{\text{dyad}}$, we use a standard argument to see that

$$\begin{aligned}
\sup_{2^k\le t<2^{k+1}} |m_j(tL)f(x)| &= \sup_{1\le t<2} |m_j(t2^k L)f(x)| \\
&\le |m(2^k L)f(x)| + \int_1^2 \left|\frac{\partial}{\partial t} m_j(t2^k L)f(x)\right| dt,
\end{aligned}$$

and we note that

$$\left\|\phi(s)(\partial/\partial t)m_j(t2^k s)\right\|_{H^\alpha} \lesssim \sum_{l=k-1}^{k+1} \|\phi(s)m_j(2^l s)\|_{H^{\alpha+1}}$$

holds uniformly for $1 \leq t \leq 2$. Hence the boundedness of $\mathcal{M}_m$ follows from the boundedness property of $\mathcal{M}_m^{\text{dyad}}$ obtained in the above. The proof is finished. □

3.4 Maximal multipliers on product spaces

In this section we study the maximal functions of multi-parameter multipliers on product spaces of stratified groups, which leads to the proof of Theorem 3.1.3 for maximal multipliers on the product spaces. As a byproduct, we obtain result of Theorem 3.1.4 for maximal functions of the joint spectral multipliers on the Heisenberg group. The main tools of this section are multi-expectation operators on product spaces and the maximal intermediate square functions introduced by Pipher [Ph] and Honzik [Ho1].

Let G be the direct product of n stratified groups $G_1, \cdots, G_n$ endowed with sub-Laplacians $L_1, \cdots, L_n$. We naturally lift the sub-Laplacians to the operators $L_1^\sharp, \cdots, L_n^\sharp$ defined on the product space G. Then $L_1^\sharp, \cdots, L_n^\sharp$ mutually commute and so their spectral measures $dE_1(\xi), \cdots, dE_n(\eta)$ also mutually commute. Thus, for a bounded function m on $\mathbb{R}_+^n$, we can define the multi-parameter multiplier

$$m(L_1^\sharp, \cdots, L_n^\sharp) = \int_{\mathbb{R}_+^n} m(\xi_1, \cdots, \xi_n)\, dE_1(\xi) \cdots dE_n(\xi).$$

In some cases we shall denote $m(L_1^\sharp, \cdots, L_n^\sharp)$ by $m(L)$ for notational simplicity. Under the assumption

$$|(\xi_1\partial_{\xi_1})^{\alpha_1} \cdots (\xi_n\partial_{\xi_n})^{\alpha_n} m(\xi_1, \cdots, \xi_n)| \leq C_\alpha \tag{3.1}$$

for all $\alpha_j \leq N$ with N large enough, Müller-Ricci-Stein [MRS] proved that $m(L_1^\sharp, \cdots, L_n^\sharp)$ is bounded on $L^p(G)$.

By Theorem 3.3.1, for each group G_k we can find a martingales $\{Q_\alpha^{k,j} : j \in \mathbb{N}_0, \alpha \in I_j\}$ with index sets I_j, satisfying the conditions of Theorem 3.3.1. For $1 \leq j \leq n$ and $k \in \mathbb{N}_0$ we define the $j-$th variable expectation $\mathcal{E}_k^j : S(G) \to S(G)$ by

$$\mathcal{E}_k^j f(x_1, \cdots, x_n) = \frac{1}{|Q_\alpha^{k,j}|} \int_{Q_\alpha^{k,j}} f(x_1, \cdots, x_n) dx_j,$$

where $\alpha \in I_j$ is a unique index such that $x_j \in Q_\alpha^{k,j}$. We shall simply denote $\mathcal{E}_0^j$ by $\mathcal{E}^j$. For $1 \leq s \leq n$, $1 \leq n_1 < n_2 \cdots < n_s \leq n$ and $(k_1, \cdots, k_s) \in \mathbb{N}_0^s$, we define

$$\mathcal{E}_{k_1,\cdots,k_s}^{n_1,\cdots,n_s} := \mathcal{E}_{k_1}^{n_1} \cdots \mathcal{E}_{k_s}^{n_s}.$$

Also it is convenient to set

$$\mathcal{E}^{n_1,\cdots,n_s}_{k_1,\cdots,k_s} = 0, \tag{3.2}$$

for any $(k_1, \cdots, k_s) \in (\mathbb{N}_0 \cup \{-1\})^s$ such that $k_t = -1$ for some $t \in \{1, \cdots, s\}$. Then we can define martingales by

$$D^{n_1,\cdots,n_s}_{k_1,\cdots,k_s} g = \sum_{\substack{a_j \in \{0,1\} \\ 1 \le j \le s}} (-1)^{a_1+\cdots+a_s}\, \mathcal{E}^{n_1,\cdots,n_s}_{k_1-a_1,\cdots,k_s-a_s} g.$$

For each $1 \le t \le s$ we have

$$\begin{aligned} D^{n_1,\cdots,n_s}_{k_1,\cdots,k_s} g &= \sum_{\substack{a_j \in \{0,1\} \\ 1 \le j \le s, j \ne t}} \sum_{a_t \in \{0,1\}} (-1)^{a_1+\cdots+a_s}\, \mathcal{E}^{n_1,\cdots,n_s}_{k_1-a_1,\cdots,k_s-a_s} g \\ &= \sum_{\substack{a_j \in \{0,1\} \\ 1 \le j \le s, j \ne t}} (-1)^{a_1+\cdots+\widehat{a_t}+\cdots+a_s} \left[\mathcal{E}^{n_1,\cdots,n_s}_{k_1-a_1,\cdots,k_t,\cdots,k_s-a_s} g - \mathcal{E}^{n_1,\cdots,n_s}_{k_1-a_1,\cdots,k_t-1,\cdots,k_s-a_s} g\right], \end{aligned}$$

where we use the notation $a_1 + \cdots + \widehat{a_t} + \cdots + a_s = a_1 + \cdots + a_{t-1} + a_{t+1} + \cdots + a_s$. Using this we can see that

$$\begin{aligned} & D^{n_1,\cdots,n_s}_{k_1,\cdots,k_t+1,\cdots,k_s} g + D^{n_1,\cdots,n_s}_{k_1,\cdots,k_j,\cdots,k_s} g \\ &\quad = \sum_{\substack{a_j \in \{0,1\} \\ 1 \le j \le s, j \ne t}} (-1)^{a_1+\cdots+\widehat{a_t}+\cdots+a_s} \Big\{ \left(\mathcal{E}^{n_1,\cdots,n_s}_{k_1-a_1,\cdots,k_t+1,\cdots,k_s-a_s} g + \mathcal{E}^{n_1,\cdots,n_s}_{k_1-a_1,\cdots,k_t,\cdots k_s-a_s} g\right) \\ &\qquad\qquad + (-1) \left(\mathcal{E}^{n_1,\cdots,n_s}_{k_1-a_1,\cdots,k_t,\cdots,k_s-a_s} g + \mathcal{E}^{n_1,\cdots,n_s}_{k_1-a_1,\cdots,k_t-1,\cdots k_s-a_s} g\right)\Big\}. \\ &\quad = \sum_{\substack{a_j \in \{0,1\} \\ 1 \le j \le s, j \ne t}} (-1)^{a_1+\cdots+\widehat{a_t}+\cdots+a_s} \left(\mathcal{E}^{n_1,\cdots,n_s}_{k_1-a_1,\cdots,k_t+1,\cdots,k_s-a_s} g - \mathcal{E}^{n_1,\cdots,n_s}_{k_1-a_1,\cdots,k_t-1,\cdots,k_s-a_s} g\right). \end{aligned}$$

Using this summation rule iteratively, we find

$$\sum_{k_t=0}^{\infty} D^{n_1,\cdots,n_s}_{k_1,\cdots,k_s} g(x) = D^{n_1,\cdots,\widehat{n_t},\cdots,n_s}_{k_1,\cdots,\widehat{k_t},\cdots,k_s} g(x), \tag{3.3}$$

where $\widehat{a}$ denotes the absence of a, i.e.,

$$(n_1, \cdots, \widehat{n_j}, \cdots, n_s) = (n_1, \cdots, n_{j-1}, n_{j+1}, \cdots, n_s) \in \mathbb{N}^{s-1}.$$

In what follows, we shall use the notation that

$$\sum_{k_{j_1},\cdots,k_{j_m}} := \sum_{k_{j_1}=0}^{\infty} \cdots \sum_{k_{j_m}=0}^{\infty}.$$

[Ch2]

For $1 \le m \le n$, we will simply denote $D^{1,2,\cdots,m}_{k_1,\cdots,k_m}$ by $D_{k_1,\cdots,k_m}$. Then, using (3.3) repeatedly we have

$$\sum_{k_1,\cdots,k_n} D_{k_1,\cdots,k_n} f(x) = f(x).$$

Set

$$\mathcal{A}f := (1-\mathcal{E}^1)\cdots(1-\mathcal{E}^n)f.$$

Denote by $\mathcal{A}(S(G))$ the image of $S(G)$ under the operator $\mathcal{A}$ and denote by $\mathcal{A}_j(S(G)) := (1-\mathcal{E}^j)(S(G))$ be the image of $S(G)$ under the operator $1-\mathcal{E}^j$. Note that, for each $1 \le j \le N$ we have

$$\mathcal{E}^j g = 0 \qquad \forall g \in \mathcal{A}_j(S(G)).$$

For $2 \le m \le n+1$ we introduce the intermediate square functions S_m and the maximal intermediate square function S^*_m defined by Honzik [Ho1] generalizing the double square functions defined by Pipher [Ph],

$$S_m f = \left(\sum_{k_1,\cdots,k_{m-1}} \left(\sum_{k_m,\cdots,k_n} D_{k_1,\cdots,k_n} f(x) \right)^2 \right)^{1/2}, \tag{3.4}$$

$$S^*_m f = \sup_r \left(\sum_{k_1,\cdots,k_{m-1}} \left(\sum_{k_m<r,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n} f(x) \right)^2 \right)^{1/2}. \tag{3.5}$$

For $m = 1$ we define the following maximal function

$$S_1 f(x) = \sup_r \left| \sum_{m_1 \le r, m_2,\cdots,m_n} D_{m_1,\cdots,m_n} f(x) \right|.$$

We then have the following lemma.

Lemma 3.4.1 ([Ph])**.** *Suppose $X^j_N = \sum_{q=0}^N d^j_q$, $j = 1,\cdots,M$ is a sequence of dyadic martingales and set*

$$S X^j_N = \left(\sum_q^N (d^j_q)^2 \right)^{1/2}$$

be the square function of X^j_N. Then

$$\int \exp\left(\sqrt{1 + \sum_{j=1}^M (X^j_N)^2} - \sum_{j=1}^M (S X^j_N)^2 \right) dx \le e.$$

Based on this lemma, Pipher [Ph] obtained a good λ inequality for product of two spaces and Honzik generalized it to product of n spaces with arbitrary $n \in \mathbb{N}$. Here we state and prove a variant version of it.

Lemma 3.4.2. *Let $2 \le m \le n$ and $x_1, \cdots, \widehat{x_m}, \cdots, x_n \in G_1 \times \cdots \widehat{G_m} \cdots \times G_n$, there exist constants $C > 0$ and $c > 0$ such that*

$$|\{x_m \in G_m : S_m^*(g(x_1, \cdots, x_n)) > 2\lambda; S_{m+1}g(x_1, \cdots, x_n) < \epsilon\lambda\}| \le Ce^{-c/\epsilon^2}|\{x_m : S_m^*g(x_1, \cdots, x_n) > \lambda\}|$$

holds for any $0 < \epsilon < 1/10$, $0 < \lambda < \infty$ and $g \in \mathcal{A}_m(S(G))$. The constants C and c are independent of $(x_1, \cdots, \hat{x}_m, \cdots, x_n)$.

Proof. Since $g \in \mathcal{A}_m(S(G))$ we have $\mathcal{E}_m g = 0$. Thus, for any $(k_1, \cdots, \widehat{k_m}, \cdots, k_n) \in \mathbb{N}_0^{n-1}$ we get

$$D_{k_1,\cdots,k_{m-1},0,k_{m+1},\cdots,k_n} g(x) = \left(D^{1,\cdots\widehat{m},\cdots,n}_{k_1,\cdots,\widehat{k_m},\cdots,k_n} \circ \mathcal{E}_m\right) g(x) = 0,$$

which leads to the equality

$$\left(\sum_{k_1,\cdots,k_{m-1}} \left(\sum_{k_m<1,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n} g(x)\right)^2\right)^{1/2} = \left(\sum_{k_1,\cdots,k_{m-1}} \left(\sum_{k_m=0,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n} g(x)\right)^2\right)^{1/2} = 0.$$

By this and definition (3.5), for each $x_m \in \{x_m \in G_m : S_m^* g(x) > \lambda\}$, we can find a minimal integer $r \ge 2$ such that

$$\left(\sum_{k_1,\cdots,k_{m-1}} \left(\sum_{k_m<r,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n} g(x)\right)^2\right)^{1/2} > \lambda. \tag{3.6}$$

By the property of martingales there exists a unique index α such that $x_m \in Q^m_{r,\alpha}$. Then (3.6) can be written as follows

$$\left(\sum_{k_1,\cdots,k_{m-1}} \left(D_{k_1,\cdots,k_{m-1}} \inf_{Q^m_{r,\alpha}} g\right)^2\right)^{1/2} > \lambda,$$

where $\inf_Q dx$ denote the average integral $\frac{1}{|Q|}\int_Q dx$. As a result, the set $\{x_m \in G_m : S_m^* g(x) > \lambda\}$ consists of such maximal martingales $\{Q^m_{r_j,\alpha_j}\}_{j\in I}$ with $r_j \ge 2$, where I is an index set and $Q^m_{r_j,\alpha_j}$'s are mutually disjoint.

Choose a set $Q^m_{r,\alpha} \subset \{Q^m_{r_j,\alpha_j}\}_{j\in I}$ such that $Q^m_{r,\alpha} \cap \{x_m \in G_m : S_{m+1}g(x) \le \epsilon\lambda\} \ne 0$. Then, we claim that, for any $x_m \in Q^m_{r,\alpha} \cap \{x_m \in G_m : S_{m+1}g(x) \le \epsilon\lambda\}$, the following holds true.

$$\left(\sum_{k_1,\cdots,k_{m-1}} \left(\sum_{k_m<r,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n} g(x)\right)^2\right)^{1/2} \le (1+\epsilon)\lambda. \tag{3.7}$$

With a view to a contradiction, we suppose that (3.7) does not holds, i.e.,

$$\left(\sum_{k_1,\cdots,k_{m-1}}\left(\sum_{k_m<r,\cdots,k_n} D_{k_1,\cdots,k_n}g(x)\right)^2\right)^{1/2} > (1+\epsilon)\lambda.$$

Since $S_{m+1}g(x) \le \epsilon\lambda$, we have

$$\left(\sum_{k_1,\cdots,k_{m-1}}\left(\sum_{k_m=r,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n}g(x)\right)^2\right)^{1/2} < \left(\sum_{k_1,\cdots,k_m}\left(\sum_{k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n}g(x)\right)^2\right)^{1/2} < \epsilon\lambda.$$

Using this we get

$$\begin{aligned}&\left(\sum_{k_1,\cdots,k_{m-1}}\left(\sum_{k_m<r-1,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n}g(x)\right)^2\right)^{1/2}\\ &>\left(\sum_{k_1,\cdots,k_{m-1}}\left(\sum_{k_m<r,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n}g(x)\right)^2\right)^{1/2} - \left(\sum_{k_1,\cdots,k_{m-1}}\left(\sum_{k_m=r,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n}g(x)\right)^2\right)^{1/2}\\ &> (1+\epsilon)\lambda - \epsilon\lambda = \lambda.\end{aligned}$$

However, this means that integer $r-1$ also satisfies the condition (3.6), which contradicts to the minimality of r. Hence the inequality (3.7) should hold.

Now, we define the subset $q^m_{r,\alpha} \subset Q^m_{r,\alpha}$ by

$$q^m_{r,\alpha} = \{x_m \in Q^m_{r,\alpha} : S_{m+1}g(x) \le \epsilon\lambda \quad \text{and} \quad S^*_m g(x) > 2\lambda\}. \tag{3.8}$$

For each $x_m \in q^m_{r,\alpha}$ we take a minimal number t_x such that

$$\left(\sum_{k_1,\cdots,k_{m-1}}\left(\sum_{k_m<t_x,\cdots,k_n} D_{k_1,\cdots,k_n}g(x)\right)^2\right)^{1/2} > 2\lambda. \tag{3.9}$$

We then make a new martingale on $Q^m_{r,\alpha}$ as follows.

$$g_{new}(x) = \begin{cases} \mathcal{E}^{t_x}_m g(x) - \mathcal{E}^r_m g(x) & \text{if} \quad x_m \in q^m_{r,\alpha}, \\ g(x) - \mathcal{E}^r_m g(x) & \text{if} \quad x_m \notin q^m_{r,\alpha}. \end{cases} \tag{3.10}$$

Then, $\mathcal{E}^r_m g_{new} = 0$, and so we can use a local version of Lemma 3.4.1 to get

$$\begin{aligned}\int_{Q^m_{r,\alpha}} \exp\Bigg[\alpha\left(\sum_{k_1,\cdots,k_{m-1}}\left(\sum_{k_m,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n}g_{new}\right)^2\right)^{1/2}&\\ -\alpha^2\sum_{k_1,\cdots,k_{m-1}}\left(\sum_{k_m}\left(\sum_{k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n}g_{new}\right)^2\right)\Bigg] &\le e|Q^m_{r,\alpha}|.\end{aligned} \tag{3.11}$$

From the construction of g_{new} we get $D_{k_1,\cdots,k_n} g_{new} = 0$ if $k_m \geq t_x$ or $k_m < r$, which implies $\sum_{k_m=1}^{\infty} D_{k_1,\cdots,k_n} g_{new} = \sum_{r\leq k_m<t} D_{k_1,\cdots,k_n} g$ and (3.11) equals to the following inequality

$$\begin{aligned}\int_{Q^m_{r,\alpha}} \exp\Bigg[\alpha\Bigg(\sum_{k_1,\cdots,k_{m-1}}\Bigg(\sum_{r\leq k_m<t_x,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n} g_{new}(x)\Bigg)^2\Bigg)^{1/2} \\ -\alpha^2 \sum_{k_1,\cdots,k_{m-1},r\leq k_m<t_x}\Bigg(\sum_{k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n} g_{new}(x)\Bigg)^2\Bigg] \leq e|Q^m_{r,\alpha}|.\end{aligned} \tag{3.12}$$

In order to bound $q^m_{r,\alpha}$ via this inequality, we are going to find a lower bound of the function in the integration when $x \in q^m_{r,\alpha}$. To this aim, we note from the definitions (3.8) and (3.4) that

$$\Bigg(\sum_{k_1,\cdots,k_{m-1},r\leq k_m<t_x}\Bigg(\sum_{k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n} g(x)\Bigg)^2\Bigg)^{1/2} \leq S_{m+1} g(x) \leq \epsilon\lambda.$$

On the other hand, combining (3.7) and (3.9) yields that

$$\begin{aligned}&\Bigg(\sum_{k_1,\cdots,k_{m-1}}\Bigg(\sum_{r\leq k_m<t_x,\cdots,k_n} D_{k_1,\cdots,k_n} g_{new}(x)\Bigg)^2\Bigg)^{1/2} \\ &\geq \Bigg(\sum_{k_1,\cdots,k_{m-1}}\Bigg(\sum_{k_m<t_x,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n} g_{new}(x)\Bigg)^2\Bigg)^{1/2} \\ &\quad -\Bigg(\sum_{k_1,\cdots,k_{n-1}}\Bigg(\sum_{k_m<r,k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n} g_{new}(x)\Bigg)^2\Bigg)^{1/2} \\ &\geq 2\lambda - (1+\epsilon)\lambda = (1-\epsilon)\lambda.\end{aligned}$$

Therefore, for any $x_m \in q^m_{r,\alpha} \subset Q^m_{r,\alpha}$ we have

$$\begin{aligned}&\alpha\Bigg(\sum_{k_1,\cdots,k_{m-1}}\Bigg(\sum_{r\leq k_m<t_x,\cdots,k_n} D_{k_1,\cdots,k_n} g_{new}(x)\Bigg)^2\Bigg)^{1/2} \\ &\quad - \alpha^2 \sum_{k_1,\cdots,k_{m-1},r\leq k_m<t}\Bigg(\sum_{k_{m+1},\cdots,k_n} D_{k_1,\cdots,k_n} g_{new}(x)\Bigg)^2 \geq \alpha(1-\epsilon)\lambda - \alpha^2\epsilon^2\lambda^2.\end{aligned}$$

Plugging this into (3.12) we get

$$|q^m_{r,\alpha}| \exp(\alpha(1-\epsilon)\lambda - \alpha^2\epsilon^2\lambda^2) \leq e|Q^m_{r,\alpha}|.$$

By taking $\alpha = \frac{1}{2\epsilon^2\lambda}$ here, we get

$$|q^m_{r,\alpha}| \exp\left(\frac{(1-\epsilon)}{2\epsilon^2} - \frac{1}{4\epsilon^2}\right) = |q^m_{r,\alpha}| \exp\left(\frac{1-2\epsilon}{4\epsilon^2}\right) \leq e|Q^m_{r,\alpha}|.$$

Note that we can attain this inequality of $Q^m_{r_j,\alpha_j}$ and $q^m_{r_j,\alpha_j}$ for each $j \in I_m$. Summing those inequalities over $j \in I_m$, we obtain

$$\begin{aligned}|\{x_m \in G : S^*_m g(x) > 2\lambda,\ S_{m+1}g(x) \le \epsilon\lambda\}| &= \sum_{j\in I} |\{x_m \in Q^m_{r_j,\alpha_j} : S^*_m g(x) > 2\lambda,\ S_{m+1}g(x) \le \epsilon\lambda\}| \\ &\le \sum_{j\in I} e^{-\frac{1}{4\epsilon^2}} |Q^m_{r_j,\alpha_j}| \\ &= e^{-\frac{1}{4\epsilon^2}} |\{x_m \in G : S^*_m g(x) > \lambda\}|.\end{aligned}$$

It completes the proof. □

For $1 \le j \le n$ we set M^j be the Hardy-Littlewood maximal function with respect to the variable of the space G_j, which acts on functions defined on $G = G_1 \times \cdots \times G_n$, and define the strongly maximal function by $M = M^1 \circ \cdots \circ M^n$.

For each $q > 1$, we let $M^j_q(f) = (M^j(f^q))^{1/q}$ and $M_q(f) := (M(f^q))^{1/q}$. Next we define $\mathcal{M}_q = M_q \circ M_q \circ M_q$ and the square function

$$G_r f(x) = \left(\sum_{k_1,\cdots,k_n} \left| \mathcal{M}_q \left(\psi_{k_1,\cdots,k_n}(L^\sharp_1, \cdots, L^\sharp_n) f \right)(x) \right|^2 \right)^{1/2}.$$

Lemma 3.4.3. *There exists a constant $\gamma = \gamma(G) > 0$ such that*

$$\mathbb{D}_{k_1,\cdots,k_n} \left(\psi_{l_1,\cdots,l_n}(L^\sharp_1, \cdots, L^\sharp_s) f(x) \right) \le 2^{-\frac{1}{nq'} \sum_{j=1}^n \left| \frac{l_j}{2} + \log(\delta_j) k_j \right|} M_q f(x).$$

Proof. By Lemma 3.3, for each $1 \le j \le n$, there exists $\gamma_j > 0$ such that

$$\begin{aligned}\mathbb{D}^j_{k_j} \left(\psi_{l_1,\cdots,l_n}(L^\sharp_1, \cdots, L^\sharp_n) f \right)(x) &\lesssim 2^{-\left| \frac{l_j}{2} + \log(\delta_j) k_j \right| \gamma_j} M^j_q f(x) \\ &\lesssim 2^{-\left| \frac{l_j}{2} + \log(\delta_j) k_j \right| \gamma_j} M_q f(x),\end{aligned}$$

where the second inequality holds from the trivial inequality $M^j_q f(x) \le M_q f(x)$. Therefore we have

$$\mathbb{D}_{k_1,\cdots,k_n} \left(\psi_{l_1,\cdots,l_n}(L^\sharp_1, \cdots, L^\sharp_n) f \right)(x) \lesssim 2^{-\left| \frac{l_j}{2} + \log(\delta_j) k_j \right| \gamma_j} M_q f(x)$$

for all $1 \le j \le n$. Set $\gamma = \frac{1}{n} \min_{1 \le j \le n}\{\gamma_j\}$ and we product the above inequalities with respect to j from 1 to n, which leads to

$$\mathbb{D}_{k_1,\cdots,k_n} \left(\psi_{l_1,\cdots,l_n}(L^\sharp_1, \cdots, L^\sharp_n) f \right)(x) \lesssim 2^{-\sum_{j=1}^n \left| \frac{l_j}{2} + \log(\delta_j) k_j \right| \cdot \gamma} M_q f(x).$$

It proves the Lemma. □

Lemma 3.4.4. *We have*

$$\psi_{l_1,\cdots,l_n}(L)m(L)f(x) \lesssim Mf(x).$$

Proof. If $m(\xi_1,\cdots,\xi_n) = m_1(\xi_1)\cdots,m_n(\xi_n)$, then the lemma follows by using Lemma 2.4 repeatedly.

In the general case, we write m in Fourier series,

$$m(\xi_1,\cdots,\xi_n) \quad = \quad \sum_{c_i\in\mathbb{Z}} e^{ic_1\xi_1}\cdots e^{ic_n\xi_n}a_{c_1,\cdots,c_n}\psi(\xi_1)\cdots\psi(\xi_n).$$

If we impose a sufficient regularity on m, the coefficients $a_{c_1,\cdots,c_n}$ decrease rapidly. Then we can use the above special case to finish the proof of the lemma. □

Lemma 3.4.5. *There exists a constant $C > 0$ such that $G_n(f)(x) \geq CS_{n+1}(m(L)f)(x)$.*

Proof. We remind that

$$S_{n+1}(m(L)f)(x) = \left(\sum_{k_1,\cdots,k_n}(D_{k_1,\cdots,k_n}(m(L)f))(x)^2\right)^{1/2}.$$

Using Lemma 3.4.3 we may deduce

$$\begin{aligned}
\left|D_{k_1,\cdots,k_n}(m(L)f)(x)\right| &= \left|D_{k_1,\cdots,k_n}(\sum_{l_1,\cdots,l_n}\psi_{l_1,\cdots,l_n}(L)^3 m(L)f)(x)\right| \\
&= \left|\sum_{l_1,\cdots,l_n} D_{k_1,\cdots,k_n}(\psi_{l_1,\cdots,l_n}(L))^2 m(L)\psi_{L_1,\cdots,l_n}(L)f)(x)\right| \\
&\lesssim \sum_{l_1,\cdots,l_n} 2^{-a\sum_{j=1}^n|\frac{l_j}{2}+\log(\delta_j)k_j|} M_q(M_q(\psi_{l_1,\cdots,l_n}(L)f))(x) \\
&\lesssim \left(\sum_{l_1,\cdots,l_n} 2^{-a\sum_{j=1}^n\left|\frac{l_j}{2}+\log(\delta_j)k_j\right|}\mathcal{M}_q(\psi_{l_1,\cdots,l_n}(L)f)^2(x)\right)^{1/2}.
\end{aligned}$$

Summing this we get,

$$\begin{aligned}
(S_{n+1}(m(L)f)(x))^2 &= \sum_{k_1,\cdots,k_n}|D_{k_1,\cdots,k_n}(m(L)f)(x)|^2 \\
&\lesssim \sum_{k_1,\cdots,k_n}\sum_{l_1,\cdots,l_n} 2^{-\gamma\sum_{j=1}^n\left|\frac{l_j}{2}+\log(\delta_j)k_j\right|}\mathcal{M}_q(\psi_{l_1,\cdots,l_n}(L)f)^2(x) \\
&\lesssim \sum_{l_1,\cdots,l_n}\mathcal{M}_q(\psi_{l_1,\cdots,l_n}(L)f)^2(x) \lesssim G_n(f)^2(x),
\end{aligned}$$

which is the asserted estimate. □

Proof of Theorem 3.1.3. Set

$$T_i^1(f) = \mathcal{A}(m_i(L)f)$$

and

$$T_i^2(f) = m_i(L)f - \mathcal{A}(m_i(L)f) = (1 - (1 - \mathcal{E}_1)\cdots(1 - \mathcal{E}_n))\,(m_i(L)f).$$

Then $m_i(f) = T_i^1(f) + T_i^2(f)$ and,

$$\sup_{1\le i\le N} |m_i(L)f(x)| \le \sup_{1\le i\le N} |T_i^1(f)(x)| + \sup_{1\le i\le N} |T_i^2(f)(x)|.$$

Let us estimate the second term $T_i^2(f)$ first. For this we employ Lemma 3.3 to get $\mathcal{E}_j(m_i(f))(x) \lesssim 2^{-N}Mf(x)$. Using this and the trivial bound $\mathcal{E}_l(f)(x) \le Mf(x)$ which holds for any $1 \le l \le n$, we may deduce that $T_i^2(f)(x) \lesssim 2^{-N}\mathcal{M}f(x)$, where $\mathcal{M} = M \circ \cdots \circ M$. Thus we get

$$\|T_i^2(f)(x)\|_{L^p} \lesssim 2^{-N}\|\mathcal{M}f\|_{L^p} \lesssim 2^{-N}\|f\|_{L^p},$$

which leads to

$$\left\|\sup_{1\le i\le N} |T_i^2(f)(x)|\right\|_{L^p} \le \sum_{i=1}^{N} \left\|T_i^2(f)\right\|_{L^p} \lesssim N2^{-N}\,\|f\|_{L^p}\,.$$

Next we focus on the main term $T_i^1(f)$. Let us begin with the level set formula

$$\left\|\sup_{1\le i\le N} |T_i^1(f)|\right\|_{L^p}^p = \int p\lambda^{p-1}|A_\lambda|d\lambda,$$

where $A_\lambda := \{x \in G : \sup_{1\le i\le n} |T_i^1(f)(x)| > \lambda\}$. Here, to obtain a sharp bound $|A_\lambda|$, we shall split A_λ into many piecies in a suitable way. First, note that

$$A_\lambda \subset \{x : \sup_i |T_i^1(f)| > \lambda,\ G_r(f)(x) \le C\epsilon^n\lambda\} \cup \{x : G_r(f)(x) > C\epsilon^n\lambda\}.$$

Since $G_n(f)(x) \ge CS_{n+1}(T_i^1 f)(x)$ we have

$$\{x : \sup_i |T_i^1(f)(x)| > \lambda,\ G_r(f)(x) \le C\epsilon^n\lambda\} \subset \bigcup_{i=1}^{N} B_{i,\lambda}$$

where $B_{i,\lambda} = \{x : |T_i^1(f)(x)| > \lambda, S_{n+1}(T_i^1 f)(x) \le \epsilon^n\lambda\}$. Then we get

$$|A_\lambda| \;\le\; \left|\left(\bigcup_{i=1}^{N} B_{i,\lambda}\right) \cup \{x : G_r f(x) > C\epsilon^n\lambda\}\right| \le \sum_{i=1}^{N} |B_{i,\lambda}| + |\{x : G_r f(x) > C\epsilon^n\lambda\}|,$$

and hence

$$\int p\lambda^{p-1}|A_\lambda|d\lambda \le \sum_{i=1}^{N}\int_0^\infty p\lambda^{p-1}|B_{i,\lambda}|d\lambda + \int_0^\infty p\lambda^{p-1}|\{x : G_r(f)(x) > C\epsilon^n\lambda\}|d\lambda. \tag{3.13}$$

For each $1 \le i \le n$, we split the set $B_{i,\lambda}$ further as follows

$$B_{i,\lambda} \subset \{x : |T_i^1(f)(x)| > \lambda,\ S_2(T_i^1 f)(x) \le \epsilon\lambda\} \cup \{S_2(T_i^1 f)(x) > \epsilon\lambda,\ S_{n+1}(T_i^1 f)(x) \le \epsilon^n\lambda\}.$$

Similarly, for each $1 \le k \le n-1$ we have

$$\begin{aligned}\{S_{k+1}(T_i^1 f)(x) > \epsilon^k\lambda,\ S_{n+1}(T_i^1 f)(x) \le \epsilon^n\lambda\}& \\ \subset \{S_{k+1}(T_i^1 f)(x) > \epsilon^k\lambda, S_{k+2}(T_i^1 f)(x) < \epsilon^{k+1}\lambda\}& \\ \cup \{S_{k+2}(T_i^1 f)(x) > \epsilon^{k+1}\lambda,\ S_{n+1}(T_i^1 f)(x) \le \epsilon^n\lambda\}.&\end{aligned}$$

Observing that the last set in the above sets is empty for $k = n-1$, we finally have

$$\begin{aligned}B_{i,\lambda} \subset \{x : |T_i^1(f)(x)| > \lambda,\ S_2(T_i^1 f)(x) < \epsilon\lambda\} \cup& \\ \bigcup_{i=2}^{n}\{x : S_k(T_i^1 f)(x) > \epsilon^{k-1}\lambda,\ S_{k+1}(T_i^1 f)(x) < \epsilon^k\lambda\}.&\end{aligned}$$

Using this and Lemma 3.4.2 we find

$$\begin{aligned}|B_{i,\lambda}| &\le |\{x : |T_i^1(f)(x)| > \lambda,\ S_2(m_i^1 f)(x) < \epsilon\lambda\}| \\ &\quad + \sum_{k=2}^{n}|\{x : S_k(T_i^1 f)(x) > \epsilon^{k-1}\lambda,\ S_{k+1}(T_i^1 f)(x) < \epsilon^k\lambda\}| \\ &\le \sum_{k=1}^{n} e^{-\frac{C}{\epsilon^2}}\left|\left\{x : |S_k^*(T_i^1 f)(x) \ge \frac{1}{2}\epsilon^{k-1}\lambda\right\}\right|.\end{aligned}$$

Applying this inequality, we estimate (3.13) as follows.

$$\begin{aligned}&\left\|\sup_{1\le i\le N}|T_i^1(f)(x)|\right\|_{L^p(G)}^p \\ &\lesssim \sum_{i=1}^{N}\sum_{k=1}^{n} e^{-\frac{C}{\epsilon^2}}\int \lambda^{p-1}\left|\left\{x : |S_k^*(T_i^1 f)(x) \ge \frac{1}{2}\epsilon^{k-1}\lambda\right\}\right|d\lambda + \epsilon^{-np}\|G_r(f)(x)\|_p^p \\ &\lesssim \sum_{i=1}^{N}\sum_{k=1}^{n} e^{-\frac{C}{\epsilon^2}}\int \epsilon^{-(k-1)p}\lambda^{p-1}\left|\left\{x : |S_k^*(T_i^1 f)(x) \ge \frac{1}{2}\lambda\right\}\right|d\lambda + \epsilon^{-np}\|G_r(f)(x)\|_p^p \\ &\lesssim Ne^{-\frac{C}{\epsilon^2}}\epsilon^{-(n-1)p}\sup_{k,i}\|S_k^*(T_i^1 f)\|_{L^p}^p + \epsilon^{-np}\|G_r(f)(x)\|_p^p \\ &\lesssim \left(Ne^{-C/\epsilon^2}\epsilon^{-(n-1)p} + \epsilon^{-np}\right)\|f\|_p^p.\end{aligned}$$

By taking $\epsilon = (\log N + 1)^{-1/2}$ here, we get

$$\left\| \sup_{1 \le i \le N} |T_i^1(f)(x)| \right\|_{L^p} \lesssim (\log N + 1)^{n/2} \|f\|_{L^p}.$$

This yields the desired inequality. □

The L^p boundedness of the joint spectral multipliers on the Heisenberg group was proved by Müller-Ricci-Stein (see [MRS, Lemma 2.1]). In order to get the desired bound for maximal functions of those mulitpliers, we shall use the transference argument of Coifmann-Weiss [CW].

Proof of Theorem 3.1.4. Let $G = \mathbb{H}_n \times \mathbb{R}$. For $f \in D(G)$ define a related function f^b defined on $\mathbb{H}_n$ by

$$f^b(z,t) = \int_{-\infty}^{\infty} f(z, t-u, u) du.$$

For $m \in L^\infty(\mathbb{R}^2)$ we consider the multiplier $m(L^\sharp, iT)$ and denote its kernel by $K \in D(G)$. Then, $K^b \in D(\mathbb{H}^n)$ equals to the kernel of $m(L, iT)$ (see [MRS, p. 207]). Thus,

$$\begin{aligned} m(L, iT)\phi(z,t) &= \phi * K^b(z,t) \\ &= \int_{\mathbb{H}_n} \phi((z,t)\cdot(z',w)^{-1}) \left[\int_{\mathbb{R}} K(z', w-u', u')du' \right] dz'dw \\ &= \int_G K(z,', t', u')\phi((z,t)\cdot(z', t'+u')^{-1}) dz'dt'du' \end{aligned}$$

Now we consider multipliers $\{m_j(L, iT)\}_{j=1}^N$ with functions $\{m_j\}_{j=1}^N$ satisfying the condition 3.2 uniformly. We shall suppose that the support of $K_j(z', t', u')$ in u' variable is in $[-M, M]$ for a fixed $M > 0$ for any $j \in \mathbb{N}$, and obtain a bound independent of $M > 0$. Then, the proof will be completed as a standard approximation argument can removes the restriction on supports.

For each $R \in \mathbb{N}$, we set χ_R be the characteristic function on $[-2R, 2R]$. If $R \ge 10M$, then for each $u \in (-R, R)$, we have

$$\begin{aligned} &\int_G K_j(z', t', u')\chi_R(u-u')\phi\left((z, t+u)\cdot(-z', -(t'+u'))\right) dz'dt'du' \\ &\qquad = \int_G K_j(z', t', u')\phi\left((z, t+u)\cdot(-z', -(t'+u'))\right) dz'dt'du'. \end{aligned}$$

Using this we deduce that

$$\begin{aligned}
&\Big\|\sup_{1\le j\le N} m_j(L,iT)\phi(z,t)\Big\|_{L^p(z,t)}\\
&\quad= \Big\|\sup_{1\le j\le N}\Big|\int_G K_j(z',t',u')(\phi((z,t)\cdot(-z',-(t'+u'))dz'dt'du'\Big|\Big\|_{L^p(z,t)}\\
&\quad\le \frac{1}{R^{1/p}}\Big\|\sup_{1\le j\le N}\Big|\int_G (K_j(z,',t',u'))(\chi_R(u-u')\phi((z,t+u)\cdot(-z',-(t'+u')))dz'dt'du')\Big|\Big\|_{L^p(z,t,u)}\\
&\quad\le \frac{1}{R^{1/p}}\Big\|\sup_{1\le j\le N}|m_j(L^\sharp,iT)|\Big\|_{L^p\to L^p}\|\chi_R(u)\phi(z,t+u)\|_{L^p(z,t,u)}\\
&\quad\le 10\,\Big\|\sup_{1\le j\le N}|m_j(L^\sharp,iT)|\Big\|_{L^p\to L^p}\|\phi\|_{L^p(\mathbb{H}_n)}.
\end{aligned}$$

Hence the desired bound of maximal functions of $m_j(L,iT)$ follows from the bound property of maximal functions of $m_j(L^\sharp,iT)$ which is obtained in Theorem 3.1.1. □

3.5 Bound of maximal multiplier on product spaces

In this section we briefly discuss how one can apply Theorem 3.1.3 to find a criterion that

$$\mathcal{M}_m f(x) := \sup_{t_1>0,\cdots,t_n>0} |m(t_1L_1,\cdots,t_nL_n)f(x)|$$

is bounded on $L^p(G)$, where $G = G_1\times\cdots\times G_n$. First, we consider the dyadic maximal operator

$$\mathcal{M}_m^{\rm dyad} f(x) := \sup_{k_1\in\mathbb{Z},\cdots,k_n\in\mathbb{Z}} \left|m(2^{k_1}L_1,\cdots,2^{k_n}L_n)f(x)\right|$$

For this we consider some constant $\alpha_0 > 0$ such that: For any $A\subset\mathbb{Z}^n$ there exists a finite set $F\subset\mathbb{Z}^n$ and an infinite set $B\subset\mathbb{Z}^n$ satisfying

- $\mathbb{Z}^n = \cup_{d\in F}(d+B)$,
- $|F|\le|A|^{\alpha_0}$,
- For any $b_1\in B$ and $b_2\in B$ such that $b_1\ne b_2$, two sets b_1+A and b_2+A are disjoint.

For each $1\le j<\infty$, we set $\mathcal{I}_j\subset\mathbb{Z}^n$ by

$$\mathcal{I}_j = \left\{(k_1,\cdots,k_n)\in\mathbb{Z}^n : w^*(2^{2^j})\le|w(k_1,\cdots,k_n)|\le w^*(2^{2^{j+1}})\right\}.$$

We split $m=\sum_{j=1}^\infty m_j$ so that m_j is supported in

$$\bigcup_{(k_1,\cdots,k_n)\in\mathcal{I}_j}\{(\xi_1,\cdots,\xi_n)\in\mathbb{R}^n : 2^{k_1-1}<|\xi_1|<2^{k_1+1},\cdots,2^{k_n-1}<|\xi_n|<2^{k_n+1}\}.$$

We note that $2^{2^{j+1}-1} \leq |\mathcal{I}_j| \leq 2^{2^{j+1}}$. By the definition of α_0, for each j find $F_j \subset \mathbb{Z}^n$ and $B_j \subset \mathbb{Z}^n$ such that $|F| \leq 2^{(2^{j+1}\alpha_0)}$, $\mathbb{Z}^n = \cup_{d\in F}(d+B)$, and for any $b_1 \in B_j$ and $b_2 \in B_j$ with $b_1 \neq b_2$, two sets $b_1 + \mathcal{I}_j$ and $b_2 + \mathcal{I}_j$ are disjoint. Then, as in the proof of Theorem 3.1.2, we can deduce

$$\begin{aligned}
\left\|\mathcal{M}^{\mathrm{dyad}}_{m_j} f(x)\right\|_p &= \left\|\sup_{K\in\mathbb{Z}^n} |T^j_K f|\right\|_p = \left\|\sup_{d\in F}\sup_{b_i\in B} |T^j_{d+b} f|\right\|_p \leq \left\|\sup_{d\in F}\left(\sum_{b_i\in B} |T^j_{d+b} f|^2\right)^{1/2}\right\|_p \\
&\leq C_p\left(\int_0^1 \left\|\sup_{d\in F}\left|\sum_{i=1}^{\infty} r_i(s) T_{b_i} f\right|\right\|_p^p ds\right)^{1/p} \qquad (3.1)\\
&\leq C_p(\log|F|)^{n/2}\omega^*(2^{2^{j+1}})\|f\|_p \\
&\leq C_p 2^{(2^{j-1}n\alpha_0)}\omega^*(2^{2^{j+1}})\|f\|_p,
\end{aligned}$$

where the result of Theorem 3.1.3 is applied. Here assuming that

$$\sum_{j=1}^{\infty} \frac{[\log(w^*(j))]^{\frac{n\alpha_0}{2}}}{j}\omega^*(j) < \infty, \qquad (3.2)$$

we easily see that $\sum_{j=1}^{\infty} 2^{(2^{j-1}n\alpha_0)}\omega^*(2^{2^{j+1}}) < \infty$, and (8.15) yields that

$$\left\|\mathcal{M}^{\mathrm{dyad}}_{m_j} f(x)\right\|_p \leq C\|f\|_p.$$

To apply the bound of $\mathcal{M}^{\mathrm{dyad}}_m$ to obtain a bound property of $\mathcal{M}_m$, one may use the formula

$$\begin{aligned}
&\sup_{\substack{2^{k_i}\leq t_i<2^{k_i+1}\\ 1\leq i\leq n}} |m_j(t_1L_1,\cdots,t_nL_n)f(x)| = \sup_{\substack{1\leq t_j<2\\ 1\leq j\leq n}} |m_j(t_1 2^{k_1}L_1,\cdots,t_n 2^{k_n}L_n)f(x)| \\
&\leq |m_j(2^{k_1}L_1,\cdots,2^{k_n}L_n)f(x)| + \int_1^2\cdots\int_1^2 \left|\frac{\partial^n}{\partial t_1\cdots\partial t_n} m_j(t_1 2^{k_1}L_1,\cdots,t_n 2^{k_n}L_n)f(x)\right| dt_1\cdots dt_n.
\end{aligned}$$

In the above argument, it is important to find the minimum value of α_0. Note that $\alpha_0 = 2$ and $n = 1$ in the case of Theorem 3.1.2. It would be interesting to find the minimum value of α_0 for product spaces.

Chapter 4

Maximal functions of multipliers on compact manifolds without boundary [Ch3]

4.1 Introduction

Let M be a compact manifold of dimension $n \geq 2$ without boundary. Consider a first order elliptic pseudo-differential operator P, which is is positive and self-adjoint with respect to a C^∞ density dx on M. By the spectral theorem, we have $L^2(M) = \sum_{j=1}^{\infty} E_j$, where E_j is an eigenspace of dimension one of the operator P with an eigenvalue λ_j such that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Denoting by e_j the projection operator onto the eigenspace E_j, we have for any $f \in L^2(M)$ that

$$f = \sum_{j=1}^{\infty} e_j(f),$$

and

$$\|f\|_{L^2(M)}^2 = \sum_{j}^{\infty} \|e_j(f)\|_{L^2(M)}^2. \tag{4.1}$$

For a function $m \in L^\infty([0,\infty))$ the multiplier operator $m(P) : L^2(M) \to L^2(M)$ is defined by

$$m(P)f = \sum_{j=1}^{\infty} m(\lambda_j) e_j(f), \quad f \in L^2(M). \tag{4.2}$$

From (4.1) we see that $m(P)$ is bounded on $L^2(M)$ for any $m \in L^\infty([0,\infty))$. On the other hand, more difficult is to say that $m(P)$ is bounded on L^p with $p \neq 2$. Under a condition on m involving that m is a C^∞ function, we have the L^p-bound of $m(P)$ for $1 < p < \infty$ (see [Tay2]). Later, Seeger and Sogge [SS] established the L^p-bound result under the Hörmander-Mikhlin type condition.

To state the result, we take a function $\beta \in C_0^\infty((1/2,2))$ such that $\sum_{-\infty}^{\infty} \beta(2^j s) = 1, s > 0$, and introduce the functional

$$[m]_s = \sup_{0\leq\alpha\leq s}\left[\sup_{\lambda>0} \lambda^{-1}\int_{-\infty}^{\infty} |\lambda^\alpha D_s^\alpha(\beta(s/\lambda)m(s))|^2 ds\right]. \tag{4.3}$$

The following theorem is due to Seeger-Sogge [SS]:

Theorem 4.1.1 ([SS])**.** *Let $s \in \mathbb{R}^+$ such that $s > \frac{n}{2}$. Then for any $m \in L^\infty([0,\infty))$ with finite $[m]_s$, we have*

$$\|m(P)f\|_{L^p(M)} \leq C_p\,[m]_s\,\|f\|_p, \qquad 1 < p < \infty, \quad \forall f \in L^p(M). \tag{4.4}$$

Here the constant C_p is independent of m and f.

In this paper we consider L^p-boundedness problem of maximal functions of multipliers on compact manifolds. Namely we shall obtain the following result.

Theorem 4.1.2. *Let $p \in (1,\infty)$. For $s > \max\left(\frac{n}{p}, \frac{2n-1}{2}\right)$ we have*

$$\|\sup_{1\leq i\leq N} |m_i(P)f|\|_{L^p(M)} \leq C_{p,s} \sup_{1\leq i\leq N}[m_i]_s \cdot (\log(N+1))^{1/2}\|f\|_p, \quad \forall f \in L^p(M),$$

where the constant $C_{p,s}$ is independent of N.

Study of multipliers on manifolds has recieved a lot of interest from many authors as it is also related to various partial differential equations on manifolds (see e.g. [BGT, BGT2]). Also many researches have been done to determine the boundedness of multipliers on manifolds in the L^p space of submanifolds (see e.g. [BGT3, HT, T1]). Our study of Theorem 4.1.2 was motivated by the study of Grafakos-Honzik-Seeger [GHS] where the maximal function of multipliers was studied on the Euclidean space.

Studying the multipliers on manifolds require some new analysis ans we need split the mulitplier $m(P)$ into two parts by using the Schrödinger propergator e^{itP}. One part will be handled by modifying the argument of [GHS] and another part will be estimated using the $L^p - L^q$ bound results of the spectral projection operators.

We organized the paper as follows. In Section 2 we review breifly the multiplier on compact manifolds. Then we split the multipliers into a main part and a remainder part by combining the dyadic decomposition and the Schördinger propagator. In Section 3 we first study the remainder part using the property of spectral projection operators. In Section 4 we shall further decompose the main part into a local operator and remainder terms which are small enough. In Section 5 we study the local operator. We shall complete the proof of Theorem 4.1.2 in Section 6.

Notations.

- We use C to denote generic constants that depend only on the manifold M.
- For given linear operators $\{T_j\}_{j=1}^\infty$ we shall use the notation $T_j = O_N(2^{-Nj})$ for $N \in \mathbb{N}$ when their kernels $K_{T_j}(x,y)$ satisfies $\sup_{x,y}|K_{T_j}(x,y)| = O_N(2^{-Nj})$.

4.2 Preliminaries

In this section we review some basic results on the spectral decomposition associated to a self-adjoint elliptic operator on a compact manifold, and the definition of the multiplier operators. Then we recall the expression of multipliers in terms of the Schrödinger propagator and some $L^p - L^q$ boundedness of the spectral projection operators. For more details we refer to the book [So2]. In the later part of this section, we shall decompose the multipliers into two parts which will be handled in different ways.

Let M be a compact manifold with a density dx and P be a first-order self-adjoint positive elliptic operator on $L^2(M, dx)$. Then, by spectral theory, the oprator P has positive eigenvalues $\lambda_1 \le \lambda_2 \le \cdots$ associated to orthonormal eigenfunctions $e_1, e_2, \cdots$. Let $E_j : L^2 \to L^2$ be the projection maps onto the one-dimensional eigenspace ε_j spanned by e_j. Then we have $P = \sum_{j=1}^{\infty} \lambda_j E_j$ and

$$E_j f(x) = e_j(x) \int_M f(y)\overline{e_j(y)}dy.$$

For a function $m \in L^\infty([0, \infty))$ we define the multiplier $m(P) : L^2(M) \to L^2(M)$ in the following way

$$m(P)f := \sum_{j=1}^{\infty} m(\lambda_j)E_j(f) = \sum_{j=1}^{\infty} m(\lambda_j)\left(\int_M f(y)e_j(y)dy\right)e_j(x).$$

Let $\mathcal{K}_m \in D(M \times M)$ be the kernel of $m(P)$. From the above we see that

$$\mathcal{K}_m(x, y) = \sum_{j=1}^{\infty} m(\lambda_j)e_j(x)e_j(y).$$

We also have the following forumula using the Schrodinger propagator

$$m(P) = \int_{-\infty}^{\infty} e^{itP}\hat{m}(t)f dt, \tag{4.1}$$

and the result on e^{itP};

Theorem 4.2.1 (see [So2, Theorem 3.2.1])**.** *Let M be a compact C^∞ manifold and let $P \in \phi^1_{cl}(M)$ be elliptic and self-adjoint with respect to a positive C^∞ density dx. Then there is an $\epsilon > 0$ such that when $|t| < \epsilon$,*

$$e^{itP} = Q(t) + R(t) \tag{4.2}$$

where the remainder has kernel $R(t, x, y) \in C^\infty([-\epsilon, \epsilon] \times M \times M)$ and the kernel $Q(t, x, y)$ is supported in a small neighborhood of the diagonal in $M \times M$.

We first perform a dyadic decomposition on multipliers. Let us take functions $\phi_0 \in C_0^\infty([0,1))$ and $\phi \in C_0^\infty(1/4, 1)$ such that $\sum_{j=0}^{\infty} \phi_j^3(s) = 1$ for all $s \geq 0$ where $\phi_j(s) := \phi(s/2^j)$ for $j \geq 1$. For given $m \in L^\infty([0,\infty))$ we set $m_j(\cdot) := m(\cdot)\phi_j(\cdot)$. Then,

$$m(P)f = \sum_{j=1}^{\infty} \phi_j(P) m_j(P) \phi_j(P) f. \tag{4.3}$$

Let us take a function $\rho \in C^\infty(\mathbb{R})$ satisfying $\rho(t) = 1, |t| \leq \frac{\epsilon}{2}$ and $\rho(t) = 0, |t| > \epsilon$, and we write

$$m_j(P) = A_j(m,P) + R_j(m,P), \tag{4.4}$$

where

$$A_j(m,P) = \int e^{itP} \widehat{m_j}(t)\rho(t)dt \quad \text{and} \quad R_j(m,P) = \int e^{itP} \widehat{m_j}(t)(1-\rho(t))dt. \tag{4.5}$$

Next we shall put $m_j(P)$ in a composition form to achieve a L^p bound for $p > 2$ and some cancellation property of its kernel (see Lemma 4.5.1 and Corollary 4.5.3). Take a C^∞ function $\widetilde{\phi}$ supported on $(\frac{1}{8}, 2)$ such that $\widetilde{\phi} = 1$ on $(\frac{1}{4}, 1)$.

We set $\widetilde{\phi}_j(\cdot) = \widetilde{\phi}(\frac{\cdot}{2^j})$, then $\widetilde{\phi}_j \cdot \phi_j = \phi_j$ and

$$m_j(P) = m_j(P)\widetilde{\phi}_j(P) = A_j(m,P) \circ \widetilde{\phi}_j(P) + R_j(m,P) \circ \widetilde{\phi}_j(P). \tag{4.6}$$

Injecting this into (4.3) we have $m(P) = A(m,P) + R(m,P)$, where

$$A(m,P)f := \sum_{j=1}^{\infty} \phi_j(P) \left[A_j(m,P) \circ \widetilde{\phi}_j(P)\right] \phi_j(P) f \tag{4.7}$$

and

$$R(m,P)f := \sum_{j=1}^{\infty} \phi_j(P) \left[R_j(m,P) \circ \widetilde{\phi}_j(P)\right] \phi_j(P) f. \tag{4.8}$$

We shall study these two operators in seperate ways. First we shall prove the following result.

Proposition 4.2.2. *For $s > \frac{2n-1}{2}$ and $1 \leq p < \infty$ we have*

$$\Big\| \sup_{1\leq i\leq N} |R(m_i,P)f| \Big\|_{L^\infty(M)} \leq C_{p,s} \sup_{1\leq i\leq N} [m_i]_s \|f\|_{L^p(M)}, \quad \forall f \in L^p(M). \tag{4.9}$$

This result will be proved in Section 3 by using the $L^p - L^q$ bound property of spectral projection operators. In the remaining sections, we shall study the operator $A(m,P)$ to prove the following result.

Proposition 4.2.3. *For $1 \leq p < \infty$ and $s > \frac{n}{p}$ we have*

$$\Big\| \sup_{1\leq i\leq N} |A(m_i,P)f| \Big\|_{L^p(M)} \leq C_{p,s} \sup_{1\leq i\leq N} [m_i]_s \cdot (\log(N+1))^{1/2} \|f\|_{L^p(M)}, \quad \forall f \in L^p(M). \tag{4.10}$$

To prove this result, we shall localize the operator $A(m, P)$ in Section 4. Then we devote Section 5 to exploit the property of the localized operator and bound it using the Hardy-Littlewood maximal function. In Section 6 we relate the operators with the martingale operators and we shall complete the proof of Proposition 4.2.3 using the exponential ineqaulity of the martingale operators.

Now we prove the main theorem assuming the above results.

Proof of Theorem 4.1.2. Consider functions $m^1, \cdots, m^N$ such that $\sup_{1\le i\le N}[m_i]_s \le C$ for some $s > \frac{n}{r}$. Let us write each multiplier $m^j(P)$ as $m^j(P) = A(m^j, P) + R(m^j, P)$ which are defined in (4.7) and (4.8). By triangle inequality we have

$$\Big\|\sup_{1\le i\le N} |m^j(P)f|\Big\|_{L^p(M)} \le \Big\|\sup_{1\le i\le N} \big|A(m_i, P)f\big|\Big\|_{L^p(M)} + \quad \Big\|\sup_{1\le i\le N} |R(m_i, P)f|\Big\|_{L^p(M)}$$

Using (4.9), (4.10) and Hölder's ineqaulity, we get

$$\begin{aligned}
&\Big\|\sup_{1\le i\le N} |m^j(P)f|\Big\|_{L^p(M)}\\
&\le C \sup_{1\le i\le N}[m_i]_s \sqrt{\log(N+1)}\, \|f\|_{L^p(M)} + C \sup_{1\le i\le N}[m_i]_s |\mathrm{vol}(M)|^{1/p'} \Big\|\sup_{1\le i\le N} |R(m_i, P)f|\Big\|_{L^\infty(M)}\\
&\le C \sup_{1\le i\le N}[m_i]_s \big(\sqrt{\log(N+1)} + |\mathrm{vol}(M)|^{1/p'}\big)\, \|f\|_{L^p(M)}\,.
\end{aligned}$$

It completes the proof. □

4.3 The proof of Proposition 4.2.2

In this section we shall prove Proposition 4.2.2. We shall use the $L^p - L^q$ boundedenss property of the spectral projection operators

$$\chi_\lambda f = \sum_{\lambda_j\in[\lambda,\lambda+1]} E_j f, \qquad \lambda \in [0, \infty).$$

We recall the following result.

Lemma 4.3.1 (see [So2, Lemma 4.2.4 and Lemma 5.1.1])**.** *Then there exists a constant $C > 0$ such that*

$$\|\chi_\lambda f\|_{L^\infty(M)} \le C(1+\lambda)^{(n-1)/2} \|f\|_{L^2(M)}, \tag{4.1}$$

and

$$\|\chi_\lambda f\|_{L^2(M)} \le C(1+\lambda)^{\frac{n}{2}-1} \|f\|_{L^1(M)}, \tag{4.2}$$

where the constant C is independent of λ.

Proof of Proposition 4.2.2. By the decomposition (4.8), it is enough to prove that

$$\left\|\phi_j(P)[R_j(m,P)\circ\widetilde{\phi}_j(P)]\phi_j(P)f\right\|_{L^\infty} \le C2^{j\left(n-\frac{1}{2}-s\right)}\|f\|_{L^1}. \tag{4.3}$$

Applying (4.1) we have

$$\left\|\phi_j(P)[R_j(m,P)\circ\widetilde{\phi}_j(P)]\phi_j(P)f\right\|_{L^\infty}^2 \le C2^{j(n-1)}\left\|\phi_j(P)[R_j(m,P)\circ\widetilde{\phi}_j(P)]\phi_j(P)f\right\|_{L^2}^2. \tag{4.4}$$

Using the fact that $|\phi_j|, |\widetilde{\phi}_j| \le 1$ and the orthogonality, we have

$$\left\|\phi_j(P)\left[R_j(m,P)\circ\widetilde{\phi}_j(P)\right]\phi_j(P)f\right\|_{L^2} \le \left\|\left[R_j(m,P)\right]f\right\|_{L^2}.$$

Let $\tau_j(r) = [(1-\rho(t))\hat{m}_j]^\vee(r)$. Then by (4.5) we see

$$R_j(m,P) = \int e^{itP}\hat{\tau}_j(t)dt = \tau_j(P).$$

Using the orthogonality and (4.2) we deduce

$$\begin{aligned}\left\|R_j(m,P)f\right\|_{L^2}^2 &\le \sum_{k=0}^\infty \sup_{r\in[k,k+1)}|\tau_j(r)|^2\|\chi_k f\|_{L^2}^2\\ &\le \sum_{k=0}^\infty \sup_{r\in[k,k+1)}|\tau_j(r)|^2(1+k)^{n-1}\|f\|_{L^1}^2.\end{aligned} \tag{4.5}$$

Here we claim that

$$\sum_{k\in[2^{j-2},2^{j+2}]}\sup_{r\in[k,k+1)}|\tau_j(r)|^2 \le C2^{j(1-2s)}. \tag{4.6}$$

To show this, we apply the fundamental theorem of calculus and the Cauchy-Schwartz inequality, then we get

$$\begin{aligned}\sum_{k\in[2^{j-2},2^{j+2}]}\sup_{r\in[k,k+1)}|\tau_j(r)|^2 &\le \int|\tau_j(r)|^2dr + \int|\tau_j'(r)|^2dr\\ &= \frac{1}{2\pi}\int|\hat{m}_j(t)(1-\rho(t))|^2dt + \frac{1}{2\pi}\int|t\hat{m}_j(t)|^2|(1-\rho(t))|^2dt.\end{aligned}$$

Note that $\rho(t)=1$ for $|t|<\epsilon/2$, so we can bound this by

$$\begin{aligned}\frac{1}{2\pi}2^{-j(1+2s)}\int|t^s\hat{m}_j(t/2^j)|^2dt &= 2^{-j(1+2s)}\int|D_r^s(2^jm_j(2^jr)|^2dr\\ &= 2^{j(1-2s)}\cdot\left\{2^{-j}\int|2^{-js}D_r^s(\phi(r/2^j)m(r))|^2dr\right\} \le 2^{j(1-2s)}[m]_s.\end{aligned}$$

It proves (4.6).

We see that $\tau_j(r) = \left[\hat{m}_j(\cdot)(1-\rho(\cdot))\right]^\wedge(r) = O\left((|r|+2^j)^{-N}\right)$ for any $N\in\mathbb{N}$ if $\tau\notin\left[2^{j-2},2^{j+2}\right]$. Combining this, (4.6) and (4.5) we obtain

$$\left\|R_j(m,P)f\right\|_{L^2}^2 \le C2^{j(n-2s)}\|f\|_{L^1(M)}^2. \tag{4.7}$$

Using this with (4.4) we get the estimate (4.3). It completes the proof. □

For later use, we modify the above proof to obtain the following result.

Lemma 4.3.2. *For $m \in L^\infty([0,\infty))$ such that $[m]_s < \infty$ for some $s \geq 0$, we have*

$$\|R_j(m,P)f\|_{L^\infty} \leq C2^{j(\frac{2n-1}{2}-s)}[m]_s\|f\|_{L^p}, \quad 1 < p < \infty.$$

Proof. We have $R_j(m,P)f = \tau_j(P)f = \sum_{k=0}^\infty \chi_k\tau_j(P)f$ where χ_k is the spectral projection operator. Using Lemma 4.3.1 we deduce that

$$\left\|\tau_j(P)f\right\|_{L^\infty} \leq \sum_{k=0}^{\infty}\left\|\chi_k\tau_j(P)f\right\|_{L^\infty(M)} \leq C\sum_{k=0}^{\infty}2^{k(\frac{n-1}{2})}\left\|\chi_k\tau_j(P)f\right\|_{L^2(M)}. \tag{4.8}$$

We have

$$\begin{aligned}\left\|\chi_k\tau_j(P)f\right\|^2_{L^2(M)} &\leq \sum_{2^k\leq r<2^{k+1}}\sup_{m\leq r<m+1}\left|\tau_j(t)\right|^2\|\chi_k f\|^2_{L^2}\\ &\leq \sum_{2^k\leq m<2^{k+1}}\sup_{m\leq r<m+1}\left|\tau_j(t)\right|^2(1+k)^{n-1}\|f\|^2_{L^1}.\end{aligned} \tag{4.9}$$

For $j-2 \leq k \leq j+2$, as in (4.6) we have

$$\sum_{2^k\leq m<2^{k+1}}\sup_{m\leq t<m+1}|\tau_j(t)|^2 \leq 2^{j(1-2s)}. \tag{4.10}$$

On the other hand, when $|r-2^j| > 2^j$ we have $\tau_j(r) = (\hat{m}_j(\cdot)(1-\rho(\cdot))^\wedge(r) = O((|r|+2^j)^{-N})$ for any $N \in \mathbb{N}$. These two estimates with (4.8) gives

$$\left\|\tau_j(P)f\right\|_{L^\infty} \leq C\sum_{k=0}^{\infty}2^{k\left(n-\frac{1}{2}-s\right)}\|f\|_{L^1}. \tag{4.11}$$

It proves the lemma. □

4.4 Localization of the operator $A(m,P)$

The aim of this section is to obtain the result of Proposition 4.4.7 where we split the operator $A(m,P)$ into a local operator and its remainder part. For this we shall localize first the operators $\phi_j(P)$ and $A_j(m,P)\circ\widetilde{\phi}_j(P)$ and we shall control uniformly the L^∞ norm of the remainder part.

We set

$$m_j^{loc}(P) = \int Q(t)\widehat{m}_j(t)\rho(t)dt.$$

For a smooth function $\psi \in C^\infty(1/2,1)$, abusing a notation a bit, we set $\psi_j(\cdot) := \psi(\cdot/2^j)$ for $j \in \mathbb{N}$ and

$$\psi_j^{loc}(P) = \int Q(s)\widehat{\psi}_j(s)\rho(s)ds.$$

We have

Lemma 4.4.1. *For any $m \in L^\infty([0,\infty))$, we have*

$$A_j(m,P) = m_j^{loc}(P) + O_N(2^{-jN}), \qquad j \geq 1. \tag{4.1}$$

Proof. We have from (4.2) that

$$A_j(m,P) = m_j^{loc}(P) + \int R(t,x,y)\rho(t)\widehat{m}_j(t)dt.$$

Note

$$\int R(t,x,y)\rho(t)\widehat{m}_j(t)dt = \int [R(\cdot,x,y)\rho(\cdot)]^\wedge(t)m(t)\phi\left(\frac{t}{2^j}\right)dt.$$

We recall that the support of $\phi(\frac{\cdot}{2^j})$ is contained in $\{t \in \mathbb{R}^+ | 2^{j-1} \leq t \leq 2^{j+1}\}$, and we have $m \in L^\infty(\mathbb{R})$ and $R(t,x,y) \in C^\infty([-\epsilon,\epsilon] \times M \times M)$. Thus, for any $N \in \mathbb{N}$, we have

$$[R(\cdot,x,y)\rho(\cdot)]^\wedge(t)m(t)\phi\left(\frac{t}{2^j}\right) = O_N(2^{-jN}) \quad j \geq 1.$$

Hence we have

$$\int R(t,x,y)\rho(t)\widehat{m}_j(t)dt = O_N(2^{-jN}) \quad j \geq 1. \tag{4.2}$$

It completes the proof. □

We denote by $K_j(x,y)$ the kernel of $m_j^{loc}(P) = \int Q(t)\widehat{m}_j(t)\rho(t)dt$. Then we have the following result.

Lemma 4.4.2 (see [So2]). *Suppose that $m \in L^\infty[0,\infty)$ satisfies the condtion (4.3) for a $s > 0$. Then for $j \in \mathbb{N}$ we have $K_j(x,y) = 2^{nj}K_j^*(2^jx, 2^jy)$ for some function $K_j^* \in C^1(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying*

$$\int |D_y^\alpha K_j^*(x,y)|^2(1+|x-y|)^{2s}dy \leq C, \quad 0 \leq |\alpha| \leq 1, \tag{4.3}$$

where the constant C is independent of x and $j \in \mathbb{N}$.

Remark 4.4.3. Applying Hölder's inequality to (4.3) and a change of variables we can deduce

$$\int |K_j(x,y)|dy = \int |K_j^*(2^jx,y)|dy \leq C \quad \text{for } j \in \mathbb{N}. \tag{4.4}$$

Now we study the properties of the kernel of the projection operators given by a smooth cut-off function.

Lemma 4.4.4. *For $\psi \in C^\infty(1/2, 1)$, the operator $\psi_j(P)$ defined by (4.6) is of the form*

$$\psi_j(P) = \psi_j^{loc}(P) + O(2^{-jN}), \quad j \in \mathbb{N}. \tag{4.5}$$

Moreover, the kernel $\mathcal{K}(\psi_j)$ of $\psi_j(P)$ satisfies uniformly for $j \in \mathbb{N}$ the estimate

$$\int \left|\mathcal{K}(\psi_j)(x, y)\right| dy \leq C. \tag{4.6}$$

Proof. Recalling (4.2) and (4.4) we have

$$\psi_j(P) = \int Q(s)\widehat{\psi}_j(s)\rho(s)ds + \int R(s)\widehat{\psi}_j(s)\rho(s)ds + \int e^{itP}\widehat{\psi}_j(s)(1 - \rho(s))ds.$$

By the same way for (4.2) we have $\int R(s)\rho(s)\widehat{\psi}_j(s)ds = O_N(2^{-jN})$. Since the smooth function $\psi \in C_0^\infty(1/8, 2)$ satisfies the condition (4.3) for any $s = N > 0$ with $N \in \mathbb{N}$, we may apply Lemma 4.3.2 to deduce

$$\int e^{itP}\widehat{\psi}_j(s)(1 - \rho(s))ds = O_N(2^{-jN}).$$

Thus (4.5) holds. To show (4.6) we let Ψ_j be the kernel of $\psi_j^{loc}(P)$. By (4.4) we have

$$\int \left|\Psi_j(x, y)\right| dy \leq C. \tag{4.7}$$

From this and using (4.5) we see that

$$\int \left|\mathcal{K}(\psi_j)(x, y)\right| dy \leq \int \left|\Psi_j(x, y)\right| dy + \int O(2^{-jN})dy \leq C, \tag{4.8}$$

which gives (4.6). Thus the lemma is proved. □

Remark 4.4.5. We note that the functions ϕ and $\widetilde{\phi}$ defined in Section 2 satisfies the assumption of the above lemma. Therefore the formula (4.5) and (4.6) hold for ϕ and $\widetilde{\phi}$.

We have the following result.

Lemma 4.4.6. *For $m \in L^\infty[0, \infty)$ we have*

$$A_j(m, P) \circ \widetilde{\phi}_j(P) = m_j^{loc}(P) \circ \widetilde{\phi}_j^{loc}(P) + O_N(2^{-jN}) \quad \forall j \in \mathbb{N}. \tag{4.9}$$

Proof. Using (4.1) we have

$$A_j(m, P) \circ \widetilde{\phi}_j(P) = m_j^{loc}(P) \circ \widetilde{\phi}_j(P) + O_N(2^{-jN})\widetilde{\phi}_j(P).$$

By (4.6) we see $O_N(2^{-jN}) \circ \widetilde{\phi}_j(P) = O_N(2^{-jN})$.

Next we use (4.5) and (4.4) to get

$$m_j^{loc}(P) \circ \widetilde{\phi}_j(P) = m_j^{loc}(P) \circ \widetilde{\phi}_j^{loc}(P) + m_j^{loc}(P) \circ O_N(2^{-jN}) = m_j^{loc}(P) \circ \widetilde{\phi}_j^{loc}(P) + O_N(2^{-Nj}).$$

Thus we have

$$A_j(m, P) \circ \widetilde{\phi}_j(m, P) = m_j^{loc}(P) \circ \widetilde{\phi}_j^{loc}(P) + O_N(2^{-jN}).$$

It completes the proof. □

We define the following local operator associated to $m(P)$;

$$m^{loc}(P) = \sum_{j=1}^{\infty} \phi_j^{loc}(P) \left[m_j^{loc}(P) \circ \widetilde{\phi}_j^{loc}(P) \right] \phi_j^{loc}(P). \tag{4.10}$$

Then we have

Proposition 4.4.7. *For $m \in L^\infty[0,\infty)$ we have*

$$A(m,P)f = m^{loc}(P)f + O(1)f. \tag{4.11}$$

Proof. Recall that

$$A(m,P) = \sum_{j=1}^{\infty} \phi_j(P) \left[A_j(m,P) \circ \widetilde{\phi}_j(P) \right] \phi_j(P)f.$$

Using (4.9), (4.4) and $\sum_{j=0}^{\infty} O(2^{-j}) = O(1)$ we get

$$\begin{aligned} \sum_{j=1}^{\infty} \phi_j(P) \left[A_j(m,P) \circ \widetilde{\phi}_j(P) \right] \phi_j(P)f &= \sum_{j=1}^{\infty} \phi_j(P) \left[m_j^{loc}(P) \circ \widetilde{\phi}_j^{loc}(P). + O(2^{-jN}) \right] \phi_j(P)f \\ &= \sum_{j=1}^{\infty} \phi_j(P) \left[m_j^{loc}(P) \circ \widetilde{\phi}_j^{loc}(P) \right] \phi_j(P)f + O(1)f. \end{aligned}$$

Using Lemma 4.4.4, we have $\phi_j(P) = \phi_j^{loc}(P) + O(2^{-jN})$. In addition the L^1-norms of the kernels of $\phi_j(P), m_j^{loc}(P)$, and $\widetilde{\phi}_j^{loc}(P)$ with respect to the second variable are bounded unfiormly for $j \in \mathbb{N}$. Thus,

$$\sum_{j=1}^{\infty} \phi_j(P) \left[m_j^{loc}(P) \circ \widetilde{\phi}_j^{loc}(P) \right] \phi_j(P)f = \sum_{j=1}^{\infty} \phi_j^{loc}(P) \left[m_j^{loc}(P) \circ \widetilde{\phi}_j^{loc}(P) \right] \phi_j^{loc}(P)f + O(1)f.$$

It completes the proof. □

4.5 Properties of the kernels and the Hardy-Littlewood maximal funtion

In this section we shall study $m^{loc}(P)$ given by (4.10) using the Hardy-Littlewood maximal funtion. We denote by H_j, K_j, Φ_j and $\tilde{\Phi}_j$ the kernels of the operators $m_j^{loc}(P) \circ \widetilde{\phi}_j^{loc}(P)$, $m_j^{loc}(P)$, $\phi_j^{loc}(P)$, and $\widetilde{\phi}_j^{loc}(P)$. Then we see

$$H_j(x,z) = \int K_j(x,y) \widetilde{\Phi}_j(y,z) dy. \tag{4.1}$$

From Theorem 4.2.1 we see that the kernels H_j, K_j, Φ_j and $\tilde{\Phi}_j$ are supported in a small neighborhood of the diagonal in $M \times M$. We also set

$$X_j^*(x,y) = 2^{-nj} X_j(2^{-j}x, 2^{-j}y), \quad \text{For } X = K, H, \Phi, \tilde{\Phi}. \tag{4.2}$$

By Lemma 4.4.2, for any $N \in \mathbb{N}$ we have

$$\sup_{j\geq 1} \int \left|\Phi_j^*(x,y)\right|^2 (1+|x-y|)^{2N} dx \leq C_N, \tag{4.3}$$

and

$$\int |D_y^\alpha \widetilde{\Phi}_j^*(x,y)|^2 (1+|x-y|)^{2N} dx \leq C_N, \quad 0 \leq |\alpha| \leq 1.$$

We have the following result.

Lemma 4.5.1. *Suppose that $m \in L^\infty[0,\infty)$ satisfies $[m]_s < \infty$ for some $s > 0$. Then, for each $q \geq 2$ we have*

$$\int_M |H_j^*(x,z)|^q (1+|x-z|)^{sq} dz \leq C_q \cdot [m]_s.$$

Proof. From (4.1) we see

$$2^{jn} H_j^*(2^j x, 2^j z) = \int 2^{jn} K_j^*(2^j x, 2^j y) 2^{jn} \widetilde{\Phi}_j^*(2^j y, 2^j z) dy = \int 2^{jn} K_j^*(2^j x, y) \widetilde{\Phi}_j^*(y, 2^j z) dy,$$

which shows

$$H_j^*(x,z) = \int K_j^*(x,y) \widetilde{\Phi}_j^*(y,z) dy.$$

Using Lemma 4.4.2 and Hölder's inequality, we have

$$\begin{aligned}
(1+|x-z|)^s |H_j^*(x,y)| &= (1+|x-z|)^s \int K_j^*(x,y) \widetilde{\Phi}_j^*(y,z) dy \\
&\leq \int K_j^*(x,y)(1+|x-y|)^s \cdot \widetilde{\Phi}_j^*(y,z)(1+|y-z|)^s dy \\
&\leq \left(\int |K_j^*(x,y)|^2 (1+|x-y|)^{2s} dy\right)^{1/2} \cdot \left(\int |\widetilde{\Phi}_j^*(y,z)|^2 (1+|y-z|)^{2s} dy\right)^{1/2} \\
&\leq C[m]_s.
\end{aligned}$$

On the other hand, we can use Lemma 4.4.2 to obtain

$$\left(\int_M |H_j^*(x,y)|^2 (1+|x-y|)^{2s} dy\right)^{1/2} \leq C[m]_s. \tag{4.4}$$

Combining the above two esimtaes, we get the desired result. □

Lemma 4.5.2. *Let $\psi \in C^\infty(1/2, 1)$ and set $\psi_j(\cdot) := \psi(\cdot/2^j)$ for $j \in \mathbb{N}$. Then we have*

$$\int \Psi_j(x, y)dx = O_N(2^{-jN}).$$

Proof. For each $j \geq 1$ we have $[\psi_j(P)1](x) = 0$ for all $x \in M$. Recall that $\psi_j(P)$ equals to

$$\psi_j(P) = \int [Q(s) + R(s)]\hat{\psi}_j(s)\rho(s)ds + \int e^{itP}\hat{\psi}_j(s)[1 - \rho(s)]ds.$$

Thus we have

$$\begin{aligned} &\left[\int [Q(s)]\hat{\psi}_j(s)\rho(s)dx\right] 1(x) \\ &\qquad = -\left[\int R(s)\hat{\psi}_j(s)\rho(s)ds\right] 1(x) - \left[\int e^{itP}\hat{\psi}_j(s)[1 - \rho(s)]dx\right] 1(x). \end{aligned} \tag{4.5}$$

Observing that $R(s)\rho(s)$ is a smooth function and $\psi_j(s)$ is supported on $[2^{j-1}, 2^{j+1}]$ we deduce

$$\int R(s)\widetilde{\phi}_j(s)\rho(s)ds = \int [R(\cdot)\rho(\cdot)]^\wedge (s)\psi_j(s)dx = O(2^{-jN}).$$

Next, we may apply Lemma 4.3.2 for ψ with any $s > 0$ since ψ is smooth. Then we have

$$\int e^{itP}\hat{\psi}_j(s)[1 - \rho(s)]ds = O_N(2^{-jN}).$$

Injecting the above two estimates into (4.5) we get

$$\left[\int Q(s)\hat{\psi}_j(s)\rho(s)ds\right] 1(x) = O_N(2^{-jN}).$$

Combining this with the identity $\int \Psi_j(x, y)dy = \int \left[Q(s)\hat{\psi}_j(s)\rho(s)dx\right] 1(x)$ we obtain the desired result. □

Corollary 4.5.3. *Suppose that $m \in L^\infty[0, \infty)$ satisfies the condition* (4.3) *for some $s > 0$. Then we have*

$$\int H_j(x, z)dz = O_N(2^{-jN})$$

for any $N \in \mathbb{N}$.

Proof. Let K_j be the kernel of $\int Q(t)\hat{m}_j(t)\rho(t)dt$. By (4.3) and Hölder's inequality we have

$$\int_M \left|K_j(x, y)\right| dx \leq C. \tag{4.6}$$

Using (4.1) and Lemma 4.5.2, we may deduce that

$$\begin{aligned}\left|\int H_j(x,z)dz\right| &= \left|\int\left[\int \widetilde{\Phi}_j(y,z)dz\right]K_j(x,y)dy\right| \\ &\le \int O_N(2^{-Nj})|K_j(x,y)|dy = O_N(2^{-Nj}).\end{aligned}$$

It completes the proof. □

We set

$$M_p f(x) = \sup_{x\in Q}\left(\frac{1}{|Q|}\int_Q |f(y)|^p dy\right)^{1/p}, \tag{4.7}$$

where Q are qubes centered at x. Now we have the following result.

Lemma 4.5.4. *Assume that $s > \frac{n}{p}$. We have*

$$\left|\left[m_j^{loc}(P)\circ\tilde{\phi}_j^{loc}(P)\right]f(x)\right| \le C[m]_s\cdot M_p f(x).$$

Proof. Let us take $q > 2$ such that $\frac{1}{q}+\frac{1}{r}=1$. By Lemma 4.4.2 we have H_j^* such that $H_j(x,z) = 2^{jn}H_j^*(2^jx, 2^jz)$ and

$$\int |x-y|^{\alpha q}|H_j^*(x,y)|^q dy \le C[m]_s^q \quad \text{for } 0\le\alpha<s. \tag{4.8}$$

Set $H_{k,l}^*(x,y) = H_k^*(x,y)\cdot 1_{\{2^{l-1}\le|x-y|<2^l\}}$ for $l\in\mathbb{N}$ and $H_{k,0}^*(x,y) = H_k^*(x,y)\cdot 1_{\{|x-y|<1\}}$. Then we deduce from (4.8) that

$$\sup_{l\ge 0} 2^{l\alpha q}\int |H_{k,l}^*(x,y)|^q dy \le C[m]_s^q \quad \text{for} \quad 0\le\alpha<s. \tag{4.9}$$

By a direct calculation we have

$$\begin{aligned}\left|\left[m_j^{loc}(P)\circ\tilde{\phi}_j^{loc}(P)\right]f(x)\right| &= \left|\int_G 2^{nk}H_k^*(2^kx,2^ky)f(y)dy\right| = \left|\sum_{l=0}^{\infty}\int_G 2^{nk}H_{k,l}^*(2^kx,2^ky)f(y)dy\right| \\ &\le \sum_{l=0}^{\infty}\left(\int_G 2^{nk}|H_{k,l}^*(2^kx,2^ky)|^q dy\right)^{1/q}\left(2^{nk}\int_{|x-y|\le 2^{l-k}}|f(y)|^r dy\right)^{1/r} \\ &\le \sum_{l=0}^{\infty} 2^{(ln/r)l}(M(|f|^r)(x))^{1/r}\left(\int_G |H_{k,l}^*(y)|^q dy\right)^{1/q}.\end{aligned}$$

Since $\frac{n}{r} < s$ we can take an $\epsilon > 0$ such that $\alpha := \frac{n}{r}+\epsilon < s$. Then we apply (4.9) to get

$$\begin{aligned}\left|\left[m_j^{loc}(P)\circ\tilde{\phi}_j^{loc}(P)\right]f(x)\right| &\le [m]_s\cdot\sum_{l=0}^{\infty}2^{ln/r}2^{-l\alpha}(M(|f|^r)(x))^{1/r} = [m]_s\cdot\sum_{l=0}^{\infty}2^{-l\epsilon}(M(|f|^r)(x))^{1/r} \\ &\le [m]_s\cdot(M(|f|^r)(x))^{1/r}.\end{aligned} \tag{4.10}$$

It proves the lemma. □

4.6 Martingale operators and the proof of Proposition 4.2.3

We introduce the following things on homogeneous space in [C2] which may be regarded as dyadic cubes on Euclidean space. Open set Q_α^k will role as dyadic cubes of sidelengths 2^{-k} (or more precisely, δ^k) with the two conventions : 1. For each k, the index α will run over some unspecified index set dependent on k. 2. For two sets with $Q_\alpha^{k+1} \subset Q_\beta^k$, we say that Q_β^k is a parent of Q_α^{k+1}, and Q_α^{k+1} a child of Q_β^k.

Theorem 4.6.1 (Theorem 14, [C2]). *Let X be a space of homogeneous type. Then there exists a family of subset $Q_\alpha^k \subset X$, defined for all integers k, and constants $\delta, \epsilon > 0, C < \infty$ such that*

- $\mu(X \setminus \cup_\alpha Q_\alpha^k) = 0 \ \forall k$
- *for any α, β, k, l with $l \geq k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$*
- *each Q_α^k has exactly one parent for all $k \geq 1$*
- *each Q_α^k has at least one child*
- *if $Q_\alpha^{k+1} \subset Q_\beta^k$ then $\mu(Q_\alpha^{k+1}) \geq \epsilon\mu(Q_\beta^k)$*
- *for each (α, k) there exists $x_{\alpha,k} \in X$ such that $B(x_{\alpha,k}, \delta^k) \subset Q_\alpha^k \subset B(x_{\alpha,k}, C\delta^k)$.*

Moreover,

$$\mu\{y \in Q_\alpha^k : \rho(y, X \setminus Q_\alpha^k) \leq t\delta^k\} \leq Ct^\epsilon \mu(Q_\alpha^k) \ for\ 0 < t \leq 1,\ for\ all\ \alpha, k. \tag{4.1}$$

We set $Q_1^0 = M$, and for $k \geq 0$ we define

$$\mathbb{E}_k f(x) = \mu(Q_\alpha^k)^{-1} \int_{Q_\alpha^k} f d\mu \quad \text{for } x \in Q_\alpha^k.$$

Then we define the martingale by $\mathbb{D}_k f(x) = \mathbb{E}_{k+1} f(x) - \mathbb{E}_k f(x)$. We also define the following square function

$$S(f) = \left(\sum_{k \geq 0} |\mathbb{D}_k f(x)|^2 \right)^{1/2}.$$

We have the following result on $\mathbb{E}_k$ and $S(f)$.

Theorem 4.6.2 (see [CW, Corollary 3.1.]). *There is a constant $C_d > 0$ such that, for any $\lambda > 0$, and $0 < \epsilon < \frac{1}{2}$, the following inequality holds.*

$$\begin{aligned} &meas(\{x : \sup_{k \geq 0} |\mathbb{E}_k g(x) - \mathbb{E}_0 g(x)| > 2\lambda, S(g) < \epsilon\lambda\}) \\ &\leq Cexp(-\frac{C_M}{\epsilon^2}) meas(\{x : \sup_{k \geq 0} |\mathbb{E}_k g(x)| > \lambda\}); \end{aligned} \tag{4.2}$$

Let us introduce the following functional

$$G_r(f) = (\sum_{k\in\mathbb{N}} (\mathcal{M}(|\phi_k^{loc}(P)f|^r))^{2/r})^{1/2},$$

where $\mathcal{M} = M_1 \circ M_1 \circ M_1$. Then we have the Fefferman-Stein inequality [FeS];

$$\|G_r(f)\|_p \le C_{p,r}\|f\|_p, \qquad 1 < r < 2, r < p < \infty. \tag{4.3}$$

We have the following result.

Lemma 4.6.3. *Let $s > \frac{d}{r}$ for some $r > 1$. Then, for $m \in L^\infty[0,\infty)$ such that $[m]_s < \infty$, we have*

$$S(m^{loc}(P)f)(x) \le A_r[m]_s G_r(f)(x) \quad \forall\, f \in L^p(M). \tag{4.4}$$

Proof. Given the result of Lemma 4.5.4 one may adapt the proof of [GHS, Lemma 3.1](see also [**?**, Lemma 3.4]) to get the inequality (4.4), so let us omit the details. □

Now we are in a position to prove Proposition 4.2.3.

Proof of Proposition 4.2.3. We set $T_i f = (m^i)^{loc}(P)$. By Proposition 4.4.7 we have $A(m^j, P)f = T_i(P)f + O(1)f$ where $\|O(1)f\|_{L^\infty(M)} \le C[m]_s\|f\|_{L^1(M)}$. Hence it is enough to bound

$$\Big\|\sup_{1\le i\le N} |T_i f|\Big\|_{L^p(M)} = \left(p4^p \int_0^\infty \lambda^{p-1}\text{meas}(\{x \in M : \sup_i |T_i f(x)| > 4\lambda\})d\lambda\right)^{1/p}$$

by a constant time of $[m]_s \cdot \sqrt{\log(N+1)}\|f\|_{L^p(M)}$. We have

$$\{x \in M : \sup_{1\le i\le N} |T_i f(x)| > 4\lambda\} \subset E_{\lambda,1} \cup E_{\lambda,2} \cup E_{\lambda,3},$$

where

$$\epsilon_N := \left(\frac{c_d}{10\log(N+1)}\right)^{1/2}$$

and

$$\begin{aligned} E_{\lambda,1} &= \{x \in M : \sup_{1\le i\le N} |T_i f(x) - \mathbb{E}_0 T_i f(x)| > 2\lambda, G_r(f)(x) \le \frac{\varepsilon_N \lambda}{A_r[m]_s}\}, \\ E_{\lambda,2} &= \{x \in M : G_r(f)(x) > \frac{\varepsilon_N \lambda}{A_r[m]_s}\}, \\ E_{\lambda,3} &= \{x \in M : \sup_{1\le i\le N} |\mathbb{E}_0 T_i f(x)| > 2\lambda\}. \end{aligned}$$

By proposition 4.6.3 we have $S(T_i f) \le A_r[m]_s G_r(f)$, and using this we obtain

$$E_{\lambda,1} \subset \bigcup_{i=1}^N \{x \in M : |T_i f(x) - \mathbb{E}_0 T_i f(x)| > 2\lambda, \quad S(T_i f) \le \varepsilon_N \lambda\}.$$

Applying (4.2) we get

$$\begin{aligned}\operatorname{meas}(E_{\lambda,1}) &\le \sum_{i=1}^{N} \operatorname{meas}(\{x \in M : |T_i f(x) - \mathbb{E}_0 T_i f(x)| > 2\lambda, S(T_i f) \le \varepsilon_N \lambda\}) \\ &\le \sum_{i=1}^{N} C \exp(-\frac{c_d}{\varepsilon_N^2}) \operatorname{meas}(\{x : \sup_{k\ge 0} |\mathbb{E}_k(T_i f)| > \lambda\}).\end{aligned}$$

Using this we estimate

$$\begin{aligned}\left(p\int_0^\infty \lambda^{p-1}\operatorname{meas}(E_{\lambda,1})d\lambda\right)^{1/p} &\le C\left(\sum_{i=1}^{N}\exp(-\frac{C_M}{\varepsilon_N^2})\left\|\sup_{k\ge0}|\mathbb{E}_k(T_i f)|\right\|_{L^p(M)}^p\right)^{1/p} \\ &\le C\left(\sum_{i=1}^{N}\exp(-\frac{C_M}{\varepsilon_N^2})\,\|T_i f\|_{L^p(M)}^p\right)^{1/p} \\ &\le C[m]_s\left(N\exp(-\frac{C_M}{\varepsilon_N^2})\right)^{1/p}\|f\|_{L^p(M)} \\ &\le C[m]_s\cdot\|f\|_{L^p(M)}.\end{aligned} \tag{4.5}$$

By a change of variables and (4.3) we have

$$\left(p\int_0^\infty \lambda^{p-1}\operatorname{meas}(E_{\lambda,2})d\lambda\right)^{1/p} = \frac{A_r[m]_s}{\varepsilon_N}\|G_r(f)\|_{L^p(M)} \le C[m]_s\sqrt{\log(N+1)}\|f\|_{L^p(M)}. \tag{4.6}$$

On the other hand, applying Corollary 4.5.3 we have $\mathbb{E}_0(T_i f)(x) \le C[m]_s\|f\|_{L^1(M)}$, and so

$$\begin{aligned}\left(p\int_0^\infty \lambda^{p-1}\operatorname{meas}(E_{\lambda,3})d\lambda\right)^{1/p} &= 2\left\|\sup_{i=1,\ldots,N}|\mathbb{E}_0(T_i f)|\right\|_{L^p(M)} \\ &\le C[m]_s|\operatorname{vol}(M)|^{1/p}\|f\|_{L^1(M)} \le C[m]_s\|f\|_{L^p(M)}.\end{aligned} \tag{4.7}$$

Combining the estimates (4.5), (4.6), and (4.7) we have

$$\left\|\sup_{1\le i\le N}|T_i f|\right\|_{L^p(M)} \le C[m]_s\sqrt{\log(N+1)}\|f\|_{L^p(M)}. \tag{4.8}$$

The proof is completed. □

Part II

Semilinear Elliptic Equations and Fractional Laplacians

Chapter 5

On strongly indefinite systems involving the fractional Laplacian [Ch4]

5.1 Introduction

In this paper we shall study the following nonlinear problem

$$\begin{cases} \mathcal{A}_s u = v^p & \text{in } \Omega, \\ \mathcal{A}_s v = u^q & \text{in } \Omega, \\ u > 0,\ v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $0 < s < 1$, $p > 1$, $q > 1$, Ω is a smooth bounded domain of $\mathbb{R}^n$ and $\mathcal{A}_s$ denotes the fractional Laplace operator $(-\Delta)^s$ in Ω with zero Dirichlet boundary values on $\partial\Omega$, defined in terms of the spectra of the Dirichlet Laplacian $-\Delta$ on Ω.

The problem (5.1) with $\Omega = R^n$ has been studied by many authors (see e.g. [CLO, CLO2, Y]). The problem was handled as an integro-differential system by inverting the operator $(-\Delta)^s$ to $(-\Delta)^{-s}$. This intepretation is particularly convenient in the case $\Omega = \mathbb{R}^n$.

Recently, Caffarelli and Silvestre [CaS] developed a local interpretation of the fractional Laplacian given in $\mathbb{R}^n$ by considering a Neumann type operator in the extended domain $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^{n+1} : t > 0\}$. This observation made a significant influence on the study of related nonlocal problems. A similar extension was devised by Cabré and Tan [CT] and Capella, Dávila, Dupaigne, and Sire [CDDS] (see Brändle, Colorado, de Pablo, and Sánchez [BCPS2] and Tan [T2] also). Based on this local interpretation, we shall derive many important properties of the solutions to the nonlocal system (5.1).

The fractional Laplacian appears in diverse areas including physics, biological modeling and mathematical finances and partial differential equations involving the fractional Laplacian have attracted the attention of many researchers. Many authors studied nonlinear problems of the form $\mathcal{A}_s u = f(u)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a certain function. When $s = \frac{1}{2}$, Cabré and Tan [CT] established the existence of positive solutions for equations having nonlinearities with the subcritical growth, their regularity, the symmetric property, and a priori estimates of the Gidas-Spruck type by employing a blow-up argument along with a Liouville type result for the square root of the Laplacian in the half-space. Brändle, Colorado, de Pablo, and Sánchez [BCPS2] dealt with a subcritical concave-convex problem. For $f(u) = u^q$ with the critical and supercritical exponents $q \geq \frac{n+2s}{n-2s}$, the nonexistence of solutions was proved in [BCPS2, T1, T2] in which the authors devised and used the Pohozaev type identities. The Brezis-Nirenberg type problem was studied in [T1] for $s = 1/2$.

When $s = 1$ the nonlinear problem (5.1) corresponds the well-known Lane-Emden system,

$$\begin{cases} -\Delta u = v^p & \text{in } \Omega, \\ -\Delta v = u^q & \text{in } \Omega, \\ u > 0,\ v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

This system is a fundamental form among strongly coupled nonlinear systems and so it has recieved a lot of interest from may authors.Generally, nonlinaer systems comes from mathematical modelling such as Gierer-Meinhardt type system and solitary waves of coupled schrodinger systems. We refer to [CFM, FF, HV, FLN] and references therein, and the book [QS] for a survey of this topic.

Before studying the problem (5.1) we shall establish a different proof for *a priori* estimate for solutions to the problem

$$\begin{cases} \mathcal{A}_s u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Theorem 5.1.1. *Let $n \geq 2$ and $0 < s < 1$. Assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $f(u) = u^p,\ 1 < p < \frac{n+2s}{n-2s}$.*

Then, there exists a constant $C(p, \Omega) > 0$ depending only on p and Ω such that every weak solution of (1.1) *satisfies*

$$\|u\|_{L^\infty(\Omega)} \leq C(p, \Omega).$$

*Moreover, the statement holds for any function $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying **Condition A** (see Section 4).*

The result of Theorem 5.1.1 was proved by Cábre-Tan [CT] for $s = 1/2$ and Tan [**?**] for $1/2 < s < 1$. They employed the blow-up argument with a combination of Liouville type results. We carry out a different approach using Pohozaev identity. This can be seen as a the non-local version of the argument in Figueiredo-Lions-Nussbaum [FLN] for the local case $s = 1$. In the non-local case, there arises some difficulty in using the Pohozaev identity which does not appear in the local case $s = 1$ (see Remark 5.4.3). This difficulty will be overcomed by using the estimates of Proposition 5.3.1.

As this approach does not require a Liouville-type result, the function $f(u)$ is not required to have a precise asymptocity as $u \to \infty$. Moreover, this approach is easily modified to obtain *a priori* estimates for the nonlinear system (5.1) (see Theorem 5.1.5 below).

Concerning the problem (5.1) we shall say that a pair of exponents (p, q) is sub-critical if $\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2s}{n}$, critical if $\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2s}{n}$, and super-critical if $\frac{1}{p+1} + \frac{1}{q+1} < \frac{n-2s}{n}$. Then we have the following existence result.

Theorem 5.1.2. *Suppose that (p, q) is sub-critical and choose $\alpha > 0$ and $\beta > 0$ such that*

$$\frac{1}{2} - \frac{1}{q+1} < \frac{\alpha}{n}, \quad \frac{1}{2} - \frac{1}{p+1} < \frac{\beta}{n}, \quad \text{and} \quad \alpha + \beta = 2s.$$

Then, the problem (5.1) has at least one positive solution $(u, v) \in H_0^{\alpha}(\Omega) \times H_0^{\beta}(\Omega)$.

We refer to Section 2 for the definition of weak solution and the Sobolev space $H_0^{\alpha}(\Omega)$. This existence theorem will follow easily by adapting the proof of the existence result for the problem (5.1) with $s = 1$ established in [FF] and [HV] independently. For such weak solutions, we shall prove an L^{∞} estimate of Brezis-Kato type and study the regularity property of the weak solutions based on the results of Cabré-Sire [CaS].

For further properties of solutions to (5.1) we shall relying on studying the extension problem of (5.1) in the sense of Caffarelli-Silvestre [CaS] and Cabré-Tan [CT], namely,

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = \operatorname{div}(t^{1-2s}\nabla V) = 0 & \text{in } C := \Omega \times [0, \infty), \\ U = V = 0 & \text{on } \partial_L C := \partial\Omega \times [0, \infty), \\ \partial_\nu^s U = V^p, \ \partial_\nu^s V = U^q & \text{on } \Omega \times \{0\}, \\ U > 0, \ V > 0 & \text{in } C. \end{cases} \tag{5.4}$$

Here U and V are called the s-harmonic extensions of u and v. We refer to Section 2, for the details. By obtaining a Pohozaev type identity on C, we shall get the following non-existence result.

Theorem 5.1.3. *Assume that the domain Ω is bounded and starshaped. Suppose that (p, q) is critical or sub-critical. Then the problem (5.1) has no bounded solution.*

Next we shall establish a moving plane argument and a maximum principle for the extended problem, to prove the following symmetry result.

Theorem 5.1.4. *Suppose that a bounded smooth domain $\Omega \subset \mathbb{R}^n$ is convex in the x_1-direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Let (u, v) be a $C^1(\bar{\Omega})$ solution of (5.1).*

Then, the functions u and v are symmetric in x_1-direction, that is, $u(-x_1, x') = u(x_1, x')$, $v(x_1, x') = v(-x_1, x')$ for all $(x_1, x') \in \Omega$. Moreover we have $\frac{\partial u}{\partial x_1} < 0$ and $\frac{\partial v}{\partial x_1} < 0$ for $x_1 > 0$.

The moving plane argument will be also useful to obtain a uniform bound for solutions near the boundary. Combining this uniform bound with the inequality of Proposition 5.3.1, and Sobolev embeddings we shall establish the following *a priori* estimate of Gidas-Spruck type.

Theorem 5.1.5. *Assume that $\Omega \subset \mathbb{R}^n$ is a smooth convex bounded domain and $p > 1$ and $q > 1$ are such that (p, q) is sub-critical. Then, there exists a constant $C(p, q, \Omega)$ depeding only on p, q and Ω such that every weak solution of* (5.1) *satisfies*

$$||u||_{L^\infty(\Omega)} + ||v||_{L^\infty(\Omega)} \leq C(p, q, \Omega).$$

The rest of this paper is organized as follows. In Section 2 we briefly review the basic results concerning the fractional Laplacian. In Section 3 we shall establish the integral estimate related to the Pohozaev type identities for solutions to (5.1) and (5.3). Having this estimate, we shall prove Theorem 5.1.1 in Section 4. The nonlinear system (5.1) will be studied throughout Section 5. We obtain the existence and the non-existence results of Theorem 5.1.2 and Theorem 5.1.3. Then we shall establish the Brezis-Kato type result and study the regularity of solutions to (5.1). Finally we shall establish the moving plane argument, and we shall complete the proofs of Theorem 5.1.4 and Theorem 5.1.5.

Notations.

We shall use the following notations in this paper.
- The letter z represents a variable in the $\mathbb{R}^{n+1}$. Also, it is written as $z = (x, t)$ with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.
- $C > 0$ is a generic constant that may vary from line to line. In particular, the generic constants are independent of solutions to (5.1) and (5.3) in the proofs of Theorem 5.1.1 and Theorem 5.1.5.
- For each $r > 0$ we set $\mathcal{I}(\Omega, r) = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq r\}$ and $\mathcal{O}(\Omega, r) = \{x \in \Omega : \text{dist}(x, \partial\Omega) < r\}$.

5.2 Preliminaries

In this section we first recall the backgrounds of the fractional Laplacian. We review the definition of fractional Sobolev spaces, the local interpretation of fractional Laplacians, and an embedding property.

5.2.1 Spectral definition of the fractional Sobolev spaces and fractional Laplacians

Let Ω be a smooth bounded domain of $\mathbb{R}^n$. Let also $\{\lambda_k, \phi_k\}_{k=1}^{\infty}$ be a sequence of the eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ in Ω with the zero Dirichlet boundary condition on $\partial\Omega$,

$$\begin{cases} -\Delta\phi_k = \lambda_k\phi_k & \text{in } \Omega, \\ \phi_k = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $\|\phi_k\|_{L^2(\Omega)} = 1$ and $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$. Then we set the fractional Sobolev space $H_0^s(\Omega)$ $(0 < s < 1)$ by

$$H_0^s(\Omega) = \left\{ u = \sum_{k=1}^{\infty} a_k\phi_k \in L^2(\Omega) : \sum_{k=1}^{\infty} a_k^2\lambda_k^s < \infty \right\}, \tag{5.1}$$

which is a Hilbert space whose inner product is given by

$$\left\langle \sum_{k=1}^{\infty} a_k\phi_k, \sum_{k=1}^{\infty} b_k\phi_k \right\rangle_{H_0^s(\Omega)} = \sum_{k=1}^{\infty} a_k b_k \lambda_k^s \qquad \text{if } \sum_{k=1}^{\infty} a_k\phi_k,\ \sum_{k=1}^{\infty} b_k\phi_k \in H_0^s(\Omega).$$

Moreover, for a function in $H_0^s(\Omega)$, we define the fractional Laplacian $\mathcal{A}_s : H_0^s(\Omega) \to H_0^s(\Omega) \simeq H_0^{-s}(\Omega)$ as

$$\mathcal{A}_s\left(\sum_{k=1}^{\infty} a_k\phi_k\right) = \sum_{k=1}^{\infty} a_k\lambda_k^s\phi_k. \tag{5.2}$$

We also consider the square root $\mathcal{A}_s^{1/2} : H_0^s(\Omega) \to L^2(\Omega)$ of the positive operator $\mathcal{A}_s$ which is in fact equal to $\mathcal{A}_{s/2}$. Note that by the above definitions, we have

$$\langle u, v\rangle_{H_0^s(\Omega)} = \int_\Omega \mathcal{A}_s^{1/2}u \cdot \mathcal{A}_s^{1/2}v = \int_\Omega \mathcal{A}_s u \cdot v \quad \text{for } u, v \in H_0^s(\Omega).$$

5.2.2 Extended problems of nonlinear systems

For functions $f : [0,\infty) \to \mathbb{R}$ and $g : [0,\infty) \to \mathbb{R}$ we consider the following extension problems

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = \operatorname{div}(t^{1-2s}\nabla V) = 0 & \text{in } C, \\ U = V = 0 & \text{on } \partial_L C, \\ \partial_\nu^s U = f(V) & \text{on } \Omega \times \{0\}, \\ \partial_\nu^s V = g(U) & \text{on } \Omega \times \{0\}, \end{cases} \tag{5.3}$$

Then, (5.3) with $f(x) = x^p$ and $g(x) = x^q$ is the extended problem of (5.1), i.e., if $(U, V) \in H_{0,L}^s(\Omega) \times H_{0,L}^s(\Omega)$ is a solution of (5.3), then their traces $u(x) := U(x,0)$ and $v(x) := V(x,0)$ becomes a solution of (5.1). Similarly the problem (5.3) is extended to the local-problem

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } C, \\ U = 0 & \text{on } \partial_L C, \\ \partial_\nu^s U = f(U) & \text{on } \Omega \times \{0\}. \end{cases} \tag{5.4}$$

5.2.3 Definition of weak solutions

Let $g \in L^{\frac{2n}{n+2s}}(\Omega)$ and consider the problem

$$\begin{cases} \mathcal{A}_s u = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

For each $0 < \alpha < 2s$ we say that a function $u \in H_0^\alpha(\Omega)$ is a weak solution of (5.5) provided

$$\int_\Omega \mathcal{A}_{\frac{\alpha}{2}} u \cdot \mathcal{A}_{\frac{(2s-\alpha)}{2}} \phi \, dx = \int_\Omega g(x)\phi(x)\, dx \tag{5.6}$$

for all $\phi \in H_0^{(2s-\alpha)}(\Omega)$.

As for the extended problem

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } C, \\ U = 0 & \text{on } \partial_L C, \\ \partial_\nu^s U = g(x) & \text{on } \Omega \times \{0\}, \end{cases} \tag{5.7}$$

we say that a function $U \in H_{0,L}^s(C)$ is a weak solution of (5.7) provided

$$\int_C t^{1-2s}\nabla U(x,t) \cdot \nabla\Phi(x,t)\, dxdt = C_s \int_\Omega g(x)\Phi(x,0)\, dx \tag{5.8}$$

holds for all $\Phi \in H_{0,L}^s(C)$.

5.2.4 The sobolev embedding

We recall the well-known weighted trace inequality (see [Lb]),

$$\left(\int_\Omega |U(x,0)|^{\frac{2n}{n-2s}} dx\right)^{\frac{n-2s}{2n}} \le C\left(\int_C t^{1-2s}|\nabla U(x,t)|^2 dxdt\right)^{\frac{1}{2}}, \quad U \in H_0^1(t^{1-2s}, C). \tag{5.9}$$

As an application, we have the following embedding result.

Lemma 5.2.1. *Let $w \in L^p(\Omega)$ for some $p < \frac{n}{2s}$.*

1. *Assume that U is a weak solution of the problem*

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \textit{in } C, \\ U = 0 & \textit{on } \partial_L C, \\ \partial_\nu^s U = w & \textit{on } \Omega \times \{0\}. \end{cases} \tag{5.10}$$

Then we have

$$\|U(\cdot,0)\|_{L^q(\Omega)} \le C_{p,q}\, \|w\|_{L^p(\Omega)}, \tag{5.11}$$

for any q such that $\frac{n}{q} \le \frac{n}{p} - 2s$.

2. *Assume that u is a weak solution of the problem*

$$\begin{cases} \mathcal{A}_s u = w & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.12}$$

Then we have

$$\|u\|_{L^q(\Omega)} \leq C_{p,q}\|w\|_{L^p(\Omega)}, \tag{5.13}$$

for any q such that $\frac{n}{q} \leq \frac{n}{q} - 2s$.

Proof. We multiply (5.10) by $|U|^{\beta-1}U$ for some $\beta > 1$ to get

$$\int_\Omega w(x)|U|^{\beta-1}U(x,0)\,dx = \beta \int_C t^{1-2s}|U|^{\beta-1}|\nabla U|^2\,dxdt. \tag{5.14}$$

Then, applying the trace embedding (5.9) and Hölder's inequality we can observe

$$\left\| |U|^{\frac{\beta+1}{2}}(\cdot,0) \right\|^2_{L^{\frac{2n}{n-2s}}(\Omega)} \leq C_\beta \left\| |U|^\beta(\cdot,0) \right\|_{L^{\frac{\beta+1}{2\beta}\cdot\frac{2n}{n-2s}}} \|w\|_p\,, \tag{5.15}$$

where p satisfies $\frac{1}{p} + \frac{(n-2s)\beta}{n(\beta+1)} = 1$. Let $q = \frac{n(\beta+1)}{n-2s}$, then (5.15) gives the desired inequality.

Let u be a weak solution of (5.12). We let U be the s-harmonic extension of u. Then, U is a solution of (5.10), and so (5.11) yields

$$\|u\|_{L^q(\Omega)} = \|U(\cdot,0)\|_{L^q(\Omega)} \leq C_{p,q}\|w\|_{L^p(\Omega)}. \tag{5.16}$$

The proof is completed. □

5.2.5 Green's functions and the Robin function

We have Green's function $G_C = G_C(z,x)$ $(z \in C,\ x \in \Omega)$ of the problem.

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla U) = 0 & \text{in } C, \\ U = 0 & \text{on } \partial_L C, \\ \partial^s_\nu U = g & \text{on } \Omega \times \{0\}, \end{cases} \tag{5.17}$$

A function U in C solving the problem (5.17) for some function g on $\Omega \times \{0\}$ is expressed as

$$U(z) = \int_\Omega G_C(z,y)g(y)dy, \quad z \in C,$$

where $u = \mathrm{tr}|_{\Omega\times\{0\}}U$. We have the following formula (see [CKL] for more details)

$$G_C((x,t),y) = G_{\mathbb{R}^{n+1}_+}((x,t),y) - H_C((x,t),y), \tag{5.18}$$

where

$$G_{\mathbb{R}^{n+1}_+}((x,t),y) := \frac{\mathfrak{a}_{n,s}}{|(x-y,t)|^{n-2s}}, \tag{5.19}$$

and the regular part $H_C : C \to \mathbb{R}$ satisfies

$$\begin{cases} \operatorname{div}\left(t^{1-2s}\nabla_{(x,t)}H_C((x,t),y)\right) = 0 & \text{in } C, \\ H_C(x,t,y) = \frac{\mathfrak{a}_{n,s}}{|(x-y,t)|^{n-2s}} & \text{on } \partial_L C, \\ \partial_\nu^s H_C((x,0),y) = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

Here $\mathfrak{a}_{n,s}$ is a positive constant determined by n and s. The existence of such a function H_C was obtained using a variational method in [CKL]. In addition, we have the followng boundedness property of $\mathcal{H}_C$.

Lemma 5.2.2 ([CKL]). *For any set $\mathcal{D} \in C$ such that dist$(\mathcal{D}, \partial_L C) > 0$, we have*

$$\sup_{y\in\Omega} \int_{\mathcal{D}} t^{1-2s}|\nabla_z H_C(z,y)|^2 dz < +\infty. \tag{5.20}$$

This lemma will be used to the integral estimate in the next section.

5.3 The integral estimates

In this section we establish useful integral estimates which hold for solutions to (5.1) and (5.3). These will be crucially used in the proof of the *a priori* estimates of Theorem 5.1.1 and Theorem 5.1.5. For each $r > 0$ we set $\mathcal{I}(\Omega, r) = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq r\}$ and $\mathcal{O}(\Omega, r) = \{x \in \Omega : \text{dist}(x, \partial\Omega) < r\}$. Then we have the following results.

Proposition 5.3.1.

1. *Suppose that $U \in H^s_{0,L}(C)$ is a solution of the problem* (5.4) *with f such that $f = F'$ for a function $F \in C^1(\mathbb{R})$. Then, for each $\delta > 0$ and $\gamma > \frac{n}{s}$ there is a constant $C = C(\delta, \gamma) > 0$ such that*

$$\begin{aligned} &\min_{r\in[\delta,2\delta]} \left| n \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} F(U)dx - \left(\frac{n-2s}{2}\right) \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} Uf(U)dx \right| \\ &\leq C \left[\left(\int_{\mathcal{O}(\Omega,2\delta)\times\{0\}} |f(U)|^\gamma dx \right)^{\frac{2}{\gamma}} + \int_{\mathcal{O}(\Omega,2\delta)\times\{0\}} |F(U)|dx + \left(\int_{\mathcal{I}(\Omega,\delta/2)\times\{0\}} |f(U)|dx \right)^2 \right]. \end{aligned} \tag{5.1}$$

2. *Suppose that $(U, V) \in H^s_{0,L}(C) \times H^s_{0,L}(C)$ is a solution of the problem* (5.3) *with (f, g) such that $f = F'$, $g = G'$ for some functions $F, G \in C^1(\mathbb{R})$. Then, for each $\delta > 0$ and $\gamma > \frac{n}{s}$ there*

is a constant $C = C(\delta, \gamma) > 0$ *such that*

$$\min_{r\in[\delta,2\delta]} \left| n \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} [F(U) + G(V)]\ dx - \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} \left[\left(\frac{n-2s}{2} - \theta \right) U f(U) + \theta V g(V) \right] dx \right|$$
$$\leq C \left(\left(\int_{O(\Omega,2\delta)\times\{0\}} (|f(U)| + |g(V)|)^{\gamma} dx \right)^{\frac{2}{\gamma}} + \int_{O(\Omega,2\delta)\times\{0\}} |F(U)| + |G(V)| dx \right.$$
$$\left. + \left(\int_{\mathcal{I}(\Omega,\delta/2)\times\{0\}} |f(U)| + |g(V)| dx \right)^2 \right). \tag{5.2}$$

Remark 5.3.2. The statement (1) of Proposition 5.3.1 was proved in [CKL]. We note that if u is a solution of (5.4), then (u, u) is a solution of (5.3) with $g = f$ and $q = p$. Thus the statement (1) follows directly from the statement (2) in Proposition 5.3.1.

Proof. As it explained above, it suffices to prove (5.2) only. Let $(U, V) \in H^s_{0,L}(C) \times H^s_{0,L}(C)$ are solutions of (5.3). Then, by a direct computation, we have the following identity

$$(n-2s)t^{1-2s}\nabla U \cdot \nabla V + \mathrm{div}[t^{1-2s}(z, \nabla V)\nabla U + t^{1-2s}(z, \nabla U)\nabla V] - \mathrm{div}(t^{1-2s}z(\nabla U \cdot \nabla V)) = 0. \tag{5.3}$$

For a given set $A \in C$, we denote $\partial^+ A = \partial A \cap \{(x,t) \in \mathbb{R}^{n+1}, t > 0\}$ and $\partial_b A = A \cap \{(x,t) \in \mathbb{R}^{n+1}, t = 0\}$. Using integration by parts we have

$$\begin{aligned} \int_A t^{1-2s}\nabla U \cdot \nabla V\ dxdt &= \int_{\partial^+ A} t^{1-2s}(\nabla U, \nu)v\ dS + \int_{\partial_b A} \partial^s_\nu U\ V(x,0)dx \\ &= \int_{\partial^+ A} t^{1-2s}(\nabla V, \nu)U\ dS + \int_{\partial_b A} \partial^s_\nu U\ V(x,0)dx. \end{aligned} \tag{5.4}$$

We also have

$$\begin{aligned} &\int_A \mathrm{div}\left[t^{1-2s}(z, \nabla V)\nabla U + t^{1-2s}(z, \nabla U)\nabla V\right] dxdt \\ &= \int_{\partial^+ A} \left[t^{1-2s}(z, \nabla V)(\nabla U, \nu) + t^{1-2s}(z, \nabla U)(\nabla V, \nu)\right] dS + \int_{\partial_b A} (x, \nabla_x V)\partial^s_\nu U + (x, \nabla_x U)\partial^s_\nu V dx, \end{aligned} \tag{5.5}$$

and

$$\int_A \mathrm{div}(t^{1-2s}z(\nabla U \cdot \nabla V))dxdt = \int_{\partial^+ A} t^{1-2s}(z, \nu)(\nabla U \cdot \nabla V)dS. \tag{5.6}$$

Now we define for each $r > 0$ the following sets:

$$D_r = \left\{z \in \mathbb{R}^{n+1}_+ : \mathrm{dist}(z, \mathcal{I}(\Omega, r) \times \{0\}) \leq r/2\right\},$$

$$\partial D_r^+ = \partial D_r \cap \left\{(x,t) \in \mathbb{R}^{n+1} : t > 0\right\} \quad \text{and} \quad E_\delta = \bigcup_{r=\delta}^{2\delta} \partial D_r^+.$$

Note that $\partial D_r = \partial D_r^+ \cup (\mathcal{I}(\Omega, r/2) \times \{0\})$. Fix a small number $\delta > 0$ and a value $\theta \in [0, n-2s]$. We integrate the identity (5.3) over D_r for each $r \in (0, 2\delta]$ to derive

$$\begin{aligned}
&\theta \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} \partial_\nu^s U \cdot V(x,0)dx + (n-2s-\theta) \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} \partial_\nu^s V \cdot U(x,0)dx \\
&\qquad + \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} \left[(x, \nabla_x V)\partial_\nu^s U + (x, \nabla_x U)\partial_\nu^s V\right](x,0)dx \\
&= -\theta \int_{\partial D_r^+} t^{1-2s}(\nabla U, \nu) V dS - (n-2s-\theta) \int_{\partial D_r^+} t^{1-2s}(\nabla V, \nu) U dS \\
&\quad + \int_{\partial D_r^+} t^{1-2s}(z,\nu)(\nabla U \cdot \nabla V) dS - \int_{\partial D_r^+} \left[t^{1-2s}(z, \nabla V)(\nabla U, \nu) + t^{1-2s}(z, \nabla U)(\nabla V, \nu)\right] dS,
\end{aligned} \tag{5.7}$$

where (5.4), (5.5), and (5.6) are used. By using $\partial_\nu^s U = f(V)$, $\partial_\nu^s V = g(V)$ and performing integration by parts, we derive

$$\begin{aligned}
&\theta \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} g(V) \cdot V dx + (n-2s-\theta) \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} f(U) \cdot U dx - \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} [nF(U) + nG(V)]\, dx \\
&= -\theta \int_{\partial D_r^+} t^{1-2s}(\nabla U, \nu) V dS - (n-2s-\theta) \int_{\partial D_r^+} t^{1-2s}(\nabla V, \nu) U dS \\
&\quad + \int_{\partial D_r^+} t^{1-2s}(z,\nu)(\nabla U \cdot \nabla V) dS - \int_{\partial D_r^+} \left[t^{1-2s}(z, \nabla V)(\nabla U, \nu) + t^{1-2s}(z, \nabla U)(\nabla V, \nu)\right] dS \\
&\quad + \int_{\partial\mathcal{I}(\Omega,r/2)\times\{0\}} (x,\nu)(F(U) + G(V))(x,0) dS_x.
\end{aligned}$$

From this identity we get

$$\begin{aligned}
&\left| \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} [\theta g(V) \cdot V - nG(V)]\, dx + \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} [(n-2s-\theta) f(U) \cdot U - nF(U)]\, dx \right| \\
&\qquad \le C \int_{\partial D_r^+} t^{1-2s}(|\nabla U|^2 + U^2 + |\nabla V|^2 + V^2) dS + \int_{\partial\mathcal{I}(\Omega,r/2)\times\{0\}} \langle x, \nu\rangle (F(U) + G(V))(x,0) dS_x.
\end{aligned}$$

We integrate this identity with respect to r over an interval $[\delta, 2\delta]$ and then use the Poincaré inequality. Then we observe

$$\begin{aligned}
&\min_{r\in[\delta,2\delta]} \left| \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} [\theta g(V) \cdot V - nG(V)]\, dx + \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} [(n-2s-\theta) f(U) \cdot U - nF(U)]\, dx \right| \\
&\qquad \le C \int_{E_\delta} t^{1-2s}(|\nabla U|^2 + U^2 + |\nabla V|^2 + V^2) dz + C \int_{O(\Omega,\delta)} |F(U)(x,0)| + |G(V)(x,0)| dx.
\end{aligned}$$

We only need to estimate the first term of the right-hand side of the previous inequality since the second term is already one of the terms which constitute the right-hand side of (5.2).

We are going to estimate $\int_{E_\delta} t^{1-2s}|\nabla U|^2 dz$. Note that

$$\nabla_z U(z) = \int_\Omega \nabla_z G_{\mathbb{R}^{n+1}_+}(z,y) f(V)(y,0)dy - \int_\Omega \nabla_z H_C(z,y) f(V)(y,0)dy \tag{5.8}$$

for $z \in E_\delta$. Let us deal with the last term of (5.8) first. Using (5.20) and Hölder's inequality, we get

$$\begin{aligned}
&\int_{E_\delta} t^{1-2s}\left(\int_\Omega |\nabla_z H_C(z,y) f(V)(y,0)|dy\right)^2 dz \\
&\le \left(\sup_{y\in\Omega}\int_{E_\delta} t^{1-2s}|\nabla_z H_C(z,y)|^2 dz\right)\left(\int_\Omega |f(V)(y,0)|dy\right)^2 \le C\left(\int_{\mathcal{I}(\Omega,\delta)\cup\mathcal{O}(\Omega,\delta)} |f(V)(y,0)|dy\right)^2 \\
&\le C\left[\left(\int_{\mathcal{O}(\Omega,2\delta)} |f(V)(y,0)|^q dy\right)^{\frac{2}{q}} + \left(\int_{\mathcal{I}(\Omega,\delta/2)} |f(V)(y,0)|dy\right)^2\right],
\end{aligned} \tag{5.9}$$

which is a part of the right-hand side of (5.2).

It remains to take into consideration of the first term of (5.8). We split the term as

$$\begin{aligned}
&\int_\Omega \nabla_z G_{\mathbb{R}^{n+1}_+}(z,y) f(V)(y,0)dy \\
&= \int_{\mathcal{O}(\Omega,2\delta)} \nabla_z G_{\mathbb{R}^{n+1}_+}(z,y) f(V)(y,0)dy + \int_{\mathcal{I}(\Omega,2\delta)} \nabla_z G_{\mathbb{R}^{n+1}_+}(z,y) f(V)(y,0)dy \\
&:= A_1(z) + A_2(z).
\end{aligned}$$

Take $q > \frac{n}{s}$ and $r > 1$ satisfying $\frac{1}{q} + \frac{1}{r} = 1$. Then

$$|A_1(z)| \le \left(\int_{\mathcal{O}(\Omega,2\delta)} |\nabla_z G_{\mathbb{R}^{n+1}_+}(z,y)|^r dy\right)^{\frac{1}{r}} \|f(V)(\cdot,0)\|_{L^q(\mathcal{O}(\Omega,2\delta))}.$$

In light of the definition of $G_{\mathbb{R}^{n+1}_+}$, it holds that

$$\begin{aligned}
\left(\int_{\mathcal{O}(\Omega,2\delta)} |\nabla_z G_{\mathbb{R}^{n+1}_+}(z,y)|^r dy\right)^{\frac{1}{r}} &\le C\left(\int_{\mathcal{O}(\Omega,2\delta)} \frac{1}{|(x-y,t)|^{(n-2s+1)r}} dy\right)^{\frac{1}{r}} \\
&\le C\max\left\{t^{\frac{n}{r}-(n-2s+1)}, 1\right\} = C\max\left\{t^{-\frac{n}{q}+2s-1}, 1\right\}.
\end{aligned}$$

Thus we have

$$|A_1(z)| \le C\max\left\{t^{-\frac{n}{q}+2s-1}, 1\right\} \|f(V)(\cdot,0)\|_{L^q(\mathcal{O}(\Omega,2\delta))}.$$

Using this we see

$$\begin{aligned}
\int_{E_\delta} t^{1-2s}|A_1(z)|^2 dz &\le C\int_0^1 \max\left\{t^{1-2s}t^{-\frac{2n}{q}+4s-2}, t^{1-2s}\right\} \|f(V)(\cdot,0)\|^2_{L^q(\mathcal{O}(\Omega,2\delta))} dt \\
&= \int_0^1 \max\left\{t^{2s-\frac{2n}{q}-1}, t^{1-2s}\right\} \|f(V)(\cdot,0)\|^2_{L^q(\mathcal{O}(\Omega,2\delta))} dt. \\
&\le C\|f(V)(\cdot,0)\|^2_{L^q(\mathcal{O}(\Omega,2\delta))}.
\end{aligned} \tag{5.10}$$

Concerning the term A_2, we note that E_δ is away from $\mathcal{I}(\Omega, 2\delta) \times \{0\}$. Thus we have

$$\sup_{z \in E_\delta, y \in \mathcal{I}(\Omega,2\delta)} |\nabla_z G_{\mathbb{R}^{n+1}_+}(z, y)| \leq C.$$

Hence

$$|A_2(z)| \leq C \int_{\mathcal{I}(\Omega,2\delta)} |f(V)(y, 0)| dy, \quad z \in E_\delta.$$

Using this we find

$$\int_{E_\delta} t^{1-2s} |A_2(z)|^2 dz \leq C \left(\int_{\mathcal{I}(\Omega,2\delta)} |f(V)(y, 0)| dy \right)^2. \tag{5.11}$$

We have obtained the desired bound of $\int_{E_\delta} t^{1-2s} |\nabla U|^2 dz$ through the estimates (5.9), (5.10) and (5.11). The estimates for $\int_{E_\delta} t^{1-2s} |\nabla V|^2 dz$, $\int_{E_\delta} t^{1-2s} |U|^2 dz$ and $\int_{E_\delta} t^{1-2s} |V|^2 dz$ can be obtained similarly. The proof is finished. □

5.4 The proof of Theorem 5.1.1

In this section, we prove Theorem 5.1.1. Let us now state the general condition on $f : [0, \infty) \to \mathbb{R}$ for which Theorem 5.1.1 holds.

Condition A: The function f satisfies

$$\liminf_{u \to \infty} \frac{f(u)}{u} > \lambda_1^s, \quad \lim_{u \to \infty} \frac{f(u)}{u^{(n+2s)/(n-2s)}} = 0,$$

with one of the following assumptions

1. We have

$$\limsup_{n \to \infty} \frac{u f(u) - \theta F(u)}{u^2 f(u)^{2s/n}} \leq 0, \quad \text{for some } \theta \in [0, \frac{2n}{n - 2s}). \tag{5.1}$$

2. Ω is convex or the function $u \to f(u) u^{-\frac{n+2s}{n-2s}}$ is nonincreasing on $(0, \infty)$.

It is direct to check that $f(u) = u^p$ with $p \in (1, \frac{n+2s}{n-2s})$ satisfies **Condition A** for clearity.

The first step for the *a priori* estimates is to obtain a uniform L^1 bound away from the boundary and a uniform L^∞ bound near the boundary for positive solutions.

Lemma 5.4.1. *Suppose that Ω is a smooth bounded domain and $f : [0, \infty) \to \mathbb{R}$ satisfies*

$$\liminf_{n \to \infty} \frac{f(u)}{u} > \lambda_1^s. \tag{5.2}$$

Then there exist a small number $r > 0$ and a constant $C = C(r, \Omega) > 0$ such that for any solution u of (5.3) *we have*

$$\int_{\mathcal{I}(\Omega,r)} f(u) dx \leq C, \tag{5.3}$$

and

$$\sup_{x\in O(\Omega,r)} u(x) \le C. \tag{5.4}$$

Proof. First we recall from (5.2) that $\mathcal{A}_s\phi_1(x) = \lambda_1^s\phi_1(x)$. Combining this and (5.3) we get

$$\int_\Omega \lambda_1^s\phi_1 u(x)dx = \int_\Omega (\mathcal{A}_s\phi_1)u(x)dx = \int_\Omega \phi_1(\mathcal{A}_s u)(x)dx = \int_\Omega \phi_1 f(u)(x)dx. \tag{5.5}$$

By the condtion (5.2) there are constants $\delta > 0$ and $C > 0$ such that $f(u) > (\lambda_1^s + \delta)u - C$ for all $u > 0$. Combining this with (5.5) shows

$$\int_\Omega \lambda_1^s\phi_1 u dx > \int_\Omega (\lambda_1^s + \delta)u\phi_1 dx - \int_\Omega C\phi_1 dx,$$

which directly gives

$$\int_\Omega \phi_1 u(x)dx \le \frac{1}{\delta}\int_\Omega C\phi_1(x)dx \le C. \tag{5.6}$$

It is well-known that there exists a constant $C = C(\Omega, r)$ such that $\phi_1(x) \ge C$ for all $x \in \mathcal{I}(\Omega, r)$. Hence (5.6) gives us that

$$\int_{\mathcal{I}(\Omega,r)} u dx \le C\int_\Omega \phi_1 u dx \le C. \tag{5.7}$$

Combining this with the identity (5.5), we get the estimate (5.3).

It remains to prove (5.4). It is standard to bound the value $u(x)$ for x near the boundary by a constant multiple of an integration of u over an inner subset of Ω. Consequently the bound of (5.7) gives the desired bound (5.4)(see [QS, Lemma 13.2]). However we shall present the argument here for the sake of completeness,

We first treat the case when Ω is strictly convex. In this case we can find constants $\alpha_0 > 0$ and $V_0 > 0$ such that for each point $x \in \partial\Omega$ there exists an open connected set $Q_x \subset S^{n-1}$ satisfying $|Q_x| > V_0$ with the properties:

- $A_x =: \{x + tw \mid 0 \le t \le \alpha_0, w \in Q_x\} \subset \Omega$,
- Dividing Ω into two parts Ω_1 and Ω_2 by the plane $P_x = \{x + tv \mid v \perp w\}$ so that $x \in \Omega_1$ and $x \notin \Omega_2$. Then, the reflection of Ω_1 with respect to the plane P_x is contained in Ω_2.

Then the moving plane argument presented in [CT, T2] guarantees that the solution u satisfies

$$u(x + t_1 w) \le u(x + t_2 w), \qquad \forall\, 0 \le t_1 \le t_2 \le \alpha_0 \quad \text{and} \quad w \in Q_x. \tag{5.8}$$

Consequently, we can find constants $\alpha_1 > 0$ and $V_1 > 0$ such that for any $x \in O(\Omega, \alpha_1)$ there exists an open connected set $\widetilde{Q}_x \subset S^{n-1}$ satisfying $|\widetilde{Q}_x| > V_1$ and $A_x = \{x+tw \mid 0 \le t \le \alpha_1,\ w \in \widetilde{Q}_x\} \subset \Omega$ and $u(x) \le u(y)$ for any $y \in A_x$. As a result, we have

$$u(x) \le \frac{1}{|A_x|}\int_{A_x} u(y)dy \le \frac{C}{V_1}\int_{\mathcal{I}(\Omega,\alpha_1)} u(y)dy, \qquad x \in O(\Omega, \alpha_1). \tag{5.9}$$

Then, the L^1 bound of (5.3) gives the desired uniform bound of $u(x)$ on $O(\Omega, \alpha_1)$.

In the case of general domains without the convexity assumption, it is difficult to adapt directly the moving plane argument to deduce the fact that u increases along any line starting from a point on $\partial\Omega$. Instead we shall argue a moving plane method after applying the Kelvin transform to v in the space $\mathbb{R}^{n+1}$, which will yield a weaker version of the increasing property.

Since Ω is smooth, for a point x_0 we can find a ball which contact x_0 from the exterior of Ω. We may assume $x_0 = 1$ and the ball is $B(0, 1)$ without loss of generality. We denote by U the s-harmonic extension of u and we set

$$w(z) = |z|^{2s-n} U\left(\frac{z}{|z|^2}\right).$$

Then, w satisfies

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla w) = 0 & \text{in } \kappa(C), \\ w > 0 & \text{in } \kappa(C), \\ w = 0 & \text{on } \kappa(\partial\Omega \times [0, \infty)), \\ \partial_\nu^s w = g(y, w) & \text{on } \kappa(\Omega \times \{0\}), \end{cases}$$

where $g(y, w) := f(|y|^{n-2s}w)/|y|^{n+2s}$ and $\kappa(A) := \{\frac{z}{|z|^2} : z \in A\}$ for any set $A \subset \mathbb{R}^{n+1}$. Here we note that $\kappa(C) \subset B(0, 1)$ because $C \cap B^0(0, 1) = \phi$. Now, for each $\lambda > 0$ we set

- $D_\lambda = \kappa(C) \cap \{z \in \mathbb{R}^{n+1}_+ : |z| \le 1,\ z_1 > 1 - \lambda\}$,
- $\partial_b D_\lambda = D_\lambda \cap \partial\mathbb{R}^{n+1}_+$,
- $T_\lambda(y) = (2 - 2\lambda - y_1, y_2, \cdots, y_{n+1})$.

Let $w_\lambda(y) = w(T_\lambda(y))$ and $\zeta_\lambda = w_\lambda - w$ defined on D_λ. We claim that $v_\lambda \ge 0$ if $\lambda > 0$ is sufficiently small. Set $v_\lambda^- = \max\{0, -v_\lambda\}$. Then,

$$\begin{aligned} 0 &= \int_{D_\lambda} \zeta_\lambda^- \mathrm{div}(t^{1-2s}\nabla\zeta_\lambda)dxdy \\ &= \int_{\partial_b D_\lambda} \zeta_\lambda^- \partial_\nu^s \zeta_\lambda dx + \int_{D_\lambda} t^{1-2s}|\nabla\zeta_\lambda^-|^2 dxdy. \end{aligned} \tag{5.10}$$

We have

$$\begin{aligned}\int_{\partial_b D_\lambda}(-\zeta_\lambda^-)\partial_\nu^s\zeta_\lambda dx &= \int_{\partial_b D_\lambda}(-\zeta_\lambda^-)(g(T_\lambda x, w_\lambda) - g(x,w))dx \\ &= \int_{\partial_b D_\lambda\cap\{w_\lambda\le w\}}(w-w_\lambda)(g(x,w)-g(T_\lambda x, w_\lambda))dx\end{aligned} \tag{5.11}$$

Since $u \to f(u)u^{-\frac{n+2s}{n-2s}}$ is nonincreasing, we see that $g(x,w) \le g(T_\lambda x, w_\lambda)$ because $|x| \ge |T_\lambda(x)|$. Using this we deduce that

$$\begin{aligned}\int_{D_\lambda} t^{1-2s}|\nabla\zeta_\lambda^-|^2 dxdy &\le \int_{\partial_b D_\lambda\cap\{w_\lambda\le w\}}(w-w_\lambda)(g(x,w)-g(x,w_\lambda))dx \\ &\le \int_{\partial_b D_\lambda\cap\{w_\lambda\le w\}}(w-w_\lambda)^2 h(x,w,w_\lambda)dx \\ &= \int_{\partial_b D_\lambda\cap\{w_\lambda\le w\}}(\zeta_\lambda^-)^2 h(x,w,w_\lambda)dx,\end{aligned} \tag{5.12}$$

where $h(x,w,w_\lambda) = \frac{g(x,w)-g(x,w_\lambda)}{w-w_\lambda}$. Since f is locally Lipschitz the function h is bounded by a constant multiple of $\sup_{\partial_b D_\lambda}(|w|+|w_\lambda|)$. By Hölder's inequality we deduce that

$$\begin{aligned}\int_{D_\lambda} t^{1-2s}|\nabla\zeta_\lambda^-|^2 dxdy &\le C\int_{\partial_b D_\lambda\cap\{w_\lambda\le w\}}(\zeta_\lambda^-)^2 dx \\ &\le C_1|\partial_b D_\lambda\cap\{w_\lambda\le w\}|^{2s/n}\|\zeta_\lambda^-(\cdot,0)\|^2_{L^{2n/(n-2s)}(\Omega)}.\end{aligned} \tag{5.13}$$

Using the trace inequality, we get

$$\|\zeta_\lambda^-(\cdot,0)\|^2_{L^{2n/(n-2s)}(\Omega)} \le C_1|\partial_b D_\lambda\cap\{w_\lambda\le w\}|^{2s/n}\|\zeta_\lambda^-(\cdot,0)\|^2_{L^{2n/(n-2s)}(\Omega)}. \tag{5.14}$$

If $\lambda > 0$ is sufficiently small so that $\lambda < \delta := C_1^{-\frac{n}{2s}}$, then we have

$$C_1|\partial_b D_\lambda\cap\{w_\lambda\le w\}|^{2s/n} \le C_1|\partial_b D_\lambda| < 1. \tag{5.15}$$

Combining this with (5.14) yields that $\zeta_\lambda^- \equiv 0$ for such λ.

Now we set

$$\eta = \sup\{\lambda > 0 : T_\lambda(D_\lambda) \subset \kappa(C)\},$$

and

$$S := \left\{0 < \lambda \le \frac{\eta}{2} : \zeta_\lambda \ge 0 \quad \text{on } D_\lambda\right\} \cup \{0\}.$$

We shall prove that $S = [0,\eta/2]$. Since ζ_λ is a continuous function of λ, the set S is closed. Thus, it is enough to show that S is also open in $[0,\eta/2]$. Note that the constant C_1 in the inequality (5.13) can be chosen uniformly for $\lambda \in [0,\eta/2]$ since $\sup_{0<\lambda<\eta/2}\sup_{\partial_b D_\lambda}[|w|+|w_\lambda|]$ is bounded.

Choose any $0 < \lambda_0 < \eta/2$ contained in S. Then we have $\zeta_{\lambda_0} \geq 0$. Since $\zeta_{\lambda_0} > 0$ on $\kappa(\partial\Omega \times [0,\infty)) \cap D_{\lambda_0}$ and $\mathrm{div}(t^{1-2s}\nabla\zeta_{\lambda_0}) \equiv 0$ in D_{λ_0}, we see that $\zeta_{\lambda_0} > 0$ in D_{λ_0} by the maximum principle (see e.g. [CS]). Thus we can find $c > 0$ such that

$$|D_{\lambda_0,c}| := \{x \in D_{\lambda_0} : \zeta_{\lambda_0} > c\}| \geq |D_{\lambda_0}| - \delta/2.$$

By continuity, there is $\epsilon > 0$ such that $\zeta_\lambda > \frac{c}{2}$ on $D_{\lambda_0,c}$ and $|D_\lambda \setminus D_{\lambda_0}| < \frac{\delta}{2}$ for $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$. For such λ we then see that

$$\begin{aligned} |\left\{x \in D_\lambda : \zeta_\lambda > \frac{c}{2}\right\}| &\geq |D_{\lambda_0,c}| \geq |D_{\lambda_0}| - \frac{\delta}{2} \\ &> |D_\lambda| - \frac{\delta}{2} - \frac{\delta}{2} = |D_\lambda| - \delta. \end{aligned}$$

This yields that

$$|\{x \in D_\lambda : \zeta_\lambda \leq 0\}| > \delta.$$

Then the inequality (5.13) again implies that $\zeta_\lambda \geq 0$ for $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$. Therefore we have that w increases in any line in Ω starting from a boundary point. Then, by definition of w we deduce a weaker version of (5.8):

$$u(x + t_1 w) \leq Cu(x + t_2 w), \qquad \forall\, 0 \leq t_1 \leq t_2 \leq \alpha_0 \quad \text{and} \quad w \in Q_x. \tag{5.16}$$

Here $C > 1$ is a constant which is determined only by the domain Ω. Having this inequality, we can argue similarly to derive the L^∞ bound near the boundary $\partial\Omega$ as in the convex case. It completes the proof. □

The next step is to derive a uniform bound of a higher order integration of u on the whole domian Ω.

Proposition 5.4.2. *Suppose that* $1 < p < \frac{n+2s}{n-2s}$ *and let* $u \in C^2(\bar{C})$ *be a solution of the equation* (5.3) *with* $f(u) = u^p$. *Then there exists a constant* $C = C(p,\Omega) > 0$ *such that*

$$\int_\Omega u^{p+1}(x)dx \leq C. \tag{5.17}$$

Generally, for any function $f : [0,\infty) \to \mathbb{R}$ *satisfying* ***Condition A***, *there exists a constant* $C = C(f,\Omega) > 0$ *such that*

$$\int_{\Omega\times\{0\}} \left\{nF(u) - \frac{n-2s}{2}uf(u)\right\} dx \leq C,$$

where $F(v) := \int_0^v f(s)ds$.

Proof. By Lemma 5.4.1 there are a number $\delta > 0$ and a constant $C = C(\delta, \Omega) > 0$ so that

$$\sup_{\mathcal{O}(\Omega,2\delta)} u(x) \leq C, \tag{5.18}$$

and

$$\int_{\mathcal{I}(\Omega,\delta)} f(u)(x)dx \leq C. \tag{5.19}$$

Then we apply these estimates to the inequality (5.20)

$$\begin{aligned} &\min_{r\in[\delta,2\delta]} \left| \int_{\mathcal{I}(\Omega,r)} nF(u) - \left(\frac{n-2s}{2}\right) uf(u)dx \right| \\ &\leq C \left[\left(\int_{\mathcal{O}(\Omega,2\delta)\times\{0\}} |f(u)|^{\gamma} dx \right)^{\frac{2}{\gamma}} + \int_{\mathcal{O}(\Omega,2\delta)\times\{0\}} |F(u)|dx + \left(\int_{\mathcal{I}(\Omega,\delta/2)\times\{0\}} |f(u)|dx \right)^2 \right]. \end{aligned} \tag{5.20}$$

Then we obtain

$$\min_{r\in[\delta,2\delta]} \left| \int_{\mathcal{I}(\Omega,r)} nF(u) - \left(\frac{n-2s}{2}\right) uf(u)dx \right| \leq C. \tag{5.21}$$

Combining this with (5.18) gives the estimate

$$\left| \int_{\Omega\times\{0\}} nF(u) - \left(\frac{n-2s}{2}\right) uf(u)dx \right| \leq C. \tag{5.22}$$

It proves the general case. Note that if $f(u) = u^p$ with $p < \frac{n+2s}{n-2s}$ we have

$$nF(u) - \left(\frac{n-2s}{2}\right) uf(u) = \left(\frac{(n+2s)-(n-2s)p}{2(p+1)}\right) u^{(p+1)}. \tag{5.23}$$

Thus (5.22) gives the bound (5.17), and so the proof is completed. □

Remark 5.4.3. In the local problem $-\Delta u = u^p$ in Ω, $u = 0$ on $\partial\Omega$, with $1 < p < \frac{n+2}{n-2}$, given the L^∞ bound (5.4) of a solution $u$near the boundary, one can use $W^{1,p}$ regularity estimate on $\mathcal{O}(\Omega, \delta)$ to get the L^∞ estimates of $|\nabla u|$ on the $\mathcal{O}(\Omega, \delta/2)$. Then, for $f(u) = u^p$ and $p < \frac{n+2}{n-2}$, the Pohozaev identity

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 (x,\nu)d\sigma = \left(\frac{n}{p+1} - \frac{n-2}{2}\right) \int_\Omega u^{p+1}dx$$

gives a uniform bound of $\int_\Omega u^{p+1}dx$. Having this bound, we can use the Sobolev embeddings iteratively to get the uniform bound of $\|u\|_{L^\infty(\Omega)}$. This is not applicable to our problem (5.3) because the Pohozaev identity is given on the extended domain $\Omega \times [0, \infty)$ as follows (see [T1, Lemma 3.1])

$$\frac{1}{2} \int_{\partial_L C} t^{1-2s} |\nabla U|^2 (z,\nu)d\sigma = \left(\frac{n}{p+1} - \frac{n-2s}{2}\right) \int_{\Omega\times\{0\}} |U|^{p+1}dx, \tag{5.24}$$

where U is the harmonic extention of u. In this case the left-hand side would not be bounded by using only the L^∞ estimate of $u(x) = U(x,0)$ near $\partial\Omega$ since the harmonic extension $U(z)$ is made of all values of $u(x)$ for $x \in \Omega$. This is the reason that we need to rely on the integral estimates of Proposition 5.3.1 in the above proof.

Given the higher order bound of Proposition 5.4.2, we shall use the Sobolev embedding iteratively to obtain the L^∞ estimate.

Proof of Theorem 5.1.1. For the sake of simplicity, first we prove the theorem for $f(u) = u^p$. Let q_1 be such that $\frac{p}{p+1} - \frac{1}{q_1} = \frac{2s}{n}$. Since $p < \frac{n+2s}{n-2s}$ we have $q_1 > p$. Using Lemma 5.2.1 we get

$$\|u\|_{q_1} \leq C\|\mathcal{A}_s u\|_{\frac{p+1}{p}} \leq C\|u^p\|_{\frac{p+1}{p}} \leq C. \tag{5.25}$$

Now we define the numbers q_k for $k \geq 2$ by the relation $\frac{p}{q_k} - \frac{1}{q_{k+1}} = \frac{2s}{n}$ and stop the sequence when we have $\frac{p}{q_N} < \frac{2s}{n}$. Then, using Lemma 5.2.1, for $k = 1, \cdots, N-1$, we deduce

$$\|u\|_{q_{k+1}} \leq C\|\mathcal{A}_s u\|_{\frac{q_k}{p}} \leq C\|u^p\|_{\frac{q_k}{p}} = C\|u\|_{q^k}^p.$$

Combining this with (5.25) we get $\|u\|_{q_N} \leq C$. Then, using Lemma 5.2.1 again we deduce that $\|u\|_{L^\infty} \leq C$. It completes the proof when the nonlinearity is given by $f(u) = u^p$, $p < \frac{n+2s}{n-2s}$.

Now we shall prove the theorem for general nonlinearity f satisfying **Condition A**. First we see from Proposition 5.4.2 that

$$\int_{\Omega\times\{0\}} \left\{ nF(v) - \frac{n-2s}{2} v f(v) \right\} dx \leq C. \tag{5.26}$$

From the condition (5.1), for any $\epsilon > 0$, we can find $C > 0$ depending on ϵ such that

$$uf(u) \leq \theta F(u) + \epsilon u^2 f(u)^{2s/n} + C. \tag{5.27}$$

Using Hölder's inequality and the Sobolev embedding we deduce that

$$\int_\Omega u^2 |f(u)|^{\frac{2s}{n}} dx \leq \|u\|^2_{\frac{2n}{n-2s}(\Omega)} \|f(u)\|^{2s/n}_{L^1(\Omega)} \leq C\|\mathcal{A}_s^{1/2} u\|_2^2, \tag{5.28}$$

and we have

$$\int_\Omega uf(u)dx = \int_\Omega u\mathcal{A}_s u\, dx = \int_\Omega \mathcal{A}_s^{1/2} u \cdot \mathcal{A}_s^{1/2} u\, dx = \|\mathcal{A}_s^{1/2} u\|_2^2. \tag{5.29}$$

From (5.26) and (5.27), we can deduce that

$$\left(\frac{n}{\theta} - \frac{n-2s}{2}\right) \int_\Omega uf(u)dx \leq \frac{\epsilon}{\theta} \int_\Omega u^2 f(u)^{\frac{2s}{n}} dx + C.$$

Choose $\epsilon = \epsilon(\theta, n) > 0$ small enough so that $\left(\frac{n}{\theta} - \frac{n-2s}{2}\right) > \frac{\epsilon}{\theta}$. Then combining (5.28) and (5.29) with the above inequality yields for a constant $C = C(\theta, n) > 0$ we have

$$\left(\frac{n}{\theta} - \frac{n-2s}{2}\right) \|\mathcal{A}_s^{1/2} u\|_2^2 \le C\frac{\epsilon}{\theta} \|A_s^{1/2} u\|_2^2 + C, \tag{5.30}$$

which implies

$$\|\mathcal{A}_s^{1/2} u\|_2^2 \le C. \tag{5.31}$$

Let $p > 1$ and $q = (p+1)\frac{n}{n-2s}$. Then

$$\begin{aligned} \left(\int_\Omega u^q dx\right)^{\frac{n-2s}{n}} &= \|u^{(p+1)/2}\|^2_{\frac{2n}{n-2s}} \\ &\le C \int_{\Omega\times(0,\infty)} |\nabla u^{\frac{(p+1)}{2}}|^2 dx = C_p \int_{\Omega\times(0,\infty)} \nabla u \cdot \nabla(u^p) dx \\ &= C_p \int_\Omega \frac{\partial u}{\partial \nu} \cdot u^p dx \\ &\le \epsilon C_p \int_\Omega u^{\frac{n+2s}{n-2s}} u^p dx + C. \end{aligned} \tag{5.32}$$

Since $p + 1 = \frac{n-2s}{n} q$ we have

$$\begin{aligned} \int_\Omega u^{\frac{n+2s}{n-2s}} u^p dx &= \int_\Omega u^{q(n-2s)/n} u^{\frac{2}{n-2s}} dx \\ &\le \left(\int_\Omega u^{q(n-2s)/n \cdot \frac{n}{n-2s}} dx\right)^{\frac{n-2s}{n}} \left(\int_\Omega u^{\frac{2}{n-2s}\cdot n} dx\right)^{\frac{2s}{n}} \\ &\le C\left(\int_\Omega u^{q(n-2s)/n \cdot \frac{n}{n-2s}} dx\right)^{\frac{n-2s}{n}} \|\mathcal{A}_s^{1/2} u\|_2^{\frac{2}{n-2s}} \\ &\le C\left(\int_\Omega u^q dx\right)^{\frac{n-2s}{n}}, \end{aligned} \tag{5.33}$$

where we used (5.31) in the last estimate. Combinig this with (5.32) yields that

$$\left(\int_\Omega u^q dx\right)^{1/q} \le C,$$

Since p is an arbitrary number, q may also become arbitrary large, and so we can use Lemma 5.2.1 again to deduce that

$$\|u\|_{L^\infty} \le C.$$

It completes the proof. □

5.5 On the nonlinear system (5.1)

In this section, we study the nonlinear system (5.1) and its extension problem (5.4). First, the existence of weak solution and Brezis-Kato type estimate will follow from the same proof of [HV]. For these part, we do not need to consider the extended problem. But for further investigation of the non-local problem (5.1) we shall heavily rely on studying the local interpretation (5.4). We shall obtain a Pohozaev type identity on the cylinder C, which yields the nonexistence of non-trivial solutions for the problem (5.1) in critical and supercritical cases. Next, a symmety result will be obtained by a moving plane argument. Lastly, we shall obtain *a priori* estimates for subcritical cases by applying the approach used for Theorem 5.1.1.

The existence result follows by applying the proof of [HV, Theorem 1] for the case $s = 1$ with only minor modifications. The idea is to consider the following sets

- $E^a(\Omega) = H^a(\Omega) \times H^{2s-a}(\Omega), \quad 0 < a < 2s,$
- $E^{\pm} = \{(u, \pm(-\Delta)^{a-2s}u) : u \in H^a(\Omega)\},$

and to find a solution (u, v) in the space $E^a(\Omega)$ for some $0 < a < 2s$ such that $H^a(\Omega) \to L^{q+1}(\Omega)$ and $H^{2s-a}(\Omega) \to L^{p+1}(\Omega)$ are compact embeddings. Such a choice of a is possible when (p, q) is sub-critical. The spaces $E^{\pm}$ are aimed to turn E^a into a direct sum of two Hilbert spaces, namely,

$$E^a(\Omega) = E^+ \oplus E^- = \{\mathbf{u} = u^+ + u^-,\ u^{\pm} \in E^{\pm}\}. \tag{5.1}$$

We easily see that $E^{\pm}$ have their orthonormal basis

$$\left\{ \frac{1}{\sqrt{2}} (\lambda_k^{-a/2}\phi_k, \pm\lambda_k^{a/2-s}\phi_k) : k = 1, 2, \cdots \right\}. \tag{5.2}$$

Let

$$L = \begin{pmatrix} 0 & (-\Delta)^{2s-a} \\ (-\Delta)^{a-2s} & 0 \end{pmatrix}. \tag{5.3}$$

Then,

$$\frac{1}{2}(L\mathbf{u}, \mathbf{u})_{E^a} = \frac{1}{2}\langle (-\Delta)^s u^+, u^- \rangle. \tag{5.4}$$

We set

$$\mathcal{H}(\mathbf{u}) = \frac{1}{q+1}\int_\Omega |u^+|^{q+1} dx + \frac{1}{p+1}\int_\Omega |u^{-1}|^{p+1} dx, \tag{5.5}$$

and

$$\mathcal{L}(\mathbf{u}) = \frac{1}{2}(L\mathbf{u}, \mathbf{u})_{E^a} - \mathcal{H}(\mathbf{u}). \tag{5.6}$$

Then we see that a critical point (u^+, u^-) of the functional $\mathcal{L}(\mathbf{u})$ is a solution of the problem (5.1). To find a critical point, we rely on the following result of Benci-Rabinowitz [BP].

Theorem 5.5.1 (Indefinite Functional Theorem)**.** *Let H be a real Hilbert sapce with $H = H_1 \oplus H_2$. Suppose $\mathcal{L} \in C^1(H, \mathbb{R})$ satisfies the Palais-smale condition, and*

1. $\mathcal{L}(\boldsymbol{u}) = \frac{1}{2}(L\boldsymbol{u}, \boldsymbol{u})_H - \mathcal{H}(\boldsymbol{u})$, *where $L : H \to H$ is bounded and self-adjoint, and L leaves H_1 and H_2 invariant;*
2. $\mathcal{H}'$ *is compact;*
3. *there exists a subspace $\bar{H} \subset H$ and sets $S \subset H$, $Q \subset \bar{H}$ and constants $\alpha > \omega$ such that*
   (a) $S \subset H_1$ *and* $\mathcal{L}\,|_S \geq \alpha$,
   (b) Q *is bounded and* $\mathcal{L} \leq \omega$ *on the boundary ∂Q of Q in $\bar{H}$,*
   (c) S *and ∂Q link.*

Then $\mathcal{L}$ possesses a critical value $c \geq \alpha$.

Proof of Theorem 5.1.2. Since (p, q) is sub-critical, we can find a value $a \in (0, 2s)$ such that

$$\frac{1}{2} - \frac{1}{q+1} < \frac{a}{n} \quad \text{and} \quad \frac{1}{2} - \frac{1}{p+1} < \frac{2s-a}{n}, \tag{5.7}$$

which guarantees that $H^a \hookrightarrow L^{q+1}$ and $H^{2s-a} \hookrightarrow L^{p+1}$ are compact embeddings. In order to find a solution of the problem (5.1), we apply Theorem 5.5.1 for functional $\mathcal{L}$ defined by (5.6) with the spaces $H = E^a(\Omega)$, $H_1 = E^+$, and $H_2 = E^-$. Then one can follow the proof of [HV, Theorem1] with slight modification to check that the conditions (1)-(3) of Theorem 5.5.1 are satisfied, which shows the existence of a weak solution (u, v). The only difference is the different ranges of index in using the Sobolev embeddings. We refer to [HV] for the calculations. □

Next we shall prove an L^∞ estimate of Brezis-Kato type L^∞.

Proposition 5.5.2. *Assume that (p, q) is critical or sub-critical. Let (u, v) be a weak solution of* (5.1). *Then we have $u \in L^\infty(\Omega)$ and $v \in L^\infty(\Omega)$.*

Proof. We consider the critical case only since the proof is applicable for sub-critical cases with a minor modification.

Letting $a = u^{p-1}$ and $b = v^{q-1}$, we have $a \in L^{\frac{p+1}{p-1}}(\Omega)$ and $b \in L^{\frac{q+1}{q-1}}(\Omega)$. Now we write (5.1) as

$$\begin{cases} \mathcal{A}_s v = a(x)u & \text{in } \Omega, \\ \mathcal{A}_s u = b(x)v & \text{in } \Omega. \end{cases} \tag{5.8}$$

Since $a(x) \in L^{\frac{p+1}{p-1}}(\Omega)$, by considering $u(x) = u(x)1_{u(x)>K} + u(x)1_{u(x)\leq K}$ for a sufficiently large $K > 1$, we may have

$$a(x)u(x) = q_\epsilon(x)u(x) + f_\epsilon(x), \tag{5.9}$$

where $f_\epsilon \in L^\infty(\Omega)$ and $\|q_\epsilon\|_{\frac{p+1}{p-1}(\Omega)} < \epsilon$, where $\epsilon > 0$ is a small number to be determined below.

From (5.8) we have

$$u(x) = (\mathcal{A}_s)^{-1}(bv)(x), \tag{5.10}$$

and

$$v(x) = (\mathcal{A}_s)^{-1}\left[q_\epsilon(\mathcal{A}_s)^{-1}(bv)\right](x) + (\mathcal{A}_s)^{-1}f_\epsilon(x). \tag{5.11}$$

By letting an operator $\mathcal{D}$ as $\mathcal{D}v = (\mathcal{A}_s)^{-1}\left[q_\epsilon(\mathcal{A}_s)^{-1}(bv)\right]$, we get from (5.11) that

$$(I - \mathcal{D})v = (\mathcal{A}_s)^{-1}f_\epsilon(x). \tag{5.12}$$

As $f_\epsilon \in L^\infty(\Omega)$ holds, we have $(\mathcal{A}_s)^{-1}f_\epsilon \in L^\infty(\Omega)$. Fix $\alpha > 1$. Then, from Lemma 5.2.1 and Hölder's inequality, we have the following embedding properties

- $w \to b(x)w$ is bounded form $L^\alpha(\Omega)$ to $L^{\alpha_1}(\Omega)$ for
$$\frac{1}{\alpha_1} = \frac{q-1}{q+1} + \frac{1}{\alpha}. \tag{5.13}$$
- $w \to (\mathcal{A}_s)^{-1}w$ is bounded from $L^{\alpha_1}(\Omega)$ to $L^{\alpha_2}(\Omega)$ for
$$2s = n\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right). \tag{5.14}$$
- $w \to q_\epsilon(x)w$ is bounded from L^{α_2} to L^{α_3} with the norm $\|q_\epsilon\|_{L^{\frac{p+1}{p-1}}(\Omega)}$ for
$$\frac{1}{\alpha_3} = \frac{p-1}{p+1} + \frac{1}{\alpha_2}. \tag{5.15}$$
- $w \to (\mathcal{A}_s)^{-1}w$ is bounded from $L^{\alpha_4}(\Omega)$ to $L^{\alpha_3}(\Omega)$ for
$$2s = n\left(\frac{1}{\alpha_3} - \frac{1}{\alpha_4}\right). \tag{5.16}$$

We compute

$$\begin{aligned}\frac{n}{\alpha_4} &= \frac{n}{\alpha_3} - 2s = n\left(\frac{p-1}{p+1} + \frac{1}{\alpha_2}\right) - 2s = n\left(\frac{p-1}{p+1}\right) + \left(\frac{n}{\alpha_1} - 2s\right) - 2s \\ &= n\left(\frac{p-1}{p+1}\right) + n\left(\frac{q-1}{q+1}\right) - 4s = \frac{n}{\alpha} + (2n - 4s) - 2n\left(\frac{1}{p+1} + \frac{1}{q+1}\right) = \frac{n}{\alpha},\end{aligned} \tag{5.17}$$

which reveals that $\alpha_4 = \alpha$. Therefore, by combining the above embedding properties, we see that the map $\mathcal{D} : w \to (\mathcal{A}_s)^{-1}\left[q_\epsilon(\mathcal{A}_s)^{-1}(bw)\right]$ is bounded from $L^\alpha(\Omega)$ to $L^\alpha(\Omega)$ for any $\alpha > 1$ with operator norm less than $C\|q_\epsilon\|_{L^{\frac{p+1}{p-1}}(\Omega)} \le C\epsilon < \frac{1}{2}$, which is guaranteed once we choose ϵ sufficiently small. Combining this and the fact that $(\mathcal{A}_s)^{-1}f_\epsilon \in L^\alpha(\Omega)$, we deduce from (5.12) that v is bounded on $L^\alpha(\Omega)$. Since α can be arbitrary large, we may use Lemma 5.2.1 to deduce that $u \in L^\infty(\Omega)$. From this, and using Lemma 5.2.1 again, we deduce that $v \in L^\infty(\Omega)$. The lemma is proved □

Now we recall the regularity result from [CS]. Consider weak solution $U \in H^s_{0,L}(C) \cap L^\infty(C)$ to the problem

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } C, \\ U = 0 & \text{on } \partial_L C, \\ \partial^s_\nu U(x,0) = g(x) & \text{on } \Omega \times \{0\}. \end{cases} \tag{5.18}$$

Then, for $g \in C^\alpha(\Omega)$ with some $\alpha \geq 0$ we have

$$\begin{cases} v \in C^{\alpha+2s}(\Omega) & \text{if } \alpha + 2s < 1, \\ v \in C^{1,\alpha+2s-1}(\Omega) & \text{if } 1 \leq \alpha + 2s < 2, \\ v \in C^{2,\alpha+2s-2}(\Omega) & \text{if } \alpha + 2s > 2. \end{cases} \tag{5.19}$$

Here $g \in C^0(\Omega)$ can be replaced by $g \in L^\infty(\Omega)$. Using this result iteratively, we can prove the following result.

Proposition 5.5.3. *Let (u, v) is a weak solution of* (5.1) *such that $u \in H^{s_1}(\Omega) \cap L^\infty(\Omega)$ and $v \in H^{s_2}(\Omega) \cap L^\infty(\Omega)$ for some $s_1 > 0$ and $s_2 > 0$. Then it holds that $u \in C^{1,\alpha}(\Omega)$ and $v \in C^{1,\gamma}(\Omega)$ for some $\gamma \in (0,1)$.*

Proof. Suppose that (u, v) is a weak solution of the problem (5.1). By Proposition 5.5.2, we have that $u \in L^\infty(\Omega)$ and $v \in L^\infty(\Omega)$. Then, we can use (5.19) to deduce that $u \in C^{2s}(\Omega)$ and $v \in C^{2s}(\Omega)$. Hence it holds that $u^q \in C^{2s}(\Omega)$ and $v^p \in C^{2s}(\Omega)$ because $q > 1$ and $p > 1$. Again, we can apply (5.19) to deduce that $u \in C^{4s}(\Omega)$. Iteratively, we can raise the regularity so that $u \in C^{1,\gamma}$ and $v \in C^{1,\gamma}$ for some $\gamma > 0$. The proof is completed. □

We shall obtain a Pohozaev type identity for the system (5.4). It will gives the nonexistence result for the critical and super-critical cases.

Theorem 5.5.4. *Suppose that $(U, V) \in H^s_{0,L}(C) \times H^s_{0,L}(C)$ satisfies*

$$\begin{cases} div(t^{1-2s}\nabla U) = div(t^{1-2s}\nabla V) = 0 & in\ C, \\ U = V = 0 & on\ \partial_L C. \end{cases} \tag{5.20}$$

Then we have

$$\begin{aligned} &\int_{\partial_L C} t^{1-2s}(z \cdot \nu)\frac{\partial U}{\partial \nu}\frac{\partial V}{\partial \nu} d\sigma \\ &\quad = -\int_{\Omega \times \{y=0\}} [(x, \nabla_x V)\partial^s_\nu U + (x, \nabla_x U)\partial^s_\nu V] dx - (n-2s)\int_C t^{1-2s}\nabla U \cdot \nabla V dx. \end{aligned} \tag{5.21}$$

Proof. Let $(U, V) \in H^s_{0,L}(C) \times H^s_{0,L}(C)$ be a solution of (5.20). Then, it follows from a direct compuation that

$$\begin{aligned} &\operatorname{div}[t^{1-2s}z \cdot \nabla V)\nabla U + t^{1-2s}z \cdot \nabla U \nabla V] \\ &\quad = (z, \nabla V)\operatorname{div}(t^{1-2s}\nabla U) + (z, \nabla U)\operatorname{div}(t^{1-2s}\nabla V) + t^{1-2s}z \cdot \nabla(\nabla U \cdot \nabla V) + 2t^{1-2s}\nabla U \cdot \nabla V. \end{aligned}$$

Then, using (5.20) we have

$$\mathrm{div}[t^{1-2s}(z,\nabla V)\nabla U + t^{1-2s}(z,\nabla U)\nabla V] = t^{1-2s}z\cdot\nabla(\nabla U\cdot\nabla V) + 2t^{1-2s}\nabla U\cdot\nabla V. \tag{5.22}$$

We also have

$$\begin{aligned}\mathrm{div}[t^{1-2s}(z)(\nabla U\cdot\nabla V)] &= (\mathrm{div}t^{1-2s}z)(\nabla U\cdot\nabla V) + t^{1-2s}z\cdot\nabla(\nabla U\cdot\nabla V)\\ &= (n+2-2s)t^{1-2s}(\nabla U\cdot\nabla V) + t^{1-2s}z\cdot\nabla(\nabla U\cdot\nabla V).\end{aligned}$$

Combining these two formulas we have

$$\mathrm{div}[t^{1-2s}(z,\nabla V)\nabla U + t^{1-2s}(z,\nabla U)\nabla V] - \mathrm{div}(t^{1-2s}z(\nabla U\cdot\nabla V)) + (n-2s)t^{1-2s}\nabla U\cdot\nabla V = 0 \tag{5.23}$$

Using the divergence theorem and the fact that $U = V = 0$ on $\partial\Omega\times[0,R)$, we get

$$\begin{aligned}&\int_{\Omega\times(0,R)}\mathrm{div}[t^{1-2s}(z,\nabla v)\nabla U + t^{1-2s}(z,\nabla U)\nabla V]dx\\ &= \int_{\Omega\times(0,R)}2t^{1-2s}(z\cdot\nu)\frac{\partial U}{\partial\nu}\cdot\frac{\partial v}{\partial\nu}d\sigma + \int_{\Omega\times\{y=0\}}[(x,\nabla_x V)\partial_\nu^s U + (x,\nabla_x U)\partial_\nu^s V]dx\\ &\quad + \int_{\Omega\times\{y=R\}}t^{1-2s}[(x,\nabla_x V)(\nabla U,\nu) + (x,\nabla_x U)(\nabla V,\nu)]dx,\end{aligned} \tag{5.24}$$

and

$$\int_{\Omega\times(0,R)}\mathrm{div}(t^{1-2s}z(\nabla U\cdot\nabla V))dx = \int_{\partial\Omega\times(0,R)}t^{1-2s}(z\cdot\nu)(\frac{\partial U}{\partial\nu}\cdot\frac{\partial V}{\partial\nu})d\sigma + \int_{\Omega\times\{y=R\}}R^{2-2s}(\nabla U\cdot\nabla V)dx. \tag{5.25}$$

Letting $R\to\infty$ we obtain

$$\begin{aligned}&\int_C \mathrm{div}[t^{1-2s}(z,\nabla V)\nabla U + t^{1-2s}(z,\nabla U)\nabla V]dx\\ &= \int_C 2t^{1-2s}(z\cdot\nu)\frac{\partial U}{\partial\nu}\cdot\frac{\partial V}{\partial\nu}d\sigma + \int_{\Omega\times\{y=0\}}[(x,\nabla_x V)\partial_\nu^s U + (x,\nabla_x U)\partial_\nu^s V]dx,\end{aligned} \tag{5.26}$$

and

$$\int_C \mathrm{div}(t^{1-2s}z(\nabla U\cdot\nabla V))dx = \int_C t^{1-2s}(z\cdot\nu)(\frac{\partial U}{\partial\nu}\cdot\frac{\partial V}{\partial\nu})d\sigma. \tag{5.27}$$

Here the limit can be justified using that $U, V\in H_{0,L}^s(C)$. We refer [**?**] for a detail. Integrating (5.23) over C and using the above two formulas we obtain

$$\int_C t^{1-2s}(z\cdot\nu)\frac{\partial U}{\partial\nu}\cdot\frac{\partial V}{\partial\nu}d\sigma + \int_{\Omega\times\{y=0\}}[(x,\nabla_x v)\partial_\nu^s U + (x,\nabla_x U)\partial_\nu^s V]dx = (n-2s)\int_C t^{1-2s}\nabla U\nabla V dx,$$

which is the desired identity (5.21). □

Proof of Theorem 5.1.3. We may assume that Ω is starshaped with respect to the origin, that is, $(x \cdot \nu) > 0$ for any $x \in \partial_L \Omega$. It easily implies that $(x \cdot \nu) > 0$ holds also for $x \in \partial_L C$.

Suppose that (u, v) is a solution of (5.1) and denote by U and V the s-harmonic extensions of u and v, which are solutions of (5.4). Because $U(x, 0) = 0$ on $\partial\Omega \times \{0\}$, we get

$$\begin{aligned}\int_{\Omega\times\{0\}} (x, \nabla_x V)\partial_\nu^s U(x,0)dx &= \int_{\Omega\times\{0\}} (x, \nabla_x v)V^p dx \\ &= \frac{1}{p+1}\int_{\Omega\times\{0\}} (x, \nabla_x V^{p+1})dx = -\frac{n}{p+1}\int_{\Omega\times\{0\}} V^{p+1}dx.\end{aligned} \tag{5.28}$$

Likewise, we have

$$\int_{\Omega\times\{0\}} (x, \nabla_x U)\partial_\nu^s V(x,0)dx = -\frac{n}{q+1}\int_{\Omega\times\{0\}} G(u)dx.$$

Applying these equalities into (5.21), for any $\theta \in (0, 1)$ we get

$$\begin{aligned}&\frac{1}{2}\int_{\partial_L C} t^{1-2s}(x\cdot\nu)\frac{\partial U}{\partial\nu}\frac{\partial V}{\partial\nu}d\sigma \\ &= \int_{\Omega\times\{y=0\}} \Big(\frac{n}{p+1} - (n-2s)\theta\Big)V^{p+1} + \Big(\frac{n}{q+1} - (n-2s)(1-\theta)\Big)U^{q+1}dx.\end{aligned} \tag{5.29}$$

Since $U = V = 0$ on $\partial_L C$ and U, V are nonnegative on C, it follows that $\frac{\partial U}{\partial\nu} \geq 0$ and $\frac{\partial V}{\partial\nu} \geq 0$ on $\partial_L C$. If (p, q) is super-critical we can find a $\theta \in (0, 1)$ such that

$$\frac{n}{p+1} - (n-2s)\theta < 0 \quad \text{and} \quad \frac{n}{q+1} - (n-2s)(1-\theta) < 0.$$

Then, we can conclude from (5.29) that $U \equiv V \equiv 0$ on $\Omega \times \{0\}$.

In the critical case, we can find a $\theta \in (0, 1)$ such that

$$\frac{n}{p+1} - (n-2s)\theta = 0 \quad \text{and} \quad \frac{n}{q+1} - (n-2s)(1-\theta) = 0.$$

Then, we deduce from (5.29) that

$$\frac{1}{2}\int_{\partial_L C} t^{1-2s}(x,\nu)\frac{\partial U}{\partial\nu}\frac{\partial V}{\partial\nu}d\sigma = 0,$$

which implies that $\frac{\partial U}{\partial\nu}(x_0) = 0$ or $\frac{\partial V}{\partial\nu}(x_0) = 0$ for a given point $x_0 \in \partial_L C$. Since $\operatorname{div}(t^{1-2s}\nabla U) = \operatorname{div}(t^{1-2s}\nabla V) = 0$ and u and v are nonnegative on C, it follows from Hopf's lemma that $U \equiv 0$ or $V \equiv 0$, which yields that $U \equiv V \equiv 0$. The proof is complete. □

Next, we shall establish the moving plane argument, which will give a symmetry result and the L^∞ bound near the boundary of positive solutions to (5.1). As a preliminary step, we need the following type of maximum principle.

Lemma 5.5.5. *Assume that $c \le 0,\ d \le 0$ and Ω is a bounded (not necessary smooth) domain of $\mathbb{R}^n$ and set $C = \Omega \times (0,\infty)$. Suppose $U, V \in C^2(\bar{C}) \cap L^\infty(C)$ is a solution of the system*

$$\begin{cases} div(t^{1-2s}\nabla U) = div(t^{1-2s}\nabla V) = 0 & in\ C, \\ U \ge 0,\ V \ge 0 & on\ \partial_L C, \\ \partial_\nu^s U + c(x)V \ge 0 & on\ \Omega \times \{0\}, \\ \partial_\nu^s V + d(x)U \ge 0 & on\ \Omega \times \{0\}, \end{cases} \tag{5.30}$$

and there is some point $x_0 \in C$ such that $U(x_0) = V(x_0) = 0$. Then, there exists $\delta > 0$ depending only on $\|c\|_{L^\infty(\Omega)}$, $\|d\|_{L^\infty}$ and n such that if

$$|\Omega \cap \{U(\cdot,0) < 0\}| \cdot |\Omega \cap \{V(\cdot,0) < 0\}| \le \delta,$$

then $U \ge 0$ and $V \ge 0$ in C.

Proof. Set $U^- = \max\{0, -U\}$ and $V^- = \max\{0, -V\}$. As $U^- = V^- = 0$ on $\partial\Omega \times [0,\infty)$, we get

$$0 = \int_C U^- \mathrm{div}(t^{1-2s}\nabla U) dxdy = \int_{\Omega\times\{0\}} U^- \partial_\nu^s U dx + \int_C t^{1-2s}|\nabla U^-|^2 dxdy.$$

Using $c \le 0$, we deduce that

$$\begin{aligned} \int_C t^{1-2s}|\nabla U^-|^2 dxdy &= -\int_{\Omega\times\{0\}} V^- \partial_\nu^s U dx \\ &= \int_{\Omega\times\{0\}} U^- cV dx \\ &\le \int_{\Omega\times\{0\}} U^-(-c)V^- dx \\ &\le |\Omega \cap \{U^-(\cdot,0) > 0\}|^{2s/n} \|c\|_{L^\infty(\Omega)} \|U^-\|_{L^{2n/(n-2s)}(\Omega)} \cdot \|V^-\|_{L^{2n/(n-2s)}(\Omega)}. \end{aligned}$$

Similarly for V^-, we get

$$\int_C t^{1-2s}|\nabla V^-|^2 dxdy \le |\Omega \cap \{U^-(\cdot,0) > 0\}|^{2s/n} \|d\|_{L^\infty(\Omega)} \|U^-\|_{L^{2n/(n-2s)}(\Omega)} \cdot \|V^-\|_{L^{2n/(n-2s)}(\Omega)}. \tag{5.31}$$

Multipliying the above two inequalities, we obtain

$$\begin{aligned} &\left(\int_C t^{1-2s}|\nabla U^-|^2 dxdy\right)\left(\int_C t^{1-2s}|\nabla V^-|^2 dxdy\right) \\ &\le |\Omega \cap \{U^-(\cdot,0) > 0\}|^{1/n} |\Omega \cap \{V^-(\cdot,0) > 0\}|^{2s/n} \|c\|_{L^\infty(\Omega)} \|d\|_{L^\infty(\Omega)} \|U^-\|^2_{L^{2n/(n-2s)}(\Omega)} \|V^-\|^2_{L^{2n/(n-2s)}(\Omega)}. \end{aligned} \tag{5.32}$$

We now use the Sobolev trace inequality

$$S_0 \|U^-(\cdot,0)\|^2_{L^{2n/(n-2s)}(\Omega)} \le \int_C |\nabla U^-|^2 dxdy \tag{5.33}$$

and

$$S_0\|V^-(\cdot,0)\|^2_{L^{2n/(n-2s)}(\Omega)} \le \int_C |\nabla V^-|^2 dxdy, \tag{5.34}$$

where $S_0 > 0$ is a constant determined by only n and s. Combining (5.33), (5.34) and (5.32) we get

$$\begin{aligned} &S_0^2\|U^-(\cdot,0)\|^2_{L^{2n/(n-2s)}(\Omega)}\|V^-(\cdot,0)\|^2_{L^{2n/(n-2s)}(\Omega)} \\ &\le |\Omega\cap\{U^-(\cdot,0)>0\}|^{2s/n}|\Omega\cap\{V^-(\cdot,0)>0\}|^{2s/n}\|c\|_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}\|U^-\|^2_{L^{2n/(n-2s)}(\Omega)}\|V^-\|^2_{L^{2n/(n-2s)}(\Omega)}. \end{aligned}$$

If we choose δ so that $S_0^2 > \delta^{1/n}\|c\|_{L^\infty(\Omega)}\|d\|_{L^\infty(\Omega)}$, then the above inequality yields that $U^- \equiv 0$ or $V^- \equiv 0$. Say $U^- \equiv 0$, then we have $\int_C |\nabla V^-|^2 dxdy = 0$ from (5.31). Thus we have $\nabla V^- \equiv 0$, and since $V(x_0) = 0$, we conclude that $V^- \equiv 0$. The proof is complete. □

For $y \in \partial\Omega$ and $\lambda > 0$ we set

$$\begin{aligned} T(y,\lambda) &:= \{x \in \mathbb{R}^n : \langle y-x, \nu(y)\rangle = \lambda\}, \\ \Sigma(y,\lambda) &:= \{x \in \Omega : \langle y-x, \nu(y)\rangle \le \lambda\}, \end{aligned}$$

and define $R(y,\lambda)$ be the reflection with respect to the hyperplane $T(y,\lambda)$. We also set $\Sigma'(y,\lambda) := R(y,\lambda)\Sigma(y,\lambda)$ and

$$\lambda_y := \sup\{\lambda > 0 : \Sigma(y,\lambda) \subset \Omega\}. \tag{5.35}$$

Lemma 5.5.6. *Suppose that $(u,v) \in C^2(\Omega)$ is a solution of* (5.1). *Then, for any $y \in \partial\Omega$ and $x \in \Sigma(y,\lambda)$, we have*

$$u(R(y,\lambda)x) \ge u(x) \quad \textit{and} \quad v(R(y,\lambda)x) \ge v(x)$$

for any $\lambda \in (0,\lambda_y]$.

Proof. We may assume that $0 \in \partial\Omega$ and $\nu = (1,0)$ is a normal direction to $\partial\Omega$ at this point. It is sufficient to prove the lemma at this point. For $\lambda > 0$ we set

$$\Sigma_\lambda = \{(x_1,x') \in \Omega : x_1 > \lambda\} \quad \text{and} \quad T_\lambda = \{(x_1,x') \in \Omega : x_1 = \lambda\}.$$

For $x \in \Sigma_\lambda$, define $x_\lambda = (2\lambda - x_1, x')$. From the defintion (5.35) we see

$$\{x_\lambda : x \in \Sigma_\lambda\} \subset \Omega \qquad \forall \lambda < \lambda_0.$$

We denote by U and V the s-harmonic extension of u and v in C. Then, $(U,V) \in C^2(\bar{C})$ satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = \operatorname{div}(t^{1-2s}\nabla V) = 0 & \text{in } C, \\ U = V = 0 & \text{on } \partial_L C, \\ \partial_\nu^s U = V^p,\ \partial_\nu^s V = U^q & \text{on } \Omega\times\{0\}, \\ U > 0,\ V > 0 & \text{in } C. \end{cases} \tag{5.36}$$

For $(x, y) \in \Sigma_\lambda \times [0, \infty)$, we set

$$U_\lambda(x, y) = U(x_\lambda, y) = U(2\lambda - x_1, x', y)$$

and

$$\alpha_\lambda(x, y) = (U_\lambda - U)(x, y), \quad \beta_\lambda(x, y) = (V_\lambda - V)(x, y).$$

Then we have $U_\lambda = V_\lambda = 0$ on $T_\lambda \times [0, \infty)$ and obtain from (5.36) that $U_\lambda > 0$ and $V_\lambda > 0$ on $(\partial\Omega \cap \bar{\Sigma}_\lambda) \times [0, \infty)$. Since $\partial\Sigma_\lambda = T_\lambda \cup (\partial\Omega \cap \bar{\Sigma}_\lambda)$ we see that $(\alpha_\lambda, \beta_\lambda)$ satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla\alpha_\lambda) = \operatorname{div}(t^{1-2s}\nabla\Delta\beta_\lambda) = 0 & \text{in } \Sigma_\lambda \times (0, \infty), \\ \alpha_\lambda \geq 0, \ \beta_\lambda \geq 0 & \text{on } (\partial\Sigma_\lambda) \times (0, \infty), \\ \partial_\nu^s \alpha_\lambda + c_\lambda(x)\beta_\lambda = 0 & \text{on } \Sigma_\lambda \times \{0\}, \\ \partial_\nu^s \beta_\lambda + d_\lambda(x)\alpha_\lambda = 0 & \text{on } \Sigma_\lambda \times \{0\}, \end{cases}$$

where

$$c_\lambda(x, 0) = -\frac{V_\lambda^p - V^p}{V_\lambda - V} \quad \text{and} \quad d_\lambda(x, 0) = -\frac{U_\lambda^p - U^p}{U_\lambda - U}.$$

Note that $c_\lambda \leq 0$ and $d_\lambda \leq 0$. Now we choose a small number $\kappa > 0$ so that the set Σ_λ has small measure for $0 < \lambda < \kappa$. We then deduce from Lemma 5.30 that, for all $\lambda \in (0, \kappa)$,

$$\alpha_\lambda \geq 0 \qquad \text{and} \quad \beta_\lambda \geq 0 \quad \text{on } \Sigma_\lambda \times (0, \infty).$$

The strong maximum principle implies that α_λ and β_λ are identically equal to zero or strictly positive in $\Sigma_\lambda \times (0, \infty)$. Since $\lambda > 0$, we have $\alpha_\lambda > 0$ and $\beta_\lambda > 0$ in $(\partial\Omega \cap \partial\Sigma_\lambda) \times (0, \infty)$, and so we deduce that $\alpha_\lambda > 0$ and $\beta_\lambda > 0$ in $\Sigma_\lambda \times (0, \infty)$.

We let $\lambda_1 = \sup\{\lambda > 0 | \alpha_\lambda \geq 0 \text{ and } \beta_\lambda \geq 0 \text{ in } \Sigma_\lambda \times (0, \infty)\}$. We claim that $\lambda_1 = \lambda_0$. With a view to contradiction, we suppose that $\lambda_1 < \lambda_0$. By continuity we have $\alpha_{\lambda_1} \geq 0$ and $\beta_{\lambda_1} \geq 0$ in $\Sigma_{\lambda_1} \times (0, \infty)$. As before, by the strong maximum principle, we have that $\alpha_{\lambda_1} > 0$ and $\beta_{\lambda_1} > 0$ in $\Sigma_{\lambda_1} \times (0, \infty)$. Next, let $\delta > 0$ be a constant and find a compact set $K \subset \Sigma_{\lambda_1}$ such that $|\Sigma_{\lambda_1} \setminus K| \leq \delta/2$. We have $\alpha_{\lambda_1} \geq \mu > 0$ and $\beta_{\lambda_1} \geq \eta > 0$ in K for some constant η, since K is compact. Thus, we obtain that $\alpha_{\lambda_1+\epsilon}(\cdot, 0) \geq 0$ and $\beta_{\lambda_1+\epsilon}(\cdot, 0) \geq 0$ in K and that $|\Sigma_{\lambda_1+\epsilon} \setminus K| \leq \delta$ for sufficiently small $\epsilon > 0$.

By applying Lemma 5.5.5 to the function $(\alpha_{\lambda_1+\epsilon}, \beta_{\lambda_1+\epsilon})$, in $\Sigma_{\lambda_1+\epsilon} \times (0, \infty)$, we deduce that $\alpha_{\lambda_1+\epsilon} \geq 0$ and $\beta_{\lambda_1+\epsilon} \geq 0$ in K. Thus $\{\alpha_{\lambda_1+\epsilon} < 0\}, \{\beta_{\lambda_1+\epsilon} < 0\} \subset \Sigma_{\lambda_1+\epsilon} \setminus K$, which have measure at most δ. We take δ to be the constant of Lemma 5.30. Then it follows that

$$\alpha_{\lambda_1+\epsilon} \geq 0 \quad \text{and} \quad \beta_{\lambda_1+\epsilon} \geq 0 \quad \text{in } \Sigma_{\lambda_1+\epsilon} \times (0, \infty).$$

This is a contradiction to the definition of λ_1. Thus, it should hold that $\lambda_1 = \lambda_0$, which proves the lemma. □

Comsidering the s-harmonic extension, the above lemma gives directly the proof of the symmetry result of Theorem 5.1.4.

We are now ready to prove Theorem 5.1.5.

Proof of Theorem 5.1.5. Since (p, q) is sub-critical, we can choose $\theta \in (0, 1)$ such that

$$\frac{n}{p+1} - (n-2s)\theta > 0 \quad \text{and} \quad \frac{n}{q+1} - (n-2s)(1-\theta) > 0.$$

Take a small value $\delta > 0$. Then, we deduce from Proposition 5.3.1 that for a fixed $\gamma > \frac{n}{2}$, we have

$$\begin{aligned}\min_{r\in[\delta,2\delta]}&\left|\int_{\mathcal{I}(\Omega,r/2)}\left[\frac{n}{p+1}-(n-2s)\theta\right]v^{p+1}\,dx-\int_{\mathcal{I}(\Omega,r/2)}\left[\frac{n}{q+1}-(n-2s)\theta\right]u^{q+1}\,dx\right|\\ &\le C\left(\left(\int_{O(\Omega,2\delta)}(u^q+v^p)^\gamma dx\right)^{\frac{2}{\gamma}}+\int_{O(\Omega,2\delta)}\left(\frac{u^{q+1}}{q+1}+\frac{v^{p+1}}{p+1}\right)dx\right.\\ &\qquad\left.+\left(\int_{\mathcal{I}(\Omega,\delta/2)}(u^q+v^p)dx\right)^2\right.\end{aligned} \tag{5.37}$$

By definition 5.2 and using (5.1) we have

$$\lambda_1{}^{2s}\int_\Omega u\phi_1 dx = \int_\Omega v^p\phi_1 dx \quad \text{and} \quad \lambda_1{}^{2s}\int_\Omega v\phi_1 dx = \int_\Omega u^q\phi_1 dx. \tag{5.38}$$

Using Jense's inequality to the right hand sides, we get

$$\lambda_1{}^{2s}\int_\Omega u\phi_1 dx \ge C(\int_\Omega v\phi_1 dx)^p \quad \text{and} \quad \lambda_1{}^{2s}\int_\Omega v\phi_1 dx \ge C(\int_\Omega u\phi_1 dx)^q,$$

which easily yields that

$$\int_\Omega v\phi_1 dx \le C \quad \text{and} \quad \int_\Omega u\phi_1 dx \le C. \tag{5.39}$$

Combining this with (5.38) we also have

$$\int_\Omega (v^p + u^q)\phi_1 dx \le C. \tag{5.40}$$

Given the result of Lemma 5.5.6, we can apply the argument used in Lemma 5.4.1 to get the L^∞ uniform estimate of u on $O(\Omega, \delta)$ for a fixed small value $\delta > 0$. Applying this bound near the boundary and (5.40) into the inequality (5.37) we obtain

$$\int_\Omega (v^{p+1} + u^{q+1})dx \le C. \tag{5.41}$$

We now use a bootstrap argument to improve the integrability of u and v. Since (p,q) is subcritical, there is a positive number $\delta > 0$ such that $\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2s}{n} + \delta$, and so

$$\begin{aligned} &\max\left[(q+1)\left(\frac{n-2s}{n} - \frac{1}{p+1}\right), (p+1)\left(\frac{n-2s}{n} - \frac{1}{q+1}\right)\right] \\ &= \max\left[1-(q+1)\delta, 1-(p+1)\delta\right] < 1. \end{aligned} \tag{5.42}$$

Hence we can find a value $\rho > 1$ such that

$$\frac{1}{\rho} > \max\left[(q+1)\left(\frac{n-2s}{n} - \frac{1}{p+1}\right), (p+1)\left(\frac{n-2s}{n} - \frac{1}{q+1}\right)\right]. \tag{5.43}$$

Now we set

$$p_k = (p+1)\rho^k \quad \text{and} \quad q_k = (q+1)\rho^k \quad \forall k \geq 0, \tag{5.44}$$

Then we can check that for any $k \geq 0$,

$$\frac{p}{(p+1)\rho^k} - \frac{1}{(q+1)\rho^{k+1}} < \frac{2s}{n} \quad \text{and} \quad \frac{q}{(q+1)\rho^k} - \frac{1}{(p+1)\rho^{k+1}} < \frac{2s}{n}. \tag{5.45}$$

Actually, it is enough to the examine the case $k = 0$,

$$\frac{p}{p+1} - \frac{1}{(q+1)\rho} < \frac{2s}{n} \quad \text{and} \quad \frac{q}{q+1} - \frac{1}{(p+1)\rho} < \frac{2s}{n},$$

which is equivalent to (5.43). Since (5.45) holds for each $k \geq 0$, we can use the Sobolev embedding of Lemma 5.2.1 to show that

$$\|u\|_{(q+1)\rho^{k+1}} \leq C\|(-\Delta)u\|_{\frac{(p+1)\rho^k}{p}} = C\|v^p\|_{\frac{(p+1)\rho^k}{p}} = C\|v\|^p_{(p+1)\rho^k}, \tag{5.46}$$

and

$$\|v\|_{(p+1)\rho^{k+1}} \leq C\|(-\Delta)u\|_{\frac{(p+1)\rho^k}{p}} = C\|v^q\|_{\frac{(q+1)\rho^k}{q}} = C\|v\|^q_{(q+1)\rho^k}. \tag{5.47}$$

Hence we have that $\|u\|_{p_{k+1}} \leq C\|v\|^p_{p_k}$ and $\|v\|_{q_{k+1}} \leq C\|u\|^q_{q_k}$ for each $k \geq 0$. Combining this with (5.41) gives *a priori* bounds of $\|u\|_{q_k}$ and $\|v\|_{p_k}$ for each $k \geq 0$, i.e.,

$$\|u\|_{q_k} \leq C \quad \text{and} \quad \|v\|_{p_k} \leq C. \tag{5.48}$$

Since $\rho > 1$ we can find $N \geq 1$ such that $\frac{p}{q_N} < \frac{2s}{n}$ and $\frac{q}{p_N} < \frac{2s}{n}$. Then, combining the bound (5.48) for $k = N$ with Lemma 5.2.1, we can conclude that $\|u\|_\infty + \|v\|_\infty < C$. The proof is completed. □

Chapter 6

Asymptotic behavior of solutions for nonlinear elliptic problems with the fractional Laplacian [CKL]

6.1 Introduction

The aim of this paper is to study the nonlocal equations:

$$\begin{cases} \mathcal{A}_s u = u^p + \epsilon u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where $0 < s < 1$, $p := \frac{n+2s}{n-2s}$, $\epsilon > 0$ is a small parameter, Ω is a smooth bounded domain of $\mathbb{R}^n$ and $\mathcal{A}_s$ denotes the fractional Laplace operator $(-\Delta)^s$ in Ω with zero Dirichlet boundary values on $\partial\Omega$, defined in terms of the spectra of the Dirichlet Laplacian $-\Delta$ on Ω. It can be understood as the nonlocal version of the Brezis-Nirenberg problem [BN].

The aim of this paper is to study the problem (6.1) when $p = \frac{n+2s}{n-2s}$ is the critical Sobolev exponent and $\epsilon > 0$ is close to zero. During this study we develop some nonlocal techniques which also have their own interests.

The first part is devoted to study least energy solutions of (6.1). To state the result, we recall from [CoT] that the sharp fractional Sobolev inequality for $n > 2s$ and $s > 0$

$$\left(\int_{\mathbb{R}^n} |f(x)|^{p+1} dx\right)^{\frac{1}{p+1}} \leq \mathcal{S}_{n,s} \left(\int_{\mathbb{R}^n} |\mathcal{A}_s^{1/2} f(x)|^2 dx\right)^{\frac{1}{2}} \quad \text{for any } f \in H^s(\mathbb{R}^n)$$

which holds with the constant

$$\mathcal{S}_{n,s} = 2^{-s}\pi^{-s/2}\left[\frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)}\right]^{\frac{1}{2}}\left[\frac{\Gamma(n)}{\Gamma(n/2)}\right]^{\frac{s}{n}}. \tag{6.2}$$

Our first result is the following.

Theorem 6.1.1. *Assume* $0 < s < 1$ *and* $n > 4s$. *For* $\epsilon > 0$, *let* u_ϵ *be a solution of* (6.1) *such that*

$$\lim_{\epsilon \to 0} \frac{\int_\Omega |\mathcal{A}_s^{1/2} u_\epsilon|^2 dx}{\left(\int_\Omega |u_\epsilon|^{p+1} dx\right)^{2/(p+1)}} = \mathcal{S}_{n,s}. \tag{6.3}$$

Then there exist a point $x_0 \in \Omega$ *and a constant* $\mathfrak{b}_{n,s} > 0$ *such that*

$$u_\epsilon \to 0 \text{ in } \begin{cases} C^\alpha_{loc}(\Omega \setminus \{x_0\}) & \text{for all } \alpha \in (0, 2s) & \text{if } s \in (0, 1/2], \\ C^{1,\alpha}_{loc}(\Omega \setminus \{x_0\}) & \text{for all } \alpha \in (0, 2s-1) & \text{if } s \in (1/2, 1), \end{cases}$$

and

$$\|u_\epsilon(x)\|_{L^\infty} u_\epsilon(x) \to \mathfrak{b}_{n,s} G(x, x_0) \text{ in } \begin{cases} C^\alpha_{loc}(\Omega \setminus \{x_0\}) & \text{for all } \alpha \in (0, 2s) & \text{if } s \in (0, 1/2], \\ C^{1,\alpha}_{loc}(\Omega \setminus \{x_0\}) & \text{for all } \alpha \in (0, 2s-1) & \text{if } s \in (1/2, 1), \end{cases}$$

as ϵ *goes to 0. The constant* $\mathfrak{b}_{n,s}$ *is explicitly computed in Section 6.3 (see* (6.31)*).*

Here the function $G = G(x, y)$ for $x, y \in \Omega$ is Green's function of $\mathcal{A}_s$ with the Dirichlet boundary condition, which solves the equation

$$\mathcal{A}_s G(\cdot, y) = \delta_y \text{ in } \Omega \quad \text{and} \quad G(\cdot, y) = 0 \text{ on } \partial\Omega. \tag{6.4}$$

The regular part of G is given by

$$H(x, y) = \frac{\mathfrak{a}_{n,s}}{|x-y|^{n-2s}} - G(x, y) \quad \text{where } \mathfrak{a}_{n,s} = \frac{1}{|S^{n-1}|} \cdot \frac{2^{1-2s}\Gamma(\frac{n-2s}{2})}{\Gamma(\frac{n}{2})\Gamma(s)}. \tag{6.5}$$

The diagonal part τ of the function H, namely, $\tau(x) := H(x, x)$ for $x \in \Omega$ is called the Robin function and it plays a crucial role for our problem.

Theorem 6.1.2. *Assume that* $0 < s < 1$ *and* $n > 4s$. *Suppose* $x_0 \in \Omega$ *is a point given by Theorem 6.1.1. Then*
(1) x_0 *is a critical point of the function* $\tau(x)$.
(2) It holds that

$$\lim_{\epsilon \to 0} \epsilon \|u_\epsilon\|^{2\frac{n-4s}{n-2s}}_{L^\infty(\Omega)} = \mathfrak{d}_{n,s} |\tau(x_0)|$$

where the constant $\mathfrak{d}_{n,s}$ *is computed in Section 6.5 (see* (6.2)*).*

These two results are motivated by the work of Han [H] and Rey [R] on the classical local Brezis-Nirenberg problem, which dates back to Brezis and Peletier [BP],

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}} + \epsilon u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.6}$$

On the other hand, in the latter part of his paper, Rey [R] constructed a family of solutions for (6.6) which asymptotically blow up at a nondegenerate critical point of the Robin function. Moreover, this result was extended in [MP], where Musso and Pistoia obtained the existence of multi-peak solutions for certain domains. In the second part of our paper, by employing the Lyapunov-Schmidt reduction method, we prove an analogous result to it for the nonlocal problem (6.1).

Theorem 6.1.3. *Suppose that* $0 < s < 1$ *and* $n > 4s$*. Let* $\Lambda_1 \subset \Omega$ *be a stable critical set of the Robin function* τ*. Then, for small* $\epsilon > 0$*, there exists a family of solutions of* (6.1) *which blow up and concentrate at the point* $x_0 \in \Lambda_1$ *as* $\epsilon \to 0$*.*

This result is an immediate consequence of the following result. Given any $k \in \mathbb{N}$, set

$$\Upsilon_k(\boldsymbol{\lambda}, \boldsymbol{\sigma}) = c_1^2 \left(\sum_{i=1}^{k} H(\sigma_i, \sigma_i)\lambda_i^{n-2s} - \sum_{\substack{i,h=1 \\ i\neq h}}^{k} G(\sigma_i, \sigma_h)(\lambda_i\lambda_h)^{\frac{n-2s}{2}} \right) - c_2 \sum_{i=1}^{k} \lambda_i^{2s} \tag{6.7}$$

for $(\boldsymbol{\lambda}, \boldsymbol{\sigma}) = (\lambda_1, \cdots, \lambda_k, \sigma_1, \cdots, \sigma_k) \in (0, \infty)^k \times \Omega^k$, where

$$c_1 = \int_{\mathbb{R}^n} w_{1,0}^p(x)dx \quad \text{and} \quad c_2 = \int_{\mathbb{R}^n} w_{1,0}^2(x)dx \tag{6.8}$$

with $w_{1,0}$ the function defined in (6.5) with $(\lambda, \xi) = (1, 0)$. Then we have

Theorem 6.1.4. *Assume* $0 < s < 1$ *and* $n > 4s$*. Given* $k \in \mathbb{N}$*, suppose that* Υ_k *has a stable critical set* Λ_k *such that*

$$\Lambda_k \subset \left\{((\lambda_1, \cdots, \lambda_k), (\sigma_1, \cdots, \sigma_k)) \in (0, \infty)^k \times \Omega^k : \sigma_i \neq \sigma_j \text{ if } i \neq j \text{ and } i, j = 1, \cdots, k\right\}.$$

Then there exist a point $((\lambda_1^0, \cdots, \lambda_k^0), (\sigma_1^0, \cdots, \sigma_k^0)) \in \Lambda_k$ *and a small number* $\epsilon_0 > 0$ *such that for* $0 < \epsilon < \epsilon_0$*, there is a family of solutions* u_ϵ *of* (6.1) *which concentrate at each point* $\sigma_1^0, \cdots, \sigma_{k-1}^0$ *and* σ_k^0 *as* $\epsilon \to 0$*.*

For the precise description of the asymptotic behavior of u_ϵ, see the proof of Theorem 6.1.4 in Subsection 6.6.3.

Here we borrowed the notion of stable critical sets from [Li2]. As in the case $s = 1$ (see [MP, EGP] for instance), we can prove that if the domain Ω is a dumbbell-shaped domain which consists of disjoint k-open sets and sufficiently narrow channels connecting them, then Υ_k has a stable critical point for each $k \in \mathbb{N}$, thereby obtaining the following result.

Theorem 6.1.5. *There exist contractible domains* Ω *such that, for* $\epsilon > 0$ *small enough,* (6.1) *possesses a family of solutions which blow up at exactly k different points of each domain* Ω *as* ϵ *converges to 0.*

For the detailed explanation, see Section 6.6.

In order to study the asymptotic behavior, we will use the fundamental observation of Caffarelli and Silvestre [CaS] and Cabré and Tan [CT] (see also [ST, CDDS, BCPS2, T2]). In particular, we study the local problem on a half-cylinder $\mathcal{C} := \Omega \times [0, \infty)$,

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C} = \Omega \times (0, \infty), \\ U > 0 & \text{in } \mathcal{C}, \\ U = 0 & \text{on } \partial_L \mathcal{C} := \partial\Omega \times (0, \infty), \\ \partial_\nu^s U = f(U) & \text{on } \Omega \times \{0\}, \end{cases} \tag{6.9}$$

where ν is the outward unit normal vector to $\mathcal{C}$ on $\Omega \times \{0\}$ and

$$\partial_\nu^s U(x,0) := -C_s^{-1} \left(\lim_{t\to 0+} t^{1-2s} \frac{\partial U}{\partial t}(x,t) \right) \quad \text{for } x \in \Omega \tag{6.10}$$

where

$$C_s := \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)} \tag{6.11}$$

Under appropriate regularity assumptions, the trace of a solution U of (6.9) on $\Omega \times \{0\}$ solves the nonlinear problem (6.1).

A key step of the proof for Theorem 6.1.1 is to get a sort of the uniform bound after rescaling the solutions $\{u_\epsilon : \epsilon > 0\}$. For this purpose, we will establish a priori L^∞-estimates by using the Moser iteration argument. Recently, such type of estimates have been established in [GQ, TX, XY]. However, they cannot be applied to our case directly, so we will derive a result which is adequate in our setting (refer to Lemmas 6.4.2 and 6.4.5). We remark that a similar argument to our proof appeared in [GQ]. One more thing which has to be stressed is that we need a bound of $\|u_\epsilon\|_{L^\infty}$ in terms of a certain negative power of $\epsilon > 0$ (Lemma 6.4.8) to apply the elliptic estimates (Lemma 6.4.5). For this, we will use an inequality which comes from a local version of Pohozaev identity on the extended domain (see Proposition 6.4.7). We refer to Section 6.3 for the details.

We also study problems having nonlinearities of slightly subcritical growth

$$\begin{cases} \mathcal{A}_s u = u^{p-\epsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.12}$$

In particular, the following two theorems will be obtained.

Theorem 6.1.6. *Assume that* $0 < s < 1$ *and* $n > 2s$. *For* $\epsilon > 0$, *let* u_ϵ *be a solution of* (6.12) *satisfying* (6.3). *Then, there exist a point* $x_0 \in \Omega$ *and a constant* $\mathfrak{b}_{n,s} > 0$ *such that*

$$u_\epsilon \to 0 \text{ in } \begin{cases} C^\alpha_{loc}(\Omega \setminus \{x_0\}) & \text{for all } \alpha \in (0, 2s) & \text{if } s \in (0, 1/2], \\ C^{1,\alpha}_{loc}(\Omega \setminus \{x_0\}) & \text{for all } \alpha \in (0, 2s-1) & \text{if } s \in (1/2, 1), \end{cases}$$

and

$$\|u_\epsilon(x)\|_{L^\infty} u_\epsilon(x) \to \mathfrak{b}_{n,s} G(x, x_0) \text{ in } \begin{cases} C^{\alpha}_{loc}(\Omega \setminus \{x_0\}) & \text{for all } \alpha \in (0, 2s) & \text{if } s \in (0, 1/2], \\ C^{1,\alpha}_{loc}(\Omega \setminus \{x_0\}) & \text{for all } \alpha \in (0, 2s-1) & \text{if } s \in (1/2, 1), \end{cases}$$

as $\epsilon \to 0$. *Moreover,*

(1) x_0 *is a critical point of the function* $\tau(x)$*.*

(2) We have

$$\lim_{\epsilon \to 0} \epsilon \|u_\epsilon\|^2_{L^\infty(\Omega)} = \mathfrak{g}_{n,s} |\tau(x_0)|.$$

Here $\mathfrak{b}_{n,s}$ *is the same constant to one given in Theorem 6.1.1 and* $\mathfrak{g}_{n,s}$ *is computed in Section 6.7 (see* (6.7)*).*

Like (6.7), we define

$$\widetilde{\Upsilon}_k(\boldsymbol{\lambda}, \boldsymbol{\sigma}) = c_1^2 \left(\sum_{i=1}^{k} H(\sigma_i, \sigma_i) \lambda_i^{n-2s} - \sum_{\substack{i,h=1 \\ i \neq h}}^{k} G(\sigma_i, \sigma_h) (\lambda_i \lambda_h)^{\frac{n-2s}{2}} \right) - \frac{c_1 (n-2s)^2}{4n} \log(\lambda_1 \cdots \lambda_k) \quad (6.13)$$

for $(\boldsymbol{\lambda}, \boldsymbol{\sigma}) = (\lambda_1, \cdots, \lambda_k, \sigma_1, \cdots, \sigma_k) \in (0, \infty)^k \times \Omega^k$, where $c_1 > 0$ is defined in (6.8). Then we have

Theorem 6.1.7. *Assume* $0 < s < 1$ *and* $n > 2s$*. Given* $k \in \mathbb{N}$*, suppose that* $\widetilde{\Upsilon}_k$ *has a stable critical set* Λ_k *such that*

$$\Lambda_k \subset \left\{ ((\lambda_1, \cdots, \lambda_k), (\sigma_1, \cdots, \sigma_k)) \in (0, \infty)^k \times \Omega^k : \sigma_i \neq \sigma_j \text{ if } i \neq j \text{ and } i, j = 1, \cdots, k \right\}.$$

Then there exist a point $((\lambda_1^0, \cdots, \lambda_k^0), (\sigma_1^0, \cdots, \sigma_k^0)) \in \Lambda_k$ *and a small number* $\epsilon_0 > 0$ *such that for* $0 < \epsilon < \epsilon_0$*, there is a family of solutions* u_ϵ *of* (6.12) *which concentrate at each point* $\sigma_1^0, \cdots, \sigma_{k-1}^0$ *and* σ_k^0 *as* $\epsilon \to 0$.

Most of the steps in the proof for Theorem 6.1.1 and Theorem 6.1.2 can be adapted in proving Theorem 6.1.6. However the order of the proof for Theorem 6.1.6 is different from that of previous theorems and some new observations have to be made. We refer to Section 6.7 for the details.

Regarding Theorem 6.1.6, it would be interesting to consider whether we can obtain a further description on the asymptotic behavior of a least energy solution of (6.12) (i.e. a solution satisfying (6.3)) as in [FW], where Flucher and Wei found that a least energy solution concentrates at a minimum of the Robin function in the local case ($s = 1$).

Moreover, we believe that even in the nonlocal case ($s \in (0, 1)$) there exist solutions of (6.12) (with the nonlinearity changed into $|u|^{p-1-\epsilon} u$) which can be characterized as sign-changing towers of bubbles. See the papers e.g. [DDM, PW, MP2, GMP] which studied the existence of bubble-towers for the related local problems.

Before concluding this introduction, we would like to mention some related results to our problem. In [DDW], the authors took into account the singularly perturbed nonlinear Schrödinger equations

$$\begin{cases} \epsilon^{2s}\mathcal{A}_s u + Vu - u^p = 0 & \text{in } \mathbb{R}^n, \\ u > 0 & \text{in } \mathbb{R}^n, \\ u \in H^{2s}(\mathbb{R}^n) \end{cases} \tag{6.14}$$

where $\epsilon > 0$ is sufficiently small, $0 < s < 1$, $p \in (1, \frac{n+2s}{n-2s})$ and V is a positive bounded $C^{1,\alpha}$ function whose value is away from 0. In particular, employing the nondegeneracy result of [FLS], they deduced the existence of various types of spike solutions, like multiple spikes and clusters, such that each of the local maxima concentrates on a critical point of V. See also the result of [ChZ] in which a single peak solution is found under stronger assumptions on (6.14) than those of [DDW] (in particular, it is assumed that $s \in (\max\{\frac{1}{2}, \frac{n}{4}\}, 1)$ in [ChZ]). As far as we know, these works are the first results to investigate concentration phenomena for singularly perturbed equations with the fractional operator $\mathcal{A}_s$ by utilizing the Lyapunov-Schmidt reduction method.

On the other hand, in [SV] and [SV2], the Brezis-Nirenberg problem is also considered when the fractional Laplace operator is defined as in a different way:

$$(-\Delta)^s u(x) = c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \quad \text{for } x \in \Omega$$

where Ω is bounded and $c_{n,s}$ is a normalization constant. (Here, we refer to an interesting paper [MN] which compares two different notions of the fractional Laplacians.) It turns out that a similar result can be deduced to one in [T1] and [BCPS], the papers aforementioned in this introduction. In this point of view, it would be interesting to obtain results for this operator corresponding to ours. As a matter of fact, we suspect that concentration points of solutions for (6.1) and (6.12) are governed by Green's function of the operator in this case too.

This paper is organized as follows. In section 6.2, we review certain notions related to the fractional Laplacian and study the regularity of Green's function of $\mathcal{A}_s$. Section 6.3 is devoted to prove Theorem 6.1.1. In section 6.5, we show Theorem 6.1.2 by finding some estimates for Green's function. In Section 6.6, multi-peak solutions is constructed by the Lyapunov-Schmidt reduction method, giving the proof of Theorem 6.1.4 and Theorem 6.1.5. On the other hand, the Lane-Emden equation (6.12) whose nonlinearity has slightly subcritical growth is considered in Section 6.7, and the proof of Theorem 6.1.6 and Theorem 6.1.7 is presented there. In Appendix 6.A, we give the proof of Proposition 6.4.7 and (6.1), respectively, while we exhibit some necessary computations for the construction of concentrating solutions in Appendix 6.B.

Notations.

Here we list some notations which will be used throughout the paper.
- The letter z represents a variable in the $\mathbb{R}^{n+1}$. Also, it is written as $z = (x, t)$ with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

- Suppose that a domain D is given and $\mathcal{T} \subset \partial D$. If f is a function on D, then the trace of f on $\mathcal{T}$ is denoted by $\mathrm{tr}|_{\mathcal{T}} f$ whenever it is well-defined.
- For a domain $D \subset \mathbb{R}^n$, the map $\nu = (\nu_1, \cdots, \nu_n) : \partial D \to \mathbb{R}^n$ denotes the outward pointing unit normal vector on ∂D.
- dS stands for the surface measure. Also, a subscript attached to dS (such as dS_x or dS_z) denotes the variable of the surface.
- $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ denotes the Lebesgue measure of $(n-1)$-dimensional unit sphere S^{n-1}.
- For a function f, we set $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$.
- Given a function $f = f(x)$, $\nabla_x f$ means the gradient of f with respect to the variable x.
- We will use big O and small o notations to describe the limit behavior of a certain quantity as $\epsilon \to 0$.
- $C > 0$ is a generic constant that may vary from line to line.
- For $k \in \mathbb{N}$, we denote by $B_k(x_0, r)$ the ball $\{x \in \mathbb{R}^k : |x - x_0| < r\}$ for each $x_0 \in \mathbb{R}^k$ and $r > 0$.

6.2 Preliminaries

In this section we first recall the backgrounds of the fractional Laplacian. We refer to [BCPS2, CT, CaS, CDDS, T2, KL] for the details. In particular, the latter part of this section is devoted to prove a C^∞ regularity property of Green's function for the fractional Laplacian with zero Dirichlet boundary condition.

Fractional Sobolev spaces, fractional Laplacians and s-harmonic extensions

Let Ω be a smooth bounded domain of $\mathbb{R}^n$. Let also $\{\lambda_k, \phi_k\}_{k=1}^\infty$ be a sequence of the eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ in Ω with the zero Dirichlet boundary condition on $\partial\Omega$,

$$\begin{cases} -\Delta\phi_k = \lambda_k\phi_k & \text{in } \Omega, \\ \phi_k = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $\|\phi_k\|_{L^2(\Omega)} = 1$ and $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$. Then we set the fractional Sobolev space $H_0^s(\Omega)$ $(0 < s < 1)$ by

$$H_0^s(\Omega) = \left\{ u = \sum_{k=1}^\infty a_k\phi_k \in L^2(\Omega) : \sum_{k=1}^\infty a_k^2\lambda_k^s < \infty \right\}, \tag{6.1}$$

which is a Hilbert space whose inner product is given by

$$\left\langle \sum_{k=1}^\infty a_k\phi_k, \sum_{k=1}^\infty b_k\phi_k \right\rangle_{H_0^s(\Omega)} = \sum_{k=1}^\infty a_k b_k \lambda_k^s \qquad \text{if } \sum_{k=1}^\infty a_k\phi_k,\ \sum_{k=1}^\infty b_k\phi_k \in H_0^s(\Omega).$$

Moreover, for a function in $H_0^s(\Omega)$, we define the fractional Laplacian $\mathcal{A}_s : H_0^s(\Omega) \to H_0^s(\Omega) \simeq H_0^{-s}(\Omega)$ as

$$\mathcal{A}_s\left(\sum_{k=1}^{\infty} a_k\phi_k\right) = \sum_{k=1}^{\infty} a_k\lambda_k^s\phi_k.$$

We also consider the square root $\mathcal{A}_s^{1/2} : H_0^s(\Omega) \to L^2(\Omega)$ of the positive operator $\mathcal{A}_s$ which is in fact equal to $\mathcal{A}_{s/2}$. Note that by the above definitions, we have

$$\langle u, v\rangle_{H_0^s(\Omega)} = \int_\Omega \mathcal{A}_s^{1/2}u \cdot \mathcal{A}_s^{1/2}v = \int_\Omega \mathcal{A}_s u \cdot v \quad \text{for } u, v \in H_0^s(\Omega).$$

If the domain Ω is the whole space $\mathbb{R}^n$, the space $H^s(\mathbb{R}^n)$ $(0 < s < 1)$ is given as

$$H^s(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n) : \|u\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} (1 + |2\pi\xi|^{2s})|\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} < \infty\right\}$$

where $\hat{u}$ denotes the Fourier transform of u, and the fractional Laplacian $\mathcal{A}_s : H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ is defined to be

$$\widehat{\mathcal{A}_s u}(\xi) = |2\pi\xi|^{2s}\hat{u}(\xi) \quad \text{for any } \xi \in \mathbb{R}^n \text{ given } u \in H^s(\mathbb{R}^n).$$

Regarding (6.9) (see also (6.4) below), we need to introduce some more function spaces on $C = \Omega \times (0, \infty)$ where Ω is either a smooth bounded domain or $\mathbb{R}^n$. If Ω is bounded, the function space $H_{0,L}^s(C)$ is defined as the completion of

$$C_{c,L}^\infty(C) := \left\{U \in C^\infty\left(\overline{C}\right) : U = 0 \text{ on } \partial_L C = \partial\Omega \times (0, \infty)\right\}$$

with respect to the norm

$$\|U\|_C = \left(\int_C t^{1-2s}|\nabla U|^2\right)^{\frac{1}{2}}. \tag{6.2}$$

Then it is a Hilbert space endowed with the inner product

$$(U, V)_C = \int_C t^{1-2s}\nabla U \cdot \nabla V \quad \text{for} \quad U,\ V \in H_{0,L}^s(C).$$

In the same manner, we define the space $H_{0,L}^s(C_\epsilon)$ and $C_{c,L}^\infty(C_\epsilon)$ for the dilated problem (6.5). Moreover, $\mathcal{D}^s(\mathbb{R}_+^{n+1})$ is defined as the completion of $C_c^\infty\left(\overline{\mathbb{R}_+^{n+1}}\right)$ with respect to the norm $\|U\|_{\mathbb{R}_+^{n+1}}$ (defined by putting $C = \mathbb{R}_+^{n+1}$ in (6.2) above). Recall that if Ω is a smooth bounded domain, it is verified that

$$H_0^s(\Omega) = \{u = \mathrm{tr}|_{\Omega\times\{0\}}U : U \in H_{0,L}^s(C)\} \tag{6.3}$$

in [CaS, Proposition 2.1] and [CDDS, Proposition 2.1] and [T2, Section 2]. Furthermore, it holds that

$$\|U(\cdot,0)\|_{H^s(\mathbb{R}^n)} \le C\|U\|_{\mathbb{R}^{n+1}_+}$$

for some $C > 0$ independent of $U \in \mathcal{D}^s(\mathbb{R}^{n+1}_+)$.

Now we may consider the fractional harmonic extension of a function u defined in Ω, where Ω is $\mathbb{R}^n$ or a smooth bounded domain. By the celebrated results of Caffarelli-Silvestre [CaS] (for $\mathbb{R}^n$) and Cabré-Tan [CT] (for bounded domains, see also [ST, CDDS, BCPS2, T2]), if we set $U \in H^s_{0,L}(C)$ (or $\mathcal{D}^s(\mathbb{R}^{n+1}_+)$) as a unique solution of the equation

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } C, \\ U = 0 & \text{on } \partial_L C, \\ U(x,0) = u(x) & \text{for } x \in \Omega, \end{cases} \tag{6.4}$$

for some fixed function $u \in H^s_0(\Omega)$ (or $H^s(\mathbb{R}^n)$), then $\mathcal{A}_s u = \partial^s_\nu U|_{\Omega\times\{0\}}$ where the operator $u \mapsto \partial^s_\nu U|_{\Omega\times\{0\}}$ is defined in (6.10). (If $\Omega = \mathbb{R}^n$, we set $\partial_L C = \emptyset$.) We call this U the s-harmonic extension of u. We remark that an *explicit* description of U is obtained in [BCPS2, T2] if Ω is bounded.

6.2.1 Sharp Sobolev and trace inequalities

Given any $\lambda > 0$ and $\xi \in \mathbb{R}^n$, let

$$w_{\lambda,\xi}(x) = \mathfrak{c}_{n,s}\left(\frac{\lambda}{\lambda^2 + |x-\xi|^2}\right)^{\frac{n-2s}{2}} \quad \text{for } x \in \mathbb{R}^n, \tag{6.5}$$

where

$$\mathfrak{c}_{n,s} = 2^{\frac{n-2s}{2}}\left(\frac{\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma\left(\frac{n-2s}{2}\right)}\right)^{\frac{n-2s}{4s}}. \tag{6.6}$$

Then the sharp Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{p+1}dx\right)^{\frac{1}{p+1}} \le \mathcal{S}_{n,s}\left(\int_{\mathbb{R}^n} |\mathcal{A}_s^{1/2}u|^2 dx\right)^{\frac{1}{2}}$$

gets the equality if and only if $u(x) = cw_{\lambda,\xi}(x)$ for any $c > 0$, $\lambda > 0$ and $\xi \in \mathbb{R}^n$, given $\mathcal{S}_{n,s}$ the value defined in (6.2) (refer to [Lb, ChL, FL]). Furthermore, it was shown in [CLO, Li3, LiZ] that if a suitable decay assumption is imposed, then $\{w_{\lambda,\xi}(x) : \lambda > 0, \xi \in \mathbb{R}^n\}$ is the set of all solutions for the problem

$$\mathcal{A}_s u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = 0. \tag{6.7}$$

We use $W_{\lambda,\xi} \in \mathcal{D}^s(\mathbb{R}^{n+1}_+)$ to denote the (unique) s-harmonic extension of $w_{\lambda,\xi}$ so that $W_{\lambda,\xi}$ solves

$$\begin{cases} \operatorname{div}(t^{1-2s} W_{\lambda,\xi}(x,t)) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ W_{\lambda,\xi}(x,0) = w_{\lambda,\xi}(x) & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{6.8}$$

It follows that for the Sobolev trace inequality

$$\left(\int_{\mathbb{R}^n} |U(x,0)|^{p+1} dx\right)^{\frac{1}{p+1}} \le \frac{\mathcal{S}_{n,s}}{\sqrt{C_s}} \left(\int_0^\infty \int_{\mathbb{R}^n} t^{1-2s} |\nabla U(x,t)|^2 dxdt\right)^{\frac{1}{2}}, \tag{6.9}$$

the equality is attained by some function $U \in \mathcal{D}^s(\mathbb{R}^{n+1}_+)$ if and only if $U(x,t) = cW_{\lambda,\xi}(x,t)$ for any $c > 0$, $\lambda > 0$ and $\xi \in \mathbb{R}^n$, where $C_s > 0$ is the constant defined in (6.11) (see [X]). In what follows, we simply denote $w_{1,0}$ and $W_{1,0}$ by w_1 and W_1, respectively.

6.2.2 Green's functions and the Robin function

Let G be Green's function of the fractional Laplacian $\mathcal{A}_s$ with the zero Dirichlet boundary condition (see (6.4)). Then it can be regarded as the trace of Green's function $G_C = G_C(z,x)$ ($z \in C$, $x \in \Omega$) for the extended Dirichlet-Neumann problem which satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla G_C(\cdot,x)) = 0 & \text{in } C, \\ G_C(\cdot,x) = 0 & \text{on } \partial_L C, \\ \partial_\nu^s G_C(\cdot,x) = \delta_x & \text{on } \Omega \times \{0\}. \end{cases} \tag{6.10}$$

In fact, if a function U in C solves

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla U) = 0 & \text{in } C, \\ U = 0 & \text{on } \partial_L C, \\ \partial_\nu^s U = g & \text{on } \Omega \times \{0\}, \end{cases}$$

for some function g on $\Omega \times \{0\}$, then we can see that U has the expression

$$U(z) = \int_\Omega G_C(z,y) g(y) dy = \int_\Omega G_C(z,y) \mathcal{A}_s u(y) dy, \quad z \in C,$$

where $u = \operatorname{tr}|_{\Omega\times\{0\}} U$. Then, by plugging $z = (x,0)$ in the above equalities, we obtain

$$u(x) = \int_\Omega G_C((x,0),y) \mathcal{A}_s u(y) dy,$$

which implies that $G_C((x,0),y) = G(x,y)$ for any $x, y \in \Omega$.

Green's function G_C on the half cylinder C can be partitioned to the singular part and the regular part. The singular part is given by Green's function

$$G_{\mathbb{R}^{n+1}_+}((x,t),y) := \frac{\mathfrak{a}_{n,s}}{|(x-y,t)|^{n-2s}} \tag{6.11}$$

on the half space $\mathbb{R}^{n+1}_+$ satisfying

$$\begin{cases} \text{div}\left(t^{1-2s}\nabla_{(x,t)}G_{\mathbb{R}^{n+1}_+}((x,t),y)\right) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial^s_\nu G_{\mathbb{R}^{n+1}_+}((x,0),y) = \delta_y(x) & \text{on } \Omega \times \{0\}, \end{cases}$$

for each $y \in \mathbb{R}^n$. Note that $\mathfrak{a}_{n,s}$ is the constant defined in (6.5). The regular part is given by the function $H_C : C \to \mathbb{R}$ which satisfies

$$\begin{cases} \text{div}\left(t^{1-2s}\nabla_{(x,t)}H_C((x,t),y)\right) = 0 & \text{in } C, \\ H_C(x,t,y) = \frac{\mathfrak{a}_{n,s}}{|(x-y,t)|^{n-2s}} & \text{on } \partial_L C, \\ \partial^s_\nu H_C((x,0),y) = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

The existence of such a function H_C can be proved using a variational method (see Lemma 6.2.2 below). We then have

$$G_C((x,t),y) = G_{\mathbb{R}^{n+1}_+}((x,t),y) - H_C((x,t),y). \tag{6.12}$$

Accordingly, the Robin function τ which was defined in the paragraph after Theorem 6.1.1 can be written as $\tau(x) := H_C((x,0),x)$. As we will see, the function τ and the relation (6.12) turn out to be very important throughout the paper.

6.2.3 Maximum principle

Here we prove a maximum principle which serves as a valuable tool in studying properties of Green's function G of $\mathcal{A}_s$.

Lemma 6.2.1. *Suppose that V is a weak solution of the following problem*

$$\begin{cases} \text{div}(t^{1-2s}\nabla V) = 0 & \text{in } C, \\ V(x,t) = B(x,t) & \text{on } \partial_L C, \\ \partial^s_\nu V(x,0) = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

for some function B on $\partial_L C$. Then we have

$$\sup_{(x,t)\in C} |V(x,t)| \leq \sup_{(x,t)\in\partial_L C} |B(x,t)|.$$

Proof. Let $S^+ = \sup_{(x,t)\in\partial_L C} B(x,t)$. Consider the function $Y(x,t) := S^+ - V(x,t)$, which satisfies

$$\begin{cases} \text{div}(t^{1-2s}\nabla Y) = 0 & \text{in } C, \\ Y(x,t) \geq 0 & \text{on } \partial_L C \\ \partial^s_\nu Y(x,0) = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

Note that $Y_-(x,t) = 0$ on $\partial_L C$. Then we get

$$0 = \int_C t^{1-2s}\nabla Y(x,t)\cdot\nabla Y_-(x,t)dxdt = -\int_C t^{1-2s}|\nabla Y_-(x,t)|^2 dxdt.$$

It proves that $Y_- \equiv 0$. Thus we have $S^+ \geq V(x,t)$ for all $(x,t) \in C$.

Similarly, if we set $S^- = \inf_{(x,t)\in\partial_L C} B(x,t)$ and define the function $Z(x,t) = V(x,t) - S^-$, we may deduce that $V(x,t) \geq S_-$ for all $(x,t) \in C$. Consequently, we have

$$S^- \leq V(x,t) \leq S^+ \quad \text{for all } (x,t) \in C.$$

It completes the proof. □

6.2.4 Properties of the Robin function

We study more on the property of the function H_C by using the maximum principle obtained in the previous subsection. We first prove the existence of the function H_C.

Lemma 6.2.2. *For each point $y \in \Omega$ the function $H_C((\cdot,\cdot),y)$ is the minimizer of the problem*

$$\min_{V\in S} \int_C t^{1-2s}|\nabla V(x,t)|^2 dxdt, \tag{6.13}$$

where

$$S = \left\{ V : \int_C t^{1-2s}|\nabla V(x,t)|^2 dxdt < \infty \quad \text{and} \quad V(x,t) = G_{\mathbb{R}^{n+1}_+}(x,t,y) \text{ on } \partial_L C \right\}.$$

Here the derivatives are defined in a weak sense.

Proof. Let $\eta \in C^\infty(\mathbb{R}^{n+1})$ be a function such that $\eta(z) = 0$ for $|z| \leq 1$ and $\eta(z) = 1$ for $|z| \geq 2$. Assuming without loss of any generality that $B_{n+1}((y,0),2) \cap \mathbb{R}^{n+1}_+ \subset C$, let V_0 be the function defined in C by

$$V_0(x,t) = G_{\mathbb{R}^{n+1}_+}(x,t,y)\eta(x-y,t).$$

Then it is easy to check that

$$\int_C t^{1-2s}|\nabla V_0(x,t)|^2 dxdt < \infty.$$

Thus S is nonempty and we can find a minimizing function V of the problem (6.13) in S. Then, for any $\Phi \in C^\infty(C)$ such that $\Phi = 0$ on $\partial_L C$, we have

$$\int_C t^{1-2s}\nabla V(x,t)\cdot\nabla\Phi(x,t)dxdt = 0.$$

Hence it holds that

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla V(x,t)) = 0 & \text{for } (x,t) \in C, \\ V(x,t) = G_{\mathbb{R}^{n+1}_+}(x,t,y) & \text{for } (x,t) \in \partial_L C, \\ \partial_\nu^s V(x,0) = 0 & \text{for } x \in \Omega. \end{cases}$$

in a weak sense. This completes the proof. □

In the same way, for a fixed point $y = (y^1, \cdots, y^n) \in \Omega$ and any multi-index $I = (i_1, i_2, \cdots, i_n) \in (\mathbb{N} \cup \{0\})^n$, we find the function $\mathcal{H}_C^I((\cdot,\cdot), y)$ satisfying

$$\begin{cases} \operatorname{div}\left(t^{1-2s}\nabla_{(x,t)}\mathcal{H}_C^I(x,t,y)\right) = 0 & \text{in } C, \\ \mathcal{H}_C^I(x,t,y) = \partial_y^I G_{\mathbb{R}^{n+1}_+}(x,t,y) & \text{on } \partial_L C, \\ \partial_\nu^s \mathcal{H}_C^I(\cdot,\cdot,y) = 0 & \text{on } \Omega \times \{0\}, \end{cases} \tag{6.14}$$

where $\partial_y^I = \partial_{y^1}^{i_1} \cdots \partial_{y^n}^{i_n}$. In the below we shall show that, for any $(x,t) \in C$, the function $H_C(x,t,y)$ is $C^\infty_{\text{loc}}(\Omega)$ and that $\partial_y^I H_C(x,t,y) = \mathcal{H}_C^I(x,t,y)$.

Lemma 6.2.3. *For each $(x,t) \in C$ the function $H_C(x,t,y)$ is continuous with respect to y. Moreover, such continuity is uniform on $(x,t,y) \in C \times \mathcal{K}$ for any compact subset $\mathcal{K}$ of Ω.*

Proof. Take points y_1 and y_2 in a compact subset $\mathcal{K}$ of Ω, sufficiently close to each other. If we apply Lemma 6.2.1 to the function $H_C(x,t,y_1) - H_C(x,t,y_2)$, then we get

$$\begin{aligned} \sup_{(x,t)\in C} |H_C(x,t,y_1) - H_C(x,t,y_2)| &\le \sup_{(x,t)\in\partial_L C} |H_C(x,t,y_1) - H_C(x,t,y_2)| \\ &= \sup_{(x,t)\in\partial_L C} \left| \frac{\mathfrak{c}_{n,s}}{|(x-y_1,t)|^{n-2s}} - \frac{\mathfrak{c}_{n,s}}{|(x-y_2,t)|^{n-2s}} \right| \\ &\le C(\mathcal{K})|y_1 - y_2|, \end{aligned}$$

where $C(\mathcal{K}) > 0$ is constant relying only on $\mathcal{K}$. It proves the lemma. □

The next lemma provides a regularity property of the function H_C. We recall that the result of Fabes, Kenig, and Serapioni [FKS] which gives that $(x,t,y) \mapsto H_C(x,t,y)$ is C^α for some $0 < \alpha < 1$.

Lemma 6.2.4. *(1) For each $(x,t) \in C$, the function $y \to H_C(x,t,y)$ is a C^∞ function. Moreover, for each multi-index $I \in (\mathbb{N} \cup \{0\})^n$, we have*

$$\partial_y^I H_C(x,t,y) = \mathcal{H}_C^I(x,t,y) \tag{6.15}$$

and $\partial_y^I H_C(x,t,y)$ is bounded on $(x,t,y) \in C \times \mathcal{K}$ for any compact set $\mathcal{K}$ of Ω.
(2) For each $y \in \Omega$, the function $x \in \Omega \mapsto H_C(x,0,y)$ is a C^∞ function. Moreover, for each multi-index $I \in (\mathbb{N} \cup \{0\})^n$, the derivative $\partial_x^I H_C(x,0,y)$ is bounded on $(x,y) \in \mathcal{K} \times \Omega$ for any compact set $\mathcal{K}$ of Ω.

Proof. For two points y_1 and y_2 in a compact subset $\mathcal{K}$ of Ω chosen to be close enough to each other, we apply Lemma 6.2.1 to the function

$$H_C(x,t,y_2) - H_C(x,t,y_1) - (y_2 - y_1) \cdot (\mathcal{H}_C^{I_1}, \cdots, \mathcal{H}_C^{I_n})(x,t,y_1)$$

where I_j is the multi-index in $(\mathbb{N}\cup\{0\})^n$ such that the j-th coordinate is 1 and the other coordinates are 0 for $1 \le j \le n$. Then we obtain

$$\begin{aligned}
&\sup_{(x,t)\in C} \left| H_C(x,t,y_2) - H_C(x,t,y_1) - (y_2 - y_1)\cdot(\mathcal{H}_C^{I_1}, \cdots, \mathcal{H}_C^{I_n})(x,t,y_1) \right| \\
&\le \sup_{(x,t)\in\partial_L C} \left| \frac{\mathfrak{c}_{n,s}}{|(x-y_2,t)|^{n-2s}} - \frac{\mathfrak{c}_{n,s}}{|(x-y_1,t)|^{n-2s}} - (y_2 - y_1)\cdot \frac{\mathfrak{c}_{n,s}(n-2s)(x-y_1)}{|(x-y_1,t)|^{n-2s+2}} \right| \\
&\le C(\mathcal{K})|y_1 - y_2|^2
\end{aligned}$$

for some $C(\mathcal{K}) > 0$ independent of the choice of y_1 and y_2. This shows that $\nabla_y H_C(x,t,y) = (\mathcal{H}_C^{I_1}, \cdots, \mathcal{H}_C^{I_n})(x,t,y)$ proving (6.15) for $|I| = 1$. We can adapt this argument inductively, which proves the first statement of the lemma.

Since $H_C(x,0,y) = H_C(y,0,x)$ holds for any $(x,y) \in \Omega\times\Omega$, the second statement follows directly from the first statement. □

Given the above results, we can prove a lemma which is essential when we deduce certain regularity properties of a sequence u_ϵ in the statement of Theorems 6.1.1 and 6.1.6. See Section 6.3.

Lemma 6.2.5. *Suppose that the functions $\tilde{u}_\epsilon$ for $\epsilon > 0$ defined in Ω are given by*

$$\tilde{u}_\epsilon(x) = \int_\Omega G(x,y)\tilde{v}_\epsilon(y)dy,$$

where the set of functions $\{\tilde{v}_\epsilon : \epsilon > 0\}$ satisfies $\sup_{\epsilon>0}\sup_{x\in\Omega}|\tilde{v}_\epsilon(x)| < \infty$. Then $\{\tilde{u}_\epsilon : \epsilon > 0\}$ are equicontinuous on any compact set.

Proof. Suppose that x_1 and x_2 are contained in a compact set $\mathcal{K}$ of Ω. We have

$$\tilde{u}_\epsilon(x) = \int_\Omega G_C(x,0,y)\tilde{v}_\epsilon(y)dy = \int_\Omega G_{\mathbb{R}^{n+1}_+}(x,0,y)\tilde{v}_\epsilon(y)dy - \int_\Omega H_C(x,0,y)\tilde{v}_\epsilon(y)dy$$

for any $x \in \Omega$. Take any number $\eta > 0$. It is well-known that the first term of the right-hand side is C^α for any $\alpha < 2s$ if $s \in (0,1/2]$ and $C^{1,\alpha}$ for any $\alpha < 2s-1$ if $s \in (1/2,1)$. Let us denote the last term by R_ϵ. Then we have

$$|R_\epsilon(x_1) - R_\epsilon(x_2)| \le \int_\Omega |H_C(x_1,0,y) - H_C(x_2,0,y)|\,|\tilde{v}_\epsilon(y)|dy.$$

By Lemma 6.2.4 (2), we can find $\eta > 0$ such that if $|x_1 - x_2| < \eta$ and $(x_1,x_2) \in \mathcal{K}\times\mathcal{K}$, then

$$\sup_{y\in\Omega}|H_C(x_1,0,y) - H_C(x_2,0,y)| \le C\eta.$$

From this, we derive that

$$|R_\epsilon(x_1) - R_\epsilon(x_2)| \le C\eta|\Omega|.$$

It proves that $\{\tilde{u}_\epsilon : \epsilon > 0\}$ are equicontinuous on any compact set. □

6.3 The asymptotic behavior

Here we prove Theorem 6.1.1 by studying the normalized functions B_ϵ of the s-harmonic extension U_ϵ of solutions u_ϵ for (6.1), given $\epsilon > 0$ sufficiently small. We first find a pointwise convergence of the functions B_ϵ. Then we will prove that the functions B_ϵ are uniformly bounded by a certain function, which is more difficult part to handle. To obtain this result, we apply the Kelvin transform in the extended problem (6.9), and then attain L^∞-estimates for its solution. In addition we also need an argument to get a bound of the supremum $\|u_\epsilon\|_{L^\infty(\Omega)}$ in terms of $\epsilon > 0$. It involves a local version of the Pohozaev identity (see Proposition 6.4.7).

Pointwise convergence

Set U_ϵ be the s-harmonic extension of u_ϵ to the half cylinder $\Omega \times [0,\infty)$, that is, U_ϵ satisfies $\mathrm{tr}|_{\Omega\times\{0\}} U_\epsilon = u_\epsilon$ and it is a solution to the problem

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla U_\epsilon) = 0 & \text{in } \mathcal{C} = \Omega \times (0,\infty), \\ U_\epsilon > 0 & \text{in } \mathcal{C}, \\ U_\epsilon = 0 & \text{on } \partial_L \mathcal{C} = \partial\Omega \times [0,\infty), \\ \partial_\nu^s U_\epsilon = U_\epsilon^p + \epsilon U_\epsilon & \text{in } \Omega \times \{0\}. \end{cases} \tag{6.1}$$

First we note the following identity

$$\begin{aligned} \int_{\mathcal{C}} t^{1-2s} |\nabla U_\epsilon(x,t)|^2 dxdt &= C_s \int_{\Omega\times\{0\}} \partial_\nu^s U_\epsilon(x,0) U_\epsilon(x,0) dx \\ &= C_s \int_{\Omega\times\{0\}} \mathcal{A}_s u_\epsilon(x) u_\epsilon(x) dx \\ &= C_s \int_{\Omega\times\{0\}} \left|\mathcal{A}_s^{1/2} u_\epsilon(x)\right|^2 dx. \end{aligned}$$

Using this with (6.3), we have

$$\frac{(\int_\Omega |U_\epsilon(x,0)|^{p+1} dx)^{1/(p+1)}}{(\int_{\mathcal{C}} t^{1-2s} |\nabla U_\epsilon(x,t)|^2 dxdt)^{1/2}} = \frac{\mathcal{S}_{n,s}}{\sqrt{C_s}} + o(1) \quad \text{as} \quad \epsilon \to 0.$$

Also, by (6.1), it holds that

$$\begin{aligned} \int_{\mathcal{C}} t^{1-2s} |\nabla U_\epsilon(x,t)|^2 dxdt &= C_s \int_{\Omega\times\{0\}} \left|\mathcal{A}_s^{1/2} u_\epsilon(x)\right|^2 dx = C_s \int_{\Omega\times\{0\}} \mathcal{A}_s u_\epsilon(x) u_\epsilon(x) dx \\ &= C_s \int_{\Omega\times\{0\}} u_\epsilon^{p+1}(x) dx + \epsilon C_s \int_{\Omega\times\{0\}} u_\epsilon^2(x) dx. \end{aligned}$$

The two equalities above give

$$(\mathcal{S}_{n,s} + o(1))^2 \left(\|U_\epsilon(\cdot,0)\|_{L^{p+1}(\Omega)}^{p+1} + \epsilon \|U_\epsilon(\cdot,0)\|_{L^2(\Omega)}^2 \right) = \|U_\epsilon(\cdot,0)\|_{L^{p+1}(\Omega)}^2.$$

From $\|U_\epsilon(\cdot,0)\|_{L^2(\Omega)} \le C(\Omega)\|U_\epsilon(\cdot,0)\|_{L^{p+1}(\Omega)}$ we obtain

$$(\mathcal{S}_{n,s} + o(1))^2 \|U_\epsilon(\cdot,0)\|_{L^{p+1}(\Omega)}^{p+1} = \|U_\epsilon(\cdot,0)\|_{L^{p+1}(\Omega)}^2,$$

which turns to be

$$\lim_{\epsilon\to 0}\int_\Omega U_\epsilon(x,0)^{p+1}dx = \mathcal{S}_{n,s}^{-\frac{n}{s}}. \tag{6.2}$$

We set

$$\mathcal{I}(\Omega, r) = \{x \in \Omega : \text{dist}(x, \partial\Omega) \ge r\} \quad \text{for } r > 0 \tag{6.3}$$

and

$$\mathcal{O}(\Omega, r) = \{x \in \Omega : \text{dist}(x, \partial\Omega) < r\} \quad \text{for } r > 0. \tag{6.4}$$

The following lemma presents a uniform bound of the solutions near the boundary.

Lemma 6.3.1. *Let u be a bounded solution of* (6.1) *with $p > 1$ and $0 < \epsilon < \lambda_1^s$, where λ_1 is the first eigenvalue of $-\Delta$ with the zero Dirichlet condition. Then, for any $r > 0$ there exists a number $C(r,\Omega) > 0$ such that*

$$\int_{\mathcal{I}(\Omega,r)} u\, dx \le C(r,\Omega). \tag{6.5}$$

Moreover, there is a constant $C > 0$ such that

$$\sup_{x\in\mathcal{O}(\Omega,r)} u(x) \le C. \tag{6.6}$$

Proof. Let ϕ_1 be a first eigenfunction of the Dirichlet Laplacian $-\Delta$ in Ω such that $\phi_1 > 0$ in Ω. We have

$$\lambda_1^s \int_\Omega \phi_1 u dx = \int_\Omega (\mathcal{A}_s\phi_1) u dx = \int_\Omega \phi_1(\mathcal{A}_s u) dx = \int_\Omega \phi_1 u^p dx + \epsilon \int_\Omega \phi_1 u dx.$$

Using the Jensen inequality we get the estimate

$$C\left(\int_\Omega \phi_1 u dx\right)^p \le \int_\Omega \phi_1 u^p dx = (\lambda_1^s - \epsilon)\int_\Omega \phi_1 u dx,$$

and hence

$$\int_\Omega \phi_1 u\, dx \le \left(\frac{\lambda_1^s - \epsilon}{C}\right)^{\frac{1}{p-1}}.$$

Because $\phi_1 \ge C$ on $\mathcal{I}(\Omega, r)$, we have

$$C\int_{\mathcal{I}(\Omega,r)} u\, dx \le \left(\frac{\lambda_1^s - \epsilon}{C}\right)^{\frac{1}{p-1}}. \tag{6.7}$$

This completes the derivation of the estimate (6.5).

If Ω is strictly convex, the moving plane argument, which is given in the proof of [CT, Theorem 7.1] for $s = 1/2$ and can be extended to any $s \in (0,1)$ with [T2, Lemma 3.6] and [CS, Corollary 4.12], yields the fact that the solution u increases along an arbitrary straight line toward inside of Ω emanating from a point on $\partial\Omega$. Then, by borrowing an averaging argument from [QS, Lemma 13.2] or [H], which heavily depends on this fact, we can bound $\sup_{x \in O(\Omega,r)} u(x)$ by a constant multiple of $\int_{\mathcal{I}(\Omega,r)} u(x)dx$. In short, estimate (6.7) gives the uniform bound (6.6) near the boundary. The general cases can be proved using the Kelvin transformation in the extended domain (see [**?**]). □

Lemma 6.3.2. *Let*

$$\mu_\epsilon = \mathfrak{c}_{n,s}^{-1} \sup_{x \in \Omega} u_\epsilon(x) \tag{6.8}$$

where the definition of $\mathfrak{c}_{n,s}$ is provided in (6.6). *(Its finiteness comes from [BCPS, Proposition 5.2].) If a point $x_\epsilon \in \Omega$ satisfies $\mu_\epsilon = \mathfrak{c}_{n,s}^{-1} u_\epsilon(x_\epsilon)$, then we have*

$$\lim_{\epsilon \to 0} \mu_\epsilon = \infty,$$

and x_ϵ converges to an interior point x_0 of Ω along a subsequence.

Proof. Suppose that u_ϵ has a bounded subsequence. As before, we let U_ϵ be the extension of u_ϵ (see (6.1)). By Lemma 6.2.5, u_ϵ are equicontinuous, and thus the Arzela-Ascoli theorem implies that u_ϵ converges to a function v uniformly on any compact set. We denote by V the extension of v. Then we see that $\lim_{\epsilon \to 0} \nabla U_\epsilon(x,t) = \nabla V(x,t)$ for any $(x,t) \in C$ from the Green's function representation. Thus we have

$$\begin{aligned} \int_C t^{1-2s} |\nabla V|^2 dxdt = \int_C t^{1-2s} \liminf_{\epsilon \to 0} |\nabla U_\epsilon|^2 dxdt &\le \liminf_{\epsilon \to 0} \int_C t^{1-2s} |\nabla U_\epsilon|^2 dxdt \\ &= \liminf_{\epsilon \to 0} C_s \int_\Omega (u_\epsilon^{p+1} + \epsilon u_\epsilon^2) dx \\ &= C_s \int_\Omega v^{p+1} dx. \end{aligned}$$

Meanwhile, using (6.2), we obtain

$$\left(\int_C t^{1-2s} |\nabla V|^2 dxdt \right)^{\frac{1}{2}} \le \frac{C_s^{1/2}}{\mathcal{S}_{n,s}} \left(\int_\Omega V^{p+1}(x,0) dx \right)^{\frac{1}{p+1}}.$$

Hence the function V attains the equality in the sharp Sobolev trace inequality (6.9), so we can deduce that $V = cW_{\lambda,\xi}$ for some $c, \lambda > 0$ and $\xi \in \mathbb{R}^n$ (see Subsection 6.2.1). However, the support of V is C by its own definition. Consequently, a contradiction arises and the supremum $\mu_\epsilon = \mathfrak{c}_{n,s}^{-1} u_\epsilon(x_\epsilon)$ diverges. Since Lemma 6.3.1 implies u_ϵ is uniformly bounded near the boundary for all small $\epsilon > 0$, the point x_ϵ converges to an interior point passing to a subsequence. □

Now, we normalize the solutions u_ϵ and their extensions U_ϵ, that is, we set

$$b_\epsilon(x) := \mu_\epsilon^{-1} u_\epsilon(\mu_\epsilon^{-\frac{2}{n-2s}} x + x_\epsilon), \quad x \in \Omega_\epsilon := \mu_\epsilon^{\frac{2}{n-2s}}(\Omega - x_\epsilon), \tag{6.9}$$

and

$$B_\epsilon(z) := \mu_\epsilon^{-1} U_\epsilon(\mu_\epsilon^{-\frac{2}{n-2s}} z + x_\epsilon), \quad z \in C_\epsilon := \mu_\epsilon^{\frac{2}{n-2s}}(C - (x_\epsilon, 0)) \tag{6.10}$$

with the value μ_ϵ defined in (6.8). It satisfies $b_\epsilon(0) = \mathfrak{c}_{n,s}$ and $0 \le b_\epsilon \le \mathfrak{c}_{n,s}$, and the domain Ω_ϵ converges to $\mathbb{R}^n$ as ϵ goes to zero. The function B_ϵ satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla B_\epsilon) = 0 & \text{in } C_\epsilon, \\ B_\epsilon > 0 & \text{in } C_\epsilon, \\ B_\epsilon = 0 & \text{on } \partial_L C_\epsilon, \\ \partial_\nu^s B_\epsilon = B_\epsilon^p + \epsilon\mu_\epsilon^{-p+1} B_\epsilon & \text{in } \Omega_\epsilon \times \{0\}. \end{cases}$$

We have

Lemma 6.3.3. *The function b_ϵ converges to the function w_1 uniformly on any compact set in a subsequence.*

Proof. Let B be the weak limit of B_ϵ in $H^s_{0,L}(C)$ and $b = \operatorname{tr}|_{\Omega\times\{0\}}B$. Then it satisfies $b(0) = \max_{x\in\mathbb{R}^n} b(x) = \mathfrak{c}_{n,s}$ and

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla B) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ B > 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_\nu^s B = B^p & \text{in } \mathbb{R}^n \times \{0\}, \end{cases}$$

as well as B is an extremal function of the Sobolev trace inequality (6.9) (see Subsection 6.2.1). Therefore $B(x,t) = W_1(x,t)$. By Lemma 6.2.5, the family of functions $\{b_\epsilon(x) : \epsilon > 0\}$ are equicontinuous on any compact set in $\mathbb{R}^n$, so by the Arzela-Ascoli theorem b_ϵ converges to a function v on any compact set. The function v should be equal to the weak limit function w_1. It proves the lemma. □

6.4 Uniform boundedness

The previous lemma tells that the dilated solution b_ϵ converges to the function w_1 uniformly on each compact set of Ω_ϵ. However it is insufficient for proving our main theorems and in fact we need a refined uniform boundedness result.

Proposition 6.4.1. *There exists a constant $C > 0$ independent of $\epsilon > 0$ such that*

$$b_\epsilon(x) \le C w_1(x). \tag{6.1}$$

By rescaling, it can be shown that it is equivalent to

$$u_\epsilon(x) \le C w_{\mu_\epsilon^{-\frac{2}{n-2s}}, x_\epsilon}(x). \tag{6.2}$$

The proof of this result follows as a combination of the Kelvin transformation, a priori L^∞-estimates, and an inequality which comes from a local Pohozaev identity for the solutions of (6.9).

We set the Kelvin transformation

$$d_\epsilon(x) = |x|^{-(n-2s)} b_\epsilon\left(\kappa(x)\right) \quad \text{for } x \in \Omega_\epsilon, \tag{6.3}$$

and

$$D_\epsilon(z) = |z|^{-(n-2s)} B_\epsilon\left(\kappa(z)\right) \quad \text{for } z \in C_\epsilon, \tag{6.4}$$

where $\kappa(x) = \frac{x}{|x|^2}$ is the inversion map. Then, inequality (6.1) is equivalent to that $d_\epsilon(x) \le C$ for all $x \in \kappa(\Omega_\epsilon)$. Because $0 < b_\epsilon(x) \le \mathfrak{c}_{n,s}$ for $x \in \Omega_\epsilon$, it is enough to find a constant $C > 0$ and a radius $r > 0$ such that

$$d_\epsilon(x) \le C \quad \text{for} \quad x \in B_n(0,r) \cap \kappa(\Omega_\epsilon) \quad \text{for all } \epsilon > 0. \tag{6.5}$$

After making elementary but tedious computations, we find that the function D_ϵ satisfies

$$\operatorname{div}(t^{1-2s}\nabla D_\epsilon) = 0 \quad \text{in } \kappa(C_\epsilon).$$

Also we have

$$\begin{aligned}
\partial_\nu^s D_\epsilon(x,0) &= \lim_{t\to 0} t^{1-2s} \frac{\partial}{\partial \nu}\left[|z|^{-(n-2s)} B_\epsilon\left(\frac{z}{|z|^2}\right)\right] \\
&= \lim_{t\to 0} t^{1-2s} |z|^{-(n-2s+2)} \frac{\partial}{\partial \nu} B_\epsilon\left(\frac{z}{|z|^2}\right) \\
&= \lim_{t\to 0} |z|^{-n-2s} \lim_{t\to 0}\left[\left(\frac{t}{|z|^2}\right)^{1-2s} \frac{\partial}{\partial \nu} B_\epsilon\left(\frac{z}{|z|^2}\right)\right] \\
&= |x|^{-n-2s} B_\epsilon^p\left(\frac{x}{|x|^2}\right) + \epsilon\mu_\epsilon^{-p+1} |x|^{-n-2s} B_\epsilon^p\left(\frac{x}{|x|^2}\right) \\
&= D_\epsilon^p(x,0) + \epsilon\mu_\epsilon^{-p+1}|x|^{-4s} D_\epsilon(x,0) \qquad \text{for } x \in \kappa(\Omega_\epsilon).
\end{aligned}$$

Hence the function D_ϵ satisfies

$$\begin{cases}
\operatorname{div}(t^{1-2s}\nabla D_\epsilon)(z) = 0 & \text{in } \kappa(C_\epsilon), \\
D_\epsilon > 0 & \text{in } \kappa(C_\epsilon), \\
D_\epsilon = 0 & \text{on } \kappa(\partial_L C_\epsilon), \\
\partial_\nu^s D_\epsilon = D_\epsilon^p + \epsilon\mu_\epsilon^{-p+1}|x|^{-4s} D_\epsilon & \text{on } \kappa(\Omega_\epsilon \times \{0\}).
\end{cases} \tag{6.6}$$

Here we record that

$$\|\mu_\epsilon^{-p+1}|x|^{-4s}\|_{L^{\frac{n}{2s}}(B_n(0,1)\cap\kappa(\Omega_\epsilon))} \le \left(\mu_\epsilon^{-\frac{2n}{n-2s}} \int_{\left\{|x|\ge \mu_\epsilon^{-\frac{p-1}{2s}}\right\}} |x|^{-2n} dx\right)^{\frac{2s}{n}} = C. \tag{6.7}$$

In order to show (6.5), we shall prove two regularity results for the problem (6.6) in Lemma 6.4.2 and Lemma 6.4.5 below.

In fact, to make (6.6) satisfy the conditions that Lemma 6.4.5 can be applicable, we need a higher order integrability of the term $\epsilon\mu_\epsilon^{-p+1}|x|^{-4s}$ than that in (6.7). Note that for $\delta > 0$ we have

$$c\epsilon\mu_\epsilon^{\frac{8s^2\delta}{n+2s\delta}} \le \|\epsilon\mu_\epsilon^{-p+1}|x|^{-4s}\|_{L^{\frac{n}{2s}+\delta}(B_n(0,1)\cap\kappa(\Omega_\epsilon))} \le C\epsilon\mu_\epsilon^{\frac{8s^2\delta}{n+2s\delta}}, \tag{6.8}$$

for some constants $C > 0$ and $c > 0$. Thus it is natural to find a bound of μ_ϵ in terms of a certain positive power of ϵ^{-1}. It will be achieved later by using Lemma 6.4.6 and an inequality derived from a local version of the Pohozaev identity (see Lemma 6.4.8).

In what follows, whenever we consider a family of functions whose domains of definition are a set $D \subset \mathbb{R}^k$, we will denote $\int_{B_k(0,r)} f = \int_{B_k(0,r)\cap D} f$ for any ball $B_k(0,r) \subset \mathbb{R}^k$ for each $r > 0$ and $k \in \mathbb{N}$.

Lemma 6.4.2. *Let V be a bounded solution of the equations:*

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla V)(z) = 0 & \text{in } \kappa(C_\epsilon), \\ V > 0 & \text{in } \kappa(C_\epsilon), \\ V = 0 & \text{on } \kappa(\partial_L C_\epsilon), \\ \partial_\nu^s V(x,0) = g(x)V(x,0) & \text{on } \kappa(\Omega_\epsilon \times \{0\}). \end{cases}$$

Fix $\beta \in (1,\infty)$. Suppose that there is a constant $r > 0$ such that

$$\|g\|_{L^{\frac{n}{2s}}(\kappa(\Omega_\epsilon\times\{0\})\cap B_n(0,2r))} \le \frac{\beta}{2S_{n,s}^2(\beta+1)^2}, \tag{6.9}$$

and

$$\int_{B_{n+1}(0,2r)} t^{1-2s} V(x,t)^{\beta+1} dxdt \le Q.$$

Then, there exists a constant $C = C(\beta, r, Q) > 0$ such that

$$\int_{B_n(0,r)} V(x,0)^{\frac{(\beta+1)(p+1)}{2}} dx \le C.$$

Remark 6.4.3. Here we imposed the condition that V is bounded for the simplicity of the proof. This is a suitable assumption for our case, because we will apply it to the function D_ϵ which is already known to be bounded for each $\epsilon > 0$. However, this lemma holds without the assumption on the boundedness. To prove this, one may use a truncated function $V_L := V \cdot 1_{\{|v|\le L\}}$ with for large $L > 0$ where the function 1_D for any set D denotes the characteristic function on D. See the proof of Lemma 6.6.1.

Proof. Choose a smooth function $\eta \in C_c^\infty(\mathbb{R}^{n+1}, [0,1])$ supported on $B_{n+1}(0, 2r) \subset \mathbb{R}^{n+1}$ satisfying $\eta = 1$ on $B_{n+1}(0, r)$. Multiplying the both sides of

$$\operatorname{div}(t^{1-2s}\nabla V) = 0 \quad \text{in } \kappa(C_\epsilon)$$

by $\eta^2 V^\beta$ and using that $V = 0$ on $\kappa(\partial_L C_\epsilon)$, we discover that

$$C_s \int_{\kappa(\Omega_\epsilon \times \{0\})} g(x) V^{\beta+1}(x,0)\eta^2(x,0)dx = \int_{\kappa(C_\epsilon)} t^{1-2s}(\nabla V)\cdot\nabla(\eta^2 V^\beta)dz. \tag{6.10}$$

Also, we can employ Young's inequality to get

$$\begin{aligned}\int_{\kappa(C_\epsilon)} t^{1-2s}(\nabla V)\cdot\nabla(\eta^2 V^\beta)dz &= \int_{\kappa(C_\epsilon)} \beta t^{1-2s}\eta^2 V^{\beta-1}|\nabla V|^2 + 2t^{1-2s}V^\beta \eta(\nabla V)\cdot(\nabla\eta)dz \\ &= \int_{\kappa(C_\epsilon)} t^{1-2s}\beta |V^{\frac{\beta-1}{2}}\eta(\nabla V)|^2 dz + 2\int_{\kappa(C_\epsilon)} t^{1-2s}V^\beta\eta(\nabla V)\cdot(\nabla\eta)dz \\ &\geq \frac{\beta}{2}\int_{\kappa(C_\epsilon)} t^{1-2s}|V^{\frac{\beta-1}{2}}\eta(\nabla V)|^2 dz - \frac{2}{\beta}\int_{\kappa(C_\epsilon)} t^{1-2s}|V^{\frac{\beta+1}{2}}(\nabla\eta)|^2 dz.\end{aligned} \tag{6.11}$$

On the other hand, applying the identity

$$\nabla(V^{\frac{\beta+1}{2}}\eta) = \frac{\beta+1}{2}V^{\frac{\beta-1}{2}}\eta(\nabla V) + V^{\frac{\beta+1}{2}}(\nabla\eta),$$

we obtain

$$2\left(\frac{\beta+1}{2}\right)^2 |V^{\frac{\beta-1}{2}}\eta(\nabla V)|^2 + 2|V^{\frac{\beta+1}{2}}(\nabla\eta)|^2 \geq |\nabla(V^{\frac{\beta+1}{2}}\eta)|^2.$$

This gives

$$|V^{\frac{\beta-1}{2}}\eta(\nabla V)|^2 \geq \frac{2}{(\beta+1)^2}\left\{|\nabla(V^{\frac{\beta+1}{2}}\eta)|^2 - 2|V^{\frac{\beta+1}{2}}(\nabla\eta)|^2\right\}.$$

Combining this with (6.10) and (6.11), and using the Sobolev trace inequality, we deduce that

$$\begin{aligned} &C_s \int_{\kappa(\Omega_\epsilon\times\{0\})} g(x)V^{\beta+1}(x,0)\eta^2(x,0)dx \\ &\geq \frac{\beta}{2}\frac{2}{(\beta+1)^2}\int_{\kappa(C_\epsilon)} t^{1-2s}|\nabla(V^{\frac{\beta+1}{2}}\eta)|^2 dz - \left(\frac{2}{\beta} + \frac{2\beta}{(\beta+1)^2}\right)\int_{\kappa(C_\epsilon)} t^{1-2s}|V^{\frac{\beta+1}{2}}(\nabla\eta)|^2 dz \\ &\geq \frac{C_s\beta}{\mathcal{S}_{n,s}^2(\beta+1)^2}\left(\int_{\kappa(\Omega_\epsilon\times\{0\})}\left(V^{\frac{\beta+1}{2}}\eta\right)^{p+1}dx\right)^{\frac{2}{p+1}} \\ &\qquad - \left(\frac{2}{\beta} + \frac{2\beta}{(\beta+1)^2}\right)\int_{\kappa(C_\epsilon)} t^{1-2s}|V^{\frac{\beta+1}{2}}(\nabla\eta)|^2 dz.\end{aligned} \tag{6.12}$$

Moreover, we use the assumption (6.9) to get

$$\int_{\kappa(\Omega_\epsilon\times\{0\})} g(x)V^{\beta+1}(x,0)\eta^2(x,0)dx \le \left(\int_{\kappa(\Omega_\epsilon\times\{0\})}(\eta V^{\frac{\beta+1}{2}})^{p+1}dx\right)^{\frac{2}{p+1}} \|g\|_{L^{\frac{p+1}{p-1}}(\kappa(\Omega_\epsilon\times\{0\})\cap B_n(0,2r))}$$
$$\le \frac{\beta}{2\mathcal{S}_{n,s}^2(\beta+1)^2}\left(\int_{\kappa(\Omega_\epsilon\times\{0\})}(\eta V^{\frac{\beta+1}{2}})^{p+1}dx\right)^{\frac{2}{p+1}}. \quad (6.13)$$

Using this estimate, we can derive from (6.12) that

$$\frac{C_s\beta}{2\mathcal{S}_{n,s}^2(\beta+1)^2}\left(\int_{\kappa(\Omega_\epsilon\times\{0\})}(V^{\beta+1}\eta^2)^{\frac{p+1}{2}}dx\right)^{\frac{2}{p+1}}$$
$$\ge \frac{C_s\beta}{\mathcal{S}_{n,s}^2(\beta+1)^2}\left(\int_{\kappa(\Omega_\epsilon\times\{0\})}(V^{\frac{\beta+1}{2}}\eta)^{p+1}dx\right)^{\frac{2}{p+1}} - \left(\frac{2}{\beta}+\frac{2\beta}{(\beta+1)^2}\right)\int_{\kappa(C_\epsilon)} t^{1-2s}|V^{\frac{\beta+1}{2}}(\nabla\eta)|^2dz.$$

We now have

$$\int_{\kappa(\Omega_\epsilon)\cap B_n(0,r)}(V^{\frac{\beta+1}{2}})^{p+1}dx \le C\left(\int_{\kappa(C_\epsilon)} t^{1-2s}|V^{\frac{\beta+1}{2}}\nabla\eta|^2dz\right)^{\frac{p+1}{2}}$$
$$\le C\left(\int_{\kappa(C_\epsilon)\cap B_{n+1}(0,2r)} t^{1-2s}|V|^{\beta+1}dz\right)^{\frac{p+1}{2}} \le C.$$

This completes the proof. □

Next, we prove the L^∞-estimate by applying the Moser iteration technique. For the proof of Lemma 6.4.5, we utilize the Sobolev inequality on weighted spaces which appeared in Theorem 1.3 of [FKS] as well as the Sobolev trace inequality (6.9). Such an approach already appeared in the proof of Theorem 3.4 in [GQ].

Proposition 6.4.4. *[FKS, Theorem 1.3] Let Ω be an open bounded set in $\mathbb{R}^{n+1}$. Then there exists a constant $C = C(n,s,\Omega) > 0$ such that*

$$\left(\int_\Omega |t|^{1-2s}|U(x,t)|^{\frac{2(n+1)}{n}}dxdt\right)^{\frac{n}{2(n+1)}} \le C\left(\int_\Omega |t|^{1-2s}|\nabla U(x,t)|^2dxdt\right)^{\frac{1}{2}} \quad (6.14)$$

holds for any function U whose support is contained in Ω whenever the right-hand side is well-defined.

Lemma 6.4.5. *Let V be a bounded solution of the equations*

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla V) = 0 & \text{in } \kappa(C_\epsilon),\\ V > 0 & \text{in } \kappa(C_\epsilon),\\ V = 0 & \text{on } \kappa(\partial_L C_\epsilon),\\ \partial_\nu^s V(x,0) = g(x)V(x,0) & \text{on } \kappa(\Omega_\epsilon\times\{0\}). \end{cases}$$

Fix $\beta_0 \in (1, \infty)$. Suppose that

$$\int_{B_{n+1}(0,r)} t^{1-2s} V(x,t)^{\beta_0+1} dxdt + \int_{B_n(0,r)} V(x,0)^{\beta_0+1} dx \le Q_1$$

and

$$\int_{\kappa(\Omega_\epsilon\times\{0\})\cap B_n(0,r)} |g(x)|^q dx \le Q_2$$

for some $r > 0$ and $q > \frac{n}{2s}$. Then there exists a constant $C = C(\beta_0, r, Q_1, Q_2) > 0$ such that

$$\|V(\cdot,0)\|_{L^\infty(B_n(0,r/2))} \le C.$$

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^{n+1})$. Then the same argument as (6.10)-(6.12) in the proof of the previous lemma gives

$$\begin{aligned} C_s &\int_{\kappa(\Omega_\epsilon\times\{0\})} g\eta^2 V^{\beta+1} dx \\ &\ge \frac{\beta}{2}\frac{2}{(\beta+1)^2}\int_{\kappa(C_\epsilon)} t^{1-2s}|\nabla(V^{\frac{\beta+1}{2}}\eta)|^2 dz - \left(\frac{2}{\beta}+\frac{2\beta}{(\beta+1)^2}\right)\int_{\kappa(C_\epsilon)} t^{1-2s}|V^{\frac{\beta+1}{2}}\nabla\eta|^2 dz. \end{aligned} \tag{6.15}$$

First, we use Hölder's inequality to estimate the left-hand side by

$$\begin{aligned} \int_{\kappa(\Omega_\epsilon\times\{0\})} g\eta^2 V^{\beta+1} dx &\le \left(\int_{\kappa(\Omega_\epsilon\times\{0\})} (V^{\beta+1}\eta^2)^{q'} dx\right)^{\frac{1}{q'}} \left(\int_{\kappa(\Omega_\epsilon\times\{0\})} |g|^q dx\right)^{\frac{1}{q}} \\ &\le C\left(\int_{\kappa(\Omega_\epsilon\times\{0\})} (V^{\beta+1}\eta^2)^{q'} dx\right)^{\frac{1}{q'}} \end{aligned}$$

where q' denotes the Hölder conjugate of q, i.e., $q' = \frac{q}{q-1}$. Since $q > \frac{p+1}{p-1}$, we have $q' < \frac{p+1}{2}$ and so the following interpolation inequality holds.

$$\begin{aligned} &\left(\int_{\kappa(\Omega_\epsilon\times\{0\})} (V^{\beta+1}\eta^2)^{q'} dx\right)^{\frac{1}{q'}} \\ &\quad\le \left(\int_{\kappa(\Omega_\epsilon\times\{0\})} (V^{\beta+1}\eta^2)^{\frac{p+1}{2}} dx\right)^{\frac{2\theta}{p+1}} \left(\int_{\kappa(\Omega_\epsilon\times\{0\})} (V^{\beta+1}\eta^2) dx\right)^{1-\theta} \\ &\quad\le \delta^{\frac{1}{\theta}}\theta\left(\int_{\kappa(\Omega_\epsilon\times\{0\})} (V^{\beta+1}\eta^2)^{\frac{p+1}{2}} dx\right)^{\frac{2}{p+1}} + \delta^{-\frac{1}{1-\theta}}(1-\theta)\int_{\kappa(\Omega_\epsilon\times\{0\})} (V^{\beta+1}\eta^2) dx, \end{aligned}$$

where $\theta \in (0,1)$ and $\delta > 0$ satisfy respectively

$$\frac{2\theta}{p+1} + (1-\theta) = \frac{1}{q'} \quad \text{and} \quad \delta = \left(\frac{1}{\theta C}\cdot\frac{\beta}{2(\beta+1)^2}\right)^\theta$$

for an appropriate number $C > 0$. Then (6.15) gives

$$\begin{aligned}&\frac{\beta}{2(\beta+1)^2}\int_{\kappa(C_\epsilon)} t^{1-2s}|\nabla(V^{\frac{\beta+1}{2}}\eta)|^2 dz\\&\leq C\beta^{\frac{\theta}{1-\theta}}\int_{\kappa(\Omega_\epsilon\times\{0\})}(V^{\beta+1}\eta^2)dx + \left(\frac{2}{\beta}+\frac{2\beta}{(\beta+1)^2}\right)\int_{\kappa(C_\epsilon)} t^{1-2s}|V^{\frac{\beta+1}{2}}\nabla\eta|^2 dz.\end{aligned} \tag{6.16}$$

Consequently the weighted Sobolev inequality (6.14), the trace inequality (6.9) and (6.16) yield that

$$\begin{aligned}&\left(\int_{\kappa(\Omega_\epsilon\times\{0\})}|V^{\frac{\beta+1}{2}}\eta|^{p+1}dx\right)^{\frac{2}{p+1}} + \left(\int_{\kappa(C_\epsilon)} t^{1-2s}|V^{\frac{\beta+1}{2}}\eta|^{\frac{2(n+1)}{n}}dxdt\right)^{\frac{n}{n+1}}\\&\leq C\int_{\kappa(C_\epsilon)} t^{1-2s}|\nabla(V^{\frac{\beta+1}{2}}\eta)|^2 dxdt\\&\leq C\beta^{\frac{1}{1-\theta}}\left[\int_{\kappa(C_\epsilon)} t^{1-2s}|V^{\frac{\beta+1}{2}}\nabla\eta|^2 dxdt + \int_{\kappa(\Omega_\epsilon\times\{0\})}|V^{\beta+1}\eta^2|dx\right].\end{aligned} \tag{6.17}$$

Now, for each $0 < r_1 < r_2$, we take a function $\eta \in C_c^\infty(\mathbb{R}^{n+1}, [0,1])$ supported on $B_{n+1}(0, r_2)$ such that $\eta = 1$ on $B_{n+1}(0, r_1)$. Then the above estimate (6.17) implies

$$\begin{aligned}&\left(\int_{B_n(0,r_1)} V^{(\beta+1)\frac{p+1}{2}}dx\right)^{\frac{2}{p+1}} + \left(\int_{B_{n+1}(0,r_1)} t^{1-2s}V^{(\beta+1)\frac{n+1}{n}}dz\right)^{\frac{n}{n+1}}\\&\leq \frac{C\beta^{\frac{1}{1-\theta}}}{(r_2-r_1)^2}\left[\left(\int_{B_n(0,r_2)} V^{\beta+1}dx\right) + \left(\int_{B_{n+1}(0,r_2)} t^{1-2s}V^{\beta+1}dz\right)\right].\end{aligned} \tag{6.18}$$

We will use this inequality iteratively. We denote $\theta_0 = \min\{\frac{p+1}{2}, \frac{n+1}{n}\} > 1$ and set $\beta_k + 1 = (\beta_0+1)\theta_0^k$ and $R_k = r/2 + r/2^k$ for $k \in \mathbb{N} \cup \{0\}$. By applying the inequality $a^\gamma + b^\gamma \geq (a+b)^\gamma$ for any $a, b > 0$ and $\gamma \in (0, 1]$ with Hölder's inequality, and then taking $\beta = \beta_k$ in (6.18), we obtain

$$\begin{aligned}&\left(\int_{B_n(0,R_{k+1})} V^{\beta_{k+1}+1}dx + \int_{B_{n+1}(0,R_{k+1})} t^{1-2s}V^{\beta_{k+1}+1}dz\right)^{\frac{1}{\beta_{k+1}+1}}\\&\leq C^{\frac{1}{(\beta_0+1)\theta_0^k}}\left[\theta_0^{\frac{k}{1-\theta}}2^{2k}\right]^{\frac{1}{(\beta_0+1)\theta_0^k}}\left(\int_{B_n(0,R_k)} V^{\beta_k+1}dx + \int_{B_{n+1}(0,R_k)} t^{1-2s}V^{\beta_k+1}dz\right)^{\frac{1}{\beta_k+1}}.\end{aligned}$$

Set

$$A_k(V) = \left(\int_{B_n(0,R_k)} V^{\beta_k+1}dx + \int_{B_{n+1}(0,R_k)} t^{1-2s}V^{\beta_k+1}dz\right)^{\frac{1}{\beta_k+1}}.$$

Then, for $D := (4\theta_0^{\frac{1}{1-\theta}})^{\frac{1}{\beta_0+1}}$, we have

$$A_{k+1} \leq C^{\frac{1}{\theta_0^k}} D^{\frac{k}{\theta_0^k}} A_k.$$

Using this we get

$$A_k \leq C^{\sum_{j=1}^\infty \frac{1}{\theta_0^j}} D^{\sum_{j=1}^\infty \frac{j}{\theta_0^j}} A_0 \leq CA_0,$$

from which we deduce that

$$\sup_{x\in B_n(0,r/2)} V(x,0) = \lim_{k\to\infty}\left(\int_{B_{n+1}(0,r/2)} V^{\beta_k+1}(x,0)dx\right)^{\frac{1}{\beta_k+1}} \le \sup_{k\in\mathbb{N}} A_k \le C.$$

This concludes the proof. ◻

As we mentioned before, we cannot use the above result to the function D_ϵ directly because the estimate (6.7) is not enough to employ this result. To overcome this difficulty, we will seek a refined estimation of the term $\epsilon\mu_\epsilon^{-p+1}|x|^{-4s}$ than (6.7), and in particular we will try to bound μ_ϵ by a constant multiple of $\epsilon^{-\alpha}$ having (6.8) in mind where $\alpha > 0$ is a sufficiently small number. We deduce the next result, which is a local invariant of the previous lemma, as the first step for this objective.

Lemma 6.4.6. *Let V be a bounded solution of the equations*

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla V) = 0 & \text{in } \kappa(C_\epsilon),\\ V > 0 & \text{in } \kappa(C_\epsilon),\\ V = 0 & \text{on } \kappa(\partial_L C_\epsilon),\\ \partial_\nu^s V(x,0) = g(x)V(x,0) + \epsilon\varphi(x)V(x,0) & \text{on } \kappa(\Omega_\epsilon\times\{0\}). \end{cases}$$

Fix $\beta\in(1,\infty)$. Suppose that φ satisfies $\|\varphi\|_{L^{\frac{n}{2s}}(\mathbb{R}^n)} \le Q_1$,

$$\int_{B_{n+1}(0,r)} t^{1-2s}V(x,t)^{\beta+1}dxdt + \int_{B_n(0,r)} V(x,0)^{\beta+1}dx \le Q_2$$

and

$$\int_{B_n(0,r)} g(x)^q dx \le Q_3,$$

for some $r > 0$ and $q > \frac{n}{2s}$. Then, for any $J > 1$, there exist constants $\epsilon_0 = \epsilon_0(Q_1, J) > 0$ and $C = C(r, Q_1, Q_2, Q_3, J) > 0$ depending on r, Q_1, Q_2, Q_3 and J such that, if $0 < \epsilon < \epsilon_0$, then we have

$$\|V(\cdot,0)\|_{L^J(B_n(0,r/2))} \le C.$$

Proof. Let $\eta\in C_c^\infty(\mathbb{R}^{n+1})$. Then the same argument for (6.12) gives

$$\begin{aligned} C_s&\int_{\kappa(\Omega_\epsilon\times\{0\})} g(x)\eta^2V^{\beta+1}(x,0)dx + \epsilon C_s\int_{\kappa(\Omega_\epsilon\times\{0\})}\varphi(x)\eta^2V^{\beta+1}(x,0)dx\\ &\ge \frac{\beta}{2}\frac{2}{(\beta+1)^2}\int_{\kappa(C_\epsilon)} t^{1-2s}|\nabla(V^{\frac{\beta+1}{2}}\eta)|^2dz - \left(\frac{2}{\beta}+\frac{2\beta}{(\beta+1)^2}\right)\int_{\kappa(C_\epsilon)} t^{1-2s}|V^{\frac{\beta+1}{2}}\nabla\eta|^2dz. \quad (6.19) \end{aligned}$$

Using Hölder's inequality we get

$$\epsilon\int_{\kappa(\Omega_\epsilon\times\{0\})}\varphi(x)\eta^2V^{\beta+1}(x,0)dx \le \epsilon\|\varphi\|_{L^{\frac{p+1}{p-1}}(\mathbb{R}^n)}\|\eta^2V^{\beta+1}(\cdot,0)\|_{L^{\frac{p+1}{2}}(\mathbb{R}^n)}.$$

If $\epsilon < \frac{\beta}{4(\beta+1)^2 \mathcal{S}_{n,s}^2 Q_1}$, from the trace inequality, we obtain

$$\epsilon \|\varphi\|_{L^{\frac{p+1}{p-1}}(\mathbb{R}^n)} \|\eta^2 V^{\beta+1}(\cdot,0)\|_{L^{\frac{p+1}{2}}(\mathbb{R}^n)} \le \frac{\beta}{4(\beta+1)^2} \int_{\kappa(C_\epsilon)} t^{1-2s} |\nabla (V^{\frac{\beta+1}{2}} \eta)|^2 dz.$$

Now we can follow the steps (6.16)-(6.18) of the previous lemma. Moreover, we can iterate it with respect to β as long as $\epsilon < \frac{\beta}{4(\beta+1)^2 \mathcal{S}_{n,s}^2 Q_1}$ holds. Thus, for $\epsilon < \frac{J}{4(J+1)^2 \mathcal{S}_{n,s}^2 Q_1}$, we can find a constant $C = C(r, C_1, C_2, C_3, J)$ such that

$$\|V(\cdot,0)\|_{L^J(B_n(0,r/2))} \le C.$$

It proves the lemma. □

To apply the previous lemma to get a bound of μ_ϵ in terms of ϵ, we also need to make the use of the Pohozaev identity of U_ϵ:

$$\frac{1}{2C_s} \int_{\partial_L C} t^{1-2s} |\nabla U_\epsilon(z)|^2 \langle z, \nu \rangle dS = \epsilon s \int_{\Omega \times \{0\}} U_\epsilon(x,0)^2 dx.$$

As a matter of fact, we will not use this identity directly, but instead we will utilize its local version to prove the following result.

Proposition 6.4.7. *Suppose that $U \in H^s_{0,L}(C)$ is a solution of problem* (6.9) *with f such that f has the critical growth and $f = F'$ for some function $F \in C^1(\mathbb{R})$. Then, for each $\delta > 0$ and $q > \frac{n}{s}$ there is a constant $C = C(\delta, q) > 0$ such that*

$$\begin{aligned} &\min_{r \in [\delta, 2\delta]} \left| n \int_{\mathcal{I}(\Omega, r/2) \times \{0\}} F(U) dx - \left(\frac{n-2s}{2}\right) \int_{\mathcal{I}(\Omega, r/2) \times \{0\}} U f(U) dx \right| \\ &\le C \left[\left(\int_{\mathcal{O}(\Omega, 2\delta) \times \{0\}} |f(U)|^q dx \right)^{\frac{2}{q}} + \int_{\mathcal{O}(\Omega, 2\delta) \times \{0\}} |F(U)| dx + \left(\int_{\mathcal{I}(\Omega, \delta/2) \times \{0\}} |f(U)| dx \right)^2 \right] \end{aligned} \tag{6.20}$$

where $\mathcal{I}$ and $\mathcal{O}$ is defined in (6.3) *and* (6.4).

We defer the proof of the proposition to Appendix 6.A. We remark that this kind of estimate was used in [**?**] for $s = 1/2$.

Now we can prove the following result.

Lemma 6.4.8. *There exist a constant $C > 0$ and $\alpha > 0$ such that*

$$\mu_\epsilon \le C \epsilon^{-\alpha} \quad \textit{for all} \ \ \epsilon > 0.$$

Proof. We denote

$$f(u) = u^p + \epsilon u \quad \text{and } F(u) = \frac{1}{p+1} u^{p+1} + \frac{1}{2} \epsilon u^2 \quad \text{for } u > 0 \tag{6.21}$$

and fix a small number $\delta > 0$ so that $\mathcal{I}(\Omega, \delta)$ has the same topology as that of Ω. For $r \in [\delta, 2\delta]$ we see that

$$\begin{aligned}\epsilon \int_{\mathcal{I}(\Omega,r)} u_\epsilon(x)^2 dx = \epsilon \int_{\mathcal{I}(\Omega,r)} \mu_\epsilon^2 b_\epsilon \left(\mu_\epsilon^{\frac{p-1}{2s}} (x - x_\epsilon)\right)^2 dx = \epsilon \mu_\epsilon^2 \mu_\epsilon^{-\frac{p-1}{2s} n} \int_{\mu_\epsilon^{\frac{p-1}{2s}} (\mathcal{I}(\Omega,r) - x_\epsilon)} b_\epsilon(x)^2 dx \\ \geq \epsilon \mu_\epsilon^{-\frac{4s}{n-2s}} \int_{B_n(0,1)} b_\epsilon^2(x) dx \geq C \epsilon \mu_\epsilon^{-\frac{4s}{n-2s}},\end{aligned} \tag{6.22}$$

where we used the fact that b_ϵ converges to w_1 uniformly on any compact set (see Lemma 6.3.3). Since U_ϵ is a solution of (6.9) with f given in (6.21), we have

$$\begin{aligned}\min_{r \in [\delta, 2\delta]} \left| n \int_{\mathcal{I}(\Omega,r)\times\{0\}} F(U_\epsilon) dx - \left(\frac{n-2s}{2}\right) \int_{\mathcal{I}(\Omega,r)\times\{0\}} U_\epsilon f(U_\epsilon) dx \right| \\ = \min_{r \in [\delta, 2\delta]} \left| \epsilon s \int_{\mathcal{I}(\Omega,r)} U_\epsilon(x,0)^2 dx \right| \geq C \epsilon \mu_\epsilon^{-\frac{4s}{n-2s}}.\end{aligned}$$

This gives a lower bound of the left-hand side of (6.20).

Now we shall find an upper bound of the right-hand side of (6.20). By Lemma 6.4.6, for any $q < \infty$, we get $\|d_\epsilon\|_{L^q(B_n(0,1))} \leq C$ with a constant $C = C(q) > 0$. Using this we have

$$\begin{aligned} C \geq \int_{\{|x|\leq 1\}} d_\epsilon^q(x) dx &= \int_{\{|x|\leq 1\}} |x|^{-(n-2s)q} b_\epsilon^q \left(\frac{x}{|x|^2}\right) dx \\ &= \int_{\{|x|\geq 1\}} |x|^{(n-2s)q} b_\epsilon^q(x) |x|^{-2n} dx \\ &= \int_{\{|x|\geq 1\}} |x|^{(n-2s)q-2n} \mu_\epsilon^{-q} u_\epsilon^q \left(\mu_\epsilon^{-\frac{p-1}{2s}} x + x_\epsilon\right) dx \\ &= \int_{\left\{|x-x_\epsilon|\geq \mu_\epsilon^{-\frac{p-1}{2s}}\right\}} \mu_\epsilon^{\frac{p-1}{2s}[(n-2s)q-2n]} \mu_\epsilon^{-q} \mu_\epsilon^{\frac{p-1}{2s} n} |x - x_\epsilon|^{(n-2s)q-2n} u_\epsilon^q(x) dx \\ &= \int_{\left\{|x-x_\epsilon|\geq \mu_\epsilon^{-\frac{p-1}{2s}}\right\}} \mu_\epsilon^{q - \frac{2n}{n-2s}} |x - x_\epsilon|^{(n-2s)q-2n} u_\epsilon^q(x) dx. \end{aligned} \tag{6.23}$$

First of all, we find a bound of $\int_\Omega u_\epsilon^p(x) dx$. Using (6.23) and Hölder's inequality we deduce that

$$\begin{aligned} &\int_{\left\{|x-x_\epsilon|\geq \mu_\epsilon^{-\frac{p-1}{2s}}\right\}} u_\epsilon^p(x) dx \\ &\leq \left(\int_{\left\{|x-x_\epsilon|\geq \mu_\epsilon^{-\frac{p-1}{2s}}\right\}} u_\epsilon^q(x) |x - x_\epsilon|^{(n-2s)q-2n} dx\right)^{\frac{p}{q}} \\ &\qquad \times \left(\int_{\left\{|x-x_\epsilon|\geq \mu_\epsilon^{-\frac{p-1}{2s}}\right\}} |x - x_\epsilon|^{-[(n-2s)q-2n]\frac{p}{q-p}} dx\right)^{\frac{q-p}{q}} \\ &\leq \mu_\epsilon^{-(q - \frac{n}{n-2s})\frac{p}{q}} \mu_\epsilon^{\frac{p-1}{2s}[((n-2s)q-2n)\frac{p}{q-p} - n]\frac{q-p}{q}}. \end{aligned}$$

Note that if $q = \infty$, then the last term is equal to $\mu_\epsilon^{-p}\mu_\epsilon^{\frac{p-1}{2s}[(n+2s)-n]} = \mu_\epsilon^{-1}$. Thus, for any $\kappa > 0$, we can find $q = q(\kappa)$ sufficiently large so that the last term of the above estimate is bounded by $\mu_\epsilon^{-1+\kappa}$. Then it follows that

$$\left(\int_{\left\{|x-x_\epsilon|\geq\mu_\epsilon^{-\frac{p-1}{2s}}\right\}} u_\epsilon^p(x)dx\right)^2 \leq \mu_\epsilon^{-2+2\kappa}. \tag{6.24}$$

On the other hand, because $u_\epsilon(x) \leq C\mu_\epsilon$, we have

$$\left(\int_{\left\{|x-x_\epsilon|\leq\mu_\epsilon^{-\frac{p-1}{2s}}\right\}} u_\epsilon^p(x)dx\right)^2 \leq C\mu_\epsilon^{2p}\mu_\epsilon^{-\frac{p-1}{2s}\cdot 2n} = C\mu_\epsilon^{\frac{4s-2n}{n-2s}} = C\mu_\epsilon^{-2}. \tag{6.25}$$

These two estimates give us the bound of $\int_\Omega u_\epsilon^p(x)dx$.

Now we turn to bound $\|f(U_\epsilon)(\cdot,0)\|_{L^q(O(\Omega,2\delta))}$. For this we again use inequality (6.23) to have

$$\int_{\{|x-x_\epsilon|\geq \mathrm{dist}(x_0,\partial\Omega)/2\}} u_\epsilon^{pq}(x)dx \leq C\mu_\epsilon^{-(pq-\frac{2n}{n-2s})} \text{ for any } q > 1.$$

Using this inequality for a sufficiently large q and Hölder's inequality we can deduce that

$$\left(\int_{\{|x-x_\epsilon|\geq \mathrm{dist}(x_0,\partial\Omega)/2\}} u_\epsilon^{pq}(x)dx\right)^{\frac{2}{q}} \leq C\left(\mu_\epsilon^{-(pq-\frac{2n}{n-2s})}\right)^{\frac{2}{q}} \leq C\mu_\epsilon^{-2p+\kappa}. \tag{6.26}$$

Similarly we have

$$\int_{O(\Omega,2\delta)} |F(u_\epsilon(x))|dx \leq C\mu_\epsilon^{-(p+1)+\kappa}.$$

Combining this estimate with (6.24), (6.25) and (6.26) gives the bound

$$\left(\int_{O(\Omega,2\delta)} |f(U_\epsilon)(x,0)|^q dx\right)^{\frac{2}{q}} + \int_{O(\Omega,2\delta)} |F(U_\epsilon(x,0)|dx + \left(\int_\Omega f(U_\epsilon)(x,0)dx\right)^2 \leq C\mu_\epsilon^{-2+2\kappa}.$$

We put this bound and (6.22) into (6.20) in the statement of Proposition 6.4.7. Then we finally get

$$\epsilon\mu_\epsilon^{-\frac{4s}{n-2s}} \leq C\mu_\epsilon^{-2+2\kappa} \tag{6.27}$$

which is equivalent to

$$\mu_\epsilon^{\frac{2n-8s}{n-2s}-2\kappa} \leq \frac{C}{\epsilon}.$$

Choose $\kappa > 0$ such that $\alpha := \frac{2n-8s}{n-2s} - 2\kappa$ is positive. Then the estimate (6.27) turns out to be

$$\mu_\epsilon \leq C\epsilon^{-\alpha},$$

which is the desired inequality. □

Proof of Proposition 6.4.1. We know that

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} t^{1-2s}|\nabla(B_\epsilon - W_1)|^2 dxdt = 0.$$

By employing the Sobolev trace embedding, we find that

$$\lim_{\epsilon\to 0}\left[\int_{\Omega_\epsilon} |b_\epsilon(x) - w_1(x)|^{p+1} dx\right] = 0. \tag{6.28}$$

Since $p = \frac{n+2s}{n-2s}$, we have the scaling invariance

$$\int_{\mathbb{R}^n} |a(x)|^{p+1} dx = \int_{\mathbb{R}^n} |x|^{-2n}|a(\kappa(x))|^{p+1} dx,$$

and

$$\int_{\mathbb{R}^{n+1}} t^{1-2s}|\nabla A(z)|^2 dz \geq C\int_{\mathbb{R}^{n+1}} t^{1-2s}|\nabla[|z|^{-(n-2s)}A(\kappa(z))]|^2 dz$$

for arbitrary functions $a : \mathbb{R}^n \to \mathbb{R}$ and $A : \mathbb{R}^{n+1} \to \mathbb{R}$ which decay sufficiently fast. Using these identities, we deduce from (6.28) that

$$\lim_{\epsilon\to 0}\left[\int_{\kappa(\Omega_\epsilon)} |d_\epsilon(x) - w_1(x)|^{p+1} dx + \int_{\kappa(C_\epsilon)} t^{1-2s}|\nabla(D_\epsilon - W_1)(x,t)|^2 dxdt\right] = 0.$$

Using the Sobolev embedding theorem and Hölder's inequality, for $\beta_0 = \min\{p, \frac{n+2}{n}\} > 1$, we get

$$\lim_{\epsilon\to 0}\int_{B_{n+1}(0,1)} t^{1-2s}|D_\epsilon(x,t) - W_1(x,t)|^{\beta_0+1} dxdt = 0. \tag{6.29}$$

Finally, estimates (6.29) and (6.7) enable us to apply Lemma 6.4.2 so that we can find $\delta > 0$ satisfying

$$\int_{\kappa(\Omega_\epsilon\times\{0\})\cap B_n(0,\delta)} \left(d_\epsilon^{p-1}\right)^{\frac{n}{2s}\frac{\beta_0+1}{2}} dx \leq C \quad \text{for any } \epsilon > 0. \tag{6.30}$$

Next, from Lemma 6.4.8 we may find $\alpha > 0$ such that $\mu_\epsilon \leq \epsilon^{-\alpha}$. Then, for $\zeta > 0$ small enough, we have

$$\begin{aligned}
\|\epsilon\mu_\epsilon^{-p+1}|x|^{-4s}\|_{L^{\frac{n}{2s}+\zeta}(\kappa(\Omega_\epsilon))} &\leq \epsilon\left[\int_{\left\{|x|\geq\mu_\epsilon^{-\frac{p-1}{2s}}\right\}} \mu_\epsilon^{-(p-1)(\frac{n}{2s}+\zeta)}|x|^{-2n-4s\zeta} dz\right]^{\frac{1}{\frac{n}{2s}+\zeta}} \\
&\leq \epsilon\left[\mu_\epsilon^{-(p-1)(\frac{n}{2s}+\zeta)}\mu_\epsilon^{\frac{p-1}{2s}(n+4s\zeta)}\right]^{\frac{1}{\frac{n}{2s}+\zeta}} \\
&= \epsilon\mu_\epsilon^{\frac{\zeta(p-1)}{\frac{n}{2s}+\zeta}} \leq \epsilon\cdot\epsilon^{-\frac{\alpha\zeta(p-1)}{\frac{n}{2s}+\zeta}} \leq 1.
\end{aligned}$$

Given this estimate and (6.30), we can apply Lemma 6.4.5 to get

$$\|d_\epsilon\|_{L^\infty(B_n(0,\delta/2))} \leq C.$$

The proof is concluded. $\square$

Proof of Theorem 6.1.1

We are now ready to prove Theorem 6.1.1.

Proof of Theorem 6.1.1. By the definition of μ_ϵ in (6.8), we have

$$\mathcal{A}_s(\|u_\epsilon\|_{L^\infty(\Omega)}u_\epsilon)(x) = \mathfrak{c}_{n,s}\left[\mu_\epsilon u_\epsilon^p(x) + \epsilon\mu_\epsilon u_\epsilon(x)\right], \quad x \in \Omega.$$

Note from $p = \frac{n+2s}{n-2s}$ that

$$\begin{aligned}\int_\Omega (\mu_\epsilon u_\epsilon^p(x) + \epsilon\mu_\epsilon u_\epsilon(x))\,dx &= \int_\Omega \mu_\epsilon^{p+1} b_\epsilon^p\left(\mu_\epsilon^{\frac{p-1}{2s}}(x-x_\epsilon)\right)dx + \epsilon\mu_\epsilon^2\int_\Omega b_\epsilon\left(\mu_\epsilon^{\frac{p-1}{2s}}(x-x_\epsilon)\right)dx \\ &= \int_{\Omega_\epsilon} b_\epsilon^p(x)dx + \epsilon\mu_\epsilon^2\mu_\epsilon^{-\frac{p-1}{2s}n}\int_{\Omega_\epsilon} b_\epsilon(x)dx.\end{aligned}$$

Note also that

$$\begin{aligned}\mu_\epsilon^2\mu_\epsilon^{-\frac{p-1}{2s}n}\int_{\Omega_\epsilon} b_\epsilon(x)dx &\le \mu_\epsilon^2\mu_\epsilon^{-\frac{p-1}{2s}n}\int_{\left\{|x|\le\mu_\epsilon^{\frac{p-1}{2s}}\right\}}\frac{C}{(1+|x|)^{n-2s}}dx \\ &\le C\mu_\epsilon^{2-\frac{p-1}{2s}n+\frac{p-1}{2s}2s} \le C.\end{aligned}$$

Given the uniform bound (6.1), we use the Lebesgue dominated convergence theorem to obtain

$$\lim_{\epsilon\to 0}\int_\Omega \mathfrak{c}_{n,s}\mu_\epsilon u_\epsilon^p(x) = \int_{\mathbb{R}^n}\mathfrak{c}_{n,s}w_1^p(x)dx = \mathfrak{b}_{n,s},$$

where

$$\mathfrak{b}_{n,s} := \frac{|S^{n-1}|}{2}\frac{\Gamma(s)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)}\mathfrak{c}_{n,s}^{p+1}. \tag{6.31}$$

For $x \neq x_0$, we have $\lim_{\epsilon\to 0}\mu_\epsilon u_\epsilon^p(x) = 0$ by (6.2). Therefore we may conclude that

$$\lim_{\epsilon\to 0}\mathcal{A}_s(\|u_\epsilon\|_{L^\infty(\Omega)}u_\epsilon)(x) = \mathfrak{b}_{n,s}\delta_{x_0}(x) \quad \text{in} \quad C(\Omega)'.$$

Set $v_\epsilon := \mathcal{A}_s(\|u_\epsilon\|_{L^\infty(\Omega)}u_\epsilon)$. Then $\lim_{\epsilon\to 0}\int_\Omega v_\epsilon dx = \mathfrak{b}_{n,s}$ and $\lim_{\epsilon\to 0} v_\epsilon(x) = 0$ uniformly on any compact set of $\Omega \setminus \{x_0\}$. We observe the formula

$$\|u_\epsilon\|_{L^\infty(\Omega)}U_\epsilon(x,t) = \int_\Omega\left[\frac{\mathfrak{a}_{n,s}}{|(x-y,t)|^{n-2s}} - H_C(x,t,y)\right]v_\epsilon(y)dy. \tag{6.32}$$

On the other hand we have $H_C(x,t,\cdot)$ is in $C^\infty_{\text{loc}}(\Omega)$ and $\|H_C(x,t,\cdot)\|_{L^{\frac{2n}{n-2s}}(\Omega)} \le C$ which holds uniformly on any compact set of $\Omega \setminus \{x_0\}$. From this we conclude that

$$\|u_\epsilon\|_{L^\infty(\Omega)}U_\epsilon(x,t) \to \mathfrak{b}_{n,s}G_C(x,t,x_0) \quad \text{in } C^0_{\text{loc}}(\overline{C}\setminus\{(x_0,0)\}).$$

Also, pointwise convergence in C is valid for the derivatives of $\|u_\epsilon\|_{L^\infty(\Omega)}U_\epsilon$ by elliptic regularity. Especially, for $t = 0$, the regularity property of the function $x \in \Omega \to H_C(x, 0, y)$ given in Lemma 6.2.4 proves that

$$\|u_\epsilon\|_{L^\infty(\Omega)}u_\epsilon(x) \to \mathfrak{b}_{n,s}G(x, x_0) \text{ in } \begin{cases} C^\alpha_{\text{loc}}(\Omega \setminus \{x_0\}) & \text{for all } \alpha \in (0, 2s) & \text{if } s \in (0, 1/2], \\ C^{1,\alpha}_{\text{loc}}(\Omega \setminus \{x_0\}) & \text{for all } \alpha \in (0, 2s-1) & \text{if } s \in (1/2, 1). \end{cases}$$

This completes the proof. □

6.5 Location of the blowup point

The objective of this section is to prove Theorem 6.1.2. For this goal, we will derive several identities related to Green's function. Throughout this section, we keep using the notations: $X_0 = (x_0, 0)$, $B_r = B_{n+1}(X_0, r) \cap \mathbb{R}^{n+1}_+$, $\partial B_r^+ = \partial B_r \cap \mathbb{R}^{n+1}_+$ and $\Gamma_r = B_n(x_0, r)$ for $r > 0$ small. We also use $G(z)$ (or $H(z)$) to denote $G_C(z, x_0)$ (or $H_C(z, x_0)$) for brevity.

The first half of this section is devoted to proving the second statement of Theorem 6.1.2.

Proof of Theorem 6.1.2 (2). It holds

$$\begin{aligned} &\lim_{\epsilon\to 0} \epsilon s C_s \mu_\epsilon^{\frac{2(n-4s)}{n-2s}} \delta \int_{\mathbb{R}^n} w_1^2(x)dx \\ &= \mathfrak{b}_{n,s}^2 \int_\delta^{2\delta} \left[\int_{\partial B_r^+} t^{1-2s} \left\langle (z - X_0, \nabla G(z))\nabla G(z) - (z - X_0)\frac{|\nabla G(z)|^2}{2}, \nu \right\rangle dS \right. \\ &\left. + \left(\frac{n-2s}{2}\right) \int_{\partial B_r^+} t^{1-2s} G(z, x_0) \frac{\partial G(z)}{\partial \nu} dS \right] dr \end{aligned} \tag{6.1}$$

for an each $\delta > 0$ small enough. We will now take a limit $\delta \to 0$. Putting

$$G(z) = \frac{\mathfrak{a}_{n,s}}{|z - X_0|^{n-2s}} - H(z) \quad \text{and} \quad \nabla G(z) = -\mathfrak{a}_{n,s}(n-2s)\frac{z - X_0}{|z - X_0|^{n+2-2s}} - \nabla H(z)$$

into the right-hand side of (6.1) and applying $\nu = \frac{z - X_0}{r}$ on ∂B_r^+, we can derive

$$\begin{aligned} &2 \lim_{\epsilon\to 0} \epsilon s C_s \mu_\epsilon^{\frac{2(n-4s)}{n-2s}} \int_{\mathbb{R}^n} w_1^2(x)dx \\ &= (n-2s)^2 \mathfrak{a}_{n,s}\mathfrak{b}_{n,s}^2 \lim_{r\to 0} \left(2\int_{\partial B_{2r}^+} \frac{t^{1-2s}}{(2r)^{n+1-2s}} H(z)dS - \int_{\partial B_r^+} \frac{t^{1-2s}}{r^{n+1-2s}} H(z)dS \right) \\ &+ \lim_{\delta\to 0} \frac{1}{\delta} \int_\delta^{2\delta} \int_{\partial B_r^+} t^{1-2s} O\left(\langle \nu, \nabla H(z) \rangle \left(\frac{1}{r^{n-2s}} + H(z) \right) + r|\nabla H(z)|^2 \right) dS\, dr. \end{aligned}$$

Since $\partial_i H_C(\cdot, x_0)$ has a bounded Hölder norm over a small neighborhood of x_0 for each $i = 1, \cdots, n$ (refer to [CDDS, Lemma 2.9]), the second term in the right-hand side tends to 0. As a result,

$$2 \lim_{\epsilon \to 0} \epsilon s C_s \mu_\epsilon^{\frac{2(n-4s)}{n-2s}} \int_{\mathbb{R}^n} w_1^2(x) dx \to (n-2s)^2 \mathcal{D}_{n,s} \mathfrak{a}_{n,s} \mathfrak{b}_{n,s}^2 \tau(x_0)$$

as $\delta \to 0$, where

$$\mathcal{D}_{n,s} := \lim_{r \to 0} \int_{\partial B_{n+1}(0,r) \cap \mathbb{R}^{n+1}_+} \frac{t^{1-2s}}{r^{n+1-2s}} dS = \int_{B_n(0,1)} \frac{1}{(1-|x|^2)^s} dx = \frac{|S^{n-1}|}{2} B\left(1-s, \frac{n}{2}\right),$$

B denoting the beta function. This proves Theorem 6.1.2 (2). We also know that the constant $\mathfrak{d}_{n,s}$ in the statement of the theorem is given by

$$\mathfrak{d}_{n,s} = \frac{\Gamma(n-2s)}{\pi^{n/s}\Gamma(\frac{n}{2}-2s)} \frac{(n-2s)^2}{2sC_s} \mathcal{D}_{n,s} \mathfrak{a}_{n,s} \mathfrak{b}_{n,s}^2 \mathfrak{c}_{n,s}^{-\frac{4s}{n-2s}}. \tag{6.2}$$

□

Next, we prove the first statement of Theorem 6.1.2, that is, $\tau'(x_0) = 0$.

Proof of Theorem 6.1.2 (1). If U is a solution to (6.9), for each $1 \le k \le n$, we have

$$\begin{aligned} \int_{\partial B_r^+} t^{1-2s} |\nabla U|^2 \nu_k dS &= \int_{B_r} t^{1-2s} \partial_k |\nabla U|^2 dz = 2 \int_{B_r} t^{1-2s} \nabla U \cdot \nabla \partial_k U dz \\ &= 2 \int_{\partial B_r^+} t^{1-2s} \langle \nabla U, \nu \rangle \partial_k U dS + 2C_s \int_{\partial \Gamma_r} F(U) \nu_k dS_x \end{aligned}$$

where $F(t) := \int_0^t f(t)dt$, ν_k is the k-th component of ν, ∂_k is the partial derivative with respect to the k-th variable and $r > 0$ small. For the last equality, we used $\int_{\Gamma_r} f(U) \partial_k U dx = \int_{\Gamma_r} (\partial_k F)(U) dx = \int_{\partial \Gamma_r} F(U) \nu_k dS_x$. Therefore putting $\|U_\epsilon(\cdot, 0)\|_{L^\infty(\Omega)} U_\epsilon$ (see (6.1)) in the place of U in the above identity, integrating the result from δ to 2δ in r and taking $\epsilon \to 0$, we obtain

$$\int_\delta^{2\delta} \int_{\partial B_r^+} t^{1-2s} |\nabla G|^2 \nu_k dS\, dr = 2 \int_\delta^{2\delta} \int_{\partial B_r^+} t^{1-2s} \langle \nabla G, \nu \rangle \partial_k G dS\, dr \tag{6.3}$$

(cf. Appendix **??**). On the other hand, a direct calculation shows that

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\int_{\delta}^{2\delta}\int_{\partial B_r^+} t^{1-2s}\langle\nabla G(z),\nu(z)\rangle\partial_k G(z)dS\,dr\\
&=\lim_{\delta\to 0}\frac{1}{\delta}\int_{\delta}^{2\delta}\int_{\partial B_r^+} t^{1-2s}\left[\frac{\mathfrak{a}_{n,s}(n-2s)}{r^{n-2s+1}}+\left\langle\frac{z-X_0}{r},\nabla H(z)\right\rangle\right]\\
&\qquad\qquad\times\left[(x_k-x_{0,k})\frac{\mathfrak{a}_{n,s}(n-2s)}{r^{n+2-2s}}+\partial_k H(z)\right]dS\,dr\\
&=\lim_{r\to 0}\int_{\partial B_r^+} t^{1-2s}\frac{\mathfrak{a}_{n,s}(n-2s)}{r^{n-2s+1}}\partial_k H(z)dS\\
&\qquad+\lim_{\delta\to 0}\frac{1}{\delta}\int_{\delta}^{2\delta}\int_{\partial B_r^+} t^{1-2s}\frac{\mathfrak{a}_{n,s}(n-2s)}{r^{n-2s+3}}(x_k-x_{0,k})\langle z-X_0,\nabla H(z)\rangle dS\,dr\\
&=(n-2s+3)(n-2s)\mathfrak{a}_{n,s}\mathcal{E}_{n,s}\partial_k\tau(x_0),
\end{aligned}\tag{6.4}$$

where x_k and $x_{0,k}$ mean the k-th component of x and x_0, respectively, and

$$\mathcal{E}_{n,s}:=\lim_{r\to 0}\int_{\partial B_{n+1}(0,r)\cap\mathbb{R}^{n+1}_+}\frac{t^{1-2s}}{r^{n-2s+3}}x_k^2dS=\frac{1}{n}\int_{B_n(0,1)}\frac{|x|^2}{(1-|x|^2)^s}dx=\frac{|S^{n-1}|}{2n}B\left(1-s,\frac{n+2}{2}\right).$$

In particular, $\mathcal{D}_{n,s}=(n-2s+2)\mathcal{E}_{n,s}$. Moreover we observe

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\int_{\delta}^{2\delta}\int_{\partial B_r^+} t^{1-2s}|\nabla G(z)|^2\nu_k(z)dS\,dr\\
&=2\lim_{r\to 0}\int_{\partial B_r^+} t^{1-2s}\frac{\mathfrak{a}_{n,s}(n-2s)}{r^{n-2s+3}}(x_k-x_{0,k})^2\partial_k H(z)dS+\lim_{\delta\to 0}\frac{1}{\delta}\int_{\delta}^{2\delta}\int_{\partial B_r^+} t^{1-2s}|\nabla H(z)|^2\nu_k(z)dS\,dr\\
&=2(n-2s)\mathfrak{a}_{n,s}\mathcal{E}_{n,s}\partial_k\tau(x_0).
\end{aligned}\tag{6.5}$$

Taking $\delta\to 0$ in (6.3) with (6.4) and (6.5) in hand gives our desired result. □

6.6 Construction of solutions for (6.1) concentrating at multiple points

In this section we prove Theorem 6.1.4 by applying the Lyapunov-Schmidt reduction method to the extended problem

$$\begin{cases}\operatorname{div}\left(t^{1-2s}\nabla U\right)=0 & \text{in } C=\Omega\times(0,\infty),\\ U>0 & \text{in } C,\\ U=0 & \text{on } \partial_L C=\partial\Omega\times(0,\infty),\\ \partial_\nu^s U=U^p+\epsilon U & \text{on } \Omega\times\{0\},\end{cases}\tag{6.1}$$

where $0 < s < 1$ and $p = \frac{n+2s}{n-2s}$. We remind that the functions $w_{\lambda,\xi}$ and $W_{\lambda,\xi}$ are defined in (6.5) and (6.8). By the result of Dávila, del Pino and Sire [DDS], it is known that the space of the bounded solutions for the linearized equation of (6.7) at $w_{\lambda,\xi}$, namely,

$$\mathcal{A}_s \phi = p w_{\lambda,\xi}^{p-1} \phi \quad \text{in } \mathbb{R}^n \tag{6.2}$$

is spanned by

$$\frac{\partial w_{\lambda,\xi}}{\partial \xi_1}, \cdots, \frac{\partial w_{\lambda,\xi}}{\partial \xi_n} \quad \text{and} \quad \frac{\partial w_{\lambda,\xi}}{\partial \lambda} \tag{6.3}$$

where $\xi = (\xi_1, \cdots, \xi_n)$ represents the variable in $\mathbb{R}^n$. From this, it also follows that the solutions of the extended problem of (6.2)

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla \Phi) = 0 & \text{in } \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty), \\ \partial_\nu^s \Phi = p w_{\lambda,\xi}^{p-1} \Phi & \text{on } \mathbb{R}^n \times \{0\}, \end{cases} \tag{6.4}$$

which are bounded on $\Omega \times \{0\}$, consist of the linear combinations of

$$\frac{\partial W_{\lambda,\xi}}{\partial \xi_1}, \cdots, \frac{\partial W_{\lambda,\xi}}{\partial \xi_n} \quad \text{and} \quad \frac{\partial W_{\lambda,\xi}}{\partial \lambda}.$$

In the proof of Theorem 6.1.4, we will often consider the dilated equation

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}_\epsilon = \Omega_\epsilon \times (0,\infty), \\ U > 0 & \text{in } \mathcal{C}_\epsilon, \\ U = 0 & \text{on } \partial_L \mathcal{C}_\epsilon = \partial\Omega_\epsilon \times (0,\infty), \\ \partial_\nu^s U = U^p + \epsilon^{1+2s\alpha_0} U & \text{on } \Omega_\epsilon \times \{0\}, \end{cases} \tag{6.5}$$

where

$$\mathcal{C}_\epsilon = \frac{\mathcal{C}}{\epsilon^{\alpha_0}} = \left\{ \frac{(x,t)}{\epsilon^{\alpha_0}} : (x,t) \in \mathcal{C} \right\}$$

and

$$\Omega_\epsilon = \frac{\Omega}{\epsilon^{\alpha_0}} = \left\{ \frac{x}{\epsilon^{\alpha_0}} : x \in \Omega \right\}$$

for some $\alpha_0 > 0$ to be determined later. If U is a solution of (6.5), then $U_\epsilon(z) := \epsilon^{-\frac{(n-2s)}{2}\alpha_0} U(\epsilon^{-\alpha_0} z)$ for $z \in \Omega$ becomes a solution of problem (6.1).

Since we want solutions to be positive, we use a well-known trick that replaces the nonlinear term U^p in (6.1) with its positive part U_+^p. Namely, we consider the following modified equation of (6.5)

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}_\epsilon, \\ U = 0 & \text{on } \partial_L \mathcal{C}_\epsilon, \\ \partial_\nu^s U = f_\epsilon(U) := U_+^p + \epsilon^{1+2s\alpha_0} U & \text{on } \Omega_\epsilon \times \{0\}. \end{cases} \tag{6.6}$$

6.6.1 Finite dimensional reduction

In order to construct a k-peak solution of (6.1) ($k \in \mathbb{N}$), we define the admissible set

$$O^{\delta_0} = \Big\{(\lambda, \boldsymbol{\sigma}) := ((\lambda_1, \cdots, \lambda_k), (\sigma_1, \cdots, \sigma_k)) \in (\mathbb{R}^+)^k \times \Omega^k : \sigma_i = (\sigma_i^1, \cdots, \sigma_i^n), \\ \mathrm{dist}(\sigma_i, \partial\Omega) > \delta_0,\ \delta_0 < \lambda_i < \frac{1}{\delta_0},\ |\sigma_i - \sigma_j| > \delta_0,\ i \neq j,\ i, j = 1, \cdots, k\Big\} \quad (6.7)$$

with some small $\delta_0 > 0$ fixed, which recodes the information of the concentration rate and the locations of points of concentration.

Let the map

$$i_\epsilon^* : L^{\frac{2n}{n+2s}}(\Omega_\epsilon) \to H^s_{0,L}(C_\epsilon)$$

be the adjoint operator of the Sobolev trace embedding

$$i_\epsilon : H^s_{0,L}(C_\epsilon) \to L^{\frac{2n}{n-2s}}(\Omega_\epsilon) \quad \text{defined by} \quad i_\epsilon(U) := \mathrm{tr}|_{\Omega_\epsilon \times \{0\}}(U) \quad \text{for } U \in H^s_{0,L}(C_\epsilon),$$

which comes from the inequality (6.9) (for the definition of $H^s_{0,L}(C_\epsilon)$, see Subsection 6.2). From its definition, $i_\epsilon^*(u) = V$ for some $u \in L^{\frac{2n}{n+2s}}(\Omega_\epsilon)$ and $V \in H^s_{0,L}(C_\epsilon)$ if and only if

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla V) = 0 & \text{in } C_\epsilon, \\ V = 0 & \text{on } \partial_L C_\epsilon, \\ \partial_\nu^s V = C_s^{-1} u & \text{on } \Omega_\epsilon \times \{0\}. \end{cases}$$

where $C_s > 0$ is the constant defined in (6.11). Therefore finding a solution $U \in H^s_{0,L}(C_\epsilon)$ of (6.5) is equivalent to solving the relation

$$i_\epsilon^*(f_\epsilon(i_\epsilon(U))) = C_s^{-1} U. \quad (6.8)$$

It is valuable to note that from (6.3) we have in fact $i_\epsilon : H^s_{0,L}(C_\epsilon) \to H^s_0(\Omega_\epsilon) \subset L^{\frac{2n}{n-2s}}(\Omega_\epsilon)$ and so $\mathcal{A}_s(i_\epsilon(U))$ makes sense. See also Sublemma 6.B.6.

We introduce the functions

$$\Psi^0_{\lambda,\xi} = \frac{\partial W_{\lambda,\xi}}{\partial \lambda}, \quad \Psi^j_{\lambda,\xi} = \frac{\partial W_{\lambda,\xi}}{\partial \xi^j}, \quad \psi^0_{\lambda,\xi} = \frac{\partial w_{\lambda,\xi}}{\partial \lambda}, \quad \psi^j_{\lambda,\xi} = \frac{\partial w_{\lambda,\xi}}{\partial \xi^j} \quad (6.9)$$

where $\xi = \left(\xi^1, \cdots, \xi^n\right) \in \mathbb{R}^n$ and $j = 1, \cdots, n$, and

$$P_\epsilon W_{\lambda,\xi} = i_\epsilon^*\left(w^p_{\lambda,\xi}\right), \quad P_\epsilon \Psi^j_{\lambda,\xi} = i_\epsilon^*\left(p w^{p-1}_{\lambda,\xi} \psi^j_{\lambda,\xi}\right) \quad \text{for } j = 0, 1, \cdots, n. \quad (6.10)$$

Furthermore, we let the functions $P_\epsilon w_{\lambda,\xi}$ and $P_\epsilon \psi^j_{\lambda,\xi}$ be

$$P_\epsilon w_{\lambda,\xi} = i_\epsilon(P_\epsilon W_{\lambda,\xi}) \quad \text{and} \quad P_\epsilon \psi^j_{\lambda,\xi} = i_\epsilon\left(P_\epsilon \Psi^j_{\lambda,\xi}\right) \quad \text{for } j = 0, \cdots, n \quad (6.11)$$

which vanish on $\partial\Omega_\epsilon$ and solve the equations $\mathcal{A}_s u = w^p_{\lambda,\xi}$ and $\mathcal{A}_s u = p w^{p-1}_{\lambda,\xi}\psi^j_{\lambda,\xi}$ in Ω_ϵ, respectively. Also, whenever $(\boldsymbol{\lambda},\boldsymbol{\sigma}) \in O^{\delta_0}$ is chosen, we denote

$$W_i = W_{\lambda_i,\sigma_i\epsilon^{-\alpha_0}}, \quad P_\epsilon W_i = P_\epsilon W_{\lambda_i,\sigma_i\epsilon^{-\alpha_0}} \quad \text{and} \quad P_\epsilon \Psi^j_i = P_\epsilon \Psi^j_{\lambda_i,\sigma_i\epsilon^{-\alpha_0}} \tag{6.12}$$

and similarly define $P_\epsilon w_i$ and $P_\epsilon \psi^j_i$ ($i = 1,\cdots,k$ and $j = 0,1,\cdots,n$) for the sake of simplicity. Set also

$$K^\epsilon_{\boldsymbol{\lambda},\boldsymbol{\sigma}} = \left\{ u \in H^1_{0,L}(C_\epsilon) : \left(u, P_\epsilon \Psi^j_i\right)_{C_\epsilon} = 0,\ i = 1,2,\cdots,k,\ j = 0,1,\cdots,n \right\} \tag{6.13}$$

for $\epsilon > 0$ and $(\boldsymbol{\lambda},\boldsymbol{\sigma}) \in O^{\delta_0}$ and define the orthogonal projection operator $\Pi^\epsilon_{\boldsymbol{\lambda},\boldsymbol{\sigma}} : H^s_{0,L}(C_\epsilon) \to K^\epsilon_{\boldsymbol{\lambda},\boldsymbol{\sigma}}$.

Now, if we set $L^\epsilon_{\boldsymbol{\lambda},\boldsymbol{\sigma}} : K^\epsilon_{\boldsymbol{\lambda},\boldsymbol{\sigma}} \to K^\epsilon_{\boldsymbol{\lambda},\boldsymbol{\sigma}}$ by

$$L^\epsilon_{\boldsymbol{\lambda},\boldsymbol{\sigma}}(\Phi) = C_s^{-1}\Phi - \Pi^\epsilon_{\boldsymbol{\lambda},\boldsymbol{\sigma}} i^*_\epsilon \left[f'_\epsilon \left(\sum_{i=1}^k P_\epsilon w_i \right) \cdot i_\epsilon(\Phi) \right], \tag{6.14}$$

then we can obtain the following lemma from the nondegeneracy result of [DDS].

Lemma 6.6.1. *Suppose that $(\boldsymbol{\lambda},\boldsymbol{\sigma})$ is contained in O^{δ_0}. Then there exists a positive constant $C = C(n,\delta_0)$ such that*

$$\|L^\epsilon_{\boldsymbol{\lambda},\boldsymbol{\sigma}}(\Phi)\|_{C_\epsilon} \geq C\|\Phi\|_{C_\epsilon} \quad \textit{for all } \Phi \in K^\epsilon_{\boldsymbol{\lambda},\boldsymbol{\sigma}} \textit{ and sufficiently small } \epsilon > 0.$$

Proof. Assume the contrary. Then there exist sequences $\epsilon_l > 0$, $\Phi_l \in K^{\epsilon_l}_{\boldsymbol{\lambda}_l,\boldsymbol{\sigma}_l}$, $H_l = L^{\epsilon_l}_{\boldsymbol{\lambda}_l,\boldsymbol{\sigma}_l}(\Phi_l)$ and $(\boldsymbol{\lambda}_l,\boldsymbol{\sigma}_l) = ((\lambda_{1l},\cdots,\lambda_{kl}),(\sigma_{1l},\cdots,\sigma_{kl})) \in O^{\delta_0}$ ($l \in \mathbb{N}$) satisfying

$$\lim_{l\to\infty} \epsilon_l = 0, \quad \|\Phi_l\|_{C_{\epsilon_l}} = 1, \quad \lim_{l\to\infty} \|H_l\|_{C_{\epsilon_l}} = 0, \quad \lim_{l\to\infty} (\boldsymbol{\lambda}_l,\boldsymbol{\sigma}_l) = (\boldsymbol{\lambda}_\infty,\boldsymbol{\sigma}_\infty) \in O^{\delta_0}. \tag{6.15}$$

Set $C_l = C_{\epsilon_l}$, $\Omega_l = \Omega_{\epsilon_l}$, $P_l w_{il} = P_{\epsilon_l} w_{\lambda_{il},\sigma_{il}}$ and $P_l \Psi^j_{il} = P_{\epsilon_l} \Psi^j_{\lambda_{il},\sigma_{il}}$ for $i = 1,\cdots,k$ and $j = 1,\cdots,n$. If we further denote $\phi_l = i_{\epsilon_l}(\Phi_l)$, then we have

$$C_s^{-1}\Phi_l - i^*_{\epsilon_l}\left[f'_{\epsilon_l}\left(\sum_{i=1}^k P_l w_{il} \right) \phi_l \right] = H_l + Q_l \quad \text{in } H^1_{0,L}(C_l) \tag{6.16}$$

where $Q_l := \sum_{i=1}^k \sum_{j=1}^n c^l_{ij} P_l \Psi^j_{il}$ for some constants $c^l_{ij} \in \mathbb{R}$. By our assumptions above and the relation

$$\lim_{l\to\infty} \left(P_l \Psi^{j_1}_{i_1 l}, P_l \Psi^{j_2}_{i_2 l} \right)_{C_l} = p \cdot C_s \lim_{l\to\infty} \int_{\Omega_l} U^{p-1}_{i_1} \psi^{j_1}_{i_1 l} P_l \psi^{j_2}_{i_2 l} = c_{j_1} \delta_{i_1 i_2} \delta_{j_1 j_2} \tag{6.17}$$

for some constant $c_{j_1} > 0$ depending on j_1 ($i_1, i_2 = 1,\cdots,k$ and $j_1, j_2 = 0,\cdots,n$), it holds that $\|Q_l\|_{C_l}$ is bounded and so is $\left|c^l_{ij}\right|$.

First we claim that

$$\lim_{l\to\infty} \|Q_l\|_{C_l} = 0.$$

Indeed, testing (6.16) with Q_l, and employing Lemmas 6.B.3, 6.B.4 and 6.B.5, the definition of the operator $i^*_{\epsilon_l}$, and the relation $\left(\Phi_l, P_l\Psi^j_{il}\right)_{C_l} = C_s \int_{\Omega_l} w_{il}^{p-1}\psi^j_{il}\phi_l = 0$ which comes from $\Phi_l \in K^{\epsilon_l}_{\lambda_l,\sigma_l}$ and (6.10), we can deduce

$$\begin{aligned}
&\|Q_l\|^2_{C_l}\\
&= -\int_{\Omega_l} f'_{\epsilon_l}\left(\sum_{i=1}^k P_l w_{il}\right)\phi_l q_l - (H_l, Q_l)_{C_l}\\
&\le \left[\left(p\left\|\left(\sum_{i=1}^k P_l w_{il}\right)^{p-1} - \sum_{i=1}^k w_{il}^{p-1}\right\|_{L^{\frac{n}{2s}}(\Omega_l)} + \epsilon^{1+2s\alpha_0}|\Omega_l|^{\frac{2s}{n}}\right)\|\Phi_l\|_{L^{\frac{2n}{n-2s}}(\Omega_l)} + \|H_l\|_{C_l}\right]\|Q_l\|_{C_l}\\
&\quad + \left(\sum_{i=1}^k \left\|f'_{\epsilon_l}(w_{il})\right\|_{L^{\frac{n}{2s}}(\Omega_l)}\right)\|\Phi_l\|_{L^{\frac{2n}{n-2s}}(\Omega_l)}\left(\sum_{i,j}\left|c^l_{ij}\right|\left\|P_l\psi^j_{il} - \psi^j_{il}\right\|_{L^{\frac{2n}{n-2s}}(\Omega_l)}\right)\\
&= o(1)\|Q_l\|_{C_l} + o(1) = o(1)
\end{aligned}$$

where $q_l := i_{\epsilon_l}(Q_l)$.

Choose now a smooth function $\chi : \mathbb{R} \to [0,1]$ such that $\chi(x) = 1$ if $|x| \le \delta_0/2$ and $\chi(x) = 0$ if $|x| \ge \delta_0$ (where δ_0 is the small number chosen in (6.7)), and set

$$\chi_l(x) = \chi(\epsilon_l^{\alpha_0} x), \quad \Phi_{hl}(x,t) = \Phi_l(x + \epsilon_l^{-\alpha_0}\sigma_{hl}, t)\chi_l(x) \quad \text{for } (x,t) \in C_l$$

and $\phi_{hl} := i_{\epsilon_l}(\Phi_{hl})$ for each $h = 1, \cdots, k$. Since $\|\Phi_{hl}\|_{\mathbb{R}^{n+1}_+}$ is bounded for each h, Φ_{hl} converges to $\Phi_{h\infty}$ weakly in $\mathcal{D}^s(\mathbb{R}^{n+1}_+)$ up to a subsequence. Using the same arguments of [MP], we can conclude that $\Phi_{h\infty}$ is a weak solution of (6.4) with $(\lambda, \xi) = (\lambda_{h\infty}, 0)$ and

$$\int_{\mathbb{R}^{n+1}_+} t^{1-2s}\nabla\Phi_{h\infty}\cdot\nabla\Psi^j_{\lambda_{h\infty},0} = 0 \quad \text{for all } j = 0, 1, \cdots, n.$$

In order to use the result of [DDS] to show $\Phi_{h\infty} = 0$, we also need to know that $\phi_{h\infty}$ is bounded where $\phi_{h\infty}(x) := \Phi_{h\infty}(x,0)$ for any $x \in \mathbb{R}^n$, and it is the next step we will be concerned with. Define $\widetilde{\Phi}_L = \min\{|\Phi_{h\infty}|, L\}$ and $\widetilde{\phi}_L = \mathrm{tr}|_{\mathbb{R}^{n+1}}\widetilde{\Phi}_L$ for any $L > 0$, and select the test function $\widetilde{\Phi}^\beta_L \in \mathcal{D}^s(\mathbb{R}^{n+1})$ for (6.16) with any $\beta > 1$ to obtain

$$\frac{4\beta}{(\beta+1)^2}\left\|\widetilde{\Phi}_L^{\frac{\beta+1}{2}}\right\|^2_{\mathbb{R}^{n+1}_+} = C_s\int_{\mathbb{R}^n} f'_0(w_{\lambda_{h\infty}})\widetilde{\phi}_L^{\beta+1}dx.$$

Then by applying the Sobolev trace embedding and taking $L \to \infty$, we can get

$$\left\|\phi_{h\infty}^{\frac{\beta+1}{2}}\right\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^{n+1}_+)} \le C_\beta\,\|\phi_{h\infty}\|^{\frac{\beta+1}{2}}_{L^{\beta+1}(\mathbb{R}^{n+1}_+)} \tag{6.18}$$

with a constant $C_\beta > 0$ which depends only on β. Since we already have that $\|\phi_{h\infty}\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^{n+1}_+)}$ is finite, we may deduce from (6.18) that for any $q > 1$, there is a constant $C_q > 0$ which relies only on the choice of q such that

$$\|\phi_{h\infty}\|_{L^q(\mathbb{R}^{n+1}_+)} \le C_q.$$

Now we note the expression

$$\begin{aligned}
&\phi_{h\infty}(x)\\
&= \int_{\mathbb{R}^n} \frac{\mathfrak{a}_{n,s}}{|x-y|^{n-2s}} f_0'(w_{\lambda_{h\infty}})(y)\phi_{h\infty}(y)dy\\
&= \int_{\{|x-y|\le 1\}} \frac{\mathfrak{a}_{n,s}}{|x-y|^{n-2s}} f_0'(w_{\lambda_{h\infty}})(y)\phi_{h\infty}(y)dy + \int_{\{|x-y|>1\}} \frac{\mathfrak{a}_{n,s}}{|x-y|^{n-2s}} f_0'(w_{\lambda_{h\infty}})(y)\phi_{h\infty}(y)dy\\
&:= I_1(x) + I_2(x) \qquad \text{for } x \in \mathbb{R}^n.
\end{aligned}$$

As for I_1, we take a very large number q so that $r := \frac{q}{q-1}$ is sufficiently close to 1. Then we get

$$\begin{aligned}
I_1 &\le C\left(\int_{\{|x-y|\le 1\}} \frac{1}{|x-y|^{(n-2s)r}} dy\right)^{\frac{1}{r}} \left(\int_{\{|x-y|\le 1\}} \left|f_0'(w_{\lambda_{h\infty}})(y)\phi_{h\infty}(y)\right|^q dy\right)^{\frac{1}{q}}\\
&\le C\|\phi_{h\infty}\|_{L^q(\mathbb{R}^{n+1})} \le C.
\end{aligned} \tag{6.19}$$

Considering I_2 we take r such that $r = \frac{n}{n-2s} + \zeta$ for a small number $\zeta > 0$. Then q is close to $\frac{n}{2s}$. We further find numbers q_1 slightly less than $\frac{n}{2s}$ and q_2 such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then we get

$$I_2 \le C\left(\int_{\{|x-y|>1\}} \frac{1}{|x-y|^{(n-2s)r}} dy\right)^{\frac{1}{r}} \left\|f_0'(w_{\lambda_{h\infty}})\right\|_{L^{q_1}(\mathbb{R}^{n+1})} \|\phi_{h\infty}\|_{L^{q_2}(\mathbb{R}^{n+1})} \le C. \tag{6.20}$$

The estimates (6.19) and (6.20) show that $\phi_{h\infty}$ is bounded. Now we may achieve that $\Phi_{h\infty} = 0$ by the classification of the solutions for the linear problem (6.4) obtained in [DDS]. In summary, we proved that

$$\lim_{l\to\infty} \Phi_{hl} = 0 \quad \text{weakly in } \mathcal{D}^s(\mathbb{R}^{n+1}_+) \quad \text{and} \quad \lim_{l\to\infty} \phi_{hl} = 0 \quad \text{strongly in } L^q(\Omega) \text{ for } 1 \le q < p+1 \tag{6.21}$$

$(h = 1, \cdots, k)$.

Consequently, (6.21) yields

$$\lim_{l\to\infty} \int_{\Omega_l} f_\epsilon'\left(\sum_{i=1}^k P_l w_{il}\right)\phi_l^2 = 0.$$

Hence by testing Φ_l to (6.16) we may deduce that

$$\lim_{l\to\infty} \|\Phi_l\|_{C_l} = 0.$$

However it contradicts to (6.15). This proves the validity of the lemma. □

For each $\epsilon > 0$ sufficiently small and $(\lambda, \sigma) \in O^{\delta_0}$ fixed, the linear operator $L^\epsilon_{\lambda,\sigma} : K^\epsilon_{\lambda,\sigma} \to K^\epsilon_{\lambda,\sigma}$ has the form Id+$\mathcal{K}$ where Id is the identity operator and $\mathcal{K}$ is a compact operator on $K^\epsilon_{\lambda,\sigma}$, because the trace operator $i_\epsilon : H^s_{0,L}(C_\epsilon) \to L^q(\Omega_\epsilon) \subset L^{p+1}(\Omega_\epsilon)$ is compact whenever $q \in [1, p+1)$. Therefore, by the Fredholm alternative, it is a Fredholm operator of index 0. However Lemma 6.6.1 implies that it is also an injective operator. Consequently, we have the following result.

Proposition 6.6.2. *The inverse $(L^{\epsilon}_{\lambda,\sigma})^{-1}$ of $L^{\epsilon}_{\lambda,\sigma} : K^{\epsilon}_{\lambda,\sigma} \to K^{\epsilon}_{\lambda,\sigma}$ exists for any $\epsilon > 0$ small and $(\boldsymbol{\lambda}, \boldsymbol{\sigma}) \in O^{\delta_0}$. Besides, its operator norm is uniformly bounded in ϵ and $(\boldsymbol{\lambda}, \boldsymbol{\sigma}) \in O^{\delta_0}$, if ϵ is small enough.*

The previous proposition gives us that

Proposition 6.6.3. *For any sufficiently small $\delta_0 > 0$ chosen fixed, we can select $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ and $(\boldsymbol{\lambda}, \boldsymbol{\sigma}) \in O^{\delta_0}$, there exists a unique $\Phi^{\epsilon}_{\lambda,\sigma} \in K^{\epsilon}_{\lambda,\sigma}$ satisfying*

$$\Pi^{\epsilon}_{\lambda,\sigma} \left\{ C_s^{-1} \left(\sum_{i=1}^{k} P_\epsilon W_i + \Phi^{\epsilon}_{\lambda,\sigma} \right) - i^*_\epsilon \left[f_\epsilon \left(\sum_{i=1}^{k} P_\epsilon w_i + i_\epsilon \left(\Phi^{\epsilon}_{\lambda,\sigma} \right) \right) \right] \right\} = 0$$

and

$$\|\Phi^{\epsilon}_{\lambda,\sigma}\|_{C_\epsilon} \le C \epsilon^{\eta_0} \quad \text{with} \quad \eta_0 := \begin{cases} \frac{1}{2} + 2s\alpha_0 & \text{if } n \ge 6s, \\ \frac{1}{2} + (1+\delta)s\alpha_0 & \text{if } 4s < n < 6s, \end{cases} \tag{6.22}$$

where $\delta > 0$ is chosen to satisfy $(4 + 2\delta)s < n$. Furthermore, the map $(\boldsymbol{\lambda}, \boldsymbol{\sigma}) \mapsto \Phi^{\epsilon}_{\lambda,\sigma}$ is $C^1(O^{\delta_0})$.

Proof. Define

$$N_\epsilon(\Phi) = \Pi^{\epsilon}_{\lambda,\sigma} \circ i^*_\epsilon \left[f_\epsilon \left(\sum_{i=1}^{k} P_\epsilon w_i + i_\epsilon(\Phi) \right) - f_\epsilon \left(\sum_{i=1}^{k} P_\epsilon w_i \right) - f'_\epsilon \left(\sum_{i=1}^{k} P_\epsilon w_i \right) i_\epsilon(\Phi) \right],$$

$$R_\epsilon = \Pi^{\epsilon}_{\lambda,\sigma} \left(i^*_\epsilon \left[f_\epsilon \left(\sum_{i=1}^{k} P_\epsilon w_i \right) \right] - C_s^{-1} \sum_{i=1}^{k} P_\epsilon W_i \right)$$

and

$$T_\epsilon(\Phi) = (L^{\epsilon}_{\lambda,\sigma})^{-1}(N_\epsilon(\Phi) + R_\epsilon) \quad \text{for } \Phi \in K^{\epsilon}_{\lambda,\sigma},$$

where the set $K^{\epsilon}_{\lambda,\sigma}$ and the operator $\Pi^{\epsilon}_{\lambda,\sigma}$ are defined in (6.13) and the sentence following it. Also, the well-definedness of the inverse of the operator $L^{\epsilon}_{\lambda,\sigma}$ is guaranteed by Proposition 6.6.2. By Lemmas 6.B.1, 6.B.3 and 6.B.5, we have $\|R_\epsilon\|_{C_\epsilon} = O(\epsilon^{\eta_0})$ as $\epsilon \to 0$, and from this we can conclude that T_ϵ is a contraction mapping on $\mathcal{K}^{\epsilon}_{\lambda,\sigma} := \{\Phi \in K^{\epsilon}_{\lambda,\sigma} : \|\Phi\|_{C_\epsilon} \le C\epsilon^{\eta_0}\}$ for some small $C > 0$, which implies the existence of a unique fixed point of T_ϵ on $\mathcal{K}^{\epsilon}_{\lambda,\sigma}$. It is easy to check that this fixed point is our desired function $\Phi^{\epsilon}_{\lambda,\sigma}$. For the detailed treatment of the argument, we refer to [MP, Proposition 1.8] (see also [DDM, Proposition 3]). □

6.6.2 The reduced problem

We set $\alpha_0 = \frac{1}{n-4s}$. Notice that equation (6.5) for each fixed $\epsilon > 0$ has the variational structure, that is, $U \in H^s_{0,L}(\Omega)$ is a weak solution of the equation if and only if it is a critical point of the energy functional

$$E_\epsilon(U) := \frac{1}{2C_s} \int_{C_\epsilon} t^{1-2s} |\nabla U|^2 - \int_{\Omega_\epsilon \times \{0\}} F_\epsilon(i_\epsilon(U)) \tag{6.23}$$

where $F_\epsilon(t) := \int_0^t f_\epsilon(t)dt$. In fact, thanks to the Sobolev trace embedding $i_\epsilon : H^s_{0,L}(C_\epsilon) \to L^{p+1}(\Omega_\epsilon)$, we can obtain that $E_\epsilon : H^s_{0,L}(C_\epsilon) \to \mathbb{R}$ is a C^1-functional and

$$E'_\epsilon(U)\Phi = \frac{1}{C_s}\int_{C_\epsilon} t^{1-2s}\nabla U \cdot \nabla\Phi - \int_{\Omega_\epsilon\times\{0\}} f_\epsilon(i_\epsilon(U))i_\epsilon(\Phi) \quad \text{for any } \Phi \in H^s_{0,L}(C_\epsilon).$$

Using Proposition 6.6.3, we can define a localized energy functional defined in the admissible set O^{δ_0} in (6.7):

$$\widetilde{E}_\epsilon(\boldsymbol{\lambda}, \boldsymbol{\sigma}) := E_\epsilon\left(\sum_{i=1}^k P_\epsilon W_{\lambda_i, \frac{\sigma_i}{\epsilon^{\alpha_0}}} + \Phi^\epsilon_{\boldsymbol{\lambda},\boldsymbol{\sigma}}\right) \tag{6.24}$$

for $(\boldsymbol{\lambda}, \boldsymbol{\sigma}) = ((\lambda_1, \cdots, \lambda_k), (\sigma_1, \cdots, \sigma_k)) \in O^{\delta_0}$. Then we can obtain the following important properties of $\widetilde{E}_\epsilon$.

Proposition 6.6.4. *Suppose $\epsilon > 0$ is sufficiently small.*
(1) If $\widetilde{E}'_\epsilon(\boldsymbol{\lambda}^\epsilon, \boldsymbol{\sigma}^\epsilon) = 0$ for some $(\boldsymbol{\lambda}^\epsilon, \boldsymbol{\sigma}^\epsilon) \in O^{\delta_0}$, then the function $U_\epsilon := \sum_{i=1}^k P_\epsilon W_{\lambda_i^\epsilon, \frac{\sigma_i^\epsilon}{\epsilon^{\alpha_0}}} + \Phi^\epsilon_{\boldsymbol{\lambda}^\epsilon,\boldsymbol{\sigma}^\epsilon}$ is a solution of (6.6). *Hence one concludes that a dilated function $V_\epsilon(z) := \epsilon^{-\frac{n-2s}{2(n-4s)}} U_\epsilon(\epsilon^{-\frac{1}{n-4s}} z)$ defined for $z \in C$ is a solution of* (6.1).
(2) Recall the number η_0 chosen in (6.22). *Then it holds that*

$$\widetilde{E}_\epsilon(\boldsymbol{\lambda}, \boldsymbol{\sigma}) = \frac{ks}{n}c_0 + \frac{1}{2}\Upsilon_k(\boldsymbol{\lambda}, \boldsymbol{\sigma})\epsilon^{\frac{n-2s}{n-4s}} + o(\epsilon^{\frac{n-2s}{n-4s}}) \tag{6.25}$$

in C^1-uniformly in $(\boldsymbol{\lambda}, \boldsymbol{\sigma}) \in O^{\delta_0}$. Here Υ_k is the function introduced in (6.7) *and*

$$c_0 = \int_{\mathbb{R}^n} w_{1,0}^{p+1}(x)dx \tag{6.26}$$

(recall that $w_{1,0}$ is the function obtained by taking $(\lambda, \xi) = (1, 0)$ in (6.5)*).*

We postpone its proof in Appendix 6.B.3.

6.6.3 Definition of stable critical sets and conclusion of the proofs of Theorems

We recall the definition of stable critical sets which was introduced by Li [Li2].

Definition 6.6.5. Suppose that $D \subset \mathbb{R}^n$ is a domain and g is a C^1 function in D. We say that a bounded set $\Lambda \subset D$ of critical points of f is a stable critical set if there is a number $\delta > 0$ such that $\|g - h\|_{L^\infty(\Lambda)} + \|\nabla(g - h)\|_{L^\infty(\Lambda)} < \delta$ for some $h \in C^1(D)$ implies the existence of a critical point of h in Λ.

Now we are ready to prove Theorem 6.1.4.

Proof of Theorem 6.1.4. By the virtue of Proposition 6.6.4 (2) and Definition 6.6.5, we can find a pair $(\boldsymbol{\lambda}^\epsilon, \boldsymbol{\sigma}^\epsilon) \in \Lambda_k$ which is a critical point of the reduced energy functional $\widetilde{E}_\epsilon$ (defined in (6.23)) given $0 < \epsilon < \epsilon_0$ for some ϵ_0 small enough. From this fact and Proposition 6.6.4 (1), we obtain a solution $v_\epsilon := i_\epsilon(V_\epsilon)$ of (6.1) for $\epsilon \in (0, \epsilon_0)$.

Also, by using the dilation invariance of (6.7) and the trace inequality (6.9), we see that $v_\epsilon = \sum_{i=1}^k P_1 w_{\epsilon^{\alpha_0}\lambda_i^\epsilon, \sigma_i^\epsilon} + \widetilde{\phi}^\epsilon_{\lambda^\epsilon,\sigma^\epsilon}$ in Ω where $\|\widetilde{\phi}^\epsilon_{\lambda^\epsilon,\sigma^\epsilon}\|_{L^{\frac{2n}{n-2s}}(\Omega)} \le C\|\Phi^\epsilon_{\lambda^\epsilon,\sigma^\epsilon}\|_{\mathcal{C}_\epsilon} = O(\epsilon^{\eta_0})$ ($\eta_0 > 0$ is chosen in (6.22)). From this fact, if we test (6.1) with $\widetilde{\phi}^\epsilon_{\lambda,\sigma}$ and use (6.11), we can deduce $\|\widetilde{\phi}^\epsilon_{\lambda,\sigma}\|_{H^s(\Omega)} = o(1)$. Furthermore, it is obvious that there exists a point $(\boldsymbol{\lambda}^0, \boldsymbol{\sigma}^0) \in \Lambda_k$ such that $(\boldsymbol{\lambda}^\epsilon, \boldsymbol{\sigma}^\epsilon) \to (\boldsymbol{\lambda}^0, \boldsymbol{\sigma}^0)$ up to a subsequence. This completes the proof of Theorem 6.1.4. □

Proof of Theorem 6.1.5. We recall that G and τ are Green's function and the Robin function of $\mathcal{A}_s$ in Ω with the zero Dirichlet boundary condition, respectively (see (6.4) and (6.5)). To emphasize the dependence of G and τ on the domain Ω, we append the subscript Ω in G and τ so that $G = G_\Omega$ and $\tau = \tau_\Omega$.

If a sequence of domains $\{\Omega_\epsilon : \epsilon > 0\}$ satisfies $\lim_{\epsilon\to0} \Omega_\epsilon = \Omega$ and $\Omega_{\epsilon_1} \subset \Omega_{\epsilon_2}$ for any $\epsilon_1 < \epsilon_2$, then τ_{Ω_ϵ} converges to τ_Ω in $C^1_{\text{loc}}(\Omega)$. In order to prove this statement, we first note that the maximum principle (Lemma 6.2.1, cf. [T2, Lemma 3.3]) ensures that τ_{Ω_ϵ} is monotone increasing as $\epsilon \to 0$ and tends to τ_Ω pointwise. Then we can deduce from Lemma 6.2.4 that it converges also in C^1 on any compact set of Ω. Similar arguments also apply to show that $G_{\Omega_\epsilon}(x, y)$ converges to $G_\Omega(x, y)$ in C^1 locally on $\{(x, y) \in \Omega_\epsilon^2 : x \neq y\}$. The rest part of the proof goes along the same way to [MP] or [EGP], where the authors considered domains Ω_ϵ consisting of k disjoint balls and thin strips liking them whose widths are ϵ. □

6.7 The subcritical problem

We are now concerned in the proofs of Theorem 6.1.6 and Theorem 6.1.7. Since many steps of the proofs for the previous theorems can be modified easily for problem (6.12), we only stress the parts where some different arguments should be introduced.

Remind that $\mu_\epsilon = \mathfrak{c}_{n,s}^{-1} \sup_{x\in\Omega} u_\epsilon(x)$ and $x_\epsilon \in \Omega$ is a point which satisfies $\mu_\epsilon = \mathfrak{c}_{n,s}^{-1} u_\epsilon(x_\epsilon)$. (See Lemma 6.3.2.) We also define the functions b_ϵ and B_ϵ with their domains Ω_ϵ and C_ϵ as in (6.9) and (6.10), replacing the scaling factor $\frac{2}{n-2s} = \frac{p-1}{2s}$ by $\frac{p-1-\epsilon}{2s}$. Then b_ϵ converges to w_1 pointwisely.

In order to get the uniform boundedness result, we first need the following bound of μ_ϵ.

Lemma 6.7.1. *There exists a constant* $C > 0$ *such that*

$$C \le \mu_\epsilon^{-(\frac{n}{2s}-1)\epsilon} \quad \textit{for all } \epsilon > 0. \tag{6.1}$$

Proof. Since b_ϵ converges to w_1 pointwise, we have

$$\int_{B_n(0,1)} b_\epsilon^{p+1-\epsilon} \ge C$$

by Fatou's lemma. Note that

$$\begin{aligned}\int_{B_n(0,1)} b_\epsilon^{p+1-\epsilon}(x)dx &= \int_{B_n(0,1)} \mu_\epsilon^{-(p+1-\epsilon)} u_\epsilon^{p+1-\epsilon}\left(\mu_\epsilon^{-\frac{p-1-\epsilon}{2s}} x + x_\epsilon\right) dx \\ &\leq \int_\Omega \mu_\epsilon^{-(p+1-\epsilon)} \mu_\epsilon^{\frac{n}{2s}(p-1-\epsilon)} u_\epsilon^{p+1-\epsilon}(x)dx \\ &\leq C\mu_\epsilon^{-(\frac{n}{2s}-1)\epsilon}.\end{aligned}$$

Combining these two estimates completes the proof. □

Next, as before we denote by d_ϵ and D_ϵ the Kelvin transforms of b_ϵ and B_ϵ (see (6.3) and (6.4)). Then the function D_ϵ satisfies

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla D_\epsilon)(z) = 0 & \text{in } \kappa(C_\epsilon), \\ \partial_\nu^s D_\epsilon = |x|^{-\epsilon(n-2s)} D_\epsilon^{p-\epsilon} & \text{in } \kappa(\Omega_\epsilon \times \{0\}), \\ D_\epsilon > 0 & \text{in } \kappa(C_\epsilon), \\ D_\epsilon = 0 & \text{on } \kappa(\partial_L C_\epsilon). \end{cases}$$

From (6.9), we have $|x| \geq C\mu_\epsilon^{-\frac{p-1-\epsilon}{2s}}$ for $x \in \kappa(\Omega_\epsilon)$, hence Lemma 6.7.1 yields

$$|x|^{-\epsilon(n-2s)} \leq \mu_\epsilon^{\frac{(p-1-\epsilon)}{2s}(n-2s)\epsilon} \leq C \quad \text{for all } x \in \kappa(\Omega_\epsilon).$$

By this fact we may use Lemma 6.4.2 and Lemma 6.4.5 and the proof of Proposition 6.4.1 to find $C > 0$ such that

$$u_\epsilon(x) \leq Cw_{\mu_\epsilon^{-\frac{2}{n-2s}}, x_\epsilon}(x) \quad \text{for all } \epsilon > 0 \text{ and } x \in \Omega. \tag{6.2}$$

Now we need to get a sharpened bound of μ_ϵ. Considering both (6.2) with Proposition 6.4.7 simultaneously, we can prove the following lemma.

Lemma 6.7.2. *(1) There exists a constant $C > 0$ such that*

$$\epsilon \leq C\mu_\epsilon^{-2-2\epsilon} \quad \textit{for } \epsilon > 0 \textit{ small.}$$

(2) We have

$$\lim_{\epsilon\to 0} \mu_\epsilon^\epsilon = 1. \tag{6.3}$$

Proof. As in the proof of Lemma 6.4.8, we take a small number $\delta > 0$. Recall also the definition of $\mathcal{I}(\Omega, r)$ and $\mathcal{O}(\Omega, r)$ (see (6.3) and (6.4)). Then we see that the left-hand side of (6.20) is bounded below, i.e.,

$$\left(\frac{n}{p+1-\epsilon} - \frac{n-2s}{2}\right) \int_{\mathcal{I}(\Omega,\delta)\times\{0\}} |U_\epsilon(x,0)|^{p+1-\epsilon} dx \geq C\epsilon \tag{6.4}$$

for some constant $C > 0$.

On the other hand, using (6.2) we deduce

$$\begin{aligned}\left(\int_\Omega u_\epsilon^{p-\epsilon} dx\right)^2 &\le C\left(\int_{\mathbb{R}^n} w^{p-\epsilon}_{\mu_\epsilon^{-\frac{2}{n-2s}},x_\epsilon}(x)dx\right)^2 \\ &\le C\mu_\epsilon^{-2(p-\epsilon)}\left(\int_{\mathbb{R}^n}\left(\mu_\epsilon^{-\frac{4}{n-2s}}+|x|^2\right)^{-\frac{n-2s}{2}(p-\epsilon)}dx\right)^2 \\ &\le C\mu_\epsilon^{-2-2\epsilon}.\end{aligned} \tag{6.5}$$

Since $x_0 \notin \mathcal{O}(\Omega, 2\delta)$ we have $w_{\mu_\epsilon,x_\epsilon}(x) \le C\mu_\epsilon^{-1}$ for $x \in \mathcal{O}(\Omega,\delta)$. It yields, for a fixed large number $q > 0$, that

$$\left(\int_{\mathcal{O}(\Omega,\delta)} u_\epsilon^{(p-\epsilon)q} dx\right)^{2/q} \le C\mu_\epsilon^{-2(p-\epsilon)}. \tag{6.6}$$

Now we inject the estimates (6.4), (6.5) and (6.6) to the inequality in the statement of Proposition 6.4.7 to get

$$C\epsilon \le C\left[\mu_\epsilon^{-2-2\epsilon}+\mu_\epsilon^{-2(p-\epsilon)}\right] \le 2C\mu_\epsilon^{-2-2\epsilon},$$

which proves the first statement of the lemma. Using Taylor's theorem, we get

$$|\mu_\epsilon^\epsilon - 1| \le \sup_{0\le t\le 1} \epsilon\mu_\epsilon^{t\epsilon}\log(\mu_\epsilon) = O(\mu_\epsilon^{-1-2\epsilon}\log(\mu_\epsilon)).$$

It proves $\lim_{\epsilon\to 0}\mu_\epsilon^\epsilon = 1$ because μ_ϵ goes to infinity. Now the proof is complete. □

We now prove Theorems 6.1.6 and 6.1.7.

Proof of Theorem 6.1.6. By definition we have

$$\mathcal{A}_s(\|u_\epsilon\|_{L^\infty(\Omega)}u_\epsilon)(x) = \mathfrak{c}_{n,s}\mu_\epsilon u_\epsilon^{p-\epsilon}(x).$$

Note from $p = \frac{n+2s}{n-2s}$ that

$$\int_\Omega \mathfrak{c}_{n,s}\mu_\epsilon u_\epsilon^{p-\epsilon}(x)dx = \int_\Omega \mathfrak{c}_{n,s}\mu_\epsilon^{p+1-\epsilon} b_\epsilon^{p-\epsilon}\left(\mu_\epsilon^{\frac{p-1-\epsilon}{2s}}(x-x_\epsilon)\right)dx = \int_{\Omega_\epsilon} \mathfrak{c}_{n,s}\mu_\epsilon^{(\frac{n}{2s}-1)\epsilon} b_\epsilon^{p-\epsilon}(x)dx.$$

Here, from Lemma 6.7.2 and the dominated convergence theorem with the fact that b_ϵ converges to w_1 pointwise, we conclude that

$$\lim_{\epsilon\to 0}\int_\Omega \mathfrak{c}_{n,s}\mu_\epsilon u_\epsilon^{p-\epsilon}(x)dx = \int_{\mathbb{R}^n}\mathfrak{c}_{n,s} w_1^p(x)dx = \mathfrak{b}_{n,s}$$

(see (6.31)). Now the first statement follows as in the proof of Theorem 6.1.1.

The proof of the second statement can be performed similarly to the proof of Theorem 6.1.2. The constant $\mathfrak{g}_{n,s}$ is given by

$$\mathfrak{g}_{n,s} = \frac{4n}{2C_s}\mathcal{S}_{n,s}^{n/s}\mathcal{D}_{n,s}\mathfrak{a}_{n,s}\mathfrak{b}_{n,s}^2. \tag{6.7}$$

The proof is complete. □

Proof of Theorem 6.1.7. This theorem can be proved in a similar way to the proof of Theorem 6.1.4. In this case, if we take $\alpha_0 = \frac{1}{n-2s}$ and $f_\epsilon(U) = U_+^{p-\epsilon}$, then an analogous result of Proposition 6.6.4 holds with $\widetilde{\Upsilon}$ (refer to (6.13)). Therefore there exists a family of solutions which concentrate at a critical point of $\widetilde{\Upsilon}$. □

Appendix

6.A Proof of Proposition 6.4.7

This section is devoted to present the proof of Proposition 6.4.7, namely, the following proposition.

Proposition 6.A.1. *Suppose that $U \in H^s_{0,L}(C)$ is a solution of problem* (6.9) *with f such that f has the critical growth and $f = F'$ for a function $F \in C^1(\mathbb{R})$. Then, for each $\delta > 0$ and $q > \frac{n}{s}$ there is a constant $C = C(\delta, q) > 0$ such that*

$$\begin{aligned} &\min_{r\in[\delta,2\delta]} \left| n \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} F(U)dx - \left(\frac{n-2s}{2}\right) \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} Uf(U)dx \right| \\ &\leq C\left[\left(\int_{\mathcal{O}(\Omega,2\delta)\times\{0\}} |f(U)|^q dx \right)^{\frac{2}{q}} + \int_{\mathcal{O}(\Omega,2\delta)\times\{0\}} |F(U)|dx + \left(\int_{\mathcal{I}(\Omega,\delta/2)\times\{0\}} |f(U)|dx \right)^2 \right] \end{aligned} \tag{6.8}$$

where $\mathcal{I}$ and $\mathcal{O}$ is defined in (6.3) *and* (6.4).

Proof. Recall the local form of the Pohozaev identity

$$\operatorname{div}\left\{ t^{1-2s}\langle z, \nabla U\rangle \nabla U - t^{1-2s}\frac{|\nabla U|^2}{2} z \right\} + \left(\frac{n-2s}{2}\right) t^{1-2s}|\nabla U|^2 = 0 \tag{6.9}$$

and define the following sets:

$$D_r = \left\{ z \in \mathbb{R}^{n+1}_+ : \operatorname{dist}(z, \mathcal{I}(\Omega, r)\times\{0\}) \leq r/2 \right\},$$

$$\partial D_r^+ = \partial D_r \cap \left\{ (x,t) \in \mathbb{R}^{n+1} : t > 0 \right\} \quad \text{and} \quad E_\delta = \bigcup_{r=\delta}^{2\delta} \partial D_r^+.$$

Note that $\partial D_r = \partial D_r^+ \cup (\mathcal{I}(\Omega, r/2)\times\{0\})$. Fix a small number $\delta > 0$. We integrate the identity (6.9) over D_r for each $r \in (0, 2\delta]$ to derive

$$\begin{aligned} \int_{\partial D_r^+} t^{1-2s} \left\langle \langle z, \nabla U\rangle \nabla U - z\frac{|\nabla U|^2}{2}, \nu \right\rangle dS + C_s \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} \langle x, \nabla_x U\rangle \partial^s_\nu U dx \\ = -\left(\frac{n-2s}{2}\right) \int_{D_r} t^{1-2s}|\nabla U|^2 dxdt. \end{aligned} \tag{6.10}$$

In view of Lemmas 4.4 and 4.5 of [CS], one can deduce that the i-th component $\partial_{x_i} U$ of $\nabla_x U$ is Hölder continuous in $\overline{D_r}$ for each $i = 1, \cdots, n$, which justifies the above formula. By using $\partial_\nu^s U = f(U)$ and performing integration by parts, we derive

$$\begin{aligned}\int_{\mathcal{I}(\Omega,r/2)\times\{0\}} \langle x, \nabla_x U\rangle \partial_\nu^s U dx &= \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} \langle x, \nabla_x F(U)\rangle dx \\ &= -n \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} F(U)dx + \int_{\partial\mathcal{I}(\Omega,r/2)\times\{0\}} \langle x, \nu\rangle F(U) dS_x\end{aligned}$$

and

$$\int_{D_r} t^{1-2s}|\nabla U|^2 dxdt = C_s \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} Uf(U)dx + \int_{\partial D_r^+} t^{1-2s} U \frac{\partial U}{\partial \nu} dS.$$

Then (6.10) is written as

$$\begin{aligned}C_s &\left\{ n \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} F(U)dx - \left(\frac{n-2s}{2}\right) \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} Uf(U)dx \right\} \\ &= \int_{\partial D_r^+} t^{1-2s} \left[\left\langle \langle z, \nabla U\rangle \nabla U - z\frac{|\nabla U|^2}{2}, \nu \right\rangle dS + \left(\frac{n-2s}{2}\right) U \frac{\partial U}{\partial \nu} \right] dS \\ &+ \int_{\partial\mathcal{I}(\Omega,r/2)\times\{0\}} \langle x, \nu\rangle F(U) dS_x.\end{aligned} \tag{6.11}$$

From this identity we get

$$\begin{aligned}&\left| n \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} F(U)dx - \left(\frac{n-2s}{2}\right) \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} Uf(U)dx \right| \\ &\qquad \le C \int_{\partial D_r^+} t^{1-2s} (|\nabla U|^2 + U^2) dS + \int_{\partial\mathcal{I}(\Omega,r/2)\times\{0\}} \langle x, \nu\rangle F(U) dS_x.\end{aligned}$$

We integrate this identity with respect to r over an interval $[\delta, 2\delta]$ and then use the Poincaré inequality. Then we observe

$$\begin{aligned}\min_{r\in[\delta,2\delta]} &\left| n \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} F(U)dx - \left(\frac{n-2s}{2}\right) \int_{\mathcal{I}(\Omega,r/2)\times\{0\}} Uf(U)dx \right| \\ &\qquad \le C \int_{E_\delta} t^{1-2s} |\nabla U|^2 dz + C \int_{\mathcal{O}(\Omega,\delta)} |F(U)(x,0)| dx.\end{aligned}$$

We only need to estimate the first term of the right-hand side of the previous inequality since the second term is already one of the terms which constitute the right-hand side of (6.8). Note that

$$\nabla_z U(z) = \int_\Omega \nabla_z G_{\mathbb{R}^{n+1}_+}(z,y) f(U)(y,0)dy - \int_\Omega \nabla_z H_{\mathcal{C}}(z,y) f(U)(y,0)dy \tag{6.12}$$

for $z \in E_\delta$.

Let us deal with the last term of (6.12) first. Admitting the estimation

$$\sup_{y\in\Omega}\int_{E_\delta} t^{1-2s}|\nabla_z H_C(z,y)|^2 dz \le C \tag{6.13}$$

for a while and using Hölder's inequality, we get

$$\begin{aligned}&\int_{E_\delta} t^{1-2s}\left(\int_\Omega |\nabla_z H_C(z,y)f(U)(y,0)|dy\right)^2 dz\\ &\le \left(\sup_{y\in\Omega}\int_{E_\delta} t^{1-2s}|\nabla_z H_C(z,y)|^2 dz\right)\left(\int_\Omega |f(U)(y,0)|dy\right)^2 \le C\left(\int_{\mathcal{I}(\Omega,\delta)\cup\mathcal{O}(\Omega,\delta)} |f(U)(y,0)|dy\right)^2 \\ &\le C\left[\left(\int_{\mathcal{O}(\Omega,2\delta)} |f(U)(y,0)|^q dy\right)^{\frac{2}{q}} + \left(\int_{\mathcal{I}(\Omega,\delta/2)} |f(U)(y,0)|dy\right)^2\right],\end{aligned} \tag{6.14}$$

which is a part of the right-hand side of (6.8).

The validity of (6.13) can be reasoned as follows. First of all, if y is a point in Ω such that $\mathrm{dist}(y,E_\delta)\le\delta/2$, then it automatically satisfies that $\mathrm{dist}(y,\partial\Omega)\ge\delta/2$ from which we know

$$\sup_{\mathrm{dist}(y,\partial\Omega)\ge\delta/2}\left(\int_{E_\delta} t^{1-2s}|\nabla_z H_C(z,y)|^2 dz\right) \le \sup_{\mathrm{dist}(y,\partial\Omega)\ge\delta/2}\left(\int_{C} t^{1-2s}|\nabla_z H_C(z,y)|^2 dz\right) \le C.$$

See the proof of Lemma 6.2.2 for the second inequality. Meanwhile, in the complementary case $\mathrm{dist}(y,E_\delta)>\delta/2$, we can assert that

$$\int_{E_\delta} t^{1-2s}|\nabla_z H_C(z,y)|^2 dz \le C\left(\int_{N(E_\delta,\delta/4)} t^{1-2s}|H_C(z,y)|^2 dz\right) \tag{6.15}$$

where $N(E_\delta,\delta/4) := \{z\in C : \mathrm{dist}(z,E_\delta)\le\delta/4\}$. To show this, we recall that H_C satisfies

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla H_C(\cdot,y)) = 0 & \text{in } C,\\ \partial_\nu^s H_C(\cdot,y) = 0 & \text{on } \Omega\times\{0\}.\end{cases} \tag{6.16}$$

Fix a smooth function $\phi\in C_0^\infty(N(E_\delta,\delta/4))$ such that $\phi=1$ on E_δ and $|\nabla\phi|^2\le C_0\phi$ holds for some $C_0>0$, and multiply $H_C(\cdot,y)\phi(\cdot)$ to (6.16). Then we have

$$\int_C t^{1-2s}|\nabla H_C(z,y)|^2\phi(z) + \int_C t^{1-2s}[\nabla H_C(z,y)\cdot\nabla\phi(z)]H_C(z,y)dz = 0.$$

From this we deduce that

$$\begin{aligned}&\int_C t^{1-2s}|\nabla H_C(z,y)|^2\phi(z)dz\\ &= -\int_C t^{1-2s}[\nabla H_C(z,y)\cdot\nabla\phi(z)]H_C(z,y)dz\\ &\le \frac{1}{2C_0}\int_C t^{1-2s}|\nabla H_C(z,y)|^2|\nabla\phi(z)|^2 dz + 2C_0\int_{N(E_\delta,\delta/4)} t^{1-2s}|H_C(z,y)|^2 dz.\end{aligned}$$

Using the property $|\nabla\phi|^2 \le C_0\phi$ we derive that

$$\int_C t^{1-2s}|\nabla H_C(z,y)|^2\phi(z)dz \le 4C_0\int_{N(E_\delta,\delta/4)} t^{1-2s}|H_C(z,y)|^2dz.$$

It verifies inequality (6.15). Since the assumption $\text{dist}(y,E_\delta) > \delta/2$ implies $\text{dist}(y,N(E_\delta,\delta/4)) > \delta/4$, it holds

$$\sup_{\text{dist}(y,E_\delta)>\delta/2}\ \sup_{z\in N(E_\delta,\delta/4)} |H_C(z,y)| \le \sup_{\text{dist}(y,E_\delta)>\delta/2}\ \sup_{z\in N(E_\delta,\delta/4)} |G_{\mathbb{R}^{n+1}_+}(z,y)| \le C.$$

Combination of this and (6.15) gives

$$\sup_{\text{dist}(y,E_\delta)>\delta/2}\left(\int_{E_\delta} t^{1-2s}|\nabla_z H_C(z,y)|^2dz\right) \le C\left(\int_{N(E_\delta,\delta/4)} t^{1-2s}dz\right) \le C.$$

This concludes the derivation of the desired uniform bound (6.13).

It remains to take into consideration of the first term of (6.12). We split the term as

$$\begin{aligned}&\int_\Omega \nabla_z G_{\mathbb{R}^{n+1}_+}(z,y)f(U)(y,0)dy\\ &= \int_{O(\Omega,2\delta)} \nabla_z G_{\mathbb{R}^{n+1}_+}(z,y)f(U)(y,0)dy + \int_{\mathcal{I}(\Omega,2\delta)} \nabla_z G_{\mathbb{R}^{n+1}_+}(z,y)f(U)(y,0)dy\\ &:= A_1(z) + A_2(z).\end{aligned}$$

Take $q > \frac{n}{s}$ and $r > 1$ satisfying $\frac{1}{q} + \frac{1}{r} = 1$. Then

$$|A_1(z)| \le \left(\int_{O(\Omega,2\delta)} |\nabla_z G_{\mathbb{R}^{n+1}_+}(z,y)|^r dy\right)^{\frac{1}{r}} \|f(U)(\cdot,0)\|_{L^q(O(\Omega,2\delta))}.$$

In light of the definition of $G_{\mathbb{R}^{n+1}_+}$, it holds that

$$\begin{aligned}\left(\int_{O(\Omega,2\delta)} |\nabla_z G_{\mathbb{R}^{n+1}_+}(z,y)|^r dy\right)^{\frac{1}{r}} &\le C\left(\int_{O(\Omega,2\delta)} \frac{1}{|(x-y,t)|^{(n-2s+1)r}}dy\right)^{\frac{1}{r}}\\ &\le C\max\left\{t^{\frac{n}{r}-(n-2s+1)}, 1\right\} = C\max\left\{t^{-\frac{n}{q}+2s-1}, 1\right\}.\end{aligned}$$

Thus we have

$$|A_1(z)| \le C\max\left\{t^{-\frac{n}{q}+2s-1}, 1\right\} \|f(U)(\cdot,0)\|_{L^q(O(\Omega,2\delta))}.$$

Using this we see

$$\begin{aligned}\int_{E_\delta} t^{1-2s}|A_1(z)|^2dz &\le C\int_0^1 \max\left\{t^{1-2s}t^{-\frac{2n}{q}+4s-2}, t^{1-2s}\right\} \|f(U)(\cdot,0)\|^2_{L^q(O(\Omega,2\delta))}dt\\ &= \int_0^1 \max\left\{t^{2s-\frac{2n}{q}-1}, t^{1-2s}\right\} \|f(U)(\cdot,0)\|^2_{L^q(O(\Omega,2\delta))}dt.\\ &\le C\|f(U)(\cdot,0)\|^2_{L^q(O(\Omega,2\delta))}.\end{aligned} \tag{6.17}$$

Concerning the term A_2, we note that E_δ is away from $\mathcal{I}(\Omega, 2\delta) \times \{0\}$. Thus we have

$$\sup_{z \in E_\delta, y \in \mathcal{I}(\Omega, 2\delta)} |\nabla_z G_{\mathbb{R}^{n+1}_+}(z, y)| \leq C.$$

Hence

$$|A_2(z)| \leq C \int_{\mathcal{I}(\Omega, 2\delta)} |f(U)(y, 0)| dy, \quad z \in E_\delta.$$

Using this we find

$$\int_{E_\delta} t^{1-2s} |A_2(z)|^2 dz \leq C \left(\int_{\mathcal{I}(\Omega, 2\delta)} |f(U)(y, 0)| dy \right)^2. \tag{6.18}$$

We have obtained the desired bound of $\int_{E_\delta} t^{1-2s} |\nabla U|^2 dz$ through the estimates (6.14), (6.17) and (6.18). The proof is complete. □

Remark 6.A.2. Estimate (6.13) can be generalized to

$$\sup_{y \in \Omega} \int_{E_\delta} t^{1-2s} |\nabla_z \partial_y^I H_C(z, y)|^2 dz \leq C, \tag{6.19}$$

for any multi-index $I \in (\mathbb{N} \cup \{0\})^n$. The proof of this fact follows in the same way as the derivation of (6.13) with an observation that $\partial_y^I H_C(\cdot, y)$ satisfies equation (6.14).

6.B Technical computations in the proof of Theorem 6.1.4

In this section, we collect technical lemmas which are necessary during the proof of Theorem 6.1.4.

6.B.1 Estimation of the projected bubbles

We recall the functions $w_{\lambda,\xi}$, $\psi^j_{\lambda,\xi}$, $P_\epsilon w_{\lambda,\xi}$ and $P_\epsilon \psi^j_{\lambda,\xi}$ defined in (6.5), (6.9) and (6.10) for any $\lambda > 0$, $\xi \in \mathbb{R}^n$ and $j = 0, \cdots, n$.

In the next two lemmas, we estimate the difference between $w_{\lambda,\xi}$ and $P_\epsilon w_{\lambda,\xi}$ (or $\psi^j_{\lambda,\xi}$ and $P_\epsilon \psi^j_{\lambda,\xi}$) in terms of Green's function G and its regular part H of the fractional Laplacian $\mathcal{A}_s$ (see (6.4) and (6.5) for their definitions).

Lemma 6.B.1. *Let $\lambda > 0$ and $\sigma = (\sigma^1, \cdots, \sigma^n) \in \Omega$. Then we have*

$$\begin{aligned}
P_\epsilon w_{\lambda,\sigma\epsilon^{-\alpha_0}}(x) &= w_{\lambda,\sigma\epsilon^{-\alpha_0}}(x) - c_1 \lambda^{\frac{n-2s}{2}} H(\epsilon^{\alpha_0} x, \sigma) \epsilon^{(n-2s)\alpha_0} + o(\epsilon^{(n-2s)\alpha_0}), \\
P_\epsilon \psi^j_{\lambda,\sigma\epsilon^{-\alpha_0}}(x) &= \psi^j_{\lambda,\sigma\epsilon^{-\alpha_0}}(x) - c_1 \lambda^{\frac{n-2s}{2}} \frac{\partial H}{\partial \sigma^j}(\epsilon^{\alpha_0} x, \sigma) \epsilon^{(n-2s+1)\alpha_0} + o(\epsilon^{(n-2s+1)\alpha_0}), \\
P_\epsilon \psi^0_{\lambda,\sigma\epsilon^{-\alpha_0}}(x) &= \psi^0_{\lambda,\sigma\epsilon^{-\alpha_0}}(x) - \frac{(n-2s)c_1}{2} \lambda^{\frac{n-2s-2}{2}} H(\epsilon^{\alpha_0} x, \sigma) \epsilon^{(n-2s)\alpha_0} + o(\epsilon^{(n-2s)\alpha_0})
\end{aligned}$$

for all $x \in \Omega_\epsilon$ where $c_1 > 0$ is the constant defined in (6.8). *Here the little o terms tend to zero as $\epsilon \to 0$ uniformly in $x \in \Omega_\epsilon$ and $\sigma \in \Omega$ provided dist$(\sigma, \partial\Omega) > \overline{C}$ for some constant $\overline{C} > 0$.*

Proof. For fixed $\lambda > 0$ and $\sigma \in \Omega$, let $\Phi_{\lambda,\sigma\epsilon^{-\alpha_0}} = W_{\lambda,\sigma\epsilon^{-\alpha_0}} - P_\epsilon W_{\lambda,\sigma\epsilon^{-\alpha_0}}$. Then $\Phi_{\lambda,\sigma\epsilon^{-\alpha_0}}$ satisfies

$$\begin{cases} \operatorname{div}\left(t^{1-2s}\nabla\Phi_{\lambda,\sigma\epsilon^{-\alpha_0}}\right) = 0 & \text{in } \mathcal{C}_\epsilon, \\ \Phi_{\lambda,\sigma\epsilon^{-\alpha_0}} = W_{\lambda,\sigma\epsilon^{-\alpha_0}} & \text{on } \partial_L \mathcal{C}_\epsilon, \\ \partial_\nu^s \Phi_{\lambda,\sigma\epsilon^{-\alpha_0}} = 0 & \text{on } \Omega_\epsilon \times \{0\}. \end{cases}$$

On the other hand, the function $\mathcal{F}(z) := c_1 \lambda^{\frac{n-2s}{2}} H_C(\epsilon^{\alpha_0} z, \sigma)\epsilon^{(n-2s)\alpha_0}$ defined for $z \in \mathcal{C}_\epsilon$ solves

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla\mathcal{F}) = 0 & \text{in } \mathcal{C}_\epsilon, \\ \mathcal{F}(z) = \epsilon^{(n-2s)\alpha_0} c_1 \lambda^{\frac{n-2s}{2}} G_{\mathbb{R}^{n+1}_+}(\epsilon^{\alpha_0} z, \sigma) & \text{on } \partial_L \mathcal{C}_\epsilon, \\ \partial_\nu^s \mathcal{F} = 0 & \text{on } \Omega_\epsilon \times \{0\}. \end{cases}$$

Note that

$$\begin{aligned} W_{\lambda,\sigma\epsilon^{-\alpha_0}}(x,t) &= \int_{\mathbb{R}^n} G_{\mathbb{R}^{n+1}_+}(x,t,y) W^p_{\lambda,\sigma\epsilon^{-\alpha_0}}(y,0)dy \\ &= \mathfrak{c}^p_{n,s} \int_{\mathbb{R}^n} G_{\mathbb{R}^{n+1}_+}(x,t,y) \frac{\lambda^{\frac{n+2s}{2}}}{|(y-\sigma\epsilon^{-\alpha_0},\lambda)|^{n+2s}} dy \\ &= \mathfrak{c}^p_{n,s} \int_{\mathbb{R}^n} \lambda^{\frac{n-2s}{2}} G_{\mathbb{R}^{n+1}_+}(x,t,\lambda y + \sigma\epsilon^{-\alpha_0}) \frac{1}{|(y,1)|^{n+2s}} dy \quad \text{for } (x,t) \in \mathbb{R}^{n+1}_+. \end{aligned}$$

For $(x,t) \in \partial_L \mathcal{C}$, we calculate

$$\begin{aligned} W_{\lambda,\sigma\epsilon^{-\alpha_0}}(x\epsilon^{-\alpha_0}, t\epsilon^{-\alpha_0}) &= \mathfrak{c}^p_{n,s} \int_{\mathbb{R}^n} \lambda^{\frac{n-2s}{2}} G_{\mathbb{R}^{n+1}_+}((x-\sigma)\epsilon^{-\alpha_0}, t\epsilon^{-\alpha_0}, \lambda y) \frac{1}{|(y,1)|^{n+2s}} dy \\ &= \epsilon^{(n-2s)\alpha_0} \mathfrak{c}^p_{n,s} \int_{\mathbb{R}^n} \lambda^{\frac{n-2s}{2}} G_{\mathbb{R}^{n+1}_+}(x-\sigma, t, \lambda y) \frac{\epsilon^{-n\alpha_0}}{|(y\epsilon^{-\alpha_0},1)|^{n+2s}} dy \\ &= \epsilon^{(n-2s)\alpha_0} c_1 \lambda^{\frac{n-2s}{2}} G_{\mathbb{R}^{n+1}_+}(x-\sigma, t, 0) + o(\epsilon^{(n-2s)\alpha_0}). \end{aligned}$$

As $\epsilon > 0$ goes to 0, the term $o(\epsilon^{(n-2s)\alpha_0})$ above converges to 0 uniformly in $(x,t) \in \partial_L \mathcal{C}$ and $\sigma \in \Omega$ satisfying $\operatorname{dist}(\sigma, \partial\Omega) > \overline{C}$.

On the other hand, we have

$$\mathcal{F}(x\epsilon^{-\alpha_0}, t\epsilon^{-\alpha_0}) = \epsilon^{(n-2s)\alpha_0} c_1 \lambda^{\frac{n-2s}{2}} G_{\mathbb{R}^{n+1}_+}(x,t,\sigma) = \epsilon^{(n-2s)\alpha_0} c_1 \lambda^{\frac{n-2s}{2}} G_{\mathbb{R}^{n+1}_+}(x-\sigma, t, 0).$$

Thus

$$\sup_{(x\epsilon^{-\alpha_0}, t\epsilon^{-\alpha_0}) \in \partial_L \mathcal{C}} |\Psi_{\lambda,\sigma\epsilon^{-\alpha_0}}(x\epsilon^{-\alpha_0}, t\epsilon^{-\alpha_0}) - \mathcal{F}(x\epsilon^{-\alpha_0}, t\epsilon^{-\alpha_0}, \sigma)| = o(\epsilon^{(n-2s)\alpha_0}).$$

By the maximum principle (Lemma 6.2.1), we get

$$\sup_{z \in \mathcal{C}_\epsilon} |\Psi_{\lambda,\sigma\epsilon^{-\alpha_0}}(z) - \mathcal{F}(z)| = o(\epsilon^{(n-2s)\alpha_0}).$$

By taking $z = (x,0)$ for $x \in \Omega_\epsilon$ we obtain $\sup_{x\in\Omega_\epsilon} |w_{\lambda,\sigma\epsilon^{-\alpha_0}}(x) - Pw_{\lambda,\sigma\epsilon^{-\alpha_0}}(x) - \mathcal{F}(x,0)| = o(\epsilon^{(n-2s)\alpha_0})$. Now the first identity follows from the definition of $\mathcal{F}$.

The second and third estimation can be proved similarly. □

From the above lemma, we immediately get the following lemma.

Lemma 6.B.2. *For any $\lambda > 0$ and $\sigma = (\sigma^1, \cdots, \sigma^n) \in \Omega$, we have*

$$\begin{aligned}
P_\epsilon w_{\lambda,\sigma\epsilon^{-\alpha_0}}(x) &= c_1 \lambda^{\frac{n-2s}{2}} G(\epsilon^{\alpha_0}x, \sigma)\epsilon^{(n-2s)\alpha_0} + o(\epsilon^{(n-2s)\alpha_0}) \\
P_\epsilon \psi^j_{\lambda,\sigma\epsilon^{-\alpha_0}}(x) &= c_1 \lambda^{\frac{n-2s}{2}} \frac{\partial G}{\partial \sigma^j}(\epsilon^{\alpha_0}x, \sigma)\epsilon^{(n-2s+1)\alpha_0} + o(\epsilon^{(n-2s+1)\alpha_0}) \\
P_\epsilon \psi^0_{\lambda,\sigma\epsilon^{-\alpha_0}}(x) &= \left(\frac{n-2s}{2}\right) c_1 \lambda^{\frac{n-2s-2}{2}} G(\epsilon^{\alpha_0}x, \sigma)\epsilon^{(n-2s)\alpha_0} + o(\epsilon^{(n-2s)\alpha_0}),
\end{aligned}$$

where the little o terms tend to zero uniformly in $x \in \Omega_\epsilon$ and $\sigma \in \Omega$ provided $|\epsilon^{\alpha_0}x - \sigma| > C$ and dist$(\epsilon^{\alpha_0}x, \partial\Omega) > C$ for a fixed constant $C > 0$. As the previous lemma, $c_1 > 0$ is the constant given in (6.8).

6.B.2 Basic estimates

Let w_i and ψ_i^j (for $i = 1, \cdots, k$ and $j = 0, \cdots, n$) be the functions given in (6.12). Then applying the definition of $w_{\lambda,\xi}$ in (6.5), Lemma 6.B.1 and the Sobolev trace inequality (6.9), we can deduce the following estimates. For the details, we refer to [MP] in which the authors deal with the case $s = 1$.

Lemma 6.B.3. *It holds that*

$$\|P_\epsilon w_i\|_{L^{\frac{2n}{n-2s}}(\Omega_\epsilon)} \le \|w_i\|_{L^{\frac{2n}{n-2s}}(\Omega_\epsilon)} \le C.$$

Also we have

$$\|P_\epsilon w_i\|_{L^{\frac{2n}{n+2s}}(\Omega_\epsilon)} \le \begin{cases} C & \text{if } n > 6s, \\ C\epsilon^{-(6s-n)\alpha_0/2}|\log \epsilon| & \text{if } n \le 6s. \end{cases}$$

Similarly,

$$\left\|P_\epsilon \psi_i^j\right\|_{L^{\frac{2n}{n-2s}}(\Omega_\epsilon)} \le C, \quad \left\|P_\epsilon \psi_i^j\right\|_{L^{\frac{2n}{n+2s}}(\Omega_\epsilon)} \le C \quad \text{if } j = 1, \cdots, n,$$

and

$$\left\|P_\epsilon \psi_i^0\right\|_{L^{\frac{2n}{n+2s}}(\Omega_\epsilon)} \le \begin{cases} C & \text{if } n > 6s, \\ C\epsilon^{-(6s-n)\alpha_0/2}|\log \epsilon| & \text{if } n \le 6s. \end{cases}$$

Lemma 6.B.4. *For $i = 1, \cdots, k$, we have*

$$\left\|P_\epsilon \psi_i^j - \psi_i^j\right\|_{L^{\frac{2n}{n-2s}}(\Omega_\epsilon)} \le C\epsilon^{\alpha_0\left(\frac{n}{2}-s+1\right)} \quad \text{if } j = 1, \cdots, n$$

and

$$\left\|P_\epsilon \psi_i^0 - \psi_i^0\right\|_{L^{\frac{2n}{n-2s}}(\Omega_\epsilon)} \le C\epsilon^{\alpha_0\frac{n-2s}{2}}.$$

Lemma 6.B.5. *It holds that*

$$\left\| \left(\sum_{i=1}^{k} P_\epsilon w_i \right)^p - \sum_{i=1}^{k} w_i^p \right\|_{L^{\frac{2n}{n+2s}}(\Omega_\epsilon)} \le \begin{cases} C\epsilon^{\frac{n+2s}{2}\alpha_0} & \text{if } n > 6s, \\ C\epsilon^{(n-2s)\alpha_0} |\log \epsilon| & \text{if } n \le 6s. \end{cases}$$

Besides,

$$\left\| \left(\sum_{i=1}^{k} P_\epsilon w_i \right)^{p-1} - \sum_{i=1}^{k} w_i^{p-1} \right\|_{L^{\frac{n}{2s}}(\Omega_\epsilon)} \le C\epsilon^{2s\alpha_0}.$$

and

$$\left\| \left[\left(\sum_{i=1}^{k} P_\epsilon w_i \right)^{p-1} - \sum_{i=1}^{k} w_i^{p-1} \right] P_\epsilon \psi_h^j \right\|_{L^{\frac{2n}{n+2s}}(\Omega_\epsilon)} \le C\epsilon^{\alpha_0 \frac{n+2s}{2}}$$

for $h = 1, \cdots, k$ *and* $j = 0, 1, \cdots, n$.

6.B.3 Proof of Proposition 6.6.4

This subsection is devoted to give a proof of Proposition 6.6.4.

Proof of Proposition 6.6.4. We first prove (1). Applying $\widetilde{E}'_\epsilon(\boldsymbol{\lambda}^\epsilon, \boldsymbol{\sigma}^\epsilon) = 0$, we can obtain after some computations that

$$\frac{\partial}{\partial \varrho} \widetilde{E}'_\epsilon(\boldsymbol{\lambda}^\epsilon, \boldsymbol{\sigma}^\epsilon) = \sum_{h=1}^{k} \sum_{j=0}^{n} c_{hj} \left[\left(P_\epsilon \Psi_h^j, \sum_{i=1}^{k} P_\epsilon \frac{\partial W_i}{\partial \varrho} \right)_{C_\epsilon} - \left(P_\epsilon \frac{\partial \Psi_h^j}{\partial \varrho}, \Phi^\epsilon_{\lambda^\epsilon, \sigma^\epsilon} \right)_{C_\epsilon} \right] = 0$$

where ϱ is one of λ_i or σ_i^j with $i = 1, \cdots, k$ and $j = 0, \cdots, n$ (see (6.7)). Using (6.17) and (6.22), we can conclude that $c_{hj} = 0$ for all h and j, which implies that the function U_ϵ defined in the statement of the proposition is a solution of (6.6). The assertion that V_ϵ is a solution of (6.1) is justified by the following sublemma provided $\epsilon > 0$ small.

Sublemma 6.B.6. *Suppose that* $U \in H^s_{0,L}(C)$ *is a solution of problem* (6.1) *with* U^p *substituted by* U_+^p *(here, the condition* $U > 0$ *in* C *is ignored). If* ϵ *is small, then there is a constant* $C > 0$ *depending only on* n *and* s*, such that the function* U *is positive.*

Proof. We multiply U_- by equation (6.6) with $\epsilon = 1$. Then we have

$$\int_C t^{1-2s} |\nabla U_-|^2 = \epsilon C_s \int_{\Omega \times \{0\}} U_-^2$$

(refer to (6.11)). By utilizing the Sobolev trace inequality and Hölder's inequality, we get

$$\|U_-(\cdot, 0)\|_{L^{\frac{2n}{n-2s}}(\Omega)} \le \epsilon C \|U_-(\cdot, 0)\|_{L^{\frac{2n}{n-2s}}(\Omega)}$$

for some $C > 0$ independent of U. Hence U_- should be zero given that ϵ is sufficiently small. The lemma is proved. ◻

The first part (1) of Proposition 6.6.4 is proved.

We continue our proof by considering the second part (2). By (6.22), there holds

$$\begin{aligned}\widetilde{E}_\epsilon(\boldsymbol{\lambda},\boldsymbol{\sigma}) &= E_\epsilon\left(\sum_{i=1}^k P_\epsilon W_{\lambda_i,\frac{\sigma_i}{\epsilon^{\alpha_0}}}\right) + O\left(\epsilon^{2\eta_0}\right) = E_\epsilon\left(\sum_{i=1}^k P_\epsilon W_i\right) + o\left(\epsilon^{(n-2s)\alpha_0}\right) \\ &= \frac{1}{2C_s}\int_{C_\epsilon} t^{1-2s}\left|\nabla\left(\sum_{i=1}^k P_\epsilon W_i\right)\right|^2 - \int_{\Omega_\epsilon} F_\epsilon\left(i_\epsilon\left(\sum_{i=1}^k P_\epsilon W_i\right)\right) + o\left(\epsilon^{(n-2s)\alpha_0}\right) \qquad (6.20)\end{aligned}$$

so it suffices to estimate each of the two terms that appear in (6.20) above.

Setting $B_i = B_n(\sigma_i, \delta_0/2) \subset \Omega$ where δ_0 is a small number chosen in the definition (6.7) of O^{δ_0}, and applying Lemma 6.B.1 and Lemma 6.B.2, we find that

$$\begin{aligned}&\int_{\Omega_\epsilon} w_i^p P_\epsilon w_i = \int_{\Omega_\epsilon} w_1^{p+1} + \int_{\Omega_\epsilon} w_i^p(P_\epsilon w_i - w_i) = c_0 - c_1^2\lambda_i^{n-2s}H(\sigma_i,\sigma_i)\epsilon^{(n-2s)\alpha_0} + o(\epsilon^{(n-2s)\alpha_0}),\\ &\int_{\Omega_\epsilon} w_h^p P_\epsilon w_i = \int_{\frac{B_i}{\epsilon^{\alpha_0}}} w_h^p P_\epsilon w_i + o(\epsilon^{(n-2s)\alpha_0}) = c_1^2(\lambda_h\lambda_i)^{\frac{n-2s}{2}}G(\sigma_h,\sigma_i)\epsilon^{(n-2s)\alpha_0} + o(\epsilon^{(n-2s)\alpha_0}),\\ &\int_{\Omega_\epsilon} w_i P_\epsilon w_i = \int_{\Omega_\epsilon} w_i^2 + o(1) = c_2\lambda_i^{2s} + o(1) \quad (\text{if } n > 4s),\\ &\int_{\Omega_\epsilon} w_h P_\epsilon w_i = o(1) \quad (\text{if } n > 4s),\end{aligned}$$

for $i, h = 1, \cdots, k$ and $i \neq h$, where G and H are the functions defined in (6.4) and (6.5), and c_1 and c_2 are positive constants given in (6.8) while c_0 is defined in (6.26).

Then the estimates obtained in the previous paragraph yield that

$$\begin{aligned}&\frac{1}{2C_s}\int_{C_\epsilon} t^{1-2s}\left|\nabla\left(\sum_{i=1}^k P_\epsilon W_i\right)\right|^2 = \frac{1}{2}\sum_{i=1}^k\int_{\Omega_\epsilon} w_i^p P_\epsilon w_i + \frac{1}{2}\sum_{\substack{i,h=1\\ i\neq h}}^k \int_{\Omega_\epsilon} w_h^p P_\epsilon w_i\\ &= \frac{kc_0}{2} + \left[\frac{c_1^2}{2}\left\{\sum_{i\neq h} G(\sigma_i,\sigma_h)(\lambda_h\lambda_i)^{\frac{n-2s}{2}} - \sum_{i=1}^k H(\sigma_i,\sigma_i)\lambda_i^{n-2s}\right\} + o(1)\right]\epsilon^{(n-2s)\alpha_0}\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega_\epsilon} F_\epsilon\left(\sum_{i=1}^k P_\epsilon w_i\right) \\
&= \sum_{h=1}^k \Big[\int_{\frac{B_h}{\epsilon^{\alpha_0}}} F_\epsilon\Big(w_h + (P_\epsilon w_h - w_h) + \sum_{\substack{i,h=1 \\ i\neq h}}^k P_\epsilon w_i\Big) - F_\epsilon(w_h)\Big] + \sum_{h=1}^k \int_{\frac{B_h}{\epsilon^{\alpha_0}}} F_\epsilon(w_h) + o(\epsilon^{(n-2s)\alpha_0}) \\
&= \sum_{h=1}^k \left[\int_{\frac{B_h}{\epsilon^{\alpha_0}}} F_\epsilon(w_h) + \int_{\frac{B_h}{\epsilon^{\alpha_0}}} f_\epsilon(w_h)(P_\epsilon w_h - w_h)\right] + \sum_{i\neq h} \int_{\frac{B_h}{\epsilon^{\alpha_0}}} f_\epsilon(w_h) P_\epsilon w_i + o(\epsilon^{(n-2s)\alpha_0}) \\
&= \frac{kc_0}{p+1} + \left[c_1^2 \left\{ \sum_{i\neq h} G(\sigma_i,\sigma_h)(\lambda_h\lambda_i)^{\frac{n-2s}{2}} - \sum_{i=1}^k H(\sigma_i,\sigma_i)\lambda_i^{n-2s} \right\} + \frac{c_2}{2}\sum_{i=1}^k \lambda_i^{2s} + o(1)\right] \epsilon^{(n-2s)\alpha_0}
\end{aligned}$$

Note that here we also used that $1+2s\alpha_0 = (n-2s)\alpha_0$ which holds owing to our choice $\alpha_0 = \frac{1}{n-4s}$. As a consequence, (6.25) holds C^0-uniformly in O^{δ_0}. Similarly, with Lemmas 6.B.3, 6.B.4 and 6.B.5, one can conclude that (6.25) has its validity in C^1-sense (see [GMP, Section 7] and [MP, Proposition 2.2]). This completes the proof. □

Chapter 7

Infinitely many solutions for semilinear nonlocal elliptic equations under noncompact settings [ChS]

7.1 Introduction

The aim of this paper is to prove the existence of infinitely many solutions to some kinds of semilinear elliptic equations involving the fractional Laplace operator $(-\Delta)^s$, which is nonlocal in nature. The fractional Laplace operator arises when we consider the infinitesimal generator of the Lévy stable diffusion process in probability theory or the fractional quantum mechanics for particles on stochastic fields. For further motivations and backgrounds, we refer to [FQT] and references therein.

Recently, the semilinear nonlocal elliptic equations, which are denoted by

$$(-\Delta)^s u = f(x,u) \quad \text{in } \Omega \subset \mathbb{R}^N, \quad 0 < s < 1, \tag{7.1}$$

have been widely studied under several contexts. In this paper, we are interested in equations of the form (7.1), which are forced to be posed on function spaces with noncompact Sobolev embedding. More precisely, we shall study the fractional Brezis-Nirenberg problems on bounded domains.

We first introduce the Fractional Brezis-Nirenberg problems. Let Ω be a smooth bounded domain in $\mathbb{R}^N$. For given $s \in (0,1)$ and $\mu > 0$, the following problem

$$\begin{cases} (-\Delta)^s u = |u|^{2^*(s)-2}u + \mu u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.2}$$

where $2^*(s) = \frac{2N}{N-2s}$, is called the fractional Brezis-Nirenberg problem since it is a fractional version of the classical Brezis-Nirenberg problem,

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \mu u & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases} \tag{7.3}$$

Due to the loss of compactness of critical Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$ and $H_0^s(\Omega) \hookrightarrow L^{\frac{2N}{N-2s}}(\Omega)$, more careful analysis is required to construct nontrivial solutions to the equations (7.2) and (7.3) than equations with sub-critical nonlinearities. In a celebrated paper [BN], Brezis and Nirenberg first studied the existence of a positive solution to (7.3). Let λ_1 and ϕ_1 respectively denote the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition on Ω and a corresponding positive eigenfunction. By testing ϕ_1 to (7.3), it is easy to see that if $\mu \geq \lambda_1$, there is no positive solution to (7.3). Also, the well-known Pohozaev's identity says that if $\mu \leq 0$ and Ω is star-shape, there is no nontrivial solutions to (7.3). Thus, one can deduce that the condition $\mu \in (0, \lambda_1)$ is necessary for (7.3) to admit a positive solution for general smooth domains Ω. Brezis and Nirenberg proved in [BN] that if $N \geq 4$, the above condition is sufficient. In other words, there is a positive least energy solution to (7.3) for all $\mu \in (0, \lambda_1)$.

Since the work of Brezis and Nirenberg, many research papers have been devoted to study the problem (7.3). One of most important works among them is due to Devillanova and Solimini who proved in [DS] the existence of infinitely many solutions for the problem (7.3) when $N > 7$ and $\mu > 0$. This work was extended to analogous problems involving p-Laplacian for $1 < p < \infty$ by Cao-Peng-Yan [CPY]. They proved that if $N > p^2 + p$, the following problem

$$-\Delta_p u = |u|^{p^*-2}u + \mu|u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $\mu > 0$ and $p^* = \frac{pN}{n-p}$, has infinitely many nontrivial solutions.

As a first result of the paper, we prove a multiplicity result for (7.2) with $s \in (0, 1)$, which extends the Devillanova and Solimini's result in [DS] to the fractional case.

Theorem 7.1.1. *Let $s \in (0, 1)$ and $\mu > 0$ be given. Suppose $N > 6s$. Then the equation* (7.2) *admits infinitely many nontrivial solutions.*

We shall prove Theorem 7.1.1 by following Devillanova and Solimini's ideas in [DS]. The main strategy in these ideas is to consider approximating subcritical problems, which can be shown that they admit infinitely many nontrivial solutions. In other words, we consider subcritical problems

$$\begin{cases} (-\Delta)^s u = |u|^{2^*(s)-2-\epsilon}u + \mu u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.4}$$

for small $\epsilon > 0$. From the sub-criticality of the problems, one can verify by using standard variational methods that for every small $\epsilon > 0$, (7.4) admits infinitely many nontrivial solutions in a fractional Sobolev space $H_0^s(\Omega)$. (We will define $H_0^s(\Omega)$ precisely in Section 2.) This tells us that the following compactness result plays a key role to obtain nontrivial solutions to our original equation (7.2).

Theorem 7.1.2. *Assume $N > 6s$. Let $\{u_n\}$ be a sequence of solutions to* (7.4) *with $\epsilon = \epsilon_n \to 0$ as $n \to \infty$ and $\sup_{n\in\mathbb{N}} \|u_n\|_{H_0^s(\Omega)} < \infty$. Then $\{u_n\}$ converges strongly in $H_0^s(\Omega)$ up to a subsequence.*

Combining Theorem 7.1.2 with the topological genus theory, we will see in Section 6 that there are infinitely many nontrivial solutions to (7.2).

Proving Theorem 7.1.2 is the main task of this paper and requires a series of delicate analysis. Moreover, it turns out from several technical reasons that studying our nonlocal equations (7.2) and (7.4) directly is not suitable for establishing Theorem 7.1.2. Instead, it is better to consider so-called s-harmonic extension problems (7.9) and (7.10), which are equivalent to (7.2) and (7.4) respectively. As we will see in Section 2, the equations (7.9) and (7.10) are local so that they are much easier to deal with than nonlocal ones, but the domain of problems are changed from Ω to the half-infinite cylinder $C := \Omega \times [0, \infty)$. This kind of localization was initiated by Caffarelli-Sylvestre [CaS] in which the domain under consideration is the whole space $\mathbb{R}^N$, and has been made for bounded domains by many authors [BCPS2, CT, T2].

By virtue of considering localized equations, one can easily obtain the concentration compactness principle of Struwe [Su] for a sequence of solutions to a local equation (7.10). This principle says that a bounded sequence of solutions to (7.10) consists of a function that the sequence weakly converges, finitely many bubbles that may possibly exist and a function that strongly converges to zero. Thus, to get the compactness, we need to get rid of possibility that bubbles appear. This will be achieved by arguing indirectly, i.e., we assume there exist bubbles in the sequence and get a contradiction. For this, an important issue is to verify a sharp bound of the solutions on some thin annuli near a bubbling point. We devote a large part of this paper to obtain it. We give a full detail of ideas for the proof for Theorem 7.1.2 in Section 3. After the proof of Theorem 7.1.2, we shall complete the proof of Theorem 7.1.1 by using a min-max principle combined with the topological genus.

The rest of the paper is organized as follows. In Section 2, we review the fractional Laplacian, s-harmonic extension and the extended local problems posed on half-infinite cylinders. We also arrange some basic lemmas which will be used throughout the paper. In Section 3, we give basic settings and ideas for the proof for Theorem 7.1.2. By following these ideas, we complete the proof of Theorem 7.1.1 and Theorem 7.1.2 in subsequent sections 4, 5 and 6. In Appendix A we prove a technical lemma which will be essentially used in Section 5. In Appendix B, we prove a lemma which corresponds a non-local version of Moser's iteration method. Finally in Appendix C, we establish so-called local Pohozaev identity for solutions to (7.10), that is a main ingredient for obtaining compactness of a sequence of solutions to (7.10).

7.2 Mathematical frameworks and preliminaries

7.2.1 Fractional Sobolev spaces, fractional Laplacians and fractional harmonic extensions

We first set Ω to be a smooth bounded domain of $\mathbb{R}^n$. Let $\{\lambda_k, \phi_k\}_{k=1}^{\infty}$ be the sequence of eigenvalues and corresponding eigenvectors of a eigenvalue problem:

$$\begin{cases} -\Delta\phi_k = \lambda_k\phi_k & \text{in } \Omega, \\ \phi_k = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $\|\phi_k\|_{L^2(\Omega)} = 1$ and $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$. Then one can define a fractional Sobolev space $H_0^s(\Omega)$ for $s \in (0,1)$ by

$$H_0^s(\Omega) = \left\{ u = \sum_{k=1}^{\infty} a_k\phi_k \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^s a_k^2 < \infty \right\}, \tag{7.1}$$

which is a Hilbert space equipped with an inner product:

$$\left\langle \sum_{k=1}^{\infty} a_k\phi_k, \sum_{k=1}^{\infty} b_k\phi_k \right\rangle_{H_0^s(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^s a_k b_k \qquad \text{if } \sum_{k=1}^{\infty} a_k\phi_k,\ \sum_{k=1}^{\infty} b_k\phi_k \in H_0^s(\Omega).$$

We define a fractional Laplace operator $(-\Delta)^s : H_0^s(\Omega) \to H_0^{-s}(\Omega)$ by

$$\langle (-\Delta)^s u, v \rangle_{H_0^{-s}(\Omega)} = \langle u, v \rangle_{H_0^s(\Omega)},$$

where $H_0^{-s}(\Omega)$ denotes the dual space of $H_0^s(\Omega)$. Then, for any function $u = \sum_{k=1}^{\infty} a_k\phi_k \in H_0^{2s}(\Omega)$, $(-\Delta)^s u$ belongs $L^2(\Omega)$ and is represented by

$$(-\Delta)^s u = \sum_{k=1}^{\infty} a_k \lambda_k^s \phi_k.$$

This implies that

$$\langle u, v \rangle_{H_0^s(\Omega)} = \int_\Omega (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} v \quad \text{for } u, v \in H_0^s(\Omega)$$

and if $u \in H_0^{2s}(\Omega)$ additionally, an integration by parts formula holds as follows:

$$\int_\Omega (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} v = \int_\Omega (-\Delta)^s u \cdot v.$$

Next, suppose that the domain Ω is the whole space $\mathbb{R}^n$. Then, the homogeneous fractional Sobolev space $D^s(\mathbb{R}^n)$ $(0 < s < 1)$ is given by

$$D^s(\mathbb{R}^n) = \left\{ u \in L^{\frac{N+2s}{N-2s}}(\mathbb{R}^n) : \|u\|_{D^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty \right\}$$

where $\hat{u}$ denotes the Fourier transform of u. Note that $D^s(\mathbb{R}^N)$ is a Hilbert space equipped with an inner product

$$\langle u, v\rangle_{D^s(\mathbb{R}^N)} = \int_{\mathbb{R}^n} |\xi|^{2s}\hat{u}(\xi)\hat{v}(\xi)\, d\xi.$$

We also define a fractional Laplace operator on the whole space, $(-\Delta)^s : D^s(\mathbb{R}^N) \to D^{-s}(\mathbb{R}^N)$ by

$$\langle(-\Delta)^s u, v\rangle_{D^{-s}(\mathbb{R}^N)} = \langle u, v\rangle_{D^s(\mathbb{R}^N)},$$

where $D^{-s}(\mathbb{R}^N)$ is the dual of $D^s(\mathbb{R}^N)$. Then, one can easily check that if $u \in D^{2s}(\mathbb{R}^N)$, we have $(-\Delta)^s u \in L^2(\mathbb{R}^N)$ such that

$$(-\Delta)^s u = \mathfrak{F}^{-1}[|\xi|^{2s}\hat{u}(\xi)]$$

where $\mathfrak{F}^{-1}$ denotes the inverse Fourier transform so that we see for $u, v \in D^s(\mathbb{R}^N)$

$$\langle u, v\rangle_{D^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (-\Delta)^{s/2}u \cdot (-\Delta)^{s/2}v$$

and assuming additionally $u \in D^{2s}(\mathbb{R}^N)$, $v \in L^2(\mathbb{R}^N)$, we can integrate by parts:

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2}u \cdot (-\Delta)^{s/2}v = \int_{\mathbb{R}^N} (-\Delta)^s u \cdot v.$$

Finally, the notation $H^s(\mathbb{R}^N)$ denotes the standard fractional Sobolev space defined as

$$H^s(\mathbb{R}^N) = D^s(\mathbb{R}^N) \cap L^2(\mathbb{R}^N).$$

Now we introduce the concept of s-harmonic extension of a function u defined in Ω, where either Ω is $\mathbb{R}^n$ or a smooth bounded domain, which provides a way to representing fractional Laplace operators as a form of Dirichlet-to-Neumann map. To do this, we need to define additional function spaces on the half infinite cylinder $C = \Omega \times (0, \infty)$.

Let the weighted Lebesgue space $L^2(t^{1-2s}, C)$ be the set of measurable functions $U : C \to \mathbb{R}$ such that

$$\|U\|_{L^2(t^{1-2s},C)} := \int_C t^{1-2s}U^2\, dxdt < \infty.$$

Then, the weighted Sobolev space $H^1(t^{1-2s}, C)$ defined by

$$H^1(t^{1-2s}, C) = \{U \in L^2(t^{1-2s}, C) : \nabla U \in L^2(t^{1-2s}, C)\}$$

is a Hilbert space equipped with an inner product

$$\langle U, V\rangle_{H^1(t^{1-2s},C)} = \int_C t^{1-2s}(\nabla U \cdot \nabla V + UV)\, dxdt.$$

Suppose that Ω is smooth and bounded. We set the lateral boundary $\partial_L C$ of C by

$$\partial_L C := \partial\Omega \times [0, \infty).$$

Then the function space $H^1_0(t^{1-2s}, C)$ defined by the completion of

$$C^\infty_{0,L}(C) := \left\{U \in C^\infty\left(\overline{C}\right) : U = 0 \text{ on } \partial_L C\right\}$$

with respect to the norm

$$\|U\|_{H^1_0(t^{1-2s},C)} = \left(\int_C t^{1-2s}|\nabla U|^2\, dxdt\right)^{1/2}, \tag{7.2}$$

is also a Hilbert space endowed with an inner product

$$(U, V)_{H^1_0(t^{1-2s},C)} = \int_C t^{1-2s}\nabla U \cdot \nabla V\, dxdt.$$

It is verified in [CaS, Proposition 2.1] and [T2, Section 2] that $H^s_0(\Omega)$ is the continuous trace of $H^1_0(t^{1-2s}, C)$, i.e.,

$$H^s_0(\Omega) = \{u = \mathrm{tr}|_{\Omega\times\{0\}}U : U \in H^1_0(t^{1-2s}, C)\}. \tag{7.3}$$

and

$$\|U(\cdot, 0)\|_{H^s_0(\Omega)} \le C\|U\|_{H^1_0(t^{1-2s},C)} \tag{7.4}$$

for some $C > 0$, independent of $U \in H^1_0(t^{1-2s}, C)$.

When $\Omega = \mathbb{R}^N$(in this case $C = \mathbb{R}^{N+1}_+$), one can define the weighted homogeneous Sobolev space $D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$ as the completion of $C^\infty_c\left(\overline{\mathbb{R}^{N+1}_+}\right)$ with respect to the norm

$$\|U\|_{D^1(t^{1-2s},\mathbb{R}^{N+1}_+)} := \left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\, dxdt\right)^{1/2}.$$

Similarly, it holds by taking trace that

$$D^s(\mathbb{R}^N) = \{u = \mathrm{tr}|_{\mathbb{R}^N\times\{0\}}U : U \in D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)\}$$

and

$$\|U(\cdot, 0)\|_{D^s(\mathbb{R}^N)} \le C\|U\|_{D^1(t^{1-2s},\mathbb{R}^{N+1}_+)} \tag{7.5}$$

for some $C > 0$ independent of $U \in D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$.

Now, we are ready to introduce s-harmonic extensions of $u \in H^s_0(\Omega)$ for bounded Ω or $u \in D^s(\mathbb{R}^N)$, that can be thought as the inverses of the trace processes above. Let $u \in H^s_0(\Omega)$ and $v \in D^s(\mathbb{R}^N)$. By works of Caffarelli-Silvestre [CaS] (for $\mathbb{R}^n$), Cabré-Tan [CT] (for bounded domains Ω, see also [ST, BCPS2, T2]), it is known that there are unique functions $U \in H^1_0(t^{1-2s}, C)$ and $V \in D^1(t^{1-2s}, \mathbb{R}^N)$ which satisfies the equation

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla U) = 0 & \text{in } C,\\ U = 0 & \text{on } \partial_L C,\\ U(x, 0) = u(x) & \text{for } x \in \Omega, \end{cases} \tag{7.6}$$

and

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla V) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ V(x,0) = v(x) & \text{for } x \in \mathbb{R}^N \end{cases} \tag{7.7}$$

respectively in distributional sense. Moreover, if u and v are compactly supported and smooth, then the following limits

$$\partial^s_\nu W(x,0) := -C_s^{-1}\left(\lim_{t\to 0+} t^{1-2s}\frac{\partial W}{\partial t}(x,t)\right) \quad \text{with } C_s := \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}, \quad W = U \text{ or } V,$$

are well defined and one must have

$$(-\Delta)^s w = \partial^s_\nu W(x,0), \quad w = u \text{ or } v. \tag{7.8}$$

We call these U and V the s-harmonic extensions of u and v. We point out that by a density argument, the relation (7.8) is satisfied in weak sense for $u \in H^s_0(\Omega)$ and $v \in D^s(\mathbb{R}^N)$. In other words, it holds that for every u and $\phi \in H^s_0(\Omega)$,

$$\langle u, \phi\rangle_{H^s_0(\Omega)} = C_s^{-1}\langle U, \Phi\rangle_{H^1_0(t^{1-2s},C)} \quad \text{where} \quad U,\ \Phi = \ s\text{-harmonic extensions of } u,\ \phi$$

and the analogous statement holds for every v and $\phi \in D^s(\mathbb{R}^N)$. Thus the trace inequalities (7.4) and (7.5) are improved as

$$\|U(\cdot,0)\|_{H^s_0(\Omega)} = C_s^{-1}\|U\|_{H^1_0(t^{1-2s},C)}, \quad \|U(\cdot,0)\|_{D^s(\mathbb{R}^N)} = C_s^{-1}\|U\|_{D^1(t^{1-2s},\mathbb{R}^{N+1}_+)}.$$

By the above discussion, one can deduce that a function $u \in H^s_0(\Omega)$ is a weak solution to the nonlocal problem (7.2) if and only if its s-harmonic extension $U \in H^1_0(t^{1-2s}, C)$ is a weak solution to the local problem

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } C, \\ U = 0 & \text{on } \partial_L C, \\ \partial^s_\nu U = |U|^{2^*(s)-2}U(x,0) + \mu U(x,0) & \text{on } \Omega\times\{0\}, \end{cases} \tag{7.9}$$

and similarly the problem (7.4) corresponds to

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } C, \\ U = 0 & \text{on } \partial_L C, \\ \partial^s_\nu U = |U|^p U(x,0) + \mu U(x,0) & \text{on } \Omega\times\{0\}, \end{cases} \tag{7.10}$$

where $1 < p < 2^*(s) - 2$. By weak solutions, we mean the following: Let $g \in L^{\frac{2N}{N+2s}}(\Omega)$. Given the problem

$$\begin{cases} (-\Delta)^s u = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.11}$$

we say that a function $u \in H_0^s(\Omega)$ is a weak solution of (7.11) provided

$$\int_\Omega (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} \phi \, dx = \int_\Omega g(x)\phi(x) \, dx \tag{7.12}$$

for all $\phi \in H_0^s(\Omega)$. Also, given the problem

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla U) = 0 & \text{in } \mathcal{C}, \\ U = 0 & \text{on } \partial_L \mathcal{C}, \\ \partial_\nu^s U = g(x) & \text{on } \Omega \times \{0\}, \end{cases} \tag{7.13}$$

we say that a function $U \in H_0^1(t^{1-2s}, \mathcal{C})$ is a weak solution of (7.13) provided

$$\int_{\mathcal{C}} t^{1-2s} \nabla U(x,t) \cdot \nabla \Phi(x,t) \, dxdt = C_s \int_\Omega g(x)\Phi(x,0) \, dx \tag{7.14}$$

for all $\Phi \in H_0^1(t^{1-2s}, \mathcal{C})$.

7.2.2 Weighted Sobolev and Sobolev-trace inequalities

Given any $\lambda > 0$ and $\xi \in \mathbb{R}^N$, let

$$w_{\lambda,\xi}(x) = \mathfrak{c}_{N,s} \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2} \right)^{\frac{N-2s}{2}} \quad \text{for } x \in \mathbb{R}^N, \tag{7.15}$$

where

$$\mathfrak{c}_{N,s} = 2^{\frac{N-2s}{2}} \left(\frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma\left(\frac{N-2s}{2}\right)} \right)^{\frac{N-2s}{4s}}. \tag{7.16}$$

Then we have the following Sobolev inequality

$$\left(\int_{\mathbb{R}^N} |u|^{2^*(s)} dx \right)^{\frac{1}{2^*(s)}} \le \mathcal{S}_{N,s} \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx \right)^{\frac{1}{2}}, \quad u \in H_0^s(\Omega),$$

which attains the equality exactly when $u(x) = c w_{\lambda,\xi}(x)$ for any $c > 0$, $\lambda > 0$ and $\xi \in \mathbb{R}^N$ (we refer to [Lb, ChL, FL]). Here,

$$\mathcal{S}_{N,s} = 2^{-2s} \pi^{-s} \frac{\Gamma\left(\frac{N-2s}{2}\right)}{\Gamma\left(\frac{N+2s}{2}\right)} \left[\frac{\Gamma(N)}{\Gamma(N/2)} \right]^{2s/N}. \tag{7.17}$$

It follows that for the Sobolev trace inequality

$$\left(\int_{\mathbb{R}^N} |U(x,0)|^{2^*(s)} dx \right)^{\frac{1}{2^*(s)}} \le \frac{\mathcal{S}_{N,s}}{\sqrt{C_s}} \left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U(x,t)|^2 dxdt \right)^{\frac{1}{2}}, \quad U \in D^1(t^{1-2s}, \mathbb{R}^{N+1}_+), \tag{7.18}$$

the equality is attained exactly by $U(x,t) = cW_{\lambda,\xi}(x,t)$, where $W_{\lambda,\xi}(x,t)$ is the s-harmonic extension of $w_{\lambda,\xi}$. By zero extension, we also have

$$\left(\int_\Omega |U(x,0)|^{2^*(s)}dx\right)^{\frac{1}{2^*(s)}} \leq \frac{S_{N,s}}{\sqrt{C_s}}\left(\int_C t^{1-2s}|\nabla U(x,t)|^2 dxdt\right)^{\frac{1}{2}}, \quad U \in H_0^1(t^{1-2s}, C). \tag{7.19}$$

As an application, we obtain the following estimate.

Lemma 7.2.1. *Let $w \in L^p(\Omega)$ for some $p < \frac{N}{2s}$. Assume that U is a weak solution of the problem*

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } C, \\ U = 0 & \text{on } \partial_L C, \\ \partial_\nu^s U = w & \text{on } \Omega \times \{0\}. \end{cases} \tag{7.20}$$

Then we have

$$\|U(\cdot,0)\|_{L^q(\Omega)} \leq C_{p,q}\,\|w\|_{L^p(\Omega)}\,, \tag{7.21}$$

for any q such that $\frac{N}{q} \leq \frac{N}{p} - 2s$.

Proof. We multiply (7.20) by $|U|^{\beta-1}U$ for some $\beta > 1$ to get

$$\int_\Omega w(x)|U|^{\beta-1}U(x,0)\,dx = \beta \int_C t^{1-2s}|U|^{\beta-1}|\nabla U|^2\,dxdt. \tag{7.22}$$

Then, applying the trace embedding (7.19) and Hölder's inequality we can observe

$$\left\||U|^{\frac{\beta+1}{2}}(\cdot,0)\right\|^2_{L^{\frac{2N}{N-2s}}(\Omega)} \leq C_\beta \left\||U|^\beta(\cdot,0)\right\|_{L^{\frac{\beta+1}{2\beta}\cdot\frac{2N}{N-2s}}} \|w\|_p\,, \tag{7.23}$$

where p satisfies $\frac{1}{p} + \frac{(N-2s)\beta}{N(\beta+1)} = 1$. Let $q = \frac{N(\beta+1)}{N-2s}$, then (7.23) gives the desired inequality. □

We will also make use of the following weighted Sobolev inequality.

Proposition 7.2.2. *[FKS, Theorem 1.3] Let Ω be an open bounded set in $\mathbb{R}^{N+1}$. Then there exists a constant $C = C(N,s,\Omega) > 0$ such that*

$$\left(\int_\Omega |t|^{1-2s}|U(x,t)|^{\frac{2(N-2s+2)}{N-2s}}dxdt\right)^{\frac{N-2s}{2(N-2s+2)}} \leq C\left(\int_\Omega |t|^{1-2s}|\nabla U(x,t)|^2 dxdt\right)^{\frac{1}{2}} \tag{7.24}$$

holds for any function U compactly supported in Ω whenever the right hand side is well-defined.

7.2.3 Useful lemmas

Here we prepare some lemmas which will be used importantly throughout the paper.

Lemma 7.2.3. *Suppose that $V \in H_0^1(t^{1-2s}, C)$ is a weak solution of the following problem*

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla V) = 0 & \text{on } C, \\ V(x,t) = 0 & \text{on } \partial_L C, \\ \partial_\nu^s V(x,0) = g(x) & \text{on } \Omega \times \{0\} \end{cases} \tag{7.25}$$

for some nonnegative g. Then V is nonnegative everywhere.

Proof. Let $V_- = \max\{0, -V\}$. By testing V_-, the definition of weak formulation implies

$$-\int_C t^{1-2s}|\nabla V_-|^2\, dxdt = C_s \int_\Omega g(x) \cdot V_-(x,0)\, dx \geq 0 \tag{7.26}$$

and thus

$$\int_C t^{1-2s}|\nabla V_-|^2(x,t) dxdt = 0.$$

It proves that $V_- \equiv 0$. The lemma is proved. □

Next we state a variant of the concentration compactness principle [Su] for the extended problems.

Lemma 7.2.4. *For $n \in \mathbb{N}$ let U_n be a solution of* (7.10) *with $p = p_n \to 2^*(s) - 2$ such that $\|U_n\|_{H_0^1(t^{1-2s}, C)} < C$ for some C independent of $n \in \mathbb{N}$. Then, for some $k \in \mathbb{N}$, there are k-sequences $\{(\lambda_n^j, x_n^j)\}_{n=1}^\infty \in \mathbb{R}_+ \times \Omega$, $1 \leq j \leq k$, a function $V^0 \in H_0^1(t^{1-2s}, C)$ and k-functions $V^j \in D^1(t^{1-2s}, \mathbb{R}_+^{N+1})$, $1 \leq j \leq k$ satisfying*

- *$U_n \rightharpoonup V^0$ weakly in $H_0^1(t^{1-2s}, C)$;*
- *$U_n - \left(V^0 + \sum_{j=1}^k \rho_n^j(V^j)\right) \to 0$ in $H_0^1(t^{1-2s}, C)$ as $n \to \infty$, where*

$$\rho_n^j(V^j) = (\lambda_n^j)^{\frac{N}{2^*(s)}} V^j(\lambda_n^j(\cdot - x_n^j));$$

- *V^0 is a solution of* (7.9), *and V^j are non-trivial solutions of*

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla V) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ \partial_\nu^s V = |V|^{2^*(s)-2} V & \text{on } \mathbb{R}^N \times \{0\}. \end{cases} \tag{7.27}$$

Moreover, we have

$$\frac{\lambda_n^i}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^i} + \lambda_n^i \lambda_n^j |x_n^i - x_n^j|^2 \to \infty \text{ as } n \to \infty \text{ for all } i \neq j. \tag{7.28}$$

Proof. The proof follows without difficulty by modifying the proof of the concentration compactness result for (7.3)(see [Su, Su2]), and we omit the details for the sake of simplicity of the paper. We refer to the paper [M] where S. Almaraz modified the argument in [Su] for studying the boundary Yamabe flow. His setting corresponds to the case $s = 1/2$ of the extended problems considered here. □

It is useful to know the decay rate of any entire solutions to (7.27).

Lemma 7.2.5. *Suppose that $V \in D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$ is a weak solution of (7.27). Then there exists a constant $C > 0$ such that*

$$|V(x,0)| \le \frac{C}{(1+|x|)^{N-2s}}.$$

Proof. We first show that V is a bounded function. For a sake of convenience, we consider a positive function $U \in D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$ such that

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \partial^s_\nu U = |V|^{\frac{N+2s}{N-2s}} & \text{on } \mathbb{R}^N \times \{0\}. \end{cases} \tag{7.29}$$

Then, it is easy to see $|V| \le U$ by Lemma 7.2.3 and

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\, dxdt \le \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla V|^2\, dxdt.$$

For $T > 0$ let $U_T = \min\{U, T\}$. Multiplying (8.3) by $U_T^{2\beta} U$ for $\beta > 1$ we obtain

$$\int_{\mathbb{R}^N} |V|^{\frac{N+2s}{N-2s}} \cdot U_T^{2\beta} \cdot U(x,0)dx = \int_{\mathbb{R}^{N+1}_+} t^{1-2s} 2\beta |\nabla U_T|^2 U^{2\beta} + t^{1-2s}|\nabla U|^2 U_T^{2\beta} dxdt.$$

On the other hand, a direct computation shows

$$|\nabla(U U_T^{\beta})|^2 = U_T^{2\beta}|\nabla U|^2 + (2\beta + \beta^2) U_T^{2\beta} |\nabla U_T|^2. \tag{7.30}$$

Thus we deduce

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla(U U_T^{\beta})|^2 dxdt \le C \int |V|^{\frac{N+2s}{N-2s}} \cdot U_T^{2\beta} U(x,0)dx,$$

and consequently, for $K > 0$ we have

$$\begin{aligned} \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla(U U_T^{\beta})|^2 dxdt &\le C \int_{U \le K} |V|^{\frac{N+2s}{N-2s}} \cdot U_T^{2\beta} U dx + C \int_{U > K} |V|^{\frac{N+2s}{N-2s}} \cdot U_T^{2\beta} U dx \\ &\le K^{2\beta} C + C \left(\int_{U>K} |V|^{\frac{2N}{N-2s}}(x,0)dx \right)^{\frac{2s}{N}} \left(\int_{\mathbb{R}^N} |U_T^{\beta} U(x,0)|^{\frac{2N}{N-2s}} dx \right)^{\frac{N-2s}{N}} \\ &\le K^{2\beta} C + C \left(\int_{U>K} |V|^{\frac{2N}{N-2s}}(x,0)dx \right)^{\frac{2s}{N}} \left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla(U U_T^{\beta})|^2 dxdt \right) \end{aligned}$$

Choosing a sufficiently large $K > 0$, we get

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla(UU_T^\beta)|^2 dxdt \leq 2K^{2\beta} C.$$

From this, using the Sobolev-trace inequality and letting $T \to \infty$, we obtain

$$\int_{\mathbb{R}^N} |V|^{2^*(s)(\beta+1)}(x,0)dx \leq \int_{\mathbb{R}^N} U^{2^*(s)(\beta+1)}(x,0)dx \leq C.$$

Here $\beta > 1$ can be chosen arbitrary. Now, we use the following kernel expression(See [CaS]),

$$U(x,t) = \int_{\mathbb{R}^N} \frac{C_{N,s}}{(|x-y|^2 + t^2)^{\frac{N-2s}{2}}} |V|^{2^*(s)-1}(y,0)\, dy$$

and Hölder's inequality to conclude that U is a bounded function. Therefore, V is a bounded function.

Next we consider the following Kelvin transform with $z = (x,t) \in \mathbb{R}^{N+1}_+$,

$$W(z) = |z|^{-(N-2s)} V\left(\frac{z}{|z|^2}\right). \tag{7.31}$$

From a direct computation, we see that the function W satisfies

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla W) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \partial_\nu^s W = |W|^{\frac{4s}{N-2s}} W & \text{on } \mathbb{R}^N \times \{0\}, \end{cases}$$

and $\|W\|_{D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)} \leq C\|V\|_{D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)} \leq C$. Then, we may apply the same argument for V to show that the function W is bounded on $\mathbb{R}^{N+1}_+$. So, we can deduce from (8.5) that

$$|V(z)| \leq C|z|^{-(N-2s)}.$$

This proves the lemma. □

7.3 Settings and Ideas for the proof of Theorem 7.1.2

Here we build basic settings and ideas for the proof of Theorem 7.1.2 for a clear exposition of the paper. The arguments introduced in this section are originally developed by Devillanova and Solimini in [DS] and also are inspired by a modified approach in the work of Cao, Peng and Yan in [CPY]. From now on, we will denote the norm of the weighted Sobolev space $H_0^1(t^{1-2s}, \mathcal{C})$ by $\|\cdot\|$ for simplicity.

Let $\{U_n\}_{n\in\mathbb{N}} \in H_0^1(t^{1-2s}, \mathcal{C})$ be a sequence of functions which are solutions of (7.10) with $p = p_n \to 2^*(s) - 2$ such that $\|U_n\|$ is bounded uniformly for $n \in \mathbb{N}$. What we want to prove is the compactness of the sequence $\{U_n\}_{n\in\mathbb{N}}$ in $H_0^1(t^{1-2s}, \mathcal{C})$. Arguing indirectly, suppose that $\{U_n\}_{n\in\mathbb{N}}$

is noncompact. Then Lemma 7.2.4 says that for some integer $k \geq 1$, there exist k sequences $\{(x_n^j, \lambda_n^j)\}_{n\in\mathbb{N}} \in \Omega \times \mathbb{R}_+$ with $\lim_{n\to\infty} \lambda_n^j = \infty$ such that (7.28) holds and

$$\begin{cases} U_n = V^0 + \sum_{j=1}^k \rho_n^j(V^j) + R_n, \\ \lim_{n\to\infty} \|R_n\| = 0, \end{cases} \tag{7.1}$$

where V^0 is a solution to (7.9) and V^j is an entire solution of (7.27) for $1 \leq j \leq k$. By taking a subsequence, we may assume without loss of generality

$$\lambda_n^1 \leq \lambda_n^2 \leq \cdots \leq \lambda_n^k \qquad \forall n \in \mathbb{N}.$$

We just denote λ_n^1 by λ_n and x_n^1 by x_n throughout the paper. In other words, we mean x_n by the slowest bubbling point and λ_n by the corresponding rate of blowup.

We shall derive a contradiction by making use a local Pohozaev identity (7.23) on concentric balls with center x_n and radii comparable to $\lambda_n^{-1/2}$. To do this, it is required to show that average(and weighted average) integrals of $|U|^q$ on appropriate annuli around x_n are uniform bounded for n whenever $q > 1$. Then it follows a sharp weighted L^2 estimates for ∇U. This will be accomplished in Section 4 and 5.

More precisely, we introduce in Section 4 a norm which reflects the effect of bubbles in sequence $\{U_n\}_{n=1}^\infty$ and show the uniform boundedness of $\{U_n\}$ with respect to this norm. Let q_1 and q_2 be real numbers such that $\frac{N}{N-2s} < q_2 < \frac{2N}{N-2s} < q_1 < \infty$. For given two functions $u_1 \in L^{p_1}(\Omega)$ and $u_2 \in L^{q_2}(\Omega)$, let $\alpha > 0$ and $\lambda > 0$ be satisfy

$$\begin{cases} \|u_1\|_{q_1} & \leq \alpha, \\ \|u_2\|_{q_2} & \leq \alpha \lambda^{\frac{N}{2^*(s)} - \frac{N}{q_2}}. \end{cases} \tag{7.2}$$

We define for given q_1, q_2, λ, a norm as follows:

$$\|u\|_{\lambda,q_1,q_2} = \inf\{\alpha > 0 : \text{there exist } u_1 \text{ and } u_2 \text{ such that } |u| \leq u_1 + u_2 \text{ and (7.2) holds }\}. \tag{7.3}$$

Then, we prove that

$$\sup_{n\in\mathbb{N}} \|U_n(\cdot, 0)\|_{\lambda_n,q_1,q_2} < \infty.$$

In section 5, we establish the uniform boundedness of the average integrals of $|U|^q$ and a sharp weighted L^2 estimate for ∇U on suitable annuli around x_n with widths comparable to $\lambda_n^{-1/2}$. We first show by combining the result in Section 4 and some delicate arguments in the work of Cao-Peng-Yan [CPY] with a nonlocal version of a lemma by Kilpenläinen-Malý [KM] that the desired average bounds are valid for at least relatively small range of q. Then a Moser's iteration type argument(Lemma 7.B.1) applies to widen the range of q to arbitrary $q > 1$.

With these estimates at hand, we make a contradiction from a local Pohozaev identity in Section 6, which completes the proof of Theorem 7.1.2.

7.4 A refined norm estimate

As explained in Section 3, we prove in this section the following result.

Proposition 7.4.1. *For $n \in \mathbb{N}$ let U_n be a solution of (7.10) with $p = p_n \to 2^*(s) - 2$ such that $\|U_n\| < C$ for some C independent of $n \in \mathbb{N}$. Consider any numbers q_1 and q_2 such that $\frac{N}{N-2s} < q_2 < \frac{2N}{N-2s} < q_1 < \infty$. Then we have*

$$\sup_n \|U_n(\cdot, 0)\|_{\lambda_n, q_1, q_2} < \infty.$$

We will prove this result through the three lemmas below, proofs of which heavily rely on Lemma 7.2.1, 7.2.3 and 7.2.5. Take a constant $A > 0$ such that $x^p + \mu x \leq 2x^{2^*(s)-1} + A$ for all $x \geq 0$ and consider a solution $\{D_n\}_{n\in\mathbb{N}}$ to the problem

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla D_n) = 0 & \text{in } C, \\ D_n = 0 & \text{on } \partial_L C, \\ \partial_\nu^s D_n = 2|U_n|^{2^*(s)-1} + A & \text{on } \Omega \times \{0\}. \end{cases} \tag{7.1}$$

Then, by Lemma 7.2.3, we see that D_n is positive and $|U_n| \leq D_n$. Moreover, by (7.1) for some $C_1 > 0$ we have

$$\partial_\nu^s D_n \leq C_1 \left(|V_0|^{2^*(s)-2} + \sum_{j=1}^{k} |\rho_n^j(V_j)|^{2^*(s)-2} + |R_n|^{2^*(s)-2} \right) |U_n| + A \quad \text{on } \Omega \times \{0\}. \tag{7.2}$$

We prepare the first lemma to control the remainder term R_n, which is known to converge to zero in $H_0^1(t^{1-2s}, C)$.

Lemma 7.4.2. *Let $a \in L^{\frac{N}{2s}}(\Omega)$ and $v \in L^\infty(\Omega)$. Suppose a function $U \in H_0^1(t^{1-2s}, C)$ satisfies*

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \textit{in } C, \\ U = 0 & \textit{on } \partial_L C, \\ \partial_\nu^s U = a(x)v & \textit{on } \Omega \times \{0\}. \end{cases}$$

Then, for any $\lambda > 0$ and $\frac{N}{N-2s} < q_1 < \frac{2N}{N-2s} < q_2 < \infty$ we have

$$\|U(\cdot, 0)\|_{\lambda, q_1, q_2} \leq C_{q_1, q_2} \|a\|_{\frac{N}{2s}} \|v\|_{\lambda, q_1, q_2}.$$

Proof. Choose arbitrary positive two functions $v_1 \in L^\infty(\Omega)$ and $v_2 \in L^\infty(\Omega)$ such that $|v(x)| \leq v_1(x) + v_2(x)$ for all $x \in \Omega$. Then, there exist functions $U_1 \in H_0^1(t^{1-2s}, C)$ and $U_2 \in H_0^1(t^{1-2s}, C)$ satisfying

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U_i) = 0 & \text{in } C, \\ U_i = 0 & \text{on } \partial_L C, \qquad i = 1, 2. \\ \partial_\nu^s U_i = |a(x)|v_i & \text{on } \Omega \times \{0\}, \end{cases}$$

We see from Lemma 7.2.3, the maximum principle that $|U| \le U_1 + U_2$. For given $\beta > 1$, one has

$$0 = \int_C \operatorname{div}(t^{1-2s}\nabla U_i)U_i^\beta dz = \int_{\Omega\times\{0\}} |a(x)|v_i(x)U_i^\beta(x,0)dx - \int_C t^{1-2s}\nabla U_i \nabla U_i^\beta dz,$$

which gives

$$\int_C t^{1-2s}|\nabla U_i^{\frac{\beta+1}{2}}|^2 dz = C_\beta \int_{\Omega\times\{0\}} a(x)v_i(x)U_i^\beta(x,0)dx.$$

Applying the Sobolev-trace inequality (7.19) and Hölder's inequality, we get

$$\|U_i^{\frac{\beta+1}{2}}(x,0)\|^2_{L^{\frac{2N}{N-2s}}(\Omega)} \le C\|a\|_{\frac{N}{2s}}\|v_i\|_{\frac{\beta+1}{2}\frac{2N}{N-2s}}\|U_i^\beta(x,0)\|_{L^{\frac{\beta+1}{2\beta}\frac{2N}{N-2s}}}. \tag{7.3}$$

For each $i \in \{1,2\}$ we take the value of β such that $q_i = \frac{\beta+1}{2}\frac{2N}{N-2s}$. Then (7.3) gives that

$$\|U_i(x,0)\|_{L^{q_i}} \le C\|a\|_{\frac{N}{2s}}\|v_i\|_{L^{q_i}} \quad \forall i = 1,2.$$

This and the definition (7.3) of $\|\cdot\|_{\lambda,q_1,q_2}$ yield

$$\|U(\cdot,0)\| \le C\|a\|_{N/2s}\|v\|_{\lambda,q_1,q_2}.$$

This proves the lemma. □

In the following lemma, we find a particular pair (q_1,q_2) such that $\|\ \|_{\lambda_n,q_1,q_2}$ is uniformly bounded.

Lemma 7.4.3. *For $n \in \mathbb{N}$, let U_n be a solution of* (7.10) *with $p = p_n \to 2^*(s) - 2$ such that $\|U_n\| < C$ for some C independent of $n \in \mathbb{N}$. Consider the sequence $\{D_n\}_{n\in\mathbb{N}}$ described in* (7.1). *Then, there exists $q_1 \in \left(\frac{2N}{N-2s},\infty\right)$ and $q_2 \in \left(\frac{N}{N-2s},\frac{2N}{N-2s}\right)$, and a constant $C > 0$ such that*

$$\sup_{n\in\mathbb{N}} \|D_n(\cdot,0)\|_{\rho_n,q_1,q_2} \le C.$$

Proof. For $1 \le i \le 3$ we consider the functions $D_n^i \in H_0^1(t^{1-2s},C)$ such that

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla D_i) = 0 \quad \text{in } C, \quad 1 \le i \le 3, \\ D_i = 0 \quad \text{on } \partial_L C, \quad 1 \le i \le 3, \\ \partial_\nu^s D_n^1 = C_1(|V_0|^{2^*(s)-2})|U_n| + A, \\ \partial_\nu^s D_n^2 = C_1(\sum_{j=1}^k |\rho_n^j(V_j)|^{2^*(s)-2})|U_n|, \\ \partial_\nu^s D_n^3 = C_1(|R_n|^{2^*(s)-2})|U_n|, \end{cases}$$

Then, from (7.2) we have $|D_n| \le D_n^1 + D_n^2 + D_n^3$ by the maximum principle. Because $\|U_n\|$ is uniformly bounded for $n \in \mathbb{N}$, the Sobolev-trace inequality gives

$$\sup_n \|U_n(\cdot,0)\|_{L^{2^*(s)}(\Omega)} \le C \sup_n \|U_n\| \le C.$$

Since V^0 is a bounded, applying Lemma 7.2.1 we have

$$\|D_n^1(\cdot,0)\|_{L^{q_2}} \le C\|U_n(\cdot,0)\|_{L^{2^*(s)}(\Omega)}, \tag{7.4}$$

where q_1 satisfies $\frac{1}{2^*(s)} - \frac{1}{q_1} = \frac{2s}{N}$. For $1 \le j \le k$ we see from Lemma 7.2.5 that $|V_j(\cdot,0)|^{p_n-1} \in L^r$ for any fixed number $r > \frac{N}{4s}$. Moreover, a calculation shows that

$$\left\|\rho_n^j (V_j)^{p_n-1}(\cdot,0)\right\|_{L^r} \le \lambda_n^{2s-\frac{N}{r}}.$$

Thus,

$$\begin{aligned}\|D_n^2(\cdot,0)\|_{L^{q_2}} &\le C\left\|\sum_{j=1}^{k} |\rho_n^j (V^j)^{2^*(s)-2}(\cdot,0)|\right\|_{L^r} \|U_n(\cdot,0)\|_{L^{2^*(s)}(\Omega)} \\ &\le C\lambda_n^{2s-\frac{N}{r}},\end{aligned} \tag{7.5}$$

where q_2 is such that $N\left(\frac{1}{r} + \frac{N-2s}{2N} - \frac{1}{q_2}\right) = 2s$. We note that $2s - \frac{N}{r} = \frac{N-2s}{2} - \frac{N}{q_2}$, and it is easy to check that $\frac{N}{N-2s} < q_2 < \frac{2N}{N-2s}$ for r sufficiently close to $\frac{N}{4s}$. In view of the definition (7.3), the estimates (7.5) and (7.4) imply

$$\|D_n^1(\cdot,0)\|_{\lambda_n,q_1,q_2} + \|D_n^2(\cdot,0)\|_{\lambda_n,q_1,q_2} \le C. \tag{7.6}$$

On the other hand, since $\|R_n\| = o(1)$ we have $\|R_n^{2^*(s)-2}(\cdot,0)\|_{L^{\frac{N}{2s}}(\Omega)} = \|R_n(\cdot,0)\|^{\frac{4s}{N-2s}}_{L^{\frac{2N}{N-2s}}(\Omega)} = o(1)$. Thus, applying Lemma 7.4.2 we get

$$\|D_n^3(\cdot,0)\|_{\lambda_n,q_1,q_2} \le o(1)\|D_n(\cdot,0)\|_{\lambda_n,q_1,q_2}. \tag{7.7}$$

Combining (7.6) and (7.7) we have

$$\begin{aligned}\|D_n(\cdot,0)\|_{\lambda_n,q_1,q_2} &\le \|D_n^1(\cdot,0)\|_{\lambda_n,q_1,q_2} + \|D_n^2(\cdot,0)\|_{\lambda_n,q_1,q_2} + \|D_n^3(\cdot,0)\|_{\lambda_n,q_1,q_2} \\ &\le C + o(1)\|D_n(\cdot,0)\|_{\lambda_n,q_1,q_2},\end{aligned}$$

which gives $\|D_n(\cdot,0)\|_{\lambda_n,q_1,q_2} \le C$ for a constant $C > 0$ independent of $n \in \mathbb{N}$. This completes the proof. □

The next lemma is for a bootstrap argument.

Lemma 7.4.4. *Consider two numbers q_1 and q_2 such that $\frac{N+2s}{N-2s} < q_2 < \frac{2N}{N-2s} < q_1 < \frac{N}{2s}\frac{N+2s}{N-2s}$. Let γ_1 and γ_2 satisfy*

$$\frac{1}{\gamma_i} = \frac{N+2s}{N-2s}\frac{1}{q_i} - \frac{2s}{N},\ i = 1,2.$$

Assume that for some $v \in L^{q_2}(\Omega)$, $U \in H_0^1(t^{1-2s}, \mathcal{C})$ solves

$$\begin{cases} \operatorname{div}(t^{1-2s}U) = 0 & \text{in } \mathcal{C}, \\ U = 0 & \text{on } \partial_L\mathcal{C}, \\ \partial_\nu^s U \le |v|^{2^*(s)-1} + A & \text{on } \Omega\times\{0\}. \end{cases}$$

Then there is a constant $C = C(q_1, q_2, \Omega)$ such that

$$\|U(\cdot,0)\|_{\lambda,\gamma_1,\gamma_2} \le C\left(\|v\|_{\lambda,q_1,q_2}^{2^*(s)-1} + 1\right).$$

Proof. Consider two positive functions $v_1 \in L^{q_1}(\Omega)$ and $v_2 \in L^{q_2}(\Omega)$ such that $|v| \le v_1 + v_2$. Then,

$$\partial_\nu^s U \le C\left(v_1^{2^*(s)-1} + v_2^{2^*(s)-1} + 1\right).$$

Let $U_1 \in H_0^1(t^{1-2s}, C)$ and $U_2 \in H_0^1(t^{1-2s}, C)$ be solutions to

$$\begin{cases} \mathrm{div}(t^{1-2s}\nabla U_i) = 0 & \text{in } C, \\ \partial_\nu^s U_i = v_i^{2^*(s)-1} & \text{on } \Omega\times\{0\}, \end{cases} \qquad \text{for } i = 1,2. \tag{7.8}$$

We note that U_i is nonnegative. Multiplying (7.8) by U_i^β for some $\beta > 1$, we have

$$\frac{4\beta}{(\beta+1)^2}\int_C t^{1-2s}|\nabla(U_i^{(\beta+1)/2})|^2\,dxdt = \int_{\Omega\times\{0\}} v_i^{2^*(s)-1}(x)U_i^\beta(x,0)\,dx.$$

Now we apply the Sobolev-trace inequality and Hölder's inequality to get

$$\|U_i^{\frac{\beta+1}{2}}(x,0)\|_{L^{\frac{2N}{N-2s}}(\Omega)} \le C\|v^{2^*(s)-1}\|_{L^r}\|U_i^\beta\|_{L^{\frac{\beta+1}{2\beta}\frac{2N}{N-2s}}},$$

where r is chosen to be such that $\frac{1}{r} + \frac{2\beta}{\beta+1}\frac{N-2s}{2N} = 1$. We take β satisfying $\gamma_i = \frac{\beta+1}{2}\frac{2N}{N-2s}$. Then one has $(2^*(s)-1)r = q_i$ so that the above inequality gives

$$\|U_i(\cdot,0)\|_{L^{\gamma_i}} \le C\|v\|_{L^{q_i}}^p \quad \text{for } i = 1,2.$$

Thus we get

$$\begin{aligned} \|U(\cdot,0)\|_{\lambda,\gamma_1,\gamma_2} &\le \|U_1(\cdot,0)\|_{L^{\gamma_1}} + \lambda^{\frac{N}{\gamma_2}-\frac{N}{2^*(s)}}\|U_i(\cdot,0)\|_{L^{\gamma_2}} + C \\ &\le \|v_1\|_{L^{q_1}}^{2^*(s)-1} + \lambda^{\frac{N}{\gamma_2}-\frac{N}{2^*(s)}}\|v_2\|_{L^{q_2}}^{2^*(s)-1} + C. \end{aligned} \tag{7.9}$$

From the fact that $\frac{1}{2^*(s)-1}\left(\frac{N}{\gamma_2} - \frac{N}{2^*(s)}\right) = \frac{N}{q_2} - \frac{N}{2^*(s)}$, the estimate (7.9) implies

$$\|U(\cdot,0)\|_{\lambda,\gamma_1,\gamma_2} \le C\left(\|v\|_{\lambda,q_1,q_2}^{2^*(s)-1} + 1\right),$$

which shows the lemma. □

Proof of Proposition 7.4.1. By the result of Lemma 7.4.3, there exists two numbers $q_1 \in \left(\frac{2N}{N-2s}, \infty\right)$ and $q_2 \in \left(\frac{N}{N-2s}, \frac{2N}{N-2s}\right)$ satisfying

$$\sup_{n\in\mathbb{N}} \|D_n(\cdot,0)\|_{\rho_n,q_1,q_2} \le C.$$

Then, by Lemma 7.4.4 we have

$$\sup_{n\in\mathbb{N}} \|D_n(\cdot,0)\|_{\rho_n,\gamma_1,\gamma_2} \le C,$$

where γ_1 and γ_2 satisfy $\frac{1}{\gamma_i} = \frac{N+2s}{N-2s}\frac{1}{q_i} - \frac{2s}{N}$ for $i = 1,2$. Iteratively applying this process with Hölder's inequality, one can conclude the desired result. □

7.5 Integral estimates

In this section we establish some sharp L^q estimates for solution sequence $\{U_n\}$ on some suitable annuli around the slowest bubbling point x_n, which play a fundamental role to prove our main theorems. Let us define several domains:

- $B^N(x,r) = \{y \in \mathbb{R}^N : |x-y| \le r\}$ for $x \in \mathbb{R}^N$ and $r > 0$.
- $B^{N+1}(x,r) = \{z \in \mathbb{R}^{N+1}_+ : |z-(x,0)| \le r\}$ for $x \in \mathbb{R}^N$ and $r > 0$.
- For $d = N, N+1$, $A^d(x,[r_1,r_2]) = B^d(x,r_2) \setminus B^d(x,r_1)$ for $x \in \mathbb{R}^d$ and $r_2 > r_1 > 0$.
- For a domain $D \in \mathbb{R}^{N+1}_+$
  $\partial_+ D = \{(x,t) \in \partial D : t > 0\}$,
  $\partial_b D = \{x \in \mathbb{R}^N : (x,0) \in \partial D \cap \mathbb{R}^N \times \{0\}\}$.

Consider the annuli $A^N(x_n,[5m\lambda_n^{-1/2},(5m+5)\lambda_n^{-1/2}])$, $1 \le m \le k+1$. By choosing a subsequence, we may assume that for some $m \in \{1,\cdots,k+1\}$, the annuli $A^N(x_n,[5m\lambda_n^{-1/2},5(m+1)\lambda_n^{-1/2}])$ does not contain any other bubbling points. Let

$$\begin{cases} \mathcal{A}_n^1(d) = A^d(x_n,[(5m+1)\lambda_n^{-1/2},(5m+4)\lambda_n^{-1/2}]) \cap C \text{ or } \Omega, \\ \mathcal{A}_n^2(d) = A^d(x_n,[(5m+2)\lambda_n^{-1/2},(5m+3)\lambda_n^{-1/2}]) \cap C \text{ or } \Omega, \end{cases} \quad \text{for} \quad n \in \mathbb{N}, \quad d = N, N+1.$$

For a measurable set $A \subset \mathbb{R}^{n+1}_+$ we define a weighted measure

$$m_s(A) = \int_A t^{1-2s}dxdt, \tag{7.1}$$

and a weighted average

$$\inf_A f(x,t)t^{1-2s}dxdt = \frac{\int_A f(x,t)t^{1-2s}dxdt}{\int_A t^{1-2s}dxdt}. \tag{7.2}$$

Now we state the result on the integral esimates of U_n on the annuli $\mathcal{A}_n^1(N)$ and $\mathcal{A}_n^1(N+1)$.

Proposition 7.5.1. *Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of solutions to* (7.10) *with $p = p_n \to 2^*(s)-2$ such that $\|U_n\| < C$ for some $C > 0$ independent of $n \in \mathbb{N}$. Then, for any $q > 1$, there exists a constant $C_q > 0$ such that*

$$\sup_{n\in\mathbb{N}} \left\{ \inf_{\mathcal{A}_n^1(N+1)} |U_n(x,t)|^q t^{1-2s}dxdt + \inf_{\mathcal{A}_n^1(N)} |U_n(x,0)|^q dx \right\} \le C_q. \tag{7.3}$$

To prove this proposition, we need the following lemma.

Lemma 7.5.2. *For $f \geq 0$, assume that $U \in H_0^1(t^{1-2s}, \mathcal{C})$ satisfies*

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}, \\ \partial_\nu^s U = f & \text{on } \Omega \times \{0\}, \\ U = 0 & \text{on } \partial_L \mathcal{C}. \end{cases}$$

For $\gamma \in \left(1, \frac{N-2s+2}{N-2s+1}\right)$, there exists a constant $C_q > 0$ such that

$$\left(\inf_{B^{N+1}(x,r)} t^{1-2s} U^\gamma dxdt\right)^{1/\gamma} \leq \inf_{B^{N+1}(x,1)} t^{1-2s} U^\gamma dxdt + C_q \int_r^1 \left(\frac{1}{\rho^{N-2s}} \int_{B^N(x,\rho)} f(y)dy\right) \frac{d\rho}{\rho}$$

holds for any $x \in \Omega$ and $r \in (0, r_0)$ where $r_0 = dist(x, \partial\Omega)$.

This lemma is analogous to Proposition C.1 in [CPY]. We refer to Appendix A for a proof.

Proof of Proposition 7.5.1. We consider the function D_n such that

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla D_n) = 0 & \text{in } \mathcal{C}, \\ D_n = 0 & \text{on } \partial_L \mathcal{C}, \\ \partial_\nu^s D_n = |U_n|^{2^*(s)-1} + C & \text{on } \Omega \times \{0\}. \end{cases} \tag{7.4}$$

Then we have $\|D_n\| \leq C\|U_n\| + C$, and also $|U_n| \leq D_n$ by the maximum principle. Choose a point $y \in \Omega$. For $\gamma \in \left(1, \frac{N-2s+2}{N-2s+1}\right)$ we claim that

$$\sup_{r \in (\lambda_n^{-1/2}, 1)} \inf_{B^{N+1}(y,r)} t^{1-2s} |D_n|^\gamma(x,t) dxdt \leq C, \tag{7.5}$$

with $C > 0$ independent of $y \in \Omega$ and $n \in \mathbb{N}$. We first note that

$$\sup_{n \in \mathbb{N}} \|D_n\| \leq C \sup_{n \in \mathbb{N}} \|U_n\| + C \leq C.$$

Thus, using the Sobolev embedding (7.24) and Hölder's inequality we deduce

$$\inf_{B^{N+1}(y,1)} t^{1-2s} |D_n|^\gamma(x,t) dxdt \leq C.$$

Combining this with Lemma 7.5.2, for each $0 < r < \operatorname{dist}(y, \partial\Omega)$ we get

$$\left(\inf_{B^{N+1}(y,r)} t^{1-2s} D_n^\gamma dxdt\right)^{1/\gamma} \leq C + C \int_r^1 \left[\frac{1}{\rho^{N-2s}} \int_{B^N(y,\rho)} \left(|U_n|^{2^*(s)-1}(x,0) + C\right) dx\right] \frac{d\rho}{\rho}. \tag{7.6}$$

In order to bound the last term on the right, we set $q_1 = \frac{N(N+2s)}{s(N-2s)}$ and $q_2 = \frac{N+2s}{N-2s}$, and apply Proposition 7.4.1 to find functions $w_n^1 \in L^{q_1}(\Omega)$ and $w_n^2 \in L^{q_2}(\Omega)$ such that $|U_n| \leq w_n^1 + w_n^2$ and

$$\|w_n^1\|_{L^{q_1}} \leq C \quad \text{and} \quad \|w_n^2\|_{L^{q_2}} \leq C\lambda_n^{N/(p+1)-N/q_2}. \tag{7.7}$$

Then,

$$\begin{aligned}&\int_{\sigma_n^{-1/2}}^1 \frac{1}{t^{N-2s+1}}\left[\int_{B_t(x_n)} U^p(y,0)dy\right]dt\\&\qquad\le C\int_r^1 \frac{1}{t^{N-2s+1}}\left[\int_{B^N(y,t)}(w_n^1)^p(x)dx\right]dt + C\int_r^1 \frac{1}{t^{N-2s+1}}\left[\int_{B^N(y,t)}(w_n^2)^p(x)dx\right]dt.\end{aligned}\tag{7.8}$$

We use (7.7) to deduce

$$\int_r^1 \frac{1}{t^{N-2s+1}}\left[\int_{B^N(y,t)}(w_n^1)^p(x)dx\right]dt \le C\int_{\sigma_n^{-1/2}}^1 \frac{1}{t^{N-s}}(t^{N(N-2s+1)/N})\|(w_n^1)^p\|_{L^{\frac{N}{s}}(\Omega)} \le C,$$

and

$$\begin{aligned}&\int_r^1 \frac{1}{t^{N-2s+1}}\left[\int_{B^N(y,t)}(w_n^2)^p(x,0)dx\right]dt\\&\qquad\le \int_{\sigma_n^{-1/2}}^1 \frac{1}{t^{N-2s+1}}\left[C\sigma_n^{\frac{N-2s}{2}-\frac{N(N-2s)}{N+2s}}\right]^{\frac{N+2s}{N-2s}}dt = \sigma_n^{(N-2s)/2}\sigma^{(N-2s)/2} \le C.\end{aligned}$$

These two estimates (7.8) and (7.6) prove the claim (7.5). As a result we have

$$\sup_{n\in\mathbb{N}}\inf_{A_n^{N+1}} |U_n(x,t)|^\gamma t^{1-2s}dxdt \le C. \tag{7.9}$$

To complete the proof, we only need to raise γ to higher orders in the above average estimate. In this regard, we set

$$\tilde U_n(z) = U_n(\lambda_n^{-\frac{1}{2}}z + (x_n,0)).$$

Then it satisfies

$$\begin{cases}\operatorname{div}(t^{1-2s}\nabla\tilde U_n) = 0, & \text{in } B^{N+1}(0,5m+5)\\ \partial_\nu^s\tilde U_n = \lambda^{-s}(\tilde U_n^{p-1}+C)\tilde U_n & \text{on } B^N(0,5m+5)\times\{0\},\end{cases}$$

and for $\gamma\in\left(1,\frac{N-2s+2}{N-2s+1}\right)$, the estimate (7.10) gives

$$\int_{A^{N+1}(0,[5m,5m+5])} t^{1-2s}\tilde U_n^\gamma dxdt \le C. \tag{7.10}$$

Moreover, since $A^N(x_n,[5m\lambda_n^{-1/2},5(m+1)\lambda_n^{-1/2}])$ does not any bubbling point of U_n, we easily get

$$\lim_{n\to\infty}\int_{A^{N+1}(0,[5m+1,5m+4])}\tilde U_n(x,0)^{p+1}dx = 0.$$

Given this and (7.10), we may apply Lemma 7.B.1 to deduce that for any $q>1$,

$$\int_{A^{N+1}(0,[5m+2,5m+3])} t^{1-2s}\tilde U_n^q dxdt + \int_{A^N(0,[5m+2,5m+3])}\tilde U_n^q dx \le C_q.$$

By writing down this inequality in terms of U_n on $\mathcal{A}_n^{N+1}$ and $\mathcal{A}_n^N$, we get the desired inequality (7.3). The proof is completed. □

Proposition 7.5.3. *Let* $\{U_n\}_{n\in\mathbb{N}}$ *be a sequence of solutions to* (7.10) *with* $p = p_n \to 2^*(s) - 2$ *such that* $\|U_n\|$ *is bounded uniformly for* $n \in \mathbb{N}$. *Then there exists* $C > 0$ *independent of* n *such that*

$$\int_{\mathcal{A}_n^2(N+1)} t^{1-2s}|\nabla U_n(x,t)|^2\,dxdt \le C\lambda_n^{\frac{2s-N}{2}}$$

Proof. Let $\phi_n \in C_0^\infty(A^{N+1}(x_n, [(5m+1)\lambda_n^{-1/2}, (5m+4)\lambda_n^{-1/2}]))$ be a sequence of cut-off functions such that $\phi_n = 1$ on $A^{N+1}(x_n, [(5m+2)\lambda_n^{-1/2}, (5m+3)\lambda_n^{-1/2}])$ and $0 \le \phi_n \le 1$, $|\nabla\phi_n| \le C\lambda_n^{1/2}$ on $A^{N+1}(x_n, [(5m+1)\lambda_n^{-1/2}, (5m+4)\lambda_n^{-1/2}])$. Then we see from (7.10) that

$$\begin{aligned}\int_{\mathcal{A}_n^1(N+1)} & t^{1-2s}\nabla U_n(x,t)\cdot\nabla\left(\phi_n^2(x,t)U_n(x,t)\right)dxdt \\ &\le C_s\int_{\mathcal{A}_n^1(N)}\left(|U_n(x,0)|^{p_n+1} + \mu|U(x,0)|\right)|\phi_n^2(x,0)U_n(x,0)|\,dx, \quad (7.11)\end{aligned}$$

which yields

$$\begin{aligned}&\int_{\mathcal{A}_n^1(N+1)} t^{1-2s}\phi_n^2(x,t)|\nabla U_n(x,t)|^2\,dxdt \\ &\le C\int_{\mathcal{A}_n^1(N)}|U_n(x,0)|^{p_n+2} + |U_n(x,0)|^2\,dx + C\int_{\mathcal{A}_n^1(N+1)} t^{1-2s}|U_n(x,t)\nabla\phi(x,t)|^2\,dxdt \\ &\le C\int_{\mathcal{A}_n^1(N)}\left(|U_n(x,0)|^{2^*(s)} + |U_n(x,0)|^2 + 1\right)dx + C\lambda_n^1\int_{\mathcal{A}_n^1(N+1)} t^{1-2s}|U_n(x,t)|^2\,dxdt.\end{aligned}$$

Then, this and Proposition 7.5.1 show that

$$\int_{\mathcal{A}_n^2(N+1)} t^{1-2s}|\nabla U(x,t)|^2\,dxdt \le C\lambda_n^{-\frac{N}{2}} + C\lambda_n^{-\frac{N+2-2s}{2}+1} \le C\lambda_n^{\frac{2s-N}{2}}.$$

The proof is completed. □

7.6 End of proofs of main theorems

We shall complete in this section the proof of Theorems 7.1.1 and 7.1.2. As we explained before, the strategy for the proof of Theorem 7.1.2 is to show there could be no bubbles in the decomposition (7.1) for any uniformly norm bounded sequence of solutions to (7.10) with $p = p_n \to 2^*(s) - 2$. Indeed, we will show a contradiction takes place if we assume that there are bubbles. This will be accomplished by using a local Pohozaev identity on concentric balls centered the bubbling point x_n, the blow up rate of which is minimal among all bubbling points.

Proof of Theorem 7.1.2. We denote

$$\mathcal{E}_n(N,l) = B^N(x_n, l\lambda_n^{-1/2}) \cap \Omega, \quad \mathcal{E}_n(N+1,l) = B^{N+1}((x_n,0), l\lambda_n^{-1/2}) \cap C$$

where $l \in (5m+2, 5m+3)$. By the local Pohozaev identity (7.23), we have

$$
\begin{aligned}
&C_s\left\{\left(\frac{N}{p_n+2}-\frac{N-2s}{2}\right)\int_{\mathcal{E}_n(N,l)}|U_n(x,0)|^{p_n+2}dx+\mu s\int_{\mathcal{E}_n(N,l)}|U_n(x,0)|^2dx\right\}\\
&=\int_{\partial\mathcal{E}_n(N,l)}\left(\frac{\mu}{2}|U_n(x,0)|^2+\frac{1}{p_n+2}|U_n(x,0)|^{p_n+2}\right)(x-x_0,\nu_x)\,dS_x\\
&\quad+\int_{\partial_+\mathcal{E}_n(N+1,l)}t^{1-2s}\left((z-z_0,\nabla U_n(z))\nabla U_n(z)-(z-z_0)\frac{|\nabla U_n(z)|^2}{2},\nu_z\right)dS_z\\
&\quad+\left(\frac{N-2s}{2}\right)\int_{\partial_+\mathcal{E}_n(N+1,l)}t^{1-2s}U_n(z)\frac{\partial U_n(z)}{\partial \nu_z}\,dS_z,
\end{aligned}
\tag{7.1}
$$

where $x_0 \in \mathbb{R}^N$ is arbitrary, $z_0 = (x_0, 0)$ and $z = (x, t)$. We decompose $\partial\mathcal{E}_n(N, l)$ as

$$\partial\mathcal{E}_n(N,l) = \partial_{\text{int}}\mathcal{E}_n(N,l) \cup \partial_{\text{ext}}\mathcal{E}_n(N,l)$$

where $\partial_{\text{int}}\mathcal{E}_n(N,l) := \partial\mathcal{E}_n(N,l) \cap \Omega$ and $\partial_{\text{ext}}\mathcal{E}_n(N,l) := \partial\mathcal{E}_n(N,l) \cap \partial\Omega$. Similarly,

$$\partial_+\mathcal{E}_n(N+1,l) = \partial_{\text{int}}\mathcal{E}_n(N+1,l) \cup \partial_{\text{ext}}\mathcal{E}_n(N+1,l)$$

where $\partial_{\text{int}}\mathcal{E}_n(N+1,l) := \partial_+\mathcal{E}_n(N+1,l) \cap C$ and $\partial_{\text{ext}}\mathcal{E}_n(N+1,l) := \partial_+\mathcal{E}_n(N+1,l) \cap \partial C$. For each x_n and l, we have two cases:

$$\text{(i) } B^N(x_n,l) \subset \Omega \quad \text{or} \quad \text{(ii) } B^N(x_n,l) \not\subset \Omega.$$

For the case (i), we take $x_0 = x_n$. For the case (ii), we take $x_0 \in \mathbb{R}^N \setminus \Omega$ such that $|x_0 - x_n| \le C\lambda_n^{-1/2}$ and $\nu_x \cdot (x - x_0) \le 0$ at all $x \in \partial_{\text{ext}}\mathcal{E}_n(N, l)$. Then, we see from the fact $\nu_z = (\nu_x, 0)$ that

$$\nu_z \cdot (z - z_0) = (\nu_x, 0) \cdot (x - x_0, t - 0) = \nu_x \cdot (x - x_0) \le 0$$

for any $z = (x,t) \in \partial_{\text{ext}}\mathcal{E}_n(N+1,l)$. Then, the fact $u_n = 0$ on $\partial_{\text{ext}}\mathcal{E}_n(N,l) \cup \partial_{\text{ext}}\mathcal{E}_n(N+1,l)$ yields

$$
\begin{aligned}
&\int_{\partial_{\text{ext}}\mathcal{E}_n(N,l)}\left(\frac{\mu}{2}|U_n(x,0)|^2+\frac{1}{p_n+2}|U_n(x,0)|^{p_n+2}\right)(x-x_0,\nu_x)\,dS_x = 0,\\
&\int_{\partial_{\text{ext}}\mathcal{E}_n(N+1,l)}t^{1-2s}U_n(z)\frac{\partial U_n(z)}{\partial\nu_z}\,dS_z = 0.
\end{aligned}
$$

Also, since $\nabla U_n = \pm|\nabla U_n|\nu_z$ on $\partial_{\text{ext}}\mathcal{E}_n(N+1,l)$, we see

$$
\begin{aligned}
&\int_{\partial_{\text{ext}}\mathcal{E}_n(N+1,l)}t^{1-2s}\left((z-z_0,\nabla U_n(z))\nabla U_n(z)-(z-z_0)\frac{|\nabla U_n(z)|^2}{2},\nu_z\right)dS_z,\\
&=\int_{\partial_{\text{ext}}\mathcal{E}_n(N+1,l)}t^{1-2s}\frac{|\nabla U_n(z)|^2}{2}\,(z-z_0,\nu_z)\;dS_z \le 0.
\end{aligned}
$$

Combining this with (7.1), we obtain

$$\begin{aligned}\int_{\mathcal{E}_n(N,l)} |U_n(x,0)|^2\,dx \le C\lambda_n^{-1/2} &\int_{\partial_{\text{int}}\mathcal{E}_n(N,l)} \left(|U_n(x,0)|^2 + |U_n(x,0)|^{p_n+2}\right) dS_x \\ &+ C\int_{\partial_{\text{int}}\mathcal{E}_n(N+1,l)} t^{1-2s}|U_n(z)||\nabla U_n(z)|\,dS_z \\ &+ C\lambda_n^{-1/2}\int_{\partial_{\text{int}}\mathcal{E}_n(N+1,l)} t^{1-2s}|\nabla U_n(z)|^2\,dS_z.\end{aligned} \tag{7.2}$$

Extending U_n to 0 on $\mathbb{R}^{N+1}\setminus C$ and integrating (7.2) with respect to l, we get

$$\begin{aligned}\int_{5m+2}^{5m+3}\int_{\mathcal{E}_n(N,l)} |U_n(x,0)|^2\,dx\,dl \le C&\int_{\mathcal{A}_n^2(N)} \left(|U_n(x,0)|^2 + |U_n(x,0)|^{p_n+2}\right) dx \\ &+ C\lambda_n^{1/2}\int_{\mathcal{A}_n^2(N+1)} t^{1-2s}|U_n(z)||\nabla U_n(z)|\,dz \\ &+ C\int_{\mathcal{A}_n^2(N+1)} t^{1-2s}|\nabla U_n(z)|^2\,dz,\end{aligned}$$

from which we deduce that

$$\int_{\mathcal{E}_n(N,(5m+2)\lambda_n^{-1/2})} |U(x,0)|^2\,dx \le \int_{5m+2}^{5m+3}\int_{\mathcal{E}_n(N,l)} |U(x,0)|^2\,dx\,dl \le C\lambda_n^{\frac{2s-N}{2}}, \tag{7.3}$$

by applying Proposition 7.5.1, Proposition 7.5.3 and Hölder inequality.

Next, we recall Lemma 7.2.4 that we have a representation

$$U_n = V^0 + \sum_{j=1}^{k}\rho_n^j(V^j) + R_n$$

with some $R_n \to 0$ in $H_0^1(t^{1-2s}, C)$. We also may assume that our slowest bubbling point x_n is x_n^1. Then, one can observe by extending $U_n = 0$ on $\mathbb{R}_+^{N+1}\setminus\Omega$ that for large n

$$\begin{aligned}&\int_{\mathcal{E}_n(N,(5m+2)\lambda_n^{-1/2})} |U_n(x,0)|^2\,dx \\ &= \int_{B^N(x_n,(5m+2)\lambda_n^{-1/2})} |U_n(x,0)|^2\,dx \ge \int_{B^N(x_n,\lambda_n^{-1})} |U_n(x,0)|^2\,dx \\ &\ge C\int_{B^N(x_n,\lambda_n^{-1})} |\rho_n^1(V^1)(x,0)|^2\,dx \\ &\qquad - C\int_{B^N(x_n,\lambda_n^{-1})} \sum_{j=2}^{k} |\rho_n^j(V^j)(x,0)|^2 + |V^0(x,0)|^2 + |R_n(x,0)|^2\,dx.\end{aligned}$$

One can compute

$$\int_{B^N(x_n,\lambda_n^{-1})} |\rho_n^1(V^1)(x,0)|^2\,dx = \left(\int_{B^N(0,1)} |V^1(x,0)|^2\,dx\right)\lambda_n^{-2s}$$

and

$$\int_{B^N(x_n,\lambda_n^{-1})} |\rho_n^j(V^j)(x,0)|^2\,dx = \left(\int_{S_n^j} |V^j(x,0)|^2\,dx\right)(\lambda_n^j)^{-2s}$$
$$= \left(\int_{S_n^j} |V^j(x,0)|^2\,dx\right)\left(\frac{\lambda_n^j}{\lambda_n}\right)^{-2s}\lambda_n^{-2s},$$

where

$$S_n^j := \lambda_n^j(B^N(x_n,\lambda_n^{-1}) - x_n^j).$$

Then, the fact

$$\frac{\lambda_n^j}{\lambda_n} + \lambda_n\lambda_n^j|x_n - x_n^j|^2 \to \infty \text{ as } n \to \infty \text{ for all } j \neq 1,$$

implies that

$$\left(\int_{S_n^j} |V^j(x,0)|^2\,dx\right)\left(\frac{\lambda_n^j}{\lambda_n}\right)^{-2s} = o(1).$$

Also, since $V^0 \in L^\infty(C)$ and $R_n = o(1)$ in $H_0^1(t^{1-2s}, C)$ as $n \to \infty$, we see

$$\int_{B^N(x_n,\lambda_n^{-1})} |V^0(x,0)|^2\,dx \le C\lambda_n^{-N} \le o(1)\lambda_n^{-2s}$$

and

$$\int_{B^N(x_n,\lambda_n^{-1})} |R_n(x,0)|^2\,dx \;\le C\left(\int_\Omega |R_n(x,0)|^{2^*(s)}\,dx\right)^{\frac{2}{2^*(s)}} {\lambda_n}^{-2s} = o(1){\lambda_n}^{-2s}$$

from the Sobolev-trace inequality (7.19). Thus we deduce

$$\int_{\mathcal{E}_n(N,(5m+2)\lambda_n^{-1/2})} |U_n(x,0)|^2\,dx \ge c\lambda_n^{-2s}. \tag{7.4}$$

Now, combining (7.3) with (7.4) we finally obtain

$$\lambda_n^{-2s} \le C\lambda_n^{\frac{2s-N}{2}}.$$

However, since $\lim_{n\to\infty}\lambda_n = \infty$, this contradicts with our assumption $N > 6s$. Thus, one can conclude that there are no bubbles in U_n so that $U_n \to V^0$ in $H_0^1(t^{1-2s}, C)$. This completes the whole proof of Theorem 7.1.2.

□

Proof of Theorem 7.1.1. We use the variational methods and a topological index theory to construct infinitely many solutions to (7.2). We have already seen that (7.2) is equivalent to (7.9). So let us define

$$I_\epsilon(u) := \frac{1}{2}\int_C t^{1-2s}|\nabla U|^2 dxdt - \frac{\mu}{2}\int_\Omega |U(x,0)|^2 dx - \frac{1}{2^*(s)-\epsilon}\int_\Omega |U(x,0)|^{2^*(s)-\epsilon} dx, \tag{7.5}$$

which is a variational functional for (7.10). Then, a variational functional for (7.9) corresponds to (7.5) with $\epsilon = 0$.

For a closed $\mathbb{Z}_2$ invariant set $X \subset H_0^1(t^{1-2s}, C)$, we denote by $\gamma(X)$ the topological genus of X which stands for the smallest integer m such that there is an odd map $\phi \in C(X, \mathbb{R}^m \setminus \{0\})$. For $k \in \mathbb{N}$ we define a family of sets F_k by

$$F_k = \{X \subset H_0^1(t^{1-2s}, C) : X \text{ is compact, } \mathbb{Z}_2\text{-invariant, and } \gamma(X) \geq k\}. \tag{7.6}$$

Consider the minimax value $c_{k,\epsilon} = \inf_{X \in F_k} \max_{u \in X} I_\epsilon(u)$. Then for any small $\epsilon > 0$, $c_{k,\epsilon}$ is a critical value of $I_\epsilon(u)$, i.e., there exists a solution $u_{k,\epsilon}$ to (7.10) such that $c_{\epsilon,k} = I_\epsilon(u_{k,\epsilon})$ (see e.g. [Gh, Corollary 7.12]). It is also well known that $c_{k,\epsilon} \to \infty$ as $k \to \infty$.

We first show that for each fixed $k \in \mathbb{N}$, $c_{k,\epsilon}$ is uniformly bounded for $\epsilon > 0$. For this we set

$$A_k := \inf_{X \in F_k} \max_{u \in X} \left[\frac{1}{2} \int_C t^{1-2s} |\nabla U|^2 dxdt - \frac{\mu}{2} \int_\Omega |U(x,0)|^2 dx - \frac{1}{2^*(s)} \int_\Omega |U(x,0)|^\sigma dx \right], \tag{7.7}$$

where $\sigma = \frac{1}{2}(2 + 2^*(s)) < 2^*(s)$. Take a constant $C > 0$ such that $\frac{1}{2^*(s)-\epsilon}|u|^{2^*(s)-\epsilon} + C \geq \frac{1}{2^*(s)}|u|^\sigma$ for all $0 < \epsilon < \sigma$ and $u \in \mathbb{R}$. Then it follows that $c_{k,\epsilon} \leq A_k + C$ for $\epsilon \in (0, \sigma)$.

On the other hand, it is easily derived from the identity $\langle I'_\epsilon(u_{k,\epsilon}), u_{k,\epsilon} \rangle \geq 0$ that

$$\int_C t^{1-2s} |\nabla U_{k,\epsilon}|^2 dxdt \leq C I_\epsilon(U_{k,\epsilon}) = C \cdot c_{k,\epsilon}, \tag{7.8}$$

where C depends only on N and s. Then, we have from the uniform boundedness of $c_{\epsilon,k}$ that

$$\sup_{\epsilon>0} \|U_{k,\epsilon}\| = \sup_{\epsilon>0} \int_C t^{1-2s} |\nabla U_{k,\epsilon}|^2 \, dxdt < \infty$$

and, consequently Theorem 7.1.2 implies that there is a subsequence of $\{U_{k,\epsilon_n}\}_{n\geq 1}$ such that U_{k,ϵ_n} converges strongly to a function U_k in $H_0^1(t^{1-2s}, C)$. It then easily follows that U_k solves the problem (7.9) and satisfies $I(U_k) = c_k = \lim_{n\to\infty} c_{k,\epsilon_n}$ up to a subsequence. Moreover, a standard argument (see e.g. [CSS]) applies to show that either $\{c_k\}_{k\in\mathbb{N}}$ has infinite number of elements or there is $m \in \mathbb{N}$ such that $c_k = c$ for all $k \geq m$ and infinitely many critical points correspond to the energy level c. Therefore the problem (7.2) is proved to have infinitely many solutions. This completes the proof of Theorem 7.1.1. □

Appendix

7.A Proof of Lemma 7.5.2

This section is devoted to prove Lemma 7.5.2. As a preliminary step, we first prove the following result.

Lemma 7.A.1. *For $f \geq 0$ we suppose that $U \in H^1_0(t^{1-2s}, C) \cap L^\infty(C)$ is a weak solution of*

$$\begin{cases} div(t^{1-2s}\nabla U) = 0 & in\, C, \\ \partial^s_\nu U(x,0) = f(x) & on\, \Omega \times \{0\}. \end{cases} \tag{7.9}$$

For $\gamma \in (1, \frac{N-2s+2}{N-2s+1})$ there exists a constant $C = C(N,\gamma)$ such that, for any $y \in \Omega$, $d > 0$ and $0 < r < \frac{1}{2}dist(y, \partial\Omega)$ we have

$$\left(d^{-\gamma} \inf_{B^{N+1}(y,r)} t^{1-2s}(U-a)^\gamma_+(x,t)dxdt\right)^{(2-\gamma)/\gamma}$$
$$\leq Cd^{-\gamma} \inf_{B^{N+1}((y,0),2r)} t^{1-2s}(U-a)^\gamma_+(x,t)dxdt + Cd^{-1}r^{-N+2s}\int_{B^N(y,r)} f(x)dx$$

provided that

$$m_s(\{(x,t) \in B^{N+1}((y,0),2r) : a < U(x,t) < d\}) \leq \frac{d^{-\gamma}}{2}\int_{B^{N+1}(x,r)} t^{1-2s}(U-a)^\gamma_+ dxdt. \tag{7.10}$$

Here the constant C is independent of a and d.

Proof. Without loss of generality, we may assume that $a = 0$. By assumption (7.10) we have

$$\int_{\{z\in B^{N+1}_x(r):U_+(z)<d\}} t^{1-2s}U^\gamma_+(z)dz \leq d^\gamma m_s\{x \in B^{N+1}(x,r) : 0 < U < d\}$$
$$\leq \frac{1}{2}\int_{B^{N+1}(z,r)} t^{1-2s}U^\gamma_+(z)dz.$$

It gives

$$\int_{\{z\in B^{N+1}(x,r):0<u(z)<d\}} t^{1-2s}u^\gamma_+(z)dz \leq 2\int_{\{z\in B^{N+1}(x,r):u>d\}} t^{1-2s}u^\gamma_+(z)dz.$$

Set $q = \frac{2\gamma}{2-\gamma}$ and

$$w = \left(1 + \frac{u_+}{d}\right)^{\gamma/q} - 1.$$

We can find a constant $C > 0$ such that $\left(\frac{u_+}{d}\right)^{\gamma} \leq Cw^q$ when $\frac{u_+}{d} \geq 1$. Using this we have

$$\int_{\{z\in B^{N+1}(x,r):u>d\}} t^{1-2s}u_+^{\gamma}(z)dz \leq Cd^{\gamma}\int_{B^{N+1}(x,r)} t^{1-2s}w^q(z)dz. \tag{7.11}$$

Let $\eta \in C^{\infty}(\mathbb{R}^{N+1})$ be a cut-off function supported on $B^{N+1}(x, 2r)$ such that $\eta(z) = 1$ on $B^{N+1}(x, r)$ and $|\nabla\eta(z)| \leq C/r$. By the Sobolev inequality we have

$$\begin{aligned}
&\left(r^{-(N+2-2s)}\int_{B^{N+1}(x,r)} t^{1-2s}w^q dz\right)^{2/q} \\
&\leq \left(r^{-(N+2-2s)}\int_{B^{N+1}(x,2r)} t^{1-2s}(\eta w)^q dz\right)^{2/q} \\
&\leq r^{-(N+2-2s)}r^2\int_{B^{N+1}(x,2r)} t^{1-2s}|\nabla(\eta w)|^2 dz \leq 2r^{-(N-2s)}\int_{B^{N+1}(x,2r)} t^{1-2s}(|\nabla w\cdot\eta|^2 + |w\nabla\eta|^2)dz.
\end{aligned} \tag{7.12}$$

We calculate

$$\nabla w = \frac{\gamma}{qd}\left(1 + \frac{u_+}{d}\right)^{\gamma/q-1}\nabla u_+.$$

In order to get a bound of $\int t^{1-2s}|\nabla w\cdot\eta|^2 dz$ we take $V := \left(1 - \left(1 + \frac{U_+}{d}\right)^{2\frac{\gamma}{q}-1}\right)\eta^2$ as a test function. Multiplying (7.9) by V and using Young's inequality we get

$$\begin{aligned}
&\int_{B^{N+1}(x,2r)} t^{1-2s}\nabla u_+\left(1 + \frac{u_+}{d}\right)^{2\left(\frac{\gamma}{q}\right)-2}\nabla u_+\eta^2 dz \\
&= \frac{2d}{1-\gamma}\int_{B^{N+1}(x,2r)} t^{1-2s}\nabla u_+\nabla\eta\left(1 - (1 + \frac{u_+}{d})^{2(\frac{\gamma}{q})-1}\right)\eta dz + \frac{C_s d}{1-\gamma}\int_{B^N(x,2r)} f(y)v(y)\eta^2(y)dy. \\
&\leq \frac{1}{2}\int_{B^{N+1}(x,2r)} t^{1-2s}|\nabla u_+\eta|^2\left(1 + \frac{u_+}{d}\right)^{2(\frac{\gamma}{q})-2} dz + Cd^2\int_{B^{N+1}(x,2r)} t^{1-2s}\left(1 + \frac{u_+}{d}\right)^{2-2\left(\frac{\gamma}{q}\right)}|\nabla\eta|^2 dz \\
&\quad + Cd\int_{B^N(x,2r)} fvdx.
\end{aligned}$$

Using $\frac{2\gamma}{q} - 2 = -\gamma$, we have

$$\begin{aligned}
\int_{B^{N+1}(x,2r)} t^{1-2s}|\nabla u_+|^2\left(1 + \frac{u_+}{d}\right)^{-\gamma}\eta^2 dz &\leq Cd^2\int_{B^{N+1}(x,2r)} t^{1-2s}\left(1 + \frac{u_+}{d}\right)^{-\gamma}|\nabla\eta|^2 dz \\
&\quad + Cd\int_{B^N(x,2r)} fvdy.
\end{aligned} \tag{7.13}$$

Applying $|\nabla\eta| \le C/r$ and condition (7.10) once more, we deduce

$$\begin{aligned}&\int_{B^{N+1}(x,2r)} t^{1-2s}\left(1+\frac{u_+}{d}\right)^{-\gamma}|\nabla\eta|^2dz\\ &\quad\le \frac{C}{r^2}\int_{B^{N+1}(x,2r)} t^{1-2s}\left(1+\frac{u_+}{d}\right)^{\gamma}dx \le \frac{C\,d^{-\gamma}}{r^2}\int_{B^{N+1}(x,2r)\cap\{u>0\}} t^{1-2s}u^{\gamma}dz.\end{aligned}$$

Combining this with (7.12) we get

$$\begin{aligned}&\left(\int_{B^{N+1}(x,r)} t^{1-2s}u_+^{\gamma}(z)dz\right)^{2/q}\\ &\le C\,r^{-(N-2s)}\left[r^{-2}\int_{B^{N+1}(x,2r)} t^{1-2s}w^2dz + r^{-2}d^{2-\gamma}\int_{B^{N+1}(x,r)} t^{1-2s}u_+{}^{\gamma}dz + d\int_{B^n(x,r)} fvdy\right].\end{aligned} \tag{7.14}$$

Using Hölder's inequality, (7.10) gives

$$\begin{aligned}\int_{B^{N+1}(x,2r)} t^{1-2s}w^2dz &\le \left(\int_{B^{N+1}(x,2r)} t^{1-2s}w^q dz\right)^{2/q}(m_s(B(x,2r)\cap\{u>0\}))^{1-2/q}\\ &\le d^{-\gamma}\int_{B^{N+1}(x,2r)\cap\{u>0\}} t^{1-2s}u^{\gamma}dx.\end{aligned}$$

Inserting this into (7.14) we have the desired inequality. The proof is completed. □

Proof of Lemma 7.5.2. We denote $r_k = 2^{-k}$ for $k \in \mathbb{N}$. Take $\delta > 0$ such that $\delta \le \frac{2m_s|B^{N+1}(x,r_k)|}{m_s|B^{N+1}(x,r_{k+1})|}$ whose value is independent of $k \in \mathbb{N}$. We set

$$a_{k+1} = a_k + \left(\frac{1}{\delta}\inf_{B^{N+1}(x,r_{k+1})} t^{1-2s}(u-a_k)_+^{\gamma}dxdt\right)^{1/\gamma}.$$

Let $d_k = a_{k+1} - a_k$. Then we have

$$\begin{aligned}\frac{1}{d_k^{\gamma}}\int_{B^{N+1}(x,r_{k+1})} t^{1-2s}(u-a_k)_+^{\gamma}dxdt &= \delta\ \cdot m_s|B^{N+1}(x,r_{k+1})|\\ &\ge 2m_s|B^{N+1}(x,r_k)|\\ &\ge 2m_s\left|\{(x,t)\in B^{N+1}(x,r_k) : u(x,t) > a_k\}\right|.\end{aligned}$$

By Lemma (7.A.1) we get

$$\begin{aligned}&\left(d_k^{-\gamma}r_k^{-(N+2-2s)}\int_{B^{N+1}(x,r_k)} t^{1-2s}(u-a_k)_+^{\gamma}(x,t)dxdt\right)^{2/q}\\ &\le Cd_k^{-\gamma}r_k^{-(n+2-2s)}\int_{B^{N+1}(x,2r_k)} t^{1-2s}(u-a_k)_+^{\gamma}(x,t)dxdt + Cd_k^{-\gamma}r_k^{-(N-2s)}\int_{B^{N+1}(x,r_k)} f(y)dy\\ &\le Cd_k^{-\gamma}r_k^{-(n+2-2s)}\int_{B^{N+1}(x,2r_k)} t^{1-2s}(u-a_{k-1})^{\gamma}(x,t)dxdt + Cd_k^{-\gamma}r_k^{-(N-2s)}\int_{B^{N+1}(x,r_k)} f(y)dy\\ &= C\delta\left[\frac{(a_k-a_{k-1})}{a_{k+1}-a_k}\right]^{\gamma} + Cd_k^{-1}r_k^{-(N-2s)}\int_{B^{N+1}(x,r_k)} f(y)dy.\end{aligned}$$

Using the definition of d_k we obtain

$$\delta^{2/q} \le C\delta \left[\frac{(a_k - a_{k-1})}{a_{k+1} - a_k} \right]^{\gamma} + C d_k^{-1} r_k^{-(N-2s)} \int_{B^{N+1}(x,r_k)} f(y) dy.$$

Note that $2/q = \frac{2-\gamma}{\gamma} < 1$. We choose $\delta > 0$ sufficiently small depending on C. Then it follows that

$$a_{k+1} - a_k \le \frac{1}{2}(a_k - a_{k-1}) + C r_k^{-(N-2s)} \int_{B^{N+1}(x,r_k)} f(y) dy.$$

Summing up this, we have

$$\begin{aligned} a_k &\le a_1 + C \sum_{j=1}^{k} r_j^{-(n-2s)} \int_{B^{N+1}(x,r_j)} f(y) dy \\ &\le a_1 + C \int_{r_k}^{1} \left(\frac{1}{w^{N-2s}} \int_{B^N(x,w)} f(y) dy \right) \frac{dw}{w}. \end{aligned}$$

For given $r > 0$ we take $k \in \mathbb{N}$ such that $r_{k+1} \le r < r_k$. Then it follows from the above inequality that

$$\left(\inf_{B^{N+1}(x,r)} t^{1-2s} u^{\gamma}\, dxdt \right)^{1/\gamma} \le \inf_{B^{N+1}(x,1)} t^{1-2s} u^{\gamma}\, dxdt \, + C \int_{r}^{1} \left(\frac{1}{w^{N-2s}} \int_{B^N(x,w)} f(y) dy \right) \frac{dw}{w}.$$

It completes the proof. □

7.B A variant of Moser's iteration method

Lemma 7.B.1. *Let $\gamma > 1$ and consider a function $U \in D^1(t^{1-2s}, \mathbb{R}^{N+1}_+)$ satisfying*

$$\begin{cases} div(t^{1-2s} \nabla U) = 0 & \text{in } B^{N+1}(0,5), \\ \partial_\nu^s U = a(x) U & \text{on } B^N(0,5). \end{cases} \tag{7.15}$$

Then, for each $q > 1$, there exists a number $\epsilon = \epsilon(q) >$ such that, if $\|a\|_{L^{\frac{N}{2s}} A_0^n(\frac{1}{2},4)} \le \epsilon$, then the following holds

$$\|U\|_{L^q(A_0^{N+1}(1,2))} + \|U(\cdot,0)\|_{L^q(A_0^n(1,2))} \le C \|U\|_{L^\gamma(A_0^{N+1}(\frac{1}{2},4))},$$

where C is a constant depending on q and γ.

Proof. We first take a smooth function $\phi \in C_c^\infty(B^{N+1}(0,5))$. Multiplying the function $|U|^{\beta-1} U \phi$ to (7.15) we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{N+1}_+} \text{div}(t^{1-2s} \nabla U) |U|^{\beta-1} U \phi^2 dxdt \\ &= -\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla U \nabla (|U|^{\beta-1} U \phi^2) dxdt + \int_{\mathbb{R}^N} (\partial_\nu^s U) |U|^{\beta-1} U \phi^2(x,0) dx. \end{aligned}$$

A simple computation gives

$$\begin{aligned}&\int_{\mathbb{R}^n} a(x)|U|^{\beta+1}\phi^2(x,0)dx\\&= \frac{4\beta}{(1+\beta)^2}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla(U^{\frac{\beta+1}{2}})|^2\phi^2 dxdt + \int_{\mathbb{R}^{N+1}} t^{1-2s}(\nabla U)|U|^{\beta}(2\phi\nabla\phi)\, dxdt.\end{aligned}\tag{7.16}$$

Using Young's inequality we see

$$|(\nabla U)|U|^{\beta-1}U\phi\nabla\phi| = \frac{2}{\beta+1}|(\nabla|U|^{\frac{\beta+1}{2}}\phi)(|U|^{\frac{\beta+1}{2}}\nabla\phi)| \le \frac{1}{\beta+1}\left(|(\nabla|U|^{\frac{\beta+1}{2}})\phi|^2 + \||U|^{\frac{\beta+1}{2}}\nabla\phi|^2\right).\tag{7.17}$$

We combine this inequality with (7.16) to deduce that

$$\begin{aligned}&\int_{\mathbb{R}^N} a(x)|U|^{\beta+1}\phi^2(x,0)dx + \frac{1}{\beta+1}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|U|^{\beta+1}|\nabla\phi|^2 dxdt\\&\qquad\ge \frac{3\beta}{(\beta+1)^2}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla(U^{\frac{\beta+1}{2}})|^2\phi^2\, dxdt.\end{aligned}\tag{7.18}$$

Note that $(\nabla|U|^{\frac{\beta+1}{2}})\phi = \nabla(|U|^{\frac{\beta+1}{2s}}\phi) - |U|^{\frac{\beta+1}{2}}\nabla\phi$. Then, using an elementary inequality $(a-b)^2 \ge \frac{a^2}{2} - 7b^2$ we deduce from (7.18) that

$$\begin{aligned}&\int_{\mathbb{R}^n} a(x)|U|^{\beta+1}\phi^2(x,0)\, dx + \frac{30\beta}{(1+\beta)^2}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}(|U|^{\frac{\beta+1}{2}}\nabla\phi)^2\, dxdt\\&\qquad\ge \frac{2\beta}{(1+\beta)^2}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}(\nabla(|U|^{\frac{\beta+1}{2}}\phi))^2\, dxdt.\end{aligned}\tag{7.19}$$

The left-hand side can be estimated using Hölder's inequality and the Sobolev-trace inequality as follows.

$$\begin{aligned}\int_{\mathbb{R}^n} a(x)U^{\beta+1}\phi^2(x,0)dx &\le \|a\|_{\frac{N}{s}}\|U^{\frac{\beta+1}{2}}\phi(\cdot,0)\|^2_{\frac{2N}{N-2s}}\\&\le C\epsilon\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla(U^{\frac{\beta+1}{2}}\phi)|^2 dxdt.\end{aligned}$$

We assume that $\epsilon < \frac{1}{C\beta}$. Then it follows from the above inequality and (7.19) that

$$\frac{30\beta}{(1+\beta)^2}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|U^{\frac{\beta+1}{2}}\nabla\phi|^2 dxdt \ge \frac{\beta}{(1+\beta)^2}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla(U^{\frac{\beta+1}{2}}\phi)|^2 dxdt.$$

Using the weighted Sobolev inequality and the Sobolev trace inequality we deduce that

$$\begin{aligned}&\frac{30\beta}{(1+\beta)^2}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|U^{\frac{\beta+1}{2}}\nabla\phi|^2 dxdt\\&\ge \frac{C\beta}{(1+\beta)^2}\left[\left(\int_{\mathrm{supp}\phi} t^{1-2s}|U|^{(\beta+1)\gamma}dxdt\right)^{\frac{2}{\gamma}} + \left(\int_{\mathrm{supp}\phi} |U|^{\frac{2n}{n-2s}\cdot\frac{\beta+1}{2}}(x,0)dx\right)^{\frac{n-2s}{n}}\right],\end{aligned}\tag{7.20}$$

where $\gamma = \frac{2(n-2s+2)}{n-2s}$. We use this estimate iteratively. Applying (7.20) with a suitable choice of β and ϕ at each step, and Hölder's inequality we can deduce that

$$\|U\|_{L^q(A_0^{N+1}(1,2))} + \|U(\cdot,0)\|_{L^q(A_0^{N}(1,2))} \leq C\,\|U\|_{L^\gamma(A_0^{N+1}(\frac{1}{2},4))}\,. \tag{7.21}$$

The proof is complete. □

7.C Local Pohozaev identity

For $D \subset \mathbb{R}^{N+1}_+$ we define the following sets $\partial_+ D = \{(x,t) \in \mathbb{R}^{N+1}_+ : (x,t) \in \partial D \quad \text{and} \quad t > 0\}$, and $\partial_b D = \partial D \cap \mathbb{R}^n \times \{0\}$. We state the following.

Lemma 7.C.1. *Let $E \subset \mathbb{R}^{N+1}_+$ and we assume that a function U is a solution of*

$$\begin{cases} div(t^{1-2s}\nabla U) = 0 & \text{in } E, \\ \partial^s_\nu U = f(U) & \text{on } \partial_b E. \end{cases} \tag{7.22}$$

Then, for $D \subset E$ we have the following identity.

$$\begin{aligned} &C_s\left\{N\int_{\partial_b D} F(U)dx - \left(\frac{N-2s}{2}\right)\int_{\partial_b D} Uf(U)dx\right\} \\ &= \int_{\partial_+ D} t^{1-2s}\left\langle (z-x_j,\nabla U)\nabla U - (z-x_j)\frac{|\nabla U|^2}{2}, \nu\right\rangle dS \\ &\quad + \left(\frac{N-2s}{2}\right)\int_{\partial_+ D} t^{1-2s}U\frac{\partial U}{\partial \nu}dS + \int_{\partial\partial_b D}(x,\nu)F(U)dS_x, \end{aligned} \tag{7.23}$$

where $F(s) = \int_0^s f(t)dt$.

Proof. We have the identity

$$\text{div}\left\{t^{1-2s}(z,\nabla U)\nabla U - t^{1-2s}\frac{|\nabla U|^2}{2}z\right\} + \left(\frac{N-2s}{2}\right)t^{1-2s}|\nabla U|^2 = 0. \tag{7.24}$$

Integrating this over the domain D, we get

$$\begin{aligned} &\int_{\partial_+ D} t^{1-2s}\left\langle (z,\nabla U)\nabla U - z\frac{|\nabla U|^2}{2}, \nu\right\rangle dS + C_s\int_{\partial_b D}(x,\nabla_x U)\partial^s_\nu U dx \\ &\qquad = -\left(\frac{N-2s}{2}\right)\int_D t^{1-2s}|\nabla U|^2 dxdt. \end{aligned} \tag{7.25}$$

By using $\partial^s_\nu U = f(U)$ and performing integration by parts, we deduce that

$$\begin{aligned} \int_{\partial_b D}(x,\nabla_x U)\partial^s_\nu U dx &= \int_{\partial_b D}(x,\nabla_x U)f(U)dx \\ &= \int_{\partial_b D} x\cdot\nabla_x F(U)dx \\ &= -N\int_{\partial_b D}F(U)dx + \int_{\partial\partial_b D}(x,\nu)F(U)dS_x \end{aligned}$$

and

$$\int_{D_r} t^{1-2s}|\nabla U|^2 dxdt = C_s \int_{\partial_b D} Uf(U)dx + \int_{\partial_+ D} t^{1-2s} U \frac{\partial U}{\partial \nu} dS.$$

Then (7.25) gives the desired identity. □

Chapter 8

Qualitative properties of multi-bubble solutions for nonlinear elliptic equations involving critical exponents [CKL2]

8.1 Introduction

In this paper, we perform a qualitative analysis on the problem

$$\begin{cases} -\Delta u = u^{p-\epsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1_ϵ}$$

where Ω is a bounded domain contained in $\mathbb{R}^n$ ($n \geq 3$), $p = (n+2)/(n-2)$, and $\epsilon > 0$ is a small parameter. When $\epsilon > 0$, the compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1-\epsilon}(\Omega)$ allows one to find its extremal function, hence a positive least energy solution $\bar{u}_\epsilon$ for (1.1_ϵ). However this does not hold anymore if $\epsilon = 0$ and in fact existence of solutions strongly depends on topological or geometric properties of the domain in this case (see for instance [D]). If $\epsilon = 0$ and Ω is star-shaped, then the supremum of $\bar{u}_\epsilon$ should diverge to ∞ as $\epsilon \to 0$ since an application of the Pohožaev identity [Ph] gives nonexistence of a nontrivial solution for (1.1_ϵ). In the work of Brezis and Peletier [BP], they deduced the precise asymptotic behavior of $\bar{u}_\epsilon$ when the domain Ω is the unit ball, and this result was extended to general domains by Han [H] and Rey [R], in which they independently proved that $\bar{u}_\epsilon$ blows-up at the unique point x_0 that is a critical point of the Robin function of the domain. Later, Grossi and Pacella [GP] investigated the related eigenvalue problem, obtaining estimates for its first $(n+2)$-eigenvalues, asymptotic behavior of the corresponding eigenvectors and the Morse index of $\bar{u}_\epsilon$. Since our result is closely related to their conclusion, we describe it in a detailed fashion.

Let us denote by $G = G(x,y)$ ($x, y \in \Omega$) the Green's function of $-\Delta$ with Dirichlet boundary condition satisfying

$$-\Delta G(\cdot, y) = \delta_y \text{ in } \Omega \quad \text{and} \quad G(\cdot, y) = 0 \text{ on } \partial\Omega,$$

by $H(x, y)$ its regular part, i.e.,

$$H(x, y) = \frac{\gamma_n}{|x-y|^{n-2}} - G(x, y) \quad \text{where } \gamma_n = \frac{1}{(n-2)\left|S^{n-1}\right|}, \tag{8.2}$$

and by τ the Robin function $\tau(x) = H(x, x)$. We also define the bubble $U_{\lambda,\xi}$ with the concentration rate $\lambda > 0$ and the center $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$,

$$U_{\lambda,\xi}(x) = \beta_n \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2}\right)^{\frac{n-2}{2}} \quad \text{for } x \in \mathbb{R}^n \quad \text{where } \beta_n = (n(n-2))^{\frac{n-2}{4}} \tag{8.3}$$

which are solutions of the equation

$$-\Delta U = U^p \quad \text{in } \mathbb{R}^n, \quad u > 0 \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \int_{\mathbb{R}^n} |\nabla U|^2 < \infty. \tag{8.4}$$

Theorem A (Grossi and Pacella [GP]). *Given $n \geq 3$, we consider the eigenvalue problem at a positive least energy solution $u_\epsilon = \bar{u}_\epsilon$ to (1.1_ϵ), that is,*

$$\begin{cases} -\Delta v = \mu(p-\epsilon)u_\epsilon^{p-1-\epsilon} v & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.5}$$

Let $\mu_{\ell\epsilon}$ be the ℓ-th eigenvalue of (8.5) *provided that the sequence of eigenvalues is arranged in nondecreasing order permitting duplication, and $v_{\ell\epsilon}$ the corresponding $L^\infty(\Omega)$-normalized eigenfunction (namely, $\|v_{\ell\epsilon}\|_{L^\infty(\Omega)} = 1$). Given the point $x_\epsilon \in \Omega$ such that $u_\epsilon(x_\epsilon) = \|u_\epsilon\|_{L^\infty(\Omega)}$ ($x_\epsilon \to x_0$ as $\epsilon \to 0$ by [H] and [R]), we also set*

$$\check{v}_{\ell\epsilon}(x) = v_{\ell\epsilon}\left(x_\epsilon + \frac{x}{\|u_\epsilon\|_{L^\infty(\Omega)}^{\frac{p-1-\epsilon}{2}}}\right) \quad \textit{for arbitrary } x \in \check{\Omega}_\epsilon = \|u_\epsilon\|_{L^\infty(\Omega)}^{\frac{p-1-\epsilon}{2}}(\Omega - x_\epsilon).$$

1. *If $2 \leq \ell \leq n+1$, then there exist nonzero vectors $(d_{\ell,1}, \cdots, d_{\ell,n}) \in \mathbb{R}^n$ and a constant $\widetilde{C}_1 > 0$ such that*

$$\check{v}_{\ell\epsilon} \to \sum_{k=1}^n d_{\ell,k} \frac{\partial U_{1,0}}{\partial \xi_k} \text{ in } C^1_{loc}(\mathbb{R}^n), \quad \epsilon^{-\frac{n-1}{n-2}} \check{v}_{\ell\epsilon} \to \widetilde{C}_1 \sum_{k=1}^n d_{\ell,k} \frac{\partial G}{\partial y_k}(\cdot, x_0) \text{ in } C^1(\Omega \setminus \{x_0\}).$$

   Moreover, if $\rho_2 \leq \rho_3 \leq \cdots \leq \rho_{n+1}$ are the eigenvalues of the Hessian $D^2\tau(x_0)$ of the Robin function at x_0, then

$$\mu_{\ell\epsilon} = 1 - \tilde{c}_0 \rho_\ell \epsilon^{\frac{n}{n-2}} + o\left(\epsilon^{\frac{n}{n-2}}\right)$$

   for some suitable $\tilde{c}_0 > 0$ as $\epsilon \to 0$.

2. *Assume $\ell = n+2$. Then*

$$\check{v}_{(n+2)\epsilon} \to d_{n+2}\frac{\partial U_{1,0}}{\partial \lambda} \text{ in } C^1_{loc}(\mathbb{R}^n) \quad \text{and} \quad \mu_{(n+2)\epsilon} = 1 + \tilde{c}_1\epsilon + o(\epsilon) \quad \text{as } \epsilon \to 0$$

for some $\tilde{c}_1 > 0$.

Consequently, if x_0 is a nondegenerate critical point of the Robin function τ, the Morse index of $\bar{u}_\epsilon$ is equal to 1 + (the Morse index of x_0 as a critical point of τ).

As the next step to understand equation (1.1_ϵ), one can imagine more general type of solutions so called *multi-bubbles*. Let $\{\epsilon_k\}_{k=1}^\infty$ be a sequence of small positive numbers such that $\epsilon_k \to 0$ as $k \to \infty$ and $\{u_{\epsilon_k}\}_{k=1}^\infty$ a bounded sequence in $H_0^1(\Omega)$ of solutions for (1.1_ϵ) with $\epsilon = \epsilon_k$, which blow-up at $m \in \mathbb{N}$ points $\{x_{10}, \cdots, x_{m0}\} \subset \overline{\Omega}^m$. Then by the work of Struwe [Su] on the representation of Palais-Smale sequences to (1.1_ϵ) for any $n \geq 3$, which employed the concentration-compactness principle [Ls], it can be written as

$$u_{\epsilon_k} = \sum_{i=1}^m \alpha_{ik} PU_{\lambda_{ik}\epsilon_k^{\alpha_0}, x_{ik}} + R_k \tag{8.6}$$

after extracting a subsequence if necessary. Here $\alpha_0 = 1/(n-2)$, $\{\alpha_{ik}\}_{k\in\mathbb{N}}$ and $\{\lambda_{ik}\}_{k\in\mathbb{N}}$ are sequences of positive numbers, and $\{x_{ik}\}_{k\in\mathbb{N}}$ is a sequence of elements in Ω for each fixed $i = 1, \cdots, m$ such that $\alpha_{ik} \to 1$, $\lambda_{ik} \to \lambda_{i0} > 0$ and $x_{ik} \to x_{i0} \in \Omega$ as $k \to \infty$. Also, the function $PU_{\lambda,\xi}$ is a projected bubble in $H_0^1(\Omega)$, namely, a solution of

$$\Delta PU_{\lambda,\xi} = \Delta U_{\lambda,\xi} \quad \text{in } \Omega, \quad PU_{\lambda,\xi} = 0 \quad \text{on } \partial\Omega \tag{8.7}$$

and R_k is a remainder term whose $H_0^1(\Omega)$-norm converges to 0 as $k \to \infty$. According to Bahri, Li and Rey [BLR], the blow-up rates and the concentration points $(\lambda_{10}, \cdots, \lambda_{m0}, x_{10}, \cdots, x_{m0}) \in (0,\infty)^m \times \Omega^m$ can be characterized as a critical point of the function

$$\Upsilon_m(\lambda_1, \cdots, \lambda_m, x_1, \cdots x_m) = c_1\left(\sum_{i=1}^m \tau(x_i)\lambda_i^{n-2} - \sum_{\substack{i,j=1\\ i\neq j}}^m G(x_i,x_j)(\lambda_i\lambda_j)^{\frac{n-2}{2}}\right) - c_2\log(\lambda_1\cdots\lambda_m) \tag{8.8}$$

in general, provided that $n \geq 4$. Here

$$c_1 = \left(\int_{\mathbb{R}^n} U_{1,0}^p\right)^2 \quad \text{and} \quad c_2 = \frac{(n-2)^2}{4n}\int_{\mathbb{R}^n} U_{1,0}^{p+1}. \tag{8.9}$$

Conversely, by applying the Lyapunov-Schmidt reduction method, Musso and Pistoia [MP] proved that if $n \geq 3$ and $(\lambda_{10}, \cdots, \lambda_{m0}, x_{10}, \cdots, x_{m0}) \in (0,\infty)^m \times \Omega^m$ is a C^1-stable critical point of H in the sense of Y. Li [Li1], then there is a multi-bubbling solution of (1.1_ϵ) having the form (9.33) which blows-up at each point x_{i0} with the rate of the concentration λ_{i0} $(i = 1, \cdots, m)$. This extends the existence result also achieved in paper [BLR], where the authors used the gradient flow of critical points at infinity to get solutions.

Our interest lies on the derivation of certain asymptotic behaviors of multiple bubbling solutions $\{u_\epsilon\}_\epsilon$ to (1.1_ϵ) satisfying (9.33) when ϵ converges to 0. (Precisely speaking, sequences of parameters ϵ_k, α_{ik}, λ_{ik} and x_{ik} in (9.33) should be substituted by ϵ, $\alpha_{i\epsilon}$, $\lambda_{i\epsilon}$ and $x_{i\epsilon}$, respectively, such that $\alpha_{i\epsilon} \to 1$, $\lambda_{i\epsilon} \to \lambda_{i0}$ and $x_{i\epsilon} \to x_{i0}$ as $\epsilon \to 0$. Hereafter, such a substitution is always assumed.) In particular, we shall examine the behavior of eigenpairs $(\mu_{\ell\epsilon}, v_{\ell\epsilon})$ to the linearized problem (8.5) at u_ϵ for $1 \le \ell \le (n+2)m$ as Grossi and Pacella did for single bubbles.

Firstly, we concentrate on behavior of the first m-eigenvalues and eigenvectors. Given $i,\ \ell \in \mathbb{N}$, $1 \le i \le m$, let $\tilde{v}_{\ell i\epsilon}$ be a dilation of $v_{\ell\epsilon}$ defined as

$$\tilde{v}_{\ell i\epsilon}(x) = v_{\ell\epsilon}\left(x_{i\epsilon} + \lambda_{i\epsilon}\epsilon^{\alpha_0} x\right) \quad \text{for each } x \in \Omega_{i\epsilon} := \left(\Omega - x_{i\epsilon}\right)/\left(\lambda_{i\epsilon}\epsilon^{\alpha_0}\right) \tag{8.10}$$

where $\alpha_0 = 1/(n-2)$ again.

Theorem 8.1.1. *Let $\epsilon > 0$ be a small parameter, $\{u_\epsilon\}_\epsilon$ a family of solutions for (1.1_ϵ) of the form (9.33), $\mu_{\ell\epsilon}$ the ℓ-th eigenvalue of problem (8.5) for some $1 \le \ell \le m$. Denote also as ρ_ℓ^1 the ℓ-th eigenvalue of the symmetric matrix $\mathcal{A}_1 = \left(\mathcal{A}_{ij}^1\right)_{1\le i,j\le m}$ given by*

$$\mathcal{A}_{ij}^1 = \begin{cases} -\left(\lambda_{i0}\lambda_{j0}\right)^{\frac{n-2}{2}} G\left(x_{i0}, x_{j0}\right) & \text{if } i \neq j, \\ -C_0 + \lambda_{i0}^{n-2}\tau(x_{i0}) & \text{if } i = j, \end{cases} \quad \text{where } C_0 = c_2/(c_1(n-2)) > 0. \tag{8.11}$$

Then we have

$$\mu_{\ell\epsilon} = \frac{n-2}{n+2} + b_1\epsilon + o(\epsilon) \quad \text{where } b_1 = \left(\frac{n-2}{n+2}\right)^2 + \frac{(n-2)^3 c_1}{4n(n+2)c_2}\rho_\ell^1 \tag{8.12}$$

as $\epsilon \to 0$. Moreover, there exists a nonzero column vector

$$\mathbf{c}_\ell = \left(\lambda_{10}^{\frac{n-2}{2}} c_{\ell 1}, \cdots, \lambda_{m0}^{\frac{n-2}{2}} c_{\ell m}\right)^T \in \mathbb{R}^m$$

such that for each $i \in \{1, \cdots, m\}$ the function $\tilde{v}_{\ell i\epsilon}$ converges to $c_{\ell i}U_{1,0}$ weakly in $H^1(\mathbb{R}^n)$. This $\mathbf{c}_\ell$ becomes an eigenvector corresponding to the eigenvalue ρ_ℓ^1 of $\mathcal{A}_1$, and it holds that $\mathbf{c}_{\ell_1}^T \cdot \mathbf{c}_{\ell_2}^T = 0$ for $1 \le \ell_1 \neq \ell_2 \le m$.

Next, we study the next mn-eigenvalues and corresponding eigenvectors. The first theorem for these eigenpairs concerns with asymptotic behaviors of the eigenvectors. Let us define a symmetric $m \times m$ matrix $\mathcal{M}_1 = \left(m_{ij}^1\right)_{1\le i,j\le m}$ by

$$m_{ij}^1 = \begin{cases} -G\left(x_{i0}, x_{j0}\right) & \text{if } i \neq j, \\ C_0\lambda_{i0}^{-(n-2)} + \tau\left(x_{i0}\right) & \text{if } i = j. \end{cases} \tag{8.13}$$

By Lemma 8.2.1 below, it can be checked that $\mathcal{M}_1$ is positive definite and in particular invertible. We denote its inverse by $\left(m_1^{ij}\right)_{1\le i,j\le m}$.

Theorem 8.1.2. *Assume that $m+1 \le \ell \le (n+1)m$. Then, for each $i \in \{1, \cdots, m\}$, there exists a vector $(d_{\ell,i,1}, \cdots, d_{\ell,i,n}) \in \mathbb{R}^n$, which is nonzero for some i, such that*

$$\tilde{v}_{\ell i\epsilon} \to -\sum_{k=1}^{n} d_{\ell,i,k} \frac{\partial U_{1,0}}{\partial x_k} \quad in\ C^1_{loc}(\mathbb{R}^n) \tag{8.14}$$

and

$$\begin{aligned}\epsilon^{-\frac{n-1}{n-2}} v_{\ell\epsilon}(x) \to C_1 \Bigg[\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} m_1^{ij} \left(-\frac{1}{2} \lambda_{j0}^{n-1} d_{\ell,j,k} \frac{\partial \tau}{\partial x_k}(x_{j0}) + \sum_{l \neq j} \lambda_{l0}^{n-1} d_{\ell,l,k} \frac{\partial G}{\partial y_k}\left(x_{j0}, x_{l0}\right) \right) G(x, x_{i0}) \\ + \sum_{i=1}^{m} \sum_{k=1}^{n} \lambda_{i0}^{n-1} d_{\ell,i,k} \frac{\partial G}{\partial y_k}(x, x_{i0}) \Bigg]\end{aligned} \tag{8.15}$$

in $C^1(\Omega \setminus \{x_{10}, \cdots, x_{m0}\})$ as $\epsilon \to 0$. Here $C_1 = \beta_n^p \left(\frac{n+2}{n}\right) \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^{(n+4)/2}} dx > 0$.

If $\mathbf{d}_\ell \in \mathbb{R}^{mn}$ denotes a nonzero vector defined by

$$\mathbf{d}_\ell = \left(\lambda_{10}^{\frac{n-2}{2}} d_{\ell,1,1}, \cdots, \lambda_{10}^{\frac{n-2}{2}} d_{\ell,1,n}, \lambda_{20}^{\frac{n-2}{2}} d_{\ell,2,1}, \cdots, \lambda_{(m-1)0}^{\frac{n-2}{2}} d_{\ell,m-1,n}, \lambda_{m0}^{\frac{n-2}{2}} d_{\ell,m,1}, \cdots, \lambda_{m0}^{\frac{n-2}{2}} d_{\ell,m,n} \right)^T, \tag{8.16}$$

then we can give a further description on it. Our next theorem is devoted to this fact as well as a quite precise estimate of the eigenvalues. Set an $m \times mn$ matrix $\mathcal{P} = (\mathcal{P}_{it})_{1 \le i \le m, 1 \le t \le mn}$ and a symmetric $mn \times mn$ matrix $\mathcal{Q} = (\mathcal{Q}_{st})_{1 \le s,t \le mn}$ as follows.

$$\mathcal{P}_{i,(j-1)n+k} = \begin{cases} \lambda_{j0}^{\frac{n}{2}} \dfrac{\partial G}{\partial y_k}(x_{i0}, x_{j0}) = \lambda_{j0}^{\frac{n}{2}} \dfrac{\partial G}{\partial x_k}(x_{j0}, x_{i0}) & \text{if } i \neq j, \\ -\lambda_{i0}^{\frac{n}{2}} \dfrac{1}{2} \dfrac{\partial \tau}{\partial x_k}(x_{i0}) & \text{if } i = j, \end{cases} \tag{8.17}$$

for $i, j \in \{1, \cdots, m\}$ and $k \in \{1, \cdots, n\}$, and

$$\mathcal{Q}_{(i-1)n+k,(j-1)n+q} = \begin{cases} \left(\lambda_{i0}\lambda_{j0}\right)^{\frac{n}{2}} \dfrac{\partial^2 G}{\partial x_k \partial y_q}\left(x_{i0}, x_{j0}\right) & \text{if } i \neq j, \\ -\dfrac{\lambda_{i0}^n}{2} \dfrac{\partial^2 \tau}{\partial x_k \partial x_q}(x_{i0}) + \lambda_{i0}^{\frac{n+2}{2}} \sum_{l \neq i} \lambda_{l0}^{\frac{n-2}{2}} \dfrac{\partial^2 G}{\partial x_k \partial x_q}(x_{i0}, x_{l0}) & \text{if } i = j, \end{cases} \tag{8.18}$$

for $i, j \in \{1, \cdots, m\}$ and $k, q \in \{1, \cdots, n\}$.

Theorem 8.1.3. *Let $\mathcal{A}_2$ be an $mn \times mn$ symmetric matrix*

$$\mathcal{A}_2 = \mathcal{P}^T \mathcal{M}_1^{-1} \mathcal{P} + \mathcal{Q}.$$

Then as $\epsilon \to 0$ we have

$$\mu_{\ell\epsilon} = 1 - c_0 \rho_\ell^2 \epsilon^{\frac{n}{n-2}} + o\left(\epsilon^{\frac{n}{n-2}}\right) \tag{8.19}$$

for some $c_0 > 0$ (whose value is computed in (8.1)) where ρ_ℓ^2 is the $(\ell - m)$-th eigenvalue of the matrix $\mathcal{A}_2$. Furthermore the vector $\mathbf{d}_\ell \in \mathbb{R}^{mn}$ is an eigenvector corresponding to the eigenvalue ρ_ℓ^2 of $\mathcal{A}_2$, which satisfies $\mathbf{d}_{\ell_1}^T \cdot \mathbf{d}_{\ell_2}^T = 0$ for $m+1 \le \ell_1 \neq \ell_2 \le (n+1)m$.

Remark 8.1.4. If the number of blow-up points is $m = 1$, then $\mathcal{P} = 0$ and so the matrix $\mathcal{A}_2$ is reduced to $\frac{1}{2}\lambda_{10}^n D^2\tau(x_{10})$ which is consistent with Theorem A. See also Remark 8.5.6.

Lastly, the ℓ-th eigenpair for $(n+1)m+1 \le \ell \le (n+2)m$ can be examined. Let $\mathcal{A}_3 = \left(\mathcal{A}_{ij}^3\right)_{1\le i,j\le m}$ be a symmetric matrix whose components are given by

$$\mathcal{A}_{ij}^3 = \begin{cases} -\left(\lambda_{i0}\lambda_{j0}\right)^{\frac{n-2}{2}} G\left(x_{i0}, x_{j0}\right) & \text{if } i \neq j, \\ C_0 + \lambda_{i0}^{n-2}\tau(x_{i0}) & \text{if } i = j. \end{cases} \tag{8.20}$$

Theorem 8.1.5. *For each $(n+1)m+1 \le \ell \le (n+2)m$, let ρ_ℓ^3 be the $(\ell - m(n+1))$-th eigenvalue of $\mathcal{A}_{ij}^3$, which will be shown be positive. Then there exist a nonzero vector*

$$\hat{\mathbf{d}}_\ell = \left(\lambda_{10}^{\frac{n-2}{2}} d_{\ell,1}, \cdots, \lambda_{m0}^{\frac{n-2}{2}} d_{\ell,m}\right)^T \in \mathbb{R}^m \tag{8.21}$$

and a positive number c_1 such that

$$\tilde{v}_{\ell i\epsilon} \rightharpoonup d_{\ell,i}\left(\frac{\partial U_{1,0}}{\partial\lambda}\right) \quad \text{weakly in } H^1(\mathbb{R}^n)$$

and

$$\mu_{\ell\epsilon} = 1 + c_1\rho_\ell^3\epsilon + o(\epsilon) \quad \text{as } \epsilon \to 0.$$

Furthermore, $\hat{\mathbf{d}}_\ell$ is a corresponding eigenvector to ρ_ℓ^3, and it holds that $\hat{\mathbf{d}}_{\ell_1}^T \cdot \hat{\mathbf{d}}_{\ell_2}^T = 0$ for $(n+1)(m+1) \le \ell_1 \neq \ell_2 \le (n+2)m$.

As a result, we obtain the following corollary.

Corollary 8.1.6. *Let $ind(u_\epsilon)$ and $ind_0(u_\epsilon)$ be the Morse index and the augmented Morse index of the solution u_ϵ to (1.1_ϵ), respectively. Also for the matrix $\mathcal{A}_2$ in Theorem 8.1.3, $ind(-\mathcal{A}_2)$ and $ind_0(-\mathcal{A}_2)$ are similarly understood. Then*

$$m \le m + ind(-\mathcal{A}_2) \le ind(u_\epsilon) \le ind_0(u_\epsilon) \le m + ind_0(-\mathcal{A}_2) \le (n+1)m$$

for sufficiently small $\epsilon > 0$. Therefore if $\mathcal{A}_2$ is nondegenerate, then so is u_ϵ and

$$ind(u_\epsilon) = m + ind(-\mathcal{A}_2) \in [m, (n+1)m].$$

Remark 8.1.7. By the discussion before, our results hold for solutions found by Musso and Pistoia in [MP]. Moreover, if $\epsilon_k \to 0$ as $k \to \infty$, any $H_0^1(\Omega)$-bounded sequence $\{u_{\epsilon_k}\}_{k=1}^\infty$ of solutions for (1.1_ϵ) with $\epsilon = \epsilon_k$ has a subsequence to which our work can be applied.

This extends the work of Bahri-Li-Rey [BLR] where the validity of the above corollary was obtained for $n \ge 4$. Besides Theorems 8.1.1, 8.1.2, 8.1.3 and 8.1.5 provide sharp asymptotic behaviors of the eigenpairs $(\mu_{\ell\epsilon}, v_{\ell\epsilon})$ as $\epsilon \to 0$ which were not dealt with in [BLR]. In this article we compute each component of the matrix $\mathcal{A}_2$ explicitly, which turns out to be complicated. Instead doing in this way, the authors of [BLR] gave an alternative neat description.

Our proof is based on the work of Grossi and Pacella [GP] which studied qualitative behaviors of single blow-up solutions of (1.1_ϵ), but requires a further inspection on the interaction between different bubbles here. In particular we have to control the decay of solutions u_ϵ and eigenfunctions $v_{\ell\epsilon}$ near each blow-up point in a careful way. In order to get the sharp decay of u_ϵ, we will utilize the method of moving spheres which has been used on equations from conformal geometry and related areas. (See for example [ChL, ChC, LiZ, Pa].) Furthermore we shall make use of the Moser-Harnack type estimate and an iterative comparison argument to find an almost sharp decay of $v_{\ell\epsilon}$.

The structure of this paper can be described in the following way. In Section 8.2, we gather all preliminary results necessary to deduce our main theorems. This section in particular includes estimates of the decay of the solutions u_ϵ or the eigenfunctions $v_{\ell\epsilon}$ outside of the concentration points $\{x_{10}, \cdots, x_{m0}\}$. In Section 8.3, we prove Theorem 8.1.1 which deals with the first m-eigenvalues and eigenfunctions of problem (8.5). A priori bounds for the first $(n+1)m$-eigenvalues and the limit behavior (8.14) of expanded eigenfunction $\tilde{v}_{\ell i\epsilon}$ are found in Section 8.4. Based on these results, we compute an asymptotic expansion (8.15) of the ℓ-th eigenvectors ($\ell = m+1, \cdots, (n+1)m$) and that of its corresponding eigenvalues (8.19) in Sections 8.5 and 8.6 respectively. The description of the vector $\mathbf{d}_\ell$ is also obtained as a byproduct during the derivation of (8.19). Section 8.7 is devoted to study the next m-eigenpairs, i.e., the ℓ-th eigenvalues and eigenfunctions ($\ell = (n+1)m+1, \cdots, (n+2)m$). Finally, we present the proof of Proposition 8.2.3 in Appendix 8.A, which is conducted with the moving sphere method.

Notations.

- Big-O notation and little-o notation are used to describe the limit behavior of a certain quantity as $\epsilon \to 0$.

- $B^n(x, r)$ is the n-dimensional open ball whose center is located at x and radius is r. Also, S^{n-1} is the $(n-1)$-dimensional unit sphere and $\left|S^{n-1}\right|$ is its surface area.

- $C > 0$ is a generic constant which may vary from line to line, while numbers with subscripts such as c_0 or C_1 have positive fixed values.

- For any number $c \in \mathbb{R}$, $c = c_+ - c_-$ where $c_+, c_- \geq 0$ are the positive or negative part of c, respectively.

- For any vector $\mathbf{v}$, its transpose is denoted as $\mathbf{v}^T$.

- Throughout the paper, the symbol α_0 always denotes $1/(n-2)$.

8.2 Preliminaries

In this section, we collect some results necessary for our analysis. For the rest of the paper, we write $x_1, \cdots, x_m$ to denote the concentration points, dropping out the subscript 0. The same omission also applies to the concentrate rates $\lambda_1, \cdots, \lambda_m$.

Lemma 8.2.1. *If we set a matrix* $\mathcal{M}_2 = \left(m_{ij}^2\right)_{1\le i,j\le m}$ *by*

$$m_{ij}^2 = \begin{cases} -G(x_i, x_j) & \text{if } i \neq j, \\ \tau(x_i) & \text{if } i = j, \end{cases} \tag{8.1}$$

then it is a non-negative definite matrix.

Proof. See Appendix A of Bahri, Li and Rey [BLR]. □

Fix any $i \in \{1, \cdots, m\}$ and decompose u_ϵ in the following way.

$$u_\epsilon = U_{\lambda_{i\epsilon}\epsilon^{\alpha_0},x_{i\epsilon}} + (PU_{\lambda_{i\epsilon}\epsilon^{\alpha_0},x_{i\epsilon}} - U_{\lambda_{i\epsilon}\epsilon^{\alpha_0},x_{i\epsilon}}) + (\alpha_{i\epsilon} - 1)PU_{\lambda_{i\epsilon}\epsilon^{\alpha_0},x_{i\epsilon}} + \sum_{j\neq i} \alpha_{j\epsilon} PU_{\lambda_{i\epsilon}\epsilon^{\alpha_0},x_{i\epsilon}} + R_\epsilon. \tag{8.2}$$

Then we rescale it to define

$$\tilde{u}_{i\epsilon}(x) = (\lambda_{i\epsilon}\epsilon^{\alpha_0})^{\sigma_\epsilon} u_\epsilon\left(x_{i\epsilon} + \lambda_{i\epsilon}\epsilon^{\alpha_0} x\right) \quad \text{where } \sigma_\epsilon = \frac{2}{p-1-\epsilon} = \frac{n-2}{2-(n-2)\epsilon/2}. \tag{8.3}$$

It immediately follows that $\{\tilde{u}_{i\epsilon}\}_\epsilon$ is a family of positive C^2-functions defined in $B^n\left(0, \epsilon^{-\alpha_0} r_0\right)$ for some $r_0 > 0$ small enough (determined in the next lemma), which are solutions of $-\Delta u = u^{p-\epsilon}$. Moreover it has the following property.

Lemma 8.2.2. *The sequence* $\{\tilde{u}_{i\epsilon}\}_\epsilon$ *satisfies* $\|\tilde{u}_{i\epsilon}\|_{L^\infty\left(B^n\left(0,\epsilon^{-\alpha_0} r_0\right)\right)} \le c$ *for some small* $r_0 > 0$ *and converges to* $U_{1,0}$ *weakly in* $H^1(\mathbb{R}^n)$ *as* $\epsilon \to 0$.

Proof. For fixed i, let us denote $\tilde{f}(x) = (\lambda_{i\epsilon}\epsilon^{\alpha_0})^{\sigma_\epsilon} f\left(x_{i\epsilon} + \lambda_{i\epsilon}\epsilon^{\alpha_0} x\right)$ for $x \in \Omega_{i\epsilon} = \left(\Omega - x_{i\epsilon}\right)/(\lambda_{i\epsilon}\epsilon^{\alpha_0})$. Set also $U_j = U_{\lambda_{j\epsilon}\epsilon^{\alpha_0},x_{j\epsilon}}$ for all $j \in \{1, \cdots, m\}$. Then $\|\tilde{f}\|_{H^1(\Omega_{i\epsilon})} = (1 + o(1))\|f\|_{H^1(\Omega)}$ and

$$\tilde{u}_{i\epsilon} - U_{1,0} = \sum_{j\neq i} \alpha_{j\epsilon}\widetilde{PU}_j + \left(\widetilde{PU}_i - \widetilde{U}_i\right) + (\alpha_{i\epsilon} - 1)\widetilde{PU}_i + \widetilde{R}_\epsilon \quad \text{in } \Omega_{i\epsilon} \tag{8.4}$$

by (8.2). Observe with the maximum principle that $0 \le PU_i \le U_i$ in Ω and

$$PU_{\lambda,\xi}(x) = U_{\lambda,\xi}(x) - C_2\lambda^{\frac{n-2}{2}} H(x,\xi) + o\left(\lambda^{\frac{n-2}{2}}\right) \quad \text{in } C^0\left(\overline{\Omega}\right), \quad C_2 := \int_{\mathbb{R}^n} U_{1,0}^p > 0$$

holds for any small $\lambda > 0$ and $\xi \in \Omega$ away from the boundary. Thus we get from (8.4) and (8.7) that

$$\begin{aligned}\|PU_i - U_i\|_{H^1(\Omega)}^2 &= \int_\Omega |\nabla PU_i|^2 - 2\int_\Omega \nabla PU_i \cdot \nabla U_i + \int_\Omega |\nabla U_i|^2 = -\int_\Omega U_i^p PU_i + \int_\Omega |\nabla U_i|^2 \\ &= -\int_\Omega U_i^p (PU_i - U_i) - \int_\Omega U_i^{p+1} + \int_\Omega |\nabla U_i|^2 = o(1)\end{aligned}$$

and

$$\|PU_i\|_{H^1(\Omega)}^2 = \int_\Omega U_i^p PU_i \le \int_{\mathbb{R}^n} U_{1,0}^{p+1}$$

so that the last three terms in the right-hand side of (8.4) go to 0 strongly in $H_0^1(\Omega_{i\epsilon}) \subset H^1(\mathbb{R}^n)$. On the other hand, we have

$$\left|\int_{\mathrm{supp}(\varphi)} \nabla\widetilde{PU}_j \cdot \nabla\varphi\right| \le \|\varphi\|_{L^\infty(\Omega)} \int_{\mathrm{supp}(\varphi)} \widetilde{U}_j^{p-\epsilon} \to 0$$

as $\epsilon \to 0$ for any test function $\varphi \in C_c^\infty(\mathbb{R}^n)$. Therefore $\tilde{u}_{i\epsilon} \rightharpoonup U_{1,0}$ weakly in $H^1(\mathbb{R}^n)$.

We now attempt to attain a priori L^∞-estimate for $\{\tilde{u}_{i\epsilon}\}_\epsilon$. Firstly we fix a sufficiently small r_0. In fact, the choice $r_0 = \frac{1}{2}\min\left\{|x_i - x_j| : i, j = 1, \cdots, m \text{ and } i \ne j\right\} > 0$ would suffice. Then for any number $\eta > 0$, one can find $r > 0$ small such that $\left\|\tilde{u}_{i\epsilon}^{p-1-\epsilon}\right\|_{L^{\frac{n}{2}}(B^n(x,r))} \le \eta$ is valid for any $|x| \le \epsilon^{-\alpha_0} r_0$ provided $\epsilon > 0$ sufficiently small. Hence the Moser iteration technique applies as in [H, Lemma 6], deducing

$$\|\tilde{u}_{i\epsilon}\|_{L^{(p+1)\frac{n}{n-2}}(B^n(x,r/2))} \le \frac{C}{r}\|\tilde{u}_{i\epsilon}\|_{L^{p+1}(B^n(x,r))} \le \frac{C}{r}\|\tilde{u}_{i\epsilon}\|_{H^1(\Omega_{i\epsilon})}$$

where the rightmost value is uniformly bounded in $\epsilon > 0$. Also it is notable that $C > 0$ is independent of x, r or $\tilde{u}_{i\epsilon}$. As a result, we observe from the elliptic regularity [H, Lemma 7] that

$$|u(x)| \le \|u\|_{L^\infty(B^n(x,r/4))} \le C\|u\|_{L^{p+1}(B^n(x,r/2))}$$

where $C > 0$ depends only on r and the supreme of $\left\{\|\tilde{u}_{i\epsilon}\|_{L^{(p+1)\frac{n}{n-2}}(B^n(x,r/2))}\right\}_\epsilon$. This completes the proof. □

This lemma will be used in a crucial way to deduce a local uniform estimate near each blow-up point $x_1, \cdots, x_m$ of u_ϵ.

Proposition 8.2.3. *There exist numbers $C > 0$ and small $\delta_0 \in (0, r_0)$ independent of $\epsilon > 0$ such that*

$$\tilde{u}_{i\epsilon}(x) \le CU_{1,0}(x) \quad \text{for all } x \in B^n\left(0, \epsilon^{-\alpha_0}\delta_0\right) \tag{8.5}$$

for all sufficiently small $\epsilon > 0$.

A closely related result to Proposition 8.2.3 appeared in [LZh] as an intermediate step to deduce the compactness property of the Yamabe equation, the problem proposed by Schoen who also gave the positive answer for conformally flat manifolds (see [Sc]). Even though the proof of this proposition, based on the moving sphere method, can be achieved by adapting the argument presented in [LZh] with a minor modification, we provide it in Appendix 8.A to promote clear understanding of the reader.

From the next lemma to Lemma 8.2.6, we study the behavior of solutions u_ϵ of (1.1_ϵ) outside the blow-up points $\{x_1, \cdots, x_m\}$. For the sake of notational convenience, we set

$$A_r = \Omega \setminus \cup_{i=1}^m B^n(x_i, r) \quad \text{for any } r > 0. \tag{8.6}$$

Lemma 8.2.4. *Suppose that* $\{u_\epsilon\}_\epsilon$ *is a family of solutions for* (1.1_ϵ) *satisfying the asymptotic behavior* (9.33). *Then for any* $r > 0$, *we have* $u_\epsilon(x) = o(1)$ *uniformly for* $x \in A_r$ *as* $\epsilon \to 0$.

Proof. Let $a_\epsilon = u_\epsilon^{p-1-\epsilon}$ so that $-\Delta u_\epsilon = a_\epsilon u_\epsilon$ in Ω. Then we see from (9.33) that

$$\begin{aligned}\|a_\epsilon\|_{L^{\frac{n}{2}}(A_{r/4})} &\leq C\left(\sum_{i=1}^m \left\|PU^{p-1-\epsilon}_{\lambda_{i\epsilon}\epsilon^{\alpha_0},x_{i\epsilon}}\right\|_{L^{\frac{n}{2}}(A_{r/4})} + \|R_\epsilon\|^{p-1-\epsilon}_{L^{p+1-\epsilon\frac{n}{2}}(A_{r/4})}\right)\\ &\leq C\left(\sum_{i=1}^m \left\|U^{p-1-\epsilon}_{\lambda_{i\epsilon}\epsilon^{\alpha_0},x_{i\epsilon}}\right\|_{L^{\frac{n}{2}}(\mathbb{R}^n\setminus B^n(x_i,r/4))} + \|R_\epsilon\|^{p-1-\epsilon}_{H^1(\Omega)}\right) = O\left(\epsilon^{2\alpha_0}\right) + o(1) = o(1).\end{aligned}$$

Therefore we can proceed the Moser iteration argument as in the proof of [H, Lemma 6] to get $\|a_\epsilon\|_{L^q(A_{r/2})} = o(1)$ for some $q > n/2$, and then the standard elliptic regularity result (see [H, Lemma 7]) implies $\|u_\epsilon\|_{L^\infty(A_r)} = o(1)$. □

We can improve this result by combining the kernel expression of u_ϵ and Proposition 8.2.3.

Lemma 8.2.5. *Fix* $r > 0$ *small. Then there holds*

$$u_\epsilon(x) = O\left(\sqrt{\epsilon}\right) \tag{8.7}$$

uniformly for $x \in A_r$.

Proof. Without any loss of generality, we may assume that $r \in (0, \delta_0)$ where $\delta_0 > 0$ is the number picked up in Proposition 8.2.3 so that (8.5) holds. Thus if we fix $i \in \{1, \cdots, m\}$, then we have the bound

$$u_\epsilon(x) = (\lambda_{i\epsilon}\epsilon^{\alpha_0})^{-\sigma_\epsilon}\tilde{u}_{i\epsilon}\left((\lambda_{i\epsilon}\epsilon^{\alpha_0})^{-1}(x - x_{i\epsilon})\right) \leq CU_{\lambda_{i\epsilon}\epsilon^{\alpha_0},x_{i\epsilon}}(x) \leq C\epsilon^{\left(\frac{n-2}{2}\right)\alpha_0}$$

valid for each x such that $r/2 \leq |x - x_i| \leq r$. It says that $u_\epsilon(x) \leq C\sqrt{\epsilon}$ for all $x \in A_{r/2} \setminus A_r$.

By Green's representation formula, one may write

$$u_\epsilon(x) = \int_{A_{r/2}} G(x,y)u_\epsilon^{p-\epsilon}(y)dy + \sum_{i=1}^m \int_{B^n(x_i,r/2)} G(x,y)u_\epsilon^{p-\epsilon}(y)dy. \tag{8.8}$$

Let us estimate each of the term in the right-hand side. If we set $b_\epsilon = \max\{u_\epsilon(x) : x \in A_r\}$, then we find

$$\int_{A_{r/2}} G(x,y)u_\epsilon^{p-\epsilon}(y)dy \leq C\int_{A_{r/2}} G(x,y)\left(b_\epsilon^{p-\epsilon} + \sqrt{\epsilon}^{p-\epsilon}\right)dy \leq C\left(b_\epsilon^{p-\epsilon} + C\sqrt{\epsilon}^{p-\epsilon}\right) \tag{8.9}$$

for any $x \in A_r$. Besides, (8.5) gives us that

$$\begin{aligned}\int_{B^n(x_i,r/2)} G(x,y)u_\epsilon^{p-\epsilon}(y)dy &\leq C(r)\int_{B^n(x_i,r/2)} u_\epsilon^{p-\epsilon}(y)dy \leq C\cdot C(r)\int_{B^n(x_i,r/2)} U^{p-\epsilon}_{\lambda_{i\epsilon}\epsilon^{\alpha_0},x_{i\epsilon}}(y)dy\\ &\leq C\cdot C(r)\epsilon^{\left(\frac{n-2}{2}\right)\alpha_0} = C\cdot C(r)\sqrt{\epsilon}\end{aligned} \tag{8.10}$$

for each i and $x \in A_r$, where $C(r) = \max\{G(x,y) : x, y \in \Omega, |x-y| \geq r/2\}$. Hence, by combining (8.9) and (8.10), we get

$$b_\epsilon \leq C\left(b_\epsilon^{p-\epsilon} + \sqrt{\epsilon}\right).$$

Since it is guaranteed by Lemma 8.2.4 that $b_\epsilon = o(1)$, this shows that $b_\epsilon \leq C\sqrt{\epsilon}$. The lemma is proved. □

The following result will be used to obtain the asymptotic formulas of the eigenvalues.

Lemma 8.2.6. *Suppose that u_ϵ satisfies equation* (1.1_ϵ) *and the asymptotic behavior* (9.33). *Then we have*

$$\epsilon^{-\frac{1}{2}} \cdot u_\epsilon(x) = C_2 \sum_{i=1}^{m} \lambda_i^{\frac{n-2}{2}} G(x, x_i) + o(1) \tag{8.11}$$

in $C^2(\Omega \setminus \{x_1, \cdots, x_m\})$. Here $C_2 = \int_{\mathbb{R}^n} U_{1,0}^p > 0$.

Proof. Take any $r > 0$ small for which Lemma 8.2.5 holds and decompose $u_\epsilon(x)$ as in (8.8) for $x \in A_r$. Then we have

$$\left|\epsilon^{-\frac{1}{2}} \int_{A_{r/2}} G(x,y) u_\epsilon^{p-\epsilon}(y)dy\right| \leq C\epsilon^{\frac{p-1-\epsilon}{2}} \left(\int_\Omega G(x,y)dy\right) = o(1). \tag{8.12}$$

Also, if we write

$$\int_{B^n(x_i,r/2)} G(x,y)u_\epsilon^{p-\epsilon}(y)dy = G(x,x_i)\int_{B^n(x_i,r/2)} u_\epsilon^{p-\epsilon}(y)dy + \int_{B^n(x_i,r/2)} (G(x,y)-G(x,x_i))u_\epsilon^{p-\epsilon}(y)dy$$

for $i \in \{1, \cdots, m\}$, it follows from Lemma 8.2.2 and the dominated convergence theorem that

$$\epsilon^{-\frac{1}{2}} \int_{B^n(x_i,r/2)} u_\epsilon^{p-\epsilon}(y)dy \to \lambda_i^{\frac{n-2}{2}} \int_{\mathbb{R}^n} U_{1,0}^p(y)dy = \lambda_i^{\frac{n-2}{2}} C_2 \tag{8.13}$$

and from the mean value theorem that

$$\begin{aligned} &\left|\epsilon^{-\frac{1}{2}} \int_{B^n(x_i,r/2)} (G(x,y)-G(x,x_i))u_\epsilon^{p-\epsilon}(y)dy\right| \leq \epsilon^{-\frac{1}{2}} \int_{B^n(x_i,r/2)} |G(x,y)-G(x,x_i)|u_\epsilon^{p-\epsilon}(y)dy \\ &\leq \epsilon^{-\frac{1}{2}} \int_{B^n(x_i,r/2)} \sup_{\substack{x\in A_r,\\ t\in(0,1)}} \left\|\nabla_y G(x, ty+(1-t)x_i)\right\| \cdot |y-x_i| u_\epsilon^{p-\epsilon}(y)dy \leq Cr. \end{aligned} \tag{8.14}$$

Therefore, combining (8.8), (8.12), (8.13) and (8.14), we confirm that

$$C_2 \sum_{i=1}^{m} \lambda_i^{\frac{n-2}{2}} G(x,x_i) - Cr \leq \liminf_{\epsilon\to 0} \epsilon^{-\frac{1}{2}} u_\epsilon(x) \leq \limsup_{\epsilon\to 0} \epsilon^{-\frac{1}{2}} u_\epsilon(x) \leq C_2 \sum_{i=1}^{m} \lambda_i^{\frac{n-2}{2}} G(x,x_i) + Cr.$$

Since $r > 0$ is arbitrary, (8.11) holds in $C^0(\Omega \setminus \{x_1, \cdots, x_m\})$. Also, the C^2-convergence comes from the elliptic regularity. This proves the lemma. □

In Lemma 8.2.7 and Lemma 8.2.8, we conduct a decay estimate for solutions of the eigenvalue problem (8.5).

Lemma 8.2.7. *For a fixed $\ell \in \mathbb{N}$, let $\{\mu_{\ell\epsilon}\}_\epsilon$ be the family of ℓ-th eigenvalues for problem* (8.5), *and $v_{\ell\epsilon}$ an $L^\infty(\Omega)$-normalized eigenfunction corresponding to $\mu_{\ell\epsilon}$. Then for any $r > 0$ the function $v_{\ell\epsilon}$ converges to zero uniformly in A_r as $\epsilon \to 0$.*

Proof. For $x \in A_r$, we write

$$\frac{v_{\ell\epsilon}(x)}{\mu_{\ell\epsilon}(p-\epsilon)} = \int_{A_{r/2}} G(x,y)u_\epsilon^{p-1-\epsilon}v_{\ell\epsilon}(y)dy + \sum_{i=1}^m \int_{B^n(x_i,r/2)} G(x,y)u_\epsilon^{p-1-\epsilon}v_{\ell\epsilon}(y)dy. \tag{8.15}$$

From Lemma 8.2.5, we have

$$\left|\int_{A_{r/2}} G(x,y)\left(u_\epsilon^{p-1-\epsilon}v_{\ell\epsilon}\right)(y)dy\right| \le C\cdot\epsilon^{\frac{p-1-\epsilon}{2}}\left(\int_\Omega G(x,y)dy\right) = O\left(\epsilon^{\frac{2}{n-2}}\right). \tag{8.16}$$

Also, we utilize (8.5) to obtain that

$$\begin{aligned}\left|\int_{B^n(x_i,r/2)} G(x,y)\left(u_\epsilon^{p-1-\epsilon}v_{\ell\epsilon}\right)(y)dy\right| &\le C(r)\int_{B^n(x_i,r/2)} u_\epsilon^{p-1-\epsilon}(y)dy \\ &\le C\cdot C(r)\int_{B^n(0,r)} U^{p-1-\epsilon}_{\lambda_{i\epsilon}\epsilon^{\alpha_0},0}(y)dy \\ &= \begin{cases} O\left(\epsilon^{\frac{2}{n-2}}\right) & \text{if } n \ge 5, \\ O(\epsilon\log\epsilon) & \text{if } n = 4, \\ O(\epsilon) & \text{if } n = 3\end{cases}\end{aligned} \tag{8.17}$$

for any $1 \le i \le m$ where the definition of $C(r)$ can be found in the sentence after (8.10). Putting estimates (8.16) and (8.17) into (8.15) validates that $v_{\ell\epsilon} = o(1)$ uniformly in A_r. □

Lemma 8.2.8. *Assume that $0 \in \Omega$, fix $\ell \in \mathbb{N}$ and set*

$$\tilde{v}_{\ell\epsilon} = v_{\ell\epsilon}(\epsilon^{\alpha_0}x) \quad \textit{and} \quad d_\epsilon(x) = dist\left(x, \{\epsilon^{-\alpha_0}x_{1\epsilon}, \cdots, \epsilon^{-\alpha_0}x_{m\epsilon}\}\right) \quad \textit{for } x \in \Omega_\epsilon := \epsilon^{-\alpha_0}\Omega.$$

Then for any $\zeta > 0$ small, we can pick a constant $C = C(\zeta) > 0$ independent of $\epsilon > 0$ such that

$$|\tilde{v}_{\ell\epsilon}(x)| \le \frac{C}{1+d_\epsilon(x)^{n-2-\zeta}} \quad \textit{for all } x \in \Omega_\epsilon. \tag{8.18}$$

In particular, if $i \in \{1,\cdots,m\}$ are given and $\{\tilde{v}_{\ell i\epsilon}\}_\epsilon$ is a family of dilated eigenfunctions for (1.1_ϵ) *defined as in* (8.10), *then*

$$|\tilde{v}_{\ell i\epsilon}(x)| \le \frac{C}{1+|x|^{n-2-\zeta}} \quad \textit{for all } |x| \le \epsilon^{-\alpha_0}r \tag{8.19}$$

and $v_\epsilon = O(\epsilon)$ in A_r for some $r > 0$ small.

Proof. One can derive the decay estimate (8.18) by adapting the proof of Lemmas A.5, B.3 and Proposition B.1 of Cao, Peng and Yan [CPY], in which the authors investigated the p-Laplacian version of the Brezis-Nirenberg problem. To account for the way to modify their argument to be suitable for our multi-bubble case, we briefly sketch the proof. Let $\tilde{u}_\epsilon = u_\epsilon(\epsilon^{\alpha_0}\cdot)$ and $\tilde{x}_{i\epsilon} = \epsilon^{-\alpha_0}x_{i\epsilon}$.

Notice that $\tilde{v}_{\ell\epsilon}$ solves

$$-\Delta\tilde{v}_{\ell\epsilon} = a_{\ell\epsilon}\tilde{v}_{\ell\epsilon} \quad \text{in } \Omega_\epsilon \quad \text{where} \quad a_{\ell\epsilon} = \mu_{\ell\epsilon}(p-\epsilon)\epsilon^{2\alpha_0}\tilde{u}_\epsilon^{p-1-\epsilon} \geq 0.$$

From Proposition 8.2.3 and Lemma 8.2.5, we realize that $a_{\ell\epsilon} \leq C|x|^{-4+(n-2)\epsilon}$ holds in each annulus $B^n(\tilde{x}_{i\epsilon}, \delta_0\epsilon^{-\alpha_0}) \setminus B^n(\tilde{x}_{i\epsilon}, R)$ provided $i \in \{1, \cdots, m\}$ and $R > 1$ large, and $a_{\ell\epsilon} \leq C\epsilon^{4\alpha_0}$ in $\Omega_\epsilon \setminus \cup_{i=1}^m B^n(\tilde{x}_{i\epsilon}, \delta_0\epsilon^{-\alpha_0})$. Hence, given any $\eta > 0$, there exists a large $R(\eta) > 1$ such that

$$\int_{\widetilde{A}_{R(\eta)}} |a_{\ell\epsilon}|^{\frac{n}{2}} dx < \eta \quad \text{where} \quad \widetilde{A}_R := \Omega_\epsilon \setminus \bigcup_{i=1}^m B^n(\tilde{x}_{i\epsilon}, R). \tag{8.20}$$

Suppose that $\zeta > 0$ is selected to be small enough. Then one can apply the Moser iteration technique to get a small number $\eta > 0$ and large $q > p+1$ such that if (8.20) holds, there is a constant $C > 0$ independent of R, η or $\tilde{v}_{\ell\epsilon}$ satisfying

$$\|\tilde{v}_{\ell\epsilon}\|_{L^q(\widetilde{A}_R)} \leq \frac{C}{(R-2R(\eta))^{\frac{n-2}{2}-\zeta}} \cdot \|\tilde{v}_{\ell\epsilon}\|_{L^{p+1}(\widetilde{A}_{2R(\eta)})}$$

for any $R > 2R(\eta)$. On the other hand, it is possible to get that $\|\tilde{v}_{\ell\epsilon}\|_{L^{p+1}(\widetilde{A}_{2R(\eta)})} \leq CR^{-2\zeta}$ by taking a smaller ζ if necessary. Thus standard elliptic regularity theory gives

$$|\tilde{v}_{\ell\epsilon}(x)| \leq \|\tilde{v}_{\ell\epsilon}\|_{L^\infty(B^n(x,1))} \leq C\|\tilde{v}_{\ell\epsilon}\|_{L^q(\widetilde{A}_{R-1})} \leq \frac{C}{(R-2R(\eta)-1)^{\frac{n-2}{2}-\zeta}} \cdot \|\tilde{v}_{\ell\epsilon}\|_{L^{p+1}(\widetilde{A}_{2R(\eta)})} \leq \frac{C}{R^{\frac{n-2}{2}+\zeta}} \tag{8.21}$$

for all $x \in \widetilde{A}_R$, $R \geq 3R(\eta)$.

Having (8.21) in mind, we now prove (8.18) by employing the comparison principle iteratively. Assume that it holds

$$|\tilde{v}_{\ell\epsilon}(x)| \leq D_j \sum_{i=1}^m \frac{1}{|x-\tilde{x}_{i\epsilon}|^{q_j}} \quad \text{for all } x \in \widetilde{A}_R, \tag{8.22}$$

some $D_j > 0$ and $0 < q_j < n-2$ to be determined soon ($j \in \mathbb{N}$). Since we have $(n-2)(p-1-\epsilon) > 3$ for small $\epsilon > 0$, Proposition 8.2.3, Lemma 8.2.5 and (8.22) tell us that there exists some $\widetilde{D}_j > 0$ whose choice is affected by only D_j, n and ℓ such that

$$-\Delta(\tilde{v}_{\ell\epsilon})_\pm(x) = \mu_{\ell\epsilon}(p-\epsilon)\tilde{u}_\epsilon^{p-1-\epsilon}(\tilde{v}_{\ell\epsilon})_\pm(x) \leq \widetilde{D}_j \sum_{i=1}^m \frac{1}{|x-\tilde{x}_{i\epsilon}|^{q_j+3}} \quad \text{for any } x \in \widetilde{A}_R.$$

Select any number $0 < \tilde{\eta} < \min(1, n-2-q_j)$ and set a function

$$\chi_j(x) = D_{j+1} \sum_{i=1}^m \frac{1}{|x-\tilde{x}_{i\epsilon}|^{q_j+\tilde{\eta}}} \quad \text{for } x \in \mathbb{R}^n$$

where $D_{j+1} > 0$ is a number so large that $\chi_j \geq |\tilde{v}_{\ell\epsilon}|$ on $\cup_{i=1}^m \partial B^n(\tilde{x}_{i\epsilon}, R)$. Then one can compute

$$\begin{aligned} -\Delta\chi_j(x) &= D_{j+1}\left(q_j + \tilde{\eta}\right)\left((n-2) - \left(q_j + \tilde{\eta}\right)\right)\sum_{i=1}^m \frac{1}{|x - \tilde{x}_{i\epsilon}|^{q_j+\tilde{\eta}+2}} \\ &\geq \widetilde{D}_j \sum_{i=1}^m \frac{1}{|x - \tilde{x}_{i\epsilon}|^{q_j+3}} \geq -\Delta(\tilde{v}_{\ell\epsilon})_\pm(x), \quad x \in \widetilde{A}_R \end{aligned} \tag{8.23}$$

by taking a larger D_{j+1} if necessary. However $\chi_j > 0$ and $\tilde{v}_{\ell\epsilon} = 0$ on $\partial\Omega_\epsilon$, whence $\chi_j \geq |\tilde{v}_{\ell\epsilon}|$ on $\partial\widetilde{A}_R$. Consequently, by (8.23) and the maximum principle, it follows that

$$|\tilde{v}_{\ell\epsilon}(x)| \leq \chi_j(x) = D_{j+1}\sum_{i=1}^m \frac{1}{|x - \tilde{x}_{i\epsilon}|^{q_j+\tilde{\eta}}}, \quad x \in \widetilde{A}_R.$$

Letting $q_1 = \frac{n-2}{2} + \zeta$ in (8.21), choosing an appropriate $D_1 > 0$ and repeating this comparison procedure, we can deduce

$$|\tilde{v}_{\ell\epsilon}(x)| \leq C\sum_{i=1}^m \frac{1}{|x - \tilde{x}_{i\epsilon}|^q}, \quad x \in \widetilde{A}_R$$

given any $1 < q < n - 2$. This proves (8.18).

Finally, (8.19) and the claim that $v_\epsilon = O(\epsilon)$ in A_r is a straightforward consequence of (8.18). The proof is completed. □

By utilizing (8.5), (8.19), (8.7), the fact that $v_\epsilon = O(\epsilon)$ in A_r and regularity theory, we immediately establish a decay estimate for the derivatives of $\tilde{u}_{i\epsilon}$ and $\tilde{v}_{\ell i\epsilon}$.

Lemma 8.2.9. *For any $k \in \{1, \cdots, n\}$, there exists a universal constant $C > 0$ such that*

$$\left|\frac{\partial\tilde{u}_{i\epsilon}(x)}{\partial x_k}\right| \leq \frac{C}{1 + |x|^{n-2}} \quad \text{and} \quad \left|\frac{\partial\tilde{v}_{\ell i\epsilon}(x)}{\partial x_k}\right| \leq \frac{C}{1 + |x|^{n-2-\zeta}} \quad \text{for all } |x| \leq \epsilon^{-\alpha_0} r$$

for $\zeta,\ r > 0$ small. Moreover we have

$$|\partial_k u_\epsilon|,\ |\partial_k\partial_l u_\epsilon| = O\left(\sqrt{\epsilon}\right) \quad \text{and} \quad |\partial_k v_{\ell\epsilon}| = O(\epsilon) \quad \text{for all } k,\ l = 1, \cdots, n$$

as $\epsilon \to 0$ in any compact subset of A_r.

Finally, we recall two well-known results. The first lemma states the nondegeneracy property of the standard bubble $U_{1,0}$. We refer to [BE] for its proof.

Lemma 8.2.10. *The space of solutions to the linear problem*

$$-\Delta v = pU_{1,0}^{p-1}v \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \int_{\mathbb{R}^n} |\nabla v|^2 < \infty$$

is spanned by

$$\frac{x_1}{(1+|x|^2)^{\frac{n}{2}}}, \cdots, \frac{x_n}{(1+|x|^2)^{\frac{n}{2}}} \quad \text{and} \quad \frac{1-|x|^2}{(1+|x|^2)^{\frac{n}{2}}}.$$

The next lemma lists some formulas regarding the derivatives of Green's function. The proof can be found in [GP, H].

Lemma 8.2.11. *For $\xi \in \Omega$, it holds that*

$$\int_{\partial\Omega} (x-\xi, \nu)\left(\frac{\partial G}{\partial \nu}(x,\xi)\right)^2 dS = (n-2)\tau(\xi),$$

$$\int_{\partial\Omega} \left(\frac{\partial G}{\partial \nu}(x,\xi)\right)^2 \nu_k(x) dS = \frac{\partial \tau}{\partial x_k}(\xi), \quad k = 1, \cdots, n$$

and

$$\int_{\partial\Omega} \frac{\partial G}{\partial x_k}(x,\xi)\frac{\partial}{\partial y_l}\left(\frac{\partial G}{\partial \nu}(x,\xi)\right) dS = \frac{1}{2}\frac{\partial^2 \tau}{\partial x_k \partial x_l}(\xi), \quad k, l = 1, \cdots, n.$$

Here ν is the outward normal unit vector to $\partial\Omega$ and dS is the surface measure $\partial\Omega$.

8.3 Proof of Theorem 8.1.1

In this section, we present estimates for the first m eigenvalues and eigenfunctions of (8.5).

For the set of the concentration points $\{x_1, \cdots, x_m\} \subset \Omega^m$, let us fix a small number $r > 0$ such that for any $1 \le i \ne j \le m$ and any $\epsilon > 0$ small the following holds:

$$B^n(x_i, 4r) \subset \Omega \quad \text{and} \quad B^n(x_i, 4r) \cap B^n(x_j, 4r) = \emptyset.$$

For each $1 \le i \le m$, we set $\phi_i(x) = \phi(x - x_i)$ where a cut-off function $\phi \in C_c^\infty(B^n(0, 3r))$ satisfies $\phi \equiv 1$ in $B^n(0, 2r)$ and $0 \le \phi \le 1$ in $B^n(0, 3r)$. Define also

$$u_{\epsilon,i} = \phi_i u_\epsilon, \quad \psi_{\epsilon,i,k} = \phi_i \frac{\partial u_\epsilon}{\partial x_k}\ (1 \le k \le n) \quad \text{and} \quad \psi_{\epsilon,i,n+1} = \phi_i \cdot \left((x - x_{i\epsilon}) \cdot \nabla u_\epsilon + \frac{2u_\epsilon}{p-1-\epsilon}\right) \tag{8.1}$$

in Ω.

The following lemma serves as a main ingredient for the proof of Theorem 8.1.1

Lemma 8.3.1. *Fix $\ell \in \mathbb{N}$. Suppose that $\{v_{\ell\epsilon}\}_\epsilon$ is a family of normalized eigenfunctions of* (8.5) *corresponding to the ℓ-th eigenvalue $\mu_{\ell\epsilon}$. Then there exists at least one $i_0 \in \{1, \cdots, m\}$ such that $\tilde{v}_{\ell i_0 \epsilon}$ (see* (8.10) *for its definition) converges to a nonzero function in the weak $H^1(\mathbb{R}^n)$-sense.*

Proof. Lemma 8.2.8 ensures that there exist a large $R > 0$ and a small $r > 0$ such that $|\tilde{v}_{\ell i\epsilon}| \le 1/2$ for $R \le |x| \le \epsilon^{-\alpha_0} r$. Suppose that $\tilde{v}_{\ell i\epsilon} \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^n)$ as $\epsilon \to 0$ for all $1 \le i \le m$. Then each $\tilde{v}_{\ell i\epsilon}$ tends to 0 uniformly in $B^n(0, R)$ by elliptic regularity. Since we already know that $v_\epsilon \to 0$ uniformly on A_r from Lemma 8.2.7, it follows that $\|v_\epsilon\|_{L^\infty(\Omega)} \le 1/2$. However $\|v_\epsilon\|_{L^\infty(\Omega)} = 1$ by its own definition, hence a contradiction arises. □

Given Lemma 8.3.1, we are now ready to start to prove Theorem 8.1.1.

Proof of Theorem 8.1.1. Let $\mathcal{V}$ be a vector space whose basis consists of $\{u_{\epsilon,i} : 1 \le i \le m\}$. By the Courant-Fischer-Weyl min-max principle, we have

$$\mu_{m\epsilon} = \min_{\substack{\mathcal{W}\subset H_0^1(\Omega),\\ \dim\mathcal{W}=m}} \max_{f\in\mathcal{W}\setminus\{0\}} \frac{\int_\Omega |\nabla f(x)|^2 dx}{(p-\epsilon)\int_\Omega \left(f^2 u_\epsilon^{p-1-\epsilon}\right)(x)dx} \le \max_{f\in\mathcal{V}\setminus\{0\}} \frac{\int_\Omega |\nabla f(x)|^2 dx}{(p-\epsilon)\int_\Omega \left(f^2 u_\epsilon^{p-1-\epsilon}\right)(x)dx}.$$

If we denote a nonzero element $f \in \mathcal{V}$ by $f = \sum_{i=1}^m a_i u_{\epsilon,i}$ for some $(a_1, \cdots, a_m) \neq 0$, then the fact that u_{ϵ,i_1} and u_{ϵ,i_2} have disjoint supports for any $1 \le i_1 \neq i_2 \le m$ implies

$$\begin{aligned}\frac{\int_\Omega |\nabla f|^2}{(p-\epsilon)\int_\Omega f^2 u_\epsilon^{p-1-\epsilon}} &= \frac{\sum_{i=1}^m \int_\Omega \left|\nabla\left(a_i u_{\epsilon,i}\right)\right|^2}{(p-\epsilon)\sum_{i=1}^m \int_\Omega \left(a_i u_{\epsilon,i}\right)^2 u_\epsilon^{p-1-\epsilon}} \le \max_{1\le i\le m} \frac{\int_\Omega |\nabla(a_i u_{\epsilon,i})|^2}{(p-\epsilon)\int_\Omega (a_i u_{\epsilon,i})^2 u_\epsilon^{p-1-\epsilon}} \\ &= \max_{1\le i\le m} \frac{\int_\Omega |\nabla(\phi_i u_\epsilon)|^2}{(p-\epsilon)\int_\Omega \phi_i^2 u_\epsilon^{p+1-\epsilon}} \to \frac{1}{p}\cdot\frac{\int_{\mathbb{R}^n} |\nabla U_{1,0}|^2}{\int_{\mathbb{R}^n} U_{1,0}^{p+1}} = \frac{1}{p} \quad \text{as } \epsilon \to 0.\end{aligned}$$

Thus we know that $\mu_{m\epsilon} \le p^{-1} + o(1)$, and particularly if we let $\mu_\ell = \lim_{\epsilon\to 0}\mu_{\ell\epsilon}$, then $\mu_\ell \le p^{-1}$ for any $1 \le \ell \le m$.

Fix $\ell \in \{1, \cdots, m\}$. By Lemma 8.3.1 there is an index $i_0 \in \{1, \cdots, m\}$ such that $\tilde{v}_{\ell i_0 \epsilon}$ converges $H^1(\mathbb{R}^n)$-weakly to a nonzero function V. A direct computation shows

$$-\Delta \tilde{v}_{\ell i_0\epsilon} = \mu_{\ell\epsilon}(p-\epsilon)\tilde{u}_{i\epsilon}^{p-1-\epsilon}\tilde{v}_{\ell i_0\epsilon} \quad \text{in } \Omega_{i_0\epsilon}$$

where the function $\tilde{u}_{i\epsilon}$ and the set $\Omega_{i_0\epsilon}$ are defined in (8.3) and (8.10), respectively. Thus it follows from Lemma 8.2.2 that $V \in H^1(\mathbb{R}^n) \setminus \{0\}$ is a solution of

$$-\Delta V = \mu_\ell p U_{1,0}^{p-1} V \quad \text{in } \mathbb{R}^n.$$

Note that $U_{1,0}$ can be characterized as a mountain pass solution to (8.4) and so has the Morse index 1. Consequently, in light of the estimate for μ_ℓ in the previous paragraph, the only possibility is $\mu_\ell = p^{-1}$.

On the other hand, for any i, we also see that $\tilde{v}_{\ell i\epsilon}$ converges to a function W weakly in $H^1(\mathbb{R}^n)$ so that W solves $-\Delta W = U_{1,0}^{p-1} W$ in $\mathbb{R}^n$. Thus there is a nonzero vector $\mathbf{c}_\ell = \left(\lambda_1^{\frac{n-2}{2}} c_{\ell 1}, \cdots, \lambda_m^{\frac{n-2}{2}} c_{\ell m}\right) \in \mathbb{R}^m$ such that $\tilde{v}_{\ell i\epsilon} \rightharpoonup c_{\ell i} U_{1,0}$ weakly in $H^1(\mathbb{R}^n)$ for each $i \in \{1, \cdots, m\}$.

Let us prove (8.12) now. Fixing i, we multiply (1.1_ϵ) (or (8.5) with $v = v_{\ell\epsilon}$) by $v_{\ell\epsilon}$ (or u_ϵ) to get the identity, say, I (or II respectively). Also we denote by $\int I$ and $\int II$ the identities which can be obtained after integrating I and II over $B^n(x_{i\epsilon}, r)$. Subtracting $\int I$ from $\int II$ and utilizing Green's identity (8.12) below, we see then

$$\int_{\partial B^n(x_{i\epsilon},r)} \left(\frac{\partial u_\epsilon}{\partial \nu} v_{\ell\epsilon} - \frac{\partial v_{\ell\epsilon}}{\partial \nu} u_\epsilon\right) dS = (\mu_{\ell\epsilon}(p-\epsilon) - 1) \int_{B^n(x_{i\epsilon},r)} \left(u_\epsilon^{p-\epsilon} v_{\ell\epsilon}\right)(x) dx \qquad (8.2)$$

for each $i \in \{1, \cdots, m\}$ and any $r > 0$ sufficiently small. Moreover, if we set the functions

$$C_2^{-1}\tilde{g}_i(x) = -\lambda_i^{\frac{n-2}{2}} H(x, x_i) + \sum_{j \neq i} \lambda_j^{\frac{n-2}{2}} G(x, x_j), \quad C_2^{-1}\tilde{h}_i(x) = -\lambda_i^{n-2} c_{\ell i} H(x, x_i) + \sum_{j \neq i} \lambda_j^{n-2} c_{\ell j} G(x, x_j)$$

which are harmonic near x_i, then (the proof of) Lemma 8.2.6 permits us to obtain that

$$\epsilon^{-\frac{1}{2}} u_\epsilon(x) = C_2 \lambda_i^{\frac{n-2}{2}} \frac{\gamma_n}{|x - x_{i\epsilon}|^{n-2}} + \tilde{g}_i(x) + o(1) \tag{8.3}$$

and

$$\epsilon^{-1} \frac{v_{\ell\epsilon}(x)}{\mu_{\ell\epsilon}(p - \epsilon)} = C_2 \lambda_i^{n-2} c_{\ell i} \frac{\gamma_n}{|x - x_{i\epsilon}|^{n-2}} + \tilde{h}_i(x) + o(1) \tag{8.4}$$

for $x \in B^n(x_{i\epsilon}, 2r)$. Therefore, by inserting (8.3) and (8.4) into (8.2), and then using the mean value formula for harmonic functions and $\nabla_\lambda \Upsilon(\lambda_1, \cdots, \lambda_m, x_1, \cdots, x_m) = 0$, one discovers

$$\begin{aligned}
&\int_{\partial B^n(x_{i\epsilon}, r)} \left[\frac{\partial \left(\epsilon^{-\frac{1}{2}} u_\epsilon\right)}{\partial \nu} \left(\epsilon^{-1} v_{\ell\epsilon}\right) - \frac{\partial \left(\epsilon^{-1} v_{\ell\epsilon}\right)}{\partial \nu} \left(\epsilon^{-\frac{1}{2}} u_\epsilon\right) \right] dS \\
&\quad = -(n-2) C_2 \gamma_n \int_{\partial B^n(x_{i\epsilon}, r)} \left[\frac{1}{|x - x_{i\epsilon}|^{n-1}} \lambda_i^{\frac{n-2}{2}} \tilde{h}_i(x) - \frac{1}{|x - x_{i\epsilon}|^{n-1}} \lambda_i^{n-2} c_{\ell i} \tilde{g}_i(x) \right] dS + o(1) \\
&\quad \to (n-2) C_2 \gamma_n \left| S^{n-1} \right| \left(\lambda_i^{n-2} c_{\ell i} \tilde{g}_i(x_i) - \lambda_i^{\frac{n-2}{2}} \tilde{h}_i(x_i) \right) \\
&\quad = c_1 \left[\lambda_i^{n-2} \left(\sum_{j \neq i} \lambda_j^{\frac{n-2}{2}} G(x_i, x_j) \right) c_{\ell i} - \sum_{j \neq i} \lambda_i^{\frac{n-2}{2}} \lambda_j^{n-2} G(x_i, x_j) c_{\ell j} \right] \\
&\quad = \left(c_1 \lambda_i^{\frac{3(n-2)}{2}} \tau(x_i) - \frac{c_2}{n-2} \lambda_i^{\frac{n-2}{2}} \right) c_{\ell i} - c_1 \sum_{j \neq i} \lambda_i^{\frac{n-2}{2}} \lambda_j^{n-2} G(x_i, x_j) c_{\ell j}
\end{aligned}$$

as $\epsilon \to 0$. Also, an application of the dominated convergence theorem with Lemmas 8.2.2 and 8.2.8, Proposition 8.2.3 and the observation that $\tilde{v}_{\ell i\epsilon} \to c_{\ell i} U_{1,0}$ pointwise give us that

$$\epsilon^{-\frac{1}{2}} \int_{B^n(x_{i\epsilon}, r)} u_\epsilon^{p-\epsilon} v_{\ell\epsilon} = \lambda_{i\epsilon}^{n - \sigma_\epsilon(p-\epsilon)} \epsilon^{(n - \sigma_\epsilon(p-\epsilon))\alpha_0 - \frac{1}{2}} \int_{B\left(0, r(\lambda_{i\epsilon}\epsilon^{\alpha_0})^{-1}\right)} \tilde{u}_{i\epsilon}^{p-\epsilon} \tilde{v}_{\ell i\epsilon} \to c_{\ell i} \lambda_i^{\frac{n-2}{2}} \frac{4nc_2}{(n-2)^2}$$

(refer to (8.9)). From these estimates, we deduce

$$\begin{aligned}
&\left(\lambda_i^{n-2} \tau(x_i) - \frac{c_2}{(n-2)c_1} \right) \left(\lambda_i^{\frac{n-2}{2}} c_{\ell i} \right) - \sum_{j \neq i} (\lambda_i \lambda_j)^{\frac{n-2}{2}} G(x_i, x_j) \left(\lambda_j^{\frac{n-2}{2}} c_{\ell j} \right) \\
&\qquad = \left(\frac{4nc_2}{(n-2)^2 c_1} \right) \cdot \lim_{\epsilon \to 0} \left(\frac{\mu_{\ell\epsilon}(p - \epsilon) - 1}{\epsilon} \right) \left(\lambda_i^{\frac{n-2}{2}} c_{\ell i} \right) := \rho_\ell^1 \left(\lambda_i^{\frac{n-2}{2}} c_{\ell i} \right),
\end{aligned}$$

or equivalently, $\mathcal{A}_1 \mathbf{c}_\ell = \rho_\ell^1 \mathbf{c}_\ell$. This justifies (8.12). We also showed that $\mathbf{c}_\ell^T$ is an eigenvector corresponding to the eigenvalue ρ_ℓ^1 at the same time.

Finally, to verify the last assertion of the theorem, we assume that $\ell_1 \neq \ell_2$. Since $v_{\ell_1\epsilon}$ and $v_{\ell_2\epsilon}$ are orthogonal each other, we have

$$\begin{aligned} 0 &= \lim_{\epsilon\to 0} \epsilon^{-1} \left(\mu_{\ell_1\epsilon}(p-\epsilon)\right)^{-1} \int_\Omega \nabla v_{\ell_1\epsilon} \cdot \nabla v_{\ell_2\epsilon} \\ &= \lim_{\epsilon\to 0} \epsilon^{-1} \left(\sum_{i=1}^m \int_{B^n(x_{i\epsilon},r)} u_\epsilon^{p-1-\epsilon} v_{\ell_1\epsilon} v_{\ell_2\epsilon} + \int_{\Omega\setminus\cup_{i=1}^m B^n(x_{i\epsilon},r)} u_\epsilon^{p-1-\epsilon} v_{\ell_1\epsilon} v_{\ell_2\epsilon} \right) \\ &= \lim_{\epsilon\to 0} \sum_{i=1}^m \lambda_{i\epsilon}^{n-2} \int_{B^n\left(0,(\lambda_{i\epsilon}\epsilon^{\alpha_0})^{-1} r\right)} \tilde{u}_{i\epsilon}^{p-1-\epsilon} \tilde{v}_{\ell_1 i\epsilon} \tilde{v}_{\ell_2 i\epsilon} = \sum_{i=1}^m \left(\lambda_i^{\frac{n-2}{2}} c_{\ell_1 i}\right)\left(\lambda_i^{\frac{n-2}{2}} c_{\ell_2 i}\right) \int_{\mathbb{R}^n} U_{1,0}^{p+1}. \end{aligned} \tag{8.5}$$

Thus $\mathbf{c}_{\ell_1}^T \cdot \mathbf{c}_{\ell_2}^T = 0$. □

8.4 Upper bounds for the ℓ-th eigenvalues and asymptotic behavior of the ℓ-th eigenfunctions, $m+1 \leq \ell \leq (n+1)m$

The objective of this section is to provide estimates of the ℓ-th eigenvalues and its corresponding eigenfunctions when $m+1 \leq \ell \leq (n+1)m$. Their refinement will be accomplished in the subsequent sections based on the results deduced in this section.

In the first half of this section, our interest will lie on achieving upper bounds of the eigenvalues $\mu_{\ell\epsilon}$ for $m+1 \leq \ell \leq (n+1)m$, as the following proposition depicts.

Proposition 8.4.1. *Suppose that $m+1 \leq \ell \leq (n+1)m$. Then*

$$\mu_{\ell\epsilon} \leq 1 + O\left(\epsilon^{\frac{n}{n-2}}\right).$$

Proof. We define a linear space $\mathcal{V}$ spanned by

$$\{u_{\epsilon,i} : 1 \leq i \leq m\} \cup \{\psi_{\epsilon,i,k} : 1 \leq i \leq m,\ 1 \leq k \leq n\}$$

(refer to (8.1)) so that any nonzero function $f \in \mathcal{V} \setminus \{0\}$ can be written as

$$f = \sum_{i=1}^m f_i \quad \text{with} \quad f_i = a_{i0} u_{\epsilon,i} + \sum_{k=1}^n a_{ik} \psi_{\epsilon,i,k}$$

where at least one number a_{ik} ($1 \leq i \leq m$ and $0 \leq k \leq n$) is nonzero. By the variational characterization of the eigenvalue $\mu_{\ell\epsilon}$, we have

$$\begin{aligned} \mu_{((n+1)m)\epsilon} &= \min_{\substack{\mathcal{W}\subset H_0^1(\Omega),\\ \dim\mathcal{W}=(n+1)m}} \max_{f\in\mathcal{W}\setminus\{0\}} \frac{\int_\Omega |\nabla f|^2}{(p-\epsilon)\int_\Omega f^2 u_\epsilon^{p-1-\epsilon}} \leq \max_{f\in\mathcal{V}\setminus\{0\}} \frac{\int_\Omega |\nabla f|^2}{(p-\epsilon)\int_\Omega f^2 u_\epsilon^{p-1-\epsilon}} \\ &\leq \max_{f\in\mathcal{V}\setminus\{0\}} \max_{1\leq i\leq m} \frac{\int_\Omega |\nabla f_i|^2}{(p-\epsilon)\int_\Omega f_i^2 u_\epsilon^{p-1-\epsilon}} := \max_{f\in\mathcal{V}\setminus\{0\}} \max_{1\leq i\leq m} \mathfrak{a}_i. \end{aligned}$$

Hence it suffices to show that each $\mathfrak{a}_i$ is bounded by $1 + O\left(\epsilon^{\frac{n}{n-2}}\right)$. As a matter of fact, this can be achieved along the line of the proof of [GP, Proposition 3.2], but we provide a brief sketch here since our argument slightly simplifies the known proof.

Fix $i \in \{1, \cdots, m\}$. For the sake of notational simplicity, we write $\mathfrak{a} = \mathfrak{a}_i$, $\phi = \phi_i$ and $a_k = a_{ik}$ for $0 \le k \le n$. Denote also $z_\epsilon = \sum_{k=1}^n a_k \frac{\partial u_\epsilon}{\partial x_k}$ so that $f_i = a_0 \phi u_\epsilon + \phi z_\epsilon$. After multiplying (1.1_ϵ) by $\phi^2 u_\epsilon$ or $\phi^2 z_\epsilon$, and integrating the both sides over Ω, one can deduce

$$\int_\Omega |\nabla(\phi u_\epsilon)|^2 = \int_\Omega |\nabla\phi|^2 u_\epsilon^2 + \int_\Omega \phi^2 u_\epsilon^{p+1-\epsilon}. \tag{8.1}$$

and

$$\int_\Omega \nabla(\phi u_\epsilon) \cdot \nabla(\phi z_\epsilon) = \int_\Omega |\nabla\phi|^2 u_\epsilon z_\epsilon + \int_\Omega \phi\nabla\phi \cdot (u_\epsilon \nabla z_\epsilon - z_\epsilon \nabla u_\epsilon) + \int_\Omega \phi^2 u_\epsilon^{p-\epsilon} z_\epsilon. \tag{8.2}$$

Similarly, testing $-\Delta z_\epsilon = (p-\epsilon) u_\epsilon^{p-1-\epsilon} z_\epsilon$ with $\phi^2 z_\epsilon$, one finds that

$$\int_\Omega |\nabla(\phi z_\epsilon)|^2 = \int_\Omega |\nabla\phi|^2 z_\epsilon^2 + (p-\epsilon)\int_\Omega \phi^2 u_\epsilon^{p-1-\epsilon} z_\epsilon^2. \tag{8.3}$$

Then (8.1)-(8.3) yields $\mathfrak{a} = 1 + \mathfrak{b}/\mathfrak{c}$ where

$$\begin{aligned}\mathfrak{b} = -(p-1-\epsilon)&\left(a_0^2 \int_\Omega \phi^2 u_\epsilon^{p+1-\epsilon} + 2a_0 \int_\Omega \phi^2 u_\epsilon^{p-\epsilon} z_\epsilon\right) + a_0^2 \int_\Omega |\nabla\phi|^2 u_\epsilon^2 \\ &+ \int_\Omega |\nabla\phi|^2 z_\epsilon^2 + 2a_0 \int_\Omega \phi\nabla\phi \cdot (u_\epsilon \nabla z_\epsilon - z_\epsilon \nabla u_\epsilon) + 2a_0 \int_\Omega |\nabla\phi|^2 u_\epsilon z_\epsilon. \end{aligned} \tag{8.4}$$

and

$$\mathfrak{c} = (p-\epsilon)\left(a_0^2 \int_\Omega \phi^2 u_\epsilon^{p+1-\epsilon} + 2a_0 \int_\Omega \phi^2 u_\epsilon^{p-\epsilon} z_\epsilon + \int_\Omega \phi^2 u_\epsilon^{p-1-\epsilon} z_\epsilon^2\right). \tag{8.5}$$

Our aim is to find an upper bound of $\mathfrak{b}$ and a lower bound of $\mathfrak{c}$. Let us estimate $\mathfrak{b}$ first. We see at once that

$$-(p-1-\epsilon) a_0^2 \int_\Omega \phi^2 u_\epsilon^{p+1-\epsilon} < -C a_0^2.$$

Also, if we let $\bar{a} = (a_1, \cdots, a_n)$, then (8.7) guarantees

$$\left| a_0 \int_\Omega \phi^2 u_\epsilon^{p-\epsilon} z_\epsilon \right| = \left| a_0 \sum_{j=1}^n a_k \int_\Omega \phi^2 u_\epsilon^{p-\epsilon} \frac{\partial u_\epsilon}{\partial x_k} \right| = \left| \frac{a_0}{p+1-\epsilon} \sum_{k=1}^n a_k \int_\Omega \frac{\partial \phi^2}{\partial x_k} u_\epsilon^{p+1-\epsilon} \right| \le C a_0 |\bar{a}| \epsilon^{\frac{p+1-\epsilon}{2}}.$$

Moreover we have that

$$a_0^2 \int_\Omega |\nabla\phi|^2 u_\epsilon^2 \le C a_0^2 \epsilon.$$

On the other hand, for $\mathcal{D}_1 = B^n(x_i, 3r) \setminus B^n(x_i, 2r)$ and $\mathcal{D}_2 = B^n(x_i, 4r) \setminus B^n(x_i, r)$, we easily discover

$$\int_\Omega |\nabla\phi|^2 z_\epsilon^2 \le C \int_{\mathcal{D}_1} z_\epsilon^2 \le C|\bar{a}|^2 \int_{\mathcal{D}_1} |\nabla u_\epsilon|^2 \le C|\bar{a}|^2 \int_{\mathcal{D}_2} \left(u_\epsilon^{p+1-\epsilon} + u_\epsilon^2\right) \le C|\bar{a}|^2 \epsilon$$

and

$$\int_{\mathcal{D}_1} |\nabla z_\epsilon|^2 \le C \int_{\mathcal{D}_2} \left(z_\epsilon^2 + u_\epsilon^{p-1-\epsilon} z_\epsilon^2\right) \le C \int_{\mathcal{D}_2} z_\epsilon^2 \le C|\bar{a}|^2 \epsilon$$

(cf. (8.1) and (8.3)), which implies

$$\left| 2a_0 \int_\Omega \phi \nabla\phi \cdot (u_\epsilon \nabla z_\epsilon - z_\epsilon \nabla u_\epsilon) + 2a_0 \int_\Omega |\nabla\phi|^2 u_\epsilon z_\epsilon \right| \le C a_0 |\bar{a}| \epsilon.$$

Utilizing these estimates and the Cauchy-Schwarz inequality we deduce

$$\mathfrak{b} \le C|\bar{a}|^2 \epsilon. \tag{8.6}$$

To obtain a lower bound of $\mathfrak{c}$, we note that

$$\left| \int_\Omega \phi^2 u_\epsilon^{p-\epsilon} \frac{\partial u_\epsilon}{\partial x_k} \right| = \left| \frac{1}{p+1-\epsilon} \int_\Omega \frac{\partial \phi^2}{\partial x_k} u_\epsilon^{p+1-\epsilon} \right| \le C \epsilon^{\frac{p+1-\epsilon}{2}}$$

and that Lemma 8.2.9 ensures

$$\int_\Omega \phi^2 u_\epsilon^{p-1-\epsilon} \frac{\partial u_\epsilon}{\partial x_k} \frac{\partial u_\epsilon}{\partial x_l} = \lambda_{i\epsilon}^{-2} \epsilon^{-\frac{2}{n-2}} \left(\frac{\delta_{kl}}{n} \int_{\mathbb{R}^n} U_{1,0}^{p-1} |\nabla U_{1,0}|^2 + o(1) \right)$$

for $1 \le k,\ l \le n$. Hence we conclude that

$$\mathfrak{c} \ge C a_0^2 - C a_0 |\bar{a}| \epsilon^{\frac{p+1-\epsilon}{2}} + C|\bar{a}|^2 \epsilon^{-\frac{2}{n-2}} \ge \frac{C}{2} |\bar{a}|^2 \epsilon^{-\frac{2}{n-2}}. \tag{8.7}$$

Consequently, a combination of (8.6) and (8.7) asserts that $\mathfrak{a} \le 1 + O\left(\epsilon^{\frac{n}{n-2}}\right)$. This completes the proof of the lemma. □

Corollary 8.4.2. *For $m + 1 \le \ell \le (n+1)m$, we have the following limit*

$$\lim_{\epsilon \to 0} \mu_{\ell\epsilon} = 1.$$

Proof. By Lemma 8.3.1 we can find $i_1 \in \{1, \cdots, m\}$ such that $\tilde{v}_{\ell i_1 \epsilon}$ converges weakly to a nonzero function V. Then, as in the proof of Theorem 8.1.1, we observe that V solves

$$-\Delta V = \mu_\ell p U_{1,0}^{p-1} V \quad \text{in } \mathbb{R}^n$$

where $\mu_\ell = \lim_{\epsilon \to 0} \mu_{\ell\epsilon}$. Also, owing to Proposition 8.4.1, we have $\mu_\ell \le 1$. Since the Morse index of $U_{1,0}$ is 1, it should hold that $\mu_\ell = p^{-1}$ or 1.

Assume that $\mu_\ell = p^{-1}$. Then the proof of Theorem 8.1.1 again gives us that there is a vector $\mathbf{b}_\ell = \left(\lambda_1^{\frac{n-2}{2}} b_{\ell 1}, \cdots, \lambda_m^{\frac{n-2}{2}} b_{\ell m}\right) \ne 0$ such that $\tilde{v}_{\ell i \epsilon} \rightharpoonup b_{\ell i} U_{1,0}$ weakly in $H^1(\mathbb{R}^n)$. Furthermore $\mathbf{b}_\ell \cdot \mathbf{c}_{\ell_1} = 0$ for any $1 \le \ell_1 \le m$, but this is impossible since $\{\mathbf{c}_1, \cdots, \mathbf{c}_m\}$ already spans $\mathbb{R}^m$. Hence $\mu_\ell = 1$, which finishes the proof. □

Next, we provide a general convergence result of the ℓ-th $L^\infty(\Omega)$-normalized eigenfunction $v_{\ell\epsilon}$. We recall its dilation $\tilde{v}_{\ell i\epsilon}$ defined in (8.10).

Lemma 8.4.3. *Suppose that $m+1 \le \ell \le (n+1)m$.*

1. *For any $i \in \{1, \cdots, m\}$ there exists a vector $(d_{\ell,i,1}, \cdots, d_{\ell,i,n+1}) \in \mathbb{R}^{n+1}$ such that the function $\tilde{v}_{\ell i\epsilon}$ converges to*
$$\sum_{k=1}^{n} d_{\ell,i,k}\left(\frac{\partial U_{1,0}}{\partial \xi_k}\right) + d_{\ell,i,n+1}\left(\frac{\partial U_{1,0}}{\partial \lambda}\right)$$
weakly in $H^1(\mathbb{R}^n)$ .In addition, there is at least one $i_1 \in \{1, \cdots, m\}$ such that $(d_{\ell,i_1,1}, \cdots, d_{\ell,i_1,n+1}) \neq 0$.

2. *As $\epsilon \to 0$ we have*
$$\epsilon^{-1} v_{\ell\epsilon} \to C_3 \sum_{i=1}^{m} d_{\ell,i,n+1}\lambda_i^{n-2} G(\cdot, x_i) \quad \text{in } C^1(\Omega \setminus \{x_1, \cdots, x_m\}) \tag{8.8}$$
where $C_3 = p\int_{\mathbb{R}^n} U_{1,0}^{p-1}\left(\frac{\partial U_{1,0}}{\partial \lambda}\right) > 0$.

Proof. It is not hard to show the first statement with Lemmas 8.3.1 and 8.2.10, and Corollary 8.4.2. Hence let us consider the second statement. For $r > 0$ fixed small, assume that a point $x \in \Omega$ belongs to A_r where A_r is the set in (8.6). According to Green's representation formula and Lemmas 8.2.5 and 8.2.7,
$$\epsilon^{-1} v_{\ell\epsilon}(x) = \epsilon^{-1}\mu_{\ell\epsilon}(p-\epsilon)\sum_{i=1}^{m}\int_{B^n(x_{i\epsilon}, r/2)} G(x,y) u_\epsilon^{p-1-\epsilon}(y) v_{\ell\epsilon}(y)dy + o(1).$$

Besides, Proposition 8.2.3 with Lemmas 8.2.8 and 8.4.3 (1) allow us to obtain
$$\begin{aligned}
&\lim_{\epsilon\to 0}\epsilon^{-1}\int_{B^n(x_{i\epsilon}, r/2)} G(x,y) u_\epsilon^{p-1-\epsilon}(y) v_{\ell\epsilon}(y)dy \\
&\quad = \lambda_i^{n-2}\lim_{\epsilon\to 0}\int_{B^n\left(0,(\lambda_{i\epsilon}\epsilon^{\alpha_0})^{-1}r/2\right)} G\left(x, x_{i\epsilon} + \lambda_{i\epsilon}\epsilon^{\alpha_0} y\right)\left(\tilde{u}_{i\epsilon}^{p-1-\epsilon}\tilde{v}_{\ell i\epsilon}\right)(y)dy \\
&\quad = d_{\ell,i,n+1}\lambda_i^{n-2} G(x, x_i)\int_{\mathbb{R}^n} U_{1,0}^{p-1}(y)\left(\frac{\partial U_{1,0}}{\partial\lambda}\right)(y)dy.
\end{aligned} \tag{8.9}$$

Thus the lemma is proved. □

In fact, we can refine the first statement of the above lemma to arrive at (8.14), which is the main result of the latter part of this section.

Proposition 8.4.4. *Let $m+1 \le \ell \le (n+1)m$. For each $i \in \{1, \cdots, m\}$ and $(d_{\ell,i,1}, \cdots, d_{\ell,i,n}) \in \mathbb{R}^n$, the function $\tilde{v}_{\ell i \epsilon}$ converges to*

$$\sum_{k=1}^{n} d_{\ell,i,k} \left(\frac{\partial U_{1,0}}{\partial \xi_k} \right) = - \sum_{k=1}^{n} d_{\ell,i,k} \left(\frac{\partial U_{1,0}}{\partial x_k} \right)$$

weakly in $H^1(\mathbb{R}^n)$.

As a preparation for its proof, we first consider the following auxiliary lemma.

Lemma 8.4.5. *Fix $1 \le i \le m$. For a small $r > 0$ (any choice of $r < \min\{dist(x_j, x_l) : 1 \le j \ne l \le m\}/2$ is available) and $1 \le j, l \le m$, we define*

$$\begin{aligned} \mathcal{I}^r_{jl;i} = \int_{\partial B^n(x_i,r)} \Bigg(\frac{\partial}{\partial \nu} \left[(x - x_i) \cdot \nabla G(x, x_j) + \left(\frac{n-2}{2} \right) G(x, x_j) \right] G(x, x_l) \\ - \left[(x - x_i) \cdot \nabla G(x, x_j) + \left(\frac{n-2}{2} \right) G(x, x_j) \right] \frac{\partial}{\partial \nu} G(x, x_l) \Bigg) dS. \end{aligned} \tag{8.10}$$

Then $\mathcal{I}^r_{jl;i}$ is independent of $r > 0$ and its value is computed as

$$\mathcal{I}^r_{jl;i} = \begin{cases} 0 & \text{if } j \ne i \text{ and } l \ne i, \\ \left(\frac{n-2}{2} \right) G(x_i, x_j) & \text{if } j \ne i \text{ and } l = i, \\ \left(\frac{n-2}{2} \right) G(x_i, x_l) & \text{if } j = i \text{ and } l \ne i, \\ -(n-2)\tau(x_i) & \text{if } j = l = i. \end{cases} \tag{8.11}$$

Proof. Assuming $0 < r_2 < r_1$ are small enough and putting $f(x) = (x - x_i) \cdot \nabla G(x, x_j) + G(x, x_j)$, $g(x) = G(x, x_l)$ and $D = B^n(x_i, r_1) \setminus B^n(x_i, r_2)$ into Green's identity

$$\int_{\partial D} \left(\frac{\partial f}{\partial \nu} g - \frac{\partial g}{\partial \nu} f \right) dS = \int_D (\Delta f \cdot g - \Delta g \cdot f) \, dx, \tag{8.12}$$

we see that $\mathcal{I}^r_{jl;i}$ is constant because

$$\Delta \left[(x - x_i) \cdot \nabla G(x, x_j) + \left(\frac{n-2}{2} \right) G(x, x_j) \right] = 0 \quad \text{and} \quad \Delta G(x, x_l) = 0 \tag{8.13}$$

for all $x \ne x_j,\ x_l$. Thus it suffices to find the value $\mathcal{I}_{jl;i} = \lim_{r \to 0} \mathcal{I}^r_{jl;i}$.

(1) If $j,\ l \ne i$, then $\mathcal{I}_{jl;i} = 0$. This follows simply by applying (8.12) for $D = B^n(x_i, r)$ since (8.13) holds for any $x \in B^n(x_i, r)$.

(2) If $j \neq i$ and $l = i$, then we have

$$\begin{aligned}\mathcal{I}_{jl;i} = \mathcal{I}_{ji;i} &= \lim_{r\to 0}\int_{\partial B^n(x_i,r)} -\left(\frac{n-2}{2}\right)G(x,x_j)\frac{\partial}{\partial\nu}G(x,x_i)dS\\ &= \lim_{r\to 0}\int_{\partial B^n(x_i,r)}\left(\frac{n-2}{2}\right)G(x,x_j)\cdot\frac{n-2}{(n-2)\left|S^{n-1}\right||x-x_i|^{n-1}}dS = \left(\frac{n-2}{2}\right)G(x_i,x_j).\end{aligned}$$

(3) Suppose that $j = i$ and $l \neq i$. In this case, we deduce

$$\begin{aligned}\mathcal{I}_{jl;i} = \mathcal{I}_{il;i} &= \lim_{r\to 0}\int_{\partial B^n(x_i,r)}\frac{\partial}{\partial\nu}\left[(x-x_i)\cdot\nabla G(x,x_i) + \left(\frac{n-2}{2}\right)G(x,x_i)\right]G(x,x_l)dS\\ &= \lim_{r\to 0}\int_{\partial B^n(x_i,r)}\frac{n-2}{2\left|S^{n-1}\right||x-x_i|^{n-1}}\cdot G(x,x_l)dS = \left(\frac{n-2}{2}\right)G(x_i,x_l).\end{aligned}$$

(4) If $k = l = j$, then the Green's identity, the fact that $G(x, x_i) = 0$ on $\partial\Omega$ and Lemma 8.2.11 lead

$$\begin{aligned}\mathcal{I}_{jl;i} = \mathcal{I}_{ii;i} = &\int_{\partial\Omega}\left(\frac{\partial}{\partial\nu}\left[(x-x_i)\cdot\nabla G(x,x_i) + \left(\frac{n-2}{2}\right)G(x,x_i)\right]G(x,x_i)\right.\\ &\qquad\left. - \left[(x-x_i)\cdot\nabla G(x,x_i) + \left(\frac{n-2}{2}\right)G(x,x_i)\right]\frac{\partial}{\partial\nu}G(x,x_i)\right)dS\\ = &-\int_{\partial\Omega}\left[(x-x_i)\cdot\nabla G(x,x_i)\right]\frac{\partial}{\partial\nu}G(x,x_i)dS = -(n-2)\tau(x_i).\end{aligned}$$

All the computations made in (1)-(4) show the validity of (8.11). □

Proof of Proposition 8.4.4. Fix $i \in \{1, \cdots, m\}$ and let

$$w_{i\epsilon}(x) = (x - x_{i\epsilon})\cdot\nabla u_\epsilon + \frac{2u_\epsilon}{p-1-\epsilon} \quad \text{for } x \in \Omega, \tag{8.14}$$

a solution of

$$-\Delta w_{i\epsilon} = (p-\epsilon)u_\epsilon^{p-\epsilon-1}w_{i\epsilon} \quad \text{in } \Omega.$$

Then by (8.12) it satisfies

$$\int_{\partial B^n(x_{i\epsilon},r)}\left(\frac{\partial w_{i\epsilon}}{\partial\nu}v_{\ell\epsilon} - \frac{\partial v_{\ell\epsilon}}{\partial\nu}w_{i\epsilon}\right)dS = (\mu_{\ell\epsilon}-1)(p-\epsilon)\int_{B^n(x_{i\epsilon},r)}u_\epsilon^{p-1-\epsilon}w_{i\epsilon}v_{\ell\epsilon} \tag{8.15}$$

for $r > 0$ small, where ν is the outward normal unit vector to the sphere $\partial B^n(x_i, r)$.

In light of Lemma 8.4.3 (1), we already know that $\tilde{v}_{\ell i\epsilon} \rightharpoonup \sum_{k=1}^n d_{\ell,i,k}\left(\frac{\partial U_{1,0}}{\partial\xi_k}\right) + d_{\ell,i,n+1}\left(\frac{\partial U_{1,0}}{\partial\lambda}\right)$ weakly in $H^1(\mathbb{R}^n)$ as $\epsilon \to 0$. Thus we only need to verify that $d_{\ell,i,n+1} = 0$ for all $i \in \{1, \cdots, m\}$ in order to establish Proposition 8.4.4. Assume to the contrary that $d_{\ell,i,n+1} \neq 0$ for some i. We will achieve a contradiction by showing that an estimate of $\mu_{l\epsilon} - 1$ obtained through (8.15) does not match to one found in Proposition 8.4.1. To reduce the notational complexity, we use d_i or $d_{\ell,i}$ to denote $d_{\ell,i,n+1}$ in this proof.

Let us observe from Lemma 8.2.6 and (8.14) that

$$\epsilon^{-\frac{1}{2}} w_{i\epsilon}(x) \to C_2 \sum_{j=1}^{m} \lambda_j^{\frac{n-2}{2}} \left[(x-x_i)\cdot \nabla G(x,x_j) + \left(\frac{n-2}{2}\right) G(x,x_j)\right] \quad \text{in } C^1(\Omega \setminus \{x_1,\cdots,x_m\}) \tag{8.16}$$

as $\epsilon \to 0$. Combining this with (8.8) we get

$$\lim_{\epsilon\to 0} \epsilon^{-\frac{3}{2}} \int_{\partial B^n(x_{i\epsilon},r)} \left(\frac{\partial w_{i\epsilon}}{\partial \nu} v_{\ell\epsilon} - \frac{\partial v_{\ell\epsilon}}{\partial \nu} w_{i\epsilon}\right) dS = C_2 C_3 \sum_{j,l=1}^{m} \lambda_j^{\frac{n-2}{2}} \lambda_l^{n-2} d_l \mathcal{I}^r_{jl;i}$$

where $\mathcal{I}^r_{jl;i}$ is the value defined in (8.10). By inserting (8.11) into the above identity, we further find that

$$\begin{aligned}
&\lim_{\epsilon\to 0} \epsilon^{-\frac{3}{2}} \int_{\partial B^n(x_{i\epsilon},r)} \left(\frac{\partial w_{i\epsilon}}{\partial \nu} v_{\ell\epsilon} - \frac{\partial v_{\ell\epsilon}}{\partial \nu} w_{i\epsilon}\right) dS \\
&= C_2C_3 \left[\left(\frac{n-2}{2}\right)\lambda_i^{\frac{n-2}{2}} \sum_{l\neq i} \lambda_l^{n-2} d_l G(x_i,x_l) + \lambda_i^{n-2} d_i \left(\sum_{j\neq i}\left(\frac{n-2}{2}\right)\lambda_j^{\frac{n-2}{2}} G(x_i,x_j) - (n-2)\lambda_i^{\frac{n-2}{2}}\tau(x_i)\right)\right] \\
&= C_2C_3 \left[\left(\frac{n-2}{2}\right)\lambda_i^{\frac{n-2}{2}} \sum_{j\neq i} \lambda_j^{n-2} d_j G(x_i,x_j) - \lambda_i^{n-2} d_i \left(\frac{n-2}{2}\right)\left(\lambda_i^{\frac{n-2}{2}}\tau(x_i) + C_0 \lambda_i^{-\frac{n-2}{2}}\right)\right].
\end{aligned}$$

Here $C_0 = c_2/((n-2)c_1) > 0$ as in (8.11), and we employed the fact that $(\lambda_1,\cdots,\lambda_m,x_1,\cdots,x_m)$ is a critical point of the functional Υ_m (see (8.8)) so as to obtain the second equality. Borrowing the notation of the matrix $\mathcal{A}_3$ in (8.20), the left-hand side of (8.15) can be described in a legible way.

$$\lim_{\epsilon\to 0} \epsilon^{-\frac{3}{2}} \int_{\partial B^n(x_{i\epsilon},r)} \left(\frac{\partial w_{i\epsilon}}{\partial \nu} v_{\ell\epsilon} - \frac{\partial v_{\ell\epsilon}}{\partial \nu} w_{i\epsilon}\right) dS = -C_2 C_3 \left(\frac{n-2}{2}\right) \sum_{j=1}^{m} \mathcal{A}^3_{ij} \left(\lambda_j^{\frac{n-2}{2}} d_j\right). \tag{8.17}$$

On the other hand, counting on Proposition 8.2.3 and Lemmas 8.2.2 and 8.4.3, we can compute its right-hand side as follows.

$$\begin{aligned}
&\lim_{\epsilon\to 0} \epsilon^{-\frac{1}{2}} \int_{B^n(x_{i\epsilon},r)} u_\epsilon^{p-1-\epsilon}(x) \left[(x-x_{i\epsilon})\cdot\nabla u_\epsilon(x) + \frac{2u_\epsilon(x)}{p-1-\epsilon}\right] v_{\ell\epsilon}(x)dx \\
&\qquad = \lim_{\epsilon\to 0} \lambda_i^{\frac{n-2}{2}} \int_{B^n\left(0,(\lambda_i\epsilon^{\alpha_0})^{-1}r\right)} \tilde{u}_{i\epsilon}^{p-1-\epsilon}(y) \left[y\cdot\nabla\tilde{u}_{i\epsilon}(y) + \frac{2\tilde{u}_{i\epsilon}(y)}{p-1-\epsilon}\right] \tilde{v}_{\ell i\epsilon}(y)dy \\
&\qquad = \lambda_i^{\frac{n-2}{2}} d_i \int_{\mathbb{R}^n} U_{1,0}^{p-1}(y) \left[y\cdot\nabla U_{1,0}(y) + \frac{2U_{1,0}(y)}{p-1}\right] \left(\frac{\partial U_{1,0}}{\partial\lambda}\right)(y)dy = -\lambda_i^{\frac{n-2}{2}} d_i C_4
\end{aligned} \tag{8.18}$$

where $C_4 = \int_{\mathbb{R}^n} U_{1,0}^{p-1} \left(\frac{\partial U_{1,0}}{\partial \lambda}\right)^2 > 0$. Consequently, (8.17), (8.18) and (8.15) enable us to deduce that

$$\mathcal{A}_3 \hat{\mathbf{d}}_\ell^1 = \frac{2pC_4}{(n-2)C_2C_3} \lim_{\epsilon\to 0}\left(\frac{\mu_{\ell\epsilon}-1}{\epsilon}\right)\hat{\mathbf{d}}_\ell^1 \quad \text{where } \hat{\mathbf{d}}_\ell^1 = \begin{pmatrix} \lambda_1^{\frac{n-2}{2}} d_{\ell,1} \\ \cdots \\ \lambda_m^{\frac{n-2}{2}} d_{\ell,m} \end{pmatrix} \neq 0. \tag{8.19}$$

Multiplying a row vector $\left(\hat{\mathbf{d}}_\ell^1\right)^T$ in the both sides yields

$$\lim_{\epsilon\to 0}\left(\frac{\mu_{\ell\epsilon}-1}{\epsilon}\right) = \frac{(n-2)^2C_2C_3}{2(n+2)C_4} \cdot \left(\frac{\left(\hat{\mathbf{d}}_\ell^2\right)^T \mathcal{M}_2 \hat{\mathbf{d}}_\ell^2}{\left|\hat{\mathbf{d}}_\ell^1\right|^2} + C_0\right) \tag{8.20}$$

where $\hat{\mathbf{d}}_\ell^2 = \left(\lambda_1^{n-2} d_{\ell,1}, \cdots, \lambda_m^{n-2} d_{\ell,m}\right)^T$ and $\mathcal{M}_2$ is the matrix introduced in Lemma 8.2.1. However the right-hand side of (8.20) is positive due to Lemma 8.2.1, and this contradicts the bound of $\mu_{\ell\epsilon}$ provided in Proposition 8.4.1. Hence it should hold that $d_{\ell,i} = 0$ for all i. The proof is finished. □

This result improves our knowledge on the limit behavior of the ℓ-th eigenvalues (see Corollary 8.4.2) for $m+1 \le \ell \le (n+1)m$, which is essential in the next section.

Corollary 8.4.6. *For $m+1 \le \ell \le (n+1)m$, one has*

$$|\mu_{\ell\epsilon} - 1| = O\left(\epsilon^{\frac{n-1}{n-2}}\right) \quad \text{as } \epsilon \to 0. \tag{8.21}$$

Proof. By Proposition 8.4.4 and Lemma 8.4.3 (1), there is $i_1 \in \{1, \cdots, m\}$ such that

$$\tilde{v}_{\ell i_1 \epsilon} \rightharpoonup \sum_{k=1}^n d_{\ell,i_1,k}\left(\frac{\partial U_{1,0}}{\partial \xi_k}\right) \quad \text{weakly in } H^1(\mathbb{R}^n)$$

where $(d_{\ell,i_1,1}, \cdots, d_{\ell,i_1,n}) \neq 0$. Without any loss of generality, we may assume that $d_{\ell,i_1,1} \neq 0$. By differentiating the both sides of (1.1_ϵ), we get

$$-\Delta \frac{\partial u_\epsilon}{\partial x_1} = (p-\epsilon) u_\epsilon^{p-1-\epsilon} \frac{\partial u_\epsilon}{\partial x_1}. \tag{8.22}$$

Let us multiply (8.22) by $v_{\ell\epsilon}$ and (8.5) by $\frac{\partial u_\epsilon}{\partial x_1}$, respectively, integrate both of them over $B^n(x_{i_1\epsilon}, r)$ for a small fixed $r > 0$ and subtract the first equation from the second to derive

$$\int_{\partial B^n(x_{i_1\epsilon},r)} \left\{\frac{\partial}{\partial \nu}\left(\frac{\partial u_\epsilon}{\partial x_1}\right) v_{\ell\epsilon} - \frac{\partial u_\epsilon}{\partial x_1}\frac{\partial v_{\ell\epsilon}}{\partial \nu}\right\} dS = (p-\epsilon)(\mu_{\ell\epsilon}-1) \int_{B^n(x_{i_1\epsilon},r)} u_\epsilon^{p-1-\epsilon} \frac{\partial u_\epsilon}{\partial x_1} v_{\ell\epsilon}. \tag{8.23}$$

By Lemma 8.2.9, its left-hand side is $O\left(\epsilon^{3/2}\right)$ while the right-hand side is computed as

$$\begin{aligned} \int_{B^n(x_{i_1\epsilon},r)} u_\epsilon^{p-1-\epsilon} \frac{\partial u_\epsilon}{\partial x_1} v_{\ell\epsilon} &= (\lambda_{i_1}\epsilon^{\alpha_0})^{n-(\sigma_\epsilon+1)-2} \int_{B^n(0,(\lambda_i\epsilon^{\alpha_0})^{-1}r)} \tilde{u}_{i_1\epsilon}^{p-1-\epsilon} \frac{\partial \tilde{u}_{i_1\epsilon}}{\partial x_1} \tilde{v}_{\ell i_1\epsilon} \\ &= -\lambda_{i_1}^{\frac{n-4}{2}} \epsilon^{\frac{n-4}{2(n-2)}} \left(d_{\ell,i_1,1} \int_{\mathbb{R}^n} U_{1,0}^{p-1} \left(\frac{\partial U_{1,0}}{\partial x_1}\right)^2 + o(1)\right). \end{aligned} \tag{8.24}$$

Therefore, if we denote $C_5 = \int_{\mathbb{R}^n} U_{1,0}^{p-1} \left(\frac{\partial U_{1,0}}{\partial x_1}\right)^2 > 0$, we deduce that

$$O\left(\epsilon^{\frac{3}{2}}\right) = -\lambda_{i_1}^{\frac{n-4}{2}} \epsilon^{\frac{n-4}{2(n-2)}} (p + o(1)) \left[\lim_{\epsilon\to 0}(\mu_{\ell\epsilon} - 1)\right] (d_{\ell,i,1} C_5 + o(1)),$$

which leads the desired estimate (8.21). □

8.5 A further analysis on asymptotic behavior of the ℓ-th eigenfunctions, $m + 1 \le \ell \le (n + 1)m$

In view of Lemma 8.4.3 and the proof of Proposition 8.4.4, we know that $\epsilon^{-1} v_{\ell\epsilon} \to 0$ as $\epsilon \to 0$ uniformly in Ω outside of the blow-up points $\{x_1, \cdots, x_m\}$. Motivated by the argument in [GGOS], we prove its improvement (8.15) here, which is stated once more in the following proposition.

Proposition 8.5.1. *Let* $\mathcal{M}_1$ *and* $\mathcal{P}$ *be the matrices defined in* (8.13) *and* (8.17)*, respectively. Also we remind a column vector* $\mathbf{d}_\ell \in \mathbb{R}^{mn}$ *in* (8.16) *and set two row vectors* $\mathcal{G}(x)$ *and* $\widetilde{\mathcal{G}}(x)$ *by*

$$\mathcal{G}(x) = (G(x, x_1), \cdots, G(x, x_m)) \in \mathbb{R}^m, \quad \widetilde{\mathcal{G}}(x) = \left(\lambda_1^{\frac{n}{2}} \nabla_y G(x, x_1), \cdots, \lambda_m^{\frac{n}{2}} \nabla_y G(x, x_m)\right) \in \mathbb{R}^{mn} \quad (8.1)$$

for any $x \in \Omega$*. If* $m + 1 \le \ell \le (n + 1)m$*, then*

$$\epsilon^{-\frac{n-1}{n-2}} v_{\ell\epsilon}(x) \to C_1 \left(\mathcal{G}(x)\mathcal{M}_1^{-1}\mathcal{P} + \widetilde{\mathcal{G}}(x)\right) \mathbf{d}_\ell, \quad (8.2)$$

in $C^1(\Omega \setminus \{x_1, \cdots, x_m\})$ *as* $\epsilon \to 0$ *where* $C_1 > 0$ *is a constant in Theorem 8.1.2.*

Remark 8.5.2. If we write (8.2) in terms of the components of the vectors $\mathcal{G}(x)$ and $\widetilde{\mathcal{G}}(x)$, and matrices $\mathcal{M}_1^{-1}$ and $\mathcal{P}$, we get (8.15).

We will present the proof by dividing it into several lemmas. The first lemma is a variant of Lemmas 8.2.6 and 8.4.3 (2).

Lemma 8.5.3. *Given a small fixed number* $r > 0$*, it holds that*

$$u_\epsilon(x) = \sum_{i=1}^m \kappa_{i0} G(x, x_{i\epsilon}) + o\left(\epsilon^{\frac{n}{2(n-2)}}\right)$$

and

$$\frac{v_{\ell\epsilon}(x)}{\mu_{\ell\epsilon}(p-\epsilon)} = \sum_{i=1}^m \left(\kappa_{i1} G(x, x_{i\epsilon}) + \boldsymbol{\kappa}_{i2} \cdot \nabla_y G(x, x_{i\epsilon})\right) + o\left(\epsilon^{\frac{n-1}{n-2}}\right) \quad (8.3)$$

in $C^1(\Omega \setminus \{x_1, \cdots, x_m\})$ *as* $\epsilon \to 0$ *where*

$$\kappa_{i0} = \int_{B^n(x_{i\epsilon}, r)} u_\epsilon^{p-\epsilon} = O\left(\sqrt{\epsilon}\right), \quad \kappa_{i1} = \int_{B^n(x_{i\epsilon}, r)} u_\epsilon^{p-1-\epsilon} v_{\ell\epsilon} = O(\epsilon)$$

and $\boldsymbol{\kappa}_{i2} = (\kappa_{i21}, \cdots, \kappa_{i2n}) \in \mathbb{R}^n$ is a row vector such that

$$\boldsymbol{\kappa}_{i2} = \int_{B^n(x_{i\epsilon}, r)} (y - x_{i\epsilon}) \left(u_\epsilon^{p-1-\epsilon} v_{\ell\epsilon}\right)(y) dy = O\left(\epsilon^{\frac{n-1}{n-2}}\right) \tag{8.4}$$

(note that κ_{i0}, κ_{i1} and $\boldsymbol{\kappa}_{i2}$ depend also on ϵ or ℓ).

Proof. The proof is similar to Lemmas 8.2.6 and 8.4.3 (2), so we just briefly sketch why (8.3) holds in $C^0(K)$ for any compact subset K of $\Omega \backslash \{x_1, \cdots, x_m\}$. For $x \in A_r$ (see (8.6)), a combination of Green's representation formula and the Taylor expansion of $G(x, y)$ in the y-variable show that

$$\begin{aligned}&\frac{v_{\ell\epsilon}(x)}{\mu_{\ell\epsilon}(p-\epsilon)} \\ &= \sum_{i=1}^m \int_{B^n(x_{i\epsilon}, r/2)} \left(G(x, x_{i\epsilon}) + (y - x_{i\epsilon}) \cdot \nabla_y G(x, x_{i\epsilon}) + O\left(|y - x_{i\epsilon}|^2\right)\right) \left(u_\epsilon^{p-1-\epsilon} v_{\ell\epsilon}\right)(y) dy + O\left(\epsilon^{\frac{n}{n-2}}\right)\end{aligned}$$

Also, by means of Proposition 8.2.3 and Lemma 8.2.8, we have

$$\begin{aligned}\int_{B^n(x_{i\epsilon}, r/2)} |y - x_{i\epsilon}|^2 \cdot \left|\left(u_\epsilon^{p-1-\epsilon} v_{\ell\epsilon}\right)(y)\right| dy &= (\lambda_{i\epsilon}\epsilon^{\alpha_0})^n \int_{B^n\left(0, (\lambda_{i\epsilon}\epsilon^{\alpha_0})^{-1} r/2\right)} |x|^2 \cdot \left|\left(\tilde{u}_\epsilon^{p-1-\epsilon} \tilde{v}_\epsilon\right)(x)\right| dx \\ &\le C\epsilon^{\frac{n}{n-2}} \int_0^{C\epsilon^{-\frac{1}{n-2}}} \frac{t^{n+1}}{1 + t^{(n+2)-(n-2)\epsilon}} dt = O\left(\epsilon^{\frac{n}{n-2}}\right)\end{aligned}$$

for each i, from which the desired result follows. The order of k_{i0}, k_{i1} and $\boldsymbol{\kappa}_{i2}$ can be computed as in (8.13) or (8.9). □

Let us write u_ϵ and $v_{\ell\epsilon}$ in the following way. For each $i = 1, \cdots, m$,

$$u_\epsilon(x) = \frac{\kappa_{i0}\gamma_n}{|x - x_{i\epsilon}|^{n-2}} + g_{i\epsilon}(x) + o\left(\epsilon^{\frac{n}{2(n-2)}}\right) \quad \text{where} \quad g_{i\epsilon}(x) = -\kappa_{i0} H(x, x_{i\epsilon}) + \sum_{j \ne i} \kappa_{j0} G(x, x_{j\epsilon}), \tag{8.5}$$

and

$$\frac{v_{\ell\epsilon}(x)}{\mu_{\ell\epsilon}(p-\epsilon)} = \frac{\kappa_{i1}\gamma_n}{|x - x_{i\epsilon}|^{n-2}} + (n-2)\gamma_n \boldsymbol{\kappa}_{i2} \cdot \frac{x - x_{i\epsilon}}{|x - x_{i\epsilon}|^n} + h_{i\epsilon}(x) + o\left(\epsilon^{\frac{n-1}{n-2}}\right) \tag{8.6}$$

where

$$h_{i\epsilon}(x) = -\left(\kappa_{i1} H(x, x_{i\epsilon}) + \boldsymbol{\kappa}_{i2} \cdot \nabla_y H(x, x_{i\epsilon})\right) + \sum_{j \ne i} \left(\kappa_{j1} G(x, x_{j\epsilon}) + \boldsymbol{\kappa}_{j2} \cdot \nabla_y G(x, x_{j\epsilon})\right). \tag{8.7}$$

Note that $g_{i\epsilon}$ an $h_{i\epsilon}$ are harmonic in a neighborhood of $x_{i\epsilon}$. With these decompositions we now compute κ_{i1}, will be shown to be $O\left(\epsilon^{\frac{n-1}{n-2}}\right)$, by applying the bilinear version of the Pohožaev identity which the next lemma describes.

Lemma 8.5.4. *For any point $\xi \in \mathbb{R}^n$, a positive number $r > 0$ and functions $f,\ g \in C^2\left(\overline{B^n(\xi, r)}\right)$, it holds that*

$$\int_{B^n(\xi,r)} [((x-\xi)\cdot\nabla f)\,\Delta g + ((x-\xi)\cdot\nabla g)\,\Delta f] = r\int_{\partial B^n(\xi,r)}\left(2\frac{\partial f}{\partial \nu}\frac{\partial g}{\partial \nu} - \nabla f\cdot\nabla g\right) + (n-2)\int_{B^n(\xi,r)}\nabla f\cdot\nabla g \tag{8.8}$$

where ν is the outward unit normal vector on $\partial B^n(\xi, r)$.

Proof. This follows from an elementary computation. See the proof of [Oh, Proposition 5.5] in which the author considered it when $n = 2$. □

Lemma 8.5.5. *Recall the definition of $\mathcal{M}_1$ in* (8.13) *and its inverse $\mathcal{M}_1^{-1} = \left(m_1^{ij}\right)_{1\le i,j\le m}$. Then it holds for $m + 1 \le \ell \le (n + 1)m$ that*

$$\epsilon^{-\frac{n-1}{n-2}}\kappa_{i1} = \sum_{j=1}^m m_1^{ij}\left(-\frac{1}{2}\epsilon^{-\frac{n-1}{n-2}}\boldsymbol{\kappa}_{j2}\cdot\nabla\tau(x_j) + \sum_{l\ne j}\epsilon^{-\frac{n-1}{n-2}}\boldsymbol{\kappa}_{l2}\cdot\nabla_y G(x_j, x_l)\right) + o(1). \tag{8.9}$$

Remark 8.5.6. If $m = 1$, one has that $\Upsilon_1(\lambda_1, x_1) = c_1\tau_1(x_1)\lambda_1^{n-2} - c_2\log\lambda_1$ (refer to (8.8)). Therefore (8.9) and $0 = \partial_{x_1}\Upsilon_1(\lambda_1, x_1) = c_1\left(\partial_{x_1}\tau\right)(x_1)\lambda_1^{n-2}$ imply $\epsilon^{-\frac{n-1}{n-2}}\kappa_{i1} = o(1)$.

Proof. Fixing a sufficiently small number $r > 0$, we take $\xi = x_{i\epsilon}$, $f = u_\epsilon$ and $g = v_{\ell\epsilon}$ for (8.8). Then from (1.1_ϵ), (8.5) and the estimate

$$\begin{aligned}&(1-\mu_{\ell\epsilon})\int_{B^n(x_{i\epsilon},r)}[(x-x_{i\epsilon})\cdot\nabla u_\epsilon]\,u_\epsilon^{p-1-\epsilon}v_{\ell\epsilon}\\ &= O\left(\epsilon^{\frac{n-1}{n-2}}\right)\cdot\epsilon^{\frac{1}{2}}\lambda_i^{\frac{n-2}{2}}\left(-\sum_{k=1}^n d_{\ell,i,k}\int_{\mathbb{R}^n}(x\cdot\nabla U_{1,0})\,U_{1,0}^{p-1}\frac{\partial U_{1,0}}{\partial x_k} + o(1)\right) = o\left(\epsilon^{\frac{n-1}{n-2}+\frac{1}{2}}\right)\end{aligned}$$

where Proposition 8.4.4 and Corollary 8.4.6 are made use of, one finds that the left-hand side of (8.8) is equal to

$$\begin{aligned}&-\int_{B^n(x_{i\epsilon},r)}(x-x_{i\epsilon})\cdot\nabla\left(u_\epsilon^{p-\epsilon}v_{\ell\epsilon}\right) + (1-\mu_{\ell\epsilon})(p-\epsilon)\int_{B^n(x_{i\epsilon},r)}[(x-x_{i\epsilon})\cdot\nabla u_\epsilon]\,u_\epsilon^{p-1-\epsilon}v_{\ell\epsilon}\\ &= n\int_{B^n(x_{i\epsilon},r)}u_\epsilon^{p-\epsilon}v_{\ell\epsilon} + o\left(\epsilon^{\frac{n-1}{n-2}+\frac{1}{2}}\right).\end{aligned}$$

As a result, (8.8) reads as

$$\begin{aligned}&r\int_{\partial B^n(x_{i\epsilon},r)}\left(2\frac{\partial u_\epsilon}{\partial\nu}\frac{\partial v_{\ell\epsilon}}{\partial\nu} - \nabla u_\epsilon\cdot\nabla v_{\ell\epsilon}\right) + (n-2)\int_{\partial B^n(x_{i\epsilon},r)}\frac{\partial u_\epsilon}{\partial\nu}v_{\ell\epsilon}\\ &= 2\int_{B^n(x_{i\epsilon},r)}u_\epsilon^{p-\epsilon}v_{\ell\epsilon} + o\left(\epsilon^{\frac{n-1}{n-2}+\frac{1}{2}}\right) \qquad (8.10)\\ &= 2\left[\mu_{\ell\epsilon}(p-\epsilon)-1\right]^{-1}\int_{\partial B^n(x_{i\epsilon},r)}\left(\frac{\partial u_\epsilon}{\partial\nu}v_{\ell\epsilon} - \frac{\partial v_{\ell\epsilon}}{\partial\nu}u_\epsilon\right)dS + o\left(\epsilon^{\frac{n-1}{n-2}+\frac{1}{2}}\right)\end{aligned}$$

where the latter equality is due to Green's identity (8.12).

We compute the rightmost side of (8.10) first. Since $g_{i\epsilon}$, $h_{i\epsilon}$ and $(x - x_{i\epsilon}) \cdot \nabla g_{i\epsilon}$ are harmonic near $x_{i\epsilon}$ (see (8.5) and (8.7) to remind their definitions), a direct computation with (8.5)-(8.7), the mean value formula and Green's identity (8.12) shows that

$$\begin{aligned}
&\int_{\partial B^n(x_{i\epsilon},r)} \left(\frac{\partial u_\epsilon}{\partial \nu} v_{\ell\epsilon} - \frac{\partial v_{\ell\epsilon}}{\partial \nu} u_\epsilon \right) dS \\
&= \mu_{\ell\epsilon}(p-\epsilon) \Bigg[(n-2)\gamma_n \left|S^{n-1}\right| (\kappa_{i1} g_{i\epsilon}(x_{i\epsilon}) - \kappa_{i0} h_{i\epsilon}(x_{i\epsilon})) + \frac{(n-2)\gamma_n}{r^n} \boldsymbol{\kappa}_{i2} \cdot \int_{\partial B^n(x_{i\epsilon},r)} (x - x_{i\epsilon}) \frac{\partial g_{i\epsilon}}{\partial \nu} dS \\
&\qquad + \frac{(n-2)(n-1)\gamma_n}{r^{n+1}} \boldsymbol{\kappa}_{i2} \cdot \int_{\partial B^n(x_{i\epsilon},r)} (x - x_{i\epsilon}) g_{i\epsilon} dS + o\left(\epsilon^{\frac{n-1}{n-2}+\frac{1}{2}}\right) \Bigg].
\end{aligned} \tag{8.11}$$

Moreover, both $g_{i\epsilon}$ and $\frac{x-x_{i\epsilon}}{|x-x_{i\epsilon}|^n}$ are harmonic in $B^n(x_{i\epsilon}, r) \setminus \{x_{i\epsilon}\}$, so Green's identity again infers that the value

$$\begin{aligned}
I_{1r} &:= \boldsymbol{\kappa}_{i2} \cdot \int_{\partial B^n(x_{i\epsilon},r)} \left(\frac{x - x_{i\epsilon}}{|x - x_{i\epsilon}|^n} \frac{\partial g_{i\epsilon}}{\partial \nu} + (n-1) \frac{x - x_{i\epsilon}}{|x - x_{i\epsilon}|^{n+1}} g_{i\epsilon} \right) dS \\
&= \boldsymbol{\kappa}_{i2} \cdot \int_{\partial B^n(x_{i\epsilon},r)} \left[\frac{x - x_{i\epsilon}}{|x - x_{i\epsilon}|^n} \frac{\partial g_{i\epsilon}}{\partial \nu} - \frac{\partial}{\partial \nu} \left(\frac{x - x_{i\epsilon}}{|x - x_{i\epsilon}|^n} \right) g_{i\epsilon} \right] dS
\end{aligned} \tag{8.12}$$

is independent of $r > 0$. Thus, taking the limit $r \to 0$ and applying the Taylor expansion of $g_{i\epsilon}$, we find that it is equal to

$$\begin{aligned}
I_{10} &:= \lim_{r\to 0} I_{1r} \\
&= \lim_{r\to 0} \sum_{k,l=1}^{n} \frac{\kappa_{i2k}}{r^{n+1}} \int_{\partial B^n(0,r)} x_k x_l \left[(\partial_l g_{i\epsilon})(x_{i\epsilon}) + O(|x|) \right] dS \\
&\quad + (n-1) \lim_{r\to 0} \sum_{k=1}^{n} \frac{\kappa_{i2k}}{r^{n+1}} \int_{\partial B^n(0,r)} x_k \left[g_{i\epsilon}(x_{i\epsilon}) + \sum_{l=1}^{n} x_l (\partial_l g_{i\epsilon})(x_{i\epsilon}) + O\left(|x|^2\right) \right] dS \\
&= n \sum_{k,l=1}^{n} \kappa_{i2k} (\partial_l g_{i\epsilon})(x_{i\epsilon}) \int_{\partial B^n(0,1)} x_k x_l dS = \left|S^{n-1}\right| \boldsymbol{\kappa}_{i2} \cdot \nabla g_{i\epsilon}(x_{i\epsilon}).
\end{aligned} \tag{8.13}$$

However the quantity $\boldsymbol{\kappa}_{i2} \cdot \nabla g_{i\epsilon}(x_{i\epsilon})$ is negligible in the sense that its order is $\epsilon^{\frac{n-1}{n-2}+\frac{1}{2}}$, because $\boldsymbol{\kappa}_{i2} = O\left(\epsilon^{\frac{n-1}{n-2}}\right)$ and that $\nabla_x \Upsilon_m(\lambda_1, \cdots, \lambda_m, x_1, \cdots, x_m) = 0$ means

$$\begin{aligned}
\lim_{\epsilon\to 0} \epsilon^{-\frac{1}{2}} \nabla g_{i\epsilon}(x_{i\epsilon}) &= -\lim_{\epsilon\to 0} \left(\epsilon^{-\frac{1}{2}} \kappa_{i0}\right) (\nabla_x H)(x_{i\epsilon}, x_{i\epsilon}) + \sum_{j\neq i} \lim_{\epsilon\to 0} \left(\epsilon^{-\frac{1}{2}} \kappa_{j0}\right) (\nabla_x G)(x_{i\epsilon}, x_{j\epsilon}) \\
&= \left(-\frac{1}{2} \lambda_i^{\frac{n-2}{2}} (\nabla_x \tau)(x_i) + \sum_{j\neq i} \lambda_j^{\frac{n-2}{2}} (\nabla_x G)(x_i, x_j) \right) C_2 = 0
\end{aligned} \tag{8.14}$$

where $C_2 = \int_{\mathbb{R}^n} U_{1,0}^p$ as before. Hence we can conclude that

$$I_{10} = o\left(\epsilon^{\frac{n-1}{n-2}+\frac{1}{2}}\right). \tag{8.15}$$

Regarding the leftmost side of (8.10), one gets in a similar fashion to the derivation of (8.11) that

$$\begin{aligned}
&\int_{\partial B^n(x_{i\epsilon},r)} \frac{\partial u_\epsilon}{\partial \nu}\frac{\partial v_{\ell\epsilon}}{\partial \nu} dS \\
&= \mu_{\ell\epsilon}(p-\epsilon)\left[\frac{(n-2)^2\gamma_n^2 \left|S^{n-1}\right| \kappa_{i0}\kappa_{i1}}{r^{n-1}} - \frac{(n-2)(n-1)\gamma_n}{r^{n+1}}\boldsymbol{\kappa}_{i2}\cdot\int_{\partial B^n(x_{i\epsilon},r)} (x-x_{i\epsilon})\frac{\partial g_{i\epsilon}}{\partial \nu} dS \right.\\
&\qquad \left. + \int_{\partial B^n(x_{i\epsilon},r)} \frac{\partial g_{i\epsilon}}{\partial \nu}\frac{\partial h_{i\epsilon}}{\partial \nu} dS + o\left(\epsilon^{\frac{n-1}{n-2}+\frac{1}{2}}\right)\right].
\end{aligned} \tag{8.16}$$

Furthermore, we have

$$\begin{aligned}
&\int_{\partial B^n(x_{i\epsilon},r)} \nabla u_\epsilon \cdot \nabla v_{\ell\epsilon} dS \\
&= \mu_{\ell\epsilon}(p-\epsilon)\left[\frac{(n-2)^2\gamma_n^2 \left|S^{n-1}\right| \kappa_{i0}\kappa_{i1}}{r^{n-1}} - \frac{n(n-2)\gamma_n}{r^{n+1}}\boldsymbol{\kappa}_{i2}\cdot\int_{\partial B^n(x_{i\epsilon},r)} (x-x_{i\epsilon})\frac{\partial g_{i\epsilon}}{\partial \nu} dS \right.\\
&\qquad \left. + \frac{(n-2)\gamma_n}{r^n}\boldsymbol{\kappa}_{i2}\cdot\int_{\partial B^n(x_{i\epsilon},r)} \nabla g_{i\epsilon} dS + \int_{\partial B^n(x_{i\epsilon},r)} \nabla g_{i\epsilon}\cdot\nabla h_{i\epsilon} dS + o\left(\epsilon^{\frac{n-1}{n-2}+\frac{1}{2}}\right)\right].
\end{aligned} \tag{8.17}$$

and

$$\begin{aligned}
&\int_{\partial B^n(x_{i\epsilon},r)} \frac{\partial u_\epsilon}{\partial \nu} v_{\ell\epsilon} dS \\
&= \mu_{\ell\epsilon}(p-\epsilon)\left[-\frac{(n-2)\gamma_n^2 \left|S^{n-1}\right| \kappa_{i0}\kappa_{i1}}{r^{n-2}} - (n-2)\gamma_n \left|S^{n-1}\right| \kappa_{i0} h_{i\epsilon}(x_{i\epsilon}) \right.\\
&\qquad \left. + \frac{(n-2)\gamma_n}{r^n}\boldsymbol{\kappa}_{i2}\cdot\int_{\partial B^n(x_{i\epsilon},r)} (x-x_{i\epsilon})\frac{\partial g_{i\epsilon}}{\partial \nu} dS + \int_{\partial B^n(x_{i\epsilon},r)} \frac{\partial g_{i\epsilon}}{\partial \nu} h_{i\epsilon} dS + o\left(\epsilon^{\frac{n-1}{n-2}+\frac{1}{2}}\right)\right].
\end{aligned} \tag{8.18}$$

Therefore putting (8.11) and (8.15)-(8.18) into (8.10) gives that

$$\begin{aligned}
&(\mu_{\ell\epsilon}(p-\epsilon)-1)\left[2r\int_{\partial B^n(x_{i\epsilon},r)} \frac{\partial g_{i\epsilon}}{\partial \nu}\frac{\partial h_{i\epsilon}}{\partial \nu} dS - \frac{(n-2)\gamma_n}{r^{n-1}}\boldsymbol{\kappa}_{i2}\cdot\int_{\partial B^n(x_{i\epsilon},r)} \nabla g_{i\epsilon} dS \right.\\
&\left. -r\int_{\partial B^n(x_{i\epsilon},r)} \nabla g_{i\epsilon}\cdot\nabla h_{i\epsilon} dS - (n-2)^2\gamma_n \left|S^{n-1}\right| \kappa_{i0} h_{i\epsilon}(x_{i\epsilon}) + (n-2)\int_{\partial B^n(x_{i\epsilon},r)} \frac{\partial g_{i\epsilon}}{\partial \nu} h_{i\epsilon} dS\right] \\
&= 2\left[(n-2)\gamma_n \left|S^{n-1}\right| (\kappa_{i1} g_{i\epsilon}(x_{i\epsilon}) - \kappa_{i0} h_{i\epsilon}(x_{i\epsilon})) + o\left(\epsilon^{\frac{n-1}{n-2}+\frac{1}{2}}\right)\right].
\end{aligned} \tag{8.19}$$

Noticing that each component of $\nabla g_{i\epsilon}$ is harmonic, we obtain

$$\frac{1}{r^{n-1}}\boldsymbol{\kappa}_{i2}\cdot\int_{\partial B^n(x_{i\epsilon},r)} \nabla g_{i\epsilon} dS = \left|S^{n-1}\right| \boldsymbol{\kappa}_{i2}\cdot\nabla g_{i\epsilon}(x_{i\epsilon}) = o\left(\epsilon^{\frac{n-1}{n-2}+\frac{1}{2}}\right),$$

where the second equality was deduced in (8.14). Also, by setting $f = g_{i\epsilon}$, $g = h_{i\epsilon}$ and $\xi = x_{i\epsilon}$ in the bilinear Pohožaev identity (8.8), one can verify that

$$r\left(\int_{\partial B^n(x_{i\epsilon},r)} 2\frac{\partial g_{i\epsilon}}{\partial \nu}\frac{\partial h_{i\epsilon}}{\partial \nu} - \nabla g_{i\epsilon}\cdot\nabla h_{i\epsilon}\right) dS + (n-2)\int_{\partial B^n(x_{i\epsilon},r)} \frac{\partial g_{i\epsilon}}{\partial \nu} h_{i\epsilon} dS = 0.$$

Subsequently, (8.19) is reduced to

$$2\kappa_{i1}\left(\epsilon^{-\frac{1}{2}}g_{i\epsilon}(x_{i\epsilon})\right) = [2-(\mu_{\ell\epsilon}(p-\epsilon)-1)(n-2)]\left(\epsilon^{-\frac{1}{2}}\kappa_{i0}\right)h_{i\epsilon}(x_{i\epsilon}) + o\left(\epsilon^{\frac{n-1}{n-2}}\right).$$

Now we employ $\nabla_\lambda \Upsilon_m(\lambda_1,\cdots,\lambda_m,x_1,\cdots,x_m)=0$ to see that

$$\epsilon^{-\frac{1}{2}}g_{i\epsilon}(x_{i\epsilon}) = C_2\left[-\tau(x_i)\lambda_i^{\frac{n-2}{2}} + \sum_{j\neq i}G(x_i,x_j)\lambda_j^{\frac{n-2}{2}}\right] + o(1) = -\frac{C_2c_2}{c_1(n-2)\lambda_i^{\frac{n-2}{2}}} + o(1)$$

and that $\epsilon^{-\frac{1}{2}}\kappa_{i0} = \lambda_i^{\frac{n-2}{2}}C_2+o(1)$, where $C_2>0$ is the constant that appeared in (8.14) and $c_1, c_2>0$ are the numbers in (8.9). Consequently, we have

$$\begin{aligned}
&\left(C_0\lambda_i^{-(n-2)}+o(1)\right)\kappa_{i1} = h_{i\epsilon}(x_{i\epsilon}) + o\left(\epsilon^{\frac{n-1}{n-2}}\right)\\
&= -\left[\kappa_{i1}\tau(x_{i\epsilon}) + \frac{1}{2}\boldsymbol{\kappa}_{i2}\cdot\nabla\tau(x_{i\epsilon})\right] + \sum_{j\neq i}\left(\kappa_{j1}G(x_{i\epsilon},x_{j\epsilon}) + \boldsymbol{\kappa}_{j2}\cdot\nabla_yG(x_{i\epsilon},x_{j\epsilon})\right) + o\left(\epsilon^{\frac{n-1}{n-2}}\right),
\end{aligned}$$

which can be rewritten as

$$(\mathcal{M}_1+o(1))\begin{pmatrix}\kappa_{11}\\ \vdots\\ \kappa_{m1}\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\boldsymbol{\kappa}_{12}\cdot\nabla\tau(x_1) + \sum_{j\neq 1}\boldsymbol{\kappa}_{j2}\cdot\nabla_yG(x_1,x_j)\\ \vdots\\ -\frac{1}{2}\boldsymbol{\kappa}_{m2}\cdot\nabla\tau(x_m) + \sum_{j\neq m}\boldsymbol{\kappa}_{j2}\cdot\nabla_yG(x_m,x_j)\end{pmatrix} + o\left(\epsilon^{\frac{n-1}{n-2}}\right).$$

This is nothing but (8.9). □

Proof of Proposition 8.5.1. According to (8.4) and Proposition 8.4.4, we have

$$\begin{aligned}
\epsilon^{-\frac{n-1}{n-2}}\kappa_{i2k} &= \epsilon^{-\frac{n-1}{n-2}}\int_{B^n(x_{i\epsilon},r)}(y-x_{i\epsilon})_k\left(u_\epsilon^{p-1-\epsilon}v_{\ell\epsilon}\right)(y)dy = \lambda_i^{n-1}d_{\ell,i,k}\left(-\int_{\mathbb{R}^n}x_1U_{1,0}^{p-1}\frac{\partial U_{1,0}}{\partial x_1}\right) + o(1)\\
&= \lambda_i^{n-1}d_{\ell,i,k}p^{-1}C_1 + o(1)
\end{aligned}$$

for any $i\in\{1,\cdots,m\}$ and $k\in\{1,\cdots,n\}$. Hence the proposition follows from (8.3), Corollary 8.4.2 (or Corollary 8.4.6) and Lemma 8.5.5. □

8.6 Characterization of the ℓ-th eigenvalues, $m+1\leq\ell\leq(n+1)m$

Our goal in this section is to perform the proof of Theorem 8.1.3. For the convenience, we restate it in the following proposition.

Proposition 8.6.1. *Let $\mathcal{A}_2$ be the matrix which was introduced in the statement of Theorem 8.1.3 and ρ_ℓ^2 the $(\ell - m)$-th eigenvalue of $\mathcal{A}_2$. For $m + 1 \le \ell \le (n + 1)m$, the ℓ-th eigenvalue $\mu_{\ell\epsilon}$ for linear problem* (8.5) *satisfies that*

$$\mu_{\ell\epsilon} = 1 - c_0\rho_\ell^2\epsilon^{\frac{n}{n-2}} + o\left(\epsilon^{\frac{n}{n-2}}\right) \quad \textit{where} \quad c_0 = (C_1C_2)/(pC_5) > 0. \tag{8.1}$$

In addition, the nonzero vector $\mathbf{d}_\ell \in \mathbb{R}^{mn}$ defined via (8.16) *is an eigenfunction of $\mathcal{A}_2$ corresponding to ρ_ℓ^2 and satisfies $\mathbf{d}_{\ell_1}^T \cdot \mathbf{d}_{\ell_2}^T = 0$ if $m + 1 \le \ell_1 \ne \ell_2 \le (n + 1)m$.*

The next lemma contains a key computation for the proof of Proposition 8.6.1.

Lemma 8.6.2. *Define*

$$\mathcal{J}^r_{jl;ik} = \int_{\partial B^n(x_i,r)} \left[\frac{\partial}{\partial\nu_x}\left(\frac{\partial G}{\partial x_k}(x, x_j)\right)G(x, x_l) - \frac{\partial G}{\partial x_k}(x, x_j)\frac{\partial G}{\partial\nu_x}(x, x_l)\right] \tag{8.2}$$

and

$$\mathcal{K}^r_{jl;ikq} = \int_{\partial B^n(x_i,r)} \left[\frac{\partial}{\partial\nu_x}\left(\frac{\partial G}{\partial x_k}(x, x_j)\right)\frac{\partial G}{\partial y_q}(x, x_l) - \frac{\partial G}{\partial x_k}(x, x_j)\frac{\partial}{\partial\nu_x}\left(\frac{\partial G}{\partial y_q}(x, x_l)\right)\right] \tag{8.3}$$

for each $i,\ j,\ l \in \{1, \cdots, m\}$ and $k,\ q \in \{1, \cdots, n\}$, where the outward unit normal derivative $\frac{\partial}{\partial\nu_x}$ acts over the x-variable of Green's function $G = G(x, y)$. Then they are the value independent of $r > 0$ and calculated as

$$\mathcal{J}^r_{jl;ik} = \begin{cases} 0 & \text{if } j \ne i \text{ and } l \ne i,\\ \dfrac{\partial G}{\partial x_k}(x_i, x_l) & \text{if } j = i \text{ and } l \ne i,\\ \dfrac{\partial G}{\partial x_k}(x_i, x_j) & \text{if } j \ne i \text{ and } l = i,\\ -\dfrac{\partial \tau}{\partial x_k}(x_i) & \text{if } j = l = i, \end{cases} \quad \textit{and} \quad \mathcal{K}^r_{jl;ikq} = \begin{cases} 0 & \text{if } j \ne i \text{ and } l \ne i,\\ \dfrac{\partial^2 G}{\partial x_k \partial y_q}(x_i, x_l) & \text{if } j = i \text{ and } l \ne i,\\ \dfrac{\partial^2 G}{\partial x_k \partial x_q}(x_i, x_j) & \text{if } j \ne i \text{ and } l = i,\\ -\dfrac{1}{2}\dfrac{\partial^2 \tau}{\partial x_k \partial x_q}(x_i) & \text{if } j = l = i. \end{cases}$$

Proof. As explained in the proof of Lemma 8.4.5, the integral $\mathcal{J}^r_{jl;ik}$ in (8.2) is independent of $r > 0$, so one may take $r \to 0$ to find its value. We compute each $\mathcal{J}^r_{jl;ik}$ by considering four mutually exclusive cases categorized according to the relation of indices j, l and i.

(1) If $j, l \ne i$, then $\mathcal{J}^r_{jl;ik}$ vanishes.

(2) Suppose that $j = i$ and $l \ne i$. Since

$$\frac{\partial}{\partial\nu_x}\left(\frac{\partial G}{\partial x_k}(x, x_i)\right) = (n-2)(n-1)\gamma_n\frac{(x - x_i)_k}{r^{n+1}} - \frac{(x - x_i)}{r} \cdot \nabla_x\left(\frac{\partial H(x, x_i)}{\partial x_k}\right)$$

on $\partial B^n(x_i, r)$ and

$$G(x, x_l) = G(x_i, x_l) + (x - x_i) \cdot \nabla_x G(x_i, x_l) + O\left(|x - x_i|^2\right)$$

near the point x_i, we discover

$$\mathcal{J}^r_{il;ik} = \int_{\partial B^n(x_i,r)} \left[\frac{\partial}{\partial \nu_x}\left(\frac{\partial G}{\partial x_k}(x,x_i)\right) G(x,x_l) - \frac{\partial G}{\partial x_k}(x,x_i)\frac{\partial G}{\partial \nu_x}(x,x_l)\right] = \frac{\partial G}{\partial x_k}(x_i,x_l).$$

(3) In the case that $j \neq i$ and $l = i$, a similar argument in (2) applies, yielding

$$\mathcal{J}^r_{ji;ik} = \frac{\partial G}{\partial x_k}(x_i,x_j).$$

(4) Assume that $j = l = i$. Then Green's identity (8.12) and Lemma 8.2.11 show that

$$\begin{aligned}\mathcal{J}^r_{ii;ik} &= \int_{\partial B^n(x_i,r)} \left[\frac{\partial}{\partial \nu_x}\left(\frac{\partial G}{\partial x_k}(x,x_i)\right) G(x,x_i) - \frac{\partial G}{\partial x_k}(x,x_i)\frac{\partial G}{\partial \nu_x}(x,x_i)\right] dS \\ &= -\int_{\partial\Omega} \frac{\partial G}{\partial x_k}(x,x_i)\frac{\partial G}{\partial \nu_x}(x,x_i) dS = -\int_{\partial\Omega}\left(\frac{\partial G}{\partial \nu_x}(x,x_i)\right)^2 \nu_k(x) dS = -\frac{\partial \tau}{\partial x_k}(x_i).\end{aligned}$$

We can deal with (8.3) in a similar manner, which we left to the reader. □

Proof of Proposition 8.6.1. We reconsider (8.23), but in this time we allow to put any $i \in \{1,\cdots,m\}$ and x_k ($k \in \{1,\cdots,n\}$) in the place of i_0 and x_1, respectively. By multiplying $\epsilon^{-\frac{1}{2}-\frac{n-1}{n-2}}$ on both sides, we obtain

$$\begin{aligned}&\int_{\partial B^n(x_{i\epsilon},r)} \left[\frac{\partial}{\partial \nu}\left\{\frac{\partial\left(\epsilon^{-\frac{1}{2}}u_\epsilon\right)}{\partial x_k}\right\}\cdot\left(\epsilon^{-\frac{n-1}{n-2}}v_{\ell\epsilon}\right) - \frac{\partial\left(\epsilon^{-\frac{1}{2}}u_\epsilon\right)}{\partial x_k}\cdot\frac{\partial\left(\epsilon^{-\frac{n-1}{n-2}}v_{\ell\epsilon}\right)}{\partial \nu}\right] dS \\ &\qquad = (p-\epsilon)\left(\frac{\mu_{\ell\epsilon}-1}{\epsilon^{\frac{n}{n-2}}}\right)\cdot\left[\epsilon^{-\frac{(n-4)}{2(n-2)}}\int_{B^n(x_{i\epsilon},r)} u_\epsilon^{p-1-\epsilon}\frac{\partial u_\epsilon}{\partial x_k}v_{\ell\epsilon}\right]. \quad (8.4)\end{aligned}$$

The right-hand side of (8.4) can be computed as in (8.24), which turns out to be

$$\left(\frac{\mu_{\ell\epsilon}-1}{\epsilon^{\frac{n}{n-2}}}\right)\left[-\lambda_i^{\frac{n-4}{2}} d_{\ell,i,k} p C_5 + o(1)\right].$$

Meanwhile, if we let $\boldsymbol{\lambda} \in \mathbb{R}^m$ be a nonzero column vector

$$\boldsymbol{\lambda} = \left(\lambda_{10}^{\frac{n-2}{2}},\cdots,\lambda_{m0}^{\frac{n-2}{2}}\right)^T,$$

then (8.11) in Lemma 8.2.6 can be written in a vectorial form as $\epsilon^{-1/2}u_\epsilon(x) \to C_2\mathcal{G}(x)\boldsymbol{\lambda}$ (see (8.1)). Hence, with the aid of Proposition 8.5.1 and Lemma 8.6.2, it is possible to take $\epsilon \to 0$ in the left-hand side of (8.4) to derive

$$\begin{aligned}&C_1C_2\boldsymbol{\lambda}^T\left[\int_{\partial B^n(x_i,r)}\left\{\left(\frac{\partial}{\partial \nu}\frac{\partial \mathcal{G}}{\partial x_k}(x)\right)^T\mathcal{G}(x) - \left(\frac{\partial \mathcal{G}}{\partial x_k}(x)\right)^T\left(\frac{\partial \mathcal{G}}{\partial \nu}(x)\right)\right\}dx\cdot\mathcal{M}_1^{-1}\mathcal{P}\right. \\ &\qquad\left. + \int_{\partial B^n(x_i,r)}\left\{\left(\frac{\partial}{\partial \nu}\frac{\partial \mathcal{G}}{\partial x_k}(x)\right)^T\widetilde{\mathcal{G}}(x) - \left(\frac{\partial \mathcal{G}}{\partial x_k}(x)\right)^T\left(\frac{\partial \widetilde{\mathcal{G}}}{\partial \nu}(x)\right)\right\}dx\right]\mathbf{d}_\ell \\ &= C_1C_2\boldsymbol{\lambda}^T\left[\mathcal{J}_{ik}\mathcal{M}_1^{-1}\mathcal{P} + \overline{\mathcal{K}}_{ik}\right]\mathbf{d}_\ell\end{aligned}$$

where $\mathcal{J}_{ik}$ is an $m \times m$ matrix having $\mathcal{J}^r_{jl;ik}$ defined in (8.2) as its components, namely, $\mathcal{J}_{ik} = \left(\mathcal{J}^r_{jl;ik}\right)_{1\le j,l\le m}$ for each fixed $i,\ k \in \{1, \cdots, m\}$, and $\overline{\mathcal{K}}_{ik} = \left(\overline{\mathcal{K}}_{jb;ik}\right)_{1\le j\le m, 1\le b\le mn}$ is an $m \times mn$ matrix whose components are

$$\overline{\mathcal{K}}_{j,(l-1)n+q;ik} = \lambda_l^{\frac{n}{2}} \mathcal{K}^r_{jl;ikq} = \begin{cases} 0 & \text{if } j \ne i \text{ and } l \ne i, \\ \lambda_l^{\frac{n}{2}} \dfrac{\partial^2 G}{\partial x_k \partial y_q}(x_i, x_l) & \text{if } j = i \text{ and } l \ne i, \\ \lambda_i^{\frac{n}{2}} \dfrac{\partial^2 G}{\partial x_k \partial x_q}(x_i, x_j) & \text{if } j \ne i \text{ and } l = i, \\ -\lambda_i^{\frac{n}{2}} \dfrac{1}{2} \dfrac{\partial^2 \tau}{\partial x_k \partial x_q}(x_i) & \text{if } j = l = i, \end{cases}$$

for $j,\ l,\ i \in \{1, \cdots, m\}$ and $q,\ k \in \{1, \cdots, n\}$. From direct computations especially using that

$$\lambda_i \left(\lambda^T \mathcal{J}_{ik}\right)_j = \begin{cases} \lambda_i^{\frac{n}{2}} \dfrac{\partial G}{\partial x_k}(x_i, x_j) & \text{if } i \ne j, \\ \lambda_i \displaystyle\sum_{l \ne i} \lambda_l^{\frac{n-2}{2}} \dfrac{\partial G}{\partial x_k}(x_i, x_j) - \lambda_i^{\frac{n}{2}} \dfrac{\partial \tau}{\partial x_k}(x_i) = -\lambda_i^{\frac{n}{2}} \dfrac{1}{2} \dfrac{\partial \tau}{\partial x_k}(x_i) & \text{if } i = j, \end{cases}$$

for $\lambda^T \mathcal{J}_{ik} = \left(\left(\lambda^T \mathcal{J}_{ik}\right)_1, \cdots, \left(\lambda^T \mathcal{J}_{ik}\right)_m\right) \in \mathbb{R}^m$, we conclude

$$\mathcal{A}_2 \mathbf{d}_\ell = \left[\mathcal{P}^T \mathcal{M}_1^{-1} \mathcal{P} + Q\right] \mathbf{d}_\ell = \left(-\frac{pC_5}{C_1 C_2}\right) \lim_{\epsilon\to 0} \left(\frac{\mu_{\ell\epsilon} - 1}{\epsilon^{\frac{n}{n-2}}}\right) \mathbf{d}_\ell = \rho_\ell^2 \mathbf{d}_\ell$$

with matrices $\mathcal{M}_1$, $\mathcal{P}$ and Q given in (8.13), (8.17) and (8.18). The claim that $\mathbf{d}^T_{\ell_1} \cdot \mathbf{d}^T_{\ell_2} = 0$ can be proved as in the proof of Theorem 8.1.1, or particularly, (8.5). The proof is done. □

8.7 Estimates for the ℓ-th eigenvalues and eigenfunctions, $(n+1)(m+1) \le \ell \le (n+2)m$

We now establish Theorem 8.1.5 by obtaining a series of lemmas. In the first lemma we will compute the limit of the ℓ-th eigenvalues as $\epsilon \to 0$ when $(n+1)(m+1) \le \ell \le (n+2)m$.

Lemma 8.7.1. *If $(n+1)(m+1) \le \ell \le (n+2)m$, we have*

$$\lim_{\epsilon\to 0} \mu_{\ell\epsilon} = 1.$$

Proof. By virtue of Corollary 8.4.2 or Corollary 8.4.6, it is enough to show that $\limsup_{\epsilon\to 0} \mu_{\ell\epsilon} \le 1$. Referring to (8.1), we let $\mathcal{V}$ be a vector space whose basis is

$$\{u_{\epsilon,i} : 1 \le i \le m\} \cup \{\psi_{\epsilon,i,k} : 1 \le i \le m,\ 1 \le k \le n+1\}.$$

If we write $f \in \mathcal{V} \setminus \{0\}$ as

$$f = \sum_{i=1}^{m} f_i \quad \text{with} \quad f_i = a_{i0} u_{\epsilon,i} + \sum_{k=1}^{n+1} a_{ik} \psi_{\epsilon,i,k}$$

for some $(a_{10}, \cdots, a_{1(n+1)}, \cdots, a_{m0}, \cdots, a_{m(n+1)}) \in \mathbb{R}^{m(n+1)} \setminus \{0\}$, then we have

$$\begin{aligned} \mu_{((n+2)m)\epsilon} &= \min_{\substack{\mathcal{W} \subset H_0^1(\Omega), \\ \dim \mathcal{W} = (n+2)m}} \max_{f \in \mathcal{W} \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2}{(p-\epsilon)\int_\Omega f^2 u_\epsilon^{p-1-\epsilon}} \le \max_{f \in \mathcal{V} \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2}{(p-\epsilon)\int_\Omega f^2 u_\epsilon^{p-1-\epsilon}} \\ &\le \max_{f \in \mathcal{V} \setminus \{0\}} \max_{1 \le i \le m} \frac{\int_\Omega |\nabla f_i|^2}{(p-\epsilon)\int_\Omega f_i^2 u_\epsilon^{p-1-\epsilon}} := \max_{f \in \mathcal{V} \setminus \{0\}} \max_{1 \le i \le m} \mathfrak{a}_i, \end{aligned}$$

so it is sufficient to check that $\mathfrak{a}_i \le 1 + o(1)$. If we denote $\mathfrak{a} = \mathfrak{a}_i$ for a fixed i and modify the definition of z_ϵ in the proof of Proposition 8.4.1 into $z_\epsilon = \sum_{k=1}^n a_k \frac{\partial u_\epsilon}{\partial x_k} + a_{n+1} w_{i\epsilon}$, then we again have $\mathfrak{a} = 1 + \mathfrak{b}/\mathfrak{c}$. (The definition of $\mathfrak{b}$, $\mathfrak{c}$ and $w_{i\epsilon}$ can be found in (8.4), (8.5) and (8.14).) Moreover computing each of the term of $\mathfrak{b}$ and $\mathfrak{c}$ as we did in the proof of Proposition 8.4.1, we find

$$\mathfrak{b} \le C\left(|\bar{a}|^2 + a_{n+1}^2\right)\epsilon \quad \text{and} \quad \mathfrak{c} \ge C\epsilon^{-\frac{2}{n-2}}|\bar{a}|^2 + Ca_{n+1}^2 \ge C\left(|\bar{a}|^2 + a_{n+1}^2\right),$$

from which one can conclude that $\mu_{((n+2)m)\epsilon} \le 1 + O(\epsilon)$. For more detailed computations, we ask for the reader to check the proof of Theorem 1.4 in [GP]. □

The following lemma is the counterpart of Proposition 8.4.4 for $(n+1)(m+1) \le \ell \le (n+2)m$.

Lemma 8.7.2. *Let $(n+1)(m+1) \le \ell \le (n+2)m$. For each $i \in \{1, \cdots, m\}$ and $d_{\ell,i,n+1} \in \mathbb{R}$, converges to*

$$\tilde{v}_{\ell i\epsilon} \rightharpoonup d_{\ell,i,n+1}\left(\frac{\partial U_{1,0}}{\partial \lambda}\right) \quad \textit{weakly in } H^1(\mathbb{R}^n).$$

Proof. Lemma 8.4.3 (1) holds in this case also by Lemma 8.7.1. Therefore it is enough to show that the vector $\mathbf{d}_\ell$ in (8.16) is zero.

As in (8.5), the orthogonality of $v_{\ell\epsilon}$ and $v_{\ell_1\epsilon}$ for $m+1 \le \ell_1 \le (n+1)m$ implies $\mathbf{d}_\ell^T \cdot \mathbf{d}_{\ell_1}^T = 0$. However, we also know from Proposition 8.6.1 that $\{\mathbf{d}_{m+1}, \cdots, \mathbf{d}_{(n+1)m}\}$ serves a basis for $\mathbb{R}^{mn}$. Hence $\mathbf{d}_\ell = 0$, concluding the proof. □

As a consequence, we reach at

Proposition 8.7.3. *Let $\mathcal{A}_3$ be the matrix* (8.20). *For $(n+1)(m+1) \le \ell \le (n+2)m$, if ρ_ℓ^3 is the $(\ell - (m+1)n)$-th eigenvalue of $\mathcal{A}_3$, then it is positive and the ℓ-th eigenvalue $\mu_{\ell\epsilon}$ to problem* (8.5) *is estimated as*

$$\mu_{\ell\epsilon} = 1 + c_1 \rho_\ell^3 \epsilon + o(\epsilon) \quad \textit{where } c_1 = \frac{(n-2)^2 C_2 C_3}{2(n+2)C_4}. \tag{8.1}$$

Furthermore, the nonzero vector $\hat{\mathbf{d}}_\ell$ in (8.21) *is a corresponding eigenvector to ρ_ℓ^3 and $\hat{\mathbf{d}}_{\ell_1}^T \cdot \hat{\mathbf{d}}_{\ell_2}^T = 0$ if $(n+1)(m+1) \le \ell_1 \ne \ell_2 \le (n+2)m$.*

Proof. Denote $d_{\ell,i} = d_{\ell,i,n+1}$ in the previous lemma. Then we can recover (8.8) from Lemma 8.7.1. Hence the arguments in the proof of Proposition 8.4.4 works, giving (8.20) and (8.19) to us again. From them, we conclude that ρ_ℓ^3 is positive, $\hat{\mathbf{d}}_\ell$ is an eigenvector corresponding to ρ_ℓ^3 and (8.1) is valid. The last orthogonality assertion is deduced in the same way as one in Theorem 8.1.1. See (8.5). □

Appendix

8.A A moving sphere argument

In this appendix, we show the following proposition by employing the moving sphere argument given in [LZh] (refer also to [ChL]). Note that it implies Proposition 8.2.3 at once.

Proposition 8.A.1. *Let $r_0 > 0$ be fixed and $p = (n+2)/(n-2)$ as above. Suppose that a family $\{u_\epsilon\}_\epsilon$ of positive C^2-functions which satisfy*

$$-\Delta u_\epsilon = u_\epsilon^{p-\epsilon} \quad \textit{in } B^n\left(0, \epsilon^{-\alpha_0} r_0\right), \quad \|u_\epsilon\|_{L^\infty\left(B^n\left(0,\epsilon^{-\alpha_0} r_0\right)\right)} \le c$$

for some $c > 0$, and

$$\lim_{\epsilon\to 0} u_\epsilon(x) = U_{1,0}(x) \quad \textit{weakly in } H^1(\mathbb{R}^n). \tag{8.2}$$

Then there are constants $C > 0$ and $0 < \delta_0 < r_0$ independent of $\epsilon > 0$ such that

$$u_\epsilon(x) \le C U_{1,0}(x) \quad \textit{for all } x \in B^n\left(0, \epsilon^{-\alpha_0}\delta_0\right).$$

Before conducting its proof, we introduce Green's function G_R of $-\Delta$ in $B^n(0,R)$ for each $R > 0$ with zero Dirichlet boundary condition. By the scaling invariance, we have

$$G_R(x,y) = G_1\left(\frac{x}{R}, \frac{y}{R}\right)\frac{1}{R^{n-2}} \quad \text{for } x, y \in B^n(0,R).$$

Thus we can decompose Green's function in $B^n(0,R)$ into its singular part and regular part as follows:

$$G_R(x,y) = \frac{\gamma_n}{|x-y|^{n-2}} - \frac{1}{R^{n-2}} H_1\left(\frac{x}{R}, \frac{y}{R}\right) \quad \text{for } x, y \in B^n(0,R). \tag{8.3}$$

See (8.2) for the definition of the normalizing constant γ_n.

Now we begin to prove Proposition 8.A.1. By (8.2) and elliptic regularity, for arbitrarily given $\zeta_1 > 0$ and any compact set $K \subset \mathbb{R}^n$, there is $\epsilon_1 > 0$ such that it holds

$$\|u_\epsilon - U_{1,0}\|_{C^2(K)} \le \zeta_1 \quad \text{for } \epsilon \in (0, \epsilon_1). \tag{8.4}$$

Let us define the Kelvin transform of u_ϵ:

$$u_\epsilon^\lambda(x) = \left(\frac{\lambda}{|x|}\right)^{n-2} u_\epsilon\left(x^\lambda\right), \quad x^\lambda = \frac{\lambda^2 x}{|x|^2} \quad \text{for } |x^\lambda| < \epsilon^{-\alpha_0} r_0 \tag{8.5}$$

and the difference $w_\epsilon^\lambda = u_\epsilon - u_\epsilon^\lambda$ between u_ϵ and it. Then we have

$$-\Delta w_\epsilon^\lambda = u_\epsilon^{p-\epsilon} - \left(\frac{\lambda}{|x|}\right)^{(n-2)\epsilon} \left(u_\epsilon^\lambda\right)^{p-\epsilon} \geq u_\epsilon^{p-\epsilon} - \left(u_\epsilon^\lambda\right)^{p-\epsilon} = \xi_\epsilon(x) w_\epsilon^\lambda \quad \text{for } |x| \geq \lambda \tag{8.6}$$

where

$$\xi_\epsilon(x) = \begin{cases} \dfrac{u_\epsilon^{p-\epsilon} - \left(u_\epsilon^\lambda\right)^{p-\epsilon}}{u_\epsilon - u_\epsilon^\lambda}(x) & \text{if } u_\epsilon(x) \neq u_\epsilon^\lambda(x), \\ (p-\epsilon) u_\epsilon^{p-1-\epsilon}(x) & \text{if } u_\epsilon(x) = u_\epsilon^\lambda(x). \end{cases}$$

Lemma 8.A.2. *For any $\zeta_2 > 0$, there exist small constants $\delta_1 > 0$ and $\epsilon_2 > 0$ such that*

$$\min_{|y|=r} u_\epsilon(y) \leq (1+\zeta_2) U_{1,0}(r) \quad \textit{for } 0 < r := |x| \leq \epsilon^{-\alpha_0}\delta_1 \textit{ and any } \epsilon \in (0, \epsilon_2). \tag{8.7}$$

Proof. We first choose a candidate $\delta_1 \in (0, r_0)$ for which (8.7) will have the validity. Fix a sufficiently small value $\eta_1 > 0$ and a number $R_0 > 0$ such that it holds

$$u_\epsilon^\lambda(x) \leq \left(1 + \frac{\zeta_2}{4}\right) \beta_n |x|^{2-n} \quad \text{for any } 0 < \lambda \leq 1 + \eta_1 \text{ and } |x| \geq R_0 \tag{8.8}$$

provided $\epsilon > 0$ small enough, where $\beta_n = (n(n-2))^{p-1}$ is the constant appeared in (**??**). Take $\lambda_1 = 1 - \eta_1$ and $\lambda_2 = 1 + \eta_1$. If $\lambda = \lambda_1$, because $U_{1,0}^\lambda = U_{\lambda^2,0}$ for any $\lambda > 0$ and $u_\epsilon \to U_{1,0}$ in C^1-uniformly over compact subsets of $\mathbb{R}^n$ as $\epsilon \to 0$, by enlarging $R_0 > 0$ if necessary, we can find a number $\eta_2 > 0$ small such that

$$w_\epsilon^{\lambda_1}(x) > 0 \quad \text{for } \lambda_1 < |x| \leq R_0, \quad u_\epsilon^{\lambda_1}(x) \leq (1 - 2\eta_2)\beta_n |x|^{2-n} \quad \text{for } |x| \geq R_0 \tag{8.9}$$

and

$$\int_{B^n(0,R_0)} u_\epsilon^{p-\epsilon}(x)dx \geq \left(1 - \frac{\eta_2}{2}\right) \int_{\mathbb{R}^n} U_{1,0}^p(x)dx \tag{8.10}$$

for sufficiently small $\epsilon > 0$. On the other hand, provided $\delta_1 > 0$ small enough, the inequality

$$u_\epsilon(x) \geq (1-\eta_2)\beta_n |x|^{2-n} \quad \text{for } R_0 \leq |x| \leq \epsilon^{-\alpha_0}\delta_1 \tag{8.11}$$

can be reasoned in the following way. If we choose a function $\hat{u}_\epsilon$ which solves

$$-\Delta \hat{u}_\epsilon = u_\epsilon^{p-\epsilon} \quad \text{in } B^n\left(0, \epsilon^{-\alpha_0}\right) \quad \text{and} \quad \hat{u}_\epsilon = 0 \quad \text{on } \left\{|x| = \epsilon^{-\alpha_0}\right\},$$

then the comparison principle tells us that $u_\epsilon \geq \hat{u}_\epsilon$. Since Green's function is always positive, we can make

$$H_1\left(\epsilon^{-\alpha_0}x, \epsilon^{-\alpha_0}y\right) \leq \frac{\eta_2\gamma_n}{4} \cdot \frac{\epsilon^{-\alpha_0(n-2)}}{|x-y|^{n-2}} \quad \text{for } x, y \in B^n\left(0, \epsilon^{-\alpha_0}\delta_1\right)$$

by taking δ_1 small, and the relation $|x-y| \le (1-1/l)|x|$ holds for $|x| \ge lR_0$ and $|y| \le R_0$ given any $l \in (1,\infty)$, we see from (8.3) and (8.10) that

$$\begin{aligned}\hat{u}_\epsilon(x) &= \int_{B^n\left(0,\epsilon^{-\alpha_0}\right)} u_\epsilon^{p-\epsilon}(y) G_{\epsilon^{-\alpha_0}}(x,y)dy \ge \left(1-\frac{\eta_2}{4}\right)\int_{B^n\left(0,\epsilon^{-\alpha_0}\delta_1\right)} u_\epsilon^{p-\epsilon}(y)\frac{\gamma_n}{|x-y|^{n-2}}dy \\ &\ge \left(1-\frac{\eta_2}{2}\right)\left(\int_{B^n(0,R_0)} u_\epsilon^{p-\epsilon}(y)dy\right)\frac{\gamma_n}{|x|^{n-2}} \ge (1-\eta_2)\left(\int_{\mathbb{R}^n} U_{1,0}^p(y)dy\right)\frac{\gamma_n}{|x|^{n-2}} \\ &= (1-\eta_2)\frac{\beta_n}{|x|^{n-2}} \qquad \text{for } lR_0 \le |x| \le \epsilon^{-\alpha_0}\delta_1\end{aligned}$$

by choosing l large enough. Also if $|x| \le lR_0$, the uniform convergence of u_ϵ to $U_{1,0}$ implies $u_\epsilon(x) \ge (1-\eta_2)\beta_n|x|^{2-n}$ for $\epsilon > 0$ sufficiently small. This shows the validity of (8.11).

Fixing $\delta_1 > 0$ for which (8.11) is valid, suppose that (8.7) does not hold on the contrary. Then there are sequences $\{\epsilon_k\}_{k=1}^\infty$ and $\{r_k\}_{k=1}^\infty$ such that $\epsilon_k \to 0$, $r_k \in (0, \epsilon^{-\alpha_0}\delta_1)$ and

$$\min_{|x|=r_k} u_{\epsilon_k}(x) > (1+\zeta_2)U_{1,0}(r_k).$$

Set $u_k = u_{\epsilon_k}$ for brevity. Since $u_k \to U_{1,0}$ uniformly on any compact set, it should hold that $r_k \to \infty$. Therefore

$$\min_{|x|=r_k} u_k(x) \ge \left(1+\frac{\zeta_2}{2}\right)\beta_n r_k^{2-n}. \tag{8.12}$$

To deduce a contradiction, let us apply the moving sphere method to $w_k^\lambda = u_k - u_k^\lambda$ for the parameters $\lambda_1 \le \lambda \le \lambda_2$. Define $\bar\lambda_k$ by

$$\bar\lambda_k = \sup\left\{\lambda \in [\lambda_1,\lambda_2] : w_k^\mu \ge 0 \text{ in } \Sigma_\mu \text{ for all } \lambda_1 \le \mu \le \lambda\right\} \text{ where } \Sigma_\mu = \{x \in \mathbb{R}^n : \mu < |x| < r_k\}.$$

We claim that $\bar\lambda_k = \lambda_2$ for sufficiently large $k \in \mathbb{N}$. First of all, putting together with (8.9) and (8.11), we discover that $w_k^{\lambda_1} > 0$ in Σ_{λ_1}, so $\bar\lambda_k \ge \lambda_1$. Recall from (8.6) that

$$-\Delta w_k^{\bar\lambda_k} + (\xi_{\epsilon_k})_- w_k^{\bar\lambda_k} \ge (\xi_{\epsilon_k})_+ w_k^{\bar\lambda_k} \ge 0 \quad \text{in } \Sigma_{\bar\lambda_k}.$$

Moreover, from (8.12) and (8.8) we have $w_k^{\bar\lambda_k} > 0$ on $\partial B^n(0,r_k)$. Thus by the maximum principle and Hopf's lemma we have

$$w_k^{\bar\lambda_k} > 0 \quad \text{in } \Sigma_{\bar\lambda_k} \quad \text{and} \quad \frac{\partial w_k^{\bar\lambda_k}}{\partial \nu} < 0 \quad \text{on } \partial B^n\left(0,\bar\lambda_k\right)$$

where ν is the unit outward normal vector. However this means that if $\bar\lambda_k < \lambda_2$, then $w_k^\mu \ge 0$ in Σ_μ even after taking a slightly larger value of μ than $\bar\lambda_k$, which contradicts the maximality of $\bar\lambda_k$. Hence our claim is justified. Consequently, taking a limit $k \to \infty$ to $w_k^{\lambda_2} \ge 0$ in Σ_{λ_2} allows one to get

$$U_{1,0}(x) \ge U_{1,0}^{\lambda_2}(x) \quad \text{in } |x| \ge \lambda_2,$$

but it cannot be possible since $\lambda_2 > 1$. Thus (8.7) should be true. □

The following lemma completes our proof of Proposition 8.A.1.

Lemma 8.A.3. *For some constant $C > 0$ and parameter $\delta_0 \in (0, \delta_1)$, we have*

$$u_\epsilon(x) \le CU_{1,0}(x) \quad \textit{for } |x| \le \epsilon^{-\alpha_0}\delta_0$$

provided that $\epsilon > 0$ is sufficiently small. Here $\delta_1 > 0$ is the number chosen in the proof of the previous Lemma.

Proof. Argue as in the proof of Lemma 2.4 in [LZh] employing Lemma 8.A.2 above. In that paper, the statement of the lemma as well as its proof are written for a sequence $\{u_{\epsilon_k}\}_{k=1}^\infty$ of solutions, but they apply to a family $\{u_\epsilon\}_\epsilon$ as well. To proceed our proof, we substitute G_k, R_k and v_k in [LZh] with Dirichlet Green's function $G_{\epsilon^{-\alpha_0}\delta_1}$ of $-\Delta$ in $B^n(0, \epsilon^{-\alpha_0}\delta_1)$, $R_\epsilon = \epsilon^{-\alpha_0}\delta_1\delta_2$ and u_ϵ where $\delta_2 \in (0, 1)$ is a sufficiently small number. □

Part III

Pseudodifferential Calculus on Carnot Manifolds

Chapter 9

Privileged Coordinates and Tangent Groupoid for Carnot Manifolds

9.1 Introduction

In this paper, we construct some natural tangent groupoids for equi-regular Carnot-Caratheodory space. For the convenience of exposition, we consider some types of manifold M whose tangent bundle equiped with a series of subbundles $0 = H_0 \subset H_1 \subset \cdots \subset H_m = TM$. Additionally we assume the dimension of each subbundle is constant through the manifold and $[H_i, H_j] \subset H_{i+j}$ for $i + j \leq m$. Let us call it equi-regular flagged manifolds. This setting is suitable for studying Carnot-Caratheordoy space where some k vector fields $X_1, \cdots X_k$ generates a basis of tangent space at each point through the Lie bracket actions $[X_{i_1}[X_{i_2} \cdots [X_{i_{r-1}}, X_{i_r}]]] \cdots]$, with an additional assumption of the equi-regular cases which means that for each $s \in \mathbb{N}$, $\dim\{[X_{i_1}[X_{i_2} \cdots [X_{i_{r-1}}, X_{i_r}]] \cdots](p) | r \leq s]\}$ is constant for $p \in \mathbb{R}^n$.

In Section 2, we review the defintiion and examples of Carnot manifolds. In Section 3, we study the tangent group bundle of carnot manifolds. Section 4 is devoted to study privileged coordinates for Carnot manifolds. In Section 4, we will see how privileged coordinates enables us to approximate at each point vector fields by vector fields that generate a nilpotent Lie algebra. In Section 6, we define the notion of Carnot coordinates, which is a intrinsic notion of the privileged coordinates. The asymptotic formular in composition of the privileged coordinates will be also given. Using that result, we will construct a tangent groupoid in Seciton 7.

9.2 Carnot Manifolds: Definitions and Main Examples

In what follows, given a manifold M and subbundles H_1 and H_2 of TM, we denote by $[H_1, H_2]$ the distribution given by

$$[H_1, H_2] := \bigsqcup_{a \in M} \left\{ [X_1, X_2](a);\ X_j \text{ section of } H_j \text{ near } a \right\}.$$

Definition 9.2.1. A Carnot manifold is a pair (M, H), where M is a manifold and $H = (H_0, \ldots, H_r)$ is a filtration of TM by subbundles $H_0 = \{0\} \subset H_1 \subset \cdots \subset H_{r-1} \subset H_r = TM$ such that

$$[H_w, H_{w'}] \subset H_{w+w'} \qquad \text{when } w + w' \leq r.$$

Let (M^n, H) be an n-dimensional Carnot manifold. For $j = 1, \ldots, n$ we set

$$w_j = \min\{w \in \{1, \ldots, r\};\, j \leq \operatorname{rk} H_w\}.$$

Definition 9.2.2. An H-frame near a point $a \in M$ is a local tangent frame $(X_1, \ldots, X_n)$ near a such that, for all $w = 1, \ldots, r$, the vector fields X_j, $w_j = w$, are sections of H_w.

Remark 9.2.3. If $(X_1, \ldots, X_n)$ is an H-frame near a, then, for all $w = 1, \ldots, r$, the vector fields X_j, $w_j \leq w$, form a local frame of H_w near a.

9.3 The Tangent Group Bundle of a Carnot Manifold

In this section, we present an intrinsic construction of the tangent group bundle of a Carnot manifold. In what follows, we let (M^n, H) be an n-dimensional Carnot manifold.

9.3.1 The tangent Lie algebra bundle $\mathfrak{g}M$

The filtration $H = (H_0, \ldots, H_r)$ has a natural grading defined as follows. For $w = 1, \ldots, r$ set $\mathfrak{g}^w M = H_w / H_{w-1}$, and define

$$\mathfrak{g}M := \mathfrak{g}^1 M \oplus \cdots \oplus \mathfrak{g}^r M. \tag{9.1}$$

Given $a \in M$ and $X \in H_{w,a}$, we shall denote by $\dot{X}$ its class in $\mathfrak{g}_a^w M$. In particular, if $(X_1, \ldots, X_n)$ is a local H-frame near a, then the classes $\dot{X}_j(a)$, $w_j = w$, form a basis of $\mathfrak{g}_a^w M$.

In what follows we let w and w' be weights in $\{1, \ldots, r\}$ such that $w + w' \leq r$.

Lemma 9.3.1. *Given $a \in M$ let X (resp., Y) be a local section of H_w (resp., $H_{w'}$) near a (which we regard as a vector field). Then the class of $[X, Y](a)$ in $\mathfrak{g}_a^{w+w'} M$ depends only on the respective classes of $X(a)$ and $Y(a)$ in $\mathfrak{g}_a^w M$ and $\mathfrak{g}_a^{w'} M$.*

Proof. Let $(X_1, \ldots, X_n)$ be an H-frame near a. Then $\{X_j; w_j \leq w\}$ and $\{X_j; w_j \leq w\}$ are local frames near a of H_w and $H_{w'}$, respectively. Therefore, near $x = a$ we may write

$$X = \sum_{w_j \leq w} b_j(x) X_j \qquad \text{and} \qquad Y = \sum_{w_k \leq w'} c_k(x) X_k,$$

where the $b_j(x)$ and $c_k(x)$ are smooth functions. Set $X_{[w]} = \sum_{w_j = w} b_j(x) X_j$ and $Y_{[w']} = \sum_{w_k = w'} c_k(x) X_k$. Then

$$X = X_{[w]} + X' \qquad \text{and} \qquad Y = Y_{[w']} + Y',$$

where X' and Y' are sections of H_{w-1} and $H_{w'-1}$, respectively. In particular, the respective classes of $X(a)$ and $Y(a)$ in $\mathfrak{g}^a_w M$ and $\mathfrak{g}^a_{w'} M$ depend only on the coordinate vectors $(b_j(a))_{w_j = w}$ and $(c_k(a))_{w_k = w'}$.

In addition, we have

$$[X, Y] = [X_{[w]}, Y_{[w']}] + [X_{[w]}, Y'] + [X', Y].$$

As $[X_{[w]}, Y']$ and $[X', Y]$ are sections of $H_{w+w'-1}$ we see that

$$[X, Y](a) = [X_{[w]}, Y_{[w']}](a) \mod H_{w+w'-1}(a).$$

We observe that $[X_{[w]}, Y_{[w']}]$ is equal to

$$\sum_{\substack{w_j = w \\ w_k = w'}} [b_j X_j, c_k X_k] = \sum_{\substack{w_j = w \\ w_k = w'}} b_j c_k [X_j, X_k] + \sum_{\substack{w_j = w \\ w_k = w'}} \Big(b_j X_j(c_k) X_k - c_k X_k(b_j) X_j \Big).$$

As all the vectors fields $b_j X_j(c_k) X_k - c_k X_k(b_j) X_j$ are sections of $H_{w+w'-1}$, we deduce that

$$[X, Y](a) = \sum_{\substack{w_j = w \\ w_k = w'}} b_j(a) c_k(a) [X_j, X_k](a) \mod H_{w+w'-1}(a).$$

Thus the class of $[X, Y](a)$ in $\mathfrak{g}^a_{w+w'} M$ depends only on the coordinate vectors $(b_j(a))_{w_j = a}$ and $(c_k(a))_{w_k = w'}$, and hence depends only on the respective classes of $X(a)$ and $Y(a)$ in $\mathfrak{g}^a_w M$ and $\mathfrak{g}^a_{w'} M$. The proof is complete. □

Let $a \in M$. It follows from Lemma 9.3.1 there is a unique bilinear map $\mathcal{L}_{w,w'}(a) : \mathfrak{g}^w_a M \times \mathfrak{g}^{w'}_a M \to \mathfrak{g}^{w+w'}_a M$ such that, for all sections X of H_w near a and sections Y of $H_{w'}$ near a, we have

$$\mathcal{L}_{w,w'}(a)\,(X(a), Y(a)) = \text{class of } [X, Y](a) \text{ in } \mathfrak{g}^{w+w'}_a M.$$

We note that this definition implies that

$$\mathcal{L}_{w,w'}(a)(X, Y) = -\mathcal{L}_{w',w}(a)(Y, X) \qquad \forall X \in \mathfrak{g}^w_a M\ \forall Y \in \mathfrak{g}^{w'}_a M. \tag{9.2}$$

The collection of the bilinear maps $\mathcal{L}_{w,w'}(a)$, $a \in M$, form a bilinear bundle map

$$\mathcal{L}_{w,w'} : \mathfrak{g}^w M \times \mathfrak{g}^{w'} M \to \mathfrak{g}^{w+w'} M.$$

We then have the following result.

Lemma 9.3.2. *$\mathcal{L}_{w,w'}$ is a smooth bilinear bundle map.*

Proof. Given $a \in M$, let $(X_1, \ldots, X_n)$ be a (smooth) H-frame near a. We know that the sections $\dot{X}_i$ with $w_i = w$ (resp., $w_i = w'$, $w_i = w + w'$) form a (smooth) local frame of $\mathfrak{g}^w M$ (resp., $\mathfrak{g}^{w'} M$, $\mathfrak{g}^{w+w'} M$) near a. Moreover, the fact that $[H_{w_i}, H_{w_j}] \subset H_{w_i+w_j}$ for $w_i + w_j \leq r$ implies that, near a, there are smooth functions $L_{ij}^k(x)$, $w_k \leq w_i + w_j$, such that, near $x = a$, we can write

$$[X_i, X_j] = \sum_{w_k \leq w_i + w_j} L_{ij}^k(x) X_k. \tag{9.3}$$

Therefore, when $w_i = w$ and $w_j = w'$, taking classes in $\mathfrak{g}^{w+w'} M$ gives

$$\mathcal{L}_{w,w'}(x)\left(\dot{X}_i(x), \dot{X}_j(x)\right) = \sum_{w_k = w+w'} L_{ij}^k(x) \dot{X}_k(x) \qquad \text{near } x = a. \tag{9.4}$$

As the coefficients $L_{ij}^k(x)$ depend smoothly on x we deduce that $\mathcal{L}_{w,w'}$ is a smooth bilinear bundle map near $x = a$. This proves the lemma. □

Definition 9.3.3. The bilinear bundle map $[\cdot,\cdot] : \mathfrak{g}M \times \mathfrak{g}M \to \mathfrak{g}M$ is defined as follows. For $a \in M$ and $X_j \in \mathfrak{g}_a^{w_j} M$, $j = 1, 2$, we set

$$[X_1, X_2]_a = \begin{cases} \mathcal{L}_{w_1,w_2}(a)(X_1, X_2) & \text{if } w_1 + w_2 \leq r, \\ 0 & \text{if } w_1 + w_2 > r. \end{cases} \tag{9.5}$$

Remark 9.3.4. We note that

$$[\mathfrak{g}^w M, \mathfrak{g}^{w'} M] \subset \mathfrak{g}^{w+w'} M \quad \text{if } w + w' \leq r \quad \text{and} \quad [\mathfrak{g}^w M, \mathfrak{g}^{w'} M] = \{0\} \quad \text{if } w + w' > r.$$

Defining recursively the commutator vector bundles $\mathfrak{g}^{[w]} M$, $w = 1, 2, \ldots$, by $\mathfrak{g}^{[1]} M = \mathfrak{g}M$ and $\mathfrak{g}^{[w+1]} M = [\mathfrak{g}M, \mathfrak{g}^{[w]} M]$, we see that

$$\mathfrak{g}^{[w]} M \subset \mathfrak{g}^{w+1} \quad \text{if } w < r \quad \text{and} \quad \mathfrak{g}^{[w]} M = \{0\} \quad \text{if } w \geq r.$$

Lemma 9.3.5. *The bilinear bundle map $[\cdot,\cdot]$ is a smooth field of Lie brackets on $\mathfrak{g}M$.*

Proof. As the restriction of $[\cdot,\cdot]$ on $\mathfrak{g}^{w_1} M \times \mathfrak{g}^{w_2} M$ either agrees with $\mathcal{L}_{w_1,w_2}$ if $w + w' \leq r$ or is zero if $w_1 + w_2 > r$, it follows from Lemma 9.3.2 that $[\cdot,\cdot]$ is a smooth bilinear bundle map. Moreover, it follows from (9.2) that $[\cdot,\cdot]$ is antisymmetric. Therefore, it only remains to check that, for any $a \in M$, the bilinear map $[\cdot,\cdot]_a$ satisfies Jacobi's identity on $\mathfrak{g}_a M$.

For $i = 1, 2, 3$ let $X_i \in \mathfrak{g}_a^{w_i} M$. If $w_1 + w_2 + w_3 > r$, then all three brackets $[X_1, [X_2, X_3]$, $[X_1, [X_2, X_3]$ and $[X_1, [X_2, X_3]$ vanish, and hence trivially satisfy Jacobi's identity. Assume that $w_1 + w_2 + w_3 \leq r$. For $i = 1, 2, 3$ let $\tilde{X}_i$ be a section of H_{w_i} near a such that $\tilde{X}_i(a)$ represents X_i in $\mathfrak{g}_a^{w_i} M$. By definition each bracket $[X_i, X_j](a)$ is represented by $[\tilde{X}_i, \tilde{X}_j](a)$ represents and each two-fold bracket $[X_i, [X_j, X_k](a)](a)$ is represented by $[\tilde{X}_i, [\tilde{X}_j, \tilde{X}_k]](a)$. Therefore, the Jacobi's identity for vector fields implies that

$$[X_1, [X_2, X_3]_a]_a + [X_2, [X_3, X_1]_a]_a + [X_3, [X_1, X_2]_a]_a = 0.$$

This shows that $[\cdot,\cdot]_a$ satisfies Jacobi's Identity on $\mathfrak{g}_a M$. The proof is complete. □

Combining the above lemma with Remark 9.3.4 gives the following result.

Proposition 9.3.6. $(\mathfrak{g}M, [\cdot,\cdot])$ *is a smooth bundle of step r nilpotent Lie algebras. Moreover, the grading (9.1) is a Lie algebra bundle grading.*

Definition 9.3.7. $(\mathfrak{g}M, [\cdot,\cdot])$ is called the tangent Lie algebra bundle of (M, H).

Remark 9.3.8. Let $(X_1, \ldots, X_n)$ be an H-frame near a point $a \in M$. For $j = 1, \ldots, n$ let us denote by $\dot{X}_j$ the class of X_j in $\mathfrak{g}^{w_j}M$. Then $(\dot{X}_1, \ldots, \dot{X}_n)$ is a local frame of $\mathfrak{g}M$ near $x = a$. The structure constants of $\mathfrak{g}M$ with respect to this frame are computed as follows. As in (9.3), there are unique smooth functions $L_{ij}^k(x)$, $w_k \leq w_i + w_j$, such that

$$[X_i, X_j] = \sum_{w_k \leq w_i + w_j} L_{ij}^k(x) X_k,$$

Then using (9.4) and (9.5) we get

$$[\dot{X}_i, \dot{X}_j] = \begin{cases} \displaystyle\sum_{w_k = w_i + w_j} L_{ij}^k(x)\dot{X}_l & \text{if } w_i + w_j \leq r, \\ 0 & \text{if } w_i + w_j > r. \end{cases} \tag{9.6}$$

9.3.2 The tangent Lie group bundle GM

The nilpotent Lie algebra bundle $\mathfrak{g}M$ is the Lie algebra bundle of a nilpotent Lie group bundle GM defined as follows. Given $a \in M$ the Lie group structure on G_aM is obtained by taking the exponential map $\exp_a : \mathfrak{g}_aM \to G_aM$ to be the identity and using the Campbell-Hausdorff formula to define the product law on G_aM. More explicitly, for $X \in \mathfrak{g}_a$, we let $\mathrm{ad}_X : \mathfrak{g}_a \to \mathfrak{g}_a$ be the adjoint endomorphism of X, i.e.,

$$\mathrm{ad}_X(Y) = [X, Y]_a \qquad \forall Y \in \mathfrak{g}_a. \tag{9.7}$$

We note that if $X \in \mathfrak{g}_a^w M$, then ad_X maps $\mathfrak{g}_a^{w'}M$ to $\mathfrak{g}_a^{w+w'}M$ if $w + w' \leq r$ and vanishes on $\mathfrak{g}_a^{(w')}$ if $w + w' > r$. Thus, ad_X is a nilpotent endomorphism of $\mathfrak{g}_aM$. Let us denote by $\mathrm{Der}(\mathfrak{g}_aM)$ the algebra generated by the adjoint endomorphisms ad_X, $X \in \mathfrak{g}_aM$. Then, any $A \in \mathrm{Der}(\mathfrak{g}_aM)$ maps $\mathfrak{g}_a^w M$ to $\mathfrak{g}_a^{w+1}M$ for all $w < r$ and vanishes on $\mathfrak{g}^r M$, so that A is a nilpotent endormphism of $\mathfrak{g}_aM$. Therefore, given any power series $f(z) = \sum_{k \geq 0} a_k z^k$, $a_k \in \mathbb{C}$, we may define

$$f(A) := \sum_{k \geq 0} a_k A^k = \sum_{0 \leq k \leq r} a_k A^k.$$

In addition, we set

$$\phi(z) = (z+1)\frac{\log(1+z)}{z} = 1 - \sum_{k \geq 1} \frac{(-1)^k}{k(k+1)} z^k.$$

Bearing this in mind, given X and Y in G_aM, the Campbell-Housdorff formula gives a formula for the the product of X and Y. Namely,

$$\begin{aligned} X \cdot Y &= X + \left(\int_0^1 \Phi(e^{\mathrm{ad}_X} e^{s\,\mathrm{ad}_Y} - I) ds \right) Y, \\ &= X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots . \end{aligned} \tag{9.8}$$

It follows from (9.8) and the smoothness of the Lie bracket of $\mathfrak{g}M$ that the above formula defines a smooth family of products $G_aM \times G_aM \rightarrow G_aM$.

Lemma 9.3.9. *Let $X \in G_aM$. Then $X^{-1} = -X$.*

Proof. As $\mathrm{ad}_X X = 0$, we see that $f(\mathrm{ad}_X)(-X) = f(0)$ for any power series $f(z)$. Bearing this in mind we have

$$X \cdot (-X) = X + \int_0^1 \Phi(e^{(1-s)\,\mathrm{ad}_X} - I)(-X) ds = X + \int_0^s (-X) ds = 0.$$

Likewise $(-X) \cdot X = 0$. Therefore $-X$ is the inverse of X. □

Given $a \in M$, the grading (9.1) defines a family of anisotropic dilations $\delta_t : x \rightarrow t \cdot x$, $t > 0$, on $\mathfrak{g}_aM$ given by

$$t \cdot X = t^w X \qquad \forall X \in \mathfrak{g}_a^w M. \tag{9.9}$$

We note that the fact that $\mathfrak{g}_aM$ is a graded Lie algebra implies that

$$[t \cdot X, t \cdot Y]_a = t \cdot [X, Y] \qquad \forall X, Y \in \mathfrak{g}_aM \ \forall t > 0. \tag{9.10}$$

The action of δ_t on $\mathrm{Der}(g_aM)$ is given by

$$\delta_t(A) := (\delta_t)_* A = \delta_t \circ A \circ \delta_t^{-1}. \tag{9.11}$$

In particular, it follows from (9.7) and (9.10) that

$$\delta_t(\mathrm{ad}\, X) = \mathrm{ad}_{t \cdot X} \qquad \forall X \in \mathfrak{g}_aM.$$

Lemma 9.3.10. *Let $a \in M$ and $t > 0$. Then*

$$t \cdot (X \cdot Y) = (t \cdot X) \cdot (t \cdot Y) \qquad \forall X, Y \in G_aM.$$

Proof. We note that if A and B are in $\mathrm{Der}(g_aM)$, then $\delta_t(AB) = \delta_t(A)\delta_t(B)$. More generally, for any 2-variable power series $g(z, y) = \sum a_{kl} z^k y^l$ we have $\delta_t\left(g(A, B)\right) = g\left(\delta_t(A), \delta_t(B)\right)$. Applying this to $g(z, y) = \int_0^1 \Phi(e^z e^{sy} - 1) ds$ and using (9.11) we see that

$$\delta_t \left(\int_0^1 \Phi(e^{\mathrm{ad}_X} e^{s\,\mathrm{ad}_Y} - I) ds \right) = \int_0^1 \Phi(e^{\mathrm{ad}_{t \cdot X}} e^{s\,\mathrm{ad}_{t \cdot Y}} - I) ds.$$

Therefore, the dilation $t \cdot (X \cdot Y) = \delta_t(X \cdot Y)$ is equal to

$$\delta_t(X) + \delta_t\left(\int_0^1 \Phi(e^{\mathrm{ad}_X} e^{s\,\mathrm{ad}_Y} - I)ds\right)\delta_t(Y) = t \cdot X + \left(\int_0^1 \Phi(e^{\mathrm{ad}_{t\cdot X}} e^{s\,\mathrm{ad}_{t\cdot Y}} - I)ds\right)(t \cdot Y)$$
$$= (t \cdot X) \cdot (t \cdot Y).$$

This proves the lemma. □

For $w = 1, \ldots, r$ set $G^w M = \mathfrak{g}^w M$. We note that $X \in G^w M$ if and only if $t \cdot X = t^w X$ for all $t > 0$. Moreover, if $w > r$ and $t \cdot X = t^w X$ for all $t > 0$, then $X = 0$. Combining this with Lemma 9.3.10 it then can be shown that

$$G^w M \cdot G^{w'} M \subset G^{w+w'} M \quad \text{if } w + w' \leq r \quad \text{and} \quad G^w M \cdot G^{w'} M = \{0\} \quad \text{if } w + w' > r.$$

We summarize the previous discussion in the following statement.

Proposition 9.3.11. *GM is a smooth graded step r nilpotent Lie group bundle.*

Definition 9.3.12. GM is called the tangent Lie group bundle of (M, H).

9.3.3 Description of $\mathfrak{g}_a M$ in terms of left-invariant vector fields

Let $(X_1, \ldots, X_n)$ be an H-frame near a point $a \in M$. As in (9.3) near a there are unique smooth functions $L_{ij}^k(x)$, $w_k \leq w_i + w_j$, such that

$$[X_i, X_j] = \sum_{w_k \leq w_i + w_j} L_{ij}^k(x) X_k.$$

For $i = 1, \ldots, n$ let $\dot{X}_i(a)$ the class of $X_i(a)$ in $\mathfrak{g}_a^{w_i} M$. Then $(\dot{X}_1(a), \ldots, \dot{X}_n(a))$ is basis of $\mathfrak{g}_a M$, and hence defines coordinates $(x_1, \ldots, x_n)$ on $\mathfrak{g}_a M$. In these coordinates the dilations (9.9) are given by

$$\delta_t(x_1, \cdots, x_n) = (t^{w_1} x_1, \cdots, t^{w_n} x_n), \qquad t > 0.$$

Let $X = \sum_{i \leq n} x_i \dot{X}_i(a)$ and $Y = \sum_{i \leq n} y_i \dot{X}_i(a)$ be in $\mathfrak{g}_a M$. Then using Remark 9.3.8 we get

$$\mathrm{ad}_X Y = \sum_{i,j=1}^n x_i y_j [\dot{X}_i(a), \dot{X}_j]_a = \sum_{i,j=1}^n \sum_{w_k = w_i + w_j} x_i y_j L_{ij}(a).$$

This shows that the matrix $A_a(x) = \left(A_a(x)_{kj}\right)_{1 \leq j,k \leq n}$ of ad_X is given by

$$A_a(x)_{jk} = \begin{cases} \displaystyle\sum_{w_i = w_k - w_j} L_{ij}^k(a) x_i & \text{if } w_j < w_k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $A_a(x)$ is an upper-triangular matrix. It then follows that in the coordinates $(x_1, \ldots, x_n)$ the product (9.8) of G_aM is given by

$$\begin{aligned} x \cdot y &= x + \left(\int_0^1 \Phi\left(e^{A_a(x)} e^{sA_a(y)} - I\right) ds \right) y \\ &= x + y + \frac{1}{2} A_a(x) y + \frac{1}{12} A_a(x)^2 y - \frac{1}{12} A_a(y) A_a(x) y + \cdots . \end{aligned} \tag{9.12}$$

We can interpret $\mathfrak{g}_aM$ as Lie algebra of left-invariant vector fields on G_aM as follows. For $i = 1, \ldots, n$, let X_i^a be the left-invariant vector field on G_aM that agree at $x = 0$ with $\dot{X}_i(a)$ under the identification $\mathfrak{g}_aM \simeq T_0G_a$. That is,

$$X_i^a f(x) = \frac{d}{dt} f(x \cdot (t\dot{X}_i(a)))\Big|_{t=0} \qquad \forall f \in C^\infty(G_aM).$$

The span of the vector fields X_i^a is a Lie algebra with same constant structures $L_{ij}^k(a)$ as $\mathfrak{g}_aM$. More precisely, as the Lie bracket $[X_i^a, X_j^a]$ is the left-invariant vector field on G_aM that agrees at $x = 0$ with $[X_i(a), X_j(a)]_a = \sum L_{ij}^k(a) X^k(a)$, we have

$$[X_i^a, X_j^a] = \sum_{w_k = w_i + w_j} L_{ij}^k(a) X_k^a.$$

Moreover, the vector fields X_i^a are homogeneous with respect to the anisotropic dilations δ_t. Indeed, for any $f \in C^\infty(M)$ and $s > 0$, we have

$$(\delta_s^* X_i^a) f(x) = \frac{d}{dt} f \circ \delta_s^{-1}\left(\delta_s(x) \cdot \left(t\dot{X}_i(a)\right)\right)\Big|_{t=0} = \frac{d}{dt} f\left(x \cdot \left(s^{-w_i} t\dot{X}_i(a)\right)\right)\Big|_{t=0} = s^{-w_i} X_i f(x). \tag{9.13}$$

Thus,

$$\delta_s^* X_i^a = s^{-w_i} X_i^a \qquad \forall s > 0. \tag{9.14}$$

The vector fields X_i^a are computed in the coordinates $x_1, \ldots, x_n$ as follows. Let $e_1 = (1, 0, \cdots, 0)$, ..., $e_n = (0, \cdots, 0, 1)$ be the respective coordinate vectors of $\dot{X}_1(a), \ldots, \dot{X}_n(a)$. In the coordinates $x_1, \ldots, x_n$ the vector field X_i^a is given by

$$X_i^a f(x) = \frac{d}{dt} f(x \cdot (te_i))|_{t=0}, \qquad f \in C^\infty(G_aM).$$

Using (9.12) we get

$$x \cdot (te_i) = x + t \left(\int_0^1 \Phi(e^{A_a(x)} e^{stA_a(e_i)} - I) ds \right) e_i.$$

Therefore, we see that $\frac{d}{dt} x \cdot (te_i)|_{t=0}$ is equal to

$$\left(\int_0^1 \Phi(e^{A_a(x)} - I) ds \right) e_i = \Phi\left(e^{A_a(x)} - I\right) e_i = \left(\frac{A_a(x)}{I - e^{-A_a(x)}} \right) e_i. \tag{9.15}$$

Define

$$B_a(x) = \frac{A_a(x)}{I - e^{-A_a(x)}} - I = A_a(x) + \frac{A_a(x)^2}{12} + \cdots .$$

Note that the coefficients of $B_a(x)$ are polynomials in x without constant coefficients, since $A_a(x)$ is a nilpotent matrix whose coefficients depends linearly on x. We then can rewrite (9.15) as

$$\frac{d}{dt} x \cdot (te_i)|_{t=0} = e_i + B_a(x)e_i = e_i + \sum_{j=1}^{n} B_a(x)_{ji} e_j.$$

Therefore, for all $f \in C^\infty(G_a M)$, we have

$$X_i^a f(x) = \left\langle df(x), \frac{d}{dt}\,(x \cdot_a (te_i))|_{t=0} \right\rangle = \partial_i f(x) + \sum_{j=1}^{n} B_a(x)_{ji} \partial_j f(x).$$

This shows that

$$X_i^a = \partial_i + \sum_{i=1}^{n} B_a(x)_{ji} \partial_j. \tag{9.16}$$

Let $s > 0$. Then using (9.14) we get

$$\delta_s^* X_i^a = \delta_s^* \partial_i + \sum_{j=1}^{n} B_a(\delta_s(x))_{ji} \delta_s^* \partial_j = s^{-w_i} \partial_j + \sum s^{-w_j} B_a(s \cdot x)_{ji} \partial_j.$$

Combining this with the homogeneity (9.13) of X_i^a we deduce that

$$B_a(s \cdot x)_{ji} = s^{w_j - w_i} B_a(x)_{ji} \qquad \forall s > 0. \tag{9.17}$$

In what follows, given a multi-order $\alpha \in \mathbb{N}_0^n$, we set

$$\langle \alpha \rangle = w_1 \alpha_1 + \cdots + w_n \alpha_n.$$

We note that the monomials that are homogeneous of a given degree w, $w \in \mathbb{N}_0$, with respect to the dilations δ_t are precisely those of the form x^α with $\langle \alpha \rangle = w$. Bearing this in mind the homogeneity (9.17) of $B_a(x)_{ji}$ and the fact that $B_a(x)_{ji}$ is a polynomial in x with no constant term imply that

- If $w_j \leq w_i$, then $B_a(x)_{ji} = 0$.
- If $w_j > w_i$, then $B_a(x)_{ij} = \sum_{\langle \alpha \rangle = w_j - w_i} b_{ji\alpha} x^\alpha$, where $b_{ji\alpha} = \frac{1}{\alpha!} \partial^\alpha \, B_a(x)_{ji}\big|_{x=0}$.

Combining this with (9.16) we arrive at the following result.

Lemma 9.3.13. *For $i = 1, \ldots, n$, the vector field X_i^a is given by*

$$X_i^a = \partial_i + \sum_{\substack{\langle \alpha \rangle = w_j - w_i \\ w_j > w_i}} b_{ji\alpha}^a x^\alpha \partial_j, \qquad \textit{where } b_{ji\alpha} = \frac{1}{\alpha!} \partial^\alpha \, B_a(x)_{ji}\Big|_{x=0}.$$

9.4 Privileged Coordinates for Carnot Manifolds

In this section, we explain how to extend to the setting of Carnot manifolds the construction of privileged coordinates by Bellaïche [Be].

In what follows we let $(X_1, \ldots, X_n)$ be an H-frame on an open neighborhood U of a given point $a \in M$. Then there are unique smooth functions $L_{ij}^k(x)$ on U such that

$$[X_i, X_j] = \sum_{w_k \leq w_i + w_j} L_{ij}^k(x) X_k. \tag{9.18}$$

In addition, given any finite sequence $I = (i_1, \ldots, i_k)$ with values in $\{1, \ldots, n\}$, we define

$$X_I = X_{i_1} \cdots X_{i_k}.$$

For such a sequence we also set $|I| = k$ and $\langle I \rangle = w_{i_1} + \cdots w_{i_k}$.

Definition 9.4.1. Let $f(x)$ be a smooth function defined near $x = a$ and N a nonnegative integer.

1. We say that $f(x)$ has order $\geq N$ at a when $X_I f(a) = 0$ whenever $\langle I \rangle < N$.
2. We say that $f(x)$ has order N at a when it has order $\geq N$ and there is a sequence $I = (i_1, \ldots, i_k)$ with values in $\{1, \ldots, n\}$ with $\langle I \rangle = N$ such that $X_I f(a) \neq 0$.

Remark 9.4.2. The above definition of the order of a function differs from that of Belaïche [Be] as Bellaïche only considers monomials in vector fields X_i with $w_i = 1$.

Lemma 9.4.3. *Let $f(x)$ be a smooth function near $x = a$. Then its order is independent of the choice of the H-frame $(X_1, \ldots, X_n)$ near a.*

Proof. Let $(Y_1, \cdots, Y_n)$ be another H-frame near a. We note that each vector field Y_i is a section of H_i. Thus, near $x = a$,

$$Y_i = \sum_{w_j \leq w_i} c_{ij}(x) X_j,$$

for some smooth functions $c_{ij}(x)$ such that $c_{ij}(a) \neq 0$ for some j such that $w_j = w_i$. More generally, given any finite sequence $I = (i_1, \ldots, i_k)$ with values in $\{1, \ldots, n\}$, near $x = a$, we may write

$$Y_I = Y_{i_1} \cdots Y_{i_k} = \Big(\sum_{w_{j_1} \leq w_{i_1}} c_{i_1 j_1}(x) X_j \Big) \cdots \Big(\sum_{w_{j_k} \leq w_{i_1}} c_{ij}(x) X_j \Big) = \sum_{\langle J \rangle \leq \langle I \rangle} c_{IJ}(x) X_J, \tag{9.19}$$

where the $c_{IJ}(x)$ are smooth functions.

Let N be the order of f with respect to the H-frame $(X_1, \ldots, X_n)$. If $\langle I \rangle < N$, then (9.19) shows that $Y_I f(a)$ is a linear combination of terms $X_J f(a)$ with $\langle J \rangle \leq \langle I \rangle < N$, which are zero.

Thus $Y_I f(a) = 0$ whenever $\langle I\rangle < N$. Suppose now that I is such that $\langle I\rangle = N$ and $X_I f(a) \neq 0$. In the same way as in (9.19), near $x = a$, we may write

$$X_I = \sum_{\langle J\rangle \leq \langle I\rangle} d_{IJ}(x) Y_J,$$

where the $d_{IJ}(x)$ are smooth functions near $x = a$. Then

$$0 \neq X_I f(a) = \sum_{\langle J\rangle \leq \langle I\rangle} d_{IJ}(a) Y_J f(a) = \sum_{\langle J\rangle = N} d_{IJ}(a) Y_J f(a).$$

Therefore, at least one of the number $Y_J f(a)$, $\langle J\rangle = N$, must be nonzero. We then deduce that f has order N at a with respect to the H-frame $(Y_1, \ldots, Y_n)$ as well. This shows that the order of f at a is independent of the choice of the H-frame. The lemma is thus proved. □

Lemma 9.4.4. *Let $f(x)$ and $g(x)$ be smooth functions near $x = a$ of respective orders N and N' at a. Then $f(x)g(x)$ has order $\geq N + N'$ at a.*

Proof. We know that $X_i(fg) = (X_i f)g + fX_i g$. More generally, given any sequence $I = (i_1, \ldots, i_k)$, we may write

$$X_I(fg) = X_{i_1} \cdots X_{i_k}(fg) = \sum_{\langle I'\rangle + \langle I''\rangle = \langle I\rangle} c_{I'J''}(X_{I'} f)(X_{I''} g), \tag{9.20}$$

for some constants c_{IJ} independent of f and g. If $\langle I'\rangle + \langle I''\rangle < N + N'$, then one the inequality $\langle I'\rangle < N$ or $\langle I''\rangle < N$ must hold. In both cases the product $(X_{I'} f)(a)(X_{I''} g)(a)$ is zero. Combining this with (9.20) we then see that $X_I(fg)(a) = 0$ whenever $\langle I\rangle < N + N'$. That is, $f(x)g(x)$ has order $\geq N + N'$ at a. The proof is complete. □

Given any multi-order $\alpha \in \mathbb{N}_0^n$ we set

$$X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n},$$

and we denote

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \langle\alpha\rangle = w_1\alpha_1 + \cdots + w_n\alpha_n. \tag{9.21}$$

We note that $X^\alpha = X_I$, where $I = (i_1, \ldots, i_k)$ is the unique nondecreasing sequence of length $k = |\alpha|$ where each index i appears with multiplicity α_i. Conversely, if $I = (i_1, \ldots, i_k)$ is a nondecreasing sequence, then $X_I = X^\alpha$ for some multi-order α with $|\alpha| = |I|$ and $\langle\alpha\rangle = \langle I\rangle$.

It is convenient to reformulate the definition of the order at a a function in terms of the sole monomials X^α. To this end we shall need the following lemma.

Lemma 9.4.5 (Compare [Be, Lemma 4.12])**.** *Let $I = (i_1, \ldots, i_m)$ be a finite sequence with values in $\{1, \ldots, n\}$ and set $w = \langle I\rangle$. Then, near $x = a$, then*

$$X_I = \sum_{\substack{\langle\alpha\rangle \leq w \\ |\alpha| \leq k}} c_{I\alpha}(x) X^\alpha, \tag{9.22}$$

where the $c_{I\alpha}(x)$ are smooth functions near $x = a$.

Proof. We shall prove this result by induction on k. For $k = 1$ the result is immediate. In order to prove that the results for for $k' \leq k$ imply the result for $k+1$ we shall need the following claims.

Claim 1. *Let $I = (i_1, \ldots, i_m)$ be a finite sequence with values in $\{1, \ldots, n\}$ and j an integer in $\{1, \ldots, n\}$. Set $w = \langle I \rangle + w_j$. Then, for $l = 1, \ldots, m$ and near $x = a$, we may write*

$$X_j X_{i_1} \cdots X_{i_m} = X_{i_1} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_m} + \sum_{\substack{\langle J \rangle \leq w \\ |J| \leq m}} c^J_{Ijl}(x) X_J \tag{9.23}$$

where the $c^J_{Ijl}(x)$ are smooth functions near $x = a$ (by convention $X_{i_{l+1}} \cdots X_{i_m} = 1$ for $l = m$).

Proof of Claim 1. We proceed by induction on m. For $m = 1$ the claim is an immediate consequence of (9.23). Assume that the claim is true for $m-1$ with $m \geq 2$ and let $l \in \{1, \ldots, m\}$. Using (9.18) we get

$$X_j X_{i_1} \cdots X_{i_m} = X_{i_1} X_j X_{i_2} \cdots X_{i_m} + \sum_{w_p \leq w_j + w_{i_1}} L^p_{ji_1}(x) X_p X_{i_2} \cdots X_{i_m}.$$

This gives (9.23) for $l = 1$. If $l \geq 2$, then, as the claim is true for $m-1$, near $x = a$, we may write

$$X_j X_{i_2} \cdots X_{i_m} = X_{i_2} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_m} + \sum_{\substack{\langle J \rangle \leq w - w_{i_1} \\ |J| \leq m-1}} c^J_{Ijl}(x) X_J, \tag{9.24}$$

where the $c^J_{Ijl}(x)$ are smooth functions near $x = a$. Thus,

$$\begin{aligned} X_{i_1} X_j X_{i_2} \cdots X_{i_m} &= X_{i_1} X_{i_2} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_m} + \sum_{\substack{\langle J \rangle \leq w - w_{i_1} \\ |J| \leq m-1}} X_{i_1}(c^J_{Ijl} X_J) \\ &= X_{i_1} X_{i_2} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_m} + \sum_{\substack{\langle J \rangle \leq w - w_{i_1} \\ |J| \leq m-1}} \Big((X_{i_1} c^J_{Ijl})(x) + c^J_{Ijl}(x) X_{i_1} \Big) X_J. \end{aligned}$$

Combining this with (9.24) we see that

$$X_j X_{i_1} \cdots X_{i_m} = X_{i_1} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_m} + \sum_{\substack{\langle J \rangle \leq w \\ |J| \leq m}} c^J_{Ijl}(x) X_J$$

where the $c^J_{Ijl}(x)$ are some smooth functions near $x = a$. Thus the claims holds for m. This completes the proof of Claim 1. ◻

Claim 2. *Let $j \in \{1, \ldots, n\}$ and $\alpha \in \mathbb{N}_0^n$ be such that $|\alpha| = k$. Set $w = w_j + \langle \alpha \rangle$. Assume that (9.22) holds for $|I| \leq k$. Then there is a multi-order β with $|\beta| = k+1$ and $\langle \beta \rangle = w$ such that, near $x = a$, we may write*

$$X_j X^\alpha = X^\beta + \sum_{\substack{\langle \gamma \rangle \leq w \\ |\gamma| \leq k}} c^\gamma_{\alpha j}(x) X^\gamma,$$

where the functions $c^\gamma_{\alpha j}(x)$ are smooth near $x = a$.

Proof of Claim 2. Let $I = (i_1, \ldots, i_k)$ be the unique nondecreasing sequence of lenght $k = |\alpha|$ with values in $\{1, \ldots, n\}$ such that each integer i has multiplicity α_i. Note that $\langle I \rangle = \langle \alpha \rangle$. Let l_0 be the the largest integer $l \in \{0, \ldots, n\}$ such that either $l = 0$ or $j_l < i_1$. Then by Claim 1, near $x = a$, we may write

$$X_j X^\alpha = X_j X_{i_1} \cdots X_{i_k} = X_{i_1} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_k} + \sum_{\substack{\langle J \rangle \leq w \\ |J| \leq k}} c^J_{Ijl}(x) X_J \tag{9.25}$$

where the $c^J_{Ijl}(x)$ are smooth functions near $x = a$. As the sequence $(i_1, \ldots, i_{l_0}, j, i_{l_0+1}, \ldots, i_k)$ is nondecreasing, there is a unique multiorder $\beta \in \mathbb{N}^n_0$ with $|\beta| = k + 1$ and $\langle \beta \rangle = w$ such that

$$X_{i_1} \cdots X_{i_l} X_j X_{i_{l+1}} \cdots X_{i_k} = X^\beta. \tag{9.26}$$

In the summation in (9.25) all the terms X_J are of the form (9.22), since by assumption (9.22) is true for $|I| \leq k$. Combining this with (9.25) and (9.26) proves the claim. □

Let us go back to the proof of Lemma 9.4.5. We assume that (9.22) holds when $|I| \leq k$. Let $I = (i_1, \ldots, i_{k+1})$ be a finite sequence of length $|I| = k + 1$. We may apply (9.22) to $X_{i_2} \cdots X_{i_{k+1}}$ to get

$$X_{i_2} \cdots X_{i_{k+1}} = \sum_{\substack{\langle \alpha \rangle \leq w - w_{i_1} \\ |\alpha| \leq k}} c_{I\alpha}(x) X^\alpha.$$

As in (9.25), near $x = a$, we can write

$$\begin{aligned} X_I = X_{i_1} X_{i_2} \cdots X_{i_{k+1}} &= \sum_{\substack{\langle \alpha \rangle \leq w - w_{i_1} \\ |\alpha| \leq k}} \left((X_{i_1} c_{I\alpha})(x) + c_{I\alpha}(x) X_{i_1} \right) X^\alpha \\ &= \sum \sum_{\substack{\langle \alpha \rangle \leq w - w_{i_1} \\ |\alpha| = k}} c_{I\alpha} X_{i_1} X^\alpha + \sum_{|J| \leq k} c_{IJ}(x) X_J, \end{aligned}$$

where the $c_{IJ}(x)$ are smooth functions near $x = a$. Combining this with Claim 2 shows that X_I can be put in the form (9.22). This establishes (9.22) for $|I| = k + 1$. The proof of Lemma 9.4.5 is complete. □

Proposition 9.4.6. *Let $f(x)$ be a smooth function defined near $x = a$. Then $f(x)$ has order N at $x = a$ if and only if the following two conditions are satisfied:*

(i) $(X^\alpha f)(a) = 0$ *for all multi-orders α such that $\langle \alpha \rangle < N$.*

(ii) $(X^\alpha f)(a) \neq 0$ *for at least one multi-order α with $\langle \alpha \rangle = N$.*

Proof. Assume that $f(x)$ has order N at $x = a$. It is immediate that (i) holds. Let $I = (i_1, \ldots, i_k)$ be a sequence with values in $\{1, \ldots, n\}$ with $\langle I\rangle = N$ and $X_I f(a) \neq 0$. By Lemma 9.4.5, near $x = a$,

$$X_I = \sum_{\langle\alpha\rangle\leq\langle I\rangle} c_{I\alpha}(x)X^\alpha = \sum_{\langle\alpha\rangle\leq N} c_{I\alpha}(x)X^\alpha$$

for some smooth functions $c_{I\alpha}(x)$. Thus,

$$0 \neq X_I f(a) = \sum_{\langle\alpha\rangle\leq N} c_{I\alpha}(a)X^\alpha f(a) = \sum_{\langle\alpha\rangle= N} c_{I\alpha}(a)X^\alpha f(a).$$

This implies that at least one of the numbers $X^\alpha f(a)$, $\langle\alpha\rangle = N$, is nonzero, i.e., (ii) holds.

Conversely, suppose that (i) and (ii) holds. Then (ii) implies that $f(x)$ has order $\leq N$ at $x = a$. Moreover, using (i) and Lemma 9.4.5 shows that $f(x)$ has order $\geq N$ at $x = a$. Thus $f(x)$ has order N at $x = a$. The proof is complete. □

Definition 9.4.7. We say that local coordinates $\{x_1, \ldots, x_n\}$ centered at a point $a \in M$ are linearly adpated to the H-frame $X_1, \ldots, X_n$ when $X_j(0) = \partial_j$ for $j = 1, \ldots, n$.

Lemma 9.4.8. *Given local coordinates $x = (x_1, \cdots, x_n)$ there is a unique affine change of coordinates $y = T_a(x)$ such that the coordinates $y = (y_1, \cdots, y_n)$ are centered at a and linearly adapted to the H-frame $X_1, \cdots X_n$.*

Proof. In the local coordinates $(x_1, \ldots, x_n)$ we may write

$$X_i = \sum_{1\leq j\leq n} b_{ij}(x)\frac{\partial}{\partial x_j}, \qquad i = 1, \ldots, n,$$

where the coefficients $b_{ij}(x)$ are smooth. Set $B(x) = (b_{ij})_{1\leq i,j\leq n} \in \mathrm{GL}_n(\mathbb{R})$. In what follows we shall use the same notation for the point a and its coordinate vector $a = (a_1, \ldots, a_n)$ with to the local coordinates $(x_1, \cdots, x_n)$.

Let $T(x) = A(x - a)$ be an affine transformation with $T(a) = 0$ and $A = (a_{jk}) \in \mathrm{GL}_n(\mathbb{R})$. Set $y = (y_1, \ldots, y_n) = T(x)$, i.e., $y_i = \sum_j a_{ij}(x_j - a_j)$, $i = 1, \ldots, n$. Then $(y_1, \ldots, y_n)$ are local coordinates centered at a. In those coordinates,

$$X_i = \sum_{1\leq j,k\leq n} b_{ij}(x)\frac{\partial y_k}{\partial x_j}\frac{\partial}{\partial y_k} = \sum_{1\leq k\leq n}\left(\sum_{1\leq j\leq n} b_{ij}(x)a_{kj}\right)\frac{\partial}{\partial y_k}. \tag{9.27}$$

Thus $X_i = \frac{\partial}{\partial y_k}$ at $y = 0$ if and only if $\sum_{1\leq j\leq n} b_{ij}(x)a_{kj} = \delta_{ik}$. We then see that the local coordinates $(y_1, \ldots, y_n)$ are linearly adapted at a if and only if $B(a)A^T = 1$, i.e., $A = \left(B(a)^T\right)^{-1}$. This shows that $T_a(x) = \left(B(a)^T\right)^{-1}(x - a)$ is the unique affine isomorphism that produces linearly adapted coordinates centered at a. The proof is complete. □

Definition 9.4.9. Local coordinates $x = (x_1, \ldots, x_n)$ centered at a are called privileged coordinates at a adapted to the H-frame $(X_1, \ldots, X_n)$ when the following two conditions hold:

(i) The coordinates are linearly adapted to $(X_1, \ldots, X_n)$ in the sense of Definition 9.4.7.

(ii) For all $j = 1, \ldots, n$, the coordinate function x_j has order $\geq w_j$ at a.

In what follows using local coordinates centered at a we may regard the vector fields $X_1, \ldots, X_n$ as vector fields defined on a neighborhood of the origin in $\mathbb{R}^n$.

Lemma 9.4.10 ([Be, Lemma 4.13])**.** *Let $h(x)$ be a homogeneous polynomial of degree k. Then*

$$(X^\alpha h)(0) = \begin{cases} \partial_x^\alpha h(0) & \text{if } |\alpha| = k, \\ 0 & \text{if } |\alpha| < k. \end{cases}$$

Remark 9.4.11. In the proof of the above result in [Be, page 40], the summation in Eq. (34) is over all multi-orders $\beta = (\beta_1, \ldots, \beta_n)$ such that $\beta \neq \alpha$ and $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$. This should be replaced by the summation over all multi-orders β such that $|\beta| \leq |\alpha|$.

Proposition 9.4.12 (Compare [Be, Lemma 4.14])**.** *Let $(x_1, \ldots, x_n)$ be local coordinates centered at a that are linearly adapted to the H-frame $(X_1, \ldots, X_n)$. Then there is a unique polynomial change of coordinates $y = \psi(x)$ such that*

1. *The local coordinates $y = (y_1, \ldots, y_n)$ are privileged coordinates at a adapted to $(X_1, \ldots, X_n)$.*

2. *For $j = 1, \ldots, n$, the component $y_j = \psi_j(x)$ is of the form,*

$$y_j = x_j + \sum_{\substack{\langle\alpha\rangle < w_j \\ 2 \leq |\alpha|}} a_{j\alpha} x^\alpha, \qquad a_{j\alpha} \in \mathbb{R}. \tag{9.28}$$

Proof. Let $y = \psi(x)$ be a change of coordinates of the form (9.28). Let j and l be indices in $\{1, \ldots, n\}$. Using Lemma 9.4.6 we get

$$X_l(y_j)\big|_{x=0} = X_l(x_j)\big|_{x=0} + \sum_{\substack{\langle\alpha\rangle < w_j \\ 2 \leq |\alpha|}} a_{j\alpha}\, X_l(x^\alpha)|_{x=0} = \delta_{jl}.$$

In particular, we see that $X_l(y_j)\big|_{x=0} = 0$ when $w_l < w_j$ and $X_j(y_j)\big|_{x=0} = 1$. Therefore, the coordinate $y_j = \psi(x)_j$ has order w_j if and only if $X^\alpha(y_j) = 0$ for all multi-order α such that $\langle\alpha\rangle < w_j$ and $|\alpha| \geq 2$. Let α be such a multi-order. Then by Lemma 9.4.12 we have

$$\begin{aligned} X^\alpha(y_j)\big|_{x=0} &= X^\alpha(x_j)\big|_{x=0} + \sum_{\substack{\langle\beta\rangle < w_j \\ 2 \leq |\beta|}} a_{j\beta}\, X^\alpha(x^\beta)\big|_{x=0} \\ &= X^\alpha(x_j)\big|_{x=0} + \sum_{\substack{\langle\beta\rangle < w_j \\ 2 \leq |\beta| < |\alpha|}} a_{j\beta}\, X^\alpha(x^\beta)\big|_{x=0} + \alpha! a_{j\alpha}. \end{aligned}$$

Thus,

$$X^{\alpha}(y_j)\Big|_{x=0} = 0 \iff \alpha! a_{j\alpha} = -\, X^{\alpha}(x_j)\Big|_{x=0} - \sum_{\substack{\langle\beta\rangle < w_j \\ 2\leq|\beta|<|\alpha|}} a_{j\beta}\, X^{\alpha}(x^{\beta})\Big|_{x=0}. \tag{9.29}$$

As the right-hand side uniquely determine the coefficients $a_{j\alpha}$, we deduce there is a unique polynomial change of variable $y = \psi(x)$ of the form (9.28) that produces privileged coordinates centered at a. The lemma is proved. □

Definition 9.4.13. Let $(x_1, \ldots, x_n)$ be the linearly adapted coordinates provided by the affine map T_a from Lemma 9.4.8. Then we denote by $\psi_a(x)$ the polynomial diffeormorphism provided by Proposition 9.4.12, i.e., $y = \psi_a(x)$ is the unique change of coordinates of the form (9.28) giving privileged coordinates at a.

We conclude this section with the following unicity result.

Proposition 9.4.14. *The coordinates $y = \psi_a(T_a x)$ are the unique privileged coordinates at a adapted to the H-frame $(X_1, \ldots, X_n)$ that are given by a change of coordinates of the form $y = \psi(T(x))$, where T is an affine map such that $T(a) = 0$ and $\psi(x)$ is a polynomial diffeomorphism of the form (9.28).*

Proof. Let $y = \phi(x)$ be privileged coordinates at a adapted to H-frame $(X_1, \ldots, X_n)$ such that $\phi(x) = \psi(T(x))$, where T is an affine map such that $T(a) = 0$ and $\psi(x)$ is a polynomial diffeomorphism of the form (9.28). The fact that the coordinates $y = \phi(x)$ are linearly adapted to the H-frame $(X_1, \ldots, X_n)$ exactly means that $\phi_* X_j(0) = \partial_j$ for $j = 1, \ldots, n$. Note that (9.28) implies that $\psi'(0) = \mathrm{id}$. Thus $\phi_* X_j(0) = \psi'(0) \circ T'(a)\big(X_j(a)\big) = T'(a)\big(X_j(a)\big) = T_* X_j(0)$, so that we see that $T_* X_j(0) = \partial_j$. This means that the coordinates $y = T(x)$ are linearly adapted to the H-frame $(X_1, \ldots, X_n)$. As $T(x)$ is an affine map, it then follows from Lemma 9.4.8 that $T(x) = T_a(x)$. Therefore, we see that $\psi(x)$ is a polynomial diffeomorphism of the form (9.28) that transforms the coordinates $y = \psi_a(x)$ into privileged coordinates at a adapted to $(X_1, \ldots, X_n)$. It then follows from Proposition 9.4.12 that $\psi(x) = \psi_a(x)$, so that $\phi(x) = \psi_a(T_a x)$. This proves the result. □

9.5 Nilpotent Approximation of Vector Fields

In this section, we explain how privileged coordinates enables us to approximate at every point $a \in M$ vector fields (and even differential operators) by vector fields that generate a nilpotent Lie algebra isomorphic to the tangent Lie algebra $\mathfrak{g}_a M$. This provides us with an extrinsic alternative construction of the tangent space to a Carnot structure at a point a.

Definition 9.5.1. Let $f(x)$ be a smooth function near the origin in $\mathbb{R}^n$. We shall say that

1. f has weight $\geq w$ when $\partial_x^{\alpha} f(0) = 0$ for all multiorders $\alpha \in \mathbb{N}_0^n$ such that $\langle\alpha\rangle < w$.

2. f has weight w when $f(x)$ has weight $\geq w$ and there is a multiorder $\alpha \in \mathbb{N}_0^n$ with $\langle\alpha\rangle = w$ such that $\partial_x^\alpha f(0) \neq 0$.

In the same way as in Section 9.3, for $t > 0$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we denote by $t \cdot x$ the anisitropic dilation,

$$t \cdot x := (t^{w_1} x_1, \cdots, t^{w_n} x_n).$$

We shall also denote by δ_t the map $x \to t \cdot x$.

Definition 9.5.2. A function $f(x)$ on $\mathbb{R}^n$ or $\mathbb{R} \setminus 0$ is *weight-homogeneous* of degree w, $w \in \mathbb{R}$, when

$$f(t \cdot x) = t^w f(x) \qquad \forall t > 0.$$

Examples 9.5.3. For any multi-order $\alpha \in \mathbb{N}_0^n$, the monomial x^α is weight-homogeneous of degree $\langle\alpha\rangle$.

Remark 9.5.4. If $f(x)$ is smooth and weight-homogeneous of degree w, then differentiating the equality $f(t \cdot x) = t^\omega f(x)$ shows that $\partial^\alpha f(t \cdot x) = t^{w - \langle\alpha\rangle} \partial^\alpha f(x)$. Thus $\partial^\alpha f(x)$ is weight-homogeneous of degree $w - \langle\alpha\rangle$. If we further assume that $f(x)$ is smooth and we choose α so that $\langle\alpha\rangle > w$, then, for all $t > 0$,

$$\partial^\alpha f(x) = t^{\langle\alpha\rangle - w} \partial^\alpha f(t \cdot x) \longrightarrow 0 \cdot \partial^\alpha f(0) = 0 \qquad \text{as } t \to 0.$$

Thus all the partial derivatives $\partial^\alpha f(x)$, $|\langle\alpha\rangle| > w$, vanish. It then follows that $f(x)$ must be polynomial function and w must be a nonnegative integer.

In what follows, by C^∞-topology on functions we mean the topology of uniform covergence on compact subsets of $\mathbb{R}^n$ of the functions and their partial derivatives of all orders.

Lemma 9.5.5. *Let $f(x)$ be a smooth function near $x = 0$ in $\mathbb{R}^n$. Then the following are equivalent:*

1. *f has weight w.*

2. *With respect to the C^∞-topology, there is an asymptotic expansion,*

$$f(t \cdot x) \simeq \sum_{l \geq w} t^l f^{[l]}(x) \qquad \text{as } t \to 0, \tag{9.30}$$

where $f^{[l]}(x)$ is a weight-homogeneous polynomial of degree l with $f^{[w]} \neq 0$.

Proof. Let $N \in \mathbb{N}$. By Taylor's formula there are smooth functions $R_{N\alpha}(x)$, $|\alpha| = N$, such that

$$f(x) = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \partial_x^\alpha f(0) x^\alpha + \sum_{|\alpha| = N+1} x^\alpha R_{N\alpha}(x),$$

As the monomial x^α is weight-homogeneous of degree $\langle\alpha\rangle$, we get

$$f(t\cdot x) = \sum_{|\alpha|\leq N} t^{\langle\alpha\rangle}\frac{1}{\alpha!}\partial_x^\alpha f(0)x^\alpha + t^{N+1}\sum_{|\alpha|=N+1} x^\alpha R_{N\alpha}(t\cdot x).$$

The smoothness of $R_{N\alpha}(x)$ implies that $R_{N\alpha}(t\cdot x)$ is $\mathrm{O}(1)$ with respect to the C^∞-topology as $t\to 0$. Therefore, we see that, with respect to the C^∞-topology,

$$f(t\cdot x) = \sum_{\langle\alpha\rangle\leq N} t^{\langle\alpha\rangle}\frac{1}{\alpha!}\partial_x^\alpha f(0)x^\alpha + \mathrm{O}(t^{N+1}) \qquad \text{as } t\to 0. \tag{9.31}$$

For $l = 0, 1, \ldots$ set $f^{[l]}(x) \simeq \sum_{\langle\alpha\rangle=l}\frac{1}{\alpha!}\partial_x^\alpha f(0)x^\alpha$. Then $f^{[l]}(x)$ is a weight-homogeneous polynomial of degree l. Moreover (9.31) shows that,with respect to the C^∞-topology,

$$f(t\cdot x) \simeq \sum_{l\geq 0} t^l f^{[l]}(x) \qquad \text{as } t\to 0.$$

This will be an asymptotic of the form (9.30) if and only if $f^{[l]} = 0$ for $l < w$ and $f^{[w]} \neq 0$. That is, $\partial_x^\alpha f(0) = 0$ for $\langle\alpha\rangle < w$ and $\partial_x^\alpha f(0) \neq 0$ for some multi-order α with $\langle\alpha\rangle = w$, i.e., f has weight w. The proof is complete. □

The notion of weight of a function extends to differential operators as follows. Given a differential operator P, for $t > 0$ we denote by $\delta_t^* P$ the pulback of P by the dilation δ_t, i.e., $(\delta_t^* P)u = (P(u\circ\delta_t^{-1}))\circ\delta_t$.

Definition 9.5.6. A differential operator P is weight-homogeneous of degree w when

$$\delta_t^* P = t^\omega P \qquad \forall t > 0.$$

Examples 9.5.7. For any multiorder α, the differential operator ∂^α is weight-homogeneous of degree $\langle\alpha\rangle$.

Definition 9.5.8. Let $P = \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha$ be a differential operator of order $\leq m$ in an neighborhood of the origin, where the $a_\alpha(x)$ are smooth functions. We say that P has weight w when

1. Each coefficient $a_\alpha(x)$ has weight $\geq w + \langle\alpha\rangle$.

2. There is one coefficient $a_\alpha(x)$ which has weight $w + \langle\alpha\rangle$.

Remark 9.5.9. The above notion of weight induces a notion of weight for vector fields considered as first order differential operators.

In what follows, by C^∞-topology on differential operators of order m, we mean the topology of uniform convergence of the coefficients and their derivatives on compact subsets of $\mathbb{R}^n$.

Lemma 9.5.10. *Let P be a differential operator of order m near the origin. Then the following are equivalent:*

1. *P has weight w.*

2. *With respect to the C^∞-topology, there is an asymptotic expansion,*

$$\delta_t^* P \simeq \sum_{l \geq w} t^l P^{[l]} \qquad \text{as } t \to 0, \tag{9.32}$$

where $P^{(l)}$ is a weight-homogeneous differential operator of degree l with $P^{[w]} \neq 0$.

Proof. Let us write $P = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha$, where the $a_\alpha(x)$ are smooth functions. Then

$$\delta_t^* P = \sum_{|\alpha| \leq m} a_\alpha(t \cdot x) \delta_t^* \partial_x^\alpha = \sum_{|\alpha| \leq m} t^{-w_j} a_j(t \cdot x) \partial_x^\alpha.$$

For $\alpha \in \mathbb{N}_0^N$, $|\alpha| \leq m$, let us denote by $w(\alpha)$ the weight of the function $a_\alpha(x)$. Using Lemma 9.5.5 we see that, with respect to the C^∞-topology,

$$\delta_t^* P \simeq \sum_{|\alpha| \leq m} \sum_{l_\alpha \geq w(\alpha)} t^{l_\alpha - w_\alpha} a_\alpha^{[l_\alpha]}(x) \partial_x^\alpha \qquad \text{as } t \to 0. \tag{9.33}$$

We note that $a_\alpha^{[l_\alpha]} \partial_x^\alpha$ is a weight-homogeneous differential operator of degree $l_\alpha - w_\alpha$. Therefore, the asymptotic (9.33) is of the form (9.32) if and only if $w(\alpha) - w_\alpha \geq w$ for all multiorders α, $|\alpha| \leq m$, and there is one such multiorder such that $w(\alpha) - w_\alpha = w$. This means that each coefficient $a_\alpha(x)$ has weight $\geq w + w_j$ and there is equality for at least one of those, that is, the differential operator P has weight w. The proof is complete. □

Remark 9.5.11. In the above proof $a_\alpha^{[l]}(x) = \sum_{\langle \beta \rangle = l} \frac{x^\beta}{\beta!} \partial^\beta a_\alpha(0)$. Therefore, we see that the weight w of P is given by

$$w = \min\{\langle \beta \rangle - \langle \alpha \rangle;\ \partial^\beta a_\alpha(0) \neq 0\}.$$

Suppose that the differential operator P does not vanish at $x = 0$. Then we see that

$$\min\{-\langle \alpha \rangle;\ |\alpha| \leq m\} \leq w \leq \min\{-\langle \alpha \rangle;\ a_\alpha(0) \neq 0\}$$

In particular, the weight w is always a negative integer.

Remark 9.5.12. The asymptotic (9.5.11) implies that, with respect to the C^∞-topology,

$$\lim_{t \to 0} t^{-w} \delta_t^* P = P^{[w]}.$$

From now on we let $X_1, \ldots, X_n$ be an H-frame near $a \in M$ and consider local coordinates $(x_1, \ldots, x_n)$ centered at a that are linearly adapated to this H-frame. Given any function, vector field or differential operator near a, we may define its weight in the local coordinates $(x_1, \ldots, x_n)$.

Definition 9.5.13. Let X be a vector field near a such that $X(a) \neq 0$. Let w be the weight of X in the local coordinates $(x_1, \ldots, x_n)$. Then the leading vector field $X^{[w]}$ in the asymptotic expansion (9.32) is called the weight-homogeneous approximation of X in the local coordinates $(x_1, \ldots, x_n)$ and is denoted $X^{(a)}$.

Remark 9.5.14. It follows from Remark 9.5.11 that the weight of a vector field with $X(a) \neq 0$ is always contained in $\{-r, -r+1, \ldots, -1\}$.

Remark 9.5.15. The definition of $X^{(a)}$ means that, in the local coordinates $(x_1, \ldots, x_n)$ and with respect to the C^∞-topology of vector fields,

$$\lim_{t \to 0} t^{-w} \delta_t^* X = X^{(a)}. \tag{9.34}$$

This provides us with an alternative notion of model vector field, which is extrinsic since it depends on the choice of the local coordinates $(x_1, \ldots, x_n)$.

Lemma 9.5.16. *Let $f \in C^\infty(M)$. Then, in any privileged coordinates $(x_1, \ldots, x_n)$ centered at a, the weight of $f(x)$ agrees with its order.*

Proof. Let us work in the privileged coordinates $(x_1, \ldots, x_n)$, so that, for $j = 1, \ldots, n$, the coordinate function x_j has order w_j. Let w be the weight of $f(x)$ in these coordinates. As $\partial^\alpha f(0) = 0$ for $\langle \alpha \rangle < w$, by Taylor's formula there are smooth functions $R_{N\alpha}(x)$, $|\alpha| = w' + 1$, such that

$$f(x) = \sum_{|\alpha| \leq w} \frac{1}{\alpha!} \partial^\alpha f(0) x^\alpha + \sum_{|\alpha| = w+1} x^\alpha R_{N\alpha}(x) = \sum_{\langle \alpha \rangle = w} \frac{1}{\alpha!} \partial^\alpha f(0) x^\alpha + \sum_{|\alpha| = w+1} x^\alpha R_{N\alpha}(x). \tag{9.35}$$

We note that by Lemma 9.4.6 that each monomial x^α has order $\langle \alpha \rangle$ and each term $x^\alpha R_{N\alpha}(x)$, $|\alpha| = w' + 1$, has order $\geq \langle \alpha \rangle \geq |\alpha| = w + 1$. Therefore, we see that $f(x)$ is a linear combination of functions of order $\geq w$, and hence has order $\geq w$.

Set $k = \max\{|\alpha|;\ \partial^\alpha f(0) \neq 0 \text{ and } \langle \alpha \rangle = w\}$. Then (9.35) can be rewritten as

$$f(x) = \sum_{\substack{\langle \alpha \rangle = w \\ |\alpha| \geq k}} \frac{1}{\alpha!} \partial^\alpha f(0) x^\alpha + \sum_{|\alpha| = w+1} x^\alpha R_{N\alpha}(x).$$

Let α be a multiorder such that $|\alpha| = k$, $\langle \alpha \rangle = w$, and $\partial^\alpha f(0) \neq 0$. Then using Lemma 9.4.6 we get

$$X^\alpha f(0) = \sum_{\substack{\langle \beta \rangle = w \\ |\beta| = k}} \frac{1}{\beta!} \partial^\beta f(0)\ \partial^\alpha (x^\beta)\Big|_{x=0} = \frac{1}{\alpha!} \partial^\alpha f(0) \neq 0.$$

It then follows that f has order w at a. This proves the lemma. □

Remark 9.5.17. The notion of weight can be defined relatively to any system of local coordinates centered at a. However, this an extrinsic notion, as a function may have different weights depending on the choice of the local coordinates. The previous proposition precisely says that, in the case of privileged coordinates, the extrinsic notion of weight agrees with the intrinsic notion of order.

Proposition 9.5.18. *Let $(x_1, \ldots, x_n)$ be local coordinates centered at a that are linearly adapted to the H-frame $(X_1, \ldots, X_n)$. Then the following are equivalent*

(i) $(x_1, \ldots, x_n)$ are privileged coordinates at a.

(ii) For all $j = 1, \ldots, n$, the vector field X_j has weight $-w_j$ in the local coordinates $(x_1, \ldots, x_n)$.

(iii) For all multiorders $\alpha \in \mathbb{N}_0^n$, the differential operator has weight $-\langle\alpha\rangle$ in the local coordinates $(x_1, \ldots, x_n)$.

Moreover, if X_j has weight $-w_j$, then its weight-homogeneous approximation in the local coordinates $(x_1, \ldots, x_n)$ takes the form,

$$X_j^{(a)} = \partial_j + \sum_{\substack{w_k - \langle\alpha\rangle = w_j \\ w_k > w_j}} b_\alpha x^\alpha \partial_k, \qquad b_\alpha \in \mathbb{R}. \tag{9.36}$$

Proof. If $X = \sum a_j(x)\partial_j$ is a vector field near the origin, then $X(x_j) = \sum a_k(x)\partial_k(x_j) = a_j(x)$ for $j = 0, \ldots, n$. More generally, if $P = \sum a_\alpha(x)\partial^\alpha$ is a differential operator near the origin such that $P(0) = 0$, then $a_\alpha(x) = P(x^\alpha)$ for $|\alpha| = 1$. Bearing this in mind, assume that $(x_1, \ldots, x_n)$ are privileged coordinates at a. The coefficient of ∂_k of X_j is $X_j(x_k)$ and has order $w_k - w_j$, and hence has weight $w_k - w_j$ by Lemma 9.5.16. It then follows that X_j has weight $-w_j$. This shows that (i) implies (ii).

Suppose that for $j = 1, \ldots, n$ the vector field X_j has weight w_j. Set $X_j = \sum a_{jk}(x)\partial_k$, for some smooth functions $a_{jk}(x)$. The fact that the coordinates $(x_1, \ldots, x_n)$ are linearly adapted at a means that $a_{jk}(0) = \delta_{jk}$. Therefore, the weight homogeneous approximation of X_j is given by

$$X_j^{(a)} = \sum_{w_k - \langle\alpha\rangle = w_j} \frac{1}{\alpha!}\partial^\alpha a_{jk}(0) x^\alpha \partial_k = \partial_j + \sum_{\substack{w_k - \langle\alpha\rangle = w_j \\ w_k > w_j}} \frac{1}{\alpha!}\partial^\alpha a_{jk}(0) x^\alpha \partial_k, \tag{9.37}$$

which proves (10.159). Let $\alpha \in \mathbb{N}_0^n$. It follows from (9.34) that, as $t \to 0$ and with respect to the C^∞-topology, we have

$$t^{\langle\alpha\rangle}\delta_t^* X^\alpha = (t^{w_1}\delta_t^* X_1)^{\alpha_1} \cdots (t^{w_n}\delta_t^* X_n)^{\alpha_n} \longrightarrow \left(X_1^{(a)}\right)^{\alpha_1} \cdots \left(X_n^{(a)}\right)^{\alpha_n}.$$

As $\left(X_1^{(a)}\right)^{\alpha_1} \cdots \left(X_n^{(a)}\right)^{\alpha_n} \neq 0$ this shows that X^α has weight $-\langle\alpha\rangle$. Thus (ii) implies (iii).

It remains to show that (iii) implies (i). Suppose that, for each $\alpha \in \mathbb{N}_0^n$, the differential operator X^α has weight $-\langle\alpha\rangle$. This implies that, for $j = 1, \ldots, n$, the coefficient of ∂_j, namely $X^\alpha(x_j)$, has weight $\geq w_j - \langle\alpha\rangle$, and hence vanishes at $x = 0$ if $\langle\alpha\rangle < w_j$. Thus x_j has order$\geq -w_j$. Moreover, as $X_j = \partial_j$ at $x = 0$ we have $X_j(x_j) = 1$ at $x = 0$. Therefore, x_j has order w_j, and so $(x_1, \ldots, x_n)$ are privileged coordinates. This shows that (iii) implies (i). The proof is complete. □

Definition 9.5.19. Given privileged coordinates at a relatively to the H-frame $(X_1, \ldots., X_n)$, we denote by $\mathfrak{g}^{(a)}$ the subspace of $T\mathbb{R}^n$ spanned by the weight-homogeneous weight vector fields $X_j^{(a)}$, $j = 1, \ldots, n$.

For $w = 1, \ldots, r$ let us denote by $\mathfrak{g}_w^{(a)}$ the subspace of $\mathfrak{g}^{(a)}$ spanned by vector fields $X_j^{(a)}$, $w_j = w$. As these vector fields are precisely the vector fields among $X_1^{(a)}, \ldots, X_n^{(a)}$ that are weight-homogeneous of degree $-w$, we see that, for any $X \in \mathfrak{g}^{(a)}$,

$$X \in \mathfrak{g}^{(a)} \Longleftrightarrow \delta_t^* X = t^{-w} X \quad \forall t > 0.$$

Moreover, we have the grading,

$$\mathfrak{g}^{(a)} = \mathfrak{g}_1^{(a)} \oplus \cdots \oplus \mathfrak{g}_r^{(a)}, \tag{9.38}$$

As $(X_1, \ldots, X_n)$ is an H-frame, there are smooth functions $L_{ij}^k(x)$, $w_k \leq w_i + w_j$, defined near a such that

$$[X_j, X_k] = \sum_{w_k \leq w_i + w_j} L_{ij}^k(x) X_k. \tag{9.39}$$

Lemma 9.5.20. *For $i, j = 1, \ldots, n$, we have*

$$[X_i^{(a)}, X_j^{(a)}] = \begin{cases} \displaystyle\sum_{w_k = w_i + w_j} L_{ij}^k(a) X_k^{(a)} & \text{if } w_i + w_j \leq r, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. With respect to the C^∞-topology, we have

$$[X_i^a, X_j^a] = \lim_{t\to 0} [t^{w_i} \delta_t^* X_i, t^{w_j} \delta_t^* X_j] = \lim_{t\to} t^{w_i + w_j} \delta_t^* [X_i, X_j].$$

Combining this (9.39) we get

$$[X_i^a, X_j^a] = \sum_{w_k \leq w_i + w_j} \lim_{t\to 0} t^{w_i + w_j} \delta_t^* (L_{ij}^k X_k) = \sum_{w_k \leq w_i + w_j} L_{ij}^k(a) \lim_{t\to 0} t^{w_i + w_j} \delta_t^* X_k.$$

Note that $\lim_{t\to 0} t^{w_i + w_j} \delta_t^* X_k = X_k^a$ if $w_k = w_i + w_j$ and $\lim_{t\to 0} t^{w_i + w_j} \delta_t^* X_k = 0$ if $w_k < w_i + w_j$. Therefore, $[X_i^a, X_j^a]$ is equal to $\sum_{w_k = w_i + w_j} L_{ij}^k(a) X_k^{(a)}$ if $w_i + w_j \leq r$ and is zero otherwise. The proof is complete. □

As an immediate consequence of Lemma 9.5.20 we obtain the following result.

Proposition 9.5.21. *With respect to the Lie bracket of vector fields, $\mathfrak{g}^{(a)}$ is a step r nilpotent Lie subalgebra of $T\mathbb{R}^n$. Moreover, the grading (9.1) is a Lie algebra grading, i.e.,*

$$[\mathfrak{g}^{(a)}_w, \mathfrak{g}^{(a)}_{w'}] \subset \mathfrak{g}^{(a)}_{w+w'} \qquad \text{for } w + w' \geq -r.$$

In fact, it follows from (9.6) and Lemma 9.5.20 that the Lie algebras $\mathfrak{g}_a M$ and $\mathfrak{g}^{(a)}$ have the same structure constants with respect to their respective bases $\{\dot{X}_j(a)\}$ and $\{X^{(a)}_j\}$. Therefore, we arrive at the following statement.

Proposition 9.5.22. *Given privileged coordinates at a relatively to the H-frame $(X_1, \ldots, X_n)$, define the linear isomorphism $\hat{L}_a : \mathfrak{g}_a M \to \mathfrak{g}^{(a)}$ by*

$$\hat{L}_a \left(x_1 \dot{X}_1(a) + \cdots + x_n \dot{X}_n(a) \right) = x_1 X^{(a)}_j + \cdots + x_n X^{(a)}_n \qquad \forall x_j \in \mathbb{R}. \tag{9.40}$$

Then $\hat{L}_a$ is a Lie algebra isomorphism from $\mathfrak{g}_a M$ onto $\mathfrak{g}^{(a)}$.

As $\mathfrak{g}^{(a)}$ is a Lie algebra of vector fields, it is natural to realize it as a Lie algebra of left-invariant vector field over a nilpotent Lie group $G^{(a)}$. As a manifold $G^{(a)}$ is $\mathbb{R}^n$. We define the Lie group structure on $G^{(a)}$ by using the Campbell-Hausdorff formula (9.8) and the exponential map $\exp^{(a)} : \mathfrak{g}^{(a)} \to G^{(a)}$ given by

$$\exp^{(a)}(x_1 X^{(a)}_1 + \cdots + x_n X^{(a)}_n) = \exp(x_1 X^{(a)}_1 + \cdots + x_n X^{(a)}_n)(0). \tag{9.41}$$

Here $\exp(x_1 X^{(a)}_1 + \cdots + x_n X^{(a)}_n)(0) = \exp(tX)(0)\big|_{t=1}$, where $\exp(tX)$ is the flow of the vector field $X = x_1 X^{(a)}_1 + \cdots + x_n X^{(a)}_n$, i.e., the solution of the initial-value problem,

$$\partial_t \exp(tX)(x) = X\left(\exp(tX)(x)\right), \qquad \exp(tX)(x)\big|_{t=0} = x.$$

Definition 9.5.23. $G^{(a)}$ is called the extrinsic tangent group at a in the privileged coordinates $x = (x_1, \ldots, x_n)$.

9.6 Carnot Coordinates

In this section, we shall refine the construction of the privileged to get a system of privileged coordinates in which notion of extrinsic and intrinsic model vector fields agree. In this section, we keep on using the notation of the previous sections. In particular, a is a point of M and $(X_1, \ldots, X_n)$ is an H-frame near a.

Following is the precise definition of Carnot privileged coordinates.

Definition 9.6.1. Local coordinates $(x_1, \ldots, x_n)$ centered at a are called Carnot (privileged) coordinates at a adapted to the H-frame $(X_1, \ldots, X_n)$ when

1. They are privileged coordinates at a adapted to $(X_1, \ldots, X_n)$.

2. In these coordinates, for any vector field X near a, the extrinsic model vector field $X^{(a)}$ agrees with the intrinsic model vector field X^a.

We shall now explain how to construct Carnot coordinates. The idea is to compose privileged coordinates with a Lie group isomorphism from the extrinsic tangent group $G^{(a)}$ onto the intrinsic tangent group G_aM. Let $(x_1, \ldots, x_n)$ be privileged coordinates at a adapted to the H-frame $(X_1, \ldots, X_n)$. In the previous section, we constructed $G^{(a)}$ associated to the Lie algebra $\mathfrak{g}^{(a)}$ spanned by the vector fields $X_j^{(a)}$ defined by (9.34). More precisely, as a manifold $G^{(a)}$ is just $\mathbb{R}^n$ and its product law is obtained by using the Campbell-Hausdorff formula and the exponential map $\exp^{(a)} : \mathfrak{g}^{(a)} \to G^{(a)}$ given by (9.41). We then define the diffeomorphism $\hat{\varepsilon}_a : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\hat{\varepsilon}_a = \left(\xi^{(a)}\right)^{-1} \circ \left(\exp^{(a)}\right)^{-1}, \tag{9.42}$$

where $\xi^{(a)} : \mathbb{R}^n \to \mathfrak{g}_a$ is the coordinate map,

$$\xi^{(a)}(x_1, \ldots, x_n) = x_1 X_1^{(a)} + \cdots + x_n X_n^{(a)} \qquad \forall x_j \in \mathbb{R}^n.$$

Definition 9.6.2. A map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is weight-homogeneous of degree w, $w \in \mathbb{R}$ when

$$\phi \circ \delta_t = \delta_{t^w} \circ \phi \qquad \forall t > 0. \tag{9.43}$$

Remark 9.6.3. Let us write $\phi(x) = (\phi_1(x), \ldots, \phi_n(x))$. The condition (9.43) exactly means that, for all $j = 1, \ldots, n$, the j-th component ϕ_j is weight-homogeneous of degree w_j. In particular, in vew of Remark 9.5.4, we see that if ϕ is smooth and weight-homogeneous, then it must be a polynomial map.

Lemma 9.6.4. *The diffeomorphism $\hat{\varepsilon}_a$ is a degree 1 weight-homogeneous polynomial diffeomorphism such that $\hat{\varepsilon}_a'(0) = \mathrm{id}$.*

Proof. Let $(x_1, \ldots, x_n) \in \mathbb{R}$ and set $X = x_1 X_1^{(a)} + \cdots + x_n X_n^{(a)}$. In addition, let $\lambda > 0$. As $\delta_{\lambda^{-1}}^* X_j^{(a)} = \lambda^{w_j} X_j^{(a)}$, we get

$$(\hat{\varepsilon}_a)^{-1}(\lambda \cdot x) = \exp\left(\lambda^{w_1} x_1 X_1^{(a)} + \cdots + \lambda^{w_n} x_n X_n^{(a)}\right)(0) = \exp\left(\delta_{\lambda^{-1}}^* X\right)(0).$$

As $\exp\left(t\delta_{\lambda^{-1}}^* X\right) = \delta_{\lambda^{-1}}^*\left(\exp(tX)\right)$, we see that

$$(\hat{\varepsilon}_a)^{-1}(\lambda \cdot x) = \delta_{\lambda^{-1}}^*\left(\exp(X)\right)(0) = \delta_\lambda \circ \exp(X)(\lambda^{-1} \cdot 0) = \lambda \cdot (\hat{\varepsilon}_a)^{-1}(x).$$

This proves that $\hat{\varepsilon}_a$ is weight-homogeneous of degree 1. As $\hat{\varepsilon}_a$ is smooth this implies this is a polynomial map. In addition, as $X_j^{(a)} = \partial_{x_j}$ at $x = 0$, we get

$$\partial_{x_j}(\hat{\varepsilon}_a)^{-1}(0) = \partial_t \exp\left(tX_j^{(a)}\right)(0)\Big|_{t=0} = X_j^{(a)}\left(\exp\left(tX_j^{(a)}\right)\right)(0)\Big|_{t=0} = X_j^{(a)}(0) = \partial_{x_j}.$$

This shows that $\hat{\varepsilon}_a'(0) = \mathrm{id}$. The proof is complete. □

Remark 9.6.5. Given $x^0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$, it follows from (9.16) that the vector field $X = x_1^0 X_1^{(a)} + \cdots + x_n^0 X_n^{(a)}$ is of the form,

$$X = \sum_{1\leq j\leq n} x_j^0 \partial_j + \sum_{\substack{w_k = w_j + \langle\alpha\rangle \\ w_j < w_k}} b_{jk\alpha} x_j^0 x^\alpha \partial_k = \sum_{1\leq k\leq n} \Big(x_k^0 + \sum_{\substack{w_j + \langle\alpha\rangle = w_k \\ w_j < w_k}} b_{jk\alpha} x_j^0 x^\alpha \Big) \partial_k.$$

Set $x(t) = \exp(tX)(0)$. Then $x(t) = (x_1(t), \ldots, x_n(t))$ is a solution of the following ODE system,

$$\frac{d}{dt} x_k(t) = x_k^0 + \sum_{\substack{w_k = w_j + \langle\alpha\rangle \\ w_j < w_k}} b_{jk\alpha} x_j^0 x(t)^\alpha, \qquad x(0) = 0.$$

An induction on k then shows that $x_k(t)$ is of the form,

$$x_k(t) = t x_k^0 + \sum_{\substack{\langle\alpha\rangle = w_k \\ |\alpha|\geq 2}} c_{k\alpha} t^{|\alpha|} (x^0)^\alpha, \qquad c_{k\alpha} \in \mathbb{R}.$$

Setting $t = 1$ and noting that $x(1) = (\hat{\varepsilon}_a)^{-1}\left(x^0\right)$ provides us with an alternative proof of Lemma 9.6.4.

As shown in Section 9.3 the H-frame $(X_1, \ldots, X_n)$ defines a global coordinate system $x = (x_1, \ldots, x_n)$ on $\mathfrak{g}_a M$ and $G_a M$. Thus using this coordinate system we may regard the diffeomorphism $\hat{\varepsilon}_a$ as a map from $\mathbb{R}^n = G^{(a)}$ onto $G_a M$.

Lemma 9.6.6. *Under the above convention, the diffeomorphism $\hat{\varepsilon}_\alpha$ is a Lie group isomorphism from $G^{(a)}$ onto $G_a M$. Incidentally,*

$$(\hat{\varepsilon}_a)_* X_j^{(a)} = X_j^a \qquad \text{for } j = 1, \ldots, n.$$

Proof. Let $\xi_a : \mathbb{R}^n \to \mathfrak{g}_a M$ be the coordinate map defined by the frame $(X_1, \ldots, X_n)$, i.e.,

$$\xi_a(x_1, \ldots, x_n) = x_1 \dot{X}_1(a) + \cdots + x_n \dot{X}_n(a) \qquad \forall x \in \mathbb{R}^n,$$

where $\dot{X}_j(a)$ is the class of $X_j(a)$ in $\mathfrak{g}_a^{w_j} M$. Then $\exp_a \circ \xi_a = \xi_a$ defines coordinates on $G_a M = \mathfrak{g}_a M$. We need to show that $\xi_a \circ \hat{\varepsilon}_a$ is a Lie group isomorphism. To this end note that $\xi^{(a)} \circ (\xi_a)^{-1} = \hat{L}_a$, where $\hat{L}_a$ is the linear map defined by (9.40). Thus,

$$\xi_a \circ \hat{\varepsilon}_a = (\exp_a \circ \xi_a) \circ \left(\xi^{(a)}\right)^{-1} \circ \left(\exp^{(a)}\right)^{-1} = \exp_a \circ \hat{L}_a^{-1} \circ \left(\exp^{(a)}\right)^{-1}.$$

As Proposition 9.5.22 asserts that $\hat{L}_a$ is a Lie algebra isomorphism, it then follows that $\xi_a \circ \hat{\varepsilon}_a$ is a Lie group isomorphism from $G^{(a)}$ onto $G_a M$.

Finally, as $\hat{\varepsilon}_a'(0) = \mathrm{id}$ and $\hat{\varepsilon}_a$ is a Lie group isomorphism, we see that, for $j = 1, \ldots, n$, the vector field $(\hat{\varepsilon}_a)_* X_j^{(a)}$ is a left-invariant vector field on $G_a M$ which at $x = 0$ agrees with $\hat{\varepsilon}_a'(0)\left(X_j^a(0)\right) = X_j^{(a)}(0) = \partial_j$. Thus $(\hat{\varepsilon}_a)_* X_j^{(a)} = X_j^a$ for $j = 1, \ldots, n$. The proof is complete. □

The following shows the existence of Carnot coordinates at a

Lemma 9.6.7. *The change of local coordinates $y = \hat{\varepsilon}(x)$ provides us with Carnot coordinates at a adapted to H-frame $(X_1, \ldots, X_n)$.*

Proof. The coordinates $x = (x_1, \ldots, x_n)$ are privileged coordinates at a adapted to the H-frame $(X_1, \ldots, X_n)$. As $\hat{\varepsilon}_a'(0) = \mathrm{id}$, we see that $(\varepsilon_a)_* X_j(0) = \varepsilon_a'(0)\left(X_j(0)\right) = \partial_j$ for $j = 1, \ldots, n$. Therefore, the coordinates $y = \hat{\varepsilon}(x)$ are linearly adapted to the H-frame $(X_1, \ldots, X_n)$. In addition, the fact that the coordinates $x = (x_1, \ldots, x_n)$ are privileged coordinates at a implies that, for $j = 1, \ldots, n$, we have

$$\delta_t^* X_j = t^{-w_j} X_j^{(a)} + \mathrm{O}\left(t^{-w_j+1}\right) \qquad \text{as } t \to 0^+.$$

Thanks to Lemma 9.6.4 and Lemma 9.6.6 we know that $\hat{\varepsilon}_a$ is a degree 1 weight-homogeneous diffeomorphism such that $(\varepsilon_a)_* X_j^{(a)} = X_j^a$. Thus,

$$\delta_t^* (\varepsilon_a)_* X_j = (\varepsilon_a)_* \delta_t^* X_j = t^{-w_j} (\varepsilon_a)_* X_j^{(a)} + \mathrm{O}\left(t^{-w_j+1}\right) = t^{-w_j} X_j^a + \mathrm{O}\left(t^{-w_j+1}\right).$$

This shows that $y = \hat{\varepsilon}(x)$ are Carnot coordinates at a. The proof is complete. □

We shall now show that $\hat{\varepsilon}$ is unique degree 1 weight-homogenous change of coordinates that provides us with Carnot coordinates. To reach this end we need the following lemma.

Lemma 9.6.8. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a degree 1 weight-homogeneous diffeomorphism such that*

$$\phi_* X_j^a = X_j^a \qquad \textit{for } j = 1, \ldots, n.$$

Then $\phi = \mathrm{id}$.

Proof. As mentioned in Remark 9.5.4, the fact that $\phi(x)$ is smooth and weight-homogeneous implies that $\phi(x)$ is a polynomial map. Thus,

$$\phi(x) = \phi'(0)x + h(x),$$

where $h(x) = (h_1(x), \ldots, h_n(x))$ is of the form,

$$h_j(x) = \sum_{\substack{\langle\alpha\rangle = w_j \\ |\alpha| \geq 2}} a_{j\alpha} x^\alpha, \qquad a_{j\alpha} \in \mathbb{R}.$$

Moreover, as $X_j^a(0) = \partial_j$, $j = 1, \ldots, n$, the equality $\phi_* X_j^a = X_j^a$ at $x = 0$ gives $\phi'(0)_* \partial_j = \partial_j$. It then follows that $\phi'(0) = 1$. Thus,

$$\phi(x) = x + h(x).$$

Set $y = \phi(x)$. We note that the function $h_j(x)$, $j = 1, \ldots, n$, does not depend on the variables x_k with $w_k \geq w_j$, and so $\partial_{x_k} h_j = 0$ for $w_k \geq w_j$. Thus, for $j = 1, \ldots, n$,

$$\partial_{y_j} = \partial_{x_j} + \sum_{1 \leq k \leq n} \partial_{x_j} h_j(x) \partial_{x_k} = \partial_{x_j} + \sum_{w_k > w_j} \partial_{x_j} h_j(x) \partial_{x_k}. \tag{9.44}$$

Each vector field X_j^a is of the form,

$$X_j^a = \partial_{x_j} + \sum_{\substack{w_k - \langle\alpha\rangle = w_j \\ w_k > w_j}} b_{jk\alpha} x^\alpha \partial_{x_k}, \qquad b_{jk\alpha} \in \mathbb{R}.$$

Then the equality $\phi_* X_j^a = X_j^a$ exactly means that

$$\partial_{x_j} + \sum_{\substack{w_k - \langle\alpha\rangle = w_j \\ w_k > w_j}} b_{jk\alpha} x^\alpha \partial_{x_k} = \partial_{y_j} + \sum_{\substack{w_k - \langle\alpha\rangle = w_j \\ w_k > w_j}} b_{jk\alpha} \phi^{-1}(x)^\alpha \partial_{y_k}.$$

Combining this with (9.44) we get

$$\partial_{y_j} = \partial_{x_j} + \sum_{w_k > w_j} \partial_{x_j} h_j(x) \partial_{x_k} = \partial_{x_j} + \sum_{\substack{w_k - \langle\alpha\rangle = w_j \\ w_k > w_j}} b_{jk\alpha} \left\{ x^\alpha \partial_{x_k} - \phi^{-1}(x)^\alpha \partial_{y_k} \right\}.$$

We shall show by induction on w that

$$\partial_{x_j} h_k(x) = 0 \qquad \text{whenever } w_k - w_j \leq w. \tag{9.45}$$

For $w = 0$ this is a consequence of the fact that $h(x)$ has no linear component, i.e, $\partial_j h_k(x) = 0$ whenever $w_j = w_k$. Suppose now that (9.45) is true up to w. Consider integers j, k and l be such that $w_j < w_k \leq w_j + w + 1$ and $w_k < w_l \leq w_j + w + 1$. Then $w_l - w_k \leq (w_j + w + 1) - (w_j + 1) = w$. By the induction hypothesis this implies that $\partial_k h_l = 0$. In view of this we see that, for $w_j < w_k \leq w_j + w + 1$,

$$\partial_{y_k} = \partial_{x_k} + \sum_{w_l \geq w_j + w + 2} \partial_k h_l \partial_{x_l} = \partial_{x_k} \quad \operatorname{mod} \operatorname{Span}\left\{ \partial_{x_l};\ w_l \geq w_j + w + 2 \right\}.$$

Combining this with (9.44) we obtain

$$\sum_{w_k > w_j} \partial_{x_j} h_j(x) \partial_{x_k} = \sum_{\substack{w_k - \langle\alpha\rangle = w_j \\ w_j < w_k \leq w_j + w + 1}} b_{jk\alpha} \left\{ x^\alpha - \phi^{-1}(x)^\alpha \right\} \partial_{x_k} \ \operatorname{mod} \operatorname{Span}\left\{ \partial_{x_l};\ w_l \geq w_j + w + 2 \right\}. \tag{9.46}$$

In particular, if $w_k = w_j + 1$, then

$$\partial_{x_j} h_k(x) = \sum_{\substack{w_k - \langle\alpha\rangle = w_j \\ w_k = w_j + w + 1}} b_{jk\alpha} \left\{ x^\alpha - \phi^{-1}(x)^\alpha \right\}$$

By assumption $\partial_{x_k} h_l = 0$ whenever $w_l - w_k \leq w$. In particular, for $w_k \leq w + 1$ it holds that $\partial_{x_l} h_k = 0$ for $l = 1, \ldots, n$, i.e., $h_k = 0$ and $y_k = x_k$. This also implies that $\phi^{-1}(x)_k = x_k$ whenever $w_k \leq w + 1$, and hence $\phi^{-1}(x)^\alpha = x^\alpha$ whenever $\langle\alpha\rangle \leq w + 1$. Combining this with (9.46) then shows that $\partial_{x_j} h_k(x) = 0$ whenever $w_k = w_j + w + 1$. This proves that (9.45) holds up to $w + 1$. It then follows from (9.45) holds to w_n, i.e., $\partial_{x_j} h_k(x) = 0$ for all $j, k = 1, \ldots, n$. Therefore, $h(x) = 0$, and so $\phi = \mathrm{id}$. The lemma is proved. □

We are now in a position to prove the first main result of this section.

Proposition 9.6.9. *Assume that $x = (x_1, \ldots, x_n)$ are privileged coordinates at a adapted to the H-frame $(X_1, \ldots, X_n)$. Then the coordinate change $y = \hat{\varepsilon}_a(x)$ is the unique degree 1 weight-homogeneous coordinate change providing us with Carnot coordinates at a adapted to $(X_1, \ldots, X_n)$.*

Proof. The fact that the coordinate change $y = \hat{\varepsilon}_a(x)$ provides us with Carnot coordinates is the contents of Lemma 9.6.7. Let $y = \phi(x)$ be another degree 1 weight-homogeneous coordinate change providing us with Carnot coordinates at a adapted to $(X_1, \ldots, X_n)$. This means that, for $j = 1, \ldots, n$, we have $\phi_* X_j^{(a)} = X_j^a = (\hat{\varepsilon}_a)_* X_j^{(a)}$, and so $\left(\phi \circ \hat{\varepsilon}_a^{-1}\right)_* X_j^a = X_j^a$. As $\phi \circ \hat{\varepsilon}_a^{-1}$ is weight-homogeneous of dgree 1, it then follows from Lemma 9.6.8 that $\phi \circ \hat{\varepsilon}_a^{-1} = \mathrm{id}$, i.e., $\phi = \hat{\varepsilon}_a$. This shows that $y = \hat{\varepsilon}_a(x)$ is the unique degree 1 weight-homogeneous coordinate change providing us with Carnot coordinates at a adapted to $(X_1, \ldots, X_n)$. The proof is complete. □

Definition 9.6.10. Given local coordinates $x = (x_1, \ldots, x_n)$ near a, the diffeomorphism $\varepsilon_a : \mathbb{R}^n \to \mathbb{R}^n$ is given by the composition,

$$\varepsilon_a = \hat{\varepsilon}_a \circ \psi_a \circ T_a,$$

where ψ_a and T_a are as in Definition 9.4.13 and Lemma 9.4.8, and $\hat{\varepsilon}_a$ is defined by (9.42) relatively to the privilege coordinates defined by $\psi_a \circ T_a$.

Using Lemma 9.6.7 and Proposition 9.6.9 we arrive at the following statement.

Proposition 9.6.11. *Given local coordinates $x = (x_1, \ldots, x_n)$ near a, the coordinate change $y = \varepsilon_a(x)$ provides us with Carnot coordinates at a adapted to $(X_1, \ldots, X_n)$.*

In the same way as there are many privileged coordinates at a given point, there are also many Carnot coordinates. However, the following result provides us with a characterization of Carnot coordinates at a. Ultimately, it shows that the coordinate change $y = \varepsilon_a(x)$ is "minimal" among the coordinate changes providing us with Carnot coordinates.

Definition 9.6.12. Let $\Theta(x) = (\Theta_1(x), \ldots, \Theta_n(x))$ be a smooth map between open neighborhoods of the origin in $\mathbb{R}^n$. We shall say that $\Theta(x)$ is $\mathrm{O}_w(x^{\mathrm{w}+1})$ and write $\Theta(x) = \mathrm{O}_w(x^{\mathrm{w}+1})$ when, near $x = 0$, each takes the form,

$$\Theta_j(x) = \sum_{\langle\alpha\rangle = w_j+1} x^\alpha \theta_{j\alpha}(x), \qquad j = 1, \ldots, n, \tag{9.47}$$

where the $\theta_{j\alpha}(x)$ are smooth functions near $x = 0$.

Remark 9.6.13. Equivalently, the condition (9.47) means that

$$t^{-1} \cdot \Theta(t \cdot x) = \mathrm{O}(t) \qquad \text{as } t \to 0^+.$$

Proposition 9.6.14. *Let $x = (x_1, \ldots, x_n)$ be local coordinates and $y = \phi(x)$ be a smooth change of coordinates near the point a. Then the following are equivalent:*

1. *The coordinates $y = \phi(x)$ are Carnot coordinates at a adapted to the H-frame $(X_1, \ldots, X_n)$.*
2. *Near the point a the coordinate change $\phi(x)$ takes the form,*

$$\phi(x) = \varepsilon_a(x) + \Theta\left(\varepsilon_a(x)\right), \tag{9.48}$$

where $\Theta(x)$ is $\mathrm{O}_w(x^{\mathbf{w}+1})$ near $x = 0$.

Proof. Suppose that the coordinates $y = \phi(x)$ are Carnot coordinates at a adapted to the H-frame $(X_1, \ldots, X_n)$. In particular, they are privileged coordinates at a, and so, for $j = 1, \ldots, n$, the coordinate $y_j = \phi_j(x)$ has order w_j. Thus, by Proposition 9.5.18 in the privileged coordinates provided by ε_a the component ϕ_j has weight w_j. That is, it takes the form,

$$\phi_j \circ \varepsilon_a^{-1}(x) = \sum_{\langle\alpha\rangle = w_j} a_{j\alpha} x^\alpha + \sum_{\langle\beta\rangle = w_j+1} x^\beta \theta_{j\beta}(x),$$

where the $a_{j\alpha}$ are constants and the $\theta_{j\beta}(x)$ are smooth functions near $x = 0$. Define $\hat{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\hat{\phi}_j(x) = \sum_{\langle\alpha\rangle = w_j} a_{j\alpha} x^\alpha. \tag{9.49}$$

Then $\hat{\phi}$ is weight-homogeneous of degree 1 and we have

$$\phi \circ \varepsilon_a^{-1}(x) = \hat{\phi}(x) + \Theta(x), \tag{9.50}$$

where $\Theta(x)$ is $\mathrm{O}_w(x^{\mathbf{w}+1})$ near $x = 0$. Moreover, as $t \to 0$,

$$\delta_t^{-1} \circ \left[\phi \circ \varepsilon_a^{-1}\right] \circ \delta_t(x) = \hat{\phi}(x) + \mathrm{O}(t).$$

Therefore, for $j = 1, \ldots, n$, we have

$$\left(\left(\phi \circ \varepsilon_a^{-1}\right)_* X_j\right)^{(a)} = \lim_{t\to 0} (\delta_t)_* \left(\left(\phi \circ \varepsilon_a^{-1}\right)_* X_j\right) = \lim_{t\to 0} \left(\delta_t^{-1} \circ \left(\phi \circ \varepsilon_a^{-1}\right) \circ \delta_t\right)_* (\delta_t)_* X_j = \hat{\phi}_* X_j^{(a)}.$$

As ϕ provides us with Carnot privileged coordinates at a, we have $(\phi)_* X_j^{(a)} = X_j^a = (\varepsilon_a)_* X_j^{(a)}$. Thus, it holds that

$$\hat{\phi}_* X_j^a = (\phi \circ \varepsilon_a^{-1})_* X_j^a = (\phi)_* X_j^{(a)} = X_j^a \quad \text{for } j = 1, \ldots, n.$$

It then follows from Lemma 9.6.8 that $\hat{\phi} = \mathrm{id}$. Combining this with (9.50) we see that

$$\phi(x) = \varepsilon_a(x) + \Theta\left(\varepsilon_a(x)\right),$$

where $\Theta(x)$ is $\mathrm{O}_w(x^{\mathbf{w}+1})$ near $x = 0$.

Conversely, assume that ϕ is of the form (9.48) near a. Then $\phi'(a) = \varepsilon_a'(a) = \mathrm{id}$, and so $\phi_* X_j(0) = (\varepsilon_a)^* X_j(0) = \partial_j$ for $j = 1, \ldots, n$. Thus the coordinates $y = \phi(x)$ are linearly adapted to $(X_1, \ldots, X_n)$. In addition, the form (9.48) precisely means that

$$\delta_t^{-1} \circ \left(\phi \circ \varepsilon_a^{-1}\right) \circ \delta_t = \mathrm{id} + \mathrm{O}(t) \qquad \text{as } t \to 0^+.$$

Recall also that as ε_a provides us with Carnot coordinates $t^{w_j} \delta_t^* (\varepsilon_a)_* X_j = X_j^a + \mathrm{O}(t)$ as $t \to 0^+$. Therefore, for $j = 1, \ldots, n$, as $t \to 0^+$ we have

$$t^{w_j} \delta_t^* \phi_* X_j = t^{w_j} \left(\delta_t^{-1} \circ \left(\phi \circ \varepsilon_a^{-1}\right) \circ \delta_t\right)_* \delta_t^* (\varepsilon_a)_* X_j = (\mathrm{id} + \mathrm{O}(t))_* \left(X_j^a + \mathrm{O}(t)\right) = X_j^a + \mathrm{O}(t).$$

This shows that $y = \phi(x)$ are Carnot coordinates at a. The proof is complete. □

The fact that $\hat{\varepsilon}_a(x)$ is weight-homogenous of degree 1 and $\hat{\varepsilon}_a'(0) = \mathrm{id}$ exactly means that the components $\hat{\varepsilon}_{a,j}(x)$, $j = 1, \ldots, n$, are of the form,

$$\hat{\varepsilon}_{a,j}(x) = x_j + \sum_{\substack{\langle \alpha \rangle = w_j \\ |\alpha| \geq 2}} a_{j\alpha} x^\alpha, \qquad a_{j\alpha} \in \mathbb{R}.$$

Combining this with (9.28) it not hard to see that $(\hat{\varepsilon}_a \circ \psi_a)_j(x)$ is of the form,

$$(\hat{\varepsilon}_a \circ \psi_a)_j(x) = x_j + \sum_{\substack{\langle \alpha \rangle \leq w_j \\ |\alpha| \geq 2}} a_{j\alpha} x^\alpha, \qquad a_{j\alpha} \in \mathbb{R}.$$

Proposition 9.4.14 states unicity result for privileged coordinates. The following is a version of that result for Carnot coordinates.

Proposition 9.6.15. *The coordinates $y = \varepsilon_a(x)$ are the unique Carnot coordinates at a adapted to the H-frame $(X_1, \ldots, X_n)$ given by a change of variable of the form $y = \hat{\phi}(Tx)$, where T is an affine map such that $Ta = 0$ and $\hat{\phi}$ is a polynomial diffeomorphism of the form (9.49).*

Proof. Let $y = \phi(x)$ be Carnot coordinates at a adapted to $(X_1, \ldots, X_n)$ such that $\phi(x) = \hat{\phi}(Tx)$, where T is an affine map such that $Ta = 0$ and $\hat{\phi}$ is a polynomial diffeomorphism of the form (9.49). In the same way as in the proof of Proposition 9.4.14 it can be shown that $T = T_a$. Set $\hat{\phi}_a = \hat{\varepsilon}_a \circ \psi_a$, so that $\varepsilon_a = \hat{\phi}_a \circ T_a$. In order to complete the proof it is enough to show that $\hat{\phi} = \hat{\phi}_a$.

Note that $\phi \circ \varepsilon_a^{-1} = (\hat{\varphi} \circ T_a) \circ \left(\hat{\phi}_a \circ T_a\right)^{-1} = \hat{\phi} \circ \hat{\phi}_a^{-1}$. Moreover, as $y = \phi(x)$ are Carnot coordinates at a, it follows from Proposition 9.6.14 that

$$\hat{\phi} \circ \hat{\phi}_a^{-1}(x) = \hat{\phi} \circ \psi_a^{-1} \circ \hat{\varepsilon}_a^{-1}(x) = x + \Theta(x),$$

where $\Theta(x)$ is of the form (9.47).

Claim. *The diffeormorphisms of the form (9.49) form a subgroup of the diffeomorphism group of $\mathbb{R}^n$.*

Proof of the claim. Let φ and ψ be diffeomorphisms of the form (9.49), so that their components $\varphi_j(x)$ and $\psi_j(x)$, $j = 1, \ldots, n$, are of the form ,

$$\varphi_j(x) = x_j + \sum_{\substack{\langle\alpha\rangle \leq w_j \\ |\alpha| \geq 2}} a_{j\alpha} x^\alpha \qquad \text{and} \qquad \psi_j(x) = x_j + \sum_{\substack{\langle\beta\rangle \leq w_j \\ |\beta| \geq 2}} b_{j\beta} x^\beta,$$

where the $a_{j\alpha}$ and $b_{j\beta}$ are real constants. Note this implies that $\varphi_j(x) - x_j$ and $\psi_j(x) - x_j$ are polynomials in the variables x_k with $w_k < w_j$. Therefore,

$$\psi_j \circ \varphi(x) = x_j + \sum_{\substack{\langle\alpha\rangle \leq w_j \\ |\alpha| \geq 2}} a_{j\alpha} x^\alpha + \sum_{\substack{\langle\beta\rangle \leq w_j \\ |\beta| \geq 2}} b_{j\beta} \prod_{w_k < w_j} \Big(x_k + \sum_{\substack{\langle\alpha\rangle \leq w_k \\ |\alpha| \geq 2}} a_{k\alpha} x^\alpha \Big)^{\beta_k},$$

which is of the form (9.49). Moreover, the equation $\psi_j \circ \varphi(x) = x_j$ gives

$$\sum_{\substack{\langle\alpha\rangle \leq w_j \\ |\alpha| \geq 2}} a_{j\alpha} x^\alpha = - \sum_{\substack{\langle\beta\rangle \leq w_j \\ |\beta| \geq 2}} b_{j\beta} \prod_{w_k < w_j} \Big(x_k + \sum_{\substack{\langle\alpha\rangle \leq w_k \\ |\alpha| \geq 2}} a_{k\alpha} x^\alpha \Big)^{\beta_k},$$

This uniquely determines the coefficients $a_{j\alpha}$. (Note this implies that $a_{j\alpha} = 0$ if $w_j = 1$.) It then follows that the inverse map ψ^{-1} of the form (9.49). This completes the proof of the claim. □

Let us go back to the proof of Proposition 9.6.15. The above claim ensures us that $\hat{\phi} \circ \hat{\phi}_a^{-1}(x)$ is of the form (9.49). Moreover, we know from (9.49) that $\hat{\phi} \circ \hat{\phi}_a^{-1}(x) = x + \mathrm{O}_w(x^{\mathbf{w}+1})$. However, as $\hat{\phi} \circ \hat{\phi}_a^{-1}$ is of the form (9.49) this is possible only if $\hat{\phi} \circ \hat{\phi}_a^{-1}(x) = x$. Thus $\hat{\phi}(x) = \hat{\phi}_a(x)$. The proof is complete. □

Definition 9.6.16. Consider two Carnot manifolds (M, H) and (M', H') with subbundles $H_0 = \{0\} \subset H_1 \subset \cdots \subset H_{r-1} \subset H_r = TM$ and $H'_0 = \{0\} \subset H'_1 \subset \cdots \subset H'_{r-1} \subset H'_r = TM'$. Then a diffeomorphism ϕ from M onto M' is called a Carnot diffeormorphism when

$$\phi_*(H_j) = H'_j \quad \forall 1 \leq j \leq r. \tag{9.51}$$

If ϕ is a Carnot diffeomorphism, by the property $\phi_*(H_j) = H'_j$, we see that ϕ' induces a smooth vector bundle isomorphism $\bar{\phi}$ from H_j/H_{j-1} onto H'_j/H'_{j-1} for each $1 \leq j \leq r$.

Definition 9.6.17. For a Carnot diffeomorhpism from (M, H) onto (M', H') we define the tangent map $\phi'_H : \mathfrak{g}M = H_1/H_0 \oplus H_r/H_{r-1} \to \mathfrak{g}M' = H'_1/H'_0 \oplus \cdots \oplus H'_r/H'_{r-1}$ by

$$\phi'_H(m)(X_1 + X_2 + \cdots + X_r) = \hat{\phi}'(m)X_1 + \cdots + \hat{\phi}'(m)X_r \tag{9.52}$$

for any $m \in M$ and $X_j \in H_j/H_{j-1}$, $1 \leq j \leq r$.

Remark 9.6.18. It is easy to see that the vector bundle isomorphism ϕ'_H is an isomorphism of graded Lie group bundles from GM onto GM'.

Another consequence of Proposition 9.6.14 is the following approximation result for Carnot diffeomorphisms in Carnot coordinates.

Proposition 9.6.19. *Let ϕ be a Carnot diffeomorphism from (M, H) onto a Carnot manifold (M', H'). Let $a \in M$ and set $a' = \phi(a)$. Then, in any Carnot coordinates at a and a', the diffeomorphism $\phi(x)$ has the following behavior near $x = 0$,*

$$\phi(x) = \phi'_H(a)x + \mathrm{O}_w(x^{\mathbf{w}+1}),$$

where $\phi'_H(a) : G_aM \to G_{a'}M'$ is the tangent map (9.52).

Proof. Let $(X_1, \ldots, X_n)$ be an H-frame near a and $y = \phi_a(x)$ Carnot coordinates at a adapted to $(X_1, \ldots, X_n)$. Likewise let $X'_1, \ldots, X'_n$ be an H'-frame and $y' = \phi_{a'}(x)$ Carnot coordinates at a' adapted to $(X'_1, \ldots, X'_n)$. As $y' = \phi_{a'}(x)$ are privileged coordinates at b, for $j = 1, \ldots, n$, the component $y'_j = \phi_{a',j}(x)$ has order w_j at b. Note that, as ϕ is a Carnot diffeomorphism $\{\phi_* X_1, \ldots, \phi_* X_n\}$ is an H'-frame near a'. Therefore, it follows from Definition 9.4.1 that, for all $\alpha \in \mathbb{N}_0^n$ with $\langle \alpha \rangle < w_j$, it holds that

$$0 = (\phi_* X)^\alpha (\phi_{a',j})(b) = X^\alpha \left(\phi_{a',j} \circ \phi\right)\left(\phi^{-1}(a')\right) = X^\alpha (\phi \circ \phi_{a',j})(a).$$

Thus $\phi \circ \phi_{a',j}$ has order w_j at a, and so by Proposition 9.6.14 it has weight w_j in the privileged coordinates $y = \phi_a(a)$. It then follows that $\phi_{a'} \circ \phi \circ \phi_a^{-1}$ takes the form,

$$\phi_{a'} \circ \phi \circ \phi_a^{-1} = \hat{\phi}(x) + \mathrm{O}_w(x^{\mathbf{w}+1}), \tag{9.53}$$

where $\hat{\phi}$ is a polynomial map which is weight-homeogeneous of degree 1.

In order to complete the proof it remains to show that $\hat{\phi} = \phi'_H(a)$. Let $j \in \{1, \ldots, n\}$. We observe that (9.53) implies that

$$\delta_t^{-1} \circ (\phi_{a'} \circ \phi \circ \phi_a^{-1}) \circ \delta_t = \hat{\phi} + \mathrm{O}(t) \qquad \text{as } t \to 0^+.$$

Moreover, as ϕ_a gives rise to Carnot privileged coordinates at a, we have

$$t^{w_j} \delta_t^* \phi_{a*} X_j = X_j^a + \mathrm{O}(t) \qquad \text{as } t \to 0^+.$$

As $\delta_t^* (\phi_{a'} \circ \phi)_* X_j = \left(\delta_t^{-1} \circ (\phi_{a'} \circ \phi \circ \phi_a^{-1}) \circ \delta_t\right)_* \delta_t^*(\phi_a)_* X_j$ we then deduce that

$$t^{w_j} \delta_t^* (\phi_{a'} \circ \phi)_* X_j = \left(\hat{\phi} + \mathrm{O}(t)\right)_* \left(X_j^a + \mathrm{O}(t)\right) = \hat{\phi}_* X_j^a + \mathrm{O}(t). \tag{9.54}$$

Note also that as $\phi_{a'}$ provides us with Carnot privileged coordinates at a', we have

$$t^{w_j} \delta_t^* (\phi_{a'} \circ \phi)_* X_j = t^{w_j} \delta_t^* (\phi_{a'})_* \left(\phi_* X_j\right)_* = \left(\phi_* X_j\right)^{a'} = \phi'_H(a)_* X_j^a.$$

Combining this with (9.54) shows that

$$(\hat{\phi} \circ \phi'_H(a)^{-1})_* X_j^a = X_j^a \qquad \text{for } j = 1, \ldots, n.$$

As $\hat{\phi} \circ \phi'_H(a)^{-1}$ is a weight-homogeneous diffeomorphism of degree 1, it then follows from Lemma 9.6.8 that $\hat{\phi} \circ \phi'_H(a)^{-1} = \mathrm{id}$, i.e., $\hat{\phi} = \phi'_H(a)$. This completes the proof. □

Remark 9.6.20. Bellaiche [Be, Prop. 7.29] proved a somewhat similar approximation result for Carnot diffeomorphisms in privileged coordinates. However, in this case the first order approximation need not agree with the tangent map $\phi'_H(a)$ and is a Lie algebra isomorphism between $\mathfrak{g}_a M$ and $\mathfrak{g}_{a'} M'$, rather than a Lie group isomorphism from $G_a M$ onto $G_{a'} M'$.

Finally, we look at the dependence of $\varepsilon_a(x)$ with respect to a. In the following proofs, we shall use the following notation: For given $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ we write $\mathrm{wt}(f) = w$ if $f(t \cdot p, t \cdot x) = t^w f(p, x) \quad \forall t > 0$, and $\mathrm{wt}(f) \geq w$ if $\sup_{|p|+|x| \leq 1} |f(t \cdot p, t \cdot x)| = O(t^w)$ as $t \to 0^+$. This notation means that we count the weight in both two variables.

Lemma 9.6.21. *Consider that we are given a privileged coordinates at a point $a_0 \in M^n$ and denote by $\cdot$ the group law of $G^{(a_0)}$. Suppose that the vector fields X_j equal to $X_j^{(a_0)}$ for each $1 \leq j \leq n$. Then the map $L_a(x) = a \cdot x$ provides a privileged coordinates at a. Moreover L_a^{-1} equals to the map $\psi_a \circ T_a$ defined in Proposition 9.4.14.*

Proof. From the assumption we may let $X_j = X_j^{(a_0)}$ in the proof. By (10.159) we have

$$X_j^{(a_0)}(x) = \sum_{j \leq i \leq n} b_{ji}(x) \frac{\partial}{\partial x_i}, \qquad j = 1, \ldots, n, \tag{9.55}$$

where smooth functions $b_{ji}(x)$ satisfy $b_{ii} \equiv 1$ for any $1 \leq i \leq n$ and $b_{ji}(0) = 0$ for $i \neq j$, and $\mathrm{wt}(b_{ji}(x)) = w_i - w_j$. On the other hand, as L_a is a left-translation map of the group $G^{(a_0)}$, it holds that

$$(L_a)_*(X_j^{(a_0)}(x)) = X_j^{(a_0)}(L_a(x)). \tag{9.56}$$

First, we take $x = 0$ here to see

$$(L_a)_* \left(\frac{\partial}{\partial x_j} \bigg|_0 \right) = X_j^{(a_0)}(a).$$

Thus the coordinates system changed by L_a is adapted to $X_j^{(a_0)}$ at the point a.

Next, combining (9.55) and (9.56) we find that

$$(L_a)_*(X_j^{(a_0)}(L_a^{-1}(x))) = \sum_{j \leq i \leq n} b_{ji}(x) \frac{\partial}{\partial x_i}, \qquad j = 1, \ldots, n,$$

and so the map $x \to L_a^{-1}(x)$ provides a privileged coordinates at a by the equivalence of (i) and (ii) in Proposition 9.5.18.

It remains to show that $L_a^{-1}(x) = a^{-1} \cdot x$ equals to $\psi_a \circ T_a(x)$. Letting $H_a(x) = L_a^{-1}(T_a^{-1}(x))$, it is equivalent to prove that $H_a(x) = \psi_a(x)$. For this aim, we only need to prove that $H_a(x)$ is of the form (9.28).

First we observe, from the homogeneity coming from the dilation law $(\lambda \cdot a^{-1}) \cdot (\lambda \cdot x) = \lambda \cdot (a^{-1} \cdot x)$ for $\lambda > 0$, that $(a^{-1} \cdot x)$ is of the form,

$$(a^{-1} \cdot x)_j = \sum_{\langle\alpha\rangle+\langle\beta\rangle=w_j} b_{\alpha,\beta}\, x^\alpha a^\beta = x_j - a_j + \sum_{\substack{\langle\alpha\rangle+\langle\beta\rangle=w_j \\ \langle\alpha\rangle<w_j}} b_{\alpha,\beta} x^\alpha a^\beta, \tag{9.57}$$

where $b_{\alpha\beta} \in \mathbb{R}$ and the second equality follows by testing the cases $a = 0$ and $x = 0$.

Next we know from Lemma 9.4.8 that $T_a(x) = A(x - a)$ where $A = \left(B(a)^T\right)^{-1}$ and $B(a) = (b_{ij}(a))$. Here we observe that we have the form $(B(a)^T x)_i = \sum_{k\leq i} b_{ki}(a)x_k$ from (9.55). Hence $B(a)^x + a$ is given by the form

$$(B(a)^T x + a)_i = \sum_{k\leq i} b_{ki}(a)x_k + a_i. \tag{9.58}$$

Plugging this into the position of x in (9.57), with observing that $(B(a)^T x + a)_i$ consists of terms whose weights are not greater than it of x_i and $H_a(0) = 0$, we can find that $H_a(x) = a^{-1} \cdot \left(B(a)^T x + a\right)$ is given by the form

$$(H_a(x))_j = \sum_{k\leq i} c_{jk} x_k + \sum_{\langle\alpha\rangle<w_j} d_\alpha x^\alpha \tag{9.59}$$

for some $c_{jk} \in \mathbb{R}$ and $d_\alpha \in \mathbb{R}$.

On the other hand, we note that

$$(L_a)_* \left(\frac{\partial}{\partial x_j}\bigg|_0 \right) = X_j(a) \quad \text{and} \quad (T^{-1})_* \left(\frac{\partial}{\partial x_j}\bigg|_0 \right) = X_j(a),$$

which implies

$$(H_a)_* \left(\frac{\partial}{\partial x_j}\bigg|_0 \right) = \frac{\partial}{\partial x_j}.$$

Hence $H_a(x)$ is of the form

$$(H_a(x))_j = x_j + \sum_{2\leq|\alpha|} a_{j\alpha} x^\alpha \tag{9.60}$$

for some $a_{j\alpha} \in \mathbb{R}$.

Now, combining (9.59) and (9.60) we can say that $H_a(x)$ is of the form,

$$(H_a(x))_j = x_j + \sum_{\substack{2\leq|\alpha| \\ \langle\alpha\rangle<w_j}} a_{j\alpha} x^\alpha,$$

which is exactly the form (9.28). Therefore we have $L_a^{-1}(T_a^{-1}(x)) = H_a(x) = \psi_a(x)$ by the uniqueness result of Proposition 9.4.12. Hence we get $L_a^{-1}(x) = \psi_a \circ T_a(x)$. The lemma is proved. □

In the next proposition, we shall see that the above result holds also true up to a small error for general vector fields. For this aim, it is convenient to introduce the notation $f(a,x) = O_w\left((a,x)^{\mathbf{w}+1}\right)$ for a function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $t^{-1} \cdot f(t \cdot a,\ t \cdot x) = (O(t), \cdots, O(t))$ as $t \to 0$. We remark that this is a natural variation of the notation $O_w(x^{\mathbf{w}+1})$ given in Definition 9.6.12.

Proposition 9.6.22. *Suppose that we are given a privileged coordinates* $(x_1, \cdots, x_n)$ *at some point* $a_0 \in M^n$ *and denote by* $\cdot$ *the group law of* $G^{(a_0)}$*. Then we have the following results.*

1. *The maps* $(a,x) \to \psi_a \circ T_a(x)$ *and* $(a,x) \to \varepsilon_a(x)$ *are smooth and we have*

$$\psi_a \circ T_a(x) = a^{-1} \cdot x + O_w((a,x)^{\mathbf{w}+1}), \tag{9.61}$$

2. *In the above, suppose further that the privileged coordinates provides a Carnot coordinates at the point* a_0*. Then we have*

$$\varepsilon_a(x) = a^{-1} \cdot x + O_w((a,x)^{\mathbf{w}+1}). \tag{9.62}$$

Proof. We shall first prove the smoothness result and we shall prove the asymptotic formula in the second part of the proof.

For proving the smoothness, with seeing that $\varepsilon_a(x) = \hat{\varepsilon}_a \circ \psi_a \circ T_a(x)$, it is enough to show that $\hat{\varepsilon}_a(x)$, $\psi_a(x)$ and $T_a(x)$ are smooth in a and x.

To show $T_a(x)$ is smooth, we recall from Lemma 9.4.8 that $T_a(x) = (B(a)^T)^{-1}(x-a)$ where $B(x) = (b_{ij})_{1\le i,j\le n} \in \mathrm{GL}_n(\mathbb{R})$ and the smooth functions $b_{ij}(x)$ are coefficients of the vector fields

$$X_i(x) = \sum_{1\le j\le n} b_{ij}(x)\frac{\partial}{\partial x_j}, \quad i = 1, \cdots, n. \tag{9.63}$$

As the vector fields consist a basis of the tangent space at any point and they are smooth, we find that the map $a \to (B(a)^T)^{-1}$ is smooth. From this we see that $T_a(x)$ is smooth with respect to a and x.

Next, in order to exploit $\psi_a(x)$ for each point $a \in M^n$, we work on the coordinates transformed by the map $x \to T_a(x)$ which is smooth also with respect to a. It implies importantly that, in this coordinates, we have $X_i(x) = \sum_{j=1}^n c_{ij}(a,x)\frac{\partial}{\partial x_j}$ for some functions $c_{ij}(a,x)$ which are smooth both in a and x. From Proposition 9.4.12 we see that $y_j = (\psi_a)_j(x)$ is given by

$$y_j = x_j + \sum_{\substack{\langle x\rangle < w_j \\ 2\le|\alpha|}} a_{j\alpha}(a)x^\alpha, \tag{9.64}$$

where $a_{j\alpha}(a)$ satisfies the formula (9.29),

$$\alpha! a_{j\alpha}(a) = -\left. X^\alpha(x_j)\right|_{x=a} - \sum_{\substack{\langle\beta\rangle < w_j \\ 2\le|\beta|<|\alpha|}} a_{j\beta}(a)\, \left. X^\alpha(x^\beta)\right|_{x=a}, \tag{9.65}$$

and it is not difficult to check that this inductive formula yields the smoothness of $a_{j\alpha}(a)$ with respect to a. Thus, the map $(a, x) \to \psi_a(x)$ is smooth in a and x.

Lastly, we shall now prove the smoothness of $\hat{\varepsilon}_a = (\xi^{(a)})^{-1} \circ (\exp^{(a)})^{-1}$. As in the previous step, we work on the coordinates system transformed by the map $x \to \psi_a \circ T_a(x)$. We recall from (9.41) that the exponential map is defined by

$$\exp^{(a)}(x_1 X_1^{(a)} + \cdots + x_n X_n^{(a)}) = \exp(x_1 X_1^{(a)} + \cdots + x_n X_n^{(a)})(0). \tag{9.66}$$

By (9.37) the vector field $X_j^{(a)}$ is given

$$X_j^{(a)}(x) = \sum_{w_k - \langle\alpha\rangle = w_j} \frac{1}{\alpha!} \partial^\alpha c_{jk}^a(0) x^\alpha \partial_k, \tag{9.67}$$

where $c_{jk}(x)$ are coefficients of the vector fields $X_j = \sum_k c_{jk}^a(x)\partial_k$. On the other hand, we know the map $(a, x) \to c_{jk}^a(x)$ is smooth in a and x since the coordinate change map $x \to \psi_a \circ T_a(x)$ is smooth with respect to a as we have just proved in the above. Therefore $a \to \partial^\alpha c_{jk}^a(0)$ is a smooth function of a, and combining this fact with (9.67) shows that $X_j^{(a)}(x)$ is smooth with respect to a and x. Hence the related exponential map $x \to \exp^{(a)} \circ(\xi^{(a)})(x)$ is smooth in a and x variables. Then, by the inverse function theorem, $\hat{\varepsilon}_a(x) = (\exp^{(a)} \circ(\xi^{(a)}))^{-1}$ is also smooth in a and x.

In the aboves, we have shown that all of $\hat{\varepsilon}_a(x)$, $\psi_a(x)$ and $T_a(x)$ are smooth in a and x. Thus $\varepsilon_a(x)$ is smooth in a and x.

Now we are only left to show the asymptotic formulas (9.61) and (9.62). For this aim, as the maps are formulated explicitly by the vector fields, we begin the proof with observing the vector fields first. Namely, given the privileged coordinates at a_0, we know by Proposition 9.5.18 that

$$\begin{aligned} X_j(a) &= \frac{\partial}{\partial x_j} + \sum_k F_{jk}(a) \frac{\partial}{\partial x_k} \\ &= \frac{\partial}{\partial x_j} + \sum_k F_{jk}^0(a) \frac{\partial}{\partial x_k} + \sum_k h_{jk}(a) \frac{\partial}{\partial x_k}, \end{aligned} \tag{9.68}$$

where F_{jk}^0 and h_{jk} satisfy $\mathrm{wt}(F_{jk}^0) = w_k - w_j$ and $\mathrm{wt}(h_{jk}) \geq w_k - w_j + 1$. Note that when $h_{jk} \equiv 0$ holds for any j and k, we have $\psi_a \circ T_a(x) = a^{-1} \cdot x$ by Lemma 9.6.21. Thus (9.61) holds in the case that $h_{jk} \equiv 0$ holds for all j and k.

In what follows, we denote by T_a^0 and ψ_a^0 the transforms T_a and ψ_a corresponding to the case $h_{jk} \equiv 0$. Then we just checked that $\psi_a^0 \circ T_a^0(x) = a^{-1} \cdot x$.

The idea to obtain the desired result for the the general case is to regard h_{jk} as a perturbation to the case $h_{jk} \equiv 0$ in which we know well. More precisely, using the condition $\mathrm{wt}(h_{jk}) \geq w_k - w_j + 1$ we shall show that

$$\psi_a \circ T_a(x) = \psi_a^0 \circ T_a^0(x) + O_w((a, x)^{\mathbf{w}+1}). \tag{9.69}$$

The only strategy is to observe the term by term expansion of the exact formulas of ψ_a and T_a which can be written using F_{jk}^0 and h_{jk} of (9.68), and to concern computing the *wt* of each term with the fact that $\mathrm{wt}(F_{jk}^0) = w_k - w_j$ and $\mathrm{wt}(h_{jk}) \geq w_k - w_j + 1$.

Claim 1. *We have* $T_a(x) = T^0_a(x) + O_w((a,x)^{\mathbf{w}+1})$.

Proof of Claim 1. Consider the matrix $B = (B_{jk})_{1\leq j,k\leq n}$ with $B_{jk}(a) = \delta_{jk} + F^0_{jk}(a) + h_{jk}(a)$ and let $A(a) = (B(a)^T)^{-1}$. Then, as in the proof of Lemma 9.4.8 we have

$$T_a(x) = A(a)(x-a). \tag{9.70}$$

By a direct matrix computation (See Lemma 9.A.1), we have

$$A_{kj}(a) = B^{-1}_{jk}(a) = \delta_{jk} + G^0_{jk}(a) + \widetilde{h}_{jk}(a),$$

where $G^0 = (G^0_{jk})_{1\leq j,k\leq n}$ satisfies $(I+F^0)(I+G^0) = I$ and $\widetilde{h}_{jk}$ satisfies $\operatorname{wt}(\widetilde{h}_{jk}) \geq w_k - w_j + 1$, and $\widetilde{h}_{jk} \equiv 0$ holds when $h_{\alpha\beta} \equiv 0$ for any α and β. Injecting this into (9.70) we get

$$\begin{aligned}(T_a)_j(x) &= \sum_k A_{jk}(a)(x_k - a_k)\\ &= \sum_k [\delta_{jk} + G^0_{kj}(a)](x_k - a_k) + \sum_k \widetilde{h}_{kj}(a)(x_k - a_k)\\ &= (T^0_a)_j(x) + \sum_k \widetilde{h}_{kj}(a)(x_k - a_k),\end{aligned} \tag{9.71}$$

and we note that

$$\operatorname{wt}(\widetilde{h}_{kj}(a)(x_k - a_k)) \geq \operatorname{wt}(\widetilde{h}_{kj}(a)) + \operatorname{wt}(x_k - a_k) \geq (w_j - w_k + 1) + w_k = w_j + 1.$$

This with (9.71) shows that $T_a(x) = T^0_a(x) + O_w((a,x)^{\mathbf{w}+1})$. □

Claim 2. *We have* $\psi_a \circ T_a(x) = \psi^0_a \circ T^0_a(x) + O_w((a,x)^{\mathbf{w}+1})$.

Proof of Claim 2. Let us work on the coordinates transformed by the map $x \to T_a(x)$. Then, by (9.27) we know

$$X_i(x) = \sum_{1\leq k\leq n}\Big(\sum_{1\leq j\leq n} B_{ij}(a + A^{-1}(a)\cdot x)A_{kj}(a)\Big)\frac{\partial}{\partial x_k}.$$

Using this and the forms of $B_{jk}(a)$ and $A_{jk}(a)$, we can see that X_i is of the form

$$X_i(x) = \frac{\partial}{\partial x_i} + \sum_k Q^0_{ik}(a,x)\frac{\partial}{\partial x_k} + \sum_k r_{ik}(a,x)\frac{\partial}{\partial x_k}, \tag{9.72}$$

where Q^0_{ik} satisfies $\operatorname{wt}(Q^0_{ik}(a,x)) = w_k - w_i$ and r_{ik} satisfies $\operatorname{wt}(r_{ik}(a,x)) \geq w_k - w_i + 1$, and $r_{ik} \equiv 0$ if $h_{\alpha\beta} \equiv 0$ holds for any α and β. On the other hand, from the proof of Proposition 9.4.12, we know that for $j = 1,\ldots,n$, the component $y_j = \psi_j(x)$ is given by the form,

$$y_j = x_j + \sum_{\substack{\langle\alpha\rangle < w_j\\ 2\leq|\alpha|}} a_{j\alpha}x^\alpha, \tag{9.73}$$

where $a_{j\alpha} \in \mathbb{R}$ are determined as (9.29) in the following relations

$$X^\alpha(y_j)\Big|_{x=0} = 0 \iff \alpha! a_{j\alpha} = -X^\alpha(x_j)\Big|_{x=0} - \sum_{\substack{\langle\beta\rangle < w_j \\ 2\leq|\beta|<|\alpha|}} a_{j\beta}\, X^\alpha(x^\beta)\Big|_{x=0}. \tag{9.74}$$

Let us denote by $a^0_{j\alpha}$ the values of $a_{j\alpha}$ corresponding to the case $h_{\alpha\beta} = 0\ \forall(\alpha,\beta)$. To see how $a_{j\alpha}$ is perturbed from $a^0_{j\alpha}$ for the general case, we just observe the expansion of $a_{j\alpha}$ explicitly obtained by applying (9.72) into (9.74). In that expansion, we note that any r_{ik} increases the *wt* at least one more than Q^0_{ik} by the fact that $\mathrm{wt}(r_{ik}(a,x)) \geq w_k - w_i + 1$ and $\mathrm{wt}(Q^0_{ik}(a,x)) = w_k - w_i$. Now we again note the important fact that $r_{ik} \equiv 0$ holds in the case that $h_{\alpha\beta} \equiv 0\ \forall(\alpha,\beta)$. Combining these two facts, one may see that $a_{j\alpha}$ is given by the form,

$$a_{j\alpha}(a) = a^0_{j\alpha}(a) + r_{j\alpha}(a), \tag{9.75}$$

with $r_{j\alpha}$ such that $\mathrm{wt}(r_{j\alpha}(a)) \geq w_j - \langle\alpha\rangle + 1$. Hence we have

$$(\psi_a)_j(x) = x_j + \sum_{\substack{\langle\alpha\rangle<w_j \\ 2\leq|\alpha|}} a^0_{j\alpha}(a)x^\alpha + \sum_{\substack{\langle\alpha\rangle<w_j \\ 2\leq|\alpha|}} r_{j\alpha}(a)x^\alpha = (\psi^0_a)_j(x) + \sum_{\substack{\langle\alpha\rangle<w_j \\ 2\leq|\alpha|}} r_{j\alpha}(a)x^\alpha,$$

and

$$\mathrm{wt}(r_{j\alpha}(a)x^\alpha) \geq \mathrm{wt}(r_{j\alpha}(a)) + \mathrm{wt}(x^\alpha) \geq (w_j - \langle\alpha\rangle + 1) + \langle\alpha\rangle = w_j + 1.$$

These means that $\psi_a(x) = \psi^0_a(x) + O_w((a,x)^{\mathbf{w}+1})$. □

By Claim 1 and Claim 2, and using the fact that $\mathrm{wt}(\psi^0_a(x)_j) = \mathrm{wt}(T^0_a(x)_j) = \mathrm{wt}(x_j)$, we finally get

$$\begin{aligned}\psi_a \circ T_a(x) &= (\psi^0_a(\cdot) + O_w((a,\cdot)^{\mathbf{w}+1})) \circ (T^0_a(x) + O_w((a,x)^{\mathbf{w}+1})) \\ &= \psi^0_a \circ T^0_a(x) + O_w((a,x)^{\mathbf{w}+1}).\end{aligned}$$

It completes the proof of (1).

To prove (2), we recall that $\varepsilon_a(x) = \widehat{\varepsilon}_a \circ \psi_a \circ T_a(x)$, where $\widehat{\varepsilon}_a$ is a group isomorphism which change a privileged coordinates to a Carnot coordinates. Since we are already on a Carnot coordinates at $a = 0$ by assumption, it holds that $\widehat{\varepsilon}_0 \equiv id$. This implies $\widehat{\varepsilon}_a(y) = y + r(a,y)$ where

$$(r(a,y))_j = \sum_{\langle\alpha\rangle = w_j} r_{j\alpha}(a)y^\alpha \tag{9.76}$$

for some $r_{j\alpha}(a) = O(a)$. Using this and the previous result of (1), we can deduce that

$$\begin{aligned}\widehat{\varepsilon}_a \circ \psi_a \circ T_a(x) &= \widehat{\varepsilon}_a\left(a^{-1}\cdot x + O_w((a,x)^{\mathbf{w}+1}\right) \\ &= \widehat{\varepsilon}_a(a^{-1}\cdot x) + O_w((a,x)^{\mathbf{w}+1}) \\ &= (1 + r(a,\cdot))(a^{-1}\cdot x) + O_w((a,x)^{\mathbf{w}+1}) \\ &= a^{-1}\cdot x + O_w((a,x)^{\mathbf{w}+1}),\end{aligned} \tag{9.77}$$

which proves (2). The proof is completed. □

Before finishing this section, we state the following lemma which is essential in the construction of Carnot groupoid with boundary topology in the next section.

Lemma 9.6.23. *In a given coordinates, we denote by ε_x a Carnot coordinates map at x. Next we take a point x_0 and change the coordinates system by the map ε_{x_0} and in the new coordinates system, for a given point X we find a Carnot coordinates map $\widetilde{\varepsilon}_X$ at the point X. Then, it holds that*

$$(\widetilde{\varepsilon}_X) \circ (\varepsilon_{x_0}) \circ (\varepsilon_{\varepsilon_{x_0}^{-1}(X)})^{-1}(y) = y + O_w(y^{\mathbf{w}+1}), \tag{9.78}$$

and

$$\varepsilon_{x_0} \circ \varepsilon^{-1}_{\varepsilon_{x_0}^{-1}(X)}(y) = (\widetilde{\varepsilon}_X)^{-1}(y) + O((X,y)^{\mathbf{w}+1}) = X \cdot y + O((X,y)^{\mathbf{w}+1}), \tag{9.79}$$

where $\cdot$ denotes the group law of $G_{x_0}M$ and the second equality is shown in Proposition 10.3.2.

Proof. In the coordinates which is given previously, we consider a Carnot coordinates map at point $\varepsilon_{x_0}^{-1}(X)$ and denote it by $\varepsilon_{\varepsilon_{x_0}^{-1}(X)}$. Next, we change the coordinates by the map ε_{x_0}, and we observe that the point recorded by $\varepsilon_{x_0}^{-1}(X)$ in the previous coordinates is recorded newly as

$$\varepsilon_{x_0}\left(\varepsilon_{x_0}^{-1}(X)\right) = X \tag{9.80}$$

in the new coordinates system.

In the new coordinates system, we find a Carnot coordinates map at X and denote it by $\widetilde{\varepsilon}_X$. Then, noting that X is recorded as $\varepsilon_{x_0}^{-1}(X)$ in the previous coordinates, we know that the composition of two coordinates change maps $\widetilde{\varepsilon}_X \circ \varepsilon_{x_0}$ provides a Carnot coordinates at $\varepsilon_{x_0}^{-1}(X)$ in the setting of the previous coordinates system. Consequently, using Proposition 9.6.19 we find that

$$\varepsilon_{\varepsilon_{x_0}^{-1}(X)} \circ (\varepsilon_{x_0})^{-1} \circ (\widetilde{\varepsilon}_X)^{-1}(y) = \varepsilon_{\varepsilon_{x_0}^{-1}(X)} \circ (\widetilde{\varepsilon}_X \circ \varepsilon_{x_0})^{-1}(y) = y + O_w(y^{\mathbf{w}+1}). \tag{9.81}$$

Inverting this, we get

$$(\widetilde{\varepsilon}_X) \circ (\varepsilon_{x_0}) \circ (\varepsilon_{\varepsilon_{x_0}^{-1}(X)})^{-1}(y) = y + O_w(y^{\mathbf{w}+1}). \tag{9.82}$$

Here, observing that $(\widetilde{\varepsilon}_X)^{-1}(y) = X \cdot y + O((X,y)^{\mathbf{w}+1})$ by Proposition 10.3.2, we can deduce that

$$\varepsilon_{x_0} \circ \varepsilon^{-1}_{\varepsilon_{x_0}^{-1}(X)}(y) = (\widetilde{\varepsilon}_X)^{-1}(y) + O_w((X,y)^{\mathbf{w}+1}). \tag{9.83}$$

It completes the proof. □

9.7 The Tangent Groupoid of a Carnot Manifold

This section is devoted to construct the tangent groupoids of a Carnot manifold (M, H).

9.7.1 Differentiable groupoids.

Here we review the definition of groupoids and present the example of Connes' tangent groupoid on Riemannian manifolds.

Definition 9.7.1. A *groupoids* consists of a set $\mathcal{G}$, a distinguished subset $\mathcal{G}^{(0)} \subset \mathcal{G}$, two maps r and s from $\mathcal{G}$ to $\mathcal{G}^{(0)}$(called the *range* and *source* maps) and a composition map,

$$\circ : \mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} \mid s(\gamma_1) = r(\gamma_2)\} \to \mathcal{G}$$

such that the following properties are satisfied:

1. $s(\gamma_1 \circ \gamma_2) = s(\gamma_2)$ and $r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$, for any $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$;
2. $s(x) = r(x) = x$ for any $x \in \mathcal{G}^{(0)}$;
3. $\gamma \circ s(\gamma) = r(\gamma) \circ \gamma = \gamma$ for any $\gamma \in \mathcal{G}$;
4. $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$;
5. each element $\gamma \in \mathcal{G}$ has a two-sided inverse γ^{-1} so that $\gamma \circ \gamma^{-1} = r(\gamma)$ and $\gamma^{-1} \circ \gamma = s(\gamma)$.

The groupoids interpolate between spaces and groups. This aspect especially pertains in the construction by Connes [Co] of the tangent groupoid $\mathcal{G} = \mathcal{G}M$ of a smooth manifold M.

At the set-theoretic level we let

$$\mathcal{G} = TM \sqcup (M \times M \times (0, \infty)) \quad \textit{and} \quad \mathcal{G}^{(0)} = M \times [0, \infty),$$

where TM denotes the (total space) of the tangent bundle of M. The inclusion ι of $\mathcal{G}^{(0)}$ into $\mathcal{G}$ is given by

$$\iota(m, t) = \begin{cases} (m, m, t) & \text{for } t > 0 \text{ and } m \in M, \\ (m, 0) \in TM & \text{for } t = 0 \text{ and } m \in M. \end{cases} \tag{9.84}$$

The range and source maps of $\mathcal{G}$ are such that

$$\begin{aligned} & r(p, q, t) = (p, t) \quad \textit{and} \quad s(p, q, t) = (q, t) \quad \text{for } t > 0 \text{ and } p, q \in M, \\ & r(p, X) = s(p, X) = (p, 0) \quad \text{for } t = 0 \text{ and } (p, X) \in TM, \end{aligned}$$

and the composition law is defined by

$$\begin{aligned} & (p, m, t) \circ (m, q, t) = (p, q, t) \quad \text{for } t > 0 \text{ and } m, p, q \in M, \\ & (p, X) \circ (p, Y) = (p, X + Y) \quad \text{for } t = 0 \text{ and } (p, X), (p, Y) \in TM. \end{aligned} \tag{9.85}$$

Actually, $\mathcal{G}M$ is a b-differentiable groupoid in the sense of the following definition.

Definition 9.7.2. A b-differentiable groupoid is a groupoid $\mathcal{G}$, so that $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are smooth manifolds with boundary and the following properties hold:

1. The inclusion of $\mathcal{G}^{(0)}$ into $\mathcal{G}$ is smooth;

2. The source and range maps are smooth submersions, so that $\mathcal{G}^{(2)}$ is a submanifold (with boundary) of $\mathcal{G} \times \mathcal{G}$;

3. The composition map $\circ : \mathcal{G}^{(2)} \to \mathcal{G}$ is smooth.

The tangent groupoid $\mathcal{G} = \mathcal{G}M$ is endowed with the topology such that:

- The inclusions of $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)} := M \times M \times (0, \infty)$ into $\mathcal{G}$ are continuous and $\mathcal{G}^{(1)}$ is an open subset of $\mathcal{G}$;

- A sequence (p_n, q_n, t_n) from $\mathcal{G}^{(1)}$ converges to $(p, X) \in TM$ if, and only if, $\lim(p_n, q_n, t_n) = (p, p, 0)$ and for any local chart κ near p we have

$$\lim_{n\to\infty} t_n^{-1}(\kappa(q_n) - \kappa(p_n)) = \kappa'(p)X.$$

One can check that this condition does not depend on the choice of a particular chart near p.

The differentiable structure of $\mathcal{G}M$ is obtained by gluing those of TM and of $\mathcal{G}^{(1)} = M \times M \times (0, \infty)$ by means of a chart of the form,

$$\gamma(p, X, t) = \begin{cases} (p, \exp_p(-tX), t) & \text{if } t > 0 \text{ and } (p, tX) \in \operatorname{dom}\exp, \\ (p, X) & \text{if } t = 0 \text{ and } (p, X) \in \operatorname{dom}\exp. \end{cases}$$

Here $\exp : \operatorname{dom}\exp \to M \times M$ denotes the exponential map associated to an arbitrary Reimannian metric on M, so that γ maps an open subset of $TM \times [0, \infty)$ onto an open neighborhood in $\mathcal{G}$ of the boundary TM (See [Co]).

9.7.2 The tangent groupoid of a Carnot manifold.

We now construct the tangent groupoid $\mathcal{G} = \mathcal{G}_H M$ of a Carnot manifold (M, H).

$$\mathcal{G} = GM \sqcup (M \times M \times (0, \infty)) \quad \textit{and} \quad \mathcal{G}^0 = M \times [0, \infty),$$

where GM denotes the total space of the tangent Lie group bundle of M. We have an inclusion $\iota : \mathcal{G}^0 \to \mathcal{G}$ as

$$\iota(m, t) = \begin{cases} (m, m, t) & \text{for} \quad t > 0 \quad \text{and} \quad m \in M, \\ (m, 0) \in GM & \text{for} \quad t = 0 \quad \text{and} \quad m \in M, \end{cases}$$

the range and source maps are given by

$$r(p,q,t) = (p,t) \quad s(p,q,t) = (q,t) \quad \text{for} \quad t > 0 \quad \text{and} \quad p,q \in M,$$
$$r(p,X) = s(p,X) = (p,0) \quad \text{for} \quad t = 0 \quad \text{and} \quad (p,X) \in GM.$$

We endow $\mathcal{G}$ with the composition

$$\begin{aligned} (p,m,t) \circ (m,q,t) &= (p,q,t) \quad \text{for} \quad t > 0 \quad \text{and} \quad m,p,q \in M, \\ (p,X) \circ (p,Y) &= (p, X \cdot Y) \quad \text{for} \quad t = 0 \quad \text{and} \quad (p,X),(p,Y) \in GM. \end{aligned} \tag{9.86}$$

The inverse map is given by

$$(p,q,t)^{-1} = (q,p,t) \quad \text{for} \quad t > 0 \quad \text{and} \quad p,q \in M,$$
$$(p,X)^{-1} = (p,X^{-1}) = (p,-X) \quad \text{for} \quad t = 0 \quad \text{and} \quad (p,X) \in GM.$$

Definition 9.7.3. The groupoid $\mathcal{G}_H M$ is called the tangent groupoid of (M,H).

We now turn the groupoid $\mathcal{G} = \mathcal{G}_H M$ into a b-differentiable groupoid. First, we endow $\mathcal{G}$ with the topology such that:

- The inculusions of $\mathcal{G}^0$ and $\mathcal{G}^{(1)} := M \times M \times (0,\infty)$ into $\mathcal{G}$ are continuous and make $\mathcal{G}^{(1)}$ an open subset of $\mathcal{G}$.
- A sequence (p_n, q_n, t_n) from $\mathcal{G}^{(1)}$ converges to $(p,X) \in GM$ if, and only if, $\lim(p_n,q_n,t_n) = (p,p,0)$ and for any local H-chart $\kappa : \operatorname{dom}\kappa \to U$ near p we have

$$\lim_{n\to\infty} t_n^{-1} \cdot \varepsilon_{\kappa(p_n)}(\kappa(q_n)) = (\varepsilon_{\kappa(p)} \circ \kappa)'_H(p)X. \tag{9.87}$$

Here *a local H-chart* means a local chart with a local H-frame of TM over its domain.

Lemma 9.7.4. *The condition* (9.87) *is independent of the choice of H-chart κ.*

Proof. Assume that (9.87) holds for a H-chart κ. Let κ_1 be another H-chart near p, and let $\phi = \kappa_1 \circ \kappa^{-1}$. Letting $x_n = \kappa(p_n)$ and $y_n = \kappa(q_n)$, we have

$$\begin{aligned} t_n^{-1} \cdot \varepsilon_{\kappa_1(p_n)}(\kappa_1(q_n)) &= t_n^{-1} \cdot \varepsilon_{\phi(x_n)}(\phi(y_n)) \\ &= t_n^{-1} \circ \varepsilon_{\phi(x_n)} \circ \phi \circ \varepsilon_{x_n}^{-1} \circ \delta_{t_n}(t_n^{-1} \cdot \varepsilon_{x_n}(y_n)). \end{aligned} \tag{9.88}$$

On the other hand, with the fact that ϕ is a Carnot diffeomorphism, we deduce by Proposition 9.6.19 that

$$\lim_{t\to 0} t^{-1} \circ \epsilon_{\phi(x)} \circ \phi \circ \epsilon_x^{-1} \circ \delta_t(y) = \partial_y(\epsilon_{\phi(x)} \circ \phi \circ \epsilon_x^{-1})_H(0)y.$$

locally uniformly with respect to x and y. Since $(x_n, y_n, t_n) \to (\kappa(p), \kappa(p), 0)$ and $t_n^{-1} \cdot \epsilon_{\kappa(p_n)}(\kappa(q_n)) \to (\epsilon_{\kappa(p)} \circ \kappa)'_H(p)X$, by combining this with (9.88) we obtain

$$\begin{aligned}\lim_{n\to\infty} t_n^{-1} \cdot \epsilon_{\kappa_1(p_n)}(\kappa_1(q_n)) &= (\epsilon_{\phi(\kappa(p))} \circ \phi \circ \epsilon_{\kappa(p)}^{-1})'_H(0)((\epsilon_{\kappa(p)} \circ \kappa)'_H(p)X) \\ &= (\epsilon_{\kappa_1}(p) \circ \kappa_1)'_H(p)X.\end{aligned}$$

The lemma is proved. □

In order to endow $\mathcal{G}_H M$ with a manifold structure we take the following local charts. Let $\kappa : \operatorname{dom}\kappa \to U$ be a local H-chart near a point $m \in M$. Then we give a local coordinates for $GM|_{\operatorname{dom}\kappa} \in \mathcal{G}$ by

$$\gamma_\kappa(x, X, t) = \begin{cases} \left(\kappa^{-1}(x), \kappa^{-1} \circ \varepsilon_x^{-1}(t \cdot X), t\right) & \text{if } t > 0 \text{ and } x, \varepsilon_x^{-1}(t \cdot X) \in U, \\ \left(\kappa^{-1}(x), (\kappa^{-1} \circ \varepsilon_x^{-1})'_H(0)X\right) & \text{if } t = 0 \text{ and } (x, X) \in U \times \mathbb{R}^{d+1}. \end{cases}$$

The map γ_k is one-to-one from an open neighborhood of the boundary $U \times \mathbb{R}^{d+1} \times 0$ in $U \times \mathbb{R}^{d+1} \times [0, \infty)$. Moreover, γ_k is continuous off the boundary. It is also continuous near any boundary point $(x, X, 0)$ because if a sequnce $(x_n, X_n, t_n) \in \operatorname{dom}\gamma_k$ with $t_n > 0$ converges to $(x, X, 0)$ then $(p_n, q_n, t_n) = \gamma_k(x_n, X_n, t_n)$ has limit $\lim_{n\to\infty}(p_n, q_n, t_n) = \gamma_\kappa(x, X, 0)$, for we have

$$\lim_{n\to\infty} t_n^{-1} \cdot \varepsilon_{k(p_n)}(k(q_n)) = \lim_{n\to\infty} X_n = X = \kappa'_H(\kappa(x))(\kappa^{-1})'_H(x)X.$$

The inverse γ_κ^{-1} is given by

$$\begin{aligned}&\gamma_\kappa^{-1}(p, q, t) = (\kappa(p), t^{-1} \cdot \varepsilon_{\kappa(p)} \circ \kappa(q), t) \quad \text{for } t > 0, \\ &\gamma_{\kappa_1}(p, X) = (\kappa(p), \kappa'_H(p)X) \quad \text{for } (p, X) \in GM \text{ in the range of } \gamma_{\kappa_1}.\end{aligned} \tag{9.89}$$

Therefore, if κ_1 is another local H-chart near m then, in terms of $\phi = \kappa_1^{-1} \circ \kappa$, the transition map $\gamma_\kappa^{-1} \circ \gamma_{\kappa_1}$ is

$$\gamma_\kappa^{-1} \circ \gamma_{\kappa_1}(x, X, t) = \begin{cases} (\phi(x), t^{-1} \cdot \varepsilon_{\phi(x)} \circ \phi \circ \varepsilon_x^{-1}(t \cdot X), t) & \text{for } t > 0, \\ (\phi(x), \phi'_H(x)X, 0) & \text{for } t = 0. \end{cases}$$

This shows that $\gamma_\kappa^{-1} \circ \gamma_{\kappa_1}(x, X, t)$ is smooth with respect to x and X and is meromorphic with respect to t with at worst a possible singularity at $t = 0$ only. However, by Proposition 9.6.19 we have

$$\lim_{t\to 0} t^{-1} \cdot \varepsilon_{\phi(x)} \circ \phi \circ \varepsilon_x^{-1}(t \cdot X) = \phi'_H(x)X,$$

so there is no singularity at $t = 0$. Hence $\gamma_\kappa^{-1} \circ \gamma_{\kappa_1}$ is a smooth diffeomorphism between open subsets of $\mathbb{R}^{d+1} \times [0, \infty)$. Therefore the coordinates system γ_κ allows us to glue together the differentiable structures of GM and $\mathcal{G}^{(1)} = M \times M \times [0, \infty)$ to turn $\mathcal{G}$ into a smooth manifold with boundary.

Next, $\mathcal{G}^{(0)} = M \times [0, \infty)$ is a manifold with boundary and the inclusion $i : \mathcal{G}^{(0)} \to \mathcal{G}$ is smooth. In addition, the range map r and the source map s are submersions off the boundary. Moreover, in a coordinates system γ_κ near the boundary of $\mathcal{G}$ the maps r and s are given by

$$r(x, X, t) = (x, t) \quad \text{and} \quad s(x, X, t) = (\varepsilon_x^{-1}(t \cdot X), t) \tag{9.90}$$

which shows that $\partial_{x,t} r$ and $\partial_{X,t} s$ are invertible near the boundary. Hence r and s are submersions on all $\mathcal{G}$.

Proposition 9.7.5. *The composition map $\circ : \mathcal{G}^2 \to \mathcal{G}$ is smooth.*

Proof. It is clear that $\circ$ is smooth off the boundary, and so it suffices to concern the case near the boundary. In view of (9.90), in a local coordinate system γ_κ near the boundary two elements (x, X, t) and (y, Y, t) can be composed if and only if we have $y = \varepsilon_x(t \cdot X)$. Then for $t > 0$ using (9.86) and (9.89) we see that $(x, X, t) \circ (\varepsilon_x^{-1}(t \cdot X), Y, t)$ is equal to

$$\begin{aligned}\gamma_\kappa^{-1}\big((\kappa^{-1}(x),\ \kappa^{-1}\varepsilon_x^{-1}(t \cdot X),\ t)\circ(\kappa^{-1}\varepsilon_x^{-1}(t \cdot X),\ \kappa^{-1} \circ \varepsilon^{-1}_{\varepsilon_x^{-1}(t\cdot X)}(t \cdot Y),\ t)\big)&\\ = \gamma_\kappa^{-1}\big((\kappa^{-1}(x),\ \kappa^{-1} \circ \varepsilon^{-1}_{\varepsilon_x^{-1}(t\cdot X)}(t \cdot Y), t)\big)&\\ = (x,\ t^{-1} \cdot \varepsilon_x \circ \varepsilon^{-1}_{\varepsilon_x^{-1}(t\cdot X)}(t \cdot Y), t)\big).&\end{aligned}$$

On the other hand, for $t = 0$ from (9.86) and (9.89) we see that $(x, X, 0) \circ (x, Y, 0)$ is equal to

$$\begin{aligned}&\gamma_\kappa^{-1}\big((\kappa^{-1},\ (\kappa^{-1} \circ \varepsilon_x^{-1})'_H(0)X) \circ (\kappa^{-1},\ (\kappa^{-1} \circ \varepsilon_x^{-1})'_H(0)Y)\big)\\ &\quad = \gamma_\kappa^{-1}\big((\kappa^{-1}(x),\ ((\kappa^{-1} \circ \varepsilon_x^{-1})'_H(0)X) \cdot ((\kappa^{-1} \circ \varepsilon_x^{-1})'_H(0)Y)\big)\\ &\quad = \gamma_\kappa^{-1}(\kappa^{-1}(x),\ (\kappa^{-1} \circ \varepsilon_x^{-1})'_H(0)(X \cdot Y))\\ &\quad = (x,\ X \cdot Y,\ 0),\end{aligned}$$

where we used the fact that $(\kappa^{-1} \circ \epsilon_x^{-1})'_H(0)$ is a morphism of Lie groups (cf. Remark 9.6.18). Therefore, we get

$$(x, X, t) \circ (\varepsilon_x^{-1}(t \cdot X), Y, t) = \begin{cases} (x, t^{-1} \cdot \varepsilon_x \circ \varepsilon^{-1}_{\varepsilon_x^{-1}(t\cdot X)}(t \cdot Y), t) & \text{for } t > 0,\\ (x, X \cdot Y, 0) & \text{for } t = 0.\end{cases}$$

This shows that $\circ$ is smooth with respect to x, X, and Y and is meromorphic with respect to t with at worst a singularity at $t = 0$. Therefore, in order to show the smoothness of $\circ$ at $t = 0$, it is enough to prove that

$$\lim_{t\to 0^+} t^{-1} \cdot \varepsilon_x \circ \varepsilon^{-1}_{\varepsilon_x^{-1}(t\cdot X)}(t \cdot Y) = X \cdot Y. \tag{9.91}$$

For proving this limit, we use Lemma 9.6.23 to see

$$\begin{aligned}\varepsilon_x \circ \varepsilon^{-1}_{\varepsilon_x^{-1}(t\cdot X)}(y) &= (t \cdot X) \cdot (t \cdot Y) + O((t \cdot X, t \cdot Y)^{\mathbf{w}+1})\\ &= t \cdot (X \cdot Y) + O((t \cdot (X \cdot Y))^{\mathbf{w}+1}),\end{aligned} \tag{9.92}$$

and using this we can derive that

$$\lim_{t\to 0^+} t^{-1}\cdot \varepsilon_x \circ \varepsilon^{-1}_{\varepsilon_x^{-1}(t\cdot X)}(t\cdot Y) = \lim_{t\to 0} t^{-1}\cdot t\cdot (X\cdot Y) + t^{-1}\cdot O((t\cdot (X\cdot Y))^{\mathbf{w}+1}) = X\cdot Y, \tag{9.93}$$

which is the desired equality (9.91). This completes the proof. □

Summarizing all this we have proved:

Theorem 9.7.6. *The groupoid $\mathcal{G}_H M$ is a b-differentiable groupoid.*

We conclude this section with a comparison of the tangent groupoids $\mathcal{G}_H M$ and $\mathcal{G}_{H'} M'$ such that manifolds (M, H) and (M', H') are diffeomorphic with a Carnot-diffeomorphism ϕ. For this, we consider the map $\Phi_H : \mathcal{G}_H M \to \mathcal{G}_{H'} M'$ such that

$$\begin{aligned} &\Phi_H(p,q,t) = (\phi(p), \phi(q), t) \quad \text{for } t>0 \text{ and } p,q\in M,\\ &\Phi_H(p,X) = (\phi(p), \phi'_H(p)X) \quad \text{for } (p,X)\in GM. \end{aligned} \tag{9.94}$$

For $t > 0$ and $p, q \in M$, we have

$$\begin{aligned} r_{M'}\circ \Phi_H(p,q,t) &= (\phi(q),t) = \Phi_H \circ r_M(p,q,t),\\ s_{M'}\circ \Phi_H(p,q,t) &= (\phi(p),t) = \Phi_H \circ s_M(p,q,t), \end{aligned}$$

and for $(p, X) \in GM$ we have

$$\begin{aligned} s_{M'}\circ \Phi_H(p,X) = r_{M'}\circ \Phi_H(p,X) &= (\phi(p),0)\\ &= \Phi_H\circ r_M(p,X) = \Phi_H\circ s_M(p,X). \end{aligned}$$

Thus we have $r_{M'} \circ \Phi_H = \Phi_H \circ r_M$ and $s_{M'} \circ \Phi_H = \Phi_H \circ s_M$. Moreover, for $t > 0$ and $m, p, q \in M$ we get

$$\begin{aligned} \Phi_H(m,p,t)\circ_{M'} \Phi_H(p,q,t) &= (\phi(m),\phi(q),t)\\ &= \Phi_H\left((m,p,t)\circ_M (p,q,t)\right), \end{aligned}$$

and for $p \in M$ and $X, Y \in G_p M$ we get

$$\begin{aligned} \Phi_H(p,X)\circ_{M'} \Phi_H(p,Y) &= (\phi(p),\phi'_H(p)(X\cdot Y))\\ &= \Phi_H\left((p,X)\circ_M \Phi_H(p,Y)\right). \end{aligned}$$

All this means that Φ_H is a morphism of groupoids. In addition, the inverse map is defined by replacing ϕ with ϕ^{-1} in (9.94), which yields that Φ_H is a groupoid isomorphism from $\mathcal{G}_H M$ onto $\mathcal{G}_{H'} M'$.

Continuity off the boundary for Φ_H follows by (9.94). In order to see what happens at the boundary we consider a sequence (p_n, q_n, t_n) which converges to $(p, X) \in GM$. Let κ be a local

H-chart for M' near $p' = \phi(p)$. By pulling back the H'-frame of κ by ϕ we turn $\kappa \circ \phi$ into a H-chart, so that setting $(p'_n, q'_n, t_n) = \Phi_H(p_n, q_n, t_n)$ we get

$$t_n^{-1} \cdot \varepsilon_{k(p'_n)}(k(q'_n)) = t_n \cdot \varepsilon_{\kappa\circ\phi(p_n)}(\kappa \circ \phi(q_n)) \to (\kappa \circ \phi)'_H(p)X = \kappa'_H(p)(\phi'_H(p)X).$$

Thus, Φ_H is continuous from $\mathcal{G}_H M$ to $\mathcal{G}_{H'} M'$. It also follows from (9.94) that Φ_H is smooth off the boundary. Moreover, if κ is a local H-chart for M' then $\Phi_H \circ \gamma_{\kappa\circ\phi}(p, X, t)$ coincides for $t > 0$ with

$$\begin{aligned}\left(\phi\left(\phi^{-1} \circ \kappa^{-1}(x)\right), \phi\left(\phi^{-1} \circ \kappa^{-1} \circ \varepsilon_x^{-1}(t \cdot X)\right), t\right) &= \left(\kappa^{-1}(x), \kappa^{-1} \circ \varepsilon_x^{-1}(t \cdot X), t\right)\\ &= \gamma_\kappa(x, X, t),\end{aligned}$$

while for $t = 0$ it is equal to

$$\begin{aligned}&\left(\phi\left(\phi^{-1} \circ \kappa^{-1}(x)\right), \phi'_H\left(\phi^{-1} \circ \kappa^{-1}(x)\right)\left((\kappa^{-1} \circ \varepsilon_x^{-1})'_H(0)X\right), 0\right)\\ &\qquad = \left(\kappa^{-1}(x), (\kappa^{-1} \circ \varepsilon_x^{-1})'_H(0)X, t\right) = \gamma_k(x, X, 0).\end{aligned}$$

Hence $\gamma_k \circ \Phi \circ \gamma_{\kappa\circ\phi} = \mathrm{id}$, which shows that Φ_H is smooth map. By similar arguments we see that Φ_H^{-1} is smooth, and so Φ_H is a diffeomorphism. We have thus proved:

Proposition 9.7.7. *The map* $\Phi_H : \mathcal{G}_H M \to \mathcal{G}_{H'} M'$ *given by* (9.94) *is an isomorphism of b-differentiable groupoids. Hence the isomorphism class of b-differentiable groupoids of* $\mathcal{G}_H M$ *depends only on the Carnot diffeomorphism clss of (M,H).*

Appendix

9.A A matrix computation for degree

In this appendix, we justify the result on the matrix computation concerning *wt* which was used in the proof of Proposition 10.3.2. As there, for $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ we write $\mathrm{wt}(f) = w$ if $f(t \cdot p, t \cdot x) = t^w f(p, x) \quad \forall t > 0$, and $\mathrm{wt}(f) \geq w$ if $\sup_{|p|+|x|\leq 1} |f(t \cdot p, t \cdot x)| = O(t^w)$ as $t \to 0^+$.

Lemma 9.A.1. *Consider an invertible $n \times n$ matrix $B(p) = (B_{jk}(p))_{1\leq j,k\leq n}$ with entry*

$$B_{jk} = \delta_{jk} + F_{jk}(p) + h_{jk}(p),$$

where F_{jk} and h_{jk} satisfy

$$\begin{cases} F_{jk}(p) \equiv 0 & \text{if } w_k \leq w_j, \\ wt(F_{jk}(p)) = w_k - w_j & \text{if } w_k > w_j, \\ wt(h_{jk}(p)) \geq w_k - w_j + 1 & \forall\ 1 \leq j, k \leq n. \end{cases} \tag{9.95}$$

Let $F(p) = (F_{jk}(p)_{1\leq j,k\leq n})$. Then we have

1. *The matrix $I + F(p)$ has an inverse matrix $I + G(p)$, with $G(p) = (G_{jk}(p))_{1\leq j,k\leq n}$ such that*

$$\begin{cases} G_{jk}(p) \equiv 0 & \text{if } w_k \leq w_j, \\ wt(G_{jk}(p)) = w_k - w_j & \text{if } w_k > w_j. \end{cases} \tag{9.96}$$

2. *Let $D(p) = (D_{jk}(p))_{1\leq j,k\leq n}$ be the inverse matrix of $B(p)$. Then,*

$$D_{jk}(p) = \delta_{jk} + G_{jk}(p) + e_{jk}(p), \tag{9.97}$$

where $e_{jk}(p)$ satisfies $wt(e_{jk}(p)) \geq \max\{w_k - w_j + 1, 1\}$ for any k and j.

Proof. Since F is strictly upper diagonal, $(I + F)$ has a unique inverse matrix $(I + G)$ given by a strictly upper diagonal matrix G, i.e., $G_{ab} \equiv F_{ab} \equiv 0$ whenever $a \geq b$. By the identity $(I + F)(I + G) = 0$ we have

$$\begin{aligned} \delta_{ij} &= \sum_k (\delta_{ik} + F_{ik})(\delta_{kj} + G_{kj}) \\ &= \delta_{ij} + G_{ij} + F_{ij} + \sum_{k\neq i,j} F_{ik}G_{kj} \\ &= \delta_{ij} + G_{ij} + F_{ij} + \sum_{i<k<j} F_{ik}G_{kj}. \end{aligned} \tag{9.98}$$

Therefore we get

$$G_{ij}(p) = -F_{ij} - \sum_{i<k<j} F_{ik}G_{kj}. \tag{9.99}$$

Let us fix a value of j. Taking $i = j-1$ in (9.99), we have $G_{j-1,j}(p) = -F_{j-1,j}$, and so $\mathrm{wt}(G_{j-1,j}(p)) = \mathrm{wt}(F_{j-1,j}(p)) = w_j - w_{j-1}$ holds. Thus (9.96) holds with $i = j-1$. Next we observe that G_{ij} involves G_{kj} only with $k > i$ in the formula (9.99). Hence we can use an induction argument with respect to i from $j-1$ to 1. Using that $\mathrm{wt}(F_{ik}G_{kj}) = \mathrm{wt}(F_{ik}) + \mathrm{wt}(G_{kj})$ it proves (9.96).

To show (9.97), we begin with letting $D_{jk}(p) = \delta_{jk} + G_{jk}(p) + e_{jk}(p)$ for some function $e_{jk}(p)$ to be determined. Then it is enough to show that $\mathrm{wt}(e_{jk}(p)) \geq w_k - w_j + 1,\ \forall\ 1 \leq j,k \leq n$. Since $B \cdot D = I$, we have

$$\begin{aligned}
\delta_{ij} &= \sum_k ((\delta_{ik} + F_{ik}) + h_{ik})\Big((\delta_{kj} + G_{kj}) + e_{kj}\Big) \\
&= \sum_k \Big[(\delta_{ik} + F_{ik})(\delta_{kj} + G_{kj}) + h_{ik}\delta_{kj} + e_{kj}\delta_{ik} + h_{ik}G_{kj} + e_{kj}F_{ik} + h_{ik}e_{kj}\Big].
\end{aligned}$$

Using this and (9.98) we get

$$\begin{aligned}
0 &= h_{ij} + e_{ij} + \sum_k \Big[h_{ik}G_{kj} + e_{kj}F_{ik} + h_{ik}e_{kj}\Big]. \\
&= h_{ij} + e_{ij} + \sum_k h_{ik}G_{kj} + \sum_{k>i} e_{kj}F_{ik} + \sum_k h_{ik}e_{kj},
\end{aligned} \tag{9.100}$$

where we also use the fact that $F_{ik} = 0$ for $w_i \leq w_k$. From this identity, using (9.95) and that $\mathrm{wt}(h_{ik}G_{kj}) \geq (w_k - w_i + 1) + (w_j - w_k) = w_j - w_i + 1$, we deduce

$$\mathrm{wt}(e_{ij}) \geq \min\Big\{w_j - w_i + 1,\ \min_{k>i} \mathrm{wt}(e_{ij} \cdot F_{ik}),\ \min_{1\leq k\leq n} \mathrm{wt}(e_{kj} \cdot h_{ik})\Big\}. \tag{9.101}$$

To show the property of $\mathrm{wt}(e_{ij})$ in (9.97), we shall fix j and use an induction argument with respect to i from $(j-1)$ to 1 via the inequality (9.101). Note that we have $e_{ab}(0) = h_{ab}(0) = 0$ for any a and b, and so $\mathrm{wt}(e_{ab}) \geq 1$ and $\mathrm{wt}(h_{ab}) \geq 1$. Then we take $i = j-1$ in (9.101) to get

$$\begin{aligned}
\mathrm{wt}(e_{j-1,j}) &\geq \min\Big\{w_j - w_{j-1} + 1,\ \min_{k>i} \mathrm{wt}(e_{ij} \cdot F_{ik}),\ \min_{1\leq k\leq n} \mathrm{wt}(e_{kj} \cdot h_{ik})\Big\} \\
&\geq \min\Big\{w_j - w_{j-1} + 1,\ 2\Big\} = w_j - w_{j-1} + 1.
\end{aligned}$$

Thus (9.97) holds for $i = j-1$.

Next, for a given $s \in [2, j-1]$, we assume that $\mathrm{wt}(e_{ij}) \geq w_j - w_i + 1$ holds for $i > s$. Combining this, (9.95) and (9.101) we see

$$\min\Big\{\min_{k>s} \mathrm{wt}(e_{kj} \cdot h_{sk}),\quad \min_{k>s} \mathrm{wt}\Big(e_{kj} \cdot F_{sk}\Big)\Big\} \geq w_j - w_s + 1.$$

Using this and that $\mathrm{wt}(h_{ab}) \geq 1$ for any a and b, we deduce from (9.101) that for each $i \leq s$,

$$\begin{aligned} \mathrm{wt}(e_{ij}) &\geq \min\left\{w_j - w_s + 1,\ \min_{k\leq i} \mathrm{wt}(e_{kj} \cdot h_{ik})\right\} \\ &\geq \min\left\{w_j - w_s + 1,\ \min_{k\leq s} \mathrm{wt}(e_{kj}) + 1\right\}, \end{aligned}$$

where we also used that $w_j - w_i + 1 \geq w_j - w_s + 1$ for $i \leq s$. Taking a minimum of this inequality with respect to $i \leq s$, we get

$$\min_{i\leq s} \mathrm{wt}(e_{ij}) \geq \min\left\{w_j - w_s + 1,\ \min_{k\leq s} \mathrm{wt}(e_{kj}) + 1\right\}.$$

From this, we easily get $\min_{i\leq s} \mathrm{wt}(e_{ij}) \geq w_j - w_s + 1$. It gives that $\mathrm{wt}(e_{sj}) \geq w_j - w_s + 1$. Thus (9.97) holds for s, and so the induction concludes that (9.97) holds for any case. The lemma is proved. □

Chapter 10

Pseudodifferential calculus

This paper is devoted to establish the pseudodiiferential calculus on Carnot manifolds based on the preliminary study of the previous chapter.

In Section 2, we begin with studying symbols of differential operators at each point. We then extend it to define the pseudodifferential operators on Carnot manifolds. Also the convolution of operators on Carnot group will be discussed. In Section 3, we establish the pseudodifferential calculus on Carnot manifolds. Namely, we obtain the asymptotic expansion formula for composition, change of coordinates, and adjoint operators. Section 4 is devoted to study the mapping property of pseudodifferential operators on L^p space. In Section 5, we recall the result on the equivalence between the Rockland condition and the invertibility. In Section 6, we study the hypoelliptic heat operators. In Appendices, we will arrange various technical computations which are essentially used in the paper.

10.1 Classes of Symbol and Pseudodifferential operators

In this section, we define the suitable classes of pseudodifferential operators on Carnot manifolds for studying hypoelliptic operators. First we will define symbol classes on open sets and define the classes of kernels by considering their inverse Fourier transform. Then we shall define the pseudodifferential operators using the kernels and Carnot coordinates map $\varepsilon_x(y)$.

As in the previous sections we consider a Carnot-Caratheodory space (M^n, H) with flagged vector fields

$$H_0 = \{0\} \subset H_1 \subset \cdots \subset H_{r-1} \subset H_r = TM \tag{10.1}$$

such that $[H_w, H_{w'}] \subset H_{w+w'}$ when $w + w' \leq r$. For $j = 1, \cdots, n$, we set

$$w_j = \min\{w \in \{1, \cdots r\}; j \leq \mathrm{rk} H_w\}. \tag{10.2}$$

The homogeneous dimension of M^n is then given by $Q = \sum_{j=1}^{n} w_j$. We shall use notation $\|\xi\|$ for

$\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^n$ to denote the quasi-norm

$$\|\xi\| = \sum_{j=1}^{n} |\xi_j|^{\frac{1}{w_j}}. \tag{10.3}$$

Note that this quasi-norm satisfies the relation

$$\|t \cdot \xi\| = t\|\xi\| \quad \forall t > 0 \quad \text{and} \quad \forall \xi \in \mathbb{R}^n, \tag{10.4}$$

where $\cdot$ denotes the isotropic dilation. We denote the usual norm by $|\xi| = \left(\sum_{j=1}^{n} |\xi_j|^2\right)^{1/2}$. Then we have the following basic property.

Remark 10.1.1. There exists a constant $C > 0$ such that $\|\xi\| \geq C|\xi|$ for any $\xi \in \mathbb{R}^n$ with $\|\xi\| \leq 1$. To see this, we note that if $\|\xi\| \leq 1$ then $|\xi_j| \leq \|\xi\| \leq 1$ holds, and so

$$\|\xi\| \geq \sum_{j=1}^{n} |\xi_j| \geq \left(\sum_{j=1}^{n} |\xi_j|^2\right)^{\frac{1}{2}} = |\xi|. \tag{10.5}$$

Locally we may assume that $M = U \subset \mathbb{R}^n$ is an open set endowed with a local tangent frame $(X_1, \cdots, X_n)$ such that the vector fields X_j, $w_j = w$, are sections of H_w for each $w = 1, \cdots, r$.

10.1.1 Definition of $\Psi_H DO$s

We first define the symbol classes on open sets.

Definition 10.1.2.

1. $S_m(U \times \mathbb{R}^n)$, $m \in \mathbb{C}$ consists of functions $p \in C^\infty(U \times (\mathbb{R}^n \setminus \{0\}))$ such that

$$p(x, \lambda \cdot \xi) = \lambda^m p(x, \xi), \quad \text{and} \quad \forall \lambda > 0 \tag{10.6}$$

holds for all $(x, \xi) \in U \times \mathbb{R}^n$.

2. $S^m(U \times \mathbb{R}^n)$, $m \in \mathbb{C}$ consists of functions $p \in C^\infty(U \times \mathbb{R}^n)$ which admit an asymptotic expansion:

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi), \quad p_k \in S_m(U \times \mathbb{R}^n), \tag{10.7}$$

in the sense that for all multi-orders α, β and all $N > 0$, it holds that

$$\left| D_x^\alpha D_\xi^\beta \Big(p(x, \xi) - \sum_{j<N} p_{m-j}(x, \xi) \Big) \right| \leq C_{\alpha\beta KN} \|\xi\|^{m-N-\langle\beta\rangle} \quad \forall x \in K, \quad |\xi| \geq 1, \tag{10.8}$$

where K is any compact subset of U and $C_{\alpha\beta KN}$ is a positive constant determined by α, β, K and N.

Next we shall define the classes of kernels. We begin with defining homogeneous kernels. For $K \in \mathcal{S}'(\mathbb{R}^n)$ and for $\lambda > 0$ we denote by K_λ the element of $\mathcal{S}'(\mathbb{R}^n)$ such that

$$\langle K_\lambda, f\rangle = \lambda^{-Q}\langle K(x), f(\lambda^{-1}\cdot x)\rangle \quad \forall f \in \mathcal{S}(\mathbb{R}^n). \tag{10.9}$$

We say that K is homogeneous of degree m, $m \in \mathbb{C}$, when $K_\lambda = \lambda^m K$ for any $\lambda > 0$.

Definition 10.1.3. $\mathcal{S}'_{reg}(\mathbb{R}^n)$ denotes the set of tempered distributions on $\mathbb{R}^n$ which are smooth outside the origin.

Definition 10.1.4.

1. $\mathcal{K}_m(U\times\mathbb{R}^n)$, $m \in \mathbb{C}$, consists of distributions $K(x,y)$ in $C^\infty(U)\widehat{\otimes}S'_{reg}(\mathbb{R}^n)$ such that for some functions $c_\alpha(x) \in C^\infty(U)$, $\langle\alpha\rangle = m$, we have
   (a) If $j \notin \mathbb{N}_0$, $K(x, \lambda\cdot y) = \lambda^j K(x,y) \quad \forall\lambda > 0$.
   (b) If $j \in \mathbb{N}_0$, $K(x,\lambda\cdot y) = \lambda^m K(x,y) + \lambda^m \log\lambda \sum_{\langle\alpha\rangle=m} c_{K,\alpha}(x)y^\alpha \quad \forall\ \lambda > 0$, where the functions $c_{K,\alpha}(x)$ are contained in $C^\infty(U)$.
2. $\mathcal{K}^m(U\times\mathbb{R}^n)$, $m \in \mathbb{C}$, consists of distributions $K \in \mathcal{D}'(U\times\mathbb{R}^n)$ with an asymptotic expansion $K \sim \sum_{j\geq 0} K_{m+j}$, $K_l \in \mathcal{K}_l(U\times\mathbb{R}^n)$, in the sense that, for any integer N, as soon as J is large enough we have
$$K - \sum_{j\leq J} K_{m+j} \in C^N(U\times\mathbb{R}^n). \tag{10.10}$$

Then the following result holds.

Lemma 10.1.5. *[BG, Prop. 15.24], [CoM, Lem. 1.4].*

1. *Any $p \in S_m(U\times\mathbb{R}^n)$ agrees on $U\times(\mathbb{R}^n\setminus 0)$ with a distribution $\gamma(x,\xi) \in C^\infty(U)\widehat{\otimes}\mathcal{S}'(\mathbb{R}^n)$ such that $\widehat{\gamma}_{\xi\to y}$ is in $\mathcal{K}_{\widehat{m}}(U\times\mathbb{R}^n)$, $\widehat{m} = -(m+Q)$.*
2. *If $K(x,y)$ belongs to $\mathcal{K}_{\widehat{m}}(U\times\mathbb{R}^n)$ then the restriction of $\widehat{K}_{y\to\xi}(x,\xi)$ to $U\times(\mathbb{R}^n\setminus 0)$ belongs to $S_m(U\times\mathbb{R}^n)$.*

Now we define the class of pseudodifferential operators.

Definition 10.1.6. The class $\Psi^m(U)$, $m \in C$ consists of continuous operators $P : C_c^\infty(U) \to C^\infty(U)$ with distribution kernel $k_P(x,y)$ such that

$$k_P(x,y) = |\varepsilon'_x| K_P(x, -\varepsilon_x(y)) + R(x,y), \tag{10.11}$$

with $K_P \in \mathcal{K}^{\widehat{m}}(U\times\mathbb{R}^n)$, $\widehat{m} = -m-Q$, and $R \in C^\infty(U\times U)$.

10.2 Convolutions on nilpotent Lie groups

We shall consider convolution operators with distributions kernels on nilpotent Lie groups and define a modified convolution. This is essential for the calculus of composition of pseudo-differential operators in the next section.

Let G be a nilpotent Lie group which is realized on $\mathbb{R}^n$ endowed with a group law $\cdot$. Then, for $K \in \mathcal{S}'(\mathbb{R}^n)$ we set the convolution operators by

$$T_K g(x) = \langle K,\ g \circ \psi_x \rangle, \qquad g \in \mathcal{S}(\mathbb{R}^n), \tag{10.12}$$

where $\psi_x(y) = (x \cdot y)^{-1}$. When K is contained in a suitable function space, we may write the above operator as

$$T_K f(x) = \int_G K(x \cdot y^{-1}) f(y) dy. \tag{10.13}$$

Note that it is difficult to define the convolution for the operators T_K with $K \in \mathcal{K}_m(\mathbb{R}^n)$ because K may have an unbounded supoort. To go around this difficulty, as in [BG, BGS], we consider almost homogenous functions.

Definition 10.2.1. For $m \in \mathbb{C}$, the set $S^{ah}_m(\mathbb{R}^n)$ consists of function $f \in C^\infty(\mathbb{R}^n)$ which is almost homogeneous of degree m in the sense that

$$\lambda^{-m} \delta_\lambda f - f \in \mathcal{S}(\mathbb{R}^n) \quad \text{for all } \lambda > 0, \tag{10.14}$$

where

$$\delta_\lambda f(\xi) = f(\lambda \cdot \xi). \tag{10.15}$$

We say that the function f has the homogeneous part $g \in \mathcal{F}_m$ if for each $N \geq 0$ and each α, it holds that

$$\lim_{|\zeta| \to \infty} \|\zeta\|^N D^\alpha [f(\zeta) - g(\zeta)] = 0, \tag{10.16}$$

and we shall write $g = \mathrm{hom}(f)$.

Proposition 10.2.2 ([BGS])**.** *If f is almost homogeneous of degree m, then it has a unique homoegeneous part.*

We denote by $\mathcal{E}'$ the space of compactly supported distributions. Then we have the following result.

Proposition 10.2.3 ([BGS])**.** *Assume that g_j is almost homogeneous of degree m_j, $j = 1, 2$. Then the inverse transofrm k_j is contained in $\mathcal{E}' + \mathcal{S}$. In addition, the function $g = (k_1 * k_2)^\wedge$ is almost homogeneous of degree $m_1 + m_2$, and the homogeneous part $f = hom(g)$ is uniquely determined by $f_j = hom(g_j)$, $j = 1, 2$.*

Definition 10.2.4. Suppose f_j belongs to $\mathcal{S}_{m_j}(\mathbb{R}^{n+1}), j = 1, 2$. Then $T(f_1, f_2)$ is the element of $\mathcal{S}_m(\mathbb{R}^{n+1})$, $m = m_1 + m_2$, which is defined by

$$T(f_1, f_2) = \hom([g_1^\sharp * g_2^\sharp]^b), \tag{10.17}$$

where the g_j are almost homogeneous with $\hom(g_j) = f_j$.

Definition 10.2.5. Let $K_1 \in \mathcal{K}_{\widehat{m}_1}(U \times \mathbb{R}^n)$ and $K_2 \in \mathcal{K}_{\widehat{m}_2}(U \times \mathbb{R}^n)$. Then we define $K_1 \underline{*} K_2 \in \mathcal{K}_{\widehat{m_1+m_2}}(U \times \mathbb{R}^n)$ as

$$K_1 \sharp K_2(x, y) = T\left(K_1(x, \cdot) \underline{*} K_2(x, \cdot)\right). \tag{10.18}$$

10.3 Pseudodifferential calculus

In this section, we establish the calculus of the pseudodifferential operators in $\Psi^m(U)$. Namely, we shall study the composition of two operators, the adjoint operators, and the invariance property.

For this aim, we define the following notations of distributions.

Definition 10.3.1. Let $w \in \mathbb{N}_0^n$ and and $\gamma \in \mathbb{N}_0^n$.

1. For given $K \in \mathcal{S}'(\mathbb{R}^n)$, we define the distribution K^w by

$$\langle K^w, f(z) \rangle = \langle K, z^w f(z) \rangle, \qquad f \in \mathcal{S}(\mathbb{R}^n). \tag{10.19}$$

2. For given $K \in \mathcal{S}'(\mathbb{R}^n)$, we define the distribution $K^{w:\gamma}$ by

$$\langle K^{w:\gamma}, f(z) \rangle = \langle K^w, (-\partial_z)^\gamma f(z) \rangle, \qquad f \in \mathcal{S}(\mathbb{R}^n). \tag{10.20}$$

3. For given $K \in \mathcal{K}_m(U \times \mathbb{R}^n)$, $m \in \mathbb{C}$, we define the distribution $(K)_\alpha \in \mathcal{K}_m(U \times \mathbb{R}^n)$ by

$$\langle (K)_\alpha, f \rangle = \partial_x^\alpha \langle K(x, \cdot), f \rangle, \qquad f \in \mathcal{S}(\mathbb{R}^n). \tag{10.21}$$

10.3.1 Composition of Pseudodifferential operators on vector fields

Proposition 10.3.2. *Consider two pseudo-differential operators $P_{K_1} \in \Psi^{m_1}(U)$ and $P_{K_2} \in \Psi^{m_2}(U)$ with $K_1 \in \mathcal{K}^{\widehat{m}_1}(U \times \mathbb{R}^n)$ and $K_2 \in \mathcal{K}^{\widehat{m}_2}(U \times \mathbb{R}^n)$. Assume that one of the operators is properly supported. Then, $P_{K_1} \circ P_{K_2} \in \Psi^{m_1+m_2}(U)$ and there exists a kernel $K \in \mathcal{K}^{\widehat{m_1+m_2}}(U \times \mathbb{R}^n)$ such that*

$$P_{K_1} \circ P_{K_2} = P_K. \tag{10.22}$$

In addition, the principal kernel of K equals to $(K_1)_{\widehat{m}_1} \sharp (K_2)_{\widehat{m}_2}$, and generally, the term kernel with homogeneous degree k in the asymptotic expansion of $P_{K_1} \circ P_{K_2}$ is given by the form

$$\sum C_{\alpha\beta\gamma\delta}((K_1)_l)^\gamma \sharp ((K_2)_t)_\alpha^{\delta:\beta} \tag{10.23}$$

where $C_{\alpha\beta\gamma\delta}$ are functions in $C^\infty(U)$ independent of the operators P_{K_1} and P_{K_2}. The sum is finite, taken over indices such that

- $-m_1 - Q \leq l$ *and* $-m_2 - Q \leq t$,
- $(-l - Q) - \langle \gamma \rangle + (-t - Q) - \langle \delta \rangle + \langle \beta \rangle = -k - Q$,
- $\langle \gamma \rangle + \langle \delta \rangle - \langle \beta \rangle \geq |\alpha| + |\beta|$.

Proof. Without loss of generality, we may assume that $P_{K_1} \in \Psi^{m_1}(U)$ and $P_{K_2} \in \Psi^{m_2}(U)$ with $K_1 \in \mathcal{K}^{ah}_{\hat{m}_1}(U \times \mathbb{R}^n)$ and $K_2 \in \mathcal{K}^{ah}_{\hat{m}_2}(U \times \mathbb{R}^n)$. Take a function $f \in \mathcal{S}(\mathbb{R}^n)$ which is compactly supported in U. Then we have

$$P_{K_1} f(x) = \left\langle K_1(x, -y),\ f(\varepsilon_x^{-1} y) \right\rangle \quad \text{and} \quad P_{K_2} f(x) = \left\langle K_2(x, -y),\ f(\varepsilon_x^{-1} y) \right\rangle. \tag{10.24}$$

Since one of P_{K_1} and P_{K_2} is properly supported, we may compose these two operators, which leads to

$$\begin{aligned} P_{K_1} \circ P_{K_2} f(x) &= \left\langle K_1(x, -y),\ (P_{K_2} f)(\varepsilon_x^{-1}(y)) \right\rangle \\ &= \left\langle K_1(x, -y),\ \left\langle K_2(\varepsilon_x^{-1}(y), -z),\ f(\varepsilon^{-1}_{\varepsilon_x^{-1}(y)}(z)) \right\rangle \right\rangle. \end{aligned} \tag{10.25}$$

Changing the variable $z \to \varepsilon_{\varepsilon_x^{-1}(y)} \circ \varepsilon_x^{-1}(z)$ we have

$$\begin{aligned} P_{K_1} \circ P_{K_2} f(x) &= \left\langle K_1(x, -y),\ \left\langle K_2(\varepsilon_x^{-1}(y), -\varepsilon_{\varepsilon_x^{-1}(y)} \circ \varepsilon_x^{-1}(z)),\ f(\varepsilon_x^{-1}(z)) \right\rangle \right\rangle \\ &= \left\langle K_1(x, -y),\ \left\langle K_2(x + a(x, y), -y \cdot z + b(x, y, z)),\ f(\varepsilon^{-1}(z)) \right\rangle \right\rangle \end{aligned} \tag{10.26}$$

where we have let $a(x, y) = \varepsilon_x^{-1}(y) - x$ and $b(x, y, z) = \varepsilon_{\varepsilon_x^{-1}(y)} \circ \varepsilon_x^{-1}(z) - y \cdot z$.

Using the Taylor expansion (see Lemma 10.C.1), we have the formal identity

$$\begin{aligned} &K_2\left(x + a(x, y),\ -y \cdot z + b(x, y, z)\right) \\ &= \sum_{|\alpha|+|\beta|<N} \frac{1}{\alpha!\beta!} \partial_1^\alpha \partial_2^\beta K_2(x,\ z \cdot y) a(x, y)^\alpha b(x, y, z)^\beta + R_N(x, a(x, y), -y \cdot z, b(x, y, z)), \end{aligned} \tag{10.27}$$

where

$$\begin{aligned} &R_N\left(x, a(x, y), z \cdot y, b(x, y, z)\right) \\ &\quad = \int_0^1 \frac{(N + 1)(1 - t)^N}{\alpha!\beta!} a(x, y)^\alpha b(x, y, z)^\beta \partial_1^\alpha \partial_2^\beta K_2(x + a(x, y)t,\ -y \cdot z + b(x, y, z)t) dt. \end{aligned} \tag{10.28}$$

This enables us to write

$$\begin{aligned} &\left\langle K_2\left(x + a(x, y), -y \cdot z + b(x, y, z)\right),\ f(\varepsilon^{-1}(z)) \right\rangle \\ &= \sum_{|\alpha|+|\beta|<N} \frac{1}{\alpha!\beta!} \left\langle \partial_1^\alpha \partial_2^\beta K_2(x,\ z \cdot y) a(x, y)^\alpha b(x, y, z)^\beta,\ f(\varepsilon^{-1}(z)) \right\rangle \\ &\qquad + \left\langle R_N(x, a(x, y), z \cdot y, b(x, y, z)),\ f(\varepsilon^{-1}(z)) \right\rangle. \end{aligned} \tag{10.29}$$

From the fact that $a(x,0)=0$ and using Lemma 9.6.23 we have

$$a(x,y)=O(y) \quad \text{and} \quad b(x,z,y)=O((z,y)^{\mathbf{w}+1}). \tag{10.30}$$

Thus we have

$$a(x,y)^{\alpha}=\sum_{|\alpha|\le\langle\gamma\rangle\le r|\alpha|} C_{\alpha\gamma}y^{\gamma}, \tag{10.31}$$

and

$$b(x,z,y)^{\beta}=\sum_{\gamma,\delta}\widetilde{C}_{\beta\gamma\delta}y^{\gamma}z^{\delta}=\sum_{\gamma,\delta}C_{\beta\gamma\delta}y^{\gamma}(y\cdot z)^{\delta}, \tag{10.32}$$

where $C_{\alpha\gamma}$, $C_{\beta\gamma\delta}$ and $\widetilde{C}_{\beta\gamma\delta}$ are constants depending on δ, γ, β and the group law, and the indices satisfy the relation

$$\langle\gamma\rangle+\langle\delta\rangle\ge|\beta|+\langle\beta\rangle. \tag{10.33}$$

In the second equality, we used the identity $z^{\gamma}=((y\cdot z)\cdot(-z))^{\gamma}=\sum_{\langle p\rangle+\langle q\rangle=\langle\gamma\rangle}C_{pq\gamma}(y\cdot z)^{p}(z)^{q}$, where $C_{pq\gamma}$ are constants determined by the group law. Combining (10.31) and (10.32) we have

$$a(x,y)^{\alpha}b(x,z,y)^{\beta}=\sum_{\gamma,\delta}C_{\alpha\beta\gamma\delta}y^{\gamma}(y\cdot z)^{\delta}, \tag{10.34}$$

where γ and δ satisfy the relation

$$\langle\gamma\rangle+\langle\delta\rangle\ge|\beta|+|\alpha|+\langle\beta\rangle. \tag{10.35}$$

Using this we may write each term in the right hand side of (10.27) as

$$\partial_1^{\alpha}\partial_2^{\beta}K_2(x,\ z\cdot y)a^{\alpha}b^{\beta}=\partial_1^{\alpha}\partial_2^{\beta}K_2(x,z\cdot y)\sum_{\gamma,\delta}C_{\alpha\beta\gamma\delta}y^{\gamma}(y\cdot z)^{\delta}. \tag{10.36}$$

which leads to

$$\begin{aligned}
&K_2\left((x+a(x,y),\ -y\cdot z+b(x,y,z))\right)\\
&=\sum_{|\alpha|+|\beta|<N}\sum_{\gamma,\delta}\frac{C_{\alpha\beta\gamma\delta}}{\alpha!\beta!}\partial_1^{\alpha}\partial_2^{\beta}K_2(x,z\cdot y)y^{\gamma}(y\cdot z)^{\delta}+R_N\left(x,a(x,y),z\cdot y,b(x,y,z)\right).
\end{aligned}$$

Now we put this into (10.26) to get

$$\begin{aligned}
&(P_{K_1}\circ P_{K_2})f(x)\\
&\quad=\sum_{|\alpha|+|\beta|<N}\sum_{\gamma}\frac{C_{\beta\delta\gamma}}{\alpha!\beta!}\left\langle y^{\gamma}K_1(x,-y),\ \left\langle\partial_1^{\alpha}\partial_2^{\beta}K_2(x,y\cdot z)(y\cdot z)^{\delta},\ f(\varepsilon_x^{-1}(z))\right\rangle_z\right\rangle_y\\
&\qquad+\left\langle K_1(x,y),\ \left\langle R_N(x,a(x,y),y\cdot z,b(x,y,z)),\ f(\varepsilon_x^{-1}(z))\right\rangle_z\right\rangle_y\\
&\quad:=\mathcal{M}f(x)+\mathcal{R}f(x).
\end{aligned}$$

Note that if we set $L_1 \in \mathcal{K}_{\widehat{m_1}+\langle\gamma\rangle}(U \times \mathbb{R}^n)$ and $L_2 \in \mathcal{K}_{\langle\delta\rangle+\widehat{m_2}-\langle\beta\rangle}(U \times \mathbb{R}^n)$ by

$$L_1(x,y) = y^\gamma K_1(x,y) \quad \text{and} \quad L_2(x,y) = y^\delta \partial_1^\alpha \partial_2^\beta K_2(x,y), \tag{10.37}$$

then we have

$$\left\langle y^\gamma K_1(x,-y),\ \left\langle \partial_1^\alpha \partial_2^\beta K_2(x, y\cdot z)(y\cdot z)^\delta,\ f(\varepsilon_x^{-1}(z))\right\rangle_z \right\rangle_y = \left\langle (L_1(x)\sharp L_2(x))(z),\ f(\varepsilon_x^{-1}(z))\right\rangle.$$

Since $\langle\gamma\rangle + \langle\delta\rangle \geq |\alpha| + |\beta| + \langle\beta\rangle$, we have

$$L_1 \,\sharp\, L_2 \in \mathcal{K}^{\widehat{m_1+m_2}+|\beta|+|\alpha|}(U \times \mathbb{R}^n). \tag{10.38}$$

This completes the first part of the proof.

Now we are only left to show the smoothness of the distribution kernel of the remainder term $\mathcal{R}$. Heuristically we can see from (10.30) that $(R_N)_\delta(x,a,z\cdot y,b)$ has also gain at least $|\alpha|+|\beta| = N$ order of $|y|$. Essentially, it explains why the remainder term becomes smooth as N becomes large. We shall justify this heuristic rigorously. Recall that

$$\begin{aligned} &R_N(x, a(x,y), y\cdot z, b(x,y,z)) \\ &\quad = \int_0^1 \frac{(N+1)(1-t)^N}{\alpha!\beta!} a(x,y)^\alpha b(x,y,z)^\beta \partial_1^\alpha \partial_2^\beta K_2(x + a(x,y)t,\ \Phi_t(x,y,z))dt. \end{aligned} \tag{10.39}$$

where we have let

$$\Phi_t(x,y,z) := z\cdot y + t\left(\varepsilon_{\varepsilon_x^{-1}(y)} \circ \varepsilon_x^{-1}(z) - y\cdot z\right). \tag{10.40}$$

Using (10.30) again and Lemma 10.B.3, we have

$$a(x,y)^\alpha b(x,y,z)^\beta = \sum_{\gamma,\delta} C_{\alpha\beta\gamma\delta}(t) y^\gamma \Phi_t(x,y,z)^\delta l \tag{10.41}$$

where δ and γ satisfy

$$\langle\delta\rangle + \langle\gamma\rangle \geq \langle\beta\rangle + |\beta| + |\alpha|. \tag{10.42}$$

Using this we write

$$\begin{aligned} &a(x,z)^\alpha b(x,z,y)^\beta \partial_1^\alpha \partial_2^\beta K_2(x + at, \Phi_t(x,z,y)) \\ &\qquad = \sum_{\gamma,\delta} C_{\alpha\beta\gamma\delta} y^\gamma \Phi_t(x,z,y)^\delta \partial_1^\alpha \partial_2^\beta K_2(x+at, \Phi_t(x,z,y)). \end{aligned} \tag{10.43}$$

Injecting this into (10.39) we get

$$\begin{aligned} \mathcal{R}f(x) = \sum_{|\alpha|+|\beta|=N} \sum_{\gamma,\delta} \int_0^1 \Bigg\langle &\frac{C_{\alpha\beta\gamma\delta}(t)}{\alpha!\beta!}\left(K_1(x,y)y^\gamma\right), \\ &\left\langle \left(\Phi_t(x,z,y)^\delta \partial_1^\alpha \partial_2^\beta K_2(x+at, \Phi_t(x,z,y))\right),\ f(\varepsilon_x^{-1}(z))\right\rangle_z \Bigg\rangle_y dt. \end{aligned} \tag{10.44}$$

By (10.42) we find that

$$\begin{aligned}&\mathrm{wt}(K_1(x,z)z^\gamma)+\mathrm{wt}(z^\delta\partial_1^\alpha\partial_2^\beta K_2(x+at,z))\\&\geq(\widehat{m_1}+\langle\gamma\rangle)+(\langle\delta\rangle-\langle\beta\rangle+\widehat{m_2})\geq|\alpha|+|\beta|+\widehat{m}_1+\widehat{m}_2=N+\widehat{m}_1+\widehat{m}_2.\end{aligned}\tag{10.45}$$

Note that we can write $\mathcal{R}f(x)$ as

$$\begin{aligned}\mathcal{R}f(x)=&\sum_{|\alpha|+|\beta|=N}\sum_{\gamma,\delta}\\&\int_0^1\left\langle\left\langle\frac{C_{\alpha\beta\gamma\delta}(t)}{\alpha!\beta!}\left(K_1(x,y)y^\gamma\right),\left(\Phi_t(x,z,y)^\delta\partial_1^\alpha\partial_2^\beta K_2(x+at,\Phi_t(x,z,y))\right)\right\rangle_y,\ f(\varepsilon_x^{-1}(z))\right\rangle_z dt,\end{aligned}$$

and it is enough to show that each integration

$$\left\langle K_1(x,y)y^\gamma,\ \Phi_t(x,z,y)^\delta\partial_1^\alpha\partial_2^\beta K_2(x+at,\Phi_t(x,z,y))\right\rangle_y\tag{10.46}$$

is a function in $C^M(U\times U)$ if N is large enough. Actually, this fact follows directly from (10.45) combining and Remark 10.A.3 and Lemma 10.C.2. The proof is completed. □

10.3.2 Invariance theorem of peudodifferential operators

Proposition 10.3.3. *Let U (resp. $\widetilde{U}$) be an open subset of $\mathbb{R}^n$ equipped with a hyperplane bundle $H\subset TU$ (resp. $\widetilde{H}\subset T\widetilde{U}$) and a H-frame of TU (resp. $\widetilde{H}$-frame of $T\widetilde{U}$). Suppose that U and $\widetilde{U}$ are Carnot diffeomorphic with $\phi:(U,H)\to(\widetilde{U},\widetilde{H})$ a Carnot diffeomorphism. Then, for $\widetilde{P}\in\Psi^m_{\widetilde{H}}(\widetilde{U})$, the following holds:*

1. *The operator $P=\phi^*\widetilde{P}$ is a $\Psi_H DO$ of order m on U.*
2. *Consider that the kernel of $\widetilde{P}$ is given by the form* (10.11) *with $K_{\widetilde{P}}\in\mathcal{K}^{\widehat{m}}(\widetilde{U}\times\mathbb{R}^n)$. Then, the kernel of P is of the form* (10.11) *with $K_P(x,y)\in\mathcal{K}^{\widehat{m}}(U\times\mathbb{R}^n)$ such that*

$$K_P(x,y)\sim\sum_{\langle\beta\rangle\geq\langle\alpha\rangle+|\alpha|}\frac{1}{\alpha!\beta!}a_{\alpha\beta}(x)y^\beta(\partial_2^\alpha K_{\widetilde{P}})(\phi(x),\phi_H'(x)y),\tag{10.47}$$

where we have let $a_{\alpha\beta}(x)=\partial_y^\beta\left[|\partial_y(\widetilde{\varepsilon}_{\phi(x)}\circ\phi\circ\varepsilon_x^{-1})(y)|(\widetilde{\varepsilon}_{\phi(x)}\circ\phi\circ\varepsilon_x^{-1}(y)-\phi_H'(x)y)^\alpha\right]|_{y=0}$ and $\widetilde{\varepsilon}_x$ denote the change to the Carnot coordinates at $\widetilde{x}\in\widetilde{U}$. Especially,

$$K_P(x,y)=|\phi_H'(x)|K_{\widetilde{P}}(\phi(x),\phi_H'(x)y)\quad mod\ \mathcal{K}^{\widehat{m}+1}(U\times\mathbb{R}^n).\tag{10.48}$$

Proof. The kernel of $\widetilde{P}$ is given by

$$k_{\widetilde{P}}(\widetilde{x},\widetilde{y})=|\widetilde{\varepsilon}_{\widetilde{x}}|K_{\widetilde{P}}(\widetilde{x},-\widetilde{\varepsilon}_x(\widetilde{y}))+\widetilde{R}(\widetilde{x},\widetilde{y}),\tag{10.49}$$

with $K_{\widetilde{P}}(\widetilde{x},\widetilde{y}) \in \mathcal{K}^{\widehat{m}}(\widetilde{U}\times\mathbb{R}^n)$ and $\widetilde{R}(\widetilde{x},\widetilde{y}) \in C^\infty(\widetilde{U}\times\widetilde{U})$.

By definition, we have

$$\begin{aligned}(\phi^*\widetilde{P})(f)(x) &= \widetilde{P}(f\circ(\phi^{-1}(\cdot)))(\phi(x)) \\ &= \int |\widetilde{\varepsilon}'_{\phi(x)}(y)| K_{\widetilde{P}}(\phi(x),\widetilde{\varepsilon}_{\phi(x)}(y)) f(\phi^{-1}(y))dy + \int \widetilde{R}(\phi(x),y) f(\phi^{-1}(y))dy \\ &= \int |\widetilde{\varepsilon}_{\phi(x)}(\phi(y))||\phi'(y)| K_{\widetilde{P}}(\phi(x),\widetilde{\varepsilon}_{\phi(x)}(\phi(y))) f(y)dy + \int |\phi'(y)|\widetilde{R}(\phi(x),\phi(y)) f(y)dy.\end{aligned}$$

Hence the kernel of $P = \phi^*\widetilde{P}$ is given by

$$k_P(x,y) = |\phi'(y)| K_{\widetilde{P}}(\phi(x),\phi(y)) = |\varepsilon'_x| K(x,-\varepsilon_x(y)) + \widetilde{R}(\phi(x),\phi(y)), \tag{10.50}$$

where K is a distribution on $\{(x,y)\in U\times\mathbb{R}^n;\ \varepsilon_x^{-1}(-y)\in U\} \subset U\times\mathbb{R}^n$ such that

$$K(x,y) = |\partial_y\Phi(x,y)| K_{\widetilde{P}}(\phi(x),\Phi(x,y)), \tag{10.51}$$

with $\Phi(x,y) = -\widetilde{\varepsilon}_{\phi(x)}\circ\phi\circ\varepsilon_x^{-1}(-y)$. By Proposition 9.6.19 we have

$$\Phi(x,y) = \phi'_H(x)(y) + \Theta(x,y), \tag{10.52}$$

where $\Theta(x,y) = O(y^{\mathbf{w}+1})$. By performing the Taylor expansion around $\widetilde{y} = \phi'_H(x)y$ we have

$$\begin{aligned}K(x,y) &= |\partial_y\Phi(x,y)| K_{\widetilde{P}}(\phi(x),\phi'_H(x)(y)+\Theta(x,y)) \\ &= \sum_{|\alpha|<N} |\partial_y\Phi(x,y)| \frac{\Theta(x,y)^\alpha}{\alpha!}(\partial_2^\alpha K_{\widetilde{P}})(\phi(x),\phi'_H(x)y) + R_N(x,y),\end{aligned} \tag{10.53}$$

where $R_N(x,y)$ equals to

$$R_N(x,y) = \sum_{|\alpha|=N} |\partial_y\Phi(x,y)| \frac{\Theta(x,y)^\alpha}{\alpha!}\int_0^1 (t-1)^{N-1}\partial_2^\alpha K_{\widetilde{P}}(\phi(x),\Phi_t(x,y))dt, \tag{10.54}$$

and we have let $\Phi_t(x,y) = \phi'_H(x)y + t\Theta(x,y)$.

Set $f_\alpha(x,y) = |\partial_y\Phi(x,y)|\Theta(x,y)^\alpha$. As $\Theta(x,y) = O(y^{\mathbf{w}+1})$, near $y=0$ we have

$$f_\alpha(x,y) = \sum_{\langle\alpha\rangle+|\alpha|\leq\langle\beta\rangle<2N} f_{\alpha\beta}(x)y^\beta + \sum_{\langle\beta\rangle\geq 2N} r_{N\alpha\beta}(x,y)y^\beta, \tag{10.55}$$

with $f_{\alpha\beta}(x) = \frac{1}{\beta!}\partial_y^\beta f_\alpha(x,0)$ and $r_{M\alpha\beta}(x,y) \in C^\infty(U\times U)$. Then,

$$K(x,y) = \sum_{\langle\alpha\rangle<N}\Big[\sum_{(\langle\alpha\rangle+|\alpha|)\leq\langle\beta\rangle<2N} K_{\alpha\beta}(x,y)\Big] + \sum_{\langle\alpha\rangle<N} R_{N\alpha}(x,y) + R_N(x,y), \tag{10.56}$$

where we have let

$$\begin{aligned} K_{\alpha\beta}(x,y) &= f_{\alpha\beta}(x)y^{\beta}(\partial_2^{\alpha}K_{\widetilde{P}})(\phi(x),\phi'_H(x)y), \\ R_{N\alpha}(x,y) &= \sum_{\langle\beta\rangle\geq 2N} r_{N\alpha\beta}(x,y)y^{\beta}(\partial_2^{\alpha}K_{\widetilde{P}})(\phi(x),\phi'_H(x)y). \end{aligned} \tag{10.57}$$

First we note that $K_{\alpha\beta}\in\mathcal{K}^{\langle\beta\rangle-\langle\alpha\rangle+\widehat{m}}(U\times\mathbb{R}^n)$ because $\phi'_H(x)(\lambda\cdot y)=\lambda\cdot(\phi'_H(x)y)$ holds for $\lambda>0$. Next, we see that $R_{N\alpha}\in\mathcal{K}^{N+\langle m\rangle}(U\times\mathbb{R}^n)$ as $\langle\beta\rangle-\langle\alpha\rangle\geq N$ holds in the summation of $R_{N\alpha}$. Thus, $R_{N\alpha}(x,y)\in C^M(U\times U)$ as soon as N is large enough. To see the smoothness of $R_N(x,y)$, we state the following lemma.

Lemma 10.3.4. *In a neighborhood of $y=0$ we have*

$$\Theta(x,y)^{\alpha}=\sum_{\langle\alpha\rangle+|\alpha|\leq\langle\beta\rangle\leq n|\alpha|} C_{\beta t}(x)\Phi_t(x,y)^{\beta}, \tag{10.58}$$

where $C_{\beta t}(x)$ are smooth functions which are bounded uniformly for $x\in U$ and $t\in[0,1]$.

Proof. As $\Phi_t(x,y)=\phi'_H(x)y+t\Theta(x,y)$ with $\Theta(x,y)=O(y^{w+1})$, one can apply Lemma 10.B.1 to see that

$$(\Phi_t(x,\cdot))^{-1}(y)=(\phi'_H(x))^{-1}y+\widetilde{\Theta}_t(x,y), \tag{10.59}$$

where $\widetilde{\Theta}_t(x,y)=O(y^{\mathrm{w}+1})$ uniformly for $t\in[0,1]$. Using this and the fact that $\Theta(x,y)=O(y^{w+1})$ again, we have

$$\Theta(x,(\Phi_t(x,\cdot))^{-1}(y))^{\alpha}=\Theta(x,(\phi'_H(x))^{-1}y+\widetilde{\Theta}_t(x,y))^{\alpha}=\sum_{\langle\alpha\rangle+|\alpha|\leq\langle\beta\rangle\leq n|\alpha|} C_{\beta}(x)y^{\beta}. \tag{10.60}$$

By taking $y\to\Phi_t(x,y)$ we get

$$\Theta(x,y)^{\alpha}=\sum_{\langle\alpha\rangle+|\alpha|\leq\langle\beta\rangle\leq n|\alpha|} C_{\alpha\beta}(x)\Phi_t(x,y)^{\beta}. \tag{10.61}$$

The lemma is proved. □

By applying this lemma, we may write (10.54) as

$$R_N(x,y)=\sum_{\langle\alpha\rangle=N}\sum_{\langle\alpha\rangle+|\alpha|\leq\langle\beta\rangle\leq n|\alpha|} C_{\beta}(x)|\partial_y\Phi(x,y)|\int_0^1(t-1)^{N-1}\Phi_t(x,y)^{\beta}\partial_{\tilde{y}}^{\alpha}K_{\widetilde{P}}(\phi(x),\Phi_t(x,y))dt. \tag{10.62}$$

Here we observe that

$$y^{\beta}\partial_2^{\alpha}K_{\widetilde{P}}(\phi(x),y)\in\mathcal{K}^{\langle\beta\rangle-\langle\alpha\rangle}(U\times\mathbb{R}^n), \tag{10.63}$$

and $\langle\beta\rangle-\langle\alpha\rangle\geq|\alpha|=N$. Hence this is contained in C^M as soon as N is large enough. Thus $R_N(x,y)\in C^M(U\times U)$ if N is large enough. The proof is completed. □

10.3.3 Adjoint of pseudodifferential operators

Proposition 10.3.5. *Let $P \in \Psi_H^m(U)$. Then the following holds:*

1. *The transpose operator P^t is a $\Psi_H DO$ of order m on U.*
2. *If we write the distribution kernel of P in the form* (10.11) *with $K_P(x,y)$ in $\mathcal{K}^{\widehat{m}}(U \times \mathbb{R}^n)$ then P^t is of the form* (10.11) *with $K_{P^t} \in \mathcal{K}^{\widehat{m}}(U \times \mathbb{R}^n)$ such that*

$$K_{P^t}(x,y) \sim \sum_{\langle\alpha\rangle+|\alpha|\leq\langle\beta\rangle} \sum_{|\gamma|\leq|\gamma|\leq n|\gamma|} a_{\alpha\beta\gamma\gamma}(x) y^{\beta+\gamma} (\partial_x^\gamma \partial_y^\alpha K_P)(x,-y), \tag{10.64}$$

where $a_{\alpha\beta\gamma\gamma}(x) = \frac{|\varepsilon_x^{-1}|}{\alpha!\beta!\gamma!\gamma!}[\partial_y^\beta(|\varepsilon'_{\varepsilon_x^{-1}(-y)}|(y-\varepsilon_{\varepsilon_x^{-1}(y)}(x))^\alpha)\partial_y^\gamma(\varepsilon_x^{-1}(-y)-x)^\gamma](x,0)$. In particular we have

$$K_{P^t}(x,y) = K_P(x,-y) \quad mod\ \mathcal{K}^{\widehat{m}+1}(U \times \mathbb{R}^n). \tag{10.65}$$

Proof. By definition 10.1.6 the kernel of P_K is given by

$$k_P(x,y) = |\varepsilon'_x| K_P(x, -\varepsilon_x(y)). \tag{10.66}$$

And the kernel of P^t is given by $k_{P^t}(x,y) = k_P(y,x)$, which is equal to

$$|\varepsilon'_y| K_P(y, -\varepsilon_y(x)) + R(y,x) = |\varepsilon'_x| K(x, -\varepsilon_x(y)) + R(y,x), \tag{10.67}$$

where

$$K(x,y) = |\varepsilon'_x|^{-1} |\varepsilon'_y| K_P\left(\varepsilon_x^{-1}(-y), -\varepsilon_{\varepsilon_x^{-1}(-y)}(x)\right). \tag{10.68}$$

Taking $z \to y$ and $y \to 0$ in (10.30) we have

$$\varepsilon_{\varepsilon_x^{-1}(-y)}(x) = -y - \Theta(x,y), \tag{10.69}$$

with $\Theta(x,y) = O(y^{\mathrm{w}+1})$. Using the Taylor expansion, we have

$$\begin{aligned} K(x,y) &= |\varepsilon'_x|^{-1} |\varepsilon'_y| K_P\left(\varepsilon_x^{-1}(-y), y + \Theta(x,y)\right) \\ &= \sum_{|\alpha|<N} |\varepsilon'_x|^{-1} |\varepsilon'_y| \frac{\Theta(x,y)^\alpha}{\alpha!} (\partial_2^\alpha K_P)(\varepsilon_x^{-1}(-y), y) + R_N(x,y). \end{aligned} \tag{10.70}$$

Here $R_N(x,y)$ is equal to

$$\sum_{|\alpha|=N} |\varepsilon'_x|^{-1} |\varepsilon'_y| \frac{\Theta(x,y)^\alpha}{\alpha!} \int_0^1 (1-t)^{N-1} (\partial_2^\alpha K_p)(\varepsilon_x^{-1}(-y), \Psi_t(x,y)), \tag{10.71}$$

where we let $\Psi_t(x,y) = y + t\Theta(x,y)$.

Let $a_\alpha(x, y) = |\varepsilon_x'|^{-1}|\varepsilon_y'|\frac{\Theta(x,y)^\alpha}{\alpha!}$. As in (10.55) we have

$$a_\alpha(x, y) = \sum_{|\alpha|+\langle\alpha\rangle\leq\langle\beta\rangle<2N} a_{\alpha\beta}(x)y^\beta + \sum_{\langle\beta\rangle=2N} r_{N\alpha}(x, y)y^\beta, \tag{10.72}$$

where $a_\alpha(x) = \frac{1}{\beta!}\partial^\beta a_\alpha(x, 0)$ and $r_{N\alpha}(x, y) \in C^\infty(U \times U)$. Plugging this into (10.70) we have

$$\begin{aligned} K(x, y) = &\sum_{|\alpha|<N} \sum_{|\alpha|+\langle\alpha\rangle\leq\langle\beta\rangle<2N} a_{\alpha\beta}(x)y^\beta(\partial_y^\alpha K_P)(\varepsilon_x^{-1}(-y), y) \\ &+ \sum_{|\alpha|<N} R_{N\alpha}(x, y) + R_N(x, y), \end{aligned} \tag{10.73}$$

where we have let

$$R_{N\alpha}(x, y) = \sum_{\langle\beta\rangle=2N} r_{N\alpha}(x, y)y^\beta(\partial_y^\alpha K_P)(\varepsilon_x^{-1}(-y), y). \tag{10.74}$$

Next, a further Taylor expansion around $(\partial_2^\alpha K_P)(x, y)$ shows

$$\begin{aligned} (\partial_y^\alpha K_P)(\varepsilon_x^{-1}(-y), y) = &\sum_{|\gamma|<N} \frac{1}{\gamma!}(\varepsilon_x^{-1}(-y) - x)^\gamma(\partial_x^\gamma\partial_y^\alpha K_P)(x, y) \\ &+ \sum_{|\gamma|=N} \int_0^1 (1 - t)^{N-1}(\partial_x^\gamma\partial_y^\alpha K_P)(\varepsilon_t(x, y), y), \end{aligned} \tag{10.75}$$

where we have let $\varepsilon_t(x, y) = x + t(\varepsilon_x^{-1}(-y) - x)$. As $\varepsilon_x^{-1}(-y) - x$ is a polynomial whose degree is at most n and $\epsilon_x^{-1}(0) - x = 0$, we have

$$\frac{1}{\gamma!}(\varepsilon_x^{-1}(-y) - x)^\gamma = \sum_{|\gamma|\leq|\gamma|\leq n|\gamma|} b_{\gamma\gamma}(x)y^\gamma, \tag{10.76}$$

where $b_{\gamma\gamma}(x) = \frac{1}{\gamma!\gamma!}\partial_y^\gamma(\varepsilon_x^{-1}(-y) - x)^\gamma\ |_{y=0}$. Thus,

$$\begin{aligned} K(x, y) = &\sum_{\alpha,\beta,\gamma,\gamma}^{(N)} K_{\alpha\beta\gamma\gamma}(x, y) + \sum_{|\alpha|<N} \sum_{|\alpha|+\langle\alpha\rangle\leq\langle\beta\rangle<2N} R_{N\alpha\beta}(x, y) \\ &+ \sum_{|\alpha|<N} R_{N\alpha}(x, y) + R_N(x, y), \end{aligned} \tag{10.77}$$

where the first summation goes over all the multi-indices α, β, γ and γ such that $|\alpha| < N$, $|\alpha| + \langle\alpha\rangle \leq \langle\beta\rangle < 2N$ and $|\gamma| \leq |\gamma| \leq n|\gamma| < nN$, and

$$K_{\alpha\beta\gamma\gamma}(x, y) = f_{\alpha\beta\gamma\gamma}(x)y^{\beta+\gamma}(\partial_x^\gamma\partial_y^\alpha K_P)(x, y), \tag{10.78}$$

with $f_{\alpha\beta\gamma\gamma}(x) = a_{\alpha\beta}(x)b_{\gamma\gamma}(x)$ and $R_{N\alpha\beta}(x, y)$ equals to

$$\sum_{|\gamma|=N} \sum_{N\leq|\gamma|\leq nN} a_{\alpha\beta\gamma\gamma}(x)y^{\beta+\gamma} \int_0^1 (1 - t)^{N-1}(\partial_x^\gamma\partial_y^\alpha K_P)(\varepsilon_t(x, y), y). \tag{10.79}$$

Here, we observe that $y^\beta \partial_y^\alpha K_P(x, y)$ belongs to $\mathcal{K}^{\widehat{m}-\langle\alpha\rangle+\langle\beta\rangle}(U \times \mathbb{R}^n)$. As $\langle\beta\rangle - \langle\alpha\rangle \geq |\alpha|$ we have that $y^\beta \partial_y^\alpha K_P(x, y)$ is in $C^J(U \times \mathbb{R}^n)$ as soon as N is large enough. It follows that all the remainder terms $R_{N\alpha}(x, y)$, $\langle\alpha\rangle < N$, belong to $C^J(U)$ as soon as N is large enough.

Similarly, if $\langle\alpha\rangle + |\alpha| \leq \langle\beta\rangle$ and $|\gamma| = N \leq |\gamma| \leq nN$ then $\widehat{m} - \langle\alpha\rangle + \langle\beta\rangle + \langle\gamma\rangle \geq \widehat{m} + \langle\gamma\rangle \geq \widehat{m} + N$, so we see that $y^{\beta+\gamma}(\partial_x^\gamma \partial_y^\alpha K_P)(x, y)$ is in $C^J(U \times \mathbb{R}^n)$ for N large enough. Therefore $R_{N\alpha\beta}(x, y)$ with $\langle\alpha\rangle < N$ and $|\alpha| + \langle\alpha\rangle \leq \langle\beta\rangle = 2N$ are all contained in $C^J(U)$ as soon as N is large enough.

To handle the last remainder term $R_N(x, y)$, we use Lemma 10.3.4 again to have

$$\Theta(x, y)^\alpha = \sum_{\langle\alpha\rangle+|\alpha|\leq\langle\beta\rangle\leq n|\alpha|} C_\beta(x)\Phi_t(x, y)^\beta. \tag{10.80}$$

Therefore $\chi(x, y)R_N(x, y)$ equals to

$$\sum_{|\alpha|=N} \sum_{\langle\beta\rangle=\langle\alpha\rangle+|\alpha|} |\varepsilon'_x|^{-1}|\varepsilon'_y| \int_0^1 r_{N\alpha\beta}(t, x, y)(y^\beta \partial_y^\alpha K_P)(\varepsilon_x^{-1}(-y), \Phi_t(x, y)), \tag{10.81}$$

for some functions $r_{N\alpha\beta}(t, x, y) \in C^\infty([0, 1] \times U \times \mathbb{R}^n)$. Since $(y^\beta \partial_y^\alpha K_P)$ is in $\mathcal{K}^{\widehat{m}-\langle\alpha\rangle+\langle\beta\rangle}(U \times \mathbb{R}^n)$ and we have $m - \langle\alpha\rangle + \langle\beta\rangle \geq m + |\alpha| = m + N$, we see that $\chi(x, y)R_N(x, y)$ is in $C^J(U)$ as soon as N is large enough. As $\chi(x, y)R_N(x, y)$ is properly supported with respect to x, it belongs to $C^J(U \times \mathbb{R}^n)$. □

10.4 Mapping properties on L^p spaces

In this section we prove the following theorem.

Theorem 10.4.1. *Let P be a $\Psi_H DO$ of order 0. Then, there P is bounded on $L^p(M)$ for any $1 < p < \infty$.*

We first prove this result for the case $p = 2$. The proof relys on the Cotlar-Stein lemma and the property of Carnot coordinates obtained in Lemma 9.6.23.

Lemma 10.4.2 (Cotlar-Stein Lemma)**.** *Let T^1, T^2, $\cdots$, be a family of bounded operators on a Hilbert space. Suppose that for a number $0 < \gamma < 1$, they satisfy the estimates*

$$\|T^j(T^k)^*\| \leq \gamma^{|j-k|} \quad \textit{and} \quad \|(T^j)^*T^k\| \leq C\gamma^{|j-k|} \tag{10.82}$$

for any k and j. Then, it holds that

$$\|\sum_{j=1}^N T^j\| \leq C \tag{10.83}$$

for some constant $C > 0$ independent of $N \in \mathbb{N}$.

Proof for the case $p = 2$. The operator P is given by

$$Pf(x) = \int_{\mathbb{R}^n} |\varepsilon_x'(y)| K(x, -\varepsilon_x(y)) f(y) dy, \tag{10.84}$$

for some $K \in \mathcal{K}^{-Q}(U)$. For each $j \in \mathbb{Z}$ we let $K_j(x, y) := K(x, y) 1_{2^{-j} \le \|y\| < 2^{-j+1}}(y)$ and

$$T_j f(x) = \int |\varepsilon_x'(y)| K_j(x, -\varepsilon_x(y)) f(y) dy. \tag{10.85}$$

By the Cotlar-Stein lemma it is enough to show that

$$\|T_j\|_{L^2 \to L^2} \le C, \tag{10.86}$$

for some $C > 0$ independent of $j \in \mathbb{Z}$ and for some $\gamma > 1$,

$$\|T_k(T_j^*) f\| \le C\gamma^{-|k-j|} \quad \text{and} \quad \|T_k^*(T_j) f\| \le C\gamma^{-|k-j|} \tag{10.87}$$

hold for any $(k, j) \in \mathbb{Z}^2$.

Since $K \in \mathcal{K}^{-Q}(U)$ there is a constant $C > 0$ such that $|K(x, y)| \le C|y|^{-Q}$. Hence we have

$$\int |\varepsilon_x'(y)||K_j(x, -\varepsilon_x(y))| dy = \int |K_j(x, y)| dy \le C \tag{10.88}$$

uniformly for j. Then we may apply Young's inequality to deduce that

$$\|T_j\|_{L^2 \to L^2} \le C, \tag{10.89}$$

uniformly for $j \in \mathbb{Z}$. This proves (10.86).

We are left to show (10.87). We shall only prove $\|T_k(T_j^*) f\| \le C\gamma^{-|k-j|}$ since the other one can be proved in a similar way. In addition we shall prove it for the case $k > j + 10$ only. The other case $k < j - 10$ can be handled in a similar manner.

For this aim, we shall estimate the kernel of the operator

$$\begin{aligned} T_k(T_j^*) f(x) &= \int |\varepsilon_x'(z)| K_k(x, -\varepsilon_x(y)) T_j^* f(z) dz \\ &= \int |\varepsilon_x'(z)| K_k(x, -\varepsilon_x(y)) \left(\int |\varepsilon_y'(z)| K_j(y, -\varepsilon_y(z)) f(y) dy \right) dz \\ &= \int \left(\int |\varepsilon_x'(z)||\varepsilon_y'(z)| K_k(x, -\varepsilon_x(z)) K_j(y, -\varepsilon_y(z)) dz \right) f(y) dy. \end{aligned} \tag{10.90}$$

Let

$$\mathbf{K}_{kj}(x, y) = \int |\varepsilon_x'(z)||\varepsilon_y'(z)| K_k(x, -\varepsilon_x(z)) K_j(y, -\varepsilon_y(z)) dz. \tag{10.91}$$

Lemma 10.4.3. *There exists a constant $C > 0$ independent of j such that*

$$|K_j(x) - K_j(y)| \leq C\|x - y\|\|x\|^{-Q-1} \tag{10.92}$$

holds for any $x, y \in \mathbb{R}^n$ satisfying $\|x - y\| \leq \frac{1}{2}\|x\|$

Proof. By definition it is enough to show that

$$\left|\phi(2^{-j} \cdot x)K(x) - \phi(2^{-j} \cdot y)K(y)\right| \leq C\|x - y\|\|x\|^{-Q-1}. \tag{10.93}$$

Changing the variables as $x \to 2^j \cdot x$ and $y \to 2^j \cdot y$, it is equivalent to

$$2^{-jQ}|\phi(x)K(x) - \phi(y)K(y)| = |\phi(x)K(2^j \cdot x) - \phi(y)K(2^j \cdot y)| \leq C2^{-jQ}\|x - y\|\|x\|^{-Q-1}, \tag{10.94}$$

where the first identity follows from the homogeneity of K.

In order to prove (10.94), we use the mean value formula to get

$$\begin{aligned} |\phi(x)K(x) - \phi(y)K(y)| &\leq |x - y| \sup_t |\nabla(\phi K)(x + t(y - x))| \\ &\leq C|x - y|, \end{aligned} \tag{10.95}$$

since ϕK is a $C^1(\mathbb{R}^n)$ function. Noting that $|\phi(x)K(x) - \phi(y)K(y)| = 0$ unless that at least one of $\frac{1}{2} \leq \|x\| \leq 2$ and $\frac{1}{2} \leq \|y\| \leq 2$ holds by definition of ϕ, we deduce from (10.95) that

$$\begin{aligned} |\phi(x)K(x) - \phi(y)K(y)| &\leq C|x - y|(\|x\|^{-Q-1} + \|y\|^{-Q-1}) \\ &\leq C_1|x - y|\|x\|^{-Q-1} \\ &\leq C_1\|x - y\|\|x\|^{-Q-1}, \end{aligned} \tag{10.96}$$

where the second inequality holds as $\frac{1}{2}\|x\| \leq \|y\| \leq \frac{3}{2}\|x\|$ and the last inequality holds by (10.5). We note that (10.96) is same with (10.94). Hence the lemma is proved. □

Letting $a(x, y, z) := |\varepsilon_x'(z)||\varepsilon_y'(z)|$ we write (10.91) as

$$\begin{aligned} \mathbf{K}_{kj}(x, y) = &\int a(x, y, z)K_k(x, -\varepsilon_x(z)) \left[K_j(y, -\varepsilon_y(z)) - K_j(y, -\varepsilon_y(x))\right] dz \\ &+ \left(\int a(x, y, z)K_k(x, -\varepsilon_x(z))dz\right) K_j(y, -\varepsilon_y(x)). \end{aligned} \tag{10.97}$$

Using Lemma 10.4.3 we have

$$|K_j(y, -\varepsilon_y(z)) - K_j(y, -\varepsilon_y(x))| \leq C_1\|\varepsilon_y(z) - \varepsilon_y(x)\|\|\varepsilon_y(x)\|^{-Q-1}. \tag{10.98}$$

In order to estimate the right hand side, we use Lemma 10.B.2 to get

$$|K_j(y, -\varepsilon_y(z)) - K_j(y, -\varepsilon_y(x))| \leq C_1 \left(\|\varepsilon_x(z)\| + \|\varepsilon_x(z)\|^{\frac{1}{m}}\|\varepsilon_y(z)\|^{1-\frac{1}{m}}\right) \|\varepsilon_y(x)\|^{-Q-1}. \tag{10.99}$$

Recall that $2^{-k-1} \leq \|\varepsilon_x(z)\| \leq 2^{-k+1}$, $2^{-j-1} \leq \|\varepsilon_y(z)\| \leq 2^{-j+1}$ if $K_{kj} \neq 0$ in (10.91). Since $k > j+10$ this also implies that $2^{-j-2} \leq \|\varepsilon_y(x)\| \leq 2^{-j+2}$. Injecting these estimates into (10.99) we get

$$|K_j(y, -\varepsilon_y(z)) - K_j(y, -\varepsilon_y(x))| \leq 4C_1\Big(2^{-k} + 2^{-\frac{k}{m}}2^{-j(1-\frac{1}{m})}\Big)2^{j(Q+1)}. \tag{10.100}$$

Using this we estimate the first integration of (10.97) as

$$\begin{aligned}
&\int_{2^{-j-2}\leq\|\varepsilon_y(x)\|\leq 2^{-j+2}} \left(\int \Big|a(x,y,z)K_k(x, -\varepsilon_x(z))\left[K_j(y,-\varepsilon_y(z)) - K_j(y, -\varepsilon_y(x))\right]\Big|dz\right)dy \\
&\leq \Big(2^{-k} + 2^{-\frac{k}{m}}2^{-j(1-\frac{1}{m})}\Big)2^{j(Q+1)} \int_{2^{-j-2}\leq\|\varepsilon_y(x)\|\leq 2^{-j+2}} \left(\int \Big|a(x,y,z)K_k(x,-\varepsilon_x(z))\Big|dz\right)dy \\
&\leq C\Big(2^{-k} + 2^{-\frac{k}{m}}2^{-j(1-\frac{1}{m})}\Big)2^{j(Q+1)}2^{-jQ}\log(2) \\
&= C(2^{-(k-j)} + 2^{-\frac{(k-j)}{m}}).
\end{aligned} \tag{10.101}$$

Next we turn to estimate the second integration of (10.97),

$$\left(\int a(x,y,z)K_k(x,-\varepsilon_x(z))dz\right)K_j(y, -\varepsilon_y(x)). \tag{10.102}$$

Since $K \in \mathcal{K}^{-Q}$ it holds that

$$\int_{\|z\|=1} K_k(x,z)dS_z = 0. \tag{10.103}$$

Using this we deduce that

$$\begin{aligned}
&\int a(x,y,z)K_k(x,-\varepsilon_x(z))dz = \int |\varepsilon_x'(z)|^{-1}a(x,y,\varepsilon_x^{-1}(z))K_k(x,z)dz \\
&= \int a(x,y,\varepsilon_x^{-1}(0))|\varepsilon_x'(0)|^{-1}K_k(x,z)dz \\
&\quad + \int \left[a(x,y,\varepsilon_x^{-1}(z))|\varepsilon_x'(z)|^{-1} - a(x,y,\varepsilon_x^{-1}(0))|\varepsilon_x(0)'|^{-1}\right]K_k(x,z)dz \\
&= \int \left[a(x,y,\varepsilon_x^{-1}(z))|\varepsilon_x'(z)|^{-1} - a(x,y,\varepsilon_x^{-1}(0))|\varepsilon_x'(0)|^{-1}\right]K_k(x,z)dz.
\end{aligned} \tag{10.104}$$

We estimate this using the mean value formula to get

$$\begin{aligned}
&\left|\int \left[a(x,y,\varepsilon_x^{-1}(z))|\varepsilon_x'(z)|^{-1} - a(x,y,\varepsilon_x^{-1}(0))|\varepsilon_x'(0)|^{-1}\right]K_k(x,z)dz\right| \\
&\qquad \leq C\int \|z\||K_k(x,z)|dz \leq C\int \psi(2^k\cdot z)\|z\|^{-Q+1}dz = C2^{-k}.
\end{aligned} \tag{10.105}$$

Combining (10.104) and (10.105) we have

$$\begin{aligned}
&\int \left|\left(\int a(x,y,z)K_k(x,-\varepsilon_x(z))dz\right)K_j(y,-\varepsilon_y(x))\right|dy \\
&\qquad \leq C2^{-k}\int |K_j(y,-\varepsilon_y(x))|dy \leq C\log(2)2^{-k} \leq C2^{-(k-j)}.
\end{aligned} \tag{10.106}$$

Collecting the estimates (10.101) and (10.106) with (10.97), we get

$$\int |\mathbf{K}_{kj}(x,y)|dy \leq C2^{-\frac{(k-j)}{m}}. \tag{10.107}$$

This estimate yields the inequality

$$\|T_k(T_j^*)f\| \leq C2^{-\frac{(k-j)}{m}}. \tag{10.108}$$

Hence the proof is finished. □

In order to extend the above result to the case $1 < p < \infty$, we recall the following result.

Theorem 10.4.4 (Coifman and Weiss)**.** *Let* $L(x,y)$ *be a function supported in* $\{(x,y) : |\phi_x(y)| \leq 1\}$ *with the properties:*

1. *T is bounded on L^2.*
2. *For some $C_1 > 0$ and C_2*

$$\int_{|\varepsilon_x(y)|>C_1|\phi_x(z)|} |L(y,z) - L(y,x)|dy \leq C_2. \tag{10.109}$$

3. *$Tg(x) = \int L(x,y)g(y)dy$ exists a.e. for all $g \in L^p$, $1 \leq p \leq 2$.*

Then T is bounded on L^p for each $1 < p \leq 2$.

Proof of Theorem 10.4.1. We shall first prove the condition (10.109). Recall that the kernel of P is given by

$$L(x,y) = |\varepsilon_x'(y)|K(x, -\varepsilon_x(y)). \tag{10.110}$$

Hence, to check (10.109), it is enough to show that

$$\int_{\|\varepsilon_x(y)\|\geq C_1\|\varepsilon_x(z)\|} |K(y, -\varepsilon_y(z)) - K(y, -\varepsilon_y(x))|dy \leq C_2 \tag{10.111}$$

for some $C_2 > 0$.

To obtain it, we apply Lemma 10.4.3 and Lemma 10.B.2 to get

$$\begin{aligned} |K(y,-\varepsilon_y(z)) - K(y,-\varepsilon_y(x))| &\leq C\|\varepsilon_y(z) - \varepsilon_y(x)\|\|\varepsilon_y(z)\|^{-Q-1} \\ &\leq C\Big(\|\varepsilon_x(z)\| + \|\varepsilon_x(z)\|^{\frac{1}{m}}\|\varepsilon_y(z)\|^{\frac{m-1}{m}}\Big)\|\varepsilon_y(z)\|^{-Q-1} \\ &= C\Big(\|\varepsilon_x(z)\|\|\varepsilon_y(z)\|^{-Q-1} + \|\varepsilon_x(z)\|^{\frac{1}{m}}\|\varepsilon_y(z)\|^{-Q-\frac{1}{m}}\Big). \end{aligned} \tag{10.112}$$

Using this, we esimate (10.111) as

$$
\begin{aligned}
\int_{\|\varepsilon_x(y)\|\geq C_1\|\varepsilon_x(z)\|} & |K(y,-\varepsilon_y(z)) - K(y,-\varepsilon_y(x))|dy \\
&\leq C\int_{\|\varepsilon_x(y)\|\geq C_1\|\varepsilon_x(z)\|}\left(\|\varepsilon_x(z)\|\|\varepsilon_y(z)\|^{-Q-1} + \|\varepsilon_x(z)\|^{\frac{1}{m}}\|\varepsilon_y(z)\|^{-Q-\frac{1}{m}}\right) \\
&\leq C\int_{\|z\|\geq\|\varepsilon_x(z)\|}\left(\|\varepsilon_x(z)\|\|z\|^{-Q-1} + \|\varepsilon_x(z)\|^{\frac{1}{m}}\|z\|^{-Q-\frac{1}{m}}\right) \\
&= C\left(\|\varepsilon_x(z)\|\|\varepsilon_x(z)\|^{-1} + \|\varepsilon_x(z)\|^{\frac{1}{m}}\|\varepsilon_x(z)\|^{-\frac{1}{m}}\right) = C.
\end{aligned}
\tag{10.113}
$$

It shows that P satisfies the condition (2) of Theorem 10.4.4. The condition (1) was shown previously. The condtion (3) can be checked by a standard argument. Hence we may adapt Theorem 10.4.4 to conclude that P is bounded on L^p for each $1 < p \leq 2$. The proof is completed. □

10.5 Rockland condition and the construction of parametrix

In this section we shall discuss on the invertibility of the pseudodifferential operators related to the Rockland condition. Basically we heavily rely on the result of [CGGP] for $\Psi_H DO$s on Carnot groups and the argument in [P2] where the result of [CGGP] is extended to Heisenberg manifolds.

We say that P satisfies the Rockland condition at a if for any nontrivial unitary irreducible representation π of G_aM the operator $\overline{\pi_{P^a}}$ is injective on $C^\infty_\pi(\varepsilon_a)$.

Theorem 10.5.1. *Let $P : C^\infty_0(M) \to C^\infty(M)$ be a $\Psi_H DO$ of order m. Then the following are equivalent:*

(i) P admits a parametrix Q in $\Psi^{-m}_H(M)$ such that $PQ = QP = 1$ mod $\Psi^{-\infty}(M)$.

(ii) The principal symbol $\sigma_m(P)$ of P is invertible with respect to the convolution product for homogeneous sybols.

(iii) P and P^t satisfy the Rockland condition at every point $a \in M$.

Proof. See [CGGP] and [P2, Section 3]. □

10.6 Heat equation

In this section, we shall study the pseudodifferential operators which are fit to study the heat equations with hypoelliptic diffusions. The main objective of this study is to calculate the asymptotic formula of the heat kernels.

We consider the variables $z = (x, t) \in \mathbb{R}^n \times \mathbb{R}$ and $\zeta = (\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}$.

We set the isotropic dilation

$$\lambda \cdot z = (\lambda \cdot x, \lambda^2 t), \quad \lambda \cdot \zeta = (\lambda \cdot \xi, \lambda^2 \tau), \quad \lambda \in \mathbb{R} \setminus 0. \tag{10.114}$$

For f a function on $\mathbb{R}^{n+1} \setminus 0$ we define $u_\lambda(z) = u(\lambda \cdot z)$. It is extended to distribution by

$$\langle g_\lambda, u \rangle = \lambda^{-Q-2} \langle g, u_{\frac{1}{\lambda}} \rangle. \tag{10.115}$$

Let $a \in U$. The group law of $G^{(a)} \times \mathbb{R}$;

$$(x,\ t) \cdot (y,\ s) = (x \cdot y,\ t + s). \tag{10.116}$$

Then the convolution is defined by

$$(u * v)(z) = \int u(w^{-1} \cdot z) v(w) dw = \int u(w) v(z \cdot w^{-1}) dw. \tag{10.117}$$

The dilations (10.114) are automorphisms of G and

$$(u * v)_\lambda = \lambda^{Q+2} u_\lambda * v_\lambda, \quad \lambda \in \mathbb{R} \setminus 0. \tag{10.118}$$

Convolution is associative and satisfies

$$(u * v)(z) = \langle u, (\tilde{v})_z \rangle, \tag{10.119}$$

$$\langle u_1 * u_2, v \rangle = \langle u_2, \tilde{u}_1 * v \rangle = \langle u_1, v * \tilde{u}_2 \rangle, \tag{10.120}$$

where

$$\langle u, v \rangle = \int uv, \quad \tilde{v}(z) = v(z^{-1}), \quad v_z(w) = v(wz^{-1}). \tag{10.121}$$

We set the following norm;

$$\|z\| = \sum_{j=1}^{n} |x_j|^{\frac{1}{w_j}} + |t|^{\frac{1}{2}}, \text{for } z = (x, t) \in G^{(a)} \times \mathbb{R}. \tag{10.122}$$

See that $\|\lambda \cdot z\| = |\lambda| \|z\|$. We say that a function or distribution f is homogeneous of degree m if and only if

$$f_\lambda = \lambda^m f, \quad \forall \lambda \in \mathbb{R} \setminus 0. \tag{10.123}$$

Definition 10.6.1. $\mathcal{S}_m(\mathbb{R}^{n+1})$ is the set of functions in $C^\infty(\mathbb{R}^{n+1} \setminus 0)$ which are homogeneous of degree m.

Set

$$\psi(z) = -(z^{-1}), \quad g^\sharp = \check{g} \circ \psi, \quad k^b = (k \circ \psi^{-1})^\wedge. \tag{10.124}$$

It is easy to check that Proposition 2.19 holds with

$$k_j = g_j^\sharp, \quad g = (k_1 * k_2)^b. \tag{10.125}$$

Note that any $f \in \mathcal{F}_m$ is the homogeneous part of an almost homogeneous g; indeed one may take $g = \chi f$, where $\chi \in C^\infty$ is $\equiv 0$ near 0 and $\equiv 1$ near ∞. Therefore the following construction is well defined.

Definition 10.6.2. Suppose f_j belongs to $\mathcal{S}_{m_j}(\mathbb{R}^{n+1}), j = 1, 2$. Then $T(f_1, f_2)$ is the element of $\mathcal{S}_m(\mathbb{R}^{n+1})$, $m = m_1 + m_2$, which is defined by

$$T(f_1, f_2) = \hom([g_1^\sharp * g_2^\sharp]^b), \tag{10.126}$$

where the g_j are almost homogeneous with $\hom(g_j) = f_j$.

Definition 10.6.3. $\mathcal{S}_{m,h}(\mathbb{R}^{n+1})$ consists of the functions $f(\xi, \tau) \in \mathcal{S}_m(\mathbb{R}^{n+1})$ which extend to $(\mathbb{R}^{n+1} \times \bar{\mathbb{C}}_-) \setminus \{0\}$ so that ts is C^∞ in all variables and holomorphic with respect to τ, $\tau \in \mathbb{C}_-$.

The extension is unique and will be denoted by f; it will continue to be homogeneous with respect to the dialtion (10.114) which act on $\mathbb{R}^n \times \bar{\mathbb{C}}_-$. We recall the following results.

Proposition 10.6.4 ([BGS])**.**

1. *If f belongs to $\mathcal{S}_{m,h}(\mathbb{R}^{n+1})$, then there is a distribution g such that g is homogeneous of degree m and g agrees with f on $\mathbb{R}^{n+1} \setminus \{0\}$.*

2. *Suppose g is a tempered distribution which is homogeneous of degree m. Then the restriction of g to $\mathbb{R}^{n+1} \setminus 0$ is smooth if and only if the restriction of $k = \hat{g}$ to $\mathbb{R}^{n+1} \setminus 0$ is smooth. If k also vanishes for $t < 0$, then the restriction of g belongs to $\mathcal{S}_{m,h}(\mathbb{R}^{n+1})$. Conversely, if f belongs to $\mathcal{S}_{m,h}(\mathbb{R}^{n+1})$, then the distribution g of Proposition 10.6.4 can be chosen so that $k = \check{g}$ vanishes for $t < 0$.*

3. *Suppose f_j belongs to $\mathcal{S}_{m_j,h}(\mathbb{R}^{n+1})$, $j = 1, 2$. Then $f = T(f_1, f_2)$ belongs to $\mathcal{S}_{m,h}(\mathbb{R}^{n+1})$, $m = m_1 + m_2$.*

We define the following class of kernels.

Definition 10.6.5. $\mathcal{K}_{v,m}(U \times \mathbb{R}^{n+1}_{(v)})$, $m \in \mathbb{Z}$, is the set of distributions $K(x, y, t)$ in $C^\infty(U) \hat{\oplus} \mathcal{S}'_{reg}(\mathbb{R}^{n+1})$ such that:

1. The support of $K(x, y, t)$ is contained in $U \times \mathbb{R}^{n+1} \times \mathbb{R}^+$;

2. $K(x, \lambda \cdot y, \lambda^v t) = (\mathrm{sign}\lambda)^Q \lambda^m K(x, y, t)$ for any $\lambda \in \mathbb{R} \setminus 0$.

Note that if $Q + m$ is odd, it should hold that $K(x, 0, t) = 0$ since $K(x, (-1) \cdot y, t) = -K(x, y, t)$ holds by the homogeneity.

Definition 10.6.6. $\mathcal{K}_v^m(U \times \mathbb{R}^{d+1})$, $m \in \mathbb{Z}$, is the set of distributions $K(x, y, t)$ in $\mathcal{D}'(U \times \mathbb{R}^{d+1})$ which admit an asymptotic expansion $K \sim \sum_{j \geq 0} K_{m+j}$ with K_{m+j} in $\mathcal{K}_{v,m+j}(U \times \mathbb{R}^{d+1})$ in the sense that, for any integer N, as soon as J is large enough we have

$$K - \sum_{j \leq J} K_{m+j} \in C^N(U \times \mathbb{R}^n). \tag{10.127}$$

Next we define the class of pseudodifferential operators.

Definition 10.6.7. The class $\Psi^m_{H,v}(U \times \mathbb{R})$ consists of the operators P whose kernel can be put in the form

$$k_P(x,t,;y,s) = |\varepsilon'_x| K_P(x, -\varepsilon_x(y), t-s) + R(x,y,t-s), \tag{10.128}$$

with K_P in $\mathcal{K}^{\hat{m}}_v(U \times \mathbb{R}^{d+1}_{(v)})$, $\hat{m} = -(m+Q+v)$, and R in $C^\infty(U \times \mathbb{R}^{d+1})$.

Then the composition formula follows directly from the result of Proposition 10.3.2.

Proposition 10.6.8. *Consider two pseudo-differential operators P_{K_1} and P_{K_2} with $K_1 \in \mathcal{K}^{\widehat{m_1}}(U \times \mathbb{R}^n \times \mathbb{R}^+)$ and $K_2 \in \mathcal{K}^{\widehat{m_2}}(U \times \mathbb{R}^n \times \mathbb{R}^+)$. Assume that one of the operators is properly supported. Then, $P_{K_1} \circ P_{K_2} \in \Psi^{\widehat{m_1+m_2}}$ and there exists a kernel $K \in \mathcal{K}^{\widehat{m_1+m_2}}(U \times \mathbb{R}^n \times \mathbb{R}^+)$ such that*

$$P_{K_1} \circ P_{K_2} = P_K, \tag{10.129}$$

and the principal symbol of K equals to $(K_1)_{\widehat{m_1}} \sharp (K_2)_{\widehat{m_2}}$. Generally the term with homogeneous degree k in the asymptotic expansion of $P_{K_1} \circ P_{K_2}$ is given by the form

$$\sum C_{\alpha\beta\gamma\delta}((K_1)_l)^\gamma \sharp ((K_2)_t)^{\delta;\beta}_\alpha \tag{10.130}$$

where $C_{\alpha\beta\gamma\delta}$ are functions in $C^\infty(U)$ independent of the operators P_{K_1} and P_{K_2}. The sum is finite, taken over indices such that

- $-m_1 - (Q+v) \le l$ *and* $-m_2 - (Q+v) \le t$,
- $(-l-(Q+v)) - \langle\gamma\rangle + (-t-(Q+v)) - \langle\delta\rangle - \langle\beta\rangle = -k-(Q+v)$,
- $\langle\gamma\rangle + \langle\delta\rangle - \langle\beta\rangle \ge |\alpha| + |\beta|$.

Theorem 10.6.9 ([BGS])**.** *Suppose that the operator $P + \partial_t$ satisfies the "Rockland" condition. Then:*

1. *The heat operator $P+\partial_t$ has an inverse $(P+\partial_t)^{-1}$ in $\Psi^{-v}_{H,v}(M \times \mathbb{R}_{(v)}, \mathcal{E})$.*
2. *Let $K_{(P+\partial_t)^{-1}}(x,y,t-s)$ denote the kernel of $(P+\partial_t)^{-1}$. Then the heat kernel $k_t(x,y)$ of P satisfies*
$$k_t(x,y) = K_{(P+\partial_t)^{-1}}(x,y,t) \quad \text{for } t>0. \tag{10.131}$$

Proposition 10.6.10. *Let $P \in \Psi^m_{H,v}(U \times \mathbb{R}_{(v)})$ have symbol $q \sim \sum_{j\ge 0} q_{m-j}$ and kernel $k_P(x,y,t-s)$. Then as $t \to 0^+$ the following asymptotics holds in $C^\infty(U)$,*

$$k_P(x,x,t) \sim t^{-\frac{2[\frac{m}{2}]+Q+v}{v}} \sum_{j\ge 0} t^{\frac{2j}{v}} (K_Q)_{-2[\frac{m}{2}]-Q-v+2j}(x,0,1). \tag{10.132}$$

10.7 Holomorphic families of $\Psi_H DO$s

In this section we consider holomorphic families of pseudodifferential operators.

Definition 10.7.1. $\mathrm{Hol}(\Omega, S^*(U \times \mathbb{R}^n))$ is the set of holomorphic familiy of symbols $(p_z)_{z\in\Omega} \subset S^*(U \times \mathbb{R}^n)$ in the sense that

(i) The order $m(z)$ of p_z is analytic in z;

(ii) For any $(x,\xi) \in U \times \mathbb{R}^n$ the function $z \to p_z(x,\xi)$ is holomorphic on Ω;

(iii) The bounds of the asymptotic expansion (10.7) for p_z are locally uniform with respect to z, i.e., we have $p_z \sim \sum_{j\geq 0} p_{z,m(z)-j}$, $p_{z,m(z)-j} \in S_{m(z)-j}(U \times \mathbb{R}^n)$, and for any integer N and any compacts $K \subset U$ and $L \subset \Omega$ we have

$$\left| D_x^\alpha D_\xi^\beta \Big(p_z - \sum_{j<N} p_{z,m(z)-j} \Big)(x,\xi) \right| \leq C_{\alpha\beta KNL} \|\xi\|^{\Re m(z)-N-\langle\beta\rangle} \quad \forall x \in K, \quad |\xi| \geq 1, \tag{10.133}$$

for $(x,z) \in K \times L$ and $\|\xi\| \geq 1$.

Definition 10.7.2. $\mathrm{Hol}(\Omega, \Psi_H^*(U))$ is the set of holomorphic familiy $(P_z)_{z\in\Omega} \subset \Psi_H^m(U)$ in the sense that it can be put into the form

$$P_z = p_z(x, -iX) + R_z \quad z \in \Omega, \tag{10.134}$$

with $(p_z)_{z\in\Omega} \in \mathrm{Hol}(\Omega, S^*(U \times \mathbb{R}^n))$ and $(R_z)_{z\in\Omega} \in \mathrm{Hol}(\Omega, \Psi^{-\infty}(U))$.

The following result can be obtained.

Lemma 10.7.3. *Consider* $(p_{j,z})_{z\in\Omega} \in Hol(\Omega, S^*(U \times \mathbb{R}^n))$ *for* $j = 1,2$. *Then* $(p_{1,z} * p_{2,z})_{z\in\Omega} \in Hol(\Omega, S^*(U \times \mathbb{R}^n))$.

Proof. See [P2, Lemma 4.3.5]. □

Proposition 10.7.4. *Consider* $(P_{j,z})_{z\in\Omega} \in Hol(\Omega, \Psi_H^*(U))$ *for* $j = 1,2$. *Assume that at least one of them is uniformly properly supported. Then the family* $(P_{1,z}P_{2,z})_{z\in\Omega}$ *is contained in* $Hol(\Omega, \Psi_H^*(U))$.

Proof. See [P2, Proposition 4.3.6]. □

10.7.1 Kernels of holomorphic $\Psi_H DOs$

Definition 10.7.5. $\mathrm{Hol}(\Omega, \mathcal{K}_{ah}^*(U \times \mathbb{R}^d))$ consists of holomorphic family $(K_z)_{z\in\Omega} \subset \mathcal{K}_{ah}^*(U \times \mathbb{R}^d)$ in the sense that

1. The degree $m(z)$ of K_z is a holomorphic function of Ω;

2. The family $(K_z)_{z\in\Omega}$ belongs to $\mathrm{Hol}(\Omega, C^\infty(U)\otimes \mathcal{D}'_{reg}(\mathbb{R}^d))$;

3. For any $\lambda > 0$ the family $\{K_z(x, \lambda\cdot y) - \lambda^{m(z)}K_z(x, y)\}_{z\in\Omega}$ is a holomorphic family with values in $C^\infty(U\times\mathbb{R}^d)$.

We also introduce the following definition.

Definition 10.7.6. $\mathrm{Hol}(\Omega, \mathcal{K}^*(U\times\mathbb{R}^d))$ consists of holomorphic family $(K_z)_{z\in\Omega} \subset \mathcal{K}^*(U\times\mathbb{R}^d)$ in the sense that

1. The order m_z of K_z is a holomorphic function of z;

2. For $j = 0, 1, \cdots$ there exists $(K_{j,z}) \in \mathrm{Hol}(\Omega, \mathcal{K}^*_{ah}(U\times\mathbb{R}^d))$ of degree $m(z) + j$ such that $K_z \sim \sum_{j\geq 0} K_{j,z}$ in the sense that, for any open $\Omega' \subset \Omega$ and and integer N, as soon as J is large enough we have

$$K_z - \sum_{j\leq J} K_{z,m_z+j} \in \mathrm{Hol}(\Omega', C^N(U\times\mathbb{R}^d)). \tag{10.135}$$

Then we have the following characterization of the kernels of holomorphic $\Psi_H DO$s.

Proposition 10.7.7. *Let $(P_z)_{z\in\Omega} \in Hol(\Omega, \Psi^m_H(U))$. Then its distribution kernel $k_{P_z}(x, y)$ can be put in the form*

$$k_{P_z}(x, y) = |\varepsilon'_x|P_z(x, -\varepsilon_x(y)) + R_z(x, y), \tag{10.136}$$

with $(K_z)_{z\in\Omega}$ in $Hol(\Omega, \mathcal{K}^*(U\times\mathbb{R}^d))$ of order $\hat{m}(z) := -(m(z)+Q)$ and $(R_z)_{z\in\Omega}$ in $Hol(\Omega, C^\infty(U\times U))$.

Proof. See the proof of Proposition 4.4.5 in [P2]. □

10.8 Complex powers of $\Psi_H DO$s

Let $P : C^\infty(M) \to C^\infty(M)$ be a selfadjoint differential operator of even order v such that P has an invertible principals symbol and is positive, i.e., $\langle Pu, v\rangle \geq 0$ for any $u \in C^\infty(M)$.

Let $\Pi_0(P)$ be the orthogonal projection onto $\ker P$ and set $P_0 = (1 - \Pi_0(P))P + \Pi_0(P)$. Then P_0 is selfadjoint with spectrum contained in $[c, \infty)$ for some $c > 0$. Thus by standard functional calculus, for any $s \in \mathbb{C}$, the power P_0^s is a well defined unbounded operator of $L^2(M)$. Then we define the power P^s, $s \in \mathbb{C}$, as

$$P^s = (1 - \Pi_0(P))P_0^s = P_0^s - \Pi_0(P), \tag{10.137}$$

so that P^s coincides with P_0^s on $(\ker(P))^\perp$ and is zero on $\ker(P)$. Particularly, it holds that $P^0 = 1 - \Pi_0(P)$ and P^{-1} is the partial inverse of P.

We use the approach of Theorem 5.3.1. in [P2] to get the following result.

Theorem 10.8.1. *Suppose that the principal symbol of $P + \partial_t$ admits an inverse in $S_{v,-v}(g^*M \times \mathbb{R}_{(v)})$. Then:*

1. *For any $s \in C$ the operator P^s defined by* (10.137) *is a $\Psi_H DO$ of order vs;*
2. *The family $(P^s)_{s\in\mathbb{C}}$ forms a holomorphic* 1*-parameter group of $\Psi_H DOs$.*

Proof. We first consider the case $\Re s > 0$. Then the function $x \to x^{-s}$ is bounded on $[c, \infty)$, and hence the operators P_0^{-s} and P^{-s} are bounded. By the Melin formula we have

$$P^{-s} = (1 - \Pi_0(P))P_0^s = \frac{1}{\Gamma(s)} \int_0^\infty t^{1-2s}(1 - \Pi_0(P))e^{-tP} dt. \tag{10.138}$$

We let

$$A_s = \int_0^1 t^{s-1} e^{-tP} dt, \tag{10.139}$$

and observe that

$$\begin{aligned} \Gamma(s)P^{-s} - A_s &= \int_0^1 t^{s-1}\Pi_0(P)e^{-tP}dt + \int_1^\infty t^{s-1}(1 - \Pi_0(P))e^{-tP}dt \\ &= \frac{1}{2}\Pi_0(P) + e^{-P/2}\left(\int_0^\infty (1 + t)^{s-1}e^{-tP}dt\right)e^{-P/2}. \end{aligned} \tag{10.140}$$

Since $\Pi_0(P)$ and $\left(\int_0^\infty (1 + t)^{s-1}e^{-tP}dt\right)$ are bounded operators on $L^2(M)$ and $e^{-P/2}$ is a smoothing operator, it holds that

$$(\Gamma(s)P^{-s} - A_s)_{\Re s>0} \in \mathrm{Hol}(\Re s > 0,\ \Psi^{-\infty}(M)). \tag{10.141}$$

Now it suffices to show that $(A_s)_{\Re s>0}$ is a holomorphic family of $\Psi_H DO_s$ such that $\mathrm{ord} A_s = -vs$.

As we assumed that the principal kernel of $P + \partial_t$ is invertible, Theorem 10.6.9 implies that $(P+\partial_t)$ has an inverse $Q_0 = (P+\partial_t)^{-1}$ in $\Psi_{H,v}^{-v}(M\times\mathbb{R}_{(v)})$ and the distribution kernel $K_{Q_0}(x, y, t-s)$ of Q_0 is related to the heat kernel k_t of P by

$$k_t(x, y) = K_{Q_0}(x, y, t), \quad t > 0. \tag{10.142}$$

From this and (10.139) we see that A_s has the distribution kernel

$$k_{A_s}(x, y) = \int_0^1 t^{s-1}k_t(x, y)dt = \int_0^1 t^{s-1}K_{Q_0}(x, y, t)dt. \tag{10.143}$$

Now Lemma 10.8.2 says that for any local Carnot chart $\kappa : U \to V$ the family $(\kappa_* A_{sU})_{\Re s>0}$ is a holomorphic family of $\Psi_H DO_s$ on V of order $-vs$. Next, we take two smooth functions ϕ and ψ on M with disjoint supports. Then, from (10.149) we know that $\phi A_s \psi$ has the distribution kernel

$$k_{\phi A_s\psi}(x, y) = \int_0^1 t^{s-1}\phi(x)K_{Q_0}(x, y, t)\psi(y)dt. \tag{10.144}$$

Since the distribution kernel of a Volterra-$\Psi_H DO$ is smooth off the diagonal of $(M\times\mathbb{R})\times(M\times\mathbb{R})$ the distribution $K_{Q_0}(x,y,t)$ is smooth on the region $\{x\neq y\}\times\mathbb{R}$. Hence (10.150) defines a holomorphic family of smooth kernels, and so

$$(\phi A_s\psi)_{\Re s>0}\in \mathrm{Hol}(\Re s>0,\ \Psi^{-\infty}(M)). \tag{10.145}$$

Thus $(A_s)_{\Re s>0}$ is a holomorphic family of $\Psi_H DOs$, and so is $(P^s)_{\Re s<0}$.

Now, for $s\in C$ we take a positive integer k such that $k>\Re s$. Then on $C^\infty(M)$ it holds that $P^s=P^{s-k}P^k$. Then, as P^{s-k} is a $\Psi_H DO$ of order $v(s-k)$ and P^k is a $\Psi_H DO$ of order ks, we see that P^s is a $\Psi_H DO$ of order vs and Proposition 10.7.4 yields that $(P^s)_{s\in\mathbb{C}}$ is a holomorphic family of $\Psi_H DOs$ with $\mathrm{ord}P^s=vs$ for each $s\in\mathbb{C}$.

□

Lemma 10.8.2. *For a Carnot chart $V\subset\mathbb{R}^{d+1}$, we take $Q\in\Psi^{-v}_{H,v}(V\times\mathbb{R}_{(v)})$ with distribution kernel $K_Q(x,y,t-s)$. For $\mathcal{R}s>0$ let $B_s: C_c^\infty(V)\to C^\infty(V)$ be given by the distribution kernel,*

$$k_{B_s}(x,y)=\int_0^1 t^{s-1}K_Q(x,y,t)dt. \tag{10.146}$$

Then $(B_s)_{\Re s>0}$ is a holomorphic family of $\Psi_H DOs$ with $ordB_s=-vs$.

Proof. Denote by ε_x the Carnot coordinates at x. By (10.128) the distribution $K_Q(x,y,t)$ is given by the form

$$K_Q(x,y,t)=|\varepsilon_x`|K(x,-\varepsilon_x(y),t)+R(x,y,t), \tag{10.147}$$

where $R\in C^\infty(V\times V\times\mathbb{R})$ and $K\in K_v^{-Q}(V\times\mathbb{R}^Q_{(v)})$ having an expansion $K\sim\sum_{j\geq 0}K_{j-(d+2)}$ with $K_l\in\mathcal{K}_{v,l}(V\times\mathbb{R}^Q_{(v)})$. Thus, given any integer N, if J is large enough, we have

$$K(x,y,t)=\sum_{j\leq J}K_{j-(d+2)}(x,y,t)+R_{NJ}(x,y,t),\quad R_{NJ}\in C^N(V\times\mathbb{R}^Q). \tag{10.148}$$

Hence, on $V\times V$ we have

$$K_{B_s}(x,y)=|\varepsilon'_x|K_s(x,\varepsilon_x(y))+R_s(x,y),\quad K_s(x,y)=\int_0^1 t^{s-1}K(x,y,t)dt, \tag{10.149}$$

with $(R_s)_{\Re s>0}$ in $\mathrm{Hol}(\Re s>0, C^\infty(V\times V))$. Moreover, $K_s(x,y)$ is of the form

$$K_s=\sum_{j\leq J}K_{j,s}+R_{NJ,s},\quad K_{j,s}(x,y)=\int_0^1 t^{s-1}K_{j-(d+2)}(x,y,t)dt, \tag{10.150}$$

with $(R_{NJ,s})_{\Re s>0}$ in $\mathrm{Hol}(\Re s>0, C^N(V\times V))$.

We note that $K_{j-(d+2)}(x,y,t)$ belongs to $C^\infty(V)\hat{\otimes}\mathcal{D}'_{reg}(\mathbb{R}^Q\times\mathbb{R})$ and is parabolic homogeneous of degree $j-(d+2)\geq -(d+2)$. Thus $(K_{j,s})_{\Re s>0}$ belongs to $\mathrm{Hol}(\Re s>0, C^\infty(V)\hat{\otimes}\mathcal{D}_{reg}(\mathbb{R}^Q))$ and for $\lambda>0$, the diffrenece $K_{j,s}(x,\lambda\cdot y)-\lambda^{vs+j-Q}K_{j,s}(x,y)$ equals to

$$\int_1^{\lambda^2} t^{s-1}K_{j-(d+2)}(x,y,t)dt \ \in\ \mathrm{Hol}(\Re s>0, C^\infty(V\times\mathbb{R}^Q)). \tag{10.151}$$

Therefore $(K_{j,s})_{\Re s>0}$ is a holomorphic family of almost homogeneous distribution of degree $vs-(d+2)+j$. This with (10.150) tells that $(K_s)_{\Re s>0}$ is contained in $\mathrm{Hol}(\Re s>0, K^*(V\times\mathbb{R}^Q))$ with order $vs-(d+2)$. Then, by (10.149) and Proposition 4.4.5 we know that the family $(B_s)_{\Re s>0}$ is a holomorphic family of $\Psi_H DOs$ with $\mathrm{ord}B_s=-(\mathrm{ord}K_s+Q)=-vs$. The proof is completed. □

10.9 Spectral asymptotics for Hypoelliptic operators

Applying the heat kernel asymptotics we have

Proposition 10.9.1. *As $t\to 0^+$ we have*

$$Tre^{-tP}\sim t^{-\frac{Q}{m}}\sum_{j\geq 0}t^{\frac{2j}{m}}A_j(P),\quad A_j(P)=\int_M a_j(P)(x), \tag{10.152}$$

where the density $a_j(P)(x)$ is the coefficient of $t^{\frac{2j-Q}{m}}$ in the heat kernel asymptotic (5.1.20) for P.

Let $\lambda_0(P)\leq\lambda_1(P)\leq\cdots$ denote the eigenvalues of P counted with multiplicity and let $N(P;\lambda)$ be the counting function of P, that is,

$$N(P;\lambda)=\sharp\{k\in\mathbb{N};\lambda_k(P)\leq\lambda\},\quad \lambda\in\mathbb{R}. \tag{10.153}$$

In what follows, for given two functions $f:[0,\infty)\to[0,\infty)$ and $g:[0,\infty)\to[0,\infty)$, we shall use the notation $f(t)\sim g(t)$ when $t\to t_0$ for some $t_0\in[0,\infty]$ if the following limit holds

$$\lim_{t\to t_0}\frac{f(t)}{g(t)}=1. \tag{10.154}$$

Now we recall a Tauberian theorem from [**?**].

Theorem 10.9.2. *Let $\phi:[0,\infty)\to[0,\infty)$ is a positive and increasing function such that $\lim_{x\to\infty}\phi(x)=\infty$. In addition, we assume that for some $\sigma>0$,*

$$\phi(x)=x^\sigma L(x), \tag{10.155}$$

with L such that $L(cx)\sim L(x)$ for every positive c. Now we consider an increasing function $\alpha(t)$, and assume that $I(y)\sim\phi(y^{-1})$ when $y\to 0$. Then, we have

$$\alpha(t)\sim\frac{\phi(t)}{\Gamma(\sigma+1)} \tag{10.156}$$

when $t\to\infty$.

Combining the results of Proposition 10.9.1 and Theorem 10.9.2 we can attain the following result.

Proposition 10.9.3.

1. *It holds that* $A_0(P) > 0$.
2. *As* $\lambda \to \infty$, *it holds that*
$$N(P;\lambda) \sim \nu_0(P)\lambda^{\frac{Q}{m}}, \quad \nu_0(P) = CA_0(P). \tag{10.157}$$
3. *As* $k \to \infty$, *it holds that*
$$\lambda_k(P) \sim \left(\frac{k}{\nu_0(P)}\right)^{\frac{m}{Q}}. \tag{10.158}$$

Appendix

10.A Review on the class of symbols and kernels given at a point

Given a point $x_0 \in M$, in a privileged coordinates at x_0, the point x_0 is recorded as 0 and we have

$$X_j = \partial_j + \sum_{\substack{\langle\alpha\rangle \geq w_k - w_j \\ w_k > w_j}} b_\alpha x^\alpha \partial_k, \qquad b_\alpha \in \mathbb{R}. \tag{10.159}$$

Thus, for a polynomial $P \in C^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned} P(X_1, \cdots, X_n)f(0) &= P(\partial_1, \cdots, \partial_n)f(0) \\ &= \int_{\mathbb{R}^n} P(\xi_1, \cdots, \xi_n)\hat{f}(\xi)d\xi. \end{aligned} \tag{10.160}$$

Hence, at a point x_0 in a privileged coordinates at x_0, the differential operators can be expressed with a polynomial P which is a sum of homogeneou polynomials P_m of degree m with respect to the dilation $\cdot$, i.e., $P_m(\lambda \cdot \xi) = \lambda^m P(\xi)$ for $m \in \mathbb{N}_0$.

In order to study some differential operators of this kind and their inverses, and to find the explicit form of their kernels we need to introduce some necessary prerequisites;

1. $\mathcal{S}_m(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n \setminus \{0\})$: Set of homogeneous functions of degree m.

2. $\mathcal{S}_m^{ah}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$: Set of almost homogeneous functions (see Definition 10.2.1).

3. $\mathcal{G}_m(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$: Set of distributions $g \in \mathcal{S}'(\mathbb{R}^n)$ such that the singular support of g is contained in $\{0\}$ and there are constants c_α satisfying

$$g_\lambda = \lambda^k g + \sum_{\langle\alpha\rangle = -k - \langle n \rangle} c_\alpha (\lambda^k \log \lambda)\gamma^{(\alpha)}, \quad \lambda > 0. \tag{10.161}$$

Next we define the main class of the symbols which model the differential operators and the parametrices of invertible differential operators.

Definition 10.A.1. $\mathcal{S}^m(\mathbb{R}^n)$, $m \in \mathbb{Z}$ denotes the subspace of $C^\infty(\mathbb{R}^n)$ consisting of functions p which have an asymptotic expansion:

$$p(\xi) \sim \sum_{j=0}^{\infty} p_{m-j}(\xi), \quad p_m \in S_m(\mathbb{R}^n), \tag{10.162}$$

in the sense that for all multi-orders α, β and all $N > 0$, it holds that

$$\left| D_\xi^\beta \Big(p(\xi) - \sum_{j<N} p_{m-j}(\xi) \Big) \right| \le C_{\alpha\beta KN} \|\xi\|^{m-N-\langle\beta\rangle} \quad \forall\, |\xi| \ge 1. \tag{10.163}$$

There are some pros and cons related to $\mathcal{S}_m$, $\mathcal{S}_m^{ah}(\mathbb{R}^n)$, and $\mathcal{G}_m(\mathbb{R}^n)$ in representing a symbol $p \in \mathcal{S}^m(\mathbb{R}^n)$. First, in the definition (10.162), we note that although a symbol $p \in S^m(\mathbb{R}^n)$ is represented by functions $p_m \in S_m(\mathbb{R}^n)$, there is an important difference between them in the sense that $p \in C^\infty(\mathbb{R}^n)$ while $p_m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ can be very singular near the zero. Actually we can go around this difficuly using almost homogeneous symbols, that is, we may have

$$p(\xi) \sim \sum_{j=0}^{\infty} p_{m-j}^{ah}(\xi), \quad p_m^{ah} \in S_m^{ah}(\mathbb{R}^n), \tag{10.164}$$

in the sense that

$$\left| D_\xi^\beta \Big(p(\xi) - \sum_{j<N} p_{m-j}^{ah}(\xi) \Big) \right| \le C_{\alpha\beta KN} (1 + \|\xi\|)^{m-N-\langle\beta\rangle} \quad \forall\, \xi \in \mathbb{R}^n. \tag{10.165}$$

In spite of this advantage of the functions $\mathcal{S}_m^{ah}(\mathbb{R}^n)$, when we want to know the shape of kernels of their multipliers, it is convenient to exploit the fourier transform of homogeneous functions in $S^m(\mathbb{R}^n)$. More precisely, we shall modify an element of $S^m(\mathbb{R}^n)$ to a distribution in $\mathcal{G}_m(\mathbb{R}^n)$ and observe their distributional fourier transform, whose property can be attained relatively easily. For this aim, we shall prove the following result with taking a function $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\phi \equiv 1$ near the origin for a normalization purpose.

Proposition 10.A.2.

1. *If $g \in \mathcal{G}_m(\mathbb{R}^n)$ then the restriction of g to $\mathbb{R}^n \setminus \{0\}$ belongs to $\mathcal{S}_m(\mathbb{R}^n)$.*
2. *If $f \in \mathcal{S}_m(\mathbb{R}^n)$ then there is a $g \in \mathcal{G}_m(\mathbb{R}^n)$ which agrees with f on $\mathbb{R}^n \setminus 0$. There is a unique such g satisfying*
$$\langle g, \xi^\alpha \phi \rangle = 0 \quad \textit{whenever } \langle\alpha\rangle = -m - \langle n \rangle. \tag{10.166}$$

Proof. The proof of this result can be found in [BG, Proposition 15.8]. Assume that $k \le -\langle n \rangle$. For $f \in \mathcal{G}_k$, we define a distribution g which agrees with f on $\mathbb{R}^n \setminus 0$ and acts on $u \in \mathcal{S}(\mathbb{R}^n)$ as

$$\langle g, u \rangle = \int f(\xi) \Big\{ u(\xi) - \sum_{0 \le \langle\alpha\rangle \le -k-\langle n \rangle} (\alpha!)^{-1} u^{(\alpha)}(0) \xi^\alpha \phi(\xi) \Big\} d\xi. \tag{10.167}$$

Then, it must holds that

$$f = g + \sum_{\langle\alpha\rangle \le -Q-k} c_\alpha \gamma^{(\alpha)}. \tag{10.168}$$

The detail of this fact can be found in the proof of [BG, Proposition 15.8]. To be completed.. □

Remark 10.A.3. The formulas (10.167) and (10.168) reveals that $g \in \mathcal{G}_k$ is a bounded linear functional from $C^{[-\langle n\rangle - k]}(\mathbb{R}^n)$ to $\mathbb{R}$, where [ ] is the greatest integer function. Therefore $K(x,\cdot) \in \mathcal{K}_k(U \times \mathbb{R}^n)$ is a family of bounded linear functional from $C^{[-\langle n\rangle - k]}(\mathbb{R}^n)$ to $\mathbb{R}$ which is smooth in $x \in U$.

Now we shall consider the fourier transform of distribtuions in $\mathcal{G}_m(\mathbb{R}^n)$.

Definition 10.A.4. $\mathcal{K}_m(\mathbb{R}^n)$, $m \in \mathbb{C}$, consists of distributions $K(y)$ in $S'_{reg}(\mathbb{R}^n)$ such that for some constants $c_\alpha(x)$, $\langle\alpha\rangle = m$, we have

1. If $m \notin \mathbb{N}_0$, $K(\lambda \cdot y) = \lambda^m K(y) \quad \forall \lambda > 0$.
2. If $m \in \mathbb{N}_0$, $K(\lambda \cdot y) = \lambda^m K(y) + \lambda^m \log \lambda \sum_{\langle\alpha\rangle = m} c_{K,\alpha} y^\alpha \quad \forall\, \lambda > 0$.

Then we have the following result.

Proposition 10.A.5. *The inverse fourier transform is a bijection from $\mathcal{G}_k(\mathbb{R}^n)$ to $\mathcal{K}_{-k-Q}(\mathbb{R}^n)$.*

Proof. See the proof of [BG, Proposition 15. 24]. □

From the relation of Definition 10.A.4 we can easily derive the following important property.

Proposition 10.A.6. *Let $K \in \mathcal{K}_m$. Then*

$$K(x) = \begin{cases} f(x) + p(x)\log\|x\|, & x \neq 0, \quad m \in \mathbb{N}_0, \\ f(x) & \quad\quad m \notin \mathbb{N}_0, \end{cases} \tag{10.169}$$

where $f \in S_m(\mathbb{R}^n)$ and p is a homoegenous polynomial of degree m.

Proof. See the proof of [BG, Proposition 15.21]. □

Proposition 10.A.7 ([BG])**.** *Suppose $f \in C^\infty(\mathbb{R}^n)$. The followings are equivalent:*

1. *f is almost homogeneous of degree m.*
2. *$f \in \mathcal{S}^m(\mathbb{R}^n)$ and f has a single term, of degree m, in its asymptotic expansion.*
3. *There is $g \in \mathcal{S}_m(\mathbb{R}^n)$ such that for any cut-off function $\chi \in C^\infty(\mathbb{R}^n)$ with $\chi \equiv 0$ near the origin and $\chi \equiv 1$ at ∞,*

$$f - \chi g \in \mathcal{S}(\mathbb{R}^n). \tag{10.170}$$

Moreover the function g in (c) is unique, given by

$$g(\xi) = \lim_{\lambda\to\infty} \lambda^{-m} f(\lambda\cdot\xi). \tag{10.171}$$

We call g the homogeneous part of f.

Similarly to Proposition 10.A.5 the fourier transform of almost homogeneous functions are contained in the following set.

Definition 10.A.8. For $m \in \mathbb{C}$ the set $\mathcal{K}_m^{ah}(\mathbb{R}^d)$ consists of almost homogeneous distributions $K(y) \in \mathcal{D}'_{reg}(\mathbb{R}^n)$ of degree m in the sense that

1. $K(y) \in C'(\mathbb{R}^n) + S(\mathbb{R}^n)$.
2. $K(y) - \lambda^m K(y) \in C^\infty(\mathbb{R}^d) \quad \text{for any } \lambda > 0.$

10.A.1 Micellaneous

Definition 10.A.9. A continuous linear map $T : C_c^\infty \to C^\infty(U)$ is properly supported when for each compact set $K \subset U$, there are two compact sets $K' \subset U$ and $K'' \subset U$ with $K \subset K'$, $K \subset K''$ and satisfying the properties

1. $\operatorname{supp} \subset K \Rightarrow \operatorname{supp} Tu \subset K'$.
2. $K'' \cap \operatorname{supp} u = \phi \Rightarrow K \cap \operatorname{supp} Tu = \phi$.

We introduce the notion of almost homogeneity for kernels.

Definition 10.A.10. For $m \in \mathbb{C}$ the set $\mathcal{K}_m^{ah}(U\times\mathbb{R}^d)$ consists of almost homogeneous distribution $K(x,y) \in C^\infty(U)\hat{\oplus}\mathcal{D}'_{reg}(\mathbb{R}^n)$ of degree m in the sense that

1. $K(x,y) \in C^\infty(U)\hat{\oplus}(C'(\mathbb{R}^n) + S(\mathbb{R}^n))$.
2. $K(x,\lambda\cdot y) - \lambda^m K(x,y) \in C^\infty(U\times\mathbb{R}^d) \quad \text{for any } \lambda > 0.$

We then easily get the following result.

Proposition 10.A.11. *Let $K(x,y) \in C^\infty(U)\hat{\oplus}\mathcal{D}'_{reg}(\mathbb{R}^n)$. Then the following are equivalent:*

1. *$K(x,y)$ belongs to $\mathcal{K}_m^{ah}(U\times\mathbb{R}^d)$.*
2. *We can put $K(x,y)$ in the form,*

$$K(x,y) = K_m(x,y) + R(x,y), \tag{10.172}$$

for some $K_m \in \mathcal{K}_m(U\times\mathbb{R}^n)$ and $R \in C^\infty(U\times\mathbb{R}^n)$.

10.B Technical computations

This appendix is devoted to prove Lemma 10.B.2 and Lemma 10.B.3. We begin with a preliminary lemma.

Lemma 10.B.1. *Suppose that a smooth family of maps $L_z : \mathbb{R}^n \to \mathbb{R}^n$ for $z \in \mathbb{R}^n$ is of the form $L_z(x) = x \cdot z + O((x,z)^{w+1})$ for (x,z) near $(0,0)$. Then we have*

$$(L_z)^{-1}(x) = x \cdot (-z) + O((x,z)^{w+1}) \tag{10.173}$$

for (x,z) near $(0,0)$.

Proof. Let us denote $T_z(x) = (L_z)^{-1}(x)$ and take a value $k \in \{1, \cdots, d\}$. As the map $(z,x) \to T_z(x)$ is smooth, by a Taylor expansion, we may let

$$(T_z)_k(x) = \sum_{l=0}^{w_k-1}\Big(\sum_{\langle a\rangle+\langle b\rangle=l} C_{kab}x^a z^b\Big) + \sum_{\langle a\rangle+\langle b\rangle\geq w_k} C_{kab}x^a z^b, \tag{10.174}$$

for some constants C_{kab}. To prove the lemma, we first aim to show that $\sum_{\langle a\rangle+\langle b\rangle=l} C_{kab}x^a z^b \equiv 0$ for any $0 \leq l \leq w_k - 1$ in the above formula, i.e.,

$$(T_z)_k(x) = \sum_{\langle a\rangle+\langle b\rangle\geq w_k} C_{kab}x^a z^b. \tag{10.175}$$

In order to prove this, we shall get a contradiction after assuming that $\sum_{\langle a\rangle+\langle b\rangle=l} C_{kab}x^a z^b \neq 0$ holds for some value $0 \leq l \leq w_k - 1$. In that case, there exists a minimum value l_0 such that $\sum_{\langle a\rangle+\langle b\rangle=l_0} C_{kab}x^a z^b \neq 0$. Then, (10.174) is written as

$$(T_z)_k(x) = \sum_{l=l_0}^{w_k-1}\Big(\sum_{\langle a\rangle+\langle b\rangle=l} C_{kab}x^a z^b\Big) + \sum_{\langle a\rangle+\langle b\rangle\geq w_k} C_{kab}x^a z^b, \tag{10.176}$$

We write the identity $(T_z)_k((L_z)(x)) = x_k$ using (10.176) to get

$$x_k = \sum_{\langle a\rangle+\langle b\rangle=l_0} C_{kab}L_z(x)^a z^b + \cdots + \sum_{\langle a\rangle+\langle b\rangle=w_k-1} C_{kab}L_z(x)^a z^b + \sum_{\langle a\rangle+\langle b\rangle\geq w_k} C_{kab}L_z(x)^a z^b \tag{10.177}$$

Injecting $L_z(x) = x \cdot z + O((x,z)^{w+1})$ here, we easily see that (RHS) of (10.177) has its lowest weight term

$$\sum_{\langle a\rangle+\langle b\rangle=l_0} C_{kab}(x \cdot z)^a z^b, \tag{10.178}$$

whose weight is equal to l_0. On the other hand, in (LHS) of (10.177), the term with the lowest weight is equal to x_k whose weight is w_k. As $l_0 < w_k$, this is a contradiction. Hence (10.175) should hold.

Now, we write (10.175) as

$$(T_z)_k(x) = \sum_{\langle a\rangle+\langle b\rangle=w_k} C_{kab}x^a z^b + \sum_{\langle a\rangle+\langle b\rangle>w_k} C_{kab}x^a z^b \tag{10.179}$$

and (10.177) becomes

$$\begin{aligned} x_k &= \sum_{\langle a\rangle+\langle b\rangle=w_k} C_{kab}L_z(x)^a z^b + \sum_{\langle a\rangle+\langle b\rangle>w_k} C_{kab}L_z(x)^a z^b \\ &= \sum_{\langle a\rangle+\langle b\rangle=w_k} C_{kab}(x\cdot z)^a z^b + \sum_{\langle a\rangle+\langle b\rangle>w_k} \widetilde{C}_{ab}x^a z^b, \\ &= \sum_{\langle a\rangle+\langle b\rangle=w_k} C_{kab}(x\cdot z)^a z^b, \end{aligned} \tag{10.180}$$

where the second equality follows using that $L_z(x) = x\cdot z + O((x,z)^{w+1})$ and the last equality holds as the left hand side is homogeneous of degree w_k. Taking $x \to x\cdot z^{-1}$ in (10.180), we get

$$(x\cdot z^{-1})_k = \sum_{\langle a\rangle+\langle b\rangle=w_k} C_{kab}x^a z^b. \tag{10.181}$$

Injecting this into (10.179) we have

$$(T_z)_k(x) = (x\cdot z^{-1})_k + \sum_{\langle a\rangle+\langle b\rangle>w_k} C_{kab}x^a z^b. \tag{10.182}$$

This proves the lemma. □

We are ready to prove Lemma 10.B.2;

Lemma 10.B.2. *There exists a constant $C > 0$ such that*

$$\|\varepsilon_y(z) - \varepsilon_y(x)\| \le C\left(\|\varepsilon_x(z)\| + \|\varepsilon_x(z)\|^{\frac{1}{m}}\|\varepsilon_y(z)\|^{1-\frac{1}{m}}\right) \tag{10.183}$$

holds for any $(x,z,y)\in(\mathbb{R}^n)^3$ such that $\|\varepsilon_y(z)\|\le 1$ and $\|\varepsilon_y(z)\|\le 1$.

Proof. Taking $z \to \varepsilon_y^{-1}(z)$ and $x \to \varepsilon_y^{-1}(x)$, we see that (10.183) is equivalent to

$$\begin{aligned} \|x-z\| &\le C\left(\|\varepsilon_{\varepsilon_y^{-1}(x)}\circ\varepsilon_y^{-1}(z)\| + \|\varepsilon_{\varepsilon_y^{-1}(x)}\circ\varepsilon_y^{-1}(z)\|^{\frac{1}{m}}\|z\|^{1-\frac{1}{m}}\right) \\ &= C\left(\|T_z(x)\| + \|T_z(x)\|^{\frac{1}{m}}\|z\|^{1-\frac{1}{m}}\right), \end{aligned} \tag{10.184}$$

where we have let $T_z(x) = \varepsilon_{\varepsilon_y^{-1}(x)}\circ\varepsilon_y^{-1}(z)$ in the last equality. By Lemma 9.6.23 we have

$$T_z(x) = x\cdot(-z) + O((x,z)^{w+1}), \tag{10.185}$$

and using Lemma 10.B.1 we have

$$(T_z)^{-1}(x) = x\cdot z + O((x,z)^{w+1}). \tag{10.186}$$

Letting $x \to (T_z)^{-1}x$ in (10.184), it is equivalent to prove that

$$\|(T_z)^{-1}(x) - z\| \le C(\|x\| + \|x\|^{\frac{1}{m}}\|z\|^{1-\frac{1}{m}}). \tag{10.187}$$

To show this inequality, we are concerned with the weight to see that $(T_z)^{-1}(x) - z$ is written as

$$\begin{aligned}(T_z)^{-1}(x) - z &= (x \cdot z - z)_j + O((x,z)^{w+1}) \\ &= \Bigg(\sum_{\langle a\rangle + \langle b\rangle \ge w_1} C_{1,ab} x^a z^b, \cdots, \sum_{\langle a\rangle + \langle b\rangle \ge w_d} C_{d,ab} x^a z^b \Bigg)\end{aligned} \tag{10.188}$$

In addition, observing that $(T_z)^{-1}(0) - z = z - z = 0$, we may get further as

$$(T_z)^{-1}(x) - z = \Bigg(\sum_{\substack{\langle a\rangle + \langle b\rangle \ge w_1 \\ |a|\ge 1}} C_{1,ab} x^a z^b, \cdots, \sum_{\substack{\langle a\rangle + \langle b\rangle \ge w_d \\ |a|\ge 1}} C_{d,ab} x^a z^b \Bigg) \tag{10.189}$$

For $\|x\| \le 2$ and $\|y\| \le 2$, using that $|x^a| \le \|x\|^{\langle a\rangle}$ and $|z^b| \le \|z\|^{\langle b\rangle}$, and Young's inequality we have

$$\begin{aligned}\Bigg\| \sum_{\substack{\langle a\rangle + \langle b\rangle \ge w_j \\ |a|\ge 1}} C_{1,ab} x^a z^b \Bigg\|^{\frac{1}{w_j}} &\le C\Bigg(\sum_{\substack{\langle a\rangle + \langle b\rangle \ge w_j \\ |a|\ge 1}} \|x\|^{\langle a\rangle}\|z\|^{\langle b\rangle} \Bigg)^{\frac{1}{w_j}} \\ &\le C \sum_{\substack{\langle a\rangle + \langle b\rangle \ge w_j \\ |a|\ge 1}} \|x\|^{\frac{\langle a\rangle}{w_j}}\|z\|^{\frac{\langle b\rangle}{w_j}} \\ &\le C(\|x\| + \|x\|^{\frac{1}{w_j}}\|z\|^{1-\frac{1}{w_j}}) \le C(\|x\| + \|x\|^{\frac{1}{m}}\|z\|^{1-\frac{1}{m}})\end{aligned} \tag{10.190}$$

From this and definition (10.3), we deduce from (10.189) and (10.190) that

$$\|(T_z)^{-1}(x) - z\| \le C\Big(\|x\| + \|x\|^{\frac{1}{m}}\|z\|^{1-\frac{1}{m}}\Big). \tag{10.191}$$

The lemma is proved. □

Next we prove the following lemma.

Lemma 10.B.3. *Let* $\Phi_t(x,z,y) = z \cdot y + t\left(\varepsilon_{\varepsilon_x^{-1}(-z)} \circ \varepsilon_x^{-1}(y) + z \cdot y\right)$. *Then we have*

$$y^w = \sum_{\langle p\rangle + \langle q\rangle \ge \langle w\rangle} a_{pqw}(t) z^p \Phi_t(x,z,y)^q, \tag{10.192}$$

where $a_{pqw}(t)$ *are constants determined by the group law* $\cdot$ *and depend on* t *smoothly.*

Proof. Let $L_{t,x,z}(y) = \Phi_t(x,z,y)$. Note that $\Phi_t(x,z,y) = z \cdot y + O((z,y)^{w+1})$ by Lemma 9.6.23. Then, we apply Lemma 10.B.1 to see

$$(L_{t,x,z})^{-1}(y) = (-z) \cdot y + O((z,y)^{w+1}). \tag{10.193}$$

Using this we have

$$\begin{aligned}(L^{-1}_{t,x,z}(y))^w &= \Big((-z)\cdot y + O((z,y)^{w+1})\Big)^w \\ &= \sum_{\langle p\rangle+\langle q\rangle\geq\langle w\rangle} a_{pqw}(t) z^p y^q,\end{aligned} \tag{10.194}$$

where $a_{pqw}(t)$ is a constant depending on t smoothly and determined by the group law $\cdot$. Injecting $y \to \Phi_t(x,z,y)$ into (10.194), we get

$$y^w = \sum_{\langle p\rangle+\langle q\rangle\geq\langle w\rangle} a_{pqw}(t) z^p \Phi_t(x,z,y)^q, \tag{10.195}$$

which is the desired result. □

10.C Some properties of distributions

In this section, we shall have some definitions related to distribution kernels and the properties.

For $K \in D'(\mathbb{R}^n)$ we define the distribution kernel $K(x+y)$ for each $y \in \mathbb{R}^n$ and $\partial^\alpha K(x)$ for each multi-index $\alpha \in \mathbb{N}_0^n$ by

1. $\langle K(x+y), f(x)\rangle := \langle K(x), f(x-y)\rangle$.
2. $\langle \partial^\alpha K(x), f(x)\rangle := \langle K(x), (-\partial)^\alpha f(x)\rangle$.

Then we have the following lemma.

Lemma 10.C.1. *We have*

$$\begin{aligned}&\Big\langle K(x+\gamma(x), y+\theta(x,y)), f(y)\Big\rangle \\ &\quad= \sum_{|\alpha|+|\beta|\leq N} \frac{1}{\alpha!\beta!}\Big\langle \partial_1^\alpha\partial_2^\beta K(x,y)\gamma(x)^\alpha\theta(x,y)^\beta, f(y)\Big\rangle + \Big\langle R_N(x,y), f(y)\Big\rangle,\end{aligned} \tag{10.196}$$

where

$$R_N(x,y) = \int_0^1 \frac{N+1}{\alpha!\beta!}\Big[\partial_1^\alpha\partial_2^\beta K(x+t\gamma(x), y+t\theta(x,y))\gamma(x)^\alpha\theta(x,y)^\beta\Big](1-t)^N dt. \tag{10.197}$$

Proof. Let us set

$$K_\delta(x,y) = (K(x,\cdot) * \phi_\delta(\cdot))(y), \tag{10.198}$$

where $*$ denotes the usual convolution on $\mathbb{R}^n$, .e., $(f*g)(x) = \langle f(y),\ g(x-y)\rangle$ for a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$. Next, we recall the Taylor expansion formula: For $r > 0$ and any smooth function $f : (-s,s) \to \mathbb{R}$ it holds that

$$f(r) = \sum_{k=0}^N \frac{f^{(k)}(0)}{k!} r^k - \int_0^1 \frac{(1-t)^N}{N!}\left(\frac{d}{dt}\right)^{N+1} f(rt)\, dt \tag{10.199}$$

for all $r \in (-s, s)$ and $N \in \mathbb{N}$.

As K_δ is contained in $C^\infty(U \times U)$, we may apply the Taylor expansion in the classical sense to get

$$\begin{aligned}&\Big\langle K_\delta(x+\gamma(x), y+\theta(x,y)), f(y)\Big\rangle\\&\quad = \sum_{|\alpha|+|\beta|\le N} \frac{1}{\alpha!\beta!}\Big\langle \partial_1^\alpha \partial_2^\beta K_\delta(x,y)\gamma(x)^\alpha \theta(x,y)^\beta, f(y)\Big\rangle + \Big\langle R_{N,\delta}(x,y), f(y)\Big\rangle,\end{aligned} \tag{10.200}$$

where

$$R_{N,\delta}(x,y) = \int_0^1 \frac{N+1}{\alpha!\beta!}\Big[\partial_1^\alpha \partial_2^\beta K_\delta(x+t\gamma(x), y+t\theta(x,y))\gamma(x)^\alpha\theta(x,y)^\beta\Big](1-t)^N dt. \tag{10.201}$$

Note that $\partial_1^\alpha \partial_2^\beta K_\delta(x,y) = \partial_1^\alpha \partial_2^\beta K(x,\cdot) * \phi_\delta(\cdot)(y)$. Letting $\delta \to 0$ in the above identity, we get

$$\begin{aligned}&\Big\langle K(x+\gamma(x), y+\theta(x,y)), f(y)\Big\rangle\\&\quad = \sum_{|\alpha|+|\beta|\le N} \frac{1}{\alpha!\beta!}\Big\langle \partial_1^\alpha \partial_2^\beta K(x,y)\gamma(x)^\alpha \theta(x,y)^\beta, f(y)\Big\rangle + \Big\langle R_N(x,y), f(y)\Big\rangle,\end{aligned} \tag{10.202}$$

The proof is completed. □

We conclude this section with proving the following lemma.

Lemma 10.C.2. *Suppose that $D_x(\cdot)$ is a distribution which is smooth in $x \in U$ such that*

$$\langle D_x(\cdot), g(\cdot)\rangle \le C \sum_{|\alpha|\le M} \sup_{x\in U} |\partial^\alpha g(x)|. \tag{10.203}$$

Consider a function $f_x(\cdot,\cdot) \in C^{M+N+2}(\mathbb{R}^n \times \mathbb{R}^n)$ which is smooth in $x \in U$. For $(x,y) \in U \times U$ we let

$$K(x,y) = \langle D_x(\cdot), f_x(y,\cdot)\rangle. \tag{10.204}$$

Then, we have $K(x,y) \in C^N(U \times U)$.

Proof. In order to show that $K(x,y)$ is defferential with respect to y, we write

$$\frac{K(x,y+he_1) - K(x,y)}{h} = \langle D_x(z), \frac{f_x(y+he_1,z) - f_x(y,z)}{h}\rangle. \tag{10.205}$$

Using the Taylor expansion, we have

$$f_x(y+he_1,z) = f_x(y,z) + h\frac{\partial}{\partial y_1} f_x(y,z) - \frac{h^2}{2!}\int_0^1 (1-t)^2 \frac{\partial^2}{(\partial y_1)^2} f_x(y+the_1,z)dt. \tag{10.206}$$

Injecting (10.206) into (10.205), we have

$$\frac{K(x,y+he_1)-K(x,y)}{h} = \langle D_x(z), \frac{\partial}{\partial y_1} f_x(y,z)\rangle - \frac{h}{2}\int_0^1 (1-t)^2 \Big\langle D_x(z), \frac{\partial^2}{(\partial y_1)^2} f_x(y+the_1,z)\Big\rangle dt,$$

where the right hand side is well defined as

$$\frac{\partial^2}{(\partial y_1)^2} f_x(y+the_1,\cdot) \in C^{M+N}(\mathbb{R}^n) \subset C^M(\mathbb{R}^n). \tag{10.207}$$

Taking the limit $h \to 0$, we get

$$\frac{\partial}{\partial y_1} K(x,y) = \lim_{h\to 0} \frac{K(x,y+he_1)-K(x,y)}{h} = \langle D_x(z), \frac{\partial}{\partial y_1} f_x(y,z)\rangle.$$

This shows that $K(x,y)$ is differentiable with respect to y_1 variable. In fact, we can adapt the above argument whenever the condition like (10.207) holds. Therefore we may show that

$$K(x,y) \in C^N(U \times U). \tag{10.208}$$

The lemma is proved. □

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국문초록

논문의 구성은 크게 다음의 세 부분으로 나누어져 있다; 선형작용소의 정밀한 계측, 반선형 타원형 방정식, 그리고 캐늣 다양체위에서의 의미분 연산. 이 주제들은 직접적이거나 간접적으로 서로 연관이 되어있다.

첫 부분의 저자의 논문 [Ch1, Ch2, Ch3] 을 바탕으로 하고 진동작용소와 분광 곱 연산자에 관한 정밀 계측을 얻는 것을 목표로 한다. 좀 더 구체적으로, 첫번째 논문 [Ch1]에서는 하이젠베르그 군에서 정의된 강한 특수성을 가진 작용소의 L^2 공간과 H^p 공간에서의 바운드를 보인다. L^2 공간 바운드를 위해 퇴화된 형태의 진동작용소 계측을 이용하고, H^p 공간 바운드를 위해서는 하디 공간의 분자 분해를 이용한다. 두번째 논문 [Ch2] 에서는 층상화된 군들에서 곱 작용소들의 최대함수들에 대한 정밀화된 L^p 바운드를 구한다. 또한 층상화된 군들의 곱형태의 군에서도 관련된 바운드를 얻고, 하나의 응용으로 하이젠베르그 군에서 결합 분광 곱 작용소들의 최대함수에 대해서도 정밀화된 L^p 바운드를 얻는다. 세번째 논문 [Ch3] 에서는 바운드가 없는 옹골한 다양체 위에서 정의된 양의 자체 수반 타원형 미분 작용소 P가 있을때, 헤르만더-미흘린 조건아래에서 이 작용소와 관련된 분광 곱 작용소들의 최대함수에 대한 정밀화된 L^p 바운드를 구한다.

두번째 부분은 반선형 타원형 방정식들에 대한 공부이고, 논문 [Ch4]와 공동 논문 [CKL, CKL2, ChS]을 기반으로 되어있다.

논문 [Ch4]에서 우리는 유한 영역내에서 분수 라플라시안을 포함하며 강하게 엮여있는 시스템에 대해서 연구한다. 구체적으로, 우리는 존재성과 비존재성에 관한 결과들을 보이고, 기다스-스프럭 형태의 선 계측, 대칭 구조에 관한 결과를 보인다. 여기서 우리는 논문 [CT, T1] 에서 보여졌던 비선형 타원형 방정식들에 대한 선 계측에 대해서 새로운 증명을 얻는다.

김승혁 박사님, 이기암 교수님과의 공동 논문인 [CKL]에서는 분수 라플라시안을 포함한 비선형 타원형 방정식들에 대해서 임계지수와 관련되어 최소 에너지 해들의 점근 행동을 공부하고, 다중으로 버블링하는 해들의 존재성을 공부한다. 이것은 Han (1991) [H] 과 Rey (1990) [R] 결과의 비국부적 버전이라고 할 수 있다.

석진명 교수님과 함께 연구한 논문 [ChS]에서 우리는 옹골성이 없는 비국부적 반선형 타원형 방정식에 대해서 공부한다. 구체적으로, 우리는 유한 영역내에서 분수 계수 버전의 브레지스-니렌버르그 문제가 무한해를 갖는다는 것을 증명한다.

이 파트의 마지막 챕터는 김승혁 박사님, 이기암 교수님과의 공동 연구 논문 [CKL2] 을 바탕으로 한다. 이 논문의 목적은 3차원 이상에서 레인-엠덴-파울러 방정식의 임계지수근처에서 다중 버블링하는 해들에 대한 질적 성질들을 얻는데 있다. 각각의 m 버블 해들에서 선형화된 문제를 공부하여, 우리는 처음 $(n+2)m$개의 고유함수와 고유치에 대해서 정확한 계측들을 보인다. 특별히, 우리는 4차원이상에서 다중 버블 해의 모스-인덱스가 그란함수, 로빈함수들의 일차, 이차 미분들로 이루어진 대칭 행렬들로 규명되된다는 Bahri-Li-Rey (1995) 에 의한 고전적인 결과에 대한 새로운 증명을 제시한다. 우리의 증명은 3차원일 경우에도 적용이 된다.

세번째 파트는 라파엘 폰즈교수님과 함께한 논문 [CP1, CP2]를 바탕으로 쓰여졌다. 논문 [CP1] 에서는 캐늣 다양체의 내부적으로 주어진 접한 군 다발들을 정의하고 우선적 좌표에

대해서 공부를 한다. 이를 통해서 캐놋 다양체의 매끈한 접 이군을 정의한다. 이러한 공부들을 바탕으로 논문 [CP2] 에서는 캐놋 다양체위에서의 의미분 작용소에 대한 공부를 합니다. 적절한 의미분 작용소들의 모임을 정의하고 이 작용소들의 계산법을 정확히 구한다. 구체적으로는, 결합, 수반 작용소, 좌표 변환에 관한 구체적인 커널 전개를 구한다. 이것을 통해 우리는 약한 타원성을 가진 미분 작용소들의 역의 구체적인 커널 전개 표현을 얻어낼 수 있다. 또한 관련된 열 미분 작용소에 대한 열 커널 전개도 얻을 수 있다. 이것의 한 응용으로 케놋 다양체위에서의 분광 밴드의 성질을 공부할 수 있다.

주요어휘: 반선형 타원형 방정식, 분수계수 라플라시안, 진동 작용소 계측, 최대 푸리에 작용소, 캐놋 다양체, 의미분 작용소 연산
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